# Optimal Contract Regulation in Selection Markets

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# ONLINE APPENDIX

The Appendix is organized as follows. Section A formally defines AG equilibrium. Section B characterizes equilibrium (all regimes). Section C discusses when each equilibrium regimes obtains. Section D discusses log-concavity and uniqueness. Section E analyzes comparative statics for Dispersive equilibria. Section F analyzes comparative statics for PPPP equilibria. Section G contains the proof of Proposition 12. Section H discusses comparative statics when the regime is Lemons. Section I discusses equilibrium characterization and other results in the presence of moral hazard. Section J discusses numerical simulations. Section K derives certainty equivalents in a setting with CARA utility.

## A AG Equilibrium

In this section we provide formal definitions of equilibrium and weak equilibrium, based on Azevedo and Gottlieb [2017].

**Definition 2.** $(p, \alpha)$ is a *weak equilibrium* if:

1. Individuals maximize: $u(\mu, x, p(x)) = \sup_{x' \in X} u(\mu, x', p(x'))$, for $\alpha - a.e.\ (\mu, x)$.
2. Each contract breaks even: $p(x) = x\mathbb{E}_\alpha[\mu \mid x]$, for $\alpha - a.e.\ x$.

**Definition 3.** Consider the economy $\mathcal{E} = [\Theta, X, P]$, and the sequence of perturbed economies $\mathcal{E}_j = [\Theta \cup \overline{X}_j, \overline{X}_j, P + \eta_j]$. Let $(\overline{X}_j)_{j \in N}$ be a sequence of finite subsets of $X$ which converge to $X$ in the sense of Haussdorf.[34] Let $(\eta_j)_{j \in N}$ be a sequence of measures, with $\eta_j$ supported on $\overline{X}_j$, strictly positive on $\overline{X}_j$, and $\eta_j(\overline{X}_j) \to 0$. Suppose there exists a sequence of pairs $(p_j, \alpha_j)_{j \in \mathbb{N}}$, satisfying the following conditions. First, $(p_j, \alpha_j)$ is a weak equilibrium of $\mathcal{E}_j$, where the *behavioral type* $x \in \overline{X}_j$ has zero cost and prefers $x$ to any other alternative regardless of price. Second, $\alpha_j \to \alpha$ weakly.[35] Third, whenever $(x_j)_{j \in \mathbb{N}}$ converges to $x \in X$ with $x_j \in \overline{X}_j$, then $p_j(x_j) \to p(x)$. Then, the pair $(p, \alpha)$ is an *equilibrium* of $\mathcal{E} = [\Theta, X, P]$.

[34] I.e., for each $x \in X$, there is $(x_n)_{n \in \mathbb{N}}$ converging to $x$ with $x_n \in \overline{X}^n$ for each $n \in \mathbb{N}$. In Levy and Veiga [2021], $X$ is not assumed to be compact but have a compactification $\overline{X}$ which $X$ embeds into, e.g., $X = [0, 1)$ which naturally embeds in $\overline{X} = [0, 1]$.

[35] That is, for each $f : \Theta \times X \to \mathbb{R}$ continuous and bounded, we have $\int f d\alpha^n \to \int f d\alpha$.

# B Equilibrium Characterization

In this section we state and prove a more general equilibrium characterization result. This section does not assume that the distribution of types $f$ is log-concave, and hence does not assume that equilibrium is unique. We also clarify that the assumption $\frac{\partial^2 g}{\partial x \partial \mu} \geq 0$, which implies that the willingness of agents to pay for more insurance is strictly increasing in type, is used to show that higher types purchase (weakly) higher coverage; in particular, see Lemma 7.

## B.1 Full Statement of Equilibrium Characterization

We now present an extended version of Proposition 1, which describes equilibrium in detail and more formally. Proposition 16 below describes all equilibria where $\alpha_X(\{x \mid x > \underline{x}\}) > 0$ (i.e., there is a positive mass of individuals choosing strictly more than the minimum coverage). This accounts for equilibrium regimes that are Dispersive, RS or PPPP, and hence in particular implies Proposition 1 ( Dispersive Equilibria), Proposition 3 (PPPP Equilibria), and Proposition 2 (RS Equilibria). We discuss Lemons equilibrium in Appendix H. We note that the equilibrium regime where all individuals buy $x = \underline{x}$ in was addressed in Proposition 18.

Let the set of strictly positive coverage $(x > 0)$ contracts purchased in equilibrium be

$$\mathcal{B}^+ = supp(\alpha_X) \cap [\underline{x}, \overline{x}].$$

That is, individuals purchase contracts in $\mathcal{B}^+$ and, in addition, possibly $x = 0$.

**Proposition 16.** *Consider any AG equilibrium $(p, \alpha)$ where $\alpha_X(\{x \mid x > \underline{x}\}) > 0$. Then:*

1. *There is a cut-off coverage $x^* \in [\underline{x}, \overline{x})$ such that $\mathcal{B}^+ = [x^*, \overline{x}]$ or $\mathcal{B}^+ = \{\underline{x}\} \cup [x^*, \overline{x}]$.*
2. *$\alpha_X$ has the same null sets as the Lebesgue measure in $[x^*, \overline{x}]$.*[36]
3. *If $\alpha_X(\{\underline{x}\}) > 0$ (i.e. if $\underline{x}$ is an atom of $\alpha_X$) then $x^* > \underline{x}$.*
4. *If $x = 0$ is an atom of $\alpha_X$, then $\underline{x}$ is as well: $\alpha_X(\{0\}) > 0 \Rightarrow \alpha_X(\{\underline{x}\}) > 0$.*[37]
5. *There is a mapping $\sigma : \Theta \to X$ that assigns to each type $\mu$, the contract $\sigma(\mu)$ that she chooses $\alpha$-a.s. Formally, $\alpha\big\{(\mu, x) \mid x = \sigma(\mu)\big\} = 1$. $\sigma$ is non-decreasing, and is strictly*

[36] Recall that the support of a measure is the smallest closed set whose complement is null.

[37] This is a particular case of Lemma 1, proven in this section, which also shows that no one purchasing (i.e., $\alpha_X(\{0\}) = 1$) is not an equilibrium.

*increasing and continuous for* $\mu \in \sigma^{-1}([x^*, \overline{x}])$. *Let the type that purchases contract* $x$ *be* $\tau(x) = \sigma^{-1}(x)$.

6. *Define the type who purchases* $x^*$ *as* $\mu^* = \tau(x^*)$. *For* $\mu \in [\mu^*, \overline{\mu}]$, *the choice rule* $\sigma$ *satisfies*

$$\overline{\mu} - \mu = \int_{\sigma(\mu)}^{\overline{x}} \frac{1}{x} \frac{\partial g}{\partial x} (\tau(x), x)\, dx, \qquad \forall \mu \in [\mu^*, \overline{\mu}]. \tag{20}$$

*Let type* $\mu_*$ *be the lowest type that purchases the minimum coverage* $\underline{x}$*:* $\mu_* = \inf \{\mu \mid \sigma(\mu) = \underline{x}\}$.

7. *The cut-offs* $(x^*, \mu^*)$ *and* $\sigma$ *satisfy:*

   *(a)* $\mathcal{B}^+ = [x^*, \overline{x}] \Leftrightarrow \mu^* = \underline{\mu}$ *(no pooling), or*

   *(b)* $\mathcal{B}^+ = \{\underline{x}\} \cup [x^*, \overline{x}] \Leftrightarrow \mu^* \in (\underline{\mu}, \overline{\mu})$ *(partial pooling), in which case*

      *i.* $\sigma(\mu) = \underline{x}$ *for* $\mu \in [\underline{\mu}, \mu^*)$

      *ii.* $\sigma(\mu) \in [x^*, \overline{x}]$ *for* $\mu \in [\mu^*, \overline{\mu}]$.

8. *Each contract breaks even. In particular:*

   *(a)* $p(0) = 0$,

   *(b)* $p(x) = x\tau(x)$, *for a.e.* $x \in [x^*, \overline{x}]$

   *(c) If* $\mu_* < \mu^*$, *then* $p(\underline{x}) = \underline{x}\mathbb{E}[\mu \mid \mu \in [\mu_*, \mu^*]]$.

9. $p(x)$ *is continuous, and is strictly increasing on* $\{x \mid p(x) > 0\}$. *Moreover,* $p(x) \leq \overline{\mu}x, \forall x \in X$.

10. $p$ *is Lebesgue-a.e. differentiable and*

$$p'(x) = \tau(x) + \frac{\partial g}{\partial x}(\tau(x), x), \qquad \textit{for a.e.} \quad x \in (x^*, \overline{x}) \tag{21}$$

11. *For* $x \in [\underline{x}, x^*)$, *price* $p(x)$ *is s.t. the cut-off type* $\mu^*$ *is indifferent between* $(x, p(x))$ *and* $(x^*, p(x^*))$, *i.e.*

$$g(\mu^*, x^*) = \mu^* x + g(\mu^*, x) - p(x), \qquad \forall x \in [\underline{x}, x^*]. \tag{22}$$

12. *Then,* $\mu_* = \underline{\mu}$ *if and only if all types prefer* $(\underline{x}, p(\underline{x}))$ *to* $(0,0)$, *i.e.*

$$-T \leq \underline{\mu}\underline{x} + g(\underline{\mu}, \underline{x}) - p(\underline{x}).$$

*In this case, all agents purchase insurance ($x = 0$ is not chosen). Otherwise (i.e., if $\mu_* > \underline{\mu}$), then $\mu_*$ is the type who is indifferent between these contracts:*

$$-T = \mu_* \underline{x} + g(\mu_*, \underline{x}) - p(\underline{x}). \tag{23}$$

## B.2 Equilibrium Characterization Results from LV21

We now recall some useful results from Levy and Veiga [2021] (LV21). LV21 assumed the set of allowed contracts was $X = [0, 1]$. The results of LV21 imply that, for the model of Section 3, if in addition one assumes $X = [0, 1]$, the equilibrium can be characterized as follows.

**Theorem 2.** *There is a unique equilibrium $(p, \alpha)$. In equilibrium:*

1. *Price $p(x)$ is continuous, and it is strictly increasing in $\{x \mid p(x) > 0\}$.*
2. *There is a continuous and strictly increasing mapping $\sigma : \Theta \to X$ that assigns to each type $\mu$, the alternative $\sigma(\mu)$ that she chooses $\alpha$-a.s.*
3. *The choice rule $\sigma(\mu)$ satisfies*

$$\overline{\mu} - \mu = \int_{\sigma(\mu)}^{1} \frac{1}{x} \frac{\partial g}{\partial x} (\tau(x), x)\, dx, \qquad \forall \mu \in [\underline{\mu}, \overline{\mu}].$$

4. *Each contract breaks even: $P - a.s.$, $p(\sigma(\mu)) = \mu\sigma(\mu)$.*
5. *Full insurance ($x = 1$) is in the support of the equilibrium and zero insurance ($x = 0$) is purchased by a set of individuals with measure zero.*
6. *Price is Lipshitz in any interval bounded away from full insurance ($x = 1$); if $\mu_\theta$ is bounded $P$-a.s., price is Lipshitz.*

*Proof.* See Levy and Veiga [2021], namely Theorem 2 and Corollary 2. □

Below we highlight important differences en route to the proofs of Propositions 3 and 16.

## B.3 Additional Machinery Results from LV21

We develop here more tools use in LV21 that apply to our setup as well. We fix a distribution $\alpha$ on $\Theta \times X$, with $X = [\underline{x}, \overline{x}]$ (e.g., $(p, \alpha)$ may be an equilibrium) and marginal $P := f d\mu$ on $\Theta = [\underline{\mu}, \overline{\mu}]$. LV21 assumed $X = [0, 1]$. Let the marginal of $\alpha$ on $X$ be $\alpha_X$.

We define analogues of the maximum and minimum risk $\mu$ purchasing each alternative $x$ (to allow for pooling). To avoid the influence of zero-measure sets of types, we use a variation of the essential supremum and infimum, defined for $x \in supp(\alpha_X)$ as[38]

$$\psi^+(x) = \lim_{\delta \to 0^+} \left[ \sup \left\{ \mu \mid \alpha\big(\{\theta \mid \mu_\theta \geq \mu\} \times (x - \delta, x + \delta)\big) > 0 \right\} \right] \tag{24}$$

$$\psi^-(x) = \lim_{\delta \to 0^+} \left[ \inf \left\{ \mu \mid \alpha\big(\{\theta \mid \mu_\theta \leq \mu\} \times (x - \delta, x + \delta)\big) > 0 \right\} \right]. \tag{25}$$

Intuitively, $\psi^+(x)$ captures the largest value of $\mu$ which purchases $x$ under $\alpha$, and $\psi^-(x)$ as the lowest such value of $\mu$.

Appendix C of LV21 establishes several claims when $(p, \alpha)$ is an equilibrium. Although they are made for the case $X = [0, 1]$, almost all hold when $X = [\underline{x}, \overline{x}]$ or $= \{0\} \cup [\underline{x}, \overline{x}]$ with the same proofs, with one exception (Lemma 7 there) which we discuss below (Lemma 11). We omit here those results which are irrelevant or trivial in the framework of this paper, which is slightly more restrictive than that in LV21.[39]

We note that it is for the following lemma in particular that we have made the the assumption $\frac{\partial^2 g}{\partial x \partial \mu} \geq 0$, which implies that the willingness of agents to pay for more insurance is strictly increasing in type.

**Lemma 7.** *$\sigma(\mu)$ is weakly increasing. Specifically, let $(\mu, x)$ refer to a type and the contract purchased by that type. It holds $\alpha$-a.s. that for each pair $(\mu_2, x_2), (\mu_1, x_1)$, $x_2 > x_1$ implies $\mu_2 \geq \mu_1$. This is also true if $X' \subseteq X$ is finite and $\alpha'$ is a weak equilibrium of the economy $[\Theta \cup X', P \cup \eta, X']$ (as discussed in Section 3, and elaborated in Appendix A), where $X'$ refers to behavioral types as well.*[40]

*Proof.* This is Lemma 2 of Section C.1. of LV21. An intuition is that lower types have lower willingness to pay. □

**Corollary 4.** *For $x < y < z$, we have $\psi^+(x) \leq \psi^-(y) \leq \psi^+(y) \leq \psi^-(z)$. This is also true if $X' \subseteq X$ is finite and $\alpha'$ is a weak equilibrium of the perturbed economy $[\Theta \cup X', P \cup \eta, X']$.*

*Proof.* Corollary 5 in LV21. □

One can then obtain the following sequence of lemmas (which are Lemma 3, 4, 6 in LV21, modified to the restriction $X = [\underline{x}, \overline{x}]$):

[38] For $\alpha$-a.e. $x \in X$, $\psi^+(x)$ (resp. $\psi^-(x)$) is the supremum (resp. infimum) of the support of $\alpha(\cdot \mid x)$. The limits exist as the terms they are taken over are monotonic.

[39] In LV21 types may differ in both risk and riskiness, although they are still unidimensional in terms of being linearly ordered w.r.t. willing to pay

[40] The conclusion then holds $\alpha'(\mid \Theta)$-a.s., i.e., holds for types in $\Theta$, not behavioral types for whom $\mu$ is not defined.

**Lemma 8.** $\psi^{-}(x) = \psi^{+}(x)$ *for* $x \in supp(\alpha_X) \cap (\underline{x}, \overline{x}]$.

Hence, denote $\psi = \psi^{-} = \psi^{+}$ in $(\underline{x}, \overline{x}]$.

**Lemma 9.** $\alpha(\{\theta, x \mid x > \underline{x}, p(x) \neq \psi(x)x\}) = 0$*, and (equivalently),* $p(x) = x \cdot \psi(x)$ *for all* $x \in supp\,(\alpha_X)$*,* $x > \underline{x}$*.*

*Proof.* From Lemma 8 and the break-even condition for price. □

**Lemma 10.** $\psi(x)$ *is strictly increasing for* $x \in supp(\alpha_X) \cap (\underline{x}, \overline{x}]$.

**Corollary 5.** *There is a mapping* $\sigma : [\underline{\mu}, \overline{\mu}] \to [\underline{x}, \overline{x}]$*, strictly increasing and continuous on* $[\underline{\mu}, \overline{\mu}] \backslash \sigma^{-1}(\{\underline{x}, 0\})$*, s.t.* $\alpha\big\{(\mu, x) \mid x = \sigma(\mu)\big\} = 1$.

*Proof.* By taking $\sigma = \psi^{-1}$ on a subset $\Phi \subseteq [\underline{\mu}, \overline{\mu}]$ which satisfies $\alpha(\{\mu \in \Phi\}\Delta\{x > \underline{x}\}) = 1$, where $\Delta$ notes the symmetric difference of sets; and then $\sigma(\mu) = \underline{x}$ for $\psi^{-}(\underline{x}) \leq \mu \leq \psi^{+}(\underline{x})$ if $\alpha_X(\{\underline{x}\}) > 0$, and similarly for $0$. This is an appropriate modification of Corollary 6 in LV21. □

**Lemma 11.** *If* $\alpha_X((\underline{x}, \overline{x}]) > 0$*, the supremum of the support of* $\alpha_X$ *is maximal insurance, i.e.,* $\sigma(\overline{\mu}) = \overline{x}$*.*

*Proof.* Lemma 7 in LV21 claims that full insurance is in the support of the equilibrium. Although the proof is insensitive to the choice of maximum coverage $\overline{x}$, it does rely on the minimum coverage being $0$, since it relies on the fact that, in equilibrium, the highest type $\overline{\mu}$ must be purchasing some contract $\check{x} > \underline{x}$ at price $\check{x}\overline{\mu}$. This is shown to be true there, when $\underline{x} = 0$. However, this need not be the case in our model if $\check{x} = \underline{x}$, as there could be pooling which would lead to $p(\underline{x}) < \overline{\mu}\underline{x}$. However, if pooling is not total (i.e., if $\check{x} > \underline{x}$), the conclusion and proof both hold. □

**Proposition 17.** *Let* $\mu_{**} < \mu^{**} \in [\underline{\mu}, \overline{\mu}]$*, and let* $(p, \alpha)$ *be an equilibrium with associated coverage function* $\sigma$ *with* $\sigma(\mu_{**}) > \underline{x}$*. Then*

$$(\mu^{**} - \mu_{**}) = \int_{\sigma(\mu_{**})}^{\sigma(\mu^{**})} \frac{1}{x} \frac{\partial g}{\partial x} (\tau(x), x)\, dx \tag{26}$$

where $\tau = (\sigma)^{-1}$. The proof (modified for our framework with $X = [\underline{x}, \overline{x}]$) also establishes that $\alpha_X$ conditional on $[x^{*}, \overline{x}]$ has the same null sets as the Lebesgue measure, and that

$$p'(x) = \tau(x) + x\tau'(x), \qquad a.e.\, x \in [\sigma(\mu_{**}), \sigma(\mu^{**})] \tag{27}$$

*Proof.* Proposition 13 in Appendix E.1 in LV21 (with $\underline{x}$ replacing $0$, and the proofs following otherwise verbatim).[41] $\square$

## B.4 Preliminary Results Towards Equilibrium Characterization

In this section $\sigma^B, p^B, \tau^B$ refer to the allocation rules and prices if $\underline{x} = 0$, i.e., $\sigma^B$follows (20) throughout $[\underline{\mu}, \overline{\mu}]$ for full separation, $\tau^B$ is its inverse, and $p^B$ is the resulting price ("B" refers to baseline).

**Lemma 12.** *Let $(p_1, \alpha_1), (p_2, \alpha_2)$ be equilibria with zero or partial (but not full) pooling at $\underline{x}$ (that is, some or all individuals purchase $x > \underline{x}$), and cut-off coverages $x_1^*, x_2^*$. If $x_1^* > x_2^*$, then $p_1 \leq p_2$, with strict inequality for $x \in [\underline{x}, x_1^*)$.*

*Proof.* Let $\mu_1^*, \mu_2^*$ be the associated cut-off coverages, and associated coverage functions $\sigma_1, \sigma_2$. By (20), $\mu_1^* > \mu_2^*$. In $[x_1^*, \overline{x}]$, $p_1 = p_2 = p^B(\cdot)$, as they both follow (20); in particular $p_1(x_1^*) = p_2(x_1^*)$. In $(x_2^*, x_1^*)$, $p_2 \equiv p^B(\cdot)$ and while

$$p_1'(x) = \mu_1^* + \frac{\partial g}{\partial x}\big(\mu_1^*, x\big) > \tau^B(x) + \frac{\partial g}{\partial x}\big(\tau^B(x), x\big) = \frac{dp^B}{dx}(x) = p_2'(x)$$

so $p_1(\cdot) < p^B(\cdot)$. Similarly, $x \in (\underline{x}, x_2^*)$,

$$p_2'(x) = \mu_2^* + \frac{\partial g}{\partial x}\big(\mu_2^*, x\big) < \mu_1^* + \frac{\partial g}{\partial x}\big(\tau^B(x), x\big) = p_1'(x)$$

Therefore, $p_1(x_1^*) = p_2(x_1^*)$ and $p_1(x) < p_2(x)$ a.e. in $(\underline{x}, x_1^*)$. $\square$

**Lemma 13.** *If there is one equilibrium with at least partial pooling at $\underline{x}$ (i.e., some or all individuals choose $x = \underline{x}$), then all equilibria have at least partial pooling at $\underline{x}$.*

*Proof.* Suppose $(p_1, \alpha_1)$ has an atom at $\underline{x}$ while $(p_2, \alpha_2)$ does not; $\mu_1^* > \underline{\mu} = \mu_2^*$.

First let's deal with the case $(p_1, \alpha_1)$ has only partial, not full, pooling. Hence, $x_1^* = \sigma_1(\mu_1^*) = \sigma_2(\mu_1^*) > x_2^*$, so by Lemma 12, $p_1 \leq p_2$ with strict inequality in $[\underline{x}, x_1^*)$. In particular, since $p_2(x_2^*) = \underline{\mu} x_2^*$, we have $p_1(x_2^*) < \underline{\mu} x_2^*$. Along with $p_1' \geq \underline{\mu}$ (its slope is a.e. the slope of some indifference curve), we have $p_1(\underline{x}) < \underline{x}\underline{\mu}$. But since $\underline{x}$ is an atom of $(p_1, \alpha_1)$, $p_1(\underline{x}) = \underline{x} E[\mu \mid \sigma_1(\mu) = \underline{x}] \geq \underline{x}\underline{\mu}$, a contradiction.

Now we deal with the case $(p_1, \alpha_1)$ displays full pooling. Then $p_1(\overline{x}) \leq \overline{\mu x} = p_2(\overline{x})$, $p_1'(x) = \overline{\mu} + \frac{\partial g}{\partial x}\big(\overline{\mu}, x\big) > \sigma_2^{-1}(x) + \frac{\partial g}{\partial x}\big(\tau^B(x), x\big) = p_2'(x)$ in $(\underline{x}, \overline{x})$, so $p_1 < p_2$ in $[x_2^*, \overline{x})$, and

[41] That proposition assumes that the distribution of risk $\mu$, conditional on the interval $(\mu_{**}, \mu^{**})$, has full support with a.e. strictly positive density w.r.t. the Lebesgue measure. In this paper we just assume that the density is everywhere strictly positive.

particular $p_1(x_2^*) < p_2(x_2^*) = \underline{\mu} x_2^*$; and the argument concludes verbatim as in the above case. $\square$

## B.5 Completing the Proof of Proposition 16

We now prove Proposition 16, which characterizes equilibrium when there is a mass of individuals choosing $x > \underline{x}$. As mentioned, this proposition implies Proposition 1 (for the regime of Diverse Equilibria), Proposition 3 (for the regime of PPPP Equilibria), and Proposition 2 (for the regime of RS Equilibria).

- Part 1 concerning $\mathcal{B}^+$ follows from Corollary 5 ($\sigma$ is increasing in the domain of types who purchase strictly above $\underline{x}$) and Lemma 11 ($\overline{x}$ is purchased).
- Part 2, concerning the null sets of $\alpha_X$ is remarked following Proposition 17.
- We return below to Part 3, which states that $x^* > \underline{x}$ if $\underline{x}$ is an atom.
- Part 4 is established in Lemma 11 in Section C.3.
- Part 5 is precisely Corollary 5.
- Part 6, which characterizes $\sigma(\mu)$ in $[\mu^*, \overline{\mu}]$ via an integral equation, follows from Proposition 17.
- Part 7, refining the shape of $\mathcal{B}^+$ in different cases, follows from Part 1 and Part 3.
- Part 8 is Lemma 9 together with the break-even condition.
- Part 9, with properties of $p(\cdot)$, follows from Theorem 1 and Lemma 9.
- Part 10, giving the derivative of price, is remarked following Proposition 17.
- Part 11 and Part 12 are by incentive compatibility.

Returning to Part 3: We show that if

$$supp(\alpha_X) \cap [\underline{x}, \overline{x}] = [\underline{x}, \overline{x}]$$

(i.e., if $x^* = \underline{x}$), then $\underline{x}$ cannot be an atom of $\alpha_X$. Indeed, if $\underline{x}$ is an atom of $\alpha_X$ but $x^* = \underline{x}$, then, since $\underline{x} = \lim_{\mu \to (\mu^*)^+} \sigma(\mu)$, we have

$$p(\underline{x}) = \underline{x} E_\alpha[\mu \mid \mu_* \leq \mu \leq \mu^*] < \underline{x}\mu^* = \lim_{\mu \to (\mu^*)^+} \mu\sigma(\mu) = \lim_{x \to \underline{x}^+} p(x)$$

contradicting the continuity of prices.

# C Determinants of Equilibrium Regimes

## C.1 Partial or Full Pooling

In this section, we prove the results regarding the pattern of pooling, which also describe under which circumstances each equilibrium regime obtains, i.e., Proposition 5.

The first result provides a necessary and sufficient condition in terms of model primitives for (full or partial) pooling to occur.

**Lemma 14.** *Pooling (full or partial) occurs if and only if*

$$\overline{\mu} - \underline{\mu} > \int_{\underline{x}}^{\overline{x}} \frac{1}{x}\frac{\partial g}{\partial x}\big(\tau(x), x\big)dx$$

*Proof.* First we show that, if the condition holds, then pooling occurs. If, by way of contradiction, there was no pooling, then $\sigma^{-1}(x^*) = \underline{\mu}$. (20) shows that

$$\int_{x^*}^{\overline{x}} \frac{1}{x}\frac{\partial g}{\partial x}\big(\tau(x), x\big)dx = \overline{\mu} - \sigma^{-1}(x^*) = \overline{\mu} - \underline{\mu} > \int_{\underline{x}}^{\overline{x}} \frac{1}{x}\frac{\partial g}{\partial x}\big(\tau(x), x\big)dx \geq \int_{x^*}^{\overline{x}} \frac{1}{x}\frac{\partial g}{\partial x}\big(\tau(x), x\big)dx,$$

a contradiction.

