If Many Seek, Ye Shall Find: Search Externalities and New Goods

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Online Appendix

A   Omitted Proofs

A.1   Preliminary Lemmas

Lemma A.1. *Consider a type-v_i consumer who has access to contract σ_j. Suppose this consumer believes that all firms not yet sampled are following the same independent production/pricing strategy, $\tilde{\psi} \in \Delta(\Sigma)$. Define $u^*(v_i)$ as the unique solution to[1]*

$$u^*(v_i) = \mathbb{E}_{\tilde{\psi}}\big[\max\{u^*(v_i), u(\sigma|v_i), 0\}\big] - s. \tag{A.1}$$

1. *If $\max\{u(\sigma_j|v_i), 0\} \geq u^*(v_i)$, then it is optimal for consumer i to accept σ_j (or, to exit the market if $0 > u(\sigma_j|v_i)$).*

2. *If $u^*(v_i) > \max\{u(\sigma_j|v_i), 0\}$, then it is optimal for consumer i to learn the contract offer of the next firm at a cost of s.*

Proof. Consumer i's decision problem is essentially one of sequentially sampling from a stationary distribution of independently and identically distributed random variables. The value of each sampled contract is the random variable $[u(\sigma|v_i) \vee 0]$. (Each sample's value is bounded below by zero since a consumer may opt out of the market.) The distribution of this value is derived from the distribution of σ, which the consumer believes to be $\tilde{\psi} \in \Delta(\Sigma)$. Since prices are non-negative, the expectation of $[u(\sigma|v_i) \vee 0]$ exists. By DeGroot (1970, Sections 13.5 and 13.9), the optimal strategy for the consumer is the above cutoff rule. $\square$

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[1]The notation in (A.1) is self-explanatory. Here, "σ" is random. Its distribution is $\tilde{\psi}$.

Lemma A.2. *Suppose there exists an equilibrium where all firms adopt the strategy ψ^*. Let $u^*(v_i)$ be the cutoff value describing a type-v_i consumer's equilibrium search strategy. Let σ^* be a contract such that:*

1. *$u(\sigma^*|\bar{v}) > \max\{u^*(\bar{v}), 0\}$ and $u(\sigma^*|\underline{v}) > \max\{u^*(\underline{v}), 0\}$; or,*

2. *$u(\sigma^*|\bar{v}) > \max\{u^*(\bar{v}), 0\}$ and $\max\{u^*(\underline{v}), 0\} > u(\sigma^*|\underline{v})$; or,*

3. *$\max\{u^*(\bar{v}), 0\} > u(\sigma^*|\bar{v})$ and $u(\sigma^*|\underline{v}) > \max\{u^*(\underline{v}), 0\}$.*

Then σ^* is not in the support of ψ^*. That is, there exists an alternative contract that gives firm j strictly greater profits than σ^* given consumers' search behavior.

Proof. We give the proof for case 1 with the remaining cases following analogously. Suppose the conditions of the lemma are satisfied and that firm j posts σ^*. This contract is accepted by both types of consumers whenever they encounter firm j. Suppose firm j offers instead the contract σ_ϵ^*, which is identical to σ^* except that it sets a price that is $\epsilon > 0$ greater. Suppose ϵ is sufficiently small so that $u(\sigma^*|\bar{v}) > u(\sigma_\epsilon^*|\bar{v}) > \max\{u^*(\bar{v}), 0\}$ and $u(\sigma^*|\underline{v}) > u(\sigma_\epsilon^*|\underline{v}) > \max\{u^*(\underline{v}), 0\}$. Clearly, σ_ϵ^* is accepted by both types of consumers whenever σ^* was. Hence σ_ϵ^* is accepted by the same expected number of consumers as σ^*. Since this number is strictly greater than zero and since σ_ϵ^* is associated with a strictly higher price, it will generate greater expected profits than σ^*. Therefore, σ^* cannot be a contract that is offered in equilibrium. $\square$

Lemma A.3. *Let $u^*(v_i)$ be the cutoff value characterizing a consumer's search strategy. (1) $u^*(\bar{v}) \geq u^*(\underline{v})$. (2) If $u^*(v_i) \geq 0$, then there exists some σ in the support of $\tilde{\psi}$ such that $u(\sigma|v_i) > u^*(v_i)$.*

Proof. Suppose $\sigma \sim \tilde{\psi}$. To simplify notation, define the real-valued random variable $X_{v_i} \stackrel{d}{=} [u(\sigma|v_i) \vee 0]$. Let $F_{v_i}(x)$ be the associated distribution function. Since $u(\sigma|\bar{v}) \geq u(\sigma|\underline{v})$ for all σ, $X_{\bar{v}}$ first-order stochastically dominates $X_{\underline{v}}$. Hence $F_{\underline{v}}(x) \geq F_{\bar{v}}(x)$ for all x. Finally, we can write $u^*(v_i) = \int \max\{u^*(v_i), x\} dF_{v_i} - s$. For this equality to hold, $F_{v_i}(u^*(v_i)) < 1$.

1. We argue by contradiction. Suppose $u^*(\bar{v}) < u^*(\underline{v})$. Since $X_{\bar{v}}$ first-order stochastically dominates $X_{\underline{v}}$, $u^*(\underline{v}) = \int \max\{u^*(\underline{v}), x\} dF_{\underline{v}} - s \leq \int \max\{u^*(\underline{v}), x\} dF_{\bar{v}} - s$. Thus,

$$u^*(\underline{v}) - u^*(\bar{v}) \leq \int \max\{u^*(\underline{v}), x\} dF_{\bar{v}} - \int \max\{u^*(\bar{v}), x\} dF_{\bar{v}}$$

$$\leq \int (u^*(\underline{v}) - u^*(\bar{v})) \mathbf{1}(x \leq u^*(\bar{v})) dF_{\bar{v}}$$

$$\leq (u^*(\underline{v}) - u^*(\bar{v})) F_{\bar{v}}(u^*(\underline{v}))$$

$$\leq (u^*(\underline{v}) - u^*(\bar{v})) F_{\underline{v}}(u^*(\underline{v}))$$

Since $F_{\underline{v}}(u^*(\underline{v})) < 1$, we arrive at a contradiction. Therefore, $u^*(\bar{v}) \geq u^*(\underline{v})$.

2. Again we proceed by contradiction. Suppose $u(\sigma|v_i) \leq u^*(v_i)$ for all σ in the support of $\tilde{\psi}$. Then, $u^*(v_i) = \mathbb{E}_{\tilde{\psi}}[\max\{u^*(v_i), u(\sigma|v_i), 0\}] - s \leq \mathbb{E}_{\tilde{\psi}}[\max\{u^*(v_i), u^*(v_i), 0\}] - s = u^*(v_i) - s$, which is a contradiction since $s > 0$.

