Supplemental Material: Buyer-Optimal Learning and Monopoly Pricing

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The goal of this note is to characterize buyer-optimal outcomes with minimal learning in environments where the seller's valuation is positive and the buyer's valuation can be negative.

1 Model

There is a seller who has an object to sell to a single buyer. The seller's value for the object is c. The buyer's valuation, v, is distributed according to the CDF F supported on $(-\infty, 1]$.[1] Let μ denote the expected valuation, that is, $\int_0^1 v dF(v) = \mu$. The buyer observes a signal s about v. The joint distribution of v and s is common knowledge. The seller then gives a take-it-or-leave-it price offer to the buyer, p. Finally, the buyer trades if and only if her expected valuation conditional on her signal weakly exceeds p. If trade occurs, the payoff of the seller is p and the payoff of the buyer is $v - p$; otherwise, both the buyer's payoff is zero and the seller's payoff is c. Both the seller and the buyer are von Neumann-Morgenstern expected payoff maximizers. In what follows we fix the CDF F and analyze those signal structures which maximize the buyer's expected payoff.

We may assume without loss of generality that each signal s provides the buyer with an unbiased estimate about her valuation, that is, $E(v|s) = s$. The reason is that the buyer only needs to know $E(v|s)$ in order to decide whether to trade at a given price, so it does not matter whether she observes s or $E(v|s)$. In what follows, we restrict attention to unbiased signal structures.

2 Buyer-optimal Signal Structures

First, we argue that the payoffs of both the buyer and the seller are determined by the unconditional distribution of the signal. To this end, let $D(G, p)$ denote the demand at price p

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[1] It is not hard but cumbersome to generalize our results for Fs which are supported on $\mathbb{R}$.

if the signal's distribution is G, that is, $D(G,p)$ is the probability of trade at p. Note that $D(G,p) = 1 - G(p) + \Delta(G,p)$ where $\Delta(G,s)$ denotes the the probability of s according to the CDF G.[2] The seller's optimal price, p, solves $\max_s (s-c)D(G,s)$ and the buyer's payoff is $\int_p^1 (s-p)dG(s)$. Therefore, the problem of designing a buyer-optimal signal structure can be reduced to identifying the unconditional signal distribution which maximizes the buyer's expected payoff subject to monopoly pricing. Of course, not every CDF corresponds to a signal distribution. In what follows, we characterize the set of distributions that do.

For each unbiased signal structure, v can be expressed as $s+\varepsilon$ for a random variable ε with $E(\varepsilon|s) = 0$. This means that G is the distribution of *some* unbiased signal about v if and only if F is a mean-preserving spread of G (see Definition 6.D.2 of Mas-Colell et al., (1995)). Let $\mathcal{G}_F$ denote the set of CDFs of which F is a mean-preserving spread. By Proposition 6.D.2 of Mas-Colell et al. (1995) this set can be defined as follows

$$\mathcal{G}_F = \left\{ G \in \mathcal{G} : \int_{-\infty}^{x} F(v)\, dv \geq \int_{-\infty}^{x} G(s)\, ds \text{ for all } x \in [0,1], \int_{-\infty}^{1} s dG(s) = \mu \right\}. \tag{1}$$

The problem of designing a buyer-optimal signal structure can be stated as follows

$$\max_{G \in \mathcal{G}_F} \int_p^1 (s-p)\, dG(s)$$
$$\text{s.t. } p \in \arg\max_s (s-c)\, D(G,s).$$

In what follows, we call a pair (G,p) an *outcome* if $G \in \mathcal{G}_F$ and $p \in \arg\max_s (s-c)\, D(G,s)$. In other words, the pair (G,p) is an outcome if there exists an unbiased signal about v which is distributed according to the CDF G and it induces the seller to set price p.

Next, we define a set of distributions and prove that a buyer-optimal signal distribution lies in this set. For each $q \in (c,1]$ and $B \in [q,1]$ let the CDF G_q^B be defined as follows:

$$G_q^B(s) = \begin{cases} 0 & \text{if } s \in [0,q), \\ 1 - \frac{q-c}{s-c} & \text{if } s \in [q,B), \\ 1 & \text{if } s \in [B,1]. \end{cases} \tag{2}$$

Observe that the support of G_q^B is $[q,B]$ and it specifies an atom of size $(q-c)/(B-c)$ at B. An important attribute of each CDF in this class is that the seller is indifferent between charging any price on its support. Notice that G_q^q is a degenerate distribution which specifies an atom of size one at q.

Proposition 1 *Suppose that (G,p) is a buyer-optimal outcome and G involves minimum learning among all buyer-optimal signal structures. Let λ denote $\int_{[-\infty,p)} s dG(s) / [G(p) - \Delta(G,p)]$ if $G(p) > \Delta(G,p)$ and zero otherwise. Then there exists a $B \in [p,1]$ such that*

(i) $G(s) = 0$ on $s \in [0,\lambda)$ and $G(s) = G(p)$ on $[\lambda,p)$.