Second, we now show that if $\overline{\mu} - \underline{\mu} \leq \int_{\underline{x}}^{\overline{x}} \frac{1}{x}\frac{\partial g}{\partial x}\big(\tau(x), x\big)dx$, then any equilibrium does not have an atom at $\underline{x}$. By Lemma 13, it is enough to present one equilibrium which does not have an atom at $\underline{x}$. Indeed, the equilibrium defined by

$$\overline{\mu} - \mu = \int_{\sigma(\mu)}^{\overline{x}} \frac{1}{x}\frac{\partial g}{\partial x}\big(\tau(x), x\big)dx$$

gives a well-defined $\sigma(\mu) \in [\underline{x}, \overline{x}]$ for any $\mu \in [\underline{\mu}, \overline{\mu}]$, by the assumed inequality, which we claim defines an an equilibrium allocation rule. Indeed, the economy satisfies the assumptions of AG's Theorem 1 (hence guaranteeing equilibrium existence), and the equilibrium we have presented is the one characterized by Proposition 16 when $\mu_* = \mu^* = \underline{\mu}$. $\square$

## C.2 Full Pooling

In this section we provide a condition that is necessary for full pooling (i.e., all individuals choose $x \in \{0, \underline{x}\}$) to occur, and another condition which is sufficient.

### C.2.1 A necessary condition for full pooling

First, we provide a condition that is necessary for full pooling (i.e., all individuals choose $x \in \{0, \underline{x}\}$) to occur. Intuitively, it is necessary that the highest type $\overline{\mu}$ prefers to purchase $\underline{x}$ at the average cost in the population $\underline{x}\mathbb{E}[\mu]$, over purchasing the contract typically assigned to type $\overline{\mu}$ when full pooling is not the case, ie $(\overline{x}, \overline{\mu x})$. Equivalently, the reverse of this inequality is sufficient for some individuals to choose $x > \underline{x}$ in equilibrium.

**Lemma 15.** *If there is full pooling (i.e., all individuals choose $x \in \{0, \underline{x}\}$), then type $\overline{\mu}$ weakly prefers $(\underline{x}, \underline{x}E[\mu])$ to $(\overline{x}, \overline{\mu x})$, i.e.,*

$$\underline{x}\overline{\mu} + g(\overline{\mu}, \underline{x}) - \underline{x}\mathbb{E}[\mu] \geq g(\overline{\mu}, \overline{x}). \tag{28}$$

*Proof.* Suppose there is such an equilibrium $(p, \alpha)$. Since agents prefer $(\underline{x}, p(\underline{x}))$ to $(\overline{x}, p(\overline{x}))$, and $p(\overline{x}) \leq \overline{\mu x}$ by Theorem 1,

$$\overline{\mu}\underline{x} + g(\overline{\mu}, \underline{x}) - p(\underline{x}) \geq \overline{\mu x} + g(\overline{\mu}, \overline{x}) - p(\overline{x}) \geq g(\overline{\mu}, \overline{x}).$$

Since there is full pooling, $p(\underline{x}) = \underline{x}E[\mu \mid \mu \geq \mu_*] \geq \underline{x}E[\mu]$ (with equality when $\mu_* = \underline{\mu}$, which is the case when the regime is PPPP), we have

$$\overline{\mu}\underline{x} + g(\overline{\mu}, \underline{x}) \geq g(\overline{\mu}, \overline{x}) + \underline{x}E[\mu]$$

from which the conclusion follows. □

### C.2.2 A sufficient condition for full pooling

The following result shows that, if $\overline{x}$ and $\underline{x}$ are sufficiently close together, then the market unravels with all individuals buying $x \in \{0, \underline{x}\}$. This is a sufficient condition for full pooling (i.e., all choose $x \in \{0, \underline{x}\}$, or a Lemons equilibrium).

**Lemma 16.** *Fix a the distribution of types on $[\underline{\mu}, \overline{\mu}]$. For each $\overline{x} > 0$, there is $\underline{x}_0 > 0$ s.t. for every $\underline{x} \in [\underline{x}_0, \overline{x})$, any equilibrium with alternatives $X = \{0\} \cup [\underline{x}, \overline{x}]$ displays full pooling (i.e., all choose $x \in \{0, \underline{x}\}$). Similarly, for each $\underline{x} > 0$ there is $\underline{x} < \overline{x}_0 \leq 1$ s.t. if $\overline{x} \in (\underline{x}, \overline{x}_0]$, any equilibrium displays full pooling.*

*Proof.* We prove the first case, as the latter follows similarly. Suppose not. Then there is a sequence of $\underline{x}_n \to \overline{x}$ and their associated equilibria $(p_n, \alpha_n)$ in which $\overline{\mu}$ buys $\overline{x}$. Let $\mu_{n*}, \mu_n^*$ be the associated cut-off types, $x_n^*$ the associated coverages purchased by $\mu_n^*$. The main part of the proof is to show that:

$$\limsup_{n\to\infty} \mu_{n*} < \overline{\mu} \tag{29}$$

Suppose not. By passing to a subsequence, we can assume $\lim_{n\to\infty} \mu_{n*} = \overline{\mu}$. Then $\lim_{n\to\infty} \mu_n^* = \overline{\mu}$ as well, so $p_n(x_n^*) = E[\mu|\mu \in [\mu_{n*}, \mu_n^*]]x_n^* \to \overline{\mu x}$. But since $\overline{\mu}$ strictly prefers $(\overline{x}, \overline{\mu x})$ to $(0,0)$, there is $\delta > 0$ and $N \in \mathbb{N}$ s.t. all $\mu > \overline{\mu} - \delta$ strictly prefer $(x_n^*, p_n(x_n^*))$ to $(0,0)$, and in particular for types in $[\overline{\mu} - \delta, \mu_{n*})$, contradicting that types below $\mu_{n*}$ prefer $(0,0)$ the most.

Hence, let $\mu_\sim = sup_n \mu_{n*}$, so for all $n \in \mathbb{N}$, $\mu_{n*} \leq \mu_\sim$ and of course $\mu_n^* \leq \overline{\mu}$ . For each $n$, $p_n(x_n^*) = E[\mu|\mu \in [\mu_{n*}, \mu_n^*]]x_n^* \leq \zeta x_n^*$ where $\zeta := E[\mu|\mu \in [\mu_\sim, \overline{\mu}]] < \overline{\mu}$. But $p_n(\overline{x}) = \overline{\mu x}$ and each $p_n(\cdot)$ is $L$-Lipshitz with $L = \overline{\mu} + g'(\overline{\mu}, 0)$ by AG Proposition 1, so

$$0 < \overline{x}|\overline{\mu} - \zeta| \leq |\overline{\mu x} - \zeta x_n^*| \leq |p_n(\overline{x}) - p_n(x_n^*)| \leq L|\overline{x} - x_n^*| \underset{n\to\infty}{\to} 0,$$

a contradiction. □

Lemma 16 does not require $f$ is log-concave, and if this assumption is dropped there may be multiplicity of equilibria. Also, in the first statement, $\underline{x}_0$, in addition to depending on the distribution of types $f$ may depend on $\overline{x}$, so it is not possible to state the condition in terms of only the difference $\overline{x} - \underline{x}$; similarly $\overline{x}_0$ depends on $\underline{x}$. It is possible to find at least coarse bounds on $\underline{x}_0, \overline{x}_0$ but we have not pursued this direction.

## C.3 Pooling at zero implies pooling at the minimum coverage (Lemma 1)

We now show that, if some agents choose not to buy, then some must also choose the minimum coverage.

**Lemma 17.** *In any AG equilibrium $(p, \alpha)$, if $x = 0$ is an atom of $\alpha_X$, then $x = \underline{x}$ is an atom of $\alpha_X$ as well.*

*Proof.* By way of contradiction, assume $\alpha_X(\{0\}) > 0$ but $\alpha_X(\{\underline{x}\}) = 0$. First we deal with the case $\alpha_X((\underline{x}, \overline{x}]) > 0$. Let $\sigma$ be the associated coverage function. By definition $\sigma(\mu^*) = x^*$, $p(x^*) = x^*\mu^*$, and $\sigma(\mu) = 0$ if and only if $\mu < \mu^*$. Since agents are strictly risk averse, $\mu^*$ strictly prefers $(x^*, p(x^*))$ to $(0,0)$, and hence so do all types $\mu \in (\mu^* - \delta, \mu^*)$ for some $\delta > 0$, contradicting the fact that types $\mu < \mu^*$ purchase $0$.

If $\alpha_X((\underline{x}, \overline{x}]) = 0$, then $\alpha_X(\{0\}) = 1$. However $p(\underline{x}) \leq \overline{\mu}\underline{x}$ and hence all types $\mu \in (\overline{\mu} - \delta, \overline{\mu})$ for some $\delta > 0$ strictly prefer $(\underline{x}, p(\underline{x}))$ to $(0,0)$, and hence $\alpha_X(\underline{x}) > 0$ as well. □

## C.4 Proof of Proposition 6

We show that there is an $\underline{x}_1 \in [\underline{x}_0, \overline{x})$ such that, $\forall \underline{x} \in (\underline{x}_0, \underline{x}_1)$ the equilibrium regime is PPPP, where $\underline{x}_0$ is defined by (8). Let $p(\cdot), \sigma(\cdot)$ be the prices and coverages in the RS equilibrium when the contract space is $\{0\} \cup [\underline{x}_0, \overline{x}]$.

**Lemma 18.** *There is $\delta > 0$ s.t. for $\underline{x} \in (\underline{x}_0, \underline{x}_0 + \delta)$, no equilibrium of the economy with contracts $X = \{0\} \cup [\underline{x}, \overline{x}]$ is a Lemons equilibrium (where, by Lemons equilibrium, we refer also to equilibria where only minimum coverage is purchased).*

*Proof.* Suppose not; then there is a decreasing sequence $\underline{x}_n \to \underline{x}_0$ and, for each $n \in \mathbb{N}$, a Lemons equilibrium supported on $\{0, \underline{x}_n\}$ with associated cut-off purchase type $\mu_{*n}$ and prices $p_n(\cdot)$. In each of these, $(\underline{x}_n, \underline{x}_n E[\mu \mid \mu \geq \mu_{*n}])$ is weakly preferred by $\overline{\mu}$ over $(\overline{x}, p_n(\overline{x}))$. W.l.o.g., $E[\mu \mid \mu \geq \mu_{*n}] \to \mu' \in [\underline{\mu}, \overline{\mu}]$ and $p_n(\overline{x}) \to p' \in [0, \overline{\mu x}]$, so $(\underline{x}_0, \underline{x}_0 \mu')$ is weakly preferred by $\overline{\mu}$ over $(\overline{x}, p')$, and hence she certainly weakly prefers $(\underline{x}_0, \underline{x}_0 \underline{\mu})$ to $(\overline{x}, \overline{\mu x})$. But since we have a fully separating equilibrium when $X = \{0\} \cup [\underline{x}_0, \overline{x}]$ and the slope of price in the RS equilibrium is strictly a.e. less than the slope of $\overline{\mu}$'s indifference curve, $\overline{\mu}$ strictly prefers $(\overline{x}, \overline{\mu x})$ to $(\underline{x}_0, \underline{x}_0 \underline{\mu})$, a contradiction. □

Now, we complete the proof of Proposition 6. Suppose, by contradiction, there is a decreasing sequence $\underline{x}_n \to \underline{x}_0$ and, for each $n \in \mathbb{N}$, a Dispersive equilibrium on $X = \{0\} \cup [\underline{x}_n, \overline{x}]$, with price $p_n(\cdot)$, and associated cut-off types $\underline{\mu} < \mu_{n*}, \mu_n^* < \overline{\mu}$. First, we contend, $x_n^* \to \underline{x}_0$; if not, w.l.o.g., we may assume $\underline{x}_0 < x^\circ := \inf x_n^*$. Hence $\inf \mu_n^* \geq \mu_0^* := \sigma^{-1}(x^\circ) > \underline{\mu}$. By Lemma 12, we have $p_n(\cdot) \leq p(\cdot)$ in $[\underline{x}_0, x^\circ]$ for each $n$, so

$$\limsup \underline{x}_0 E[\mu \mid \mu_{n*} \leq \mu \leq \mu_n^*] = \limsup p_n(\underline{x}_n) \leq \lim p(\underline{x}_n) = \underline{\mu} \underline{x}_0$$

i.e., $E[\mu \mid \mu \leq \mu_0^*] \leq \limsup E[\mu \mid \mu_{n*} \leq \mu \leq \mu_n^*] \leq \underline{\mu}$, a contradiction.

So $x_n^* \to \underline{x}_0$, and hence $\mu_{n*}, \mu_n^* \to \underline{\mu}$, $p_n(\underline{x}_n) \to p(\underline{x}_0)$. Since the equilibria in the sequence are Dispersive, $\mu_{n*}$ is indifferent between $(0,0)$ and $(\underline{x}_n, p_n(\underline{x}_n))$; hence $\underline{\mu}$ is indifferent between $(0,0)$ and $(\underline{x}_0, p(\underline{x}_0))$. However, since $p(\underline{x}_0) = \underline{\mu} \underline{x}_0$ and agents are risk averse, $\underline{\mu}$ strictly prefers $(\underline{x}_0, p(\underline{x}_0))$ to $(0,0)$, contradiction.

## C.5 Partial converse to the first part of Proposition 5

We also note here a partial converse to the first part of Proposition 5.

**Proposition 18.** *Suppose in an equilibrium $(p, \alpha)$, all agents are purchasing a positive level of insurance, i.e., $\alpha_X(\{0\}) = 0$. If at least some are purchasing strictly above $\underline{x}$ , then $\overline{\mu}$ strictly prefers $(\overline{x}, \overline{\mu x})$ to $(\underline{x}, \underline{x} E[\mu])$, i.e. (5) holds.*

*Proof.* Suppose some are purchasing strictly above $\underline{x}$, and yet $\overline{\mu}$ weakly prefers $(\underline{x}, \underline{x}E[\mu])$ to $(\overline{x}, \overline{\mu x})$. We have $p(\underline{x}) = \underline{x}E[\mu \mid \mu \leq \mu^*] < \underline{x}E[\mu]$. Hence types near $\overline{\mu}$ strictly prefer $(\underline{x}, p(\underline{x}))$ to $(\overline{x}, \overline{\mu x})$, contradicting that these types purchase near $(\overline{x}, \overline{\mu x})$. □

We note that all agents purchase positive levels of coverage if it is mandated, i.e., if the cost $T$ of not purchasing is high.

# D Log-Concavity and Uniqueness

## D.1 Log-Concavity

Recall that we denote

$$E(\mu_*, \mu^*) = \mathbb{E}[\mu \mid \mu \in (\mu_*, \mu^*)],$$

$$\phi(\mu_*, \mu^*) = \mu^* - E(\mu^*, \mu_*) \geq 0,$$

$$\psi(\mu_*, \mu^*) = E(\mu^*, \mu_*) - \mu_* \geq 0.$$

We now prove Lemma 2, which stated that, for log-concave $f$, then

$$\frac{\partial \psi}{\partial \mu^*} \in (0,1), \qquad \frac{\partial \phi}{\partial \mu^*} \in (0,1), \qquad \frac{\partial \phi}{\partial \mu_*} \in (-1,0), \qquad \frac{\partial \psi}{\partial \mu_*} \in (-1,0).$$

*Proof.* First, by definition, and since $f > 0$ in $[\underline{\mu}, \overline{\mu}]$, $\frac{\partial \phi}{\partial \mu_*} < 0$, $\frac{\partial \psi}{\partial \mu^*} > 0$. Second, Bagnoli and Bergstrom [2005] show that log-concavity implies $\frac{\partial \phi}{\partial \mu^*} > 0$, $\frac{\partial \psi}{\partial \mu_*} < 0$.[42] Third, observe that $\phi = \mu^* - \mu_* - \psi$. This identity, together with the previous result, imply the bounds

$$\frac{\partial \psi}{\partial \mu^*} < 1, \qquad \frac{\partial \phi}{\partial \mu^*} < 1, \qquad \frac{\partial \psi}{\partial \mu_*} > -1, \qquad \frac{\partial \phi}{\partial \mu_*} > -1.$$

□

Notice that log-concavity also implies $\frac{\partial E}{\partial \mu^*} \in (0,1)$ and $\frac{\partial E}{\partial \mu_*} \in (0,1)$.

## D.2 Uniqueness (Proposition 8)

We now show Proposition 8 (if $f$ log-concave equilibrium is unique). When $\underline{x} = 0$, this has already been shown in Levy and Veiga [2021] (under $\overline{x} = 1$, but the proof follows

[42] Bagnoli and Bergstrom [2005] actually only states $\frac{\partial \phi}{\partial \mu^*} \geq 0$ and $\frac{\partial \psi}{\partial \mu_*} \leq 0$, but an examination of the proofs there, in particular the last two inequalities on p. 467, show that strict inequalities actually hold.

verbatim for any $\overline{x} > 0$), and can be seen by the solution satisfying (20) of Proposition 16. Hence, we proceed under the assumption that $\underline{x} > 0$.

Uniqueness will follow from the properties established in Proposition 16, as well as other lemmas established in Appendix B, and by checking the various cases.

First, we already know if there are some agents choosing $x = 0$, then there are also some purchasing $x = \underline{x}$ (Lemma 1). Second, we will show that if one equilibrium has at least partial pooling (i.e., not all individuals choose $x > \underline{x}$), then all equilibria do. Third, we will show that if neither of two equilibria has pooling (i.e., if all individuals choose $x > \underline{x}$), or if both have pooling with price coinciding at $\underline{x}$, then they are the same equilibrium. Fourth, we will show that if each equilibrium has at least partial pooling but different prices of $\underline{x}$, we will deduce a contradiction. It is only in this last step that log-concavity is used. The details are as follows:

*Proof.* As mentioned, we show the case $\underline{x} > 0$. We show uniqueness of equilibrium by checking several cases. Let $(p, \alpha)$ and $(q, \beta)$ be two different equilibria.

Lemma 1 shows that if there is pooling at $0$, then there must also be pooling at $\underline{x}$. By Lemma 13, if one of the equilibria has at least partial pooling at $\underline{x}$, then they all do. If neither has pooling, both satisfy $\mu^* = \mu_* = \underline{\mu}$, they coincide as they both follow (20). If both have at least partial pooling and $p(\underline{x}) = q(\underline{x})$, then Lemma 12 shows both must have the same cut-off type $\mu^*$, and must have the same cut-off participation type $\mu_*$, and hence they coincide.

The only remaining case is they both have either partial or full pooling (possibly one of each), but differ with their price at $\underline{x}$. WLOG, $q(\underline{x}) > p(\underline{x})$. Associate with $(\alpha, p)$ the usual cut-offs $\mu^*, \mu_*$, and denote the cut-offs $(\beta, q)$ by $\omega^*, \omega_*$. Under $(\beta, q)$ agents are less willing to prefer $(\underline{x}, q(\underline{x}))$ over $(0, 0)$, so $\omega_* \geq \mu_*$. (It may be that $\mu_* = \underline{\mu}$.) We claim we must have $\omega^* \leq \mu^*$. If $(\alpha, q)$ (and possibly also $(\beta, q)$) has full pooling, then $\mu^* = \overline{\mu} \geq \omega^*$. If only $(\beta, q)$ has full pooling, then since $q(\overline{x}) \leq \overline{\mu x} = p(\overline{x})$ and $q' = \overline{\mu} + \frac{\partial g}{\partial x}(\overline{\mu}, x) > p'$ in $(\underline{x}, \overline{x})$, we have we must have $q(\underline{x}) < p(\underline{x})$, a contradiction. If they both have only partial pooling, we also know by Lemma 12, that if $\omega^* > \mu^*$, then we would have $q \leq p$ with strict inequality below the intersection of supports of $\alpha_X, \beta_X$; hence, $q(\underline{x}) < p(\underline{x})$, a contradiction.

Now either $\omega_* = \mu_*$ or $\omega_* > \mu_*$. In the former case,

$$\underline{x}E[\mu \mid \omega_* < \mu < \omega^*] = q(\underline{x}) \leq p(\underline{x}) = E[\mu \mid \mu_* < \mu < \mu^*]$$

since $\omega^* \leq \mu^*$; but this contradicts $q(\underline{x}) > p(\underline{x})$. (If both equilibria are PPPP, this ends the proof, as $\omega_* = \mu_* = \underline{\mu}$.) In the other case, i.e., $\omega^* \leq \mu^*$ with $\omega_* > \mu_*$, we have[43]

$$E[\mu \mid \omega_* < \mu < \omega^*] - \omega_* = \psi(\omega_*, \omega^*) < \psi(\mu_*, \mu^*) = E[\mu \mid \mu_* < \mu < \mu^*] - \mu_*$$

and hence, multiplying both sides by $\underline{x}$, we have $q(\underline{x}) - \omega_* \underline{x} < p(\underline{x}) - \mu_* \underline{x}$, or $\omega_* \underline{x} - q(\underline{x}) > \mu_* \underline{x} - p(\underline{x})$. However, since these are equilibria, the agent $\omega_*$ is indifferent between $(0,0)$ and $(\underline{x}, q(\underline{x}))$ and the agent $\mu_*$ at least weakly prefers $(\underline{x}, p(\underline{x}))$ to $(0,0)$ (perhaps strictly if $\mu_* = \underline{\mu}$), so we know that

$$\mu_* \underline{x} + g(\mu_*, \underline{x}) - p(\underline{x}) \geq g(\mu_*, 0) - T = g(\omega_*, 0) - T = \omega_* \underline{x} + g(\omega_*, \underline{x}) - q(\underline{x})$$

a contradiction. $\square$

# E Comparative Statics: Dispersive Regime

In this section we consider comparative statics when the equilibrium regime is Dispersive (some buy $x > \underline{x}$, some buy $\underline{x}$ and some choose not to buy). We discuss how changes in $(\underline{x}, \overline{x}, T)$ affect equilibrium and welfare. Recall that, for the comparative statics exercises, we assume $g(\mu, x) = g(x)$ is independent of $\mu$ (Assumption 3).

We begin with the following lemma, which will be used in the subsections to follow as well as in later sections of the appendix.

**Lemma 19.** *$g(x)/x$ is decreasing in $x$.*

*Proof.* Recall $g(0) = 0$. The derivative of $\frac{g(x)}{x}$ has the sign of $g'(x)x - g(x) \leq 0$. This is negative by the concavity of $g(x)$, since

$$\frac{g(x)}{x} = \frac{g(x) - g(0)}{x - 0} \geq g'(x).$$

$\square$

## E.1 Setup

We assume a "Dispersive" equilibrium (i.e., $\underline{\mu} < \mu_* < \mu^* < \overline{\mu}$). First, we write the system of equations that defines equilibrium in matrix form. Then, we implicitly differentiate this matrix equation with respect to an exogenous parameter $\eta$ (which can later be taken

[43] This is the only point at which the log-concavity is used in en route to proving Proposition 8.

to be any of the regulatory parameters $\underline{x}, \overline{x}, T$). We then use these results to discuss how these parameters affect welfare.