$\square$

A.2 Proof of Theorem 1

Lemma A.4 is used in the proof of Theorem 1, which follows.

Lemma A.4. *There does not exist an equilibrium where only good 1 is produced.*

Proof. We divide the argument into three cases. Throughout we assume that if a firm produces good 1, it sets a price in the interval $[c, \bar{v}]$. Prices below c and above $\bar{v}$ lead to negative profits.

1. Suppose there exists an equilibrium where all firms offer $\sigma^* = \langle 1, p \rangle$, $p \in [c, \bar{v}]$. Since all firms are offering the same contract, $u^*(v_i) = \max\{u(\sigma^*|v_i) - s, -s\}$ for $v_i \in \{\underline{v}, \bar{v}\}$. There are two sub-cases, each generating a contradiction.

   (a) Suppose $p \leq \underline{v}$. If $p < \underline{v}$, then $u(\sigma^*|\underline{v}) > \max\{u^*(\underline{v}), 0\}$ and $u(\sigma^*|\bar{v}) > \max\{u^*(\bar{v}), 0\}$. By Lemma A.2 this is a contradiction. Hence, $p = \underline{v}$. Therefore, the profits of a typical firm must be $\underline{v} - c - \phi$.

   Suppose firm j offers instead the contract $\langle 0, z \rangle$. This contract is acceptable to all type-$\underline{v}$ consumers and firm j ensures itself an expected profit of at least $(1 - \gamma)z$. By Assumption A-1, $(1 - \gamma)z > \underline{v} - c - \phi$. Hence, this alternative contract is more profitable—a contradiction.

   (b) Suppose instead that $p \in (\underline{v}, \bar{v}]$. By reasoning analogous to the preceding case we conclude that $p = \bar{v}$. Thus, a typical firm's profit $\gamma(\bar{v} - c) - \phi$. Now, the contract $\langle 0, z \rangle$ is acceptable by both types of consumer and will yield an expected profit of z. Noting Assumption A-1, $z > \gamma(\bar{v} - c) - \phi$. Again, this is a contradiction.

2. Suppose there exists an equilibrium with price dispersion where all firms produce good 1 but consumers *do not* search. Thus, each firm sells its product to at most one consumer.

   Suppose $\sigma = \langle 1, p \rangle$ and $\hat{\sigma} = \langle 1, \hat{p} \rangle$, $c \leq p < \hat{p} \leq \bar{v}$, are two contracts available in equilibrium. Since a type-$\bar{v}$ consumer does not search, $\max\{u^*(\bar{v}), 0\} \leq u(\hat{\sigma}|\bar{v}) < u(\sigma|\bar{v})$. If $\max\{u^*(\underline{v}), 0\} \leq u(\hat{\sigma}|\underline{v})$, then both contracts would be accepted by both types of consumers on the equilibrium path. Such an outcome is not compatible with profit equalization across contracts in equilibrium. Hence, $u(\hat{\sigma}|\underline{v}) < \max\{u^*(\underline{v}), 0\} \leq u(\sigma|\underline{v})$.

   If $\max\{u^*(\underline{v}), 0\} < u(\sigma|\underline{v})$, then Lemma A.2 implies a contradiction. Thus, $\max\{u^*(\underline{v}), 0\} = u(\sigma|\underline{v})$. Similarly, $\max\{u^*(\bar{v}), 0\} = u(\hat{\sigma}|\bar{v})$. Since the preceding reasoning applies to all pairs of contracts there are at most two contracts available in this equilibrium.

   If $u^*(\underline{v}) \geq 0$, then by Lemma A.3 there must exist some available contract, say $\tilde{\sigma} = \langle 1, \tilde{p} \rangle$, such that $u(\tilde{\sigma}|\underline{v}) > u^*(\underline{v})$. But this is a contradiction. Hence, $u^*(\underline{v}) < 0$. Therefore, $u(\sigma|\underline{v}) = 0$, which implies $p = \underline{v}$. Thus, the expected profit of a typical firm in equilibrium is $\underline{v} - c - \phi < (1 - \gamma)z$. But then any firm offering σ can increase its expected profit by posting $\langle 0, z \rangle$ instead—a contradiction.

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3. Suppose there exists an equilibrium with price dispersion where all firms produce good 1 but (some) consumers do search on the equilibrium path. If a consumer ever searches, he will certainly also search whenever he matches with the most expensive contract in the economy. Let $\hat{\sigma} = \langle 1, \hat{p} \rangle$ be that contract. Note that both types of consumers cannot search on the equilibrium path after matching with $\hat{\sigma}$. Otherwise, any firm offering it would earn strictly negative profits. Hence, there are two cases.

   (a) Suppose a type-$\underline{v}$ consumer searches on the equilibrium path. Thus, $u^*(\underline{v}) \geq 0$ and $u(\hat{\sigma}|\underline{v}) \leq u^*(\underline{v})$. Since $\hat{\sigma}$ must be acceptable to some type of consumer, $\max\{u^*(\bar{v}), 0\} \leq u(\hat{\sigma}|\bar{v})$. Since $u^*(\underline{v}) \geq 0$, by Lemma A.3 there exists some contract in the support of the firm's strategy, $\sigma = \langle 1, p \rangle$, such that $u^*(\underline{v}) < u(\sigma|\underline{v})$. But, without loss of generality, this also implies $0 \leq u^*(\underline{v}) \leq u^*(\bar{v}) < u(\sigma|\bar{v})$, which contradicts Lemma A.2.

   (b) Suppose a type-$\bar{v}$ consumer searches on the equilibrium path. Thus, $u^*(\bar{v}) \geq 0$, $u(\hat{\sigma}|\bar{v}) \leq u^*(\bar{v})$, and $\max\{u^*(\underline{v}), 0\} \leq u(\hat{\sigma}|\underline{v})$. By Lemma A.3 there exists some contract in the support of the firm's strategy, $\sigma = \langle 1, p \rangle$, such that $u^*(\bar{v}) < u(\sigma|\bar{v})$. But this also implies $\max\{u^*(\underline{v}), 0\} < u(\sigma|\underline{v})$. Again, we contradict Lemma A.2.

The three cases considered are exhaustive of the situations that may occur if only good 1 is produced. $\square$

Proof of Theorem 1. We first argue that the proposed situation is an equilibrium. If every firm offers $\sigma^* = \langle 0, z \rangle$ and consumers believe this to be the case, no consumer searches. Firm profits are z. No firm can gain by offering $\langle 0, p \rangle$, $p < z$. Since no consumer is searching, this alternative contract implies lower profits. If instead a firm offers $\langle 1, p \rangle$, its expected profits are bounded above by $\max\{\underline{v} - c - \phi, \gamma(\bar{v} - c) - \phi\} < z$ (Assumption A-1). Hence, this too is not a profitable deviation. Therefore, the market is in equilibrium.