(ii) $G_p^B(s) = [G(s) - G(p)] / [1 - G(p)]$ if $s \geq p$.

[2] Formally, $\Delta(G,s) = G(s) - \sup_{x<s} G(x)$. If G has no atom at s then $\Delta(G,s) = 0$.

Part (i) states that there is a single signal, λ, which the buyer might observe below p. In other words, with some probability, the buyer learns that her valuation is below the price but receives no additional information regarding her valuation. If the buyer's value is very likely to be above the seller's cost c, then the buyer might never observes such a signal and under the buyer-optimal signal trade occurs with probability one. Otherwise, λ is the buyer's expected value conditional on $s < p$. Part (ii) says that, conditional on observing a signal above p, the signal is distributed according to G_p^B for some B, that is, the seller is indifferent between any price on $[p, B]$.

Proof. Let (G, p) be a buyer-optimal outcome. We first argue that there exists a unique $B \geq p$ such that G_p^B generates the same expected value as $G(\cdot|s \geq p)$. Since $p \in \arg\max_s (s-c) D(G, s)$, $(s-c) D(G,s) \leq (p-c) D(G,p)$ for all $s \in [0,1]$. Using the definitions of $D(G,p)$, this inequality can be rewritten as

$$1 - \frac{p-c}{s-c} D(G,p) + \Delta(G,s) \leq G(s).$$

Since $\Delta(G,s) \in [0,1]$, the previous inequality implies

$$1 - \frac{p-c}{s-c} D(G,p) \leq G(s),$$

or equivalently,

$$\frac{1}{D(G,p)} - \frac{p-c}{s-c} \leq \frac{G(s)}{D(G,p)}.$$

Subtracting $[G(p) - \Delta(G,p)]/D(G,p)$ from both sides yields

$$1 - \frac{p-c}{s-c} \leq \frac{G(s) - G(p) + \Delta(G,p)}{D(G,p)}. \tag{3}$$

Note that the right-hand side of this inequality is the signal distribution conditional on $s \geq p$, so the previous inequality is just

$$G_p^1(s) \leq G(s|s \geq p), \tag{4}$$

that is, G_p^1 first-order stochastically dominates $G(\cdot|s \geq p)$. This implies that

$$\int_p^1 s dG_p^1(s) \geq \int_p^1 s dG(s|s \geq p) \geq pD(G,p) = \int_p^1 s dG_p^p(s),$$

where the first inequality follows from first-order stochastic dominance, and the last equality from G_p^p specifying an atom of size one at $s = p$. Since $\int_p^1 s dG_p^B(s)$ is continuous and strictly increasing in B, the Intermediate Value Theorem implies that there exists a unique $B \in [p, 1]$ such that

$$\int_p^1 s dG_p^B(s) = \int_p^1 s dG(s|s \geq p). \tag{5}$$

For each buyer-optimal outcome (G, p) we can define a signal distribution $\widetilde{G}$ as follows

$$\widetilde{G}(s) = \begin{cases} 0 & \text{if } s \in [0, \lambda), \\ G(p) - \Delta(G,p) & \text{if } s \in [\lambda, p), \\ D(G,p)\left(1 - \frac{p-c}{s-c}\right) + G(p) - \Delta(G,p) & \text{if } s \in [p, B) \\ 1 & \text{if } s \in [B, 1], \end{cases}$$

where $\lambda = \int_{[-\infty,p)} sdG(s) / [G(p) - \Delta(G,p)]$ and B is defined so that (5) is satisfied. Note that $\Delta\left(\widetilde{G}, p\right) = 0$ and $\widetilde{G}(p) = G(p) - \Delta(G,p)$.

First, we argue that the expected value of the signal conditional on $s < p$ is the same according to both distributions, G and $\widetilde{G}$. Note that

$$\frac{\int_{[-\infty,p)} sd\widetilde{G}(s)}{\widetilde{G}(p)} = \frac{\lambda [G(p) - \Delta(G,p)]}{\widetilde{G}(p)} = \lambda, \tag{6}$$

where the second equality follows from $\widetilde{G}(p) = G(p) - \Delta(G,p)$.

Second, we show that

$$\int sd\widetilde{G}(s) = \mu. \tag{7}$$

To see this, note that

$$\begin{aligned} \int sd\widetilde{G}(s) &= \widetilde{G}(p) \frac{\int_{[-\infty,p)} sd\widetilde{G}(s)}{\widetilde{G}(p)} + \left[1 - \widetilde{G}(p)\right] \frac{\int_{[p,B]} sd\widetilde{G}(s)}{1 - \widetilde{G}(p)} \\ &= [G(p) - \Delta(G,p)] \lambda + [1 - G(p) + \Delta(G,p)] \frac{\int_{[p,1]} sdG(s)}{1 - G(p) + \Delta(G,p)} = \int sdG(s) = \mu, \end{aligned}$$

where the first equality follows from (5) and (6) and the last equality follows from $G \in \mathcal{G}_F$.