Recall the definitions

$$\phi\left(\mu^{*}, \mu_{*}\right)=\mu^{*}-\mathbb{E}\left[\mu \mid \mu \in\left[\mu_{*}, \mu^{*}\right)\right],$$

$$\psi\left(\mu^{*}, \mu_{*}\right)=\mathbb{E}\left[\mu \mid \mu \in\left[\mu_{*}, \mu^{*}\right)\right]-\mu_{*}.$$

We assume that $f$ is log-concave, so equilibrium is unique. From Section 5, log-concavity of $f$ implies

$$\frac{\partial \phi}{\partial \mu^{*}}>0, \qquad \frac{\partial \phi}{\partial \mu_{*}}<0, \qquad \frac{\partial \psi}{\partial \mu^{*}}>0, \qquad \frac{\partial \psi}{\partial \mu_{*}}<0 \tag{30}$$

Equilibrium is characterized by 3 equations. Each of them is potentially a function of the three endogenous parameters $(x^{*}, \mu^{*}, \mu_{*})$ and the generic exogenous regulatory parameter $\eta$. First, (2) implies:

$$H\left(x^{*}, \mu^{*}, \eta\right)=\overline{\mu}-\mu^{*}-\int_{x^{*}}^{\overline{x}} \frac{g^{\prime}(x)}{x} d x=0.$$

Second, the indifference condition of type $\mu^{*}$ is:

$$\begin{aligned} G\left(x^{*}, \mu^{*}, \mu_{*}, \eta\right) &= x^{*} \mu^{*}+g\left(x^{*}\right)-x^{*} \mu^{*}-\left(\underline{x} \mu^{*}+g(\underline{x})-\underline{x} \mathbb{E}\left[\mu \mid \mu \in\left[\mu_{*}, \mu^{*}\right)\right]\right). \\ &= g(x^{*})-g(\underline{x})-\underline{x} \phi\left(\mu^{*}, \mu_{*}\right)=0 \end{aligned}$$

Third, the indifference condition of type $\mu_{*}$ (recalling that $g(0)=0$ and that choosing $x=0$ implies paying the fee $T$) is:

$$\begin{aligned} K\left(\mu^{*}, x^{*}, \eta\right) &= \underline{x} \mu_{*}+g(\underline{x})-\underline{x} \mathbb{E}\left[\mu \mid \mu \in\left[\mu_{*}, \mu^{*}\right)\right]+T \\ &= g(\underline{x})-\underline{x} \psi\left(\mu^{*}, \mu_{*}\right)+T=0 \end{aligned}$$

This can be written in matrix form as

$$\left[\begin{array}{c} H\left(x^{*}, \mu^{*}, \eta\right) \\ G\left(x^{*}, \mu^{*}, \mu_{*}, \eta\right) \\ K\left(\mu^{*}, x^{*}, \eta\right) \end{array}\right]=\left[\begin{array}{l} 0 \\ 0 \\ 0 \end{array}\right].$$

Total differentiation with respect to a generic exogenous parameter $\eta$ and writing in matrix form implies

$$\begin{bmatrix} \frac{\partial H}{\partial \mu^*} & \frac{\partial H}{\partial x^*} & \frac{\partial H}{\partial \mu_*} \\ \frac{\partial G}{\partial \mu^*} & \frac{\partial G}{\partial x^*} & \frac{\partial G}{\partial \mu_*} \\ \frac{\partial K}{\partial \mu^*} & \frac{\partial K}{\partial x^*} & \frac{\partial K}{\partial \mu_*} \end{bmatrix} \begin{bmatrix} \frac{\partial \mu^*}{\partial \eta} \\ \frac{\partial x^*}{\partial \eta} \\ \frac{\partial \mu_*}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -\frac{\partial H}{\partial \eta} \\ -\frac{\partial G}{\partial \eta} \\ -\frac{\partial K}{\partial \eta} \end{bmatrix}.$$

We now compute the terms in the square matrix above and the vector on the RHS. First, we have

$$\frac{\partial H}{\partial \mu^*} = -1, \qquad \frac{\partial H}{\partial x^*} = \frac{g'(x^*)}{x^*}, \qquad \frac{\partial H}{\partial \mu_*} = 0$$

$$\frac{\partial H}{\partial \underline{x}} = 0 \qquad \frac{\partial H}{\partial \overline{x}} = -\frac{g'(\overline{x})}{\underline{x}}$$

Second, we have

$$\frac{\partial G}{\partial \mu^*} = -\underline{x}\frac{d\phi}{d\mu^*}, \qquad \frac{\partial G}{\partial x^*} = g'(x^*), \qquad \frac{\partial G}{\partial \mu_*} = -\underline{x}\frac{\partial \phi}{\partial \mu_*}$$

$$\frac{\partial G}{\partial \underline{x}} = -g'(\underline{x}) - \phi \qquad \frac{\partial G}{\partial \overline{x}} = 0$$

Third, we have

$$\frac{\partial K}{\partial x^*} = 0, \qquad \frac{\partial K}{\partial \mu^*} = -\underline{x}\frac{\partial \psi}{\partial \mu^*}, \qquad \frac{\partial K}{\partial \mu_*} = -\underline{x}\frac{\partial \psi}{\partial \mu_*} \geq 0$$

$$\frac{\partial K}{\partial \underline{x}} = -\psi + g'(\underline{x}) \qquad \frac{\partial K}{\partial \overline{x}} = 0$$

The determinant of the square matrix is

$$D \;=\; g'(x^*)\underline{x}\left[\frac{\partial \psi}{\partial \mu_*} - \frac{\underline{x}}{x^*}\left(\frac{\partial \phi}{\partial \mu^*}\frac{\partial \psi}{\partial \mu_*} - \frac{\partial \phi}{\partial \mu_*}\frac{\partial \psi}{\partial \mu^*}\right)\right]. \tag{31}$$

Then, for a generic parameter $\eta$, by Cramer's Rule

$$\frac{\partial \mu^*}{\partial \eta}D \;=\; -\frac{\partial H}{\partial \eta}\frac{\partial G}{\partial x^*}\frac{\partial K}{\partial \mu_*} + \frac{\partial H}{\partial x^*}\left(\frac{\partial G}{\partial \eta}\frac{\partial K}{\partial \mu_*} - \frac{\partial G}{\partial \mu_*}\frac{\partial K}{\partial \eta}\right).$$

Also, for a generic parameter $\eta$, by Cramer's Rule

$$
\begin{aligned}
\frac{\partial \mu_*}{\partial \eta} D &= \frac{\partial H}{\partial \mu^*}\left(\frac{\partial G}{\partial \eta}\frac{\partial K}{\partial x^*} - \frac{\partial G}{\partial x^*}\frac{\partial K}{\partial \eta}\right) \\
&\quad + \frac{\partial H}{\partial x^*}\left(\frac{\partial G}{\partial \mu^*}\frac{\partial K}{\partial \eta} - \frac{\partial G}{\partial \eta}\frac{\partial K}{\partial \mu^*}\right) \\
&\quad + \frac{\partial H}{\partial \eta}\left(\frac{\partial G}{\partial x^*}\frac{\partial K}{\partial \mu^*} - \frac{\partial G}{\partial \mu^*}\frac{\partial K}{\partial x^*}\right)
\end{aligned}
$$

Finally, for a generic parameter $\eta$ and again by Cramer's Rule

$$
\begin{aligned}
\frac{\partial x^*}{\partial \eta} D &= -\frac{\partial H}{\partial \mu^*}\left(\frac{\partial G}{\partial \eta}\frac{\partial K}{\partial \mu_*} - \frac{\partial K}{\partial \eta}\frac{\partial G}{\partial \mu_*}\right) \\
&\quad - \frac{\partial H}{\partial \eta}\left(\frac{\partial G}{\partial \mu^*}\frac{\partial K}{\partial \mu_*} - \frac{\partial K}{\partial \mu^*}\frac{\partial G}{\partial \mu_*}\right)
\end{aligned}
$$

## E.2 Sign of Determinant $D$

**Lemma 20.** *If $f$ log-concave, then $D < 0$.*

*Proof.* Let $E = \mathbb{E}[\mu \mid \mu \in [\mu_*, \mu^*)]$. Notice that $\frac{\partial E}{\partial \mu^*} > 0$ and $\frac{\partial E}{\partial \mu_*} > 0$ (in both cases, the change involves conditioning on larger values of $\mu$). Recall that $\phi = \mu^* - E$ and $\psi = E - \mu_*$. Then the determinant $D$ from (31) can be written as

$$
\begin{aligned}
D &= g'(x^*)\underline{x}\left[\frac{\partial E}{\partial \mu_*} - 1 - \frac{\underline{x}}{x^*}\left(\left[1 - \frac{\partial E}{\partial \mu^*}\right]\left[\frac{\partial E}{\partial \mu_*} - 1\right] - \left[-\frac{\partial E}{\partial \mu_*}\right]\left[\frac{\partial E}{\partial \mu^*}\right]\right)\right] \\
&= g'(x^*)\frac{\underline{x}}{x^*}\left[\left(\frac{\partial E}{\partial \mu_*} - 1\right)(x^* - \underline{x}) - \underline{x}\frac{\partial E}{\partial \mu^*}\right].
\end{aligned}
$$

Recall $\frac{\partial E}{\partial \mu^*} > 0$ and $x^* - \underline{x} \geq 0$. If $f$ log-concave, then $\frac{\partial \psi}{\partial \mu_*} = \frac{\partial E}{\partial \mu_*} - 1 < 0$. Therefore, $f$ log-concave is a sufficient (but not necessary) condition for $D < 0$. For instance, if $\mu$ is uniform (and thus log-concave), then $D = -\frac{1}{2}g'(x^*)\underline{x} < 0$. □

## E.3 Effect of Regulatory Parameters on the Allocation, in the Region of Full Separation

We now discuss the effect of the regulatory parameters $\underline{x}, \overline{x}, T$ on the allocation $\sigma$ in the region of those types for which there is full separation, i.e., $\mu \in [\mu^*, \overline{\mu}]$. From Lemma 1, for $\mu \in [\mu^*, \overline{\mu}]$, the allocation $\sigma$ is independent of $\underline{x}, T$. Therefore,

$$\frac{\partial \sigma}{\partial \underline{x}} = \frac{\partial \sigma}{\partial T} = 0, \qquad \forall \mu \in [\mu^*, \overline{\mu}].$$

We now prove that $\frac{\partial \sigma}{\partial \overline{x}} > 0, \forall \mu \in [\mu^*, \overline{\mu}]$ for $\overline{x} < 1$.

*Proof.* Recall that, in the region of full separation ($\mu \in [\mu^*, \overline{\mu}]$), $\sigma$ satisfies (20), i.e.

$$\overline{\mu} - \mu - \int_{\sigma(\mu)}^{\overline{x}} \frac{g'(x)}{x} dx = 0, \qquad \forall \mu \in [\mu^*, \overline{\mu}].$$

The function $\sigma(\cdot)$ is locally independent of $\underline{x}$. That is, if $\mu > \mu^*(\underline{x}, \overline{x})$, then the above equation shows that $\sigma(\mu, \underline{x}, \overline{x}) = \sigma(\mu, \underline{x}', \overline{x})$ for $\underline{x}'$ close to $\underline{x}$. By implicit differentiation, for $\overline{x} < 1$,

$$\frac{\partial \sigma}{\partial \overline{x}} = -\left(-\frac{g'(\overline{x})}{\overline{x}}\right) \frac{1}{-\left(-\frac{g'(\sigma(\mu, \underline{x}, \overline{x}))}{\sigma(\mu)}\right)} = \frac{g'(\overline{x})}{\overline{x}} \frac{\sigma(\mu, \underline{x}, \overline{x})}{g'(\sigma(\mu, \underline{x}, \overline{x}))} > 0, \qquad \forall \mu \in [\mu^*, \overline{\mu}].$$

□

## E.4 Adjusting $T$: Equilibrium

We now prove Lemma 3, which concerns the effect of $T$ on $\mu^*, \mu_*$. We use the equations of Section E. Notice that $T$ is relevant only for one of the equilibrium conditions, namely

$$K(\mu^*, x^*, \eta) = g(\underline{x}) - \underline{x}\psi(\mu^*, \mu_*) + T = 0.$$

In this context,

$$\frac{\partial G}{\partial T} = \frac{\partial H}{\partial T} = 0, \qquad \frac{\partial K}{\partial T} = 1.$$

The partial derivatives of $G, H, K$ w.r.t. $\mu^*, \mu_*, x^*$ are unchanged. From the calculations above,

$$\begin{aligned} \frac{\partial \mu^*}{\partial T} D &= -\frac{\partial H}{\partial T}\frac{\partial G}{\partial x^*}\frac{\partial K}{\partial \mu_*} + \frac{\partial H}{\partial x^*}\left(\frac{\partial G}{\partial T}\frac{\partial K}{\partial \mu_*} - \frac{\partial G}{\partial \mu_*}\frac{\partial K}{\partial T}\right) \\ &= -\frac{\partial H}{\partial x^*}\frac{\partial G}{\partial \mu_*} \\ &= \frac{g'(x^*)}{x^*}\underline{x}\frac{\partial \phi}{\partial \mu_*} \end{aligned}$$

Since $f$ log-concave, then $\frac{\partial\phi}{\partial\mu_*} < 0$ and $D < 0$. Therefore

$$\frac{\partial\mu^*}{\partial T} > 0$$

and since $\overline{x}$ is unchanged, also $\frac{\partial x^*}{\partial T} > 0$. Moreover,

$$\frac{\partial\mu_*}{\partial T} D \quad = \quad -\frac{\partial H}{\partial\mu^*}\frac{\partial G}{\partial x^*} + \frac{\partial H}{\partial x^*}\frac{\partial G}{\partial\mu^*} = g'(x^*)\left[1 - \frac{\underline{x}}{x^*}\frac{d\phi}{d\mu^*}\right]$$

Notice that $\frac{\partial\phi}{\partial\mu^*} < 1$ and $\underline{x} < x^*$, so $1 - \frac{\underline{x}}{x^*}\frac{d\phi}{d\mu^*} < 0$. Since $f$ log-concave, then $D < 0$. Therefore,

$$\frac{\partial\mu_*}{\partial T} < 0.$$

## E.5 Adjusting $T$: Welfare

We now prove Proposition 9, which provides conditions under which welfare is increasing in $T$. We also assume that we are in the domain of dispersive equilibria (some buy $x > \underline{x}$, some buy $\underline{x}$, and some do not buy). The derivative of welfare with respect to $T$ is

$$\frac{\partial W}{\partial T} = -\frac{1}{D}g'(x^*)g(\underline{x})\left[f(\mu^*)\frac{\underline{x}}{x^*}\frac{\partial\phi}{\partial\mu_*}\left[\frac{g(x^*)}{g(\underline{x})} - 1\right] + f(\mu_*)\left[1 - \frac{\underline{x}}{x^*}\frac{\partial\phi}{\partial\mu^*}\right]\right]$$

Log-concave $f$ implies $-\frac{1}{D} > 0$. Therefore,

$$\frac{\partial W}{\partial T} > 0 \Leftrightarrow f(\mu^*)\frac{\underline{x}}{x^*}\frac{\partial\phi}{\partial\mu_*}\left[\frac{g(x^*)}{g(\underline{x})} - 1\right] + f(\mu_*)\left[1 - \frac{\underline{x}}{x^*}\frac{\partial\phi}{\partial\mu^*}\right] > 0 \tag{32}$$

Recall the notation $E = \mathbb{E}\left[\mu \mid \mu \in [\mu_*, \mu^*]\right]$. By Lemma 2, $\frac{\partial\phi}{\partial\mu_*} > -1$ and $\frac{\partial\phi}{\partial\mu^*} < 1$. Also, $\frac{\partial\phi}{\partial\mu_*} < 0$. In fact, since $f > 0$, $\frac{\partial\phi}{\partial\mu^*} = 1 - \frac{\partial E}{\partial\mu^*} < 1$. Notice that $\frac{g(x^*)}{g(\underline{x})} - 1 > 0$, $f > 0$, $x^* \geq \underline{x} > 0$. Recall $\frac{\partial\phi}{\partial\mu_*} \in [-1, 0]$. Also, since $g(x)/x$ is decreasing (Lemma 19), then

$$\frac{g(x^*)/x^*}{g(\underline{x})/\underline{x}} \leq 1.$$

We now derive bounds on each term of (32). First,

$$
\begin{aligned}
f(\mu^*)\frac{\underline{x}}{x^*}\frac{\partial\phi}{\partial\mu_*}\left[\frac{g(x^*)}{g(\underline{x})}-1\right] &= f(\mu^*)\frac{\partial\phi}{\partial\mu_*}\left[\frac{g(x^*)x^*}{g(\underline{x})\underline{x}}-\frac{\underline{x}}{x^*}\right] \\
&\geq f(\mu^*)\frac{\partial\phi}{\partial\mu_*}\left[1-\frac{\underline{x}}{x^*}\right] \\
&> -f(\mu^*)\left[1-\frac{\underline{x}}{x^*}\right].
\end{aligned}
$$

Moreover,

$$
f(\mu_*)\left[1-\frac{\underline{x}}{x^*}\frac{\partial\phi}{\partial\mu^*}\right] \geq f(\mu_*)\left[1-\frac{\underline{x}}{x^*}\right].
$$

Hence, a sufficient condition for (32) to hold is

$$
-f(\mu^*)\left[1-\frac{\underline{x}}{x^*}\right]+f(\mu_*)\left[1-\frac{\underline{x}}{x^*}\right]\geq 0 \tag{33}
$$

Since $1-\frac{\underline{x}}{x^*}>0$, (33) holds if and only if $f(\mu_*)\geq f(\mu^*)$, and if this condition holds, inequality is strictly.

Note that, if $f$ is weakly decreasing for $\mu\in[\mu_*,\overline{\mu}]$, then for $T=\hat{T}$, $\frac{\partial W}{\partial T}\mid_{T=\hat{T}}>0$.

## E.6 Adjusting $\overline{x}$: Equilibrium

We now prove Lemma 10, which concerns the effect of $\overline{x}$ on $\mu^*,\mu_*$. Assume $\overline{x}<1$. If $f$ log-concave, then $D<0$ (Appendix E.2). Moreover,

$$
\frac{\partial H}{\partial\mu_*}=\frac{\partial H}{\partial\underline{x}}=\frac{\partial G}{\partial\overline{x}}=\frac{\partial K}{\partial x^*}=\frac{\partial K}{\partial\overline{x}}=0.
$$

Recall also that $\frac{\partial\psi}{\partial\mu^*}>0$ and $\frac{\partial\psi}{\partial\mu_*}<0$. Using the notation above, the generic parameter $\eta$ refers in this case to the maximum coverage $\overline{x}$.

Using the results above, we obtain

$$
\begin{aligned}
\frac{\partial\mu^*}{\partial\overline{x}} &= -\frac{1}{D}\frac{\partial H}{\partial\overline{x}}\frac{\partial G}{\partial x^*}\frac{\partial K}{\partial\mu_*} \\
&= -\frac{1}{D}g'(\overline{x})g'(x^*)\frac{\partial\psi}{\partial\mu_*}<0
\end{aligned}
$$

Moreover,

$$\begin{aligned}\frac{\partial\mu_*}{\partial\overline{x}} &= \frac{1}{D}\frac{\partial H}{\partial\overline{x}}\frac{\partial G}{\partial x^*}\frac{\partial K}{\partial\mu^*} \\ &= \frac{1}{D}g'(\overline{x})g'(x^*)\frac{\partial\psi}{\partial\mu^*} < 0.\end{aligned}$$

We can also obtain

$$\frac{\partial x^*}{\partial\overline{x}}D = -g'(\overline{x})\underline{x}\left(\frac{d\phi}{d\mu^*}\frac{\partial\psi}{\partial\mu_*} - \frac{\partial\psi}{\partial\mu^*}\frac{\partial\phi}{\partial\mu_*}\right)$$

Recall the notation, $E = \mathbb{E}[\mu \mid \mu \in [\mu_*\mu^*]]$, $\phi = \mu^* - E$ and $\psi = E - \mu_*$. Then, the expression above can be written as

$$\frac{\partial x^*}{\partial\overline{x}}D = -g'(\overline{x})\underline{x}\left(\frac{\partial E}{\partial\mu_*} + \frac{\partial E}{\partial\mu^*} - 1\right).$$

If $f$ is log-concave, then $D < 0$, so $\frac{\partial x^*}{\partial\overline{x}}$ has the same sign as $\frac{\partial E}{\partial\mu_*} + \frac{\partial E}{\partial\mu^*} - 1$. If $f(\mu)$ uniform, then $\frac{\partial E}{\partial\mu_*} = \frac{\partial E}{\partial\mu^*} = \frac{1}{2}$ so $\frac{\partial x^*}{\partial\overline{x}} = 0$.

In general, the effect of $\overline{x}$ on the threshold contract $x^*$ is ambiguous. A change in $\overline{x}$ changes the cutoff type $\mu^*$ but also has a direct effect on the shape of $\sigma$ in the region of full separation.

## E.7 Adjusting $\underline{x}$: Equilibrium

We now prove a result concerning the effect of $\underline{x}$ on $\mu^*, \mu_*$. This result requires the strong assumption that $f$ is uniform, so it is mentioned only briefly in the body of the paper. Recall that

$$\frac{\partial H}{\partial\mu_*} = \frac{\partial H}{\partial\underline{x}} = \frac{\partial G}{\partial\overline{x}} = \frac{\partial K}{\partial x^*} = \frac{\partial K}{\partial\overline{x}} = 0.$$

Given $f(\mu)$ is log-concave, recall (11),

$$\frac{\partial\psi}{\partial\mu^*} \in (0,1), \qquad \frac{\partial\phi}{\partial\mu^*} \in (0,1), \qquad \frac{\partial\phi}{\partial\mu_*} \in (-1,0), \qquad \frac{\partial\psi}{\partial\mu_*} \in (-1,0).$$

We now take the generic parameter $\eta$ to be the minimum coverage $\underline{x}$.

Using the results above,

$$\frac{\partial \mu^*}{\partial \underline{x}} D = \frac{\partial H}{\partial x^*} \left( \frac{\partial G}{\partial \underline{x}} \frac{\partial K}{\partial \mu_*} - \frac{\partial G}{\partial \mu_*} \frac{\partial K}{\partial \underline{x}} \right).$$

After some algebra, this can be written as

$$\frac{\partial \mu^*}{\partial \underline{x}} D = \frac{g'(x^*)}{x^*} \underline{x} \left( -g'(\underline{x}) + \frac{\partial E}{\partial \mu_*} [\mu^* - \mu_*] - \phi \right)$$

In general this effect is not signed. However, if $f(\mu)$ is uniform, then $\phi = \psi = \frac{1}{2}(\mu^* - \mu_*)$ and $\frac{\partial E}{\partial \mu_*} = \frac{1}{2}$, so

$$\frac{\partial \mu^*}{\partial \underline{x}} = -\frac{1}{D} g'(x^*) g'(\underline{x}) \frac{\underline{x}}{x^*} > 0.$$

We can then show that, for $f$ uniform, $\frac{\partial x^*}{\partial \underline{x}} > 0$ follows similarly, or by using $\frac{\partial \mu^*}{\partial \underline{x}} > 0$ and the fact that (2) implies that $-\frac{\partial \mu^*}{\partial \underline{x}} = -\frac{\partial x^*}{\partial \underline{x}} \frac{g'(x^*)}{x^*}$.

We now compute

$$\frac{\partial \mu_*}{\partial \underline{x}} D = \frac{\partial G}{\partial x^*} \frac{\partial K}{\partial \underline{x}} + \frac{\partial H}{\partial x^*} \left( \frac{\partial G}{\partial \mu^*} \frac{\partial K}{\partial \underline{x}} - \frac{\partial G}{\partial \underline{x}} \frac{\partial K}{\partial \mu^*} \right)$$

After some algebra, this can be written as

$$\frac{\partial \mu_*}{\partial \underline{x}} = \frac{1}{D} g'(x^*) \left[ [-\psi + g'(\underline{x})] + \frac{\underline{x}}{x^*} \left( -g'(\underline{x}) - \frac{\partial E}{\partial \mu^*} [\mu^* - \mu_*] + \psi \right) \right]$$

In particular, if $f(\mu)$ is uniform, then

$$\frac{\partial \mu_*}{\partial \underline{x}} = \frac{1}{D} g'(x^*) \left[ -\psi + g'(\underline{x}) \left[ 1 - \frac{\underline{x}}{x^*} \right] \right].$$

We have not been able to sign $\frac{\partial \mu_*}{\partial \underline{x}}$.

### E.8 Adjusting $T$: Welfare (a Generalization)

We now describe a generalization of Proposition 9. First, we prove an auxiliary result (Proposition 19). Then, we use this result to generalize Proposition 9.

**Proposition 19.** *Let $X$ be an absolutely continuous random variable with a pdf $f$ concentrated on some interval $[a, b]$. Assume $f$ is either monotonic or single-peaked (and attains* *a maximum); and that $f(a) > 0$. Denote $\rho = \frac{\max_{x\in[a,b]} f(x)}{f(a)}$. Then there exists an absolutely continuous random variable $Y$ concentrated on $[a, b]$, which (first-order) stochastically dominates $X$, and such that there exist $z \in [a, b]$, $v_1, v_2 > 0$ with $\rho = \frac{v_2}{v_1}$, such that the pdf $g$ of $Y$ satisfies $g(x) = v_1$ (resp. $v_2$) if $x \in [a, z]$ (resp. $\in [z, b]$).*

Note that taking $z \in \{a, b\}$ allows for the possibility that $Y$ is uniformly distributed on $[a, b]$.

*Proof.* We deal with 3 cases: $f$ monotonically decreasing; $f$ monotonically increasing; and $f$ single-peaked, where the peak may be a continuum

If $f$ is monotonically decreasing, then the uniform distribution stochastically dominates $X$. We note for later use that in this case the pdf of $Y$ is a constant constant function $g(\cdot) < f(a)$.

If $f$ is monotonically increasing, then set $v_1 = f(a)$, $v_2 = f(b) = \max_{x\in[a,b]} f(x)$, and set $z \in (a, b)$ to be such that

$$\int_a^z [f(x) - f(a)]dx = \int_z^b [f(b) - f(x)]dx \Longleftrightarrow \int_a^b f(x)dx = f(a)(z - a) + f(b)(b - z)$$

which exists since $f(a) \leq \int_a^b f(x)dx \leq f(b)$ in this case. Let $Y$ have pdf $g$ which is a step function, $g(\cdot) \equiv v_1$ in $[a, z]$ and $\equiv v_2$ in $[z, b]$.

For the final case, $f$ being single-peaked, first let's deal with the case that $f$ is a step function with 3 steps: constant $\alpha$ in an interval $[a, u]$, constant $\beta > \alpha$ in an interval $[u, v]$, and constant $\gamma < \beta$ in an interval $[v, b]$. Heuristically, we raise the third step to be equal to the second, and then renormalize; formally, denoting $\delta = (\beta - \gamma)(b - v)$, and defining $Y$ with pdf

$$g(x) = \begin{cases} \frac{1}{1+\delta}\alpha & \text{if } x \in [a, u] \\ \frac{1}{1+\delta}\beta & \text{if } x \in [u, b] \end{cases}$$

it can be verified that $Y$ stochastic dominates $X$, and clearly $\rho = \frac{\max_{x\in[a,b]} g(x)}{g(a)}$.

Finally, for general single-peaked $f$, let $x^*$ be a point where a maximum is obtained. The handling above for the case of monotonic functions can now be applied separately to two random variables - $X$ conditional on $[a, x^*]$ and $X$ conditional on $[x^*, b]$ - to obtain a random variable which stochastically dominates $f$, with a pdf $\tilde{f}$ which is a step function, with 3 steps, satisfying $\frac{\max_{x\in[a,b]} \tilde{f}(x)}{\tilde{f}(a)}$, and $\tilde{f}$ is either monotonically increasing or the middle step is highest. $\square$

We now turn to the generalization of Proposition 9. We continue to assume $f$ log-concave. This means that $f$ is either monotonic or single peaked. Intuitively, the generalization says that welfare is increasing in $T$ if there is a relatively large mass of the lowest-risk types. That is, if the density does not increase much above $f\left(\underline{\mu}\right)$ before decreasing. Proposition 9 is a special case, since states that welfare increases with $T$ if $f$ decreasing.

As we discuss below, our generalization shows that welfare is increasing in $T$ if $x^*$ is sufficiently close to $\underline{x}$ (although of course $x^*$ is endogenous). We then use that part of the result to provide a lower bound on $\underline{x}$ such that welfare is increasing for all $T$ (with the bound depending on moments of the distribution $f$).

We begin by defining the following moment of the type distribution $f$. Let

$$\rho := \frac{1}{f(\underline{\mu})}\left[\max_{\mu} f(\mu)\right] \tag{34}$$

The moment $\rho$ is the ratio of the density at the modal (most common) risk type to the density of the lowest risk type, $\underline{\mu}$. In general, $\rho \geq 1$. If $f\left(\mu\right)$ is (even weakly) decreasing, then $\rho = 1$. Then, define the following monotonic transformation of $\rho$:

$$\Gamma\left(\rho\right) = \frac{\sqrt{\rho}}{\sqrt{\rho}+1}. \tag{35}$$

Notice that $\Gamma\left(\rho\right)$ is increasing and $\Gamma(\rho) < 1$. If $f$ is uniform or decreasing, then $\rho = 1$ which corresponds to the minimum of $\Gamma\left(1\right) = 1/2$.