Lemma A.4 shows that there cannot exist an equilibrium where only good 1 is produced. Thus, to establish uniqueness it is sufficient to show that there is no other equilibrium where only good 0 is produced. Let ψ^* be the equilibrium strategy of a typical firm in a symmetric equilibrium where only good 0 is produced. In this equilibrium, each firm must earn strictly positive profits. To see this fact, note that firm j can offer the contract $\sigma_j = \langle 0, \epsilon \rangle$ where $0 < \epsilon < s$. Even if consumer $i = j$ believed all other firms were offering good 0 at a price of zero, he would still prefer to accept σ_j without any search. This gives firm j a profit of at least ϵ. Since a firm's equilibrium profit is strictly greater than zero, every contract in the support of ψ^* must satisfy $u(\sigma|v_i) \geq \max\{u^*(v_i), 0\}$. (Note that $u^*(\bar{v}) = u^*(\underline{v})$ since only good 0 is available.) By Lemma A.2, this implies $u(\sigma|v_i) = \max\{u^*(v_i), 0\}$. Hence, there is at most one contract offered in this equilibrium. But this implies $u(\sigma|v_i) = 0$ and so $z = p$. $\square$

A.3 Proof of Theorem 2

Theorem 2 is proved by Lemmas A.5–A.12. In each lemma we assume that there exists a symmetric equilibrium with product differentiation. ψ^* denotes the equilibrium strategy of

a typical firm. $\mathcal{C}_k$ is the set of contracts in the support of ψ^* that provide good k.

Lemma A.5. *In an equilibrium with product differentiation, $u^*(\bar{v}) \geq 0$.*

Proof. Suppose $0 > u^*(\bar{v})$. By Lemma A.3, $0 > u^*(\bar{v}) \geq u^*(\underline{v})$. No consumer finds it worthwhile search. Any firm offering $\langle 0, p \rangle$ will have an incentive to increase p until $p = z$ since both types of consumers accept $\langle 0, p \rangle$ for all $p \leq z$. Hence, in equilibrium each firm earns z in profits. Since no consumers search, the maximal expected profits of a firm offering good 1 are bounded above by $\max\{\gamma(\bar{v} - c) - \phi, \underline{v} - c - \phi\} < z$, contradicting the equality of expected profits across equilibrium contracts. $\square$

Lemma A.6. *There exists $\sigma^* \in \mathcal{C}_1$ such that $u(\sigma^*|\bar{v}) > u^*(\bar{v})$.*

Proof. Suppose the contrary. If $u(\sigma|\bar{v}) \leq u^*(\bar{v})$ for all $\sigma \in \mathcal{C}_1$, then by Lemma A.3 there exists $\tilde{\sigma} \in \mathcal{C}_0$ such that $u(\tilde{\sigma}|\bar{v}) > u^*(\bar{v}) \geq 0$. Since $u(\tilde{\sigma}|\bar{v}) = u(\tilde{\sigma}|\underline{v})$ and $u^*(\bar{v}) \geq u^*(\underline{v})$, $u(\tilde{\sigma}|\underline{v}) > \max\{u^*(\underline{v}), 0\}$. But this contradicts Lemma A.2. $\square$

Lemma A.7. *There exists $\sigma^* \in \mathcal{C}_1$ such that $u(\sigma^*|\underline{v}) \geq \max\{u^*(\underline{v}), 0\}$.*

Proof. Suppose $\max\{u^*(\underline{v}), 0\} > u(\sigma|\underline{v})$ for all $\sigma \in \mathcal{C}_1$. By Lemmas A.5 and A.6 there exists $\tilde{\sigma} \in \mathcal{C}_1$ such that $u(\tilde{\sigma}|\bar{v}) > \max\{u^*(\bar{v}), 0\}$. But then $\max\{u^*(\underline{v}), 0\} > u(\tilde{\sigma}|\underline{v})$, which contradicts Lemma A.2. $\square$

Lemma A.8. *The expected profit of a firm in an equilibrium with product differentiation is strictly positive.*

Proof. Negative profits cannot be a feature of equilibrium. Thus, suppose they are zero. This implies that every contract of the form $\langle 0, p \rangle$, $0 < p < z$ must not be acceptable to either type of consumer. In particular, this implies that $0 < z \leq u^*(\underline{v})$.

By Lemma A.3, there exists a contract, σ^*, offered in equilibrium such that $u^*(\underline{v}) < u(\sigma^*|\underline{v})$. If $u^*(\bar{v}) < u(\sigma^*|\bar{v})$, then we would contradict Lemma A.2. Hence $u^*(\bar{v}) \geq u(\sigma^*|\bar{v})$. There are now two possibilities.

1. If $u^*(\bar{v}) = u(\sigma^*|\bar{v})$, then by Lemma A.3 there exists an alternative contract available in equilibrium, σ^{**}, such that $u(\sigma^{**}|\bar{v}) > u^*(\bar{v})$. But in this case a type-$\underline{v}$ consumer would also benefit from σ^{**}. Thus, $u(\sigma^{**}|\underline{v}) > u^*(\underline{v})$, contradicting Lemma A.2.

2. If $u^*(\bar{v}) > u(\sigma^*|\bar{v})$, then we contradict Lemma A.2 immediately.

In either case we arrive at a contradiction. $\square$

Lemma A.9. *For all $\sigma \in \mathcal{C}_0$, $u(\sigma|\underline{v}) \geq \max\{u^*(\underline{v}), 0\}$.*

Proof. Suppose there exists $\sigma \in \mathcal{C}_0$ such that $\max\{u^*(\underline{v}), 0\} > u(\sigma|\underline{v})$. Since $u(\sigma|\bar{v}) = u(\sigma|\underline{v}) < \max\{u^*(\underline{v}), 0\} \leq u^*(\bar{v})$, σ is not accepted by any consumers. Therefore, the profits of a firm offering σ must be zero contradicting Lemma A.8. $\square$

Lemma A.10. $|\mathcal{C}_0| \leq 2$.

Proof. Let $\sigma = \langle 0, p \rangle$ and $\hat{\sigma} = \langle 0, \hat{p} \rangle$ and suppose $\sigma, \hat{\sigma} \in \mathcal{C}_0$. Without loss of generality, suppose $0 < p < \hat{p} \leq z$. By Lemma A.9, $u(\sigma|\underline{v}) > u(\hat{\sigma}|\underline{v}) \geq \max\{u^*(\underline{v}), 0\}$. Since both contracts must generate the same expected profits, $\hat{\sigma}$ must not be accepted by all type-$\bar{v}$ consumers. Thus, $u^*(\bar{v}) \geq u(\hat{\sigma}|\bar{v})$.