Third, we prove that

$$\widetilde{G}(s) \leq G(s)$$

if $s \in [p, B]$. Note that

$$\widetilde{G}(s) = D(G,p)\left(1 - \frac{p-c}{s-c}\right) + G(p) - \Delta(G,p) \leq G(s), \tag{8}$$

where the inequality follows from (3).

Next, we show that G is a mean-preserving spread of $\widetilde{G}$. If $s \leq \lambda$ then

$$\int_{-\infty}^{s} G(x)\,dx \geq \int_{-\infty}^{s} \widetilde{G}(x)\,dx = 0. \tag{9}$$

If $s \in [\lambda, p)$ then

$$\begin{aligned} \frac{\int_{-\infty}^{s} G(x)\,dx}{G(p) - \Delta(G,p)} &= 1 - \lambda - \frac{\int_{s}^{p} G(x)\,dx}{G(p) - \Delta(G,p)} \\ \geq\ 1 - \lambda - \frac{\int_{s}^{p} G(p) - \Delta(G,p)\,dx}{G(p) - \Delta(G,p)} &= 1 - \lambda - \frac{\int_{s}^{p} \widetilde{G}(x)\,dx}{\widetilde{G}(p)} = \frac{\int_{-\infty}^{s} \widetilde{G}(x)\,dx}{G(p) - \Delta(G,p)}, \end{aligned}$$

where the inequality follows from $G(s) \leq G(p) - \Delta(G,p)$ for $s < p$, the second equality follows from $\widetilde{G}(x) = G(p) - \Delta(G,p)$ on $[\lambda, p)$ and the last equality follows from (6). The previous displayed inequality chain implies that if $s \in [\lambda, p)$ then

$$\int_{-\infty}^{s} G(x)\,dx \geq \int_{-\infty}^{s} \widetilde{G}(x)\,dx. \tag{10}$$

If $s \in [p, B]$, then

$$\int_{-\infty}^{s} G(x)\,dx = \int_{[-\infty,p)} G(x)\,dx + \int_{[p,s]} G(x)\,dx \geq \int_{[-\infty,p)} \widetilde{G}(x)\,dx + \int_{[p,s]} G(x)\,dx \quad (11)$$
$$\geq \int_{[-\infty,p)} \widetilde{G}(x)\,dx + \int_{[p,s]} \widetilde{G}(x)\,dx,$$

where the first inequality follows from (10) and the second inequality follows from (8).

If $s \geq B$ then

$$\int_{-\infty}^{s} G(x)\,dx = 1-\mu-\int_{s}^{1} G(x)\,dx \geq 1-\mu-(1-s) = 1-\mu-\int_{s}^{1} \widetilde{G}(x)\,dx = \int_{-\infty}^{s} \widetilde{G}(x)\,dx, \quad (12)$$

where the first inequality follows from $G(x) \leq 1$, the second equality from $\widetilde{G}(x) = 1$ if $x \geq B$ and the third equality from (7). Inequalities (9)-(12) imply that G is a mean-preserving spread of $\widetilde{G}$.

So far, we have proved that G is a mean-preserving spread of $\widetilde{G}$, so $\widetilde{G}$ involves less learning than G. In fact, $\widetilde{G}$ is strictly less informative than G unless $G = \widetilde{G}$. Note that the seller is indifferent between any price on $[p, B]$. If the seller charges p, his profit is $(p-c)\left(1-\widetilde{G}(p)\right) = (p-c)\,D(G,p)$, that is, the same as in outcome (G, p). The buyer's payoff is

$$\int_{p}^{B} (s-p)\,d\widetilde{G}(s) = \int_{p}^{B} s d\widetilde{G}(s) - \left(1-\widetilde{G}(p)\right)p = \int_{[p,1]} s dG(s) - D(G,p)\,p,$$

where the second equality follows from (5) and $\widetilde{G}(p) = G(p) - \Delta(G,p)$. The previous inequality chain implies that the buyer's payoff is also the same as in the outcome (G, p). Also note that the seller must strictly prefer setting price p to price λ because otherwise the buyer would be better off in the outcome $\left(\widetilde{G}, \lambda\right)$ than in (G, p), contradicting the assumption that (G, p) is buyer-optimal. To summarize, if (G, p) is a buyer-optimal outcome and induces minimal learning then $G = \widetilde{G}$ which implies parts (i) and (ii) of the statement of the proposition. ■

Next, we characterize the minimally informative buyer-optimal signal structure in an example and show that trade is neither efficient nor does it occur for sure if v is likely to be smaller than c.