**Proposition 20.** *Suppose $f$ is log-concave, with $f(\underline{\mu}) > 0$. Suppose that, for $T = \hat{T} \geq 0$, the equilibrium is Dispersive (i.e., some individuals buy $x > \underline{x}$, some buy $x = \underline{x}$ and some buy $x = 0$). Suppose the associated threshold coverage is $x^*$. Then*

$$\frac{g(x^*)}{g(\underline{x})} < \frac{1}{\Gamma\left(\rho\right)} \implies \frac{\partial W}{\partial T}\mid_{T=\hat{T}} > 0.$$

*Proof.* Since $f$ is log-concave, and since we have already proven $\frac{\partial W}{\partial T} > 0$ if $f$ is decreasing in $[\mu_*, \overline{\mu}]$, we may assume that $f$ is either increasing, or at least that it is increasing in some interval containing $[\underline{\mu}, \mu_*]$. Recall that a sufficient condition for welfare to be increasing, by (32), is

$$f(\mu^*)\frac{\underline{x}}{x^*}\frac{\partial\phi}{\partial\mu_*}\left[\frac{g(x^*)}{g(\underline{x})} - 1\right] + f(\mu_*)\left[1 - \frac{\underline{x}}{x^*}\frac{\partial\phi}{\partial\mu^*}\right] > 0 \tag{36}$$

Since $\frac{\underline{x}}{x^*} \leq 1$ and $\frac{\partial\phi}{\partial\mu^*} > 0$, a more demanding sufficient condition is

$$f(\mu^*)\frac{\partial\phi}{\partial\mu_*}\left[\frac{g(x^*)}{g(\underline{x})} - 1\right] + f(\mu_*)\left[1 - \frac{\partial\phi}{\partial\mu^*}\right] > 0 \tag{37}$$

Now, recall that $\frac{\partial\phi}{\partial\mu_*} = -\frac{\partial E}{\partial\mu_*}$, and $1 - \frac{\partial\phi}{\partial\mu^*} = \frac{\partial}{\partial\mu^*}(\mu^* - \phi) = \frac{\partial E}{\partial\mu^*}$. Let $s(x^*) = \frac{g(x^*)}{g(\underline{x})} - 1 > 0$. Then the condition above is equivalent to

$$-f(\mu^*)\frac{\partial E}{\partial\mu_*}s(x^*) + f(\mu_*)\frac{\partial E}{\partial\mu^*} > 0 \tag{38}$$

Recall $\phi(\mu_*, \mu^*) = \mu^* - E(\mu^*, \mu_*) \geq 0$ and $\psi(\mu_*, \mu^*) = E(\mu^*, \mu_*) - \mu_* \geq 0$. Now, consider the term $E(\mu_*, \mu^*) = \frac{\int_{\mu_*}^{\mu^*} xf(x)dx}{\int_{\mu_*}^{\mu^*} f(x)dx}$. Differentiating with respect to $(\mu^*, \mu_*)$ yields

$$\begin{aligned}
\frac{\partial E}{\partial\mu^*} &= \frac{\mu^* f(\mu^*)}{\int_{\mu_*}^{\mu^*} f(x)dx} - \frac{f(\mu^*)\int_{\mu_*}^{\mu^*} xf(x)dx}{(\int_{\mu_*}^{\mu^*} f(x)dx)^2} \\
&= \frac{f(\mu^*)}{\int_{\mu_*}^{\mu^*} f(x)dx}\left[\mu^* - E(\mu_*, \mu^*)\right] = \frac{f(\mu^*)}{\int_{\mu_*}^{\mu^*} f(x)dx}\phi
\end{aligned}$$

$$\begin{aligned}
\frac{\partial E}{\partial\mu_*} &= -\frac{\mu_* f(\mu_*)}{\int_{\mu_*}^{\mu^*} f(x)dx} + \frac{f(\mu_*)\int_{\mu_*}^{\mu^*} xf(x)dx}{(\int_{\mu_*}^{\mu^*} f(x)dx)^2} \\
&= \frac{f(\mu_*)}{\int_{\mu_*}^{\mu^*} f(x)dx}\left[E(\mu_*, \mu^*) - \mu_*\right] = \frac{f(\mu_*)}{\int_{\mu_*}^{\mu^*} f(x)dx}\psi
\end{aligned}$$

Using these expressions, (38) is equivalent to

$$\frac{1}{1+s(x^*)}\mu^* + \frac{s(x^*)}{1+s(x^*)}\mu_* \geq E(\mu_*, \mu^*) = \frac{\int_{\mu_*}^{\mu^*} xf(x)dx}{\int_{\mu_*}^{\mu^*} f(x)dx} \tag{39}$$

The LHS is a weight average of $\mu^*, \mu_*$, where the relative weights are $s(x^*) = \frac{g(x^*)}{g(\underline{x})} - 1 > 0$ and $1$. Welfare is increasing if this average is always greater than the average risk on the interval $[\mu_*, \mu^*]$. Intuitively, when there is little pooling at $x = \underline{x}$, then $x^* \approx \underline{x}$, so $s(x^*) \approx 0$, and then (39) will hold since $\mu^* > E(\mu_*, \mu^*)$. Below we characterize more precisely a bound on the difference between $\underline{x}$ and $x^*$ sufficient for welfare to be increasing.

We now describe a condition on $s(x^*)$ under which (39) holds. Define a family of measures $P_z$, one for each $z \in (\mu_*, \mu^*)$, concentrated on $[\mu_*, \mu^*]$, with PDF $f_z$ being a step function taking two values $v_1 < v_2$, where $\frac{v_2}{v_1} = \rho$. Let $f_z = v_1$ for $\mu \in [\mu_*, z]$ and $f_z = v_2$ for $\mu \in [z, \mu^*]$. Recall the definition $\rho := \frac{1}{f(\underline{\mu})}\left[\max_\mu f(\mu)\right]$. Since $f$ is increasing in some interval containing $[a, \mu_*]$, then

$$\rho \geq \frac{1}{f(\mu_*)}\left[\max_{\mu' \geq \mu_*} f(\mu')\right].$$

There exists, by Proposition 19 below, $z$ s.t. $P_z$ first order stochastically dominates $P$, where $P$ is the distribution of types in the model. For such $z$, $E_{P_z}(\mu_*, \mu^*) \geq E_P(\mu_*, \mu^*)$. Hence, it suffices to show that for all $z \in (\mu_*, \mu^*)$,

$$\frac{1}{1+s(x^*)}\mu^* + \frac{s(x^*)}{1+s(x^*)}\mu_* \geq E_{P_z}(\mu_*, \mu^*)$$

where $E_{P_z}(\mu_*, \mu^*)$ is the average of risk $\mu$ on the interval $(\mu_*, \mu^*)$, when weighted by the distribution $P_z$.

Let

$$\Gamma(\rho) = \max_{z\in[0,1]} \frac{1}{2}\frac{z^2+\rho(1-z^2)}{z+\rho(1-z)}.$$

We show below that $\Gamma(\rho)$ has the form of its definition in (35).

By renormalizing the interval WLOG, $\mu_* = 0$, $\mu^* = 1$, so it suffices to require

$$\frac{1}{1+s(x^*)} \geq \max_{z\in[0,1]} E_{P_z}(0,1) = \max_{z\in[0,1]} \frac{\int_0^1 x f_z(x)dx}{\int_0^1 f_z(x)dx} = \max_{z\in[0,1]} \frac{\frac{1}{2}v_1 z^2 + \frac{1}{2}v_2(1-z^2)}{zv_1+(1-z)v_2} = \Gamma(\rho)$$

Therefore, for $\frac{\partial W}{\partial T} \geq 0$, it is sufficient that

$$\frac{1}{1+s(x^*)} \geq \Gamma(\rho) \Leftrightarrow \frac{1}{\Gamma(\rho)} \geq \frac{g(x^*)}{g(\underline{x})}$$

We now show that $\Gamma(\rho)$ has the form of its definition in (35). The maximization problem $\Gamma(\rho)$ attains a maximum somewhere in the interior of $[0,1]$, since when $z = 0$ or $z = 1$ the quotient is $\frac{1}{2}$. Therefore, the first order condition is a sufficient condition for a maximum. The FOC is

$$\rho + (\rho-1)z^2 - 2\rho z = 0.$$

The relevant solution (since it must lie in $[0,1]$) is $z^\star = \frac{\rho-\sqrt{\rho}}{\rho-1} = \frac{\sqrt{\rho}}{\sqrt{\rho}+1}$. Using this solution, we can obtain

$$\begin{aligned}\Gamma(\rho) = \max_{z\in[0,1]} \frac{1}{2}\frac{z^2+\rho(1-z^2)}{z+\rho(1-z)} &= \frac{1}{2}\frac{\frac{1}{(\sqrt{\rho}+1)^2}}{\frac{1}{\sqrt{\rho}+1}}\frac{\rho+\rho((\sqrt{\rho}+1)^2-\rho)}{\sqrt{\rho}+\rho(\sqrt{\rho}+1-\sqrt{\rho})} \\ =&\frac{1}{2}\frac{1}{\sqrt{\rho}+1}\frac{\rho+\rho(2\sqrt{\rho}+1)}{\sqrt{\rho}+\rho} = \frac{1}{2}\frac{1}{(\sqrt{\rho}+1)^2\sqrt{\rho}}2\rho(1+\sqrt{\rho}) = \frac{\sqrt{\rho}}{\sqrt{\rho}+1}\end{aligned}$$

□

Notice that Proposition 20 implies Proposition 9: if $f(\mu)$ is decreasing, then $\rho = 1$ and $\Gamma(\rho) = 1/2$, so the condition becomes

$$\frac{g(x^*)}{g(\underline{x})} < \frac{1}{\Gamma(\rho)} \Leftrightarrow \frac{g(x^*)}{g(\underline{x})} < 2 \Leftrightarrow \frac{g(x^*)/x^*}{g(\underline{x})/\underline{x}} < 2\frac{\underline{x}}{x^*} < 2$$

This always hold since $g(x)/x$ decreasing by Lemma 19, so $\frac{g(x^*)/x^*}{g(\underline{x})/\underline{x}} \leq 1$.

The condition required by Proposition 20 depends on the endogenous quantity $x^*$. To discuss this further, we derive the following corollary, which places a bound on $\underline{x}$ (with the bound depending on $\rho$) which guarantees that welfare is increasing in $T$. This bound also assumes that $g'(x) = 1 - x$, as in the simulations (see Appendix J).

**Corollary 6.** *Assume $g'(x) = 1 - x$. Given $\rho$ defined by (34), if $\underline{x} \geq 1 - \rho^{-1/4}$, then $\frac{\partial W}{\partial T} > 0, \forall T$.*

*Proof.* Let $\Delta = \frac{1}{\Gamma(\rho)} - 1$. Recall $s(x^*) = \frac{g(x^*)}{g(\underline{x})} - 1 > 0$. To use the result from Proposition 20, we want to show that $\frac{g(x^*)}{g(\underline{x})} < \frac{1}{\Gamma(\rho)} \Leftrightarrow s(x^*) \leq \Delta$.

Recall $g(\cdot)$ is concave. Therefore, $g(x^*) \leq g(\underline{x}) + (x - \underline{x})g'(\underline{x})$. Then, $s(x^*) \leq \Delta$, it suffices to require $g'(\underline{x})(x^* - \underline{x}) \leq \Delta$, so if $g'(\underline{x}) = 1 - \underline{x}$, then we require $x^* \leq \frac{\Delta}{1-\underline{x}} + \underline{x}$. We are interested in the minimal value of $\underline{x}$, denoted $\underline{x}^{min}$, such that welfare is increasing for all $T$. A sufficient condition is $1 = \frac{\Delta}{1-\underline{x}^{min}} + \underline{x}^{min} \Leftrightarrow 1 - \sqrt{\Delta} = \underline{x}^{min}$. We can also write this as $1 - \sqrt{\frac{1}{\Gamma(\rho)} - 1} = \underline{x}^{min}$ or, alternatively, $\underline{x} \geq 1 - \sqrt{\frac{1}{\Gamma(\rho)} - 1} = 1 - \sqrt{\frac{\sqrt{\rho}+1}{\sqrt{\rho}} - 1} = 1 - \rho^{-1/4}$. $\square$

For instance, for $\rho = 2$, the we have $\Gamma(\rho) \approx 1.18$, so $\Delta \approx 0.69$. Then $1 - \sqrt{0.69} \approx 0.17$. That is if $\underline{x} \geq 0.17$, then for $s(x^*) \leq \Delta$ holds for any $x^*$, so welfare is always increasing with $T$. Figure 6 illustrates the relationship between $\rho$ and $x^{min}$.

# F Comparative Statics: PPPP Regime

This section contains results regarding comparative statics when the equilibrium regime is PPPP. We recall the definitions

$$\phi(\mu^*, \mu_*) = \mu^* - \mathbb{E}[\mu \mid \mu \in [\mu_*, \mu^*)],$$

$$\psi(\mu^*, \mu_*) = \mathbb{E}[\mu \mid \mu \in [\mu_*, \mu^*)] - \mu_*.$$

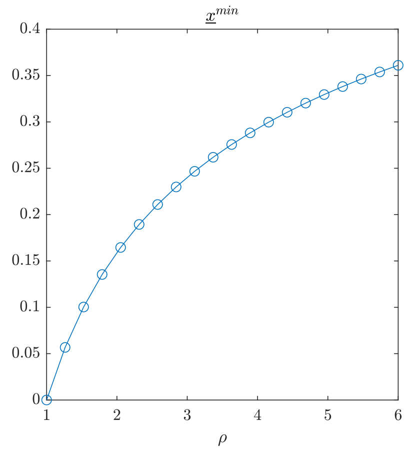


Figure 6: The value of $\underline{x}^{min}$ for each value of $\rho$. Given $\rho$, if $\underline{x} \geq \underline{x}^{min}$, then welfare increases in $T$ for all $T$.

We note that, even without the assumption that $f$ is log-concave - only that it has full support in $[\underline{\mu}, \overline{\mu}]$ - it still holds that

$$\frac{\partial \phi}{\partial \mu_*} < 0, \qquad \frac{\partial \psi}{\partial \mu^*} > 0 \tag{40}$$

although the other two inequalities from (11) need not hold.

## F.1 Setup

The way we obtain comparative statics when the equilibrium regime is PPPP is similar to the procedure we used for Dispersive equilibria. In this case, the equilibrium structure is simpler as it is characterized by only 2 equations. Let $\eta$ be a generic exogenous parameter (which can later be taken to be $\underline{x}, \overline{x}$). Each equation contains the endogenous variables $\mu^*, x^*$ and the exogenous regulatory parameter $\eta$, which can then be taken to be $\underline{x}, \overline{x}$. Since the equilibrium regime is PPPP all individuals purchase $x > 0$ so the fee $T$ has no effect on equilibrium.

The first equilibrium condition is the relationship between $\mu^*$ and $x^*$ described by (20), or

$$H\left(\mu^*, x^*, \eta\right) = \overline{\mu} - \mu^* - \int_{x^*}^{\overline{x}} \frac{g'(x)}{x} dx = 0.$$

Recall that $\underline{p} = p\left(\underline{x}\right) = \underline{x}\mathbb{E}\left[\mu \mid \mu < \mu^*\right]$. The second equilibrium condition is type $\mu^*$'s indifference condition:

$$G\left(\mu^*, x^*, \eta\right) = g(x^*) - g(\underline{x}) - \underline{x}\phi(\mu^*) = 0.$$

Following Appendix E, these two conditions can be written in matrix form as

$$\begin{bmatrix} H\left(\mu^*, x^*, \eta\right) \\ G\left(\mu^*, x^*, \eta\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Implicit differentiation with respect to $\eta$ implies

$$\begin{bmatrix} \frac{\partial H}{\partial \mu^*} & \frac{\partial H}{\partial x^*} \\ \frac{\partial G}{\partial \mu^*} & \frac{\partial G}{\partial x^*} \end{bmatrix} \begin{bmatrix} \frac{\partial \mu^*}{\partial \eta} \\ \frac{\partial x^*}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -\frac{\partial H}{\partial \eta} \\ -\frac{\partial G}{\partial \eta} \end{bmatrix}. \tag{41}$$

Following Appendix E, the terms in the square matrix are

$$\frac{\partial H}{\partial \mu^*} = -1, \qquad \frac{\partial H}{\partial x^*} = \frac{g'(x^*)}{x^*},$$

$$\frac{\partial G}{\partial \mu^*} = -\underline{x}\frac{d\phi}{d\mu^*}, \qquad \frac{\partial G}{\partial x^*} = g'(x^*).$$

The determinant of the square matrix is

$$D = g'(x^*)\left[\frac{\underline{x}}{x^*}\frac{d\phi}{d\mu^*} - 1\right].$$

**Lemma 21.** *If the equilibrium regime is PPPP, then $D < 0$.*

*Proof.* Recall that $\phi(\mu^*) = \mu^* - \mathbb{E}\left[\mu \mid \mu < \mu^*\right] > 0$. From Lemma, 2 $\frac{d\phi}{d\mu^*} < 1$. We have $D < 0$ because $\underline{x} < x^*$ in the presence of partial pooling and $\frac{d\phi}{d\mu^*} \leq 1$. □

## F.2 Adjusting $\overline{x}$: Equilibrium

We now prove $\frac{\partial \mu^*}{\partial \overline{x}} < 0$ (Proposition 4) for $\overline{x} < 1$.

*Proof.* Recall the equilibrium conditions

$$H\left(\mu^*, x^*, \eta\right) = \overline{\mu} - \mu^* - \int_{x^*}^{\overline{x}} \frac{g'(x)}{x} dx = 0.$$

$$G\left(\mu^*, x^*, \eta\right) = g(x^*) - g(\underline{x}) - \underline{x}\phi(\mu^*) = 0.$$

Taking the generic parameter $\eta$ to be the maximum coverage $\overline{x}$, we have

$$-\frac{\partial H}{\partial \overline{x}} = -\left[-\frac{g'(\overline{x})}{\overline{x}}\right] = \frac{g'(\overline{x})}{\overline{x}} > 0, \qquad \frac{\partial G}{\partial \overline{x}} = 0$$

Therefore

$$\frac{\partial \mu^*}{\partial \overline{x}} = \frac{1}{D}\left[\left(-\frac{\partial H}{\partial \eta}\right)\frac{\partial G}{\partial x^*} - \frac{\partial H}{\partial x^*}\left(-\frac{\partial G}{\partial \eta}\right)\right]$$
$$= \frac{1}{D}\underbrace{\left(-\frac{\partial H}{\partial \overline{x}}\right)}_{+}\underbrace{\frac{\partial G}{\partial x^*}}_{+} < 0$$

□

We now prove that, if $f$ is log-concave, then $\frac{\partial x^*}{\partial \overline{x}} < 0$. This result was omitted from the main text because it is not used to sign the effect of $\overline{x}$ on welfare.

**Lemma 22.** *If $f$ is log-concave, then $\frac{\partial x^*}{\partial \overline{x}} < 0$.*

*Proof.* First recall that

$$\frac{\partial G}{\partial \overline{x}} = 0, \qquad \frac{\partial H}{\partial \overline{x}} = -\frac{g(\overline{x})}{\overline{x}}.$$

Then, by applying Cramer's rule to (41),

$$\begin{aligned}\frac{\partial x^*}{\partial \overline{x}} &= \frac{1}{D}\left[\frac{\partial H}{\partial \mu^*}\left(-\frac{\partial G}{\partial \overline{x}}\right) - \frac{\partial G}{\partial \mu^*}\left(-\frac{\partial H}{\partial \overline{x}}\right)\right] \\ &= \frac{1}{D}\left[\frac{\partial G}{\partial \mu^*}\frac{\partial H}{\partial \overline{x}}\right] = \frac{1}{D}[(-\underline{x}\frac{d\phi}{d\mu^*})(-\frac{g(\overline{x})}{\overline{x}})] \\ &= \frac{1}{D}\frac{\underline{x}}{\overline{x}}\frac{g(\overline{x})}{\overline{x}}\frac{d\phi}{d\mu^*} < 0\end{aligned}$$

since, when $f$ is log-concave, $\frac{d\phi}{d\mu^*} > 0$. □

## F.3 Adjusting $\overline{x}$: Welfare

We now prove Proposition 4, which concerns the effect on welfare of changes in $\overline{x}$. Again, fix $\overline{x} < 1$. There are three possible cases of interest: full pooling, no pooling and partial pooling. We discuss these in turn.

Second, if there is no pooling (all individuals buy $x > \underline{x}$), then $\mu^* = \underline{\mu}$ so welfare is $W(\underline{x}, \overline{x}) = \int_{\underline{\mu}}^{\overline{\mu}} g(\sigma(\mu, \underline{x}, \overline{x}))f(\mu)d\mu$. Then, $\frac{\partial \sigma}{\partial \overline{x}} \geq 0, \forall \mu \in [\underline{\mu}, \overline{\mu}]$ (Lemma 4), so an increase in $\overline{x}$ increases welfare and an increase in $\underline{x}$ has no effect on welfare.

Third, we discuss partial pooling. In this case, welfare is

$$W(\underline{x}, \overline{x}) = F(\mu^*)g(\underline{x}) + \int_{\mu^*}^{\overline{\mu}} g(\sigma(\mu, \underline{x}, \overline{x}))f(\mu)d\mu,$$

From Lemma 4, an increase in $\overline{x}$ lowers the cutoff type $\mu^*$, which we can write as $\frac{d\mu^*}{d\overline{x}} < 0$. Then, the derivative of welfare is

$$\frac{\partial W}{\partial \overline{x}} = -\underbrace{\frac{d\mu^*}{d\overline{x}}}_{-}\left[g(x^*) - g(\underline{x})\right] f(\mu^*) + \int_{\mu^*}^{\overline{\mu}} g'(\sigma(\mu, \underline{x}, \overline{x}))\underbrace{\frac{\partial \sigma}{\partial \overline{x}}}_{+} f(\mu)d\mu > 0$$

This term is signed since $g'(x) \geq 0$, and $\frac{\partial \sigma}{\partial \overline{x}} > 0$ for all $\mu > \mu^* = \mu^*(\underline{x}, \overline{x})$, by Lemma 4.

## F.4 Adjusting $\underline{x}$: Equilibrium

We now prove Lemma 11, which states that $\frac{\partial x^*}{\partial \underline{x}} \geq 0$.

Lemma 11 implies that, for fixed $\underline{x}$, the domain of $\overline{x}$ for which partial pooling occurs is connected, and $x^*$ is continuous in $(\underline{x}, \overline{x})$. We now take the generic parameter $\eta$ to be $\underline{x}$. We compute the terms on the RHS of (41) as

$$\frac{\partial H}{\partial \underline{x}} = 0, \qquad \frac{\partial G}{\partial \underline{x}} = -g'(\underline{x}) - \phi(\mu^*) \leq 0.$$

From the analysis above, Cramer's rule gives

$$\begin{aligned}
\frac{\partial x^*}{\partial \underline{x}} &= \frac{1}{D}\left[\frac{\partial H}{\partial \mu^*}\left(-\frac{\partial G}{\partial \underline{x}}\right) - \frac{\partial G}{\partial \mu^*}\left(-\frac{\partial H}{\partial \underline{x}}\right)\right] \\
&= -\frac{1}{D}\left[g'\left(x\right) + \phi(\mu^*)\right]
\end{aligned}$$

We have $g'\left(x\right) > 0$ in $(0, 1)$, $\phi(\mu^*) \geq 0$ and $D < 0$ from $f$ log-concave. Therefore, $\frac{\partial x^*}{\partial \underline{x}} \geq 0$.

## F.5 Adjusting $\underline{x}$ (Proposition 11)

We now provide a sufficient condition for welfare to increase with $\underline{x}$ when the equilibrium regime is PPPP (Proposition 11).

First, we establish the following preliminary results. By log-concavity, $\frac{d\phi}{d\mu^*} \in (0, 1)$. Therefore, $\frac{1}{x^* - \underline{x}} \geq \frac{1}{x^* - \underline{x}\frac{d\phi}{d\mu^*}}$. By concavity of $g(\cdot)$, since $x^* > \underline{x}$, $g\left(\underline{x}\right) + g'\left(\underline{x}\right)\left(x^* - \underline{x}\right) \geq g\left(x^*\right)$. Also by concavity of $g(\cdot)$, $\frac{g'(\underline{x})}{g(x^*) - g(\underline{x})} \geq \frac{1}{x^* - \underline{x}}$. Using the results above, we obtain

$$\frac{\partial\mu^*}{\partial\underline{x}} = -\frac{1}{D}\frac{g'(x^*)}{x^*}\left(g'\left(\underline{x}\right)+\phi\left(\mu^*\right)\right) > 0$$

The indifference condition of type $\mu^*$ implies $\frac{g(x^*)-g(\underline{x})}{\underline{x}} = \phi\left(\mu^*\right)$. Also, $D = g'(x^*)\left[\frac{\underline{x}}{x^*}\frac{d\phi}{d\mu^*}-1\right]$. Therefore the expression for $\frac{\partial\mu^*}{\partial\underline{x}}$ can be written as

$$\frac{\partial\mu^*}{\partial\underline{x}} = \frac{1}{\left[x^*-\underline{x}\frac{d\phi}{d\mu^*}\right]}\left(g'\left(\underline{x}\right)+\frac{g(x^*)-g(\underline{x})}{\underline{x}}\right)$$

Since $g$ is concave and $x^* > \underline{x}$, then $g'\left(\underline{x}\right)\left(x^*-\underline{x}\right) \geq g\left(x^*\right)-g\left(\underline{x}\right)$. Therefore we can write

$$\frac{\partial\mu^*}{\partial\underline{x}} \leq \frac{1}{\left[x^*-\underline{x}\frac{d\phi}{d\mu^*}\right]}g'\left(\underline{x}\right)\frac{x^*}{\underline{x}}$$

This implies the weaker bound

$$\frac{\partial\mu^*}{\partial x} \leq \frac{1}{\left[x^*-\underline{x}\frac{d\phi}{d\mu^*}\right]}\frac{g'\left(\underline{x}\right)}{\underline{x}}$$

Since the equilibrium regime is PPPP, welfare is $W = F(\mu^*)g(\underline{x})+\int_{\mu^*}^{\overline{\mu}} g(\sigma(\mu))f(\mu)d\mu$. The effect of $\underline{x}$ on welfare is

$$\frac{\partial W}{\partial\underline{x}} = -f(\mu^*)\frac{\partial\mu^*}{\partial\underline{x}}\left[g(x^*)-g(\underline{x})\right]+F(\mu^*)g'(\underline{x}).$$

Then, $\frac{\partial W}{\partial\underline{x}} > 0$ if and only if

$$\frac{F(\mu^*)}{f(\mu^*)}\frac{g'(\underline{x})}{g(x^*)-g(\underline{x})} > \frac{\partial\mu^*}{\partial\underline{x}}$$

Given the bounds on $\frac{\partial\mu^*}{\partial\underline{x}}$, a more demanding sufficient condition is

$$\frac{F(\mu^*)}{f(\mu^*)}\frac{g'(\underline{x})}{g(x^*)-g(\underline{x})} > \frac{1}{\left[x^*-\underline{x}\frac{d\phi}{d\mu^*}\right]}\frac{g'\left(\underline{x}\right)}{\underline{x}}$$

Since $g$ is concave, then $\frac{g'(\underline{x})}{g(x^*)-g(\underline{x})} \geq \frac{1}{x^*-\underline{x}}$. So a more demanding sufficient condition is

$$\frac{F(\mu^*)}{x^* - \underline{x}} > \frac{f(\mu^*)}{x^* - \underline{x}\frac{d\phi}{d\mu^*}} \frac{\nu g'\left(\underline{x}\right) x^*}{\underline{x}}$$

By log concavity of $f$, $\frac{1}{x^*-\underline{x}}\left[x^* - \underline{x}\frac{d\phi}{d\mu^*}\right] \geq 1$. Therefore a more demanding sufficient condition is

$$\frac{F(\mu^*)}{f(\mu^*)} > \frac{g'\left(\underline{x}\right)}{\underline{x}}.$$

If $F$ is log-concave, then the LHS is increasing with $\mu^*$ and therefore with $\underline{x}$ since $\frac{\partial \mu^*}{\partial \underline{x}}$. The RHS is decreasing with $\underline{x}$. Therefore, if this inequality holds for some $\hat{\underline{x}}$, then it holds for all $\underline{x} > \hat{\underline{x}}$.