Suppose $u^*(\bar{v}) = u(\hat{\sigma}|\bar{v})$. If in equilibrium a type-$\bar{v}$ consumer does not accept $\hat{\sigma}$ with probability 1, then any firm offering $\hat{\sigma}$ can decrease its price by $\epsilon > 0$, and induce all type-$\bar{v}$ consumers to accept. This infinitesimal decrease in price leads to a discrete increase in demand and would be a profitable deviation. Therefore, $u^*(\bar{v}) > u(\hat{\sigma}|\bar{v})$. If $u(\hat{\sigma}|\underline{v}) > \max\{u^*(\underline{v}), 0\}$, then there would be a contradiction with Lemma A.2. Thus,

$$u(\hat{\sigma}|\underline{v}) = \max\{u^*(\underline{v}), 0\}. \tag{A.2}$$

Now consider σ. If $u(\sigma|\bar{v}) > u^*(\bar{v})$, then again Lemma A.2 is contradicted. If $u(\sigma|\bar{v}) < u^*(\bar{v})$, then it is only accepted by type-$\underline{v}$ consumers—contradicting the equality of profits between σ and $\hat{\sigma}$. Therefore,

$$u(\sigma|\bar{v}) = u^*(\bar{v}). \tag{A.3}$$

Therefore, whenever there are at least two distinct contracts in $\mathcal{C}_0$, they must satisfy (A.2) and (A.3). Since these equations must hold for any pair of distinct contracts in $\mathcal{C}_0$, there are at most two elements in $\mathcal{C}_0$. $\square$

Lemma A.11. *For all $\sigma \in \mathcal{C}_1$, $u(\sigma|\bar{v}) \geq u^*(\bar{v})$.*

Proof. We argue by contradiction. Suppose there exists $\hat{\sigma} \in \mathcal{C}_1$ such that $u(\hat{\sigma}|\bar{v}) < u^*(\bar{v})$. Without loss of generality we may assume that $\hat{\sigma}$ is the most expensive contract in $\mathcal{C}_1$. If type-$\bar{v}$ consumers do not accept $\hat{\sigma}$, type-$\underline{v}$ consumers must find it acceptable, $u(\hat{\sigma}|\underline{v}) \geq \max\{u^*(\underline{v}), 0\}$; else, a firm would earn negative profits. If the preceding inequality is strict, Lemma A.2 would be violated. Hence, $u(\hat{\sigma}|\underline{v}) = \max\{u^*(\underline{v}), 0\}$. Since $\hat{\sigma}$ was the most expensive contract in $\mathcal{C}_1$, a type-$\underline{v}$ consumer must accept all contracts in $\mathcal{C}_1$. By Lemma A.9, a type-$\underline{v}$ consumer also accepts all contracts in $\mathcal{C}_0$. Hence, a type-$\underline{v}$ consumer does search in equilibrium.

Suppose $u(\hat{\sigma}|\underline{v}) = 0$. Then the profits of a firm offering $\hat{\sigma}$ are at most $(1-\gamma)(\underline{v}-c)-\phi$. Noting Assumption A-1, such a firm can strictly increase its profit by offering the contract $\langle 0, z \rangle$ instead since $(1-\gamma)(\underline{v}-c)-\phi \leq (1-\gamma)(\underline{v}-c-\phi) \leq (1-\gamma)z$. Hence, $u(\hat{\sigma}|\underline{v}) = u^*(\underline{v}) > 0$.

Next, suppose there exists $\tilde{\sigma} \in \mathcal{C}_0$ such that $u(\tilde{\sigma}|\bar{v}) > u^*(\bar{v})$. Since $u^*(\bar{v}) \geq u^*(\underline{v})$, $u(\tilde{\sigma}|\bar{v}) = u(\tilde{\sigma}|\underline{v}) > u^*(\underline{v})$. This, however, contradicts Lemma A.2. Therefore, $u(\sigma|\bar{v}) \leq u^*(\bar{v})$ for all $\sigma \in \mathcal{C}_0$. Thus, noting Lemma A.3, we conclude that there exists $\check{\sigma} \in \mathcal{C}_1$ such that $u(\check{\sigma}|\bar{v}) > u^*(\bar{v}) \geq u^*(\underline{v}) \geq 0$. But this also contradicts Lemma A.2. $\square$

Lemma A.12. $|\mathcal{C}_1| \leq 2$.

Proof. Let $\sigma = \langle 1, p \rangle$ and $\hat{\sigma} = \langle 1, \hat{p} \rangle$ and suppose $\sigma, \hat{\sigma} \in \mathcal{C}_1$. Without loss of generality, suppose $0 < p < \hat{p} \leq \bar{v}$. Moreover, suppose σ is the least-expensive contract in $\mathcal{C}_1$. By Lemma A.11, both contracts are accepted by type-$\bar{v}$ consumers:

$$u(\sigma|\bar{v}) > u(\hat{\sigma}|\bar{v}) \geq \max\{u^*(\bar{v}), 0\}. \tag{A.4}$$

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If $u(\sigma|\underline{v}) < \max\{u^*(\underline{v}), 0\}$, Lemma A.2 is violated. Thus,

$$u(\sigma|\underline{v}) \geq \max\{u^*(\underline{v}), 0\}. \tag{A.5}$$

If the inequality in (A.5) was strict, Lemma A.2 would be violated. Thus, $u(\sigma|\underline{v}) = \max\{u^*(\underline{v}), 0\} > u(\hat{\sigma}|\underline{v})$. But then Lemma A.2 implies that $u(\hat{\sigma}|\bar{v}) = \max\{u^*(\bar{v}), 0\}$. Since $\hat{\sigma}$ was an arbitrary contract in $\mathcal{C}_1 \setminus \{\sigma\}$, we conclude that there are at most two distinct elements in $\mathcal{C}_1$. □

A.4 Proof of Theorem 3

Theorem 3 draws on three preliminary results: Lemma 1, which is stated in the main text, Lemma A.13, and Lemma A.14. The proof of Corollary 1 follows the proof of Theorem 3.

Proof of Lemma 1. Suppose a typical firm follows the mixed strategy ψ^* where it selects σ_1 with probability λ and σ_0 with probability $1 - \lambda$. By Lemma A.11, σ_1 must be accepted by a type-$\bar{v}$ consumer: $u(\sigma_1|\bar{v}) \geq \max\{u^*(\bar{v}), 0\}$. By Lemma A.7, σ_1 must also be accepted by a type-$\underline{v}$ consumer: $u(\sigma_1|\underline{v}) \geq \max\{u^*(\underline{v}), 0\}$. By Lemma A.9, σ_0 must be accepted by a type-$\underline{v}$ consumer: $u(\sigma_0|\underline{v}) \geq \max\{u^*(\underline{v}), 0\}$.