Example 1. Suppose that

$$F(v) = \begin{cases} 0 & \text{if } v < 0 \\ 1/2 & \text{if } v \in [0,1) \\ 1 & \text{if } v \geq 1, \end{cases}$$

that is, the buyer's value is either zero or one with equal probabilities. Let $c \in [0,\, 1/2]$.

For, $c > 0$, efficiency requires for trade to occur if and only if $v = 1$. So, the only signal structure that is compatible with efficient trade is the fully informative one, with $s = v$. But then, the seller charges $p = 1$ and the buyer's payoff is zero. The seller's profit is $1/2(1-c) > 0$. If the signal is pure noise, i.e. $s \equiv \mu$, the seller charges the price $p = \mu = 1/2$. The buyer's payoff is zero and the seller's profit is $1/2 - c$.

In general, for $c > 0$, the seller never charges a price smaller than c. Hence, any information structure that results in sure trade must satisfy $s \geq c$, $\forall s$ that occur with positive probability.

Next, we characterize the buyer-optimal signal structure, G_c^*, in this example. Proposition 1 implies that $s = 0$ with probability $\alpha \in [0, 1)$ and, with the remaining probability $1 - \alpha$, the distribution of the signal is generated by G_p^B. In what follows, we pin down α, p and B for some values of c. To this end, given c, let $\mu_c(p, B)$ denote the expected value generated by G_p^B. Note that $G_c^* \in \mathcal{G}_F$ implies $(1-\alpha)\mu_c(p, B) = 1/2$, and hence

$$(1-\alpha) = \frac{1}{2\mu_c(p, B)}.$$

Therefore, the buyer's payoff is

$$(1-\alpha)[\mu_c(p, B) - p] = \frac{1}{2} - \frac{p}{2\mu_c(p, B)}. \tag{13}$$

Note that since $\mu_c(p, B)$ is increasing in B, the buyer-optimal B is one. Therefore, solving for the buyer-optimal signal structure is reduced to the following problem

$$\begin{aligned} \max_{p \in [c,1]} \quad & \frac{\mu_c(p, 1)}{p} \\ \text{subject to} \quad & \mu_c(p, 1) \geq \frac{1}{2}. \end{aligned} \tag{14}$$

Next, we derive an explicit formula for $\mu_c(p, 1)$:

$$\begin{aligned} \mu_c(p,1) &= \int_p^1 \frac{p-c}{(s-c)^2} s ds + \frac{p-c}{1-c} = (p-c)\left[\int_p^1 \frac{s-c}{(s-c)^2} ds + \int_p^1 \frac{c}{(s-c)^2} ds + \frac{1}{1-c}\right] \\ &= (p-c)\left[\log(s-c)|_p^1 - \frac{c}{s-c}\Big|_p^1 + \frac{1}{1-c}\right] \\ &= (p-c)\left[\log\left(\frac{1-c}{p-c}\right) - \frac{c}{1-c} + \frac{c}{p-c} + \frac{1}{1-c}\right] = (p-c)\log\left(\frac{1-c}{p-c}\right) + p. \end{aligned}$$

Therefore, the maximization problem in (14) can be rewritten as

$$\begin{aligned} \max_{p \in [c,1]} \quad & \frac{(p-c)\log\left(\frac{1-c}{p-c}\right)}{p} \\ \text{subject to} \quad & (p-c)\log\left(\frac{1-c}{p-c}\right) + p \geq \frac{1}{2}. \end{aligned}$$

The following table summarizes the solution to this problem $c \in \{0,\ 1/8,\ 1/4,\ 1/2\}$:

To summarize, for $c = 0$ or low costs, under the buyer-optimal signal structure the seller is indifferent between all prices $[p_c^*, 1]$ and trade occurs with probability one.[3] For higher costs, for the buyer-optimal signal structure, the buyer learns that her valuation is zero with probability $\alpha^* > 0$; with the remaining probability, the signal s is generated by the CDF $G_{p^*}^1$.

[3]For the distribution in the example the upper bound on costs for which this holds is around ≈ 0.133.

c	p^*	α^*
0	≈ 0.19	0
$\frac{1}{8}$	≈ 0.25	0
$\frac{1}{4}$	≈ 0.40	≈ 0.22
$\frac{1}{2}$	≈ 0.64	≈ 0.39

We could have reached similar conclusions, i.e. trade is inefficient and does not occur for sure even if $c = 0$ but the buyer's value can be negative. The analysis of the next example is analogous to the previous one, so we omit it.

Example 2. Suppose that $c = 0$

$$F(v) = \begin{cases} 0 & \text{if } v < -1 \\ 1/2 & \text{if } v \in [-1, 1) \\ 1 & \text{if } v \geq 1, \end{cases}$$

that is, the buyer's value is either minus or one with equal probabilities.