# G Adjusting $\underline{x}$ (Proposition 12)

Fix $\overline{x} \in (0, 1]$,and let $\underline{z}$ be such that[44]

$$\overline{\mu} - \underline{\mu} = \int_{\underline{z}}^{\overline{x}} \frac{g(x)}{x} dx. \tag{42}$$

i.e., $\underline{z}$ is the largest $\underline{x}$ for which there is a fully separating equilibrium. Observe that when $\underline{x} \approx \underline{z}$ (i.e., when $\underline{x}$ is close to $\underline{z}$), we have $x^* \approx \underline{z}$ and $\mu^* \approx \underline{\mu}$, by the continuity of equilibria. Throughout the proof, denote $f_0 = f(\underline{\mu})$, $f_1 = f'(\underline{\mu})$.

## G.1 First Derivative

In a PPPP regime, from Appendix F.5, we have

$$\frac{\partial \mu^*}{\partial \underline{x}} = \frac{1}{\left[x^* - \underline{x}\frac{d\phi}{d\mu^*}\right]} \left(g'\left(\underline{x}\right) + \frac{g(x^*) - g(\underline{x})}{\underline{x}}\right).$$

Welfare is

$$\frac{\partial W}{\partial \underline{x}} = -f(\mu^*)\frac{\partial \mu^*}{\partial \underline{x}}\left[g(x^*) - g(\underline{x})\right] + F(\mu^*)g'(\underline{x})$$

When $\underline{x} \approx \underline{z}$, we have $x^* \approx \underline{z}$ and $\mu^* \approx \underline{\mu}$, and hence $g(x^*) - g(\underline{x}) \approx 0$, $F(\mu^*) \approx 0$. Therefore

$$\frac{\partial W}{\partial \underline{x}} \underset{\underline{x}\approx\underline{z}}{\approx} 0$$

[44] In the main text we had denoted it $\underline{x}_0$, but we change to $\underline{z}$ here to ease exposition.

## G.2 $\phi$ and its Derivatives

Recall that, $\frac{1}{1+x} \approx 1-x$ when $x$ is small. Since $f(\mu) = f_0 + f_1\delta\mu + o(\delta\mu)$ , where $\delta\mu = \mu - \underline{\mu}$,

$$
\begin{aligned}
\phi(\mu) = \underline{\mu} + \delta\mu - & \frac{\int_0^{\delta\mu} \mu' f(\underline{\mu} + \mu')d\mu'}{\int_0^{\delta\mu} f(\underline{\mu} + \mu')d\mu'} \\
=& \underline{\mu} + \delta\mu - \frac{\int_0^{\delta u}[\mu'(f_0 + f_1\mu') + o(\delta\mu'^2)]d\mu'}{\int_0^{\delta u}(f_0 + f_1\mu' + o(\delta\mu'))d\mu'} \\
=& \underline{\mu} + \delta\mu - \frac{\int_0^{\delta u}[\mu'(f_0 + f_1\mu')]d\mu' + o(\delta\mu^2)}{\int_0^{\delta u}(f_0 + f_1\mu')d\mu' + o(\delta\mu)} \\
=& \underline{\mu} + \delta\mu - \frac{\frac{1}{2}f_0(\delta\mu)^2 + \frac{1}{3}f_1(\delta\mu)^3 + o(\delta\mu^3)}{f_0\delta\mu + \frac{1}{2}f_1(\delta\mu)^2 + o(\delta\mu^2)} = \\
=& \underline{\mu} + \delta\mu - \frac{\frac{1}{2}\delta\mu + \frac{1}{3}\frac{f_1}{f_0}(\delta\mu)^2 + o(\delta\mu^2)}{1 + \frac{1}{2}\frac{f_1}{f_0}\delta\mu + o(\delta\mu)} \\
=& \underline{\mu} + \delta\mu - \big[\frac{1}{2}\delta\mu + \frac{1}{3}\frac{f_1}{f_0}(\delta\mu)^2 + o(\delta\mu^2)\big]\big[1 - \frac{1}{2}\frac{f_1}{f_0}\delta\mu + o(\delta\mu))\big] \\
=& \underline{\mu} + \delta\mu - \big[\frac{1}{2}\delta\mu + \frac{1}{3}\frac{f_1}{f_0}(\delta\mu)^2 - \frac{1}{4}\frac{f_1}{f_0}(\delta\mu)^2 + o(\delta\mu^2)\big] \\
=& \underline{\mu} + \frac{1}{2}\delta\mu - \frac{1}{12}\frac{f_1}{f_0}(\delta\mu)^2 + o(\delta\mu^2)
\end{aligned}
$$

Therefore,

$$
\frac{d\phi}{d\mu}|_{\mu\approx\underline{\mu}} \approx \frac{1}{2}, \qquad \frac{d^2\phi}{d\mu^2}|_{\mu\approx\underline{\mu}} \approx -\frac{1}{6}\frac{f_1}{f_0}
$$

(if $f$ was uniform, we have $\frac{d\phi}{d\mu} \equiv \frac{1}{2}$).

## G.3 Second Derivative

We now consider the second derivative of welfare. For $x^* \in (\underline{z}, \overline{x})$,

$$
\overline{\mu} - \mu^* = \int_{x^*}^{\overline{x}} \frac{g(x)}{x}dx,
$$

so we have

$$
\frac{\partial x^*}{\partial \underline{x}} = \frac{x^*}{g'(x^*)}\frac{\partial \mu^*}{\partial \underline{x}} = \frac{x^*}{g'(x^*)}\frac{1}{\left[x^* - \underline{x}\frac{d\phi}{d\mu^*}\right]}\left(g'\left(\underline{x}\right) + \frac{g(x^*) - g(\underline{x})}{\underline{x}}\right)
$$

When $\underline{x} \approx \underline{z}$, we have $x^* \approx \underline{z}$ and $\mu^* \approx \underline{\mu}$, as well as $\frac{d\phi}{d\mu}|_{\mu\approx\underline{\mu}} \approx \frac{1}{2}$, and hence $g(x^*) - g(\underline{x}) \approx 0$, $F(\mu^*) \approx 0$, and

$$\frac{\partial x^*}{\partial \underline{x}} \underset{\underline{x}\approx\underline{z}}{\approx} = \frac{\underline{z}}{g'(\underline{z})\left[\underline{z} - \underline{z}\frac{d\phi}{d\mu^*}\right]}\left[g'(\underline{z}) + \frac{g(\underline{z}) - g(\underline{z})}{\underline{z}}\right] = \frac{\underline{z}g'(\underline{z})}{\underline{z}g'(\underline{z})\frac{1}{2}} = 2$$

Therefore,

$$\frac{\partial \mu^*}{\partial \underline{x}} = \frac{g'(x^*)}{x^*}\frac{\partial x^*}{\partial \underline{x}} \underset{\underline{x}\approx\underline{z}}{\approx} 2\frac{g'(\underline{z})}{\underline{z}}$$

Now, the the second derivative of welfare is

$$\begin{aligned}\frac{\partial^2 W}{\partial \underline{x}^2} &= -f'(\mu^*)(\frac{\partial \mu^*}{\partial \underline{x}})^2\left[g(x^*) - g(\underline{x})\right] - f(\mu^*)\frac{\partial^2 \mu^*}{\partial \underline{x}^2}\left[g(x^*) - g(\underline{x})\right] \\ &\quad - f(\mu^*)\frac{\partial \mu^*}{\partial \underline{x}}[\frac{\partial x^*}{\partial \underline{x}}g'(x^*) - g'(\underline{x})] + g''(\underline{x})F(\mu^*) + g'(\underline{x})f(\mu^*)\frac{\partial \mu^*}{\partial \underline{x}} \\ &= - f'(\mu^*)(\frac{\partial \mu^*}{\partial \underline{x}})^2\left[g(x^*) - g(\underline{x})\right] - f(\mu^*)\frac{\partial^2 \mu^*}{\partial \underline{x}^2}\left[g(x^*) - g(\underline{x})\right] \\ &\quad - f(\mu^*)\frac{\partial \mu^*}{\partial \underline{x}}\frac{\partial x^*}{\partial \underline{x}}g'(x^*) + g''(\underline{x})F(\mu^*) + 2g'(\underline{x})f(\mu^*)\frac{\partial \mu^*}{\partial \underline{x}}\end{aligned}$$

Using $g(x^*) - g(\underline{x}) \approx 0$, $F(\mu^*) \approx 0$, and $\frac{\partial x^*}{\partial \underline{x}} \approx 2$, we obtain

$$\begin{aligned}\frac{\partial^2 W}{\partial \underline{x}^2} \underset{\underline{x}\approx\underline{z}}{\approx} &- f_1(\frac{\partial \mu^*}{\partial \underline{x}})^2\left[g(x^*) - g(\underline{x})\right] - f_0\frac{\partial^2 \mu^*}{\partial \underline{x}^2}\left[g(x^*) - g(\underline{x})\right] \\ &- f_0\frac{\partial \mu^*}{\partial \underline{x}}\frac{\partial x^*}{\partial \underline{x}}g'(x^*) + g''(\underline{x})F(\mu^*) + 2g'(\underline{x})f_0\frac{\partial \mu^*}{\partial \underline{x}} \\ = &- f_0\frac{\partial \mu^*}{\partial \underline{x}}2g'(x^*) + 2g'(\underline{x})f_0\frac{\partial \mu^*}{\partial \underline{x}} = 0\end{aligned}$$

## G.4 Third Derivative: Preliminaries 1

We now consider the third derivative of welfare. First, we compute the term $\frac{\partial^2 \mu^*}{\partial \underline{x}^2}$. We obtain

$$
\begin{aligned}
\frac{\partial^2 \mu^*}{\partial \underline{x}^2} =& \frac{\partial}{\partial \underline{x}}\Big(\frac{1}{\left[x^* - \frac{1}{2}\underline{x}\right]}\left(g'\left(\underline{x}\right) + \frac{g(x^*) - g(\underline{x})}{\underline{x}}\right)\Big)\\
=& \left(g'\left(\underline{x}\right) + \frac{g(x^*) - g(\underline{x})}{\underline{x}}\right)\frac{\partial}{\partial \underline{x}}\Big(\frac{1}{\left[x^* - \frac{1}{2}\underline{x}\right]}\Big)\\
&+ \frac{1}{\left[x^* - \frac{1}{2}\underline{x}\right]}\frac{\partial}{\partial \underline{x}}\left(g'\left(\underline{x}\right) + \frac{g(x^*) - g(\underline{x})}{\underline{x}}\right)
\end{aligned}
$$

First, we compute

$$
g'\left(\underline{x}\right) + \frac{g(x^*) - g(\underline{x})}{\underline{x}} \underset{\underline{x}\approx\underline{z}}{\approx} g'(\underline{z})
$$

Next, since when $\underline{x} \approx \underline{z}$, we have $x^* \approx \underline{z}$ and $\mu^* \approx \underline{\mu}$, as well as $\frac{d\phi}{d\mu}|_{\mu\approx\underline{\mu}} \approx \frac{1}{2}$, we have

$$
\frac{1}{\left[x^* - \frac{d\phi}{d\mu^*}\underline{x}\right]} \underset{\underline{x}\approx\underline{z}}{\approx} \frac{2}{\underline{z}}
$$

Now, evaluating at the point where $\underline{x} \approx \underline{z}$, and using the results above (namely $\frac{\partial x^*}{\partial \underline{x}} \underset{\underline{x}\approx\underline{z}}{\approx} 2$), we obtain

$$
\begin{aligned}
\frac{\partial}{\partial \underline{x}}\Big(\frac{1}{\left[x^* - \underline{x}\frac{d\phi}{d\mu^*}\right]}\Big) &= -\frac{\frac{\partial x^*}{\partial \underline{x}} - \frac{d\phi}{d\mu^*} - \underline{x}\frac{d^2\phi}{d(\mu^*)^2}\frac{\partial \mu^*}{\partial \underline{x}}}{\left[x^* - \frac{d\phi}{d\mu^*}\underline{x}\right]^2}\\
&\underset{\underline{x}\approx\underline{z}}{\approx} -\frac{2-\frac{1}{2}}{(\frac{1}{2}\underline{z})^2} + \frac{\underline{z}}{(\frac{1}{2}\underline{z})^2}\big(-\frac{1}{6}\frac{f_1}{f_0}\big)\big(2\frac{g'(\underline{z})}{\underline{z}}\big)\\
&\underset{\underline{x}\approx\underline{z}}{\approx} -\frac{\frac{3}{2}}{\frac{1}{4}}\frac{1}{(\underline{z})^2} + \frac{-\frac{1}{3}\frac{f_1}{f_0}g'(\underline{z})}{\frac{1}{4}(\underline{z})^2}\\
&= -\frac{\frac{3}{2}}{\frac{1}{4}}\frac{1}{(\underline{z})^2} - \frac{4}{3}\frac{1}{(\underline{z})^2}\frac{f_1}{f_0}g'(\underline{z})\\
&= -\frac{6}{(\underline{z})^2} - \frac{4}{3}\frac{g'(\underline{z})}{(\underline{z})^2}\frac{f_1}{f_0}
\end{aligned}
$$

Next we compute the term

$$
\frac{\partial}{\partial \underline{x}}\left(g'\left(\underline{x}\right) + \frac{g(x^*) - g(\underline{x})}{\underline{x}}\right) = g''(\underline{x}) + \frac{\underline{x}\big(g'(x^*)\frac{\partial x^*}{\partial \underline{x}} - g'(\underline{x})\big) - (g(x^*) - g(\underline{x}))}{(\underline{x})^2} \underset{\underline{x}\approx\underline{z}}{\approx} g''(\underline{z}) + \frac{g'(\underline{z})}{\underline{z}}
$$

Again evaluating at $\underline{x} \approx \underline{z}$, we have $g(x^*) - g(\underline{x}) \approx 0$ and $\frac{\partial x^*}{\partial \underline{x}} \underset{\underline{x}\approx\underline{z}}{\approx} 2$, we obtain

$$\frac{\partial}{\partial \underline{x}}\left(g'\left(\underline{x}\right)+\frac{g(x^*)-g(\underline{x})}{\underline{x}}\right)\underset{\underline{x}\approx\underline{z}}{\approx}g''(\underline{x})+\frac{\underline{x}\left(g'(x^*)\frac{\partial x^*}{\partial \underline{x}}-g'(\underline{x})\right)}{(\underline{x})^2}$$

$$\frac{\partial}{\partial \underline{x}}\left(g'\left(\underline{x}\right)+\frac{g(x^*)-g(\underline{x})}{\underline{x}}\right)\underset{\underline{x}\approx\underline{z}}{\approx}g''(\underline{z})+\frac{\left(g'(\underline{z})2-g'(\underline{z})\right)}{(\underline{x})}$$

$$\frac{\partial}{\partial \underline{x}}\left(g'\left(\underline{x}\right)+\frac{g(x^*)-g(\underline{x})}{\underline{x}}\right)\underset{\underline{x}\approx\underline{z}}{\approx}g''(\underline{z})+\frac{g'(\underline{z})}{\underline{z}}$$

Plugging all these back into $\frac{\partial^2\mu^*}{\partial \underline{x}^2}$ gives gives,

$$\frac{\partial^2\mu^*}{\partial \underline{x}^2}=\left(g'\left(\underline{x}\right)+\frac{g(x^*)-g(\underline{x})}{\underline{x}}\right)\frac{\partial}{\partial \underline{x}}\Big(\frac{1}{\left[x^*-\frac{d\phi}{d\mu^*}\underline{x}\right]}\Big)+\frac{1}{\left[x^*-\frac{d\phi}{d\mu^*}\underline{x}\right]}\frac{\partial}{\partial \underline{x}}\left(g'\left(\underline{x}\right)+\frac{g(x^*)-g(\underline{x})}{\underline{x}}\right)$$

$$\frac{\partial^2\mu^*}{\partial \underline{x}^2}\underset{\underline{x}\approx\underline{z}}{\approx}g'(\underline{z})\left(-6\frac{1}{(\underline{z})^2}-\frac{4}{3}\frac{g'(\underline{z})}{(\underline{z})^2}\frac{f_1}{f_0}\right)+\frac{2}{\underline{z}}\left(g''(\underline{z})+\frac{g'(\underline{z})}{\underline{z}}\right)$$

$$\frac{\partial^2\mu^*}{\partial \underline{x}^2}\underset{\underline{x}\approx\underline{z}}{\approx}-6\frac{g'(\underline{z})}{(\underline{z})^2}+\frac{2}{\underline{z}}g''(\underline{z})+\frac{2}{\underline{z}}\frac{g'(\underline{z})}{\underline{z}}-\frac{4}{3}\frac{1}{(\underline{z})^2}\frac{f_1}{f_0}(g'(\underline{z}))^2$$

$$\frac{\partial^2\mu^*}{\partial \underline{x}^2}\underset{\underline{x}\approx\underline{z}}{\approx}\left(\frac{2g''(\underline{z})}{\underline{z}}-\frac{4g'(\underline{z})}{(\underline{z})^2}\right)-\frac{4}{3}\frac{1}{(\underline{z})^2}\frac{f_1}{f_0}(g'(\underline{z}))^2 \tag{43}$$

## G.5 Third Derivative: Preliminaries 2

Recall that

$$\frac{\partial x^*}{\partial \underline{x}}=\frac{x^*}{g'(x^*)}\frac{\partial\mu^*}{\partial \underline{x}}$$

Now we compute the term

$$\begin{aligned}\frac{\partial^2 x^*}{\partial \underline{x}^2}=\frac{\partial}{\partial \underline{x}}(\frac{\partial x^*}{\partial \underline{x}})=\\ \frac{\partial}{\partial \underline{x}}(\frac{x^*}{g'(x^*)}\frac{\partial\mu^*}{\partial \underline{x}})\\ =&\frac{x^*}{g'(x^*)}\frac{\partial^2\mu^*}{\partial \underline{x}^2}+\frac{\partial\mu^*}{\partial \underline{x}}\frac{\partial x^*}{\partial \underline{x}}\frac{\partial}{\partial x^*}(\frac{x^*}{g'(x^*)})\\ =&\frac{x^*}{g'(x^*)}\frac{\partial^2\mu^*}{\partial \underline{x}^2}+\frac{\partial\mu^*}{\partial \underline{x}}\frac{\partial x^*}{\partial \underline{x}}(\frac{1}{g'(x^*)}-\frac{x^*g''(x^*)}{(g'(x^*))^2})\end{aligned}$$

Then evaluating at $\underline{x} \approx \underline{z}$ we obtain

$$\frac{\partial^2 x^*}{\partial \underline{x}^2} \underset{\underline{x}\approx\underline{z}}{\approx} \frac{\underline{z}}{g'(\underline{z})}(\frac{2}{\underline{z}}g''(\underline{z}) - 4g'(\underline{z})\frac{1}{(\underline{z})^2} - \frac{4}{3}\frac{1}{(\underline{z})^2}\frac{f_1}{f_0}(g'(\underline{z}))^2) + (2\frac{g'(\underline{z})}{\underline{z}})(2)(\frac{1}{g'(\underline{z})} - \frac{\underline{z}g''(\underline{z})}{(g'(\underline{z}))^2})$$

$$\frac{\partial^2 x^*}{\partial \underline{x}^2} \underset{\underline{x}\approx\underline{z}}{\approx} \frac{1}{g'(\underline{z})}(2g''(\underline{z}) - \frac{4g'(\underline{z})}{(\underline{z})}) + 4\frac{g'(\underline{z})}{\underline{z}}(\frac{1}{g'(\underline{z})} - \frac{\underline{z}g''(\underline{z})}{(g'(\underline{z}))^2}) - \frac{4}{3}\frac{1}{\underline{z}}\frac{f_1}{f_0}g'(\underline{z})$$

$$\frac{\partial^2 x^*}{\partial \underline{x}^2} \underset{\underline{x}\approx\underline{z}}{\approx} 2\frac{g''(\underline{z})}{g'(\underline{z})} - \frac{4}{(\underline{z})} + \frac{4}{\underline{z}} - 4\frac{g''(\underline{z})}{(g'(\underline{z}))} - \frac{4}{3}\frac{1}{\underline{z}}\frac{f_1}{f_0}g'(\underline{z})$$

$$\frac{\partial^2 x^*}{\partial \underline{x}^2} \underset{\underline{x}\approx\underline{z}}{\approx} 2\frac{g''(\underline{z})}{g'(\underline{z})} - 4\frac{g''(\underline{z})}{(g'(\underline{z}))} - \frac{4}{3}\frac{1}{\underline{z}}\frac{f_1}{f_0}g'(\underline{z})$$

$$\frac{\partial^2 x^*}{\partial \underline{x}^2} \underset{\underline{x}\approx\underline{z}}{\approx} -2\frac{g''(\underline{z})}{g'(\underline{z})} - \frac{4}{3}\frac{1}{\underline{z}}\frac{f_1}{f_0}g'(\underline{z}) \tag{44}$$

## G.6 Third Derivative

Differentiating again (the derivatives of the four non-red terms in $\frac{\partial^2 W}{\partial \underline{x}^2}$ are grouped into four terms of parenthesis, the first set of red terms comes from differentiating the red term in $\frac{\partial^2 W}{\partial \underline{x}^2}$, and the last pair of red terms come from differentiating $f(\mu^*)$ in the non-red terms),

$$\begin{aligned}
\frac{\partial^3 W}{\partial \underline{x}^3} = &\Bigg( -\frac{\partial\left(f'(\mu^*)(\frac{\partial\mu^*}{\partial\underline{x}})^2\right)}{\partial\underline{x}}\left[g(x^*) - g(\underline{x})\right] - \left[f'(\mu^*)(\frac{\partial\mu^*}{\partial\underline{x}})^2\right]\left(\frac{\partial x^*}{\partial\underline{x}}g'(x^*) - g'(\underline{x})\right)\Bigg) \\
&+\Bigg( - f(\mu^*)\frac{\partial^3\mu^*}{\partial\underline{x}^3}\left[g(x^*) - g(\underline{x})\right] - f(\mu^*)\frac{\partial^2\mu^*}{\partial\underline{x}^2}\left[\frac{\partial x^*}{\partial\underline{x}}g'(x^*) - g'(\underline{x})\right]\Bigg) \\
&+\Bigg( - f(\mu^*)\frac{\partial^2\mu^*}{\partial\underline{x}^2}\frac{\partial x^*}{\partial\underline{x}}g'(x^*) - f(\mu^*)\frac{\partial\mu^*}{\partial\underline{x}}\frac{\partial^2 x^*}{\partial\underline{x}^2}g'(x^*) - f(\mu^*)\frac{\partial\mu^*}{\partial\underline{x}}(\frac{\partial x^*}{\partial\underline{x}})^2 g''(x^*)\Bigg) \\
&\Bigg(+g'''(\underline{x})F(\mu^*) + g''(\underline{x})f(\mu^*)\frac{\partial\mu^*}{\partial\underline{x}}\Bigg) + \\
&\Bigg(2g''(\underline{x})f(\mu^*)\frac{\partial\mu^*}{\partial\underline{x}} + 2g'(\underline{x})f(\mu^*)\frac{\partial^2\mu^*}{\partial\underline{x}^2}\Bigg) \\
&+f'(\mu^*)\frac{\partial\mu^*}{\partial\underline{x}}\Big( - \frac{\partial^2\mu^*}{\partial\underline{x}^2}\left[g(x^*) - g(\underline{x})\right] - \frac{\partial\mu^*}{\partial\underline{x}}\frac{\partial x^*}{\partial\underline{x}}g'(x^*) + 2g'(\underline{x})\frac{\partial\mu^*}{\partial\underline{x}}\Big)
\end{aligned}$$