Suppose $u(\sigma_1|\underline{v}) > \max\{u^*(\underline{v}), 0\}$. Then, $u(\sigma_1|\bar{v}) = \max\{u^*(\bar{v}), 0\}$, else Lemma A.2 is violated. Since $u(\sigma_1|\bar{v}) > 0$, $u^*(\bar{v}) > 0$. By Lemma A.3, $u(\sigma_0|\bar{v}) > u^*(\bar{v})$, which implies $u(\sigma_0|\underline{v}) > \max\{u^*(\underline{v}), 0\}$. However, this conclusion contradicts Lemma A.2. Therefore, $u(\sigma_1|\underline{v}) = \max\{u^*(\underline{v}), 0\}$. Now there are two cases.

1. Suppose $u(\sigma_0|\underline{v}) > \max\{u^*(\underline{v}), 0\}$. Then if $u(\sigma_0|\bar{v}) \neq \max\{u^*(\bar{v}), 0\}$, Lemma A.2 is contradicted. Hence, $u(\sigma_0|\bar{v}) = \max\{u^*(\bar{v}), 0\}$ and so $u(\sigma_1|\bar{v}) > \max\{u^*(\bar{v}), 0\}$ by Lemma A.3. Therefore, on the equilibrium path no consumers search and the profit of a typical firm is $p_0 = p_1 - c - \phi$.

   Suppose a firm offered the contract $\tilde{\sigma}_0 = \langle 0, \tilde{p}_0 \rangle$ where $\tilde{p}_0 = p_1 - \underline{v} + z$. Since $u(\tilde{\sigma}_0|\underline{v}) = z - \tilde{p}_0 = \underline{v} - p_1 = u(\sigma_1|\underline{v})$, this is a feasible contract and it is accepted by all type-$\underline{v}$ consumers. Therefore, it gives the firm profits of at least $(1 - \gamma)\tilde{p}_0$. However, from Assumption A-1 and the fact that $p_1 \leq \underline{v}$ we conclude that

   $$\begin{aligned}
   (1 - \gamma)z > \underline{v} - c - \phi &\implies (1 - \gamma)z > (1 - \gamma)\underline{v} + \gamma p_1 - c - \phi \\
   &\iff (1 - \gamma)(z - \underline{v}) + (1 - \gamma)p_1 > (1 - \gamma)p_1 + \gamma p_1 - c - \phi \\
   &\iff (1 - \gamma)(p_1 + z - \underline{v}) > p_1 - c - \phi \\
   &\iff (1 - \gamma)\tilde{p}_0 > p_1 - c - \phi
   \end{aligned}$$

   Hence, a firm has a profitable deviation in offering $\tilde{\sigma}$. Hence, $u(\sigma_0|\underline{v}) > \max\{u^*(\underline{v}), 0\}$ is not compatible with an equilibrium.

2. Suppose $u(\sigma_0|\underline{v}) = \max\{u^*(\underline{v}), 0\}$. Then $u(\sigma_1|\underline{v}) = u(\sigma_0|\underline{v})$. Hence, $u^*(\underline{v}) \leq 0$ and $0 = u(\sigma_1|\underline{v}) = u(\sigma_0|\underline{v})$. But this implies $p_1 = \underline{v}$ and $p_0 = z$, as required.

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□

Lemma A.13. *Suppose all firms are following the strategy ψ^* where a firm selects σ_1 with probability λ and σ_0 with probability $1 - \lambda$. Suppose all consumers adopt an optimal search strategy holding correct beliefs, $\tilde{\psi} = \psi^*$. Suppose $u(\sigma_0|\underline{v}) \geq \max\{u^*(\underline{v}), 0\}$, $u(\sigma_1|\underline{v}) \geq \max\{u^*(\underline{v}), 0\}$, and $u(\sigma_1|\bar{v}) \geq \max\{u^*(\bar{v}), 0\} > u(\sigma_0|\bar{v})$. Suppose firm j deviates from ψ^* and offers σ_j such that $u(\sigma_j|\bar{v}) \geq \max\{u^*(\bar{v}), 0\}$. Then the expected number of consumers with indices $i < j$ who accept firm j's contract is $\gamma(\frac{1}{\lambda} - 1)$.*

Proof. Given the contracts available, type-$\underline{v}$ consumers do not search. Type-$\bar{v}$ consumers search when they match to σ_0. Otherwise, they accept σ_1 or σ_j should they reach firm j.

First, we calculate the probability that exactly q consumers with indices $i < j$ accept firm j's contract. Let $k \geq q$. Conditional on all firms $j' \in \{j-k, \ldots, j-1\}$ offering the contract σ_0 and firm $j - k - 1$ offering the contract σ_1, for q consumers with indices $i < j$ to accept the contract offered by firm j, exactly q consumers with indices $j - k \leq i \leq j - 1$ must be of type-$\bar{v}$. In this case, no consumers with indices $i \leq j - k - 1$ will search long enough to become acquainted with the contract offer of firm j. Thus, conditional on such a contract offer by firms, the probability that q consumers with indices $i < j$ accept the contract offered by firm j is $\binom{k}{q}\gamma^q(1-\gamma)^{k-q}$. For each k, the events described above are disjoint. Thus, the probability that exactly q consumers with indices $i < j$ accept firm j's contract is

$$\sum_{k=q}^{\infty} \lambda(1-\lambda)^k \left[\binom{k}{q}\gamma^q(1-\gamma)^{k-q} \right] = (1-\lambda)^q \lambda \gamma^q (\lambda + \gamma - \gamma\lambda)^{-1-q}.$$

We can use this probability to compute the expected number of consumers with indices $i < j$ accepting j's contract offer. This calculation simplifies conveniently:
$\sum_{q=1}^{\infty} q \cdot [(1-\lambda)^q \lambda \gamma^q (\lambda + \gamma - \gamma\lambda)^{-1-q}] = \gamma\left(\frac{1}{\lambda} - 1\right)$. □

Lemma A.14. *There exists a unique $\lambda^* \in (0,1)$ such that $\left(1 - \gamma + \frac{\gamma}{\lambda^*}\right)(\underline{v}-c) - \phi = (1-\gamma)z$. In fact, $\lambda^* = \frac{\gamma(\underline{v}-c)}{\phi - (1-\gamma)(\underline{v}-c-z)}$.*

Proof. The function $f(\lambda) = \left(1 - \gamma + \frac{\gamma}{\lambda}\right)(\underline{v} - c) - \phi$ is continuous on $(0,1]$ and strictly decreasing. Moreover, $\lim_{\lambda \to 0^+} f(\lambda) = \infty$ and $f(1) = \underline{v} - c - \phi < (1-\gamma)z$. Hence, there exists a unique λ^* such that $f(\lambda^*) = (1 - \gamma)z$. Solving this equation gives $\lambda^* = \frac{\gamma(\underline{v}-c)}{\phi - (1-\gamma)(\underline{v}-c-z)}$. □

Proof of Theorem 3. Suppose a typical firm follows the mixed strategy ψ^* where it selects σ_1 with probability λ_1^* and σ_0 with probability $1 - \lambda_1^*$. When a consumer's beliefs are that $\tilde{\psi} = \psi^*$, his search strategy is defined by the cutoff values $u^*(\underline{v}) = -s$ and $u^*(\bar{v}) = \bar{v} - \underline{v} - s/\lambda_1^*$. Condition (2) implies $u^*(\bar{v}) = \bar{v} - \underline{v} - s/\lambda_1^* \geq 0$.[2] Hence, a type-$\bar{v}$ consumer will search for σ_1 in lieu of accepting σ_0 or opting out of the market. He will stop his search once he encounters a contract $\tilde{\sigma}$ such that $u(\tilde{\sigma}|\bar{v}) \geq u^*(\underline{v})$. A type-$\underline{v}$ consumer does not search.