In the example above, the allocation resulting from the buyer-optimal information structure is inefficient because some buyers whose valuation is below the cost of the seller end up buying the object. A natural question to ask is: Can the buyer-optimal information structure induce some buyers to refrain from trading even if their valuations are above the production cost? We prove that such an inefficiency can never arise. To this end, we first prove that each buyer with a valuation above the price buys the object.

Lemma 1 *Suppose that (G, p) is a buyer-optimal outcome and $G(p) - \Delta(G, p) > 0$. Then $F(p|s < p) = 1$.*

This lemma states that, if trade does not happen surely, then the probability that the buyer's true valuation is below the price conditional on receiving a signal smaller than the price is one. In other words, if a buyer does not trade at price p then her true valuation is below the price p.

Proof. It is without loss of generality to assume that the buyer receives a single signal strictly below p. Let λ denote the value of this signal, that is, $\lambda = E_G(s|s = \lambda)$. Furthermore, let $F_\lambda(v)$ denote $F(v|s < p) = F(v|s = \lambda)$ for all v. Note that $\int v dF_\lambda(v) = \lambda$. Finally, let α denote the probability of receiving λ, that is, $\alpha = G(\lambda)$.

We prove the statement of the proposition by way of contradiction. That is, we assume that $F_\lambda(p) = F(p|s < p) < 1$ and construct an outcome in which the payoff of the buyer is higher than in (G, p). In particular, we modify the signal distribution G conditional on $s = \lambda$ and show that the buyer's payoff strictly increases. To this end, fix a $B \in (p, 1]$ such that $F_\lambda(B) < 1$. This is possible if $F_\lambda(p) < 1$. Let the signal distribution G^ε be defined such that the signal is generated by G_p^B with probability $\varepsilon\,(>0)$. With the remaining probability $1-\varepsilon$, the signal is λ^ε such that

$$\lambda = \varepsilon \int s dG_p^B(s) + (1-\varepsilon)\lambda^\varepsilon. \quad (15)$$

Note that G^ε converges to one on $[\lambda, 1]$ as ε goes to zero. Since $F_\lambda(B) < 1$, ε can be chosen small enough so that $F_\lambda < G^\varepsilon$ on $[\lambda^\varepsilon, 1]$. In addition, observe that $G^\varepsilon = 0 \leq F_\lambda$ on $[0, \lambda^\varepsilon)$. This implies that F_λ is a mean-preserving spread of G^ε.

Consider now the following modification of G denoted by $\widetilde{G}$. The signal is generated by $G(.|s>p)$ with probability $1-\alpha$. With the remaining probability α, the signal is generated by G^ε. In other words, instead of observing the signal λ, the buyer observes a more informative signal distributed according to G^ε with probability α. Since F_λ is a mean-preserving spread of G^ε and G^ε is a mean-preserving spread of $G(.|s=p)$ by (15), the signal structure $\widetilde{G}$ can be interpreted in the following sequential manner. First, the buyer receives a signal according to G. Conditional receiving the signal λ, the buyer receives an additional signal regarding her valuation according to G^ε. By construction, F is a mean-preserving spread of $\widetilde{G}$, so $\widetilde{G} \in \mathcal{G}_F$. In what follows, $\left(\widetilde{G}, p\right)$ is an outcome, that is, p is a profit-maximizing price and and $U\left(\widetilde{G}, p\right) > U(G, p)$.

To show that $\left(\widetilde{G}, p\right)$ is an outcome, first note that the seller would never set a price below p. To see this, note that the only signal below p is λ^ε, so λ^ε generates higher profit than any other price on $[0, p)$. However, if the seller finds it optimal to set price λ^ε then he would also find it optimal to set price λ when the buyer's signal is distributed according to G. Therefore, we can restrict attention to prices on $[p, 1]$. Note that

$$\begin{aligned}\max_{v\geq p} D\left(\widetilde{G}, v\right)(v-c) &= \max_{v\geq p}\left[\alpha D\left(G^\varepsilon, v\right) + (1-\alpha) D\left(G(.|s>p), v\right)\right](v-c) \\ &\geq \alpha \max_{v\geq p} D\left(G^\varepsilon, v\right)(v-c) + (1-\alpha)\max_{v\geq p} D\left(G(.|s>p), v\right)(v-c).\end{aligned}$$

Note that p is a solution to both problems on the right-hand side and hence, p is also a solution to the problem on the left-hand side. Therefore, $\left(\widetilde{G}, p\right)$ is indeed an outcome.

Finally, observe that

$$U(G,p) = (1-\alpha)\int s dG(s|s>p) < (1-\alpha)\int s dG(s|s>p) + \alpha\varepsilon \int s dG_p^B(s) = U\left(\widetilde{G}, p\right),$$

that is, the buyer's payoff is larger in the outcome $\left(\widetilde{G}, p\right)$ than in (G, p). This contradicts the hypothesis that (G, p) is buyer-optimal. ■

We are ready to prove that a buyer with valuation above the seller's cost always trades in a buyer-optimal outcome.