Then using $\underline{x} \approx \underline{z}$, $x^* \approx \underline{z}$ and $\mu^* \approx \underline{\mu}$, $F(\mu^*) \approx 0, g(x^*) - g(\underline{x}) \approx 0$, $\frac{\partial x^*}{\partial \underline{x}} \underset{\underline{x}\approx\underline{z}}{\approx} 2$, the last line becomes 0 as $\underline{x} \approx \underline{z}$,so we obtain along with $\frac{\partial \mu^*}{\partial \underline{x}} \underset{\underline{x}\approx\underline{z}}{\approx} 2\frac{g'(\underline{z})}{\underline{z}}$,

$$\begin{aligned}\frac{\partial^3 W}{\partial \underline{x}^3} \underset{\underline{x}\approx\underline{z}}{\approx} & -f_1(2\frac{g'(\underline{z})}{\underline{z}})^2 g'(\underline{z}) + \Big( - f_0 \frac{\partial^3 \mu^*}{\partial \underline{x}^3}\left[g(x^*) - g(\underline{x})\right] - f_0 \frac{\partial^2 \mu^*}{\partial \underline{x}^2}\big[\frac{\partial x^*}{\partial \underline{x}} g'(x^*) - g'(\underline{x})\big]\Big) \\ & \Big( - f_0 \frac{\partial^2 \mu^*}{\partial \underline{x}^2}\frac{\partial x^*}{\partial \underline{x}} g'(x^*) - f_0 \frac{\partial \mu^*}{\partial \underline{x}}\frac{\partial^2 x^*}{\partial \underline{x}^2} g'(x^*) - f_0 \frac{\partial \mu^*}{\partial \underline{x}}(\frac{\partial x^*}{\partial \underline{x}})^2 g''(x^*)\Big) \\ & \Big( + g'''(\underline{x}) F(\mu^*) + g''(\underline{x}) f_0 \frac{\partial \mu^*}{\partial \underline{x}}\Big) + \\ & \Big( 2g''(\underline{x}) f_0 \frac{\partial \mu^*}{\partial \underline{x}} + 2g'(\underline{x}) f_0 \frac{\partial^2 \mu^*}{\partial \underline{x}^2}\Big)\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 W}{\partial \underline{x}^3}\frac{1}{f_0} \underset{\underline{x}\approx\underline{z}}{\approx} & \left[-\frac{\partial^2 \mu^*}{\partial \underline{x}^2}\frac{\partial x^*}{\partial \underline{x}} - \frac{\partial \mu^*}{\partial \underline{x}}\frac{\partial^2 x^*}{\partial \underline{x}^2} + 2\frac{\partial^2 \mu^*}{\partial \underline{x}^2} + \frac{\partial^2 \mu^*}{\partial \underline{x}^2} - \frac{\partial^2 \mu^*}{\partial \underline{x}^2}\frac{\partial x^*}{\partial \underline{x}}\right] g'(\underline{z}) \\ & + \left[-\frac{\partial \mu^*}{\partial \underline{x}}(\frac{\partial x^*}{\partial \underline{x}})^2 + \frac{\partial \mu^*}{\partial \underline{x}} + 2\frac{\partial \mu^*}{\partial \underline{x}}\right] g''(\underline{z}) - \frac{f_1}{f_0}(2\frac{g'(\underline{z})}{\underline{z}})^2 g'(\underline{z})\end{aligned}$$

$$\frac{\partial^3 W}{\partial \underline{x}^3}\frac{1}{f_0} \underset{\underline{x}\approx\underline{z}}{\approx} \left[-2\frac{\partial^2 \mu^*}{\partial \underline{x}^2}\frac{\partial x^*}{\partial \underline{x}} - \frac{\partial \mu^*}{\partial \underline{x}}\frac{\partial^2 x^*}{\partial \underline{x}^2} + 3\frac{\partial^2 \mu^*}{\partial \underline{x}^2}\right] g'(\underline{z}) + \left[3 - (\frac{\partial x^*}{\partial \underline{x}})^2\right]\frac{\partial \mu^*}{\partial \underline{x}} g''(\underline{z}) - \frac{f_1}{f_0}(2\frac{g'(\underline{z})}{\underline{z}})^2 g'(\underline{z})$$

$$\frac{\partial^3 W}{\partial \underline{x}^3}\frac{1}{f_0} \underset{\underline{x}\approx\underline{z}}{\approx} \left[3 - 2\frac{\partial x^*}{\partial \underline{x}}\right]\frac{\partial^2 \mu^*}{\partial \underline{x}^2} g'(\underline{z}) + \left[-\frac{\partial \mu^*}{\partial \underline{x}}\frac{\partial^2 x^*}{\partial \underline{x}^2}\right] g'(\underline{z}) + \left[3 - (\frac{\partial x^*}{\partial \underline{x}})^2\right]\frac{\partial \mu^*}{\partial \underline{x}} g''(\underline{z}) - \frac{f_1}{f_0}(2\frac{g'(\underline{z})}{\underline{z}})^2 g'(\underline{z})$$

$$\frac{\partial^3 W}{\partial \underline{x}^3}\frac{1}{f_0} \underset{\underline{x}\approx\underline{z}}{\approx} [3 - 2\times 2]\frac{\partial^2 \mu^*}{\partial \underline{x}^2} g'(\underline{z}) + \left[-\frac{\partial \mu^*}{\partial \underline{x}}\frac{\partial^2 x^*}{\partial \underline{x}^2}\right] g'(\underline{z}) + [3 - 4]\frac{\partial \mu^*}{\partial \underline{x}} g''(\underline{z}) - \frac{f_1}{f_0}(2\frac{g'(\underline{z})}{\underline{z}})^2 g'(\underline{z})$$

$$\frac{\partial^3 W}{\partial \underline{x}^3}\frac{1}{f_0} \underset{\underline{x}\approx\underline{z}}{\approx} \left[-\frac{\partial^2 \mu^*}{\partial \underline{x}^2} - \frac{\partial \mu^*}{\partial \underline{x}}\frac{\partial^2 x^*}{\partial \underline{x}^2}\right] g'(\underline{z}) - \frac{\partial \mu^*}{\partial \underline{x}} g''(\underline{z}) - \frac{f_1}{f_0}(2\frac{g'(\underline{z})}{\underline{z}})^2 g'(\underline{z})$$

Since $\frac{\partial \mu^*}{\partial \underline{x}} \underset{\underline{x}\approx\underline{z}}{\approx} 2\frac{g'(\underline{z})}{\underline{z}}$

$$\frac{1}{f_0}\frac{\partial^3 W}{\partial \underline{x}^3} \underset{\underline{x}\approx\underline{z}}{\approx} -2g''(\underline{z})\frac{g'(\underline{z})}{\underline{z}} - g'(\underline{z})\frac{\partial^2 \mu^*}{\partial \underline{x}^2} - 2\frac{(g'(\underline{z}))^2}{\underline{z}}\frac{\partial^2 x^*}{\partial \underline{x}^2} - \frac{f_1}{f_0}(2\frac{g'(\underline{z})}{\underline{z}})^2 g'(\underline{z})$$

Using (43) and (44)

$$\begin{aligned}\frac{1}{f_0}\frac{\partial^3 W}{\partial \underline{x}^3} \underset{\underline{x}\approx\underline{z}}{\approx} & -2g''(\underline{z})\frac{g'(\underline{z})}{\underline{z}} - g'(\underline{z})(\frac{2}{\underline{z}}g''(\underline{z}) - 4g'(\underline{z})\frac{1}{(\underline{z})^2} - \frac{4}{3}\frac{1}{(\underline{z})^2}\frac{f_1}{f_0}(g'(\underline{z}))^2) \\ & -2\frac{(g'(\underline{z}))^2}{\underline{z}}(-2\frac{g''(\underline{z})}{g'(\underline{z})} - \frac{4}{3}\frac{1}{\underline{z}}\frac{f_1}{f_0}g'(\underline{z})) - \frac{f_1}{f_0}(2\frac{g'(\underline{z})}{\underline{z}})^2 g'(\underline{z})\end{aligned}$$

The terms without $f_1$ sum to

$$-2\frac{g''(\underline{z})g'(\underline{z})}{\underline{z}} - 2\frac{g''(\underline{z})g'(\underline{z})}{\underline{z}} + 4\frac{(g'(\underline{z}))^2}{(\underline{z})^2} + 4\frac{g''(\underline{z})g'(\underline{z})}{\underline{z}} = 4\frac{(g'(\underline{z}))^2}{(\underline{z})^2}$$

and the terms with $f_1$ sum to

$$g'(\underline{z})\frac{4}{3}\frac{1}{(\underline{z})^2}\frac{f_1}{f_0}(g'(\underline{z}))^2 + 2\frac{(g'(\underline{z}))^2}{\underline{z}}\frac{4}{3}\frac{1}{\underline{z}}\frac{f_1}{f_0}g'(\underline{z}) - \frac{f_1}{f_0}(2\frac{g'(\underline{z})}{\underline{z}})^2 g'(\underline{z}) = \frac{f_1}{f_0}g'(\underline{z})\left(\frac{g'(\underline{z})}{\underline{z}}\right)^2\left[\frac{4}{3} + 2\frac{4}{3} - 4\right] = 0$$

Hence,

$$\frac{1}{f_0}\frac{\partial^3 W}{\partial \underline{x}^3} \underset{\underline{x}\approx\underline{z}}{\approx} 4\frac{(g'(\underline{z}))^2}{(\underline{z})^2} > 0$$

# H Market for Lemons

## H.1 Effect of Coverage on the Mass of Buyers

We now prove Lemma 5, which concerns the effects of changes in coverage $\underline{x}$ on the mass of buyers.

If the equilibrium regime is Lemons, there is a type $\mu_*$ s.t. that types $\mu \geq \mu_*$ purchase $x = \underline{x}$ and types $\mu < \mu_*$ purchase $x = 0$. Each contract breaks even: $p(\underline{x}) = \underline{x} \cdot \mathbb{E}[\mu \mid \mu > \mu_*]$ and $p(0) = 0$. If $\mu_* > \underline{\mu}$, then type $\mu_*$ is indifferent between contract $(\underline{x}, p(\underline{x}))$ and $(0, 0)$, i.e.:

$$\mu_*\underline{x} + g(\mu_*, \underline{x}) - \underline{x}\mathbb{E}[\mu \mid \mu > \mu_*] = 0. \tag{45}$$

*Proof.* Implicitly differentiating (45) implies

$$\frac{\partial \mu_*}{\partial \underline{x}} = \frac{1}{\psi'(\mu_*)}\left[\frac{g(x)}{x}\right]' > 0.$$

Recall that $\left[\frac{g(x)}{x}\right]' \leq 0$ by Lemma 19. Also, if $f$ is log-concave, $\psi'(\mu_*) < 0$. Notice that $\frac{\partial q}{\partial \mu_*} = -f(\mu^*)$. Then, this implies

$$\frac{\partial \underline{x}}{\partial q} = \frac{1}{\frac{\partial q}{\partial \mu_*}\frac{\partial \mu_*}{\partial \underline{x}}} = \frac{[-\psi'(\mu_*)]}{f(\mu_*)\left[\frac{g(x)}{x}\right]'} < 0.$$

□

## H.2 Socially optimal Coverage

We now prove Proposition 13, which shows that the socially optimal level of coverage $\underline{x}$ is interior, assuming $\underline{\mu}$ strictly prefers $(0,0)$ to $(1, E[\mu])$.

Recall that, in this setting, welfare is $W(x) = q(x)g(x)$, where $q(x) = 1 - F(\mu_*(x))$. Then, using Lemma 5,

$$\begin{aligned} W'(x) =& q'(x)g(x) + q(x)g'(x) = -f(\mu_*)g(x)\frac{\partial \mu_*}{\partial \underline{x}} + q(x)g'(x) \\ =& \frac{1}{\psi'(\mu_*)}\left(-\frac{g'(x)}{x} + \frac{g(x)}{x^2}\right) f(\mu_*)g(x) + (1 - F(\mu_*))g'(x) \\ =& \frac{f(\mu_*)}{\psi'(\mu_*)}\left(\frac{g(x)}{x}\right)^2 + g'(x)\left((1 - F(\mu^*)) - \frac{f(\mu_*)}{\psi'(\mu_*)}\frac{g(x)}{x}\right) \end{aligned}$$

We note that $\mu_*(1) < \overline{\mu}$, since risk aversion implies that even if $\underline{x} = 1$ is given the highest price $\overline{\mu}$, a positive mass of agents will want to buy it; so $\psi'(\mu_*(1)) < 0$. Also $\mu_*(1) > \underline{\mu}$, since by assumption, not all types are purchasing. Since $g'(1) = 0$, $g(1) \neq 0$, and $\psi'(\mu) < 0$, $f(\mu) > 0$ for any $\mu \in (\underline{\mu}, \overline{\mu})$, the result follows.

## H.3 Welfare Quasi-concave

We now prove that log-concavity of demand implies welfare is quasi-concave.

**Lemma 23.** *If demand $q(\underline{x}) = 1 - F(\mu_*(\underline{x}))$ is log-concave, then welfare is quasi-concave in $\underline{x}$.*

*Proof.* If $q(\underline{x})$ is log concave, then $[\ln q(\underline{x})]'' = [q'(\underline{x})/q]' \leq 0$. Also, $g'(x)/g(\underline{x})$ is decreasing by Lemma 19. Then (15) can be written as $g'/g = -q'/q$. The RHS is increasing and the LHS is decreasing so the FOC holds at a unique point. □

One limitation of Lemma 23 is that it places conditions on endogenous objects, namely $\mu_*$. The following result provides sufficient conditions on primitives such that $q(\underline{x})$ is log-concave.

**Lemma 24.** *If $\mu \sim \mathcal{U}[\underline{\mu}, \overline{\mu}]$ and $g(x) = \frac{1}{2}(1 - (1-x)^2)$, then $q(\underline{x})$ is log-concave.*

This structure of $g(\cdot)$ results in the CARA-Normal model, Appendix K.

*Proof.* We can write

$$\frac{\partial q}{\partial \underline{x}}\frac{1}{q} = \frac{f(\mu_*)}{1-F(\mu^*)}\left[\frac{g(x)}{x}\right]' \frac{1}{[-\psi'(\mu_*)]}$$

Given the assumptions, then $\psi = \frac{1}{2}(\overline{\mu} - \mu_*)$ and $\frac{f(\mu_*)}{1-F(\mu^*)} = \frac{1}{\mu^*}$ and $g(x)/x = 1 - \frac{x}{2}$. Therefore,

$$\frac{\partial q}{\partial \underline{x}}\frac{1}{q} = -\frac{1}{\mu_*}$$

Then, since $\frac{\partial \mu_*}{\partial \underline{x}} \geq 0$, this implies $\frac{\partial}{\partial \underline{x}}\left[\frac{\partial q}{\partial \underline{x}}\frac{1}{q}\right] \leq 0$, so $q(\underline{x})$ is log-concave. □

# I Moral Hazard

## I.1 Equilibrium Characterization

We now discuss a model with MH. The model is as in Section 7 unless otherwise stated. We assume that $g(\mu, x) = g(x)$ independent of $\mu$, for simplicity. The contract space is $X = \{0\} \cup [\underline{x}, \overline{x}]$, where we may have $\underline{x} = 0$ as well. For an equilibrium $(\alpha, p)$, denote the set of purchased contracts with strictly positive coverage by

$$\mathcal{B}^+ = supp(\alpha_X) \cap [\underline{x}, \overline{x}].$$

Recall the notation of the social surplus,

$$s(\mu, x) = g(x) + w(\mu, x) - k(\mu, x). \tag{46}$$

We allow a non-purchasing tax $T \geq 0$, where as usual we assume $T = 0$ if $\underline{x} = 0$. The following proposition (together with Proposition 22 below) imply Proposition 14. Let the set of strictly positive coverage $(x > 0)$ contracts purchased in equilibrium be

$$\mathcal{B}^+ = supp(\alpha_X) \cap [\underline{x}, \overline{x}].$$

That is, individuals purchase contracts in $\mathcal{B}^+$ and, in addition, possibly $x = 0$.

**Proposition 21.** *Suppose $X = \{0\} \cup [\underline{x}, \overline{x}]$ and Assumption 4 holds. In any AG equilibrium $(p, \alpha)$ with $\alpha_X(\{x \mid x > \underline{x}\}) > 0$,*

1. *$p(x)$ is Lipshitz, and is strictly increasing on $\{x \mid p(x) > 0\}$, and $p(x) \leq c(\overline{\mu}, x)$ for all $x \in X$.*

2. *The equilibrium allocation function $\sigma$ is non-decreasing, and is strictly increasing on $\sigma^{-1}((\underline{x}, \overline{x}])$. Let $\tau = \sigma^{-1}$ be defined on $\mathcal{B}^+ \backslash \{\underline{x}\}$. Denote $\mu^*$ the lowest type which purchases $> \underline{x}$, and by $\mu_*$ the lowest type that purchases $\underline{x}$; and $x^* = \sigma(\mu^*)$. Denote also $\tilde{x} = \sup_\mu \sigma(\mu)$ to be the maximum coverage.*

3. *If $\overline{x} = 1$, the maximal purchase $\tilde{x}$ satisfies $\tilde{x} < 1$.*

4. *If, furthermore, if Assumption 4 holds, then: $\sigma$ is continuous, there is $x^* \in [\underline{x}, \tilde{x})$ s.t. $\mathcal{B}^+ = [x^*, \tilde{x}]$; if $\underline{x} > 0$ and it is an atom of the equilibrium, then $x^* > \underline{x}$.*

5. *Under Assumption 4, if $\underline{x} > 0$ and $0$ is an atom, then so is $\underline{x}$.*

6. *Each contract breaks even. In particular $p(0) = 0$ and*

$$p(x) = c(\tau(x), x), \qquad \textit{a.e.} \quad x \in [x^*, \tilde{x}] \tag{47}$$

$$p(\underline{x}) = \mathbb{E}\left[c(\mu, \underline{x}) \mid \mu \in [\mu_*, \mu^*]\right], \tag{48}$$

*where (48) is vacuous if $\mu_* = \mu^*$; if $\tilde{x} < \overline{x}$, then in $[\tilde{x}, \overline{x}]$, $p(\cdot)$ follows the indifference curve of $\overline{\mu}$ through $(\tilde{x}, c(\overline{\mu}, \tilde{x}))$; and in $x \in [\underline{x}, x^*)$, price $p(x)$ follows the indifference curve of $\mu^*$ through $(x^*, c(\mu^*, x^*))$.*

7. *Under Assumption 4, the equation*

$$\frac{\partial s}{\partial x}(\overline{\mu}, x^\star) = 0 \tag{49}$$

*has precisely one solution $x^\star$ in $(\underline{x}, 1]$,[45] and*

$$\tilde{x} = \begin{cases} \overline{x} & x^\star > \overline{x} \\ x^\star & \tilde{x} \in (\underline{x}, \overline{x}] \end{cases}$$

8. *If the model satisfies Assumption 4, then $\tau$ is Lebesgue-a.e. different in $\mathcal{B}^+$, with*

$$\tau'(x) = \frac{1}{x + \frac{\partial k}{\partial \mu}(\tau(x), x)} \frac{\partial s}{\partial x}(\tau(x), x) \qquad \textit{for a.e.} \quad x \in (x^*, \tilde{x}) \tag{50}$$

[45] The existence of such a solution relies also on our assumption that an equilibrium with $\alpha_X(\{x \mid x > \underline{x}\}) > 0$ exists.

As a corollary to Part 7 above, we note the following:

**Corollary 7.** *If (49) has a solution in $[0, \underline{x}]$ or no solution in $[0, 1]$, then any equilibrium has each agent either not purchasing at all or purchasing $\underline{x}$; i.e., no one buys above $\underline{x}$.*

Notice that the characterization of equilibrium in Proposition 21 assumes that some individuals choose $x > \underline{x}$. Corollary 7 emphasizes that if (49) has no "well behaved" solution, then such an equilibrium cannot exist.

#### I.1.1 Analysis of the Surplus Function

Note that the function $s(\cdot, \cdot)$ satisfies, for $x > 0$:

$$s(\mu, x) = u(\mu, x, c(\mu, x)) = (\mu x + w(\mu, x) + g(x)) - (\mu x + k(\mu, x)) \tag{51}$$

Extending the function to $0$, we have $s(\mu, x) = u(\mu, 0, 0) + T$. We observe that under Assumption 4, since $g'' < 0$, $\frac{\partial^2 s}{\partial x^2} < 0$, so for each $\mu$ there is at most one $x^{\star\star} = x^{\star\star}(\mu)$ where $\frac{\partial s}{\partial x}(\mu, x^{\star\star}) = 0$. Furthermore, implicit differentiation gives $(x^{\star\star})'(\mu) = -\frac{\frac{\partial^2 s}{\partial \mu \partial x}}{\frac{\partial^2 s}{\partial x^2}}(\mu, x^{\star\star}(\mu)))$, and since we see that under Assumption 4, $\frac{\partial^2 s}{\partial \mu \partial x} \leq 0$ we have $(x^{\star\star})'(\cdot) \leq 0$. Hence, since by the definition of the maximum coverage $\tilde{x}$, $\tilde{x} = \min[x^{\star\star}(\overline{\mu}), \overline{x}]$, we see that:

**Lemma 25.** *For any $x < \tilde{x}$ and any $\mu$, $\frac{\partial s}{\partial x}(\mu, x) > 0$.*

#### I.1.2 Proofs of Parts 1, 2 and 3 of Proposition 21

Proof of Part $1$ is Proposition 1 in AG, along with the fact that $c(\mu, ) \leq c(\overline{\mu}, )$ for all types.

To sketch a proof of Part 2, recall the definitions of $\psi^+(x), \psi^-(x)$ from (24), (25) of Appendix B.3, which heuristically express the highest and lowest types purchasing a contract $x$. Lemma 7, Corollary 4, and Lemma 8 of that section show $\psi^- \equiv \psi^+$ in $(0, 1] \cap supp(\alpha_X)$; the proof works in the restricted domain $(\underline{x}, \overline{x}]$, and with the modified MH utility and costs functions, as well. Here we've denoted $\tau$ to be this common function, and $\sigma = \tau^{-1}$ on $(\underline{x}, \overline{x}]$, with $\sigma(\mu) = \underline{x}$ (resp. $= 0$) for all those types $\mu$ purchasing $\underline{x}$ (resp. $0$).[46]

To prove Part 3, observe that by Theorem 1, for the amount $\tilde{x}$ purchased by $\overline{\mu}$, $p(\tilde{x}) = c(\overline{\mu}, \tilde{x})$, but $p(\cdot) \leq c(\overline{\mu}, )$. So if $\tilde{x} = 1$, then by Assumption 4 and since $g'(1) = 0$,

$$p'(1) \geq \frac{\partial c}{\partial x}(\overline{\mu}, 1) = \overline{\mu} + \frac{\partial k}{\partial x}(\overline{\mu}, 1) > \overline{\mu} + \frac{\partial w}{\partial x}(\overline{\mu}, 1) + g'(1) \equiv \frac{\partial u}{\partial x}(\overline{\mu}, 1, \cdot)$$

[46]Lemma 10 there shows that $\psi$ is strictly increasing, but the proof relies on the cost form with no MH; we will handle the matter differently ahead.

and hence $\overline{\mu}$ would be incentivized to lower their coverage to slightly below $1$.

### I.1.3 Proof of Parts 4, 5, 6 of Proposition 21

Subsequent sections assume Assumption 4. We have established that $\sigma$ is increasing, strictly so in $\sigma^{-1}((\mu^*, \overline{\mu}])$. Suppose it had a jump discontinuity at some $\mu^\circ$, with $0 < a := \limsup_{\mu \to (\mu^\circ)^-} \sigma(\mu) < b := \liminf_{\mu \to (\mu^\circ)^+} \sigma(\mu)$. Near $a, b$ there are masses of types close to $\mu^\circ$ purchasing. Hence $p(a) = c(\mu^\circ, a)$, $p(b) = c(\mu^\circ, b)$, with $u(\mu^\circ, a, p(a)) = u(\mu^\circ, b, p(b))$, i.e., $s(\mu^\circ, a) = s(\mu^\circ, b)$, contradicting Lemma 25.

To demonstrate the conclusion that if $\underline{x} > 0$ is an atom of the equilibrium then $x^* > \underline{x}$, we observe that if by way of contradiction $\mathcal{B}^+ = [\underline{x}, \tilde{x}]$, $\underline{x}$ being an atom would lead to a discontinuity in prices, almost verbatim to the proof as with no MH present, as done in Appendix B.5.

To prove Part 5: Suppose $0$ is an atom, but $\underline{x}$ is not (so $\underline{\mu} < \mu_* = \mu^* < \overline{\mu}$). Similar to above, we see that this implies $p(\mu_*) = c(\mu_*, x_*)$; of course $p(0) = 0 = c(\mu_*, 0)$, and that type $\mu_*$ is indifferent between $(0, p(0))$ and $(x^*, p(x^*))$, so $u(\mu_*, 0, c(\mu_*, 0)) = u(\mu_*, \underline{x}, c(\mu_*, \underline{x}))$. So all together, $s(\mu_*, x^*) = s(\mu_*, 0) - T$, and once again, Lemma 25 is contradicted.

Part 6 now follows from the general properties of AG equilibrium.

### I.1.4 Proof of Part 7 of Proposition 21

We've already mentioned $\frac{\partial s}{\partial x}(\overline{\mu}, x^\star) = 0$ has at most one solution in $[0, 1)$. An adaption of the proof of Part 3 above shows that such $\max[\min[\overline{x}, x^\star], \underline{x}]$ is the only candidate for the maximal purchase, and since the proposition assumes an equilibrium with purchase strictly above $\underline{x}$, there must be such $\underline{x} < x^\star = \tilde{x}$.

### I.1.5 Proof of Part 8 of Proposition 21

$\tau$ is a.e. differentiable in $[x^*, \tilde{x}]$ as $p(x) = c(\tau(x), x)$, $p$ is Lipshitz and $\frac{\partial c}{\partial x}(\mu, x) \geq \mu$. Since in $[x^*, \tilde{x}]$,

$$p(x) = c(\tau, x) = x\tau(x) + k(\tau(x), x)$$

so it holds a.e.

$$p'(x) = \tau(x) + x\tau'(x) + \frac{\partial k}{\partial \mu}(\tau(x), x)\tau'(x) + \frac{\partial k}{\partial x}(\tau(x), x)$$

If each individual is choosing her optimal contract, then the following First Order Condition holds for a.e. $x \in [x^*, \tilde{x}]$,

$$\tau(x) + g'(x) + \frac{\partial w}{\partial x}(\tau(x), x) - p'(x) = 0$$

Combining these gives

$$\tau(x) + g'(x) + \frac{\partial w}{\partial x}(\tau(x), x) - \tau(x) - x\tau'(x) - \frac{\partial k}{\partial \mu}(\tau(x), x)\tau'(x) - \frac{\partial k}{\partial x}(\tau(x), x) = 0$$

Cancelling out $\tau(x)$, combining the terms in $\tau'(x)$ to the RHS, gives (50).

## I.2 Uniqueness

### I.2.1 Uniqueness when $\underline{x} = 0$ (and proof of Proposition 15)

First, we state the uniqueness results when $\underline{x} = 0$; this proposition implies Proposition 15.