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[2]To confirm this fact, it is sufficient to verify that $z - c - \phi/(1 - \gamma + \gamma/\lambda_1^*) \geq \underline{v}$. If we suppose the contrary and substitute for λ_1^* we observe that $\underline{v} > z + c + \phi(\underline{v} - c)/(\phi + z(1-\gamma)) > z + c + \underline{v} - c$, which is a contradiction.

We next compute a firm's expected profits. If firm j offers $\tilde{\sigma}$, which is acceptable to both types of consumers, the expected total number of consumers who accept firm j's offer is $1 - \gamma + \frac{\gamma}{\lambda_1^*}$. Consumer $i = j$ accepts the contract without engaging in search. To this one consumer, we add $\gamma/\lambda_1^* - \gamma$ (Lemma A.13). This is the expected number of type-$\bar{v}$ consumers who end up accepting firm j's contract offer. In particular, when firm j offers σ_1, its expected profits are $\Pi_j(\sigma_1) = \left(1 + \frac{\gamma}{\lambda_1^*} - \gamma\right)(\underline{v} - c) - \phi$. If firm j offers σ_0, its expected profits are $\Pi_j(\sigma_0) = (1 - \gamma)z$. An equilibrium requires $\Pi_j(\sigma_0) = \Pi_j(\sigma_1)$. By Lemma A.14, there exists a unique $\lambda_1^* \in (0,1)$ ensuring this equality: $\lambda_1^* = \frac{\gamma(\underline{v}-c)}{\phi-(1-\gamma)(\underline{v}-c-z)}$. Thus, both σ_0 and σ_1 give a firm the same expected profits.

The final step is to verify that no firm can profitably deviate. There are two relevant classes of deviations. A firm may offer good 0 at a price less than z or good 1 at a price more than $\underline{v}$. Other deviations are clearly not optimal.

1. Suppose firm j offers $\tilde{\sigma} = \langle 0, \tilde{p} \rangle$, $\tilde{p} < z$. If $\tilde{p}$ is very close to z, the firm's expected profits will decline. For $\tilde{\sigma}$ to increase the firm's profits, it must entice type-$\bar{v}$ consumers who otherwise would search to stop and to accept it. Thus, $z - \tilde{p} \geq \bar{v} - \underline{v} - s/\lambda_1^*$. To maximize its expected profit from this deviation, the firm would set $\tilde{p} = z - \bar{v} + \underline{v} + s/\lambda_1^*$. Hence, the maximal expected profits associated with this alternative strategy are $\Pi_j(\tilde{\sigma}) = (1 + \gamma/\lambda_1^* - \gamma)(z - \bar{v} + \underline{v} + s/\lambda_1^*)$. Rearranging condition (2) we conclude that $\Pi_j(\sigma_1) \geq \Pi_j(\tilde{\sigma})$. Hence, firm j cannot increase its profits through this deviation.

2. Suppose firm j offers $\tilde{\sigma} = \langle 1, \tilde{p} \rangle$ where $\tilde{p} > \underline{v}$. In this case, no type-$\underline{v}$ consumers will accept the contract. A type-$\bar{v}$ consumer will accept $\tilde{\sigma}$ if and only if $\bar{v} - \tilde{p} \geq \bar{v} - \underline{v} - s/\lambda_1^*$. The maximum price which will be accepted is $\tilde{p} = \underline{v} + s/\lambda_1^*$. When $\tilde{p} = \underline{v} + s/\lambda_1^*$, the expected profit is $\Pi_j(\tilde{\sigma}) = \left(\gamma + \frac{\gamma}{\lambda_1^*} - \gamma\right)(\tilde{p} - c) - \phi = \frac{\gamma}{\lambda_1^*}\left(\underline{v} - c + \frac{s}{\lambda_1^*}\right) - \phi$. However, $\Pi_j(\tilde{\sigma}) - \Pi_j(\sigma_1) = \frac{\gamma s}{(\lambda_1^*)^2} - (1-\gamma)(\underline{v}-c) \leq 0$. Where the inequality follows from condition (3).

$\square$

Proof of Corollary 1. From Theorem 3 it follows that if ψ^* is the equilibrium strategy of a firm when the search costs are $\bar{s}$, ψ^* continues to be an equilibrium if search costs are $s \in (0, \bar{s})$.

The expected profit of a typical firm in the equilibrium of Theorem 3 is $(1 - \gamma)z$. All type-$\underline{v}$ consumers expect a payoff of zero. Type-$\bar{v}$ consumers earn an expected payoff of $\lambda_1^*(\bar{v} - \underline{v}) + (1 - \lambda_1^*)(\bar{v} - \underline{v} - s/\lambda_1^*)$. Hence, total expected surplus for a firm-consumer pair is $(1 - \gamma)z + \gamma(\lambda_1^*(\bar{v} - \underline{v}) + (1 - \lambda_1^*)(\bar{v} - \underline{v} - s/\lambda_1^*))$. This expression must exceed z. Taking the limit $s \to 0^+$, we see that such an outcome is possible if and only if $\bar{v} - \underline{v} \geq z$. $\square$

A.5 Multiple Goods Trading at Multiple Prices

Proof of Lemma 2. The result follows from Lemmas A.5–A.12 in the proof of Theorem 2.

1. It is sufficient to establish that $u(\sigma_0|\underline{v}) > u(\hat{\sigma}_0|\underline{v}) = \max\{u^*(\underline{v}), 0\}$ and $u(\sigma_0|\bar{v}) = u^*(\bar{v})$. By Lemma A.9, σ_0 and $\hat{\sigma}_0$ must be acceptable to a type-$\underline{v}$ consumer. Thus, $u(\sigma_0|\underline{v}) > u(\hat{\sigma}_0|\underline{v}) = \max\{u^*(\underline{v}), 0\}$. The equality is due to (A.2) in the proof of Lemma A.10. From (A.3) we have $u(\sigma_0|\bar{v}) = u^*(\bar{v})$ as needed.