Proposition 2 *Suppose that* (G, p) *is a buyer-optimal outcome and* $G(p) - \Delta(G, p) > 0$. *Then* $F(c|s<p) = 1$.

This proposition states that if some buyer does not trade in the buyer-optimal outcome then her valuation must be smaller than the production cost. In other words, any efficiency loss in a buyer-optimal outcome is due to too much trade, that is, a buyer might purchase the good even if her valuation below the cost.

Proof. By Lemma 1, $F(p|s<p)=1$. It remains to show that $F(p|s<p)=F(c|s<p)$, that is, each buyer whose valuation is in the interval $[c,p]$ ends up buying the object. Note that it is without loss of generality to assume that those buyers who receive $s<p$ learn their valuation. The reason is that such a modification of the information structure can only lower the price, which leads to more efficient purchasing decision of the buyer. We prove the statement of the proposition by way of contradiction. We assume that there is a positive mass of buyers whose valuation is in $[c,p]$ and who receive a signal smaller than p, that is, $F(p|s<p)>F(c|s<p)$. Since every buyer with $s<p$ observes her valuation, this inequality is equivalent to $G(p)>G(c)$.

Let $\widetilde{F}$ denote the distribution obtained from G by conditioning on $[c,1]$ that is,

$$\widetilde{F}(v)=\frac{G(v)-G(c)}{1-G(c)}.$$

Note that if the buyer's value distribution is given by $\widetilde{F}$ and the buyer learns her valuation, the seller finds it optimal to set price p, that is, $\left(\widetilde{F},p\right)$ is an outcome. Furthermore, since $G(p)>G(c)$, trade is not efficient in this outcome. Let $\left(\widetilde{G}^{*},\widetilde{p}^{*}\right)$ denote the buyer-optimal outcome if the buyer's valuation is distributed by $\widetilde{F}$. Also observe that if the buyer's value-distribution is given by $\widetilde{F}$, then $v>c$ with probability one and hence, efficiency requires trade for sure. The analysis of our paper applies to this case. By Theorem 1 of the paper, the outcome $\left(\widetilde{G}^{*},\widetilde{p}^{*}\right)$ induces efficient trade and a strictly larger payoff than the inefficient outcome $\left(\widetilde{F},p\right)$, that is,

$$U\left(\widetilde{G}^{*},\widetilde{p}^{*}\right)>U\left(\widetilde{F},p\right). \tag{16}$$

Let us now define a signal z as follows. If the buyer observed $s<c$ then let $z=s$. If $s>c$ then let the value of z determined by $\widetilde{G}^{*}$. Let $\overline{G}$ denote the resulting CDF. Note that since F is a mean-preserving spread of G and $\widetilde{G}^{*}$ is a mean-preserving spread of $\widetilde{F}=G(.|s>c)$, F is also a mean-preserving spread of $\overline{G}$. Therefore, $\overline{G}$ is a feasible signal distribution. Also note that $\left(\overline{G},\widetilde{p}^{*}\right)$ is an outcome because $\overline{G}(.|z>c)=\widetilde{G}^{*}$ and $\left(\widetilde{G}^{*},\widetilde{p}^{*}\right)$ is an outcome. Finally, note that

$$U(G,p)=(1-G(c))\,U\left(\widetilde{F},p\right)<(1-G(c))\,U\left(\widetilde{G}^{*},\widetilde{p}^{*}\right)=U\left(\overline{G},\widetilde{p}^{*}\right),$$

where the inequality follows from (16). This inequality chain implies that (G,p) is not a buyer-optimal outcome which is a contradiction. ■

3 Payoff Characterization

This section is devoted to the analysis of the combinations of those consumer and producer surplus which can arise as an equilibrium outcome for *some* signal s. First, we show that our analysis can be used to characterize those equilibrium payoff profiles which are efficient. Second, we characterize the set of *all* payoff-profiles that can be implemented by some signal structure Example 1.

Efficient Outcome Characterization.— Efficiency requires a buyer with value v to trade if and only if $v \geq c$. We call an outcome (G, p) efficient if it induces efficient trade, that is, a buyer with valuation v receives a signal $s \geq p$ if and only if $v \geq c$. If the buyer's valuation is always weakly larger than c, $F(c) = \Delta(F, c)$, then the analysis of our paper directly applies with the only modification that the set of distributions to be considered is defined by (2) instead of the equal-revenue distributions considered in the paper. In particular, the set of payoff-profiles that can be implemented by some signal structure can still be described by Figure 1, except that μ must be replaced by $\mu - c$ and p^* must be replaced by $p^* - c$. Observe that among all these payoff combinations, only the line segment connecting the points $(0, \mu - c)$ and $(\mu - p^*, p^* - c)$ correspond to efficient outcomes.