**Proposition 22.** *Under Assumption 4, when $\underline{x} = 0$, equilibrium is unique. If $\frac{\partial s}{\partial x}(\overline{\mu}, 0) > 0$, then all buy positive levels of coverage. If $\frac{\partial s}{\partial x}(\overline{\mu}, 0) \leq 0$ then no one purchases.*

*Proof.* Heuristically, by Corollary 7, if $\frac{\partial s}{\partial x}(\overline{\mu}, 0) > 0$, then since $\frac{\partial s}{\partial x}(\overline{\mu}, 1) < 0$, (49) has a solution in $(0, 1)$, which is the level $\overline{\mu}$ purchases, and the solution follows (50) for the others type. If $\frac{\partial s}{\partial x}(\overline{\mu}, 0) \leq 0$, then since $\frac{\partial s}{\partial x}(\overline{\mu}, \cdot)$ is monotonically decreasing, (49) has no solution in $(0, 1]$, so no one buys insurance. A rigorous argument is as follows:

If $\frac{\partial s}{\partial x}(\overline{\mu}, 0) \leq 0$, then $\frac{\partial s}{\partial x}(\overline{\mu}, ) < 0$ in $[0, \overline{x}]$, so $\overline{\mu}$ cannot have a solution $\hat{x} \in (0, \overline{x}]$ to (18), so we cannot have positive levels of purchase by Proposition 21. If $\frac{\partial s}{\partial x}(\overline{\mu}, 0) > 0$, then we do have such a unique solution so the highest type purchases $\tilde{x}$ which satisfies $\frac{\partial s}{\partial x}(\overline{\mu}, \tilde{x}) = 0$ (or $\overline{x}$ if the solution to this equation is too high); it is not incentive compatible in this case for $\overline{\mu}$ to purchase $0$. From there, $\tau(x)$ obeys (19) for all types who purchase positive amounts of coverage.

We will show that in this case, all agents purchase positive levels of coverage. Assume by way of contradiction that some do buy $0$. Let $x^*, \mu^*$ denote the associated cut-offs, i.e., $\mu$ purchases $0$ iff $\mu < \mu^*$, and $x^* = \sigma(\mu^*)$, where $\sigma$ is the associated allocation rule. There are two cases we need to eliminate: $x^* > 0$ and $x^* = 0$.

If $x^* > 0$, we have

$$s(\mu^*, x^*) = u(\mu^*, x^*, c(\mu^*, x^*)) = u(\mu^*, 0, c(\mu^*, 0)) = s(\mu^*, 0) - T$$

so $s(\mu^*, x^*) = s(\mu^*, 0) - T$, contradicting Lemma 25.

Now assume $x^* = 0$. The inverse $\tau$ of $\sigma$ satisfies, in $(0, \tilde{x})$, (50), this is

$$\tau'(x) = \frac{1}{x + \frac{\partial k}{\partial \mu}(\tau(x), x)} \frac{\partial s}{\partial x}(\tau(x), x) \geq \frac{1}{x + \frac{\partial k}{\partial \mu}(\tau(x), x)} \frac{\partial s}{\partial x}(\overline{\mu}, x)$$

where the inequality results from Assumption 4, $\frac{\partial^2 s}{\partial \mu \partial x} \leq 0$. Since $\frac{\partial s}{\partial x}(\overline{\mu}, 0) > 0$, there is $\delta > 0$ s.t. in $(0, \delta)$, $\frac{\partial s}{\partial x}(\overline{\mu}, x) > \frac{1}{2}\frac{\partial s}{\partial x}(\overline{\mu}, 0)$. Also, denoting $M = \max_{\mu \in [\underline{\mu}, \overline{\mu}], x \in [0,1]} |\frac{\partial^2 k}{\partial x \partial \mu}(\mu, x)|$, since $\frac{\partial k}{\partial \mu}(\mu, 0) \equiv 0$, it follows that $\frac{\partial k}{\partial \mu}(\mu, x) \leq Mx$ for all $\mu, x$.[47] Hence, for $x \in (0, \delta)$,

$$\ln(\tau(\delta)) \geq \ln(\tau(x)) + \frac{1}{2(1+M)} \frac{\partial s}{\partial x}(\overline{\mu}, 0) \int_x^\delta \frac{dx}{x} \geq \ln(\mu^*) + \frac{1}{2(1+M)} \frac{\partial s}{\partial x}(\overline{\mu}, 0)(\ln(\delta) - \ln(x)) \underset{x \to 0}{\rightarrow} \infty$$

a contraction. The uniqueness now follows since (50) obeys standard unique conditions for differential equations. □

In particular, in the model of Example 1, equilibrium is unique.

### I.2.2 Uniqueness when $\underline{x} > 0$ (Proposition 23)

Next, we consider the more complex case, where $\underline{x} > 0$:

**Proposition 23.** *Under Assumption 4, when $\underline{x} > 0$, if $w(,), k(,)$ are independent of riskiness $\mu$ and the distribution of types $f$ is log-concave, then equilibrium is unique.*

In the model of Example 2 below, equilibrium need not be unique when $\underline{x} > 0$.

We begin with two lemmas, which were established with no MH. It's worth noting that these lemmas do not use the assumption that $w(,), k(,)$ are independent of type, nor do they use log-concavity of the type distribution.

**Lemma 26.** *Let $(p_1, \alpha_1), (p_2, \alpha_2)$ be equilibria with zero or partial (but not full) pooling at $\underline{x}$ (that is, some individuals purchase $x > \underline{x}$), and cut-off coverages $x_1^*, x_2^*$. If $x_1^* > x_2^*$, then $p_1 \leq p_2$, with strict inequality for $x \in [\underline{x}, x_1^*)$.*

*Proof.* Same proof as Lemma 12 (the version without MH), with appropriate modifications. □

**Lemma 27.** *If there is one equilibrium with at least partial pooling at $\underline{x}$, then all equilibria have at least partial pooling at $\underline{x}$.*

[47] This is the only point at which we use the fact that the function $k(,)$ can be extended twice continuous differentiably to a neighborhood of $[\underline{\mu}, \overline{\mu}] \times [0, 1]$.

*Proof.* The proof is similar to the proof of Lemma 13 (the version without MH) but with some modifications. Suppose $(p_1, \alpha_1)$ has an atom at $\underline{x}$ while $(p_2, \alpha_2)$ does not; $\mu_1^* > \underline{\mu} = \mu_2^*$. Note that in either case, in $(\underline{x}, x_2^*)$, a standard approximating-by-sequence-of-weak equilibria argument (as per the AG definition) shows that $p_2(\cdot) \leq c(\underline{\mu}, )$.

First let's deal with the case $(p_1, \alpha_1)$ has only partial, not full, pooling. Hence, $x_1^* = \sigma_1(\mu_1^*) = \sigma_2(\mu_1^*) > x_2^*$, so by Lemma 26, $p_1 \leq p_2$ with strict inequality in $[\underline{x}, x_1^*)$. Hence $p_1(\underline{x}) < p_2(\underline{x}) \leq c(\underline{\mu}, \underline{x})$. However,

$$p_1(\underline{x}) = E[c(\mu, \underline{x}) \mid \mu_{1*} \leq \mu \leq \mu_1^*] > c(\underline{\mu}, \underline{x})$$

a contradiction.

Now, we deal with the case $(p_1, \alpha_1)$ has full pooling. Then $p_1(\tilde{x}) \leq c(\overline{\mu}, \tilde{x}) = p_2(\tilde{x})$. In $(x_2^*, \tilde{x})$,

$$p_1'(x) = \frac{\partial u}{\partial x}(\overline{\mu}, x, \cdot) > \frac{\partial u}{\partial x}(\tau(x), x, \cdot) = p_2'(x)$$

and in $(\underline{x}, x_2^*)$

$$p_1'(x) = \frac{\partial u}{\partial x}(\overline{\mu}, x, \cdot) > \frac{\partial u}{\partial x}(\mu_2^*, x, \cdot) = p_2'(x)$$

so either way, $p_1(\underline{x}) < p_2(\underline{x}) \leq c(\underline{\mu}, \underline{x})$, and a contradiction is arrived as above. □

**Lemma 28.** *If Assumption 4 holds,* $\underline{x} > 0$*, and* $w(,)$*,*$k(,)$ *are independent of type, then if one equilibrium has no one purchasing, then it is the unique equilibrium.*

*Proof.* Let $(p, \alpha)$ and $(q, \beta)$ be two different equilibria, with $\alpha(\{0\}) = 1$. In that case, $p(\cdot)$ follows the indifference curve of $\overline{\mu}$ through $(0, 0)$, i.e., for $x \in [\underline{x}, \overline{x}]$

$$g(0) - T = \overline{\mu}x + g(x) + w(x) - p(x)$$

and also

$$p(x) \leq c(\overline{\mu}, x) = \overline{\mu}x + k(x)$$

Either $\beta(\{\underline{x}\}) > 0$ or $\beta(\{\underline{x}\}) = 0$. We deal first with the case $\beta(\{\underline{x}\}) > 0$. Combining the above equations evaluated at $x = \underline{x}$

$$\overline{\mu}\underline{x} + g(\underline{x}) + w(\underline{x}) = -T + g(0) + p(\underline{x}) \leq -T + g(0) + \overline{\mu}\underline{x} + k(\underline{x})$$

i.e.,

$$(g(\underline{x}) - g(0)) + w(\underline{x}) - k(\underline{x}) \leq -T$$

Examining $(\beta, q)$, letting $\omega_*, \omega^*$ denote its cut-off types, we have

$$g(0) - T \leq \omega_* \underline{x} + g(\underline{x}) + w(\underline{x}) - E[\mu \mid \omega_* \leq \mu \leq \omega^*]\underline{x} - k(\underline{x})$$

and using the above inequality,

$$-T \leq -(E[\mu \mid \omega_* \leq \mu \leq \omega^*] - \omega_*)\underline{x} + (g(\underline{x}) - g(0)) + w(\underline{x}) - k(\underline{x}) \leq -T - (E[\mu \mid \omega_* \leq \mu \leq \omega^*] - \omega_*)\underline{x}$$

and therefore $E[\mu \mid \omega_* \leq \mu \leq \omega^*] - \omega_* \leq 0$, a contradiction.

We now deal with the case $\beta(\{\underline{x}\}) = 0$. Since under this equilibrium there is some purchasing, by Proposition 21 all types are purchasing strictly above $\underline{x}$, with full separation of types. Let $\tilde{y}, y^*$ denote the highest and lowest coverages purchased, and let $\phi(\cdot)$ be the indifference curve of $\overline{\mu}$ through $(\tilde{x}, q(\tilde{x}))$. Since $p(\tilde{x}) \leq c(\overline{\mu}, \tilde{x}) = q(\tilde{x})$, and yet in $(\alpha, p)$ $\overline{\mu}$ at least weakly prefers $(0, 0)$ to $(\tilde{x}, p(\tilde{x}))$, we must have $p(\tilde{x}) = q(\tilde{x}) = \phi(\tilde{x})$ and $\phi(0) = 0$. By incentive compatibility, $q(\cdot) \geq \phi(\cdot)$, and in $(0, y^*)$,

$$q'(x) = \underline{\mu} x + w'(x) + g'(x) < \overline{\mu} x + w'(x) + g'(x) = \phi'(x)$$

and hence $q(0) > \phi(0) = 0$, a contradiction. □

We can now prove Proposition 23, which mimics the proof of Proposition 8:

*Proof.* Recall the definition (from Section 5):

$$\psi(\mu_*, \mu^*) = E[\mu \mid \mu^* > \mu > \mu_*] - \mu_*$$

As mentioned previously, when $f(\cdot)$ is log-concave, then $\psi$ is strictly decreasing in $\mu_*$. Clearly also, $\psi$ is strictly increasing in $\mu^*$.

Like in the proof of Proposition 8, to show uniqueness of equilibrium, we check several cases. Let $(p, \alpha)$ and $(q, \beta)$ be two different equilibria. By Lemma 28, we can assume that in each of those, at least some agents purchase some coverage. Like in the proof of Proposition 8, we eliminate all cases except the case is which both have either partial or full pooling (possibly one of each), but differ in their price at $\underline{x}$. Indeed, this elimination is done verbatim, except using Lemmas 26 and 27 above instead of their non-moral-hazard versions.

Hence, assume that they both have either partial or full pooling (possibly one of each), but w.l.o.g, $q(\underline{x}) > p(\underline{x})$. Associate with $(\alpha, p)$ the usual cut-offs $\mu^*, \mu_*$, and denote the cut-offs $(\beta, q)$ by $\omega^*, \omega_*$. Under $(\beta, q)$ is less willing to prefer $(\underline{x}, q(\underline{x}))$ over $(0, 0)$, so $\omega_* \geq \mu_*$. Like there, we claim we must have $\omega^* \leq \mu^*$. The cases in which $(\alpha, q)$ (and possibly also $(\beta, q)$) has full pooling, or they both have only partial pooling, are dealt with in the same way; if only $(\beta, q)$ has full pooling, then denote $\tilde{x}$ be the maximum coverage under $(p, \alpha)$. We have $q(\tilde{x}) \leq c(\overline{\mu}, \tilde{x}) = p(\tilde{x})$. Furthermore, in $(\underline{x}, \tilde{x})$, $q' > p'$, just like the second case[48] in the proof of Lemma 27. Together, we see that $q(\underline{x}) < p(\underline{x})$, a contradiction.

We note that, up to this point, we have not made use of the log-concavity; now we will, and again the proof is similar to the corresponding stage of the proof of Proposition 8, with appropriate modifications. (We have only made use of the fact that $w(\cdot, \cdot)$ and $k(\cdot, \cdot)$ are independent to apply Lemma 28; we will make further use below.)

Now, along with $\omega^* \leq \mu^*$, we have either $\omega_* = \mu_*$ or $\omega_* > \mu_*$. In the former case,

$$q(\underline{x}) = \underline{x}E[\mu \mid \omega_* < \mu < \omega^*] + k(\underline{x}) \leq \underline{x}E[\mu \mid \mu_* < \mu < \mu^*] + k(\underline{x}) = p(\underline{x})$$

since $\omega^* \leq \mu^*$; but this contradicts $q(\underline{x}) > p(\underline{x})$. In the other case, i.e., $\omega^* \leq \mu^*$ with $\omega_* > \mu_*$. we have

$$E[\mu \mid \omega_* < \mu < \omega^*] - \omega_* = \psi(\omega_*, \omega^*) < \psi(\mu_*, \mu^*) = E[\mu \mid \mu_* < \mu < \mu^*] - \mu_*$$

and hence, multiplying both sides by $-\underline{x}$ , we have $-q(\underline{x}) + \omega_*\underline{x} > -p(\underline{x}) + \mu_*\underline{x}$. However, since these are equilibria, the agents $\omega_*$ are indifferent between $(0, 0)$ and $(\underline{x}, q(\underline{x}))$ and the agent $\mu_*$ at least weakly prefers $(\underline{x}, p(\underline{x}))$ to $(0, 0)$ (perhaps strictly if $\mu_* = \underline{\mu}$), so we know that

$$\mu_*\underline{x} + g(\underline{x}) + w(\underline{x}) - p(\underline{x}) \geq g(0) - T = \omega_*\underline{x} + g(\underline{x}) + w(\underline{x}) - q(\underline{x})$$

a contradiction. □

## I.3 Proof of Lemma 6

Since $g'(1) = 0$, $\frac{\partial s}{\partial x}(\mu, 1) < 0$ by Assumption 4. Since $\frac{\partial^2 s}{\partial x^2} < 0$, if $x^{\star\star}(\mu) > 0$ (which occurs iff $\frac{\partial s}{\partial x}(\mu, 0) > 0$) then, $x^{\star\star}(\mu)$ is the unique $x^{\star\star} \in (0, \overline{x})$ that satisfies (17). By implicit differentiation of (17), $\frac{\partial x^{\star\star}}{\partial \mu} = -\frac{\partial^2 s}{\partial x \partial \mu}\left(\frac{\partial^2 s}{\partial x^2}\right)^{-1} \leq 0$, signed by Assumption 4.

[48] $(q, \beta)$ instead of $(p_1, \alpha_1)$ and $(p, \alpha)$ instead of $(p_2, \alpha_2)$.

## I.4 Moral Hazard Reduces Insurance (Corollary 2)

Let $\tau_B, \tau_{MH}$ denote the assignment rules in the baseline and moral hazard models, respectively, with the highest types purchasing $\overline{x}, \tilde{x}$ respectively and lowest types purchasing $x^*_B, x^*_{MH}$. Since $\frac{\partial s}{\partial x}(\tau(x), x) \leq \frac{\partial g}{\partial x}(x)$ by Assumption 4 for all $x \in [0, \tilde{x}]$, and $\frac{\partial k}{\partial \mu} \geq 0$, we have $\tau'_B(x) \geq \tau'_{MH}(x) > 0$ in $(max[x^*_{MH}, x^*_B], \tilde{x})$ by (2) and (19). Moreover, $\tilde{x} \leq \overline{x}$, possibly with strict inequality; denote $\tilde{\mu} = \tau_B(\tilde{x})$. Then we have for $\mu \in [\underline{\mu}, \tilde{\mu}]$

$$\int_{\sigma_{MH}(\mu)}^{\tilde{x}} \tau'_{MH}(x)dx = \overline{\mu} - \mu = \int_{\sigma_B(\mu)}^{\overline{x}} \tau'_B(x)dx \geq \int_{\sigma_B(\mu)}^{\tilde{x}} \tau'_B(x)dx \geq \int_{\sigma_B(\mu)}^{\tilde{x}} \tau'_{MH}(x)dx$$

so $\sigma_{MH}(\mu) \leq \sigma_B(\mu)$; and by definition, for $\mu \in [\tilde{\mu}, \overline{\mu}]$, $\sigma_B(\mu) \geq \tilde{x} \geq \sigma_{MH}(\mu)$.

## I.5 Effect on $\overline{x}$ on coverage

We have the following comparative static, which states that increasing $\overline{x}$ increases the coverage purchased by any non-pooled types. It is a more precise and general statement of Corollary 3.

**Corollary 8.** *Fix $\underline{x}$. Under Assumption 4, and under the uniqueness assumptions of Propositions 22 or 23, if $\sigma(\mu \mid \overline{x}, \underline{x})$ denotes the allocation assigned to $\mu$ in the unique equilibrium on $X = \{0\} \cup [\underline{x}, \overline{x}]$, then for each $\mu \in [\underline{\mu}, \overline{\mu}]$, $\sigma(\mu \mid \overline{x}, \underline{x})$ is strictly increasing in $\overline{x}$ in the domain $\{\overline{x} \in (\underline{x}, 1) \mid \sigma(\overline{\mu} \mid \overline{x}, \underline{x}) = \overline{x}, \sigma(\mu \mid \overline{x}, \underline{x}) > \underline{x}\}$.*

*Proof.* $\sigma(\overline{\mu} \mid \overline{x}_1, \underline{x}) = \overline{x}_1 < \overline{x}_2 = \sigma(\overline{\mu} \mid \overline{x}_2, \underline{x})$. Suppose the conclusion of Corollary 8 did not hold for some $\mu^\circ$, then it would imply that for some $\overline{x}_1 < \overline{x}_2$ in this domain, $\sigma(\mu^\circ \mid \overline{x}_1, \underline{x}) = \sigma(\mu^\circ \mid \overline{x}_2, \underline{x})$; denoting this common point $x^\circ$, we see that $\tau(x^\circ \mid \overline{x}_1, \underline{x}) = \tau(x^\circ \mid \overline{x}_2, \underline{x}) = \mu^\circ$ (with $\tau(|,)$ corresponding to $\sigma(|,)$, $\sigma(\tau(x \mid \underline{x}, \overline{x}) \mid \underline{x}, \overline{x}) = x$). (19) is well-behaved enough to satisfy standard uniqueness of solution assumptions for differential equations; hence, $\tau(\cdot \mid \overline{x}_1, \underline{x}) = \tau(\cdot \mid \overline{x}_2, \underline{x})$ at least above $x^\circ$, which clearly cannot be the case as $\tau(\overline{x}_1 \mid \underline{x}, \overline{x}_1) = \overline{\mu} > \tau(\overline{x}_1 \mid \overline{x}_2, \underline{x})$. □

## I.6 Two Examples of Moral Hazard Parameterizations

We now describe two examples of parameterizations of the functions $w(\mu, x), k(\mu, x)$, which may allow future researchers to explore the model with MH in more detail.

**Example 1.** Azevedo and Gottlieb [2017], Einav et al. [2013] model ex-post MH by assuming

$$w(\mu, x) = \frac{x^2}{2}M, \qquad k(\mu, x) = x^2 M$$

for a parameter $M > 0$ which captures the propensity for MH.[49] Assume $\underline{x} = 0$. Then, (17) becomes

$$\frac{g'(x^{\star\star})}{x^{\star\star}} = M,$$

so the optimal coverage is the same for all types $\mu$. Moreover, (18) can be solved to show $\sigma(\mu)$ satisfies

$$\overline{\mu} - \mu = \int_{\sigma(\mu)}^{\tilde{x}} \left[ \frac{g'(x)}{x} - M \right] dx.$$

It is possible to include heterogeneity in the propensity for MH, by replacing the parameter $M$ with a function $\tilde{M}(\mu) \geq 0$. To satisfy Assumption 4, we must have $\tilde{M}'(\mu) \geq 0$.

Since $\frac{\partial s}{\partial x}(\overline{\mu}, 0) = g'(0) > 0$, and $\underline{x} = 0$, by Proposition 22 all types purchase positive amounts of coverage in any equilibrium. For each type $\mu$, $\sigma(\mu)$ is decreasing in $M$, but $\sigma(\mu) > 0, \forall M$. In this example, or more generally when $k(,), w(,)$ are independent of type, equilibrium is unique even when $\underline{x} > 0$ (Proposition 23).

**Example 2.** An alternative model of MH would set

$$w(\mu, x) \equiv 0, \qquad k(\mu, x) = \alpha \mu x$$

so that utility follows (1) and cost is $c(\mu, x) = (1 + \alpha) x\mu$. Then, (17) becomes

$$g'(x^{\star\star}) = \alpha\mu,$$

so $x^{\star\star}(\mu)$ is strictly decreasing in $\mu$. Moreover, (18) can be solved to show that $\sigma(\mu)$ satisfies

$$\tilde{x}^{-\frac{\alpha}{1+\alpha}} \overline{\mu} - x^{-\frac{\alpha}{1+\alpha}} \mu = \frac{1}{1+\alpha} \int_{\sigma(\mu)}^{\tilde{x}} x^{-\frac{2\alpha+1}{\alpha+1}} g'(x) dx.$$

In this example, $\frac{\partial s}{\partial x}(\overline{\mu}, 0) = g'(0) - \alpha\overline{\mu}$. When $\underline{x} = 0$, by Proposition 15, equilibria is unique; if $g'(0) > \alpha\overline{\mu}$, all types purchase positive coverage; while if $g'(0) \leq \alpha\overline{\mu}$, no one purchases insurance. If $\underline{x} = 0$, for each type $\mu$, $\sigma(\mu)$ is decreasing in $\alpha$, and for sufficiently large $\alpha$, $\sigma(\cdot) \equiv 0$. However, in this example, when $\underline{x} > 0$, there may be multiple equilibria (compare with Proposition 23); see below.

[49] These expressions for cost and utility can be obtained from assuming a specific form of utility function and consumers who optimally choose their level of health expenditures based on their type, the reimbursement rate $x$ and the parameter $M \geq 0$.

### I.7 Non-Uniqueness Example

We show that in Example 2 equilibria need not be unique if $\underline{x} > 0$. (This contrasts the conditions of Proposition 23, which are unfulfilled. Specifically, $k(,)$ is not independent of type.) Let $\mu_* \in (\underline{\mu}, \overline{\mu})$. We describe an equilibrium in which types $\mu > \mu_*$ purchase $\underline{x}$ and types $\mu < \mu_*$ purchase $0$. The cut-off $\mu_*$ must satisfy

$$\mu_* \underline{x} + g(\underline{x}) = (1+\alpha)\underline{x} E[\mu \mid \mu \geq \mu_*]$$

i.e.,

$$\frac{g(\underline{x})}{\underline{x}} = (1+\alpha)E[\mu \mid \mu \geq \mu_*] - \mu_* = (E[\mu \mid \mu \geq \mu_*] - \mu_*) + \alpha E[\mu \mid \mu \geq \mu_*]$$

If types distribute uniformly, this yields

$$\frac{g(\underline{x})}{\underline{x}} = \frac{(1+\alpha)}{2}\overline{\mu} + \frac{(\alpha-1)}{2}\mu_*$$

We see that if $\alpha = 1$ and $\overline{\mu} = \frac{g(\underline{x})}{\underline{x}}$ this equality holds, regardless of the value of $\mu_*$. If $\overline{x}$ is close enough to $\underline{x}$, this will indeed be an equilibrium (this can be shown similarly to as in Proposition 5, which showed that as $\underline{x}, \overline{x}$ approach each other, there are no equilibria with purchase above $\underline{x}$). Since this was for any $\mu_* \in (\underline{\mu}, \overline{\mu})$ there are a continuum of equilibria. (We remark that for smaller $\alpha$, there can be multiple equilibria for different distributions.)

# J Numerical simulations

## J.1 Setup

For all numerical simulations, we parameterize the surplus from insurance as

$$g(x) = \frac{1}{2}\left(1 - (1-x)^2\right)\nu. \tag{52}$$

That is, we assume surplus independent of type $\mu$. This implies $g'(x) = 1 - x$. This parameterization can be obtained by assuming individuals with CARA preferences exposed to Gaussian wealth shocks.The parameter $\nu$ scales up the surplus and is meant to capture the product of CARA risk aversion and the variance of wealth shocks.[50]

[50]This specification is used, for instance, by Veiga and Weyl [2016], Levy and Veiga [2021], Weyl and Veiga [2016]. See Appendix (K).

For most simulations, we assume types are uniformly distributed as $\mu \sim \mathcal{U}[20, 150]$. However, we will also consider other distributions, which have a similar range of types $\mu$. For most cases, we simulated approximately 2000 individuals.[51] In all simulations, we use $\nu = 25$.

Welfare is measured as the mean of $g(\sigma(\mu))$ across all simulated types. In the figures, the first panel is a histogram of the types distribution $f$. The second panel shows the behavior of the thresholds $\mu^*, \mu_*$ as we vary the regulatory parameters of interest. The third panel shows the behavior of welfare. The "baseline" values of the regulatory parameters are $T = 0, \underline{x} = 0.1, \overline{x} = 0.95$: as one regulatory parameter varies, the other ones are held fixed at these values.