2. It is sufficient to establish that $u(\sigma_1|\underline{v}) = \max\{u^*(\underline{v}), 0\}$ and $u(\sigma_1|\bar{v}) > u^*(\bar{v}) = u(\hat{\sigma}_1|\bar{v})$. By Lemma A.11, σ_1 and $\hat{\sigma}_1$ must be acceptable to a type-$\bar{v}$ consumer. Thus, $u(\sigma_1|\bar{v}) > u(\hat{\sigma}_1|\bar{v}) = u^*(\bar{v})$ where the equality follows from the proof of Lemma A.12. Also from the proof of Lemma A.12 we have $u(\sigma_1|\underline{v}) = \max\{u^*(\underline{v}), 0\}$.

$\square$

Proof of Lemma 3. By Theorem 2, only four classes of equilibria may exist in our economy. We analyzing the case of product differentiation and no price dispersion, we showed that a typical firm's profits are $(1 - \gamma)z$. Therefore, if a Pareto improvement is to occur at least one good must trade at multiple prices.

1. Suppose there is an equilibrium where firms offer only σ_0, σ_1, and $\hat{\sigma}_0$ with positive probability. That is, only good 0 trades at multiple prices. From Lemma 2 we know that $u(\sigma_0|\underline{v}) > u(\hat{\sigma}_0|\underline{v}) = \max\{u^*(\underline{v}), 0\}$ and $u(\sigma_0|\bar{v}) = u^*(\bar{v}) > u(\hat{\sigma}_0|\bar{v})$. Hence $\hat{\sigma}_0$ is acceptable only to type-$\underline{v}$ consumers and is not accepted by type-$\bar{v}$ consumers. Moreover, type-$\underline{v}$ consumers always accept σ_0.

   Suppose that type-$\underline{v}$ consumers always accept σ_1. Then the expected profit of a firm offering $\hat{\sigma}_0$ is at most $(1-\gamma)z$, which is not reflective of a Pareto improvement. Therefore, a type-$\underline{v}$ consumer must search with positive probability if his only available contract is σ_1. That is, $u(\sigma_1|\underline{v}) \leq u^*(\underline{v})$. There are two relevant sub cases:

   (a) Suppose $u(\sigma_1|\underline{v}) < u^*(\underline{v})$. Therefore, only type-$\bar{v}$ consumers accept σ_1. If $u(\sigma_1|\bar{v}) = \bar{v} - p_1 > \max\{u^*(\bar{v}), 0\}$, then a typical firm offering σ_1 could slightly increase its price without affecting its demand. Therefore, in equilibrium the price of good 1 must satisfy $\bar{v} - p_1 = \max\{u^*(\bar{v}), 0\}$. But this leads to a contradiction. Specifically, if $\bar{v} - p_1 = u(\sigma_1|\bar{v}) = u^*(\bar{v}) \geq 0$, then using Lemmas A.1 and 2 we see that $u^*(\bar{v}) = \lambda_0 u^*(\bar{v}) + \hat{\lambda}_0 u^*(\bar{v}) + \lambda_1 \max\{u^*(\bar{v}), 0\} - s$, which is impossible. On the other hand, if $u^*(\bar{v}) < 0$ then the profits generated by the contract σ_1 are at most $\gamma(\bar{v} - c) - \phi$, which is less than z by Assumption A-1. Hence, this outcome cannot lead to a Pareto improvement.

   (b) Suppose $u(\sigma_1|\underline{v}) = u^*(\underline{v})$. In this case, a type-$\underline{v}$ consumer is indifferent between searching for an alternative offer and accepting σ_1. Any firm offering σ_1 can slightly decrease its price (which is strictly positive) and discretely increase its expected profits by encouraging type-$\underline{v}$ consumers to accept its offer rather than search. Hence, this case is not compatible with an equilibrium.

2. Suppose there is an equilibrium where firms offer only σ_0, σ_1, and $\hat{\sigma}_1$ with positive probability. That is, only good 1 trades at multiple prices. From Lemma 2 we know

that $u(\sigma_1|\underline{v}) = \max\{u^*(\underline{v}), 0\} > u(\hat{\sigma}_1|\underline{v})$ and $u(\sigma_1|\bar{v}) > u(\hat{\sigma}_1|\bar{v}) = u^*(\bar{v})$. Therefore, $\hat{\sigma}_1$ is only acceptable to type-$\bar{v}$ consumers. If all type-$\bar{v}$ consumers accept σ_0, then the expected profits of a typical firm offering $\hat{\sigma}_1$ are at most $\gamma(\bar{v} - c) - \phi$, which is less than z by Assumption A-1. Hence, some type-$\bar{v}$ consumers must search if σ_0 is their only available contract. That is, $u(\sigma_0|\bar{v}) \leq u^*(\bar{v})$. There are two sub cases to consider.

(a) If $u(\sigma_0|\bar{v}) < u^*(\bar{v})$, then no type-$\bar{v}$ consumers accept σ_0. For σ_0 to generate an expected profit greater than z, $u(\sigma_0|\underline{v}) > \max\{u^*(\underline{v}), 0\}$, so as to entice some type-$\underline{v}$ consumers to search. (The strict inequality is necessary.) However, this implies that any firm offering σ_0 can strictly increase its price slightly without affecting its demand, contradicting the equilibrium condition.

(b) If $u(\sigma_0|\bar{v}) = u^*(\bar{v})$, then any firm offering σ_0 can strictly increase its profit by reducing p_0 slightly thereby encouraging type-$\bar{v}$ consumers to accept σ_0 in lieu of searching for a good 1 contract. This contradicts the equilibrium condition.

Hence, three of the four possible classes of equilibria are not compatible with a Pareto-improving outcome. Therefore, if there is an equilibrium with the desired characteristics, it must involve two goods trading at multiple prices.

Second, the requisite pattern of consumer search follows from Lemma 2. First, $u^*(\bar{v}) > 0$ since $u^*(\bar{v}) = u(\sigma_0|\bar{v}) = z - p_0 > z - \hat{p}_0 = u(\hat{\sigma}|\underline{v}) = \max\{u^*(\bar{v}), 0\} \geq 0$. Second, if $u^*(\underline{v}) < 0$, then $\hat{\sigma}_0$ is acceptable only to a type-$\underline{v}$ consumer. But this implies that $\Pi(\hat{\sigma}_0)) = (1 - \gamma)z$. Thus, firms' equilibrium profits fall below the benchmark case. $\qquad\square$

Remark A.1. In any equilibrium where a consumer is indifferent between accepting a contract and continued search, he must accept the contract. Else, a firm offering the contract in question can lower the price slightly prompting a discrete increase in expected profits.