For the general case, in which the buyer's valuation may be smaller than c, let F_c denote the distribution of the buyer's valuation conditional on the valuation being larger than c, that is, $F_c = F(.|v \geq c)$. Let μ_c denote the expectation generated by F_c, that is, $\mu_c = \int v dF_c(v)$. Furthermore, let p_c^* denote the equilibrium price in the buyer-optimal outcome (characterized by Theorem 1 of the paper) if the buyer's value distribution is F_c.

Proposition 3 *There exists an efficient outcome such that the buyer's payoff is u and the seller's profit is π if and only if there exists an $\gamma \in [0, 1]$ such that*

$$\begin{aligned} u &= (1 - F(c))(1 - \gamma)(\mu_c - p_c^*) \text{ and} \\ \pi &= (1 - F(c))[\gamma(\mu_c - c) + (1 - \gamma)(p_c^* - c)]. \end{aligned}$$

This proposition states that the set of efficient equilibrium profiles for value distribution F can be obtained by characterizing the set of efficient payoff profiles for value distribution F_c and by multiplying each of these profiles by $(1 - F(c))$. The two extreme points in this set are supported by the following information structures. Consider first the outcome $(0, (1 - F(c))(\mu_c - c))$. This profile arises if the buyer only learns whether her valuation is above or below c. The seller sets price μ_c and captures all the surplus. Consider now the point $((1 - F(c))(\mu_c - p_c^*), (1 - F(c)(p_c^* - c))$. This payoff profile arises if the buyer receives the buyer-optimal signal for F_c conditional on $v \geq c$ and receives a different signal otherwise. The seller optimally sets price p_c^*.

Proof. First note that if the buyer's value distribution was given by F_c then the combinations of those consumer and producer surplus which can arise as an equilibrium outcome can be characterized as follows. There exists an outcome where the buyer's payoff is u' and the seller's profit is π' if and only if there exists an $\gamma \in [0, 1]$ such that

$$u' = (1 - \gamma)(\mu_c - p_c^*) \text{ and } \pi' = [\gamma(\mu_c - c) + (1 - \gamma)(p_c^* - c)]. \tag{17}$$

This directly follows from the paragraph explaining Figure 1 in the paper.

Observe that if (G,p) is an efficient outcome that the buyer must learn in equilibrium whether or not her valuation is above the seller's cost. Also note that the seller's price only depends on $G(.|s \geq c)$ if the signal is distributed according to G. Therefore, an outcome (G,p) is efficient if and only if (i) a buyer with valuation v observes a signal $s \geq c$ if and only if $v \geq c$ and (ii) $(G(.|s \geq c), p)$ corresponds to an efficient outcome if the buyer's value-distribution is F_c. Finally, observe that

$$U(G,p) = (1 - F(c))\, U(G(.|s \geq c), p) \tag{18}$$

and

$$pD(G,p) = p(1 - F(c))\, D(G(.|s \geq c), p), \tag{19}$$

in other words, the payoffs of both the buyer and the seller in the outcome (G,p) is $(1 - F(c))$ as large as in the outcome $(G(.|s \geq c), p)$ given that the buyer's value-distribution is F_c. Noting that $G(.|s \geq c)$ can be any CDF which yields an efficient outcome if the buyer's valuation is distributed according to F_c, equations (17)–(19) yield the statement of the proposition. ■

Payoff Characterization in Example 1.—

Returning to our previous example, we now characterize those payoff-profiles of buyer-surplus and seller-profit that can arise as an equilibrium outcome for some signal structure in the setting of Example 1. Recall that under pure noise, the payoff profile is $(0, 1/2 - c)$ and under the fully informative signal it is $(0, 1/2 - 1/2c)$. Notice that the minimal price that can be achieved with any information structure is the minimal price that can be achieved with an information structure that induces trade with probability one. This price $\tilde{p}_c$ is given as the solution to

$$\mu_c(\tilde{p}_c, 1) = \frac{1}{2}. \tag{20}$$

The resulting payoff of the buyer is $u(\tilde{p}_c) = 1/2 - \tilde{p}_c$ and the payoff of the seller is $\pi(\tilde{p}_c) = \tilde{p}_c - c$.

The following proposition characterizes the set of all implementable payoff profiles.

Proposition 4 *The set of implementable payoff profiles is*

$$\{(x, \pi(p)) : \ x \in [0, u(p)] \ \ and \, p \in [\tilde{p}_c, 1]\}, \tag{21}$$

with

$$u(p) = \frac{1}{2} - \frac{p}{2\left((p-c)\log\left(\frac{1-c}{p-c}\right) + p\right)}, \ \ and \ \pi(p) = \frac{p-c}{2\left((p-c)\log\left(\frac{1-c}{p-c}\right) + p\right)}.$$

This result is illustrated in Figure 1, which shows the regions of the payoff-profiles that are achievable for some signal structure of the buyer. The decreasing part of each curve corresponds to $\{(u(p), \pi(p))\}_{\tilde{p}_c}^{1}$ for different values of c. The proposition states that the set of implementable payoff profiles coincide with the convex hull of these curves.