## J.2 Algorithm

We now describe the algorithm we use to find equilibrium. Given (52), following (2) and (20), the cut-off type $\mu^*$ satisfies

$$\mu^* = \overline{\mu} - \nu\left[\ln(\overline{x}) - \ln(x^*) - (\overline{x} - x^*)\right] = \tau(x^*) \tag{53}$$

where $x^* = \sigma(\mu^*)$ is the cut-off coverage. Notice that $\tau(x^*)$ depends only on $x^*$ and known exogenous parameters.

The indifference of type $\mu^*$ between $x^*$ and $\underline{x}$ gives the following expression for the price of coverage $x = \underline{x}$, which we denote $\underline{p}^1$:

$$\underline{p}^1 = \mu^* \underline{x} + \left[g(\underline{x}) - g(x^*)\right]\nu \tag{54}$$

Notice that $\underline{p}^1$ depends only on $\mu^*$, $x^*$ and exogenous parameters.

Recall that $g(0) = 0$ and that individuals who choose $x = 0$ must pay a fee $T \geq 0$. Then, (23) (the indifference of type $\mu_*$ between $x = \underline{x}$ and $x = 0$) becomes

$$\mu_* = \frac{1}{\underline{x}}\left[p(\underline{x}) - g(\underline{x})\nu - T\right] = \lambda(p(\underline{x})) \tag{55}$$

Notice that $\lambda(p(\underline{x}))$ depends only on the endogenous price $p(\underline{x})$ and known exogenous parameters.

We will also use an alternative definition of the price of coverage $x = \underline{x}$, since it must be such that the contract $x = \underline{x}$ breaks even. We denote this $\underline{p}^2$:

[51] In same cases, we simulate more individuals to reduce the noise in the graphs.

$$\underline{p}^2 = \mathbb{E}[\mu \mid \mu \in [\mu_*, \mu^*]] \tag{56}$$

Notice that $\underline{p}^2$ depends only on the endogenous quantities $\mu_*, \mu^*$ and exogenous parameters.

To find the equilibrium values of $(\mu^*, \mu_*, x^*)$ we proceed as follows.

1. First, we check the conditions of Proposition (5) to determine if there is pooling at $x = \underline{x}$. If there isn't, the equilibrium is RS.

2. For every level of coverage $y$ in a fine grid of $y \in [\underline{x}, \overline{x}]$, we proceed as if $y$ was the cutoff level of coverage $x^*$ and compute the following:

   (a) the value of the threshold type $\mu^* = \tau(y)$ that would be the the case if $y$ were the cut-off coverage, using (53) (which requires only $y$).

   (b) the price $\underline{p}^1(y)$ that would obtain if the cutoff coverage was $y$ and the threshold type was $\mu^* = \tau(y)$, using (54) (which requires $y, \mu^* = \tau(y)$)

   (c) the cut-off participation type $\mu_* = \lambda(\underline{p}^1(y))$ that would result if the price of coverage $x = \underline{x}$ was $\underline{p}^1(y)$, using (55) (which requires only $\underline{p}^1(y)$). We set $\mu_* = \underline{\mu}$ if the solution to (55) is $\lambda(\underline{p}^1(y)) < \underline{\mu}$.

   (d) the price $\underline{p}^2(y)$ that would obtain if the threshold type was $\mu^* = \tau(y)$ and the threshold participation type was $\mu_* = \lambda(\underline{p}^1(y))$ as computed above, but now we compute the price using the break even condition (56) (which requires $\mu^* = \tau(y), \mu_* = \lambda(\underline{p}^1(y))$).

3. The two method of computing the price of the minimum coverage, $\underline{p}^1(y), \underline{p}^2(y)$ must coincide in equilibrium. The equilibrium value of $x^*$ is the value of $y$ that solves $\left|\underline{p}^1(y) - \underline{p}^2(y)\right| = 0$. If this distance does not vanish (i.e., there is no solution), then there is full pooling (all individuals choose $x \in \{0, \underline{x}\}$). In this case, the equilibrium regime is Lemons, and we describe below the algorithm for computing $\mu_*$ (Section J.5).

4. Once we've found the equilibrium value of $x^*$, we compute $\mu^* = \tau(x^*)$ and $\mu_* = \lambda(\underline{p}^1(x^*))$, using (53) and (55). Then, we compute the allocation to each type using (2).

In the event of full pooling (i.e., $\mu^* = \overline{\mu}$), we define $x^* = \overline{x}$, to preserve continuity in the graphs.

## J.3 Dispersive Equilibrium

Several numerical simulations have already been presented in Figure 5 to demonstrate the effects of changes of the regulatory parameter $\underline{x}$ in the regime of Dispersive equilibrium. We now present a numerical example where we consider changes to $\overline{x}$ in that regime.

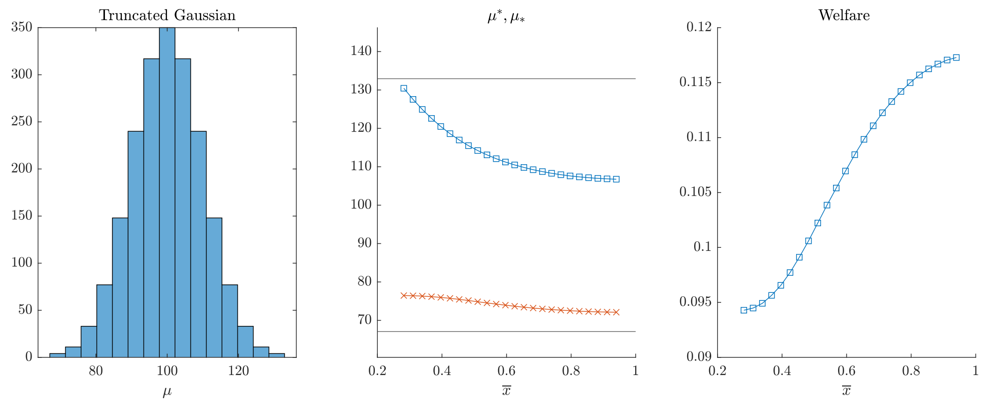


Figure 7: In a Dispersive equilibrium, an increase in $\overline{x}$ always increases welfare. Notice that the graph includes only values of the regulatory parameter for which the equilibrium regime is Dispersive.

## J.4 PPPP Equilibrium

We now consider settings where the equilibrium regime is PPPP. That is, all individuals purchase $x > 0$. To do this, we simulate market outcomes where the level of the non-purchase fee $T$ is sufficiently high so that, given all other parameter values, no individual chooses $x = 0$. From Proposition 3, the equilibrium is unique in this setting for all distributions of types $f$.

Our focus here is on the effect of changes in the minimal coverage $\underline{x}$. For clarity, we show only the range of $\underline{x}$ for which the regime is PPPP. From Proposition 12, welfare is increasing in $\underline{x}$ at the point when bunching at the minimum coverage begins (i.e., the left-most region of the third panel, which describes how welfare changes with $\underline{x}$). For all log-concave distributions $f$, in our numerical simulations, we find that $\frac{\partial W}{\partial \underline{x}} \geq 0$, but have not shown this result analytically.

We also present one example (Figure 13) where the distribution of types is not log-concave. Here, welfare is increasing in $\underline{x}$ initially (Proposition 12) but then falls over a range of values of $\underline{x}$.

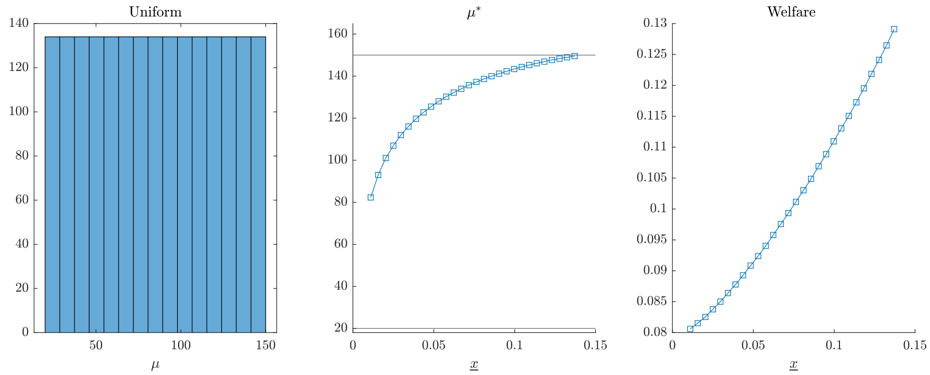


Figure 8: In a PPPP equilibrium, the effect of $\underline{x}$. Notice that the graph includes only values of the regulatory parameter for which the equilibrium regime is PPPP. In this example, $f$ follows a uniform distribution (which is log-concave).

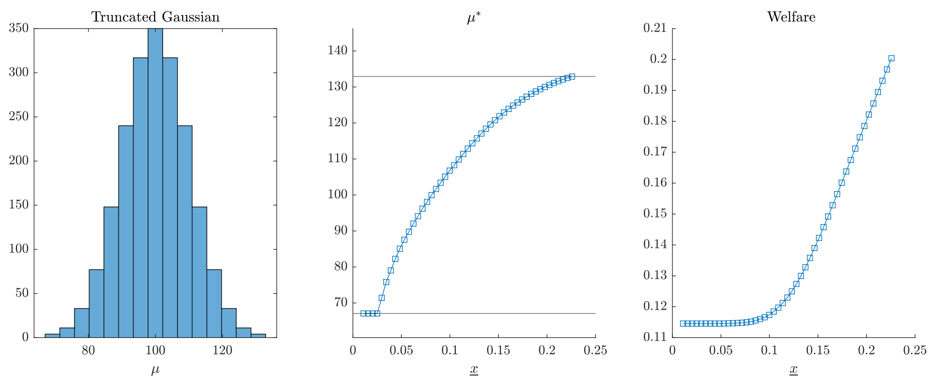


Figure 9: In a PPPP equilibrium, the effect of $\underline{x}$. Notice that the graph includes only values of the regulatory parameter for which the equilibrium regime is PPPP. In this example, $f$ follows a truncated Gaussian distribution (which is log-concave).

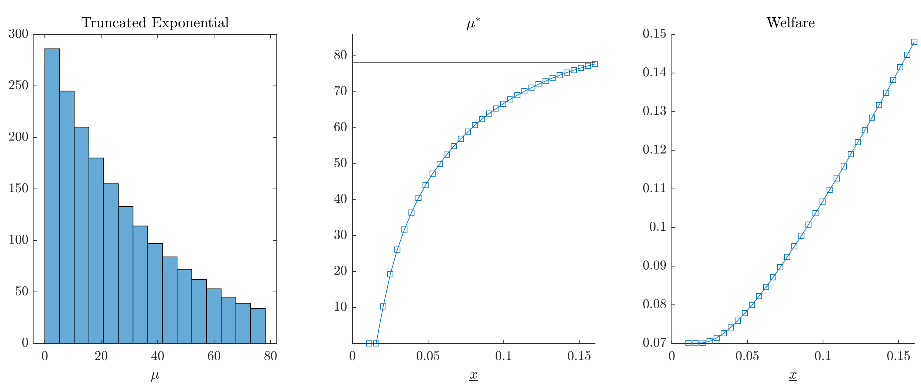


Figure 10: In a PPPP equilibrium, the effect of $\underline{x}$. Notice that the graph includes only values of the regulatory parameter for which the equilibrium regime is PPPP. In this example, $f$ follows a truncated Exponential distribution (which is log-concave).

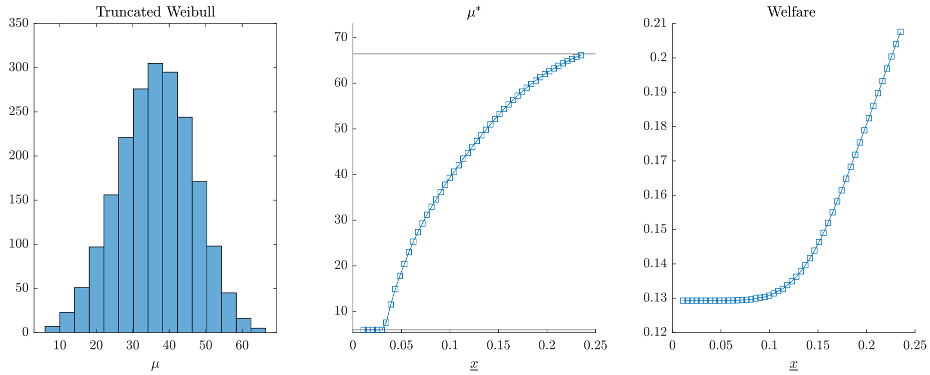


Figure 11: In a PPPP equilibrium, the effect of $\underline{x}$. Notice that the graph includes only values of the regulatory parameter for which the equilibrium regime is PPPP. In this example, $f$ follows a truncated Weibull distribution (which is log-concave).

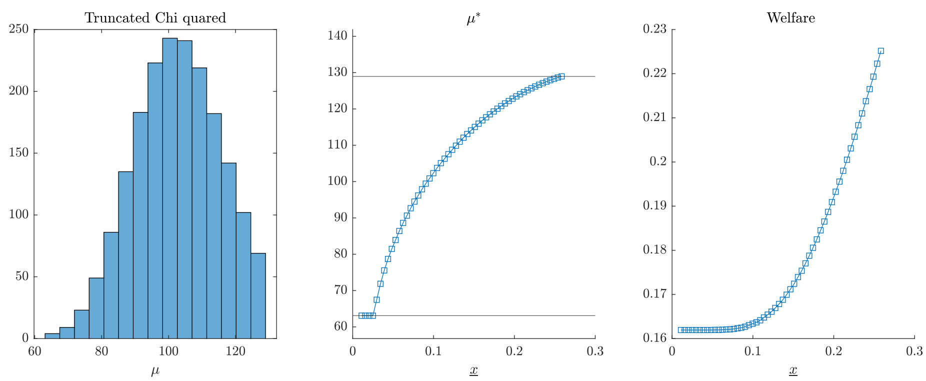


Figure 12: In a PPPP equilibrium, the effect of $\underline{x}$. Notice that the graph includes only values of the regulatory parameter for which the equilibrium regime is PPPP. In this example, $f$ follows a truncated Chi Squared distribution (which is log-concave).

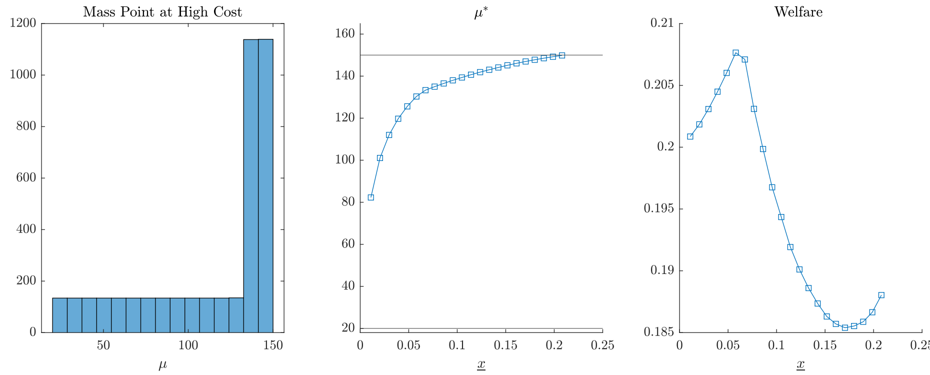


Figure 13: In a PPPP equilibrium, the effect of $\underline{x}$. Notice that the graph includes only values of the regulatory parameter for which the equilibrium regime is PPPP. In this example, $f$ follows a distribution which is uniform over most of its range, but has a large mass of high cost types. This distribution is not log concave (however, since the regime is PPPP, the equilibrium is unique). The figure illustrates that welfare is initially increasing in $\underline{x}$ (Proposition 12), but is decreasing in $\underline{x}$ for some of its range.

## J.5 Lemons Equilibrium

We now consider situations where the equilibrium regime is Lemons. Figure 14 illustrates the effects of changing $\underline{x}$ in this setting. We assume that the contract space is $X = \{0, \underline{x}\}$, so that the equilibrium regime is always Lemons, for all values of $\underline{x}$. In this simulation, welfare is quasi-concave (Lemma 23) and has a maximum for an interior level of $\underline{x}$ (Proposition 13).[52]

[52]For $f$ uniform and $g(x) = \frac{1}{2}\left(1 - (1-x)^2\right)$, if some agents do not purchase coverage (i.e., some choose $x = 0$), then $\frac{\partial \mu_*}{\partial \underline{x}} = \nu$, which explains why $\mu_*$ changes linearly with $\underline{x}$ in Figure 14.

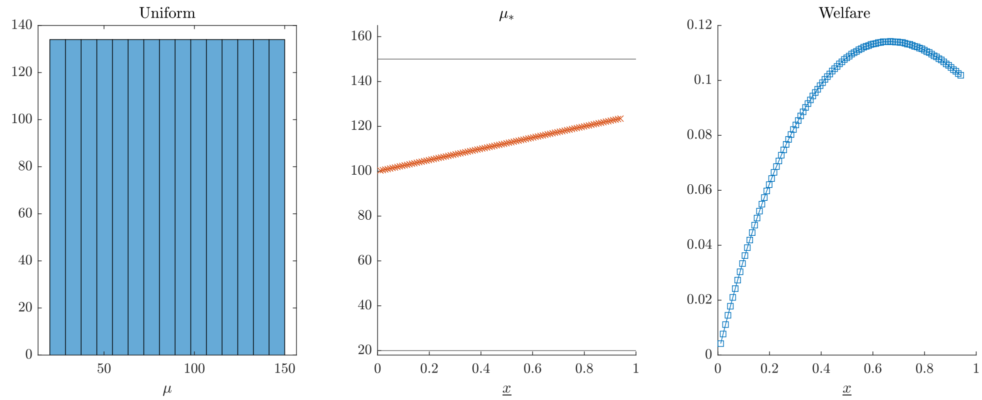


Figure 14: In a Lemons equilibrium, the effects of $\underline{x}$. The distribution is uniform. To produce this graph, it was assumed that $X = \{0, \underline{x}\}$ for all values of $\underline{x}$, so the regime is Lemons for all values of $\underline{x}$. The figure illustrates that the optimal value of $\underline{x}$ is interior (above zero and below 1).

# K Certainty Equivalent Representation

## K.1 CARA-Gaussian

In this appendix, we derive certainty equivalents when utility exhibits constant absolute risk aversion (CARA) and individuals are exposed to Gaussian wealth shocks. Suppose that an individual has CARA utility $U = -e^{-\psi c}$, where $c$ is final consumption is $\psi$ is the CARA coefficient. The individual is exposed to Gaussian wealth shocks $Z \sim \mathcal{N}(\mu, \sigma_Z^2)$. Initial wealth is $w$. Final wealth is $c = w_0 - (1-x)Z - p$, which has a log-normal distribution. Then, the log of expected utility satisfies (1) with $g(x) = \frac{1}{2}\left(1-(1-x)^2\right)\psi\sigma_Z^2$.

This model should not be taken literally. In most real world scenarios, insurance is unlikely to apply to negative wealth shocks (despite the fact the Gaussian distribution has full support). That is, insurance companies are unlikely to absorb windfalls. We intend this model to be an approximation. Indeed, if $\mu$ is large relative to $\sigma_Z^2$, then the probability of negative shocks in a Gaussian distribution becomes small. In Section K.2 we discuss more general distributions.

First, we establish the following fact about lognormal distributions. Suppose that $K$ is a standard normal random variable, i.e. $K \sim \mathcal{N}(0,1)$. Then $X = \exp(M + SK)$ has a log-normal distribution, with mean $\mathbb{E}[X] = \exp\left(M + \frac{1}{2}S^2\right)$.

The expected utility of type $\mu$ buying coverage $x$ is

$$\begin{aligned} u(\mu, x, p) &= \int_{-\infty}^{+\infty} e^{-\psi(w_0-(1-x)K-p)} \times \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(K-\mu)^2}{\sigma_Z^2}} dK \\ &= -e^{-\psi\left(w_0-\mu(1-x)-\psi\frac{1}{2}(1-x)^2\sigma_Z^2-p\right)} \end{aligned}$$

The certainty equivalent of type $\mu$ for buying quality $x$ at price $p$ is

$$u(\mu, x, p) = w_0 + \mu x - \mu - \psi \frac{1}{2}(1-x)^2 \sigma_Z^2 - p.$$

For a given type $\mu$, for any pairwise comparison of two contracts, the initial wealth $w$ and the term $-\mu$ "cancel out". Therefore, we define

$$u(\mu, x, p) = \mu x - \psi \frac{1}{2}(1-x)^2 \sigma_Z^2 - p.$$

## K.2 CARA and General Wealth Shocks

We now derive properties of the certainty equivalent with CARA utility but a general distribution of wealth shocks.

Consider an individual with a utility function $U(y)$, where $y$ is final consumption. Suppose that an individual has CARA utility $U = -e^{-\psi y}$, where $y$ is final consumption is $\psi$ is the CARA coefficient. Then, $U'(y) > 0$ and $U''(y) < 0$. Initial wealth is $w_0$ and the individual is subject to a wealth loss $Z = \mu + \epsilon$, where $\mathbb{E}[\epsilon] = 0$. That is, $\mu = \mathbb{E}[Z]$ is the expected loss. Let the PDF of $\epsilon$ be $\xi(\epsilon) > 0$. The individual obtains a coinsurance contract $x$, so that an individual who purchases insurance coverage $x$ is only exposed to a loss of $(1-x)Z$.

**Lemma 29.** *The certainty equivalent is*

$$CE = w_0 - \mu - p + x\mu - \frac{1}{\psi} \log\left(\int e^{\psi(1-x)\epsilon} \xi(\epsilon)\, d\epsilon\right).$$

*Proof.* Expected utility is

$$\begin{aligned} EU &= \int \left(-e^{-\psi(w_0-(1-x)(\mu+\epsilon)-p)}\right) \xi(\epsilon)\, d\epsilon \\ &= -e^{-\psi w_0} e^{\psi p} \int e^{\psi(1-x)(\mu+\epsilon)} \xi(\epsilon)\, d\epsilon \end{aligned}$$

The definition of the certainty equivalent $CE$ is $EU = U(CE) = -e^{-\psi CE}$. Solving for $CE$ yields ,

$$CE = w_0 - p - \frac{1}{\psi}\left[\log\left(e^{\psi(1-x)\mu}\int e^{\psi(1-x)\epsilon}\xi(\epsilon)\,d\epsilon\right)\right]$$

which implies the result. $\square$

Now let $g(x) = -\frac{1}{\psi}k(x)$, where

$$k(x) = \log\left(\int e^{\psi(1-x)\epsilon}\zeta(\epsilon)\,d\epsilon\right).$$

Notice that $k(x)$ does not depend on the first moment of the loss distribution ($\mu$).

**Lemma 30.** $g'(x) > 0$.

*Proof.* This statement is equivalent to $k'(x) < 0$. We compute

$$k'(x) = \frac{1}{\int e^{\psi(1-x)\epsilon}\zeta(\epsilon)\,d\epsilon}\int\left(-\psi\epsilon e^{\psi(1-x)\epsilon}\zeta(\epsilon)\right)d\epsilon.$$

We now use the identity $\text{Cov}[XY] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$, and recall that $E[\epsilon] = 0$, to obtain

$$k'(x) = -\psi\frac{\text{Cov}\left[\epsilon, e^{\psi(1-x)\epsilon}\right]}{\mathbb{E}\left[e^{\psi(1-x)\epsilon}\right]}$$

Since $\epsilon, e^{\psi(1-x)\epsilon}$ are both strictly increasing in $\epsilon$, then $\text{Cov}\left[\epsilon, e^{\psi(1-x)\epsilon}\right] > 0$. Moreover, $\mathbb{E}\left[e^{\psi(1-x)\epsilon}\right] > 0$. Therefore, $k'(x) < 0$ as required. $\square$

**Lemma 31.** $g'(1) = 0$.

*Proof.* $k'(1) = -\psi\text{Cov}[\epsilon, 1] = 0$. This implies the result. $\square$

**Lemma 32.** *If* $\mathbb{V}[\epsilon] > 0$*, then* $g'' < 0$.

*Proof.* This is equivalent to $k'' > 0$. Recall $k'(x) = -\psi\frac{\mathbb{E}\left[\epsilon e^{\psi(1-x)\epsilon}\right]}{\mathbb{E}\left[e^{\psi(1-x)\epsilon}\right]}$. We compute

$$k''(x) = \psi^2\frac{\mathbb{E}\left[\epsilon^2 e^{\psi(1-x)\epsilon}\right]\mathbb{E}\left[e^{\psi(1-x)\epsilon}\right] - \left(\mathbb{E}\left[\epsilon e^{\psi(1-x)\epsilon}\right]\right)^2}{\left(\mathbb{E}\left[e^{\psi(1-x)\epsilon}\right]\right)^2}.$$

A sufficient condition for $k'' > 0$ is

$$\mathbb{E}\left[\epsilon^2 e^{\psi(1-x)\epsilon}\right]\mathbb{E}\left[e^{\psi(1-x)\epsilon}\right] > \left(\mathbb{E}\left[\epsilon e^{\psi(1-x)\epsilon}\right]\right)^2.$$

We show that this holds (as long as $\varepsilon$ is not concentrated at $0$, i.e., as long the variance $\mathbb{V}[\epsilon] > 0$). Fix $x$. Denote $M = \mathbb{E}\left[e^{\psi(1-x)\epsilon}\right] > 0$. Dividing by $M^2$ gives

$$\mathbb{E}\left[\epsilon^2 \frac{1}{M} e^{\psi(1-x)\epsilon}\right] > \left(\mathbb{E}\left[\epsilon \frac{1}{M} e^{\psi(1-x)\epsilon}\right]\right)^2$$

Now, if $P$ denotes the distribution of $\epsilon$, since $E\left[\frac{1}{M} e^{\psi(1-x)\epsilon}\right] = 1$, we see that the distribution $P'$ defined by $dP'(x) = \frac{1}{M} e^{\psi(1-x)\epsilon} dP(x)$ is a well-defined probability distribution, and the above inequality is equivalent to

$$\mathbb{E}'\left[\epsilon^2\right] > \left(\mathbb{E}'\left[\epsilon\right]\right)^2$$

where $\mathbb{E}'$ is the expectation operator associated with $P'$. This inequality is equivalent to $\mathbb{V}'(\epsilon) > 0$, where $\mathbb{V}'$ is the associated variance. The only way to have $\mathbb{V}'(\epsilon) = 0$ is to have $\epsilon \equiv c$ a.s. for some $c \in \mathbb{R}$, which must be $0$ since $\mathbb{E}(\epsilon) = 0$. $\square$