Proof. First, we prove that any point in the set (21) can be implemented by a signal structure. Note that for any $p \in [\tilde{p}_c, 1]$, $\mu_c(p, 1) \geq 1/2$. Therefore, the signal structure where s is distributed

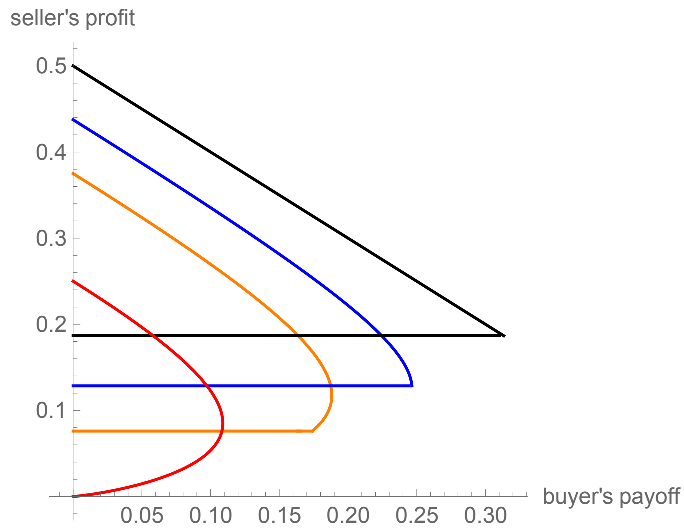

Figure 1: Illustration of outcome regions depending on costs c: $c = 0$ region within the black triangle; $c = 1/8$ region within the blue curve; $c = 1/4$ region within the orange curve ; $c = 1/2$ region within the red curve.

according to G_p^1 with probability $1/\left[2\mu_c\left(p,1\right)\right]$ and $s = 0$ otherwise is feasible. Let G_p denote this distribution. Given this signal structure, the seller finds it optimal to set price p, so (G_p, p) is an outcome. Notice that $U\left(G_p, p\right) = u\left(p\right)$ and $\Pi\left(G\right) = \pi\left(p\right)$. It remains to show that $\left(x, \pi\left(p\right)\right)$ is implementable for each $x \in \left[0, u\left(p\right)\right]$. If the buyer's signal is distributed according to G_p then the seller is indifferent between any price on $[p, 1]$. Depending on which of these prices the seller charges, the buyer's payoff can be anything on $[0, u(p)]$.

It remains to show that any point which is implementable is in (21). To this end, suppose that (G, p) is an outcome. We show that $\left(U\left(G,p\right), \Pi\left(G\right)\right)$ is in the set (21). By the proof of Proposition 1, there is a signal structure where the buyer receives signal 0 with probability α and receives a signal distributed G_p^B with probability $1-\alpha$ and the payoffs of both the seller and the buyer are the same as in the outcome (G, p), that is,

$$\Pi\left(G\right) = \frac{p-c}{2\mu_c\left(p,B\right)} \text{ and } U\left(G,p\right) = \frac{1}{2} - \frac{p}{2\mu_c\left(p,B\right)}.$$

Recall that, given price p the seller's minimum profit is

$$\pi(p) = \frac{p-c}{2\mu_c(p,1)} = \frac{p-c}{2\left((p-c)\log\left(\frac{1-c}{p-c}\right)+p\right)},$$

which is increasing in p. Thus, the seller's minimum profit is $\pi\left(\tilde{p}_c\right)$ and the maximum profit is $\pi\left(1\right)$. Therefore, by continuity, there exists a $\overline{p} \in \left[\tilde{p}_c, 1\right]$ such that $\Pi\left(G\right) = \pi\left(\overline{p}\right)$. In order to show that $\left(U\left(G,p\right), \Pi\left(G\right)\right) = \left(U\left(G,p\right), \pi\left(\overline{p}\right)\right)$ is in the set (21), we must prove that $U\left(G,p\right) \leq u\left(\overline{p}\right)$.

First, observe that

$$\pi\left(p\right)=\frac{p-c}{2\mu_c\left(p,1\right)}\leq\frac{p-c}{2\mu_c\left(p,B\right)}=\frac{\overline{p}-c}{2\mu_c\left(\overline{p},1\right)}=\pi\left(\overline{p}\right),$$

where the inequality follows from $\mu_c\left(p,B\right)$ being increasing in B. Since π is increasing in p, we conclude from this inequality chain that $p\leq\overline{p}$, therefore, $\mu_c\left(\overline{p},1\right)\geq\mu_c\left(p,B\right)$, and hence $\frac{p}{2\mu_c(p,B)}\geq\frac{\overline{p}}{2\mu_c(\overline{p},1)}$. Using (13), this implies that $U\left(G,p\right)\leq u\left(\overline{p}\right)$. ■