Proof of Lemma 4. Since $u^*(\bar{v}) \geq u^*(\underline{v}) \geq 0$, Lemma 2 implies that $z - p_0 = u^*(\bar{v}) > z - \hat{p}_0$ and $\bar{v} - p_1 > \bar{v} - \hat{p}_1 = u^*(\bar{v})$. We can write (A.1) as

$$
\begin{aligned}
u^*(\bar{v}) &= \lambda_0(z - p_0) + \hat{\lambda}_0 \max\{z - \hat{p}_0, 0, u^*(\bar{v})\} + \lambda_1(\bar{v} - p_1) + \hat{\lambda}_1(\bar{v} - \hat{p}_1) - s \\
&= \lambda_0 u^*(\bar{v}) + \hat{\lambda}_0 u^*(\bar{v}) + \lambda_1(\bar{v} - p_1) + \hat{\lambda}_1 u^*(\bar{v}) - s \\
&= (1 - \lambda_1)u^*(\bar{v}) + \lambda_1(\bar{v} - p_1) - s \\
\Longleftrightarrow u^*(\bar{v}) &= \bar{v} - p_1 - s/\lambda_1
\end{aligned}
$$

The argument for $u^*(\underline{v})$ is analogous.

To derive the relationships among prices we use equalities from Lemma 2 and the imme-

diately preceding computations.

$$u(\hat{\sigma}_0|\underline{v}) = u^*(\underline{v}) \implies z - \hat{p}_0 = z - p_0 - \frac{s}{\lambda_0} \implies \hat{p}_0 = p_0 + \frac{s}{\lambda_0}$$

$$u(\sigma_1|\underline{v}) = u^*(\underline{v}) \implies \underline{v} - p_1 = z - p_0 - \frac{s}{\lambda_0} \implies p_1 = p_0 + \frac{s}{\lambda_0} + \underline{v} - z$$

$$u(\sigma_0|\bar{v}) = u^*(\bar{v}) \implies z - p_0 = \bar{v} - p_1 - \frac{s}{\lambda_1} \implies p_0 = p_1 + \frac{s}{\lambda_1} + z - \bar{v}$$

$$u(\hat{\sigma}_1|\bar{v}) = u^*(\bar{v}) \implies \bar{v} - \hat{p}_1 = \bar{v} - p_1 - \frac{s}{\lambda_1} \implies \hat{p}_1 = p_1 + \frac{s}{\lambda_1}$$

<div align="right">□</div>

Proof of Lemma 5. From Section IV.B, we can easily describe consumer search behavior. A type-$\underline{v}$ consumer searches for an alternative offer if $\hat{\sigma}_1$ is the only contract that is available. He accepts σ_0, $\hat{\sigma}_0$, and σ_1. A type-$\bar{v}$ consumer searches for an alternative offer if $\hat{\sigma}_0$ is the only contract that is available. He accepts σ_0, σ_1, and $\hat{\sigma}_1$.

1. The contract $\sigma_0 = \langle 0, p_0 \rangle$ is acceptable to both types of consumers. We must compute the expected number of consumers with indices $i < j$ who accept firm j's offer.

   Suppose for the moment that at least one type-$\bar{v}$ consumer with an index $i < j$ accepts σ_0 from firm j. This implies that firm $j-1$ must be offering the contract $\hat{\sigma}_0 = \langle 0, \hat{p}_0 \rangle$. But then no type-$\underline{v}$ consumers with indices $i < j$ will accept firm j's offer. Any type-$\underline{v}$ consumer who is searching would accept $\hat{\sigma}_0$ from firm $j-1$ before learning the offer of firm j. Likewise, if at least one type-$\underline{v}$ consumer with an index $i < j$ accepts σ_0 from firm j, then no type-$\bar{v}$ consumers with indices $i < j$ will accept the offer from firm j. Thus, firm j sells its product to either q consumers of type-$\bar{v}$ who are searching or to q' consumers of type-$\underline{v}$ who are searching. Never does it sell its product to a mixed group of buyers with indices $i < j$.

   Following the reasoning in the proof of Lemma A.13, the probability that exactly q type-$\bar{v}$ consumers who are searching accept firm j's offer is $\sum_{k=q}^{\infty} \hat{\lambda}_0^k (1 - \hat{\lambda}_0) \binom{k}{q} \gamma^q (1 - \gamma)^{k-q} = \frac{(1-\hat{\lambda}_0)\hat{\lambda}_0^q \gamma^q}{(1-\hat{\lambda}_0(1-\gamma))^{q+1}}$. Similarly, the probability that exactly q type-$\underline{v}$ consumers who are searching accept firm j's offer is $\sum_{k=q}^{\infty} \hat{\lambda}_1^k (1 - \hat{\lambda}_1) \binom{k}{q} (1 - \gamma)^q \gamma^{k-q} = \frac{(1-\hat{\lambda}_1)\hat{\lambda}_1^q(1-\gamma)^q}{(1-\hat{\lambda}_1\gamma)^{q+1}}$. We can compute the expected number of consumers with indices $i < j$ who accept firm j's offer of σ_0 to be

   $$\sum_{q=1}^{\infty} \left[\frac{(1-\hat{\lambda}_0)\hat{\lambda}_0^q \gamma^q}{(1-\hat{\lambda}_0(1-\gamma))^{q+1}} + \frac{(1-\hat{\lambda}_1)\hat{\lambda}_1^q(1-\gamma)^q}{(1-\hat{\lambda}_1\gamma)^{q+1}} \right] q = \frac{\gamma\hat{\lambda}_0}{1-\hat{\lambda}_0} + \frac{(1-\gamma)\hat{\lambda}_1}{1-\hat{\lambda}_1}.$$

   Hence, accounting for consumer $i = j$ who also accepts σ_0, the expected profit of firm j is $\Pi_j(\sigma_0) = \left(1 + \frac{\gamma\hat{\lambda}_0}{1-\hat{\lambda}_0} + \frac{(1-\gamma)\hat{\lambda}_1}{1-\hat{\lambda}_1} \right) p_0 = \left(\frac{\gamma}{1-\hat{\lambda}_0} + \frac{1-\gamma}{1-\hat{\lambda}_1} \right) p_0.$

2. The reasoning is identical to the preceding case.

<div align="center">12</div>

3. The contract $\hat{\sigma}_0 = \langle 0, \hat{p}_0 \rangle$ is accepted only by type-$\underline{v}$ consumers. Therefore, noting the above calculations, the expected number of type-$\underline{v}$ consumers who accept $\langle 0, \hat{p}_0 \rangle$ is $\left[(1 - \gamma) + \frac{(1-\gamma)\hat{\lambda}_1}{1-\hat{\lambda}_1}\right] = \frac{1-\gamma}{1-\hat{\lambda}_1}$. Hence, the expected profit is $\Pi_j(\hat{\sigma}_0) = \frac{1-\gamma}{1-\hat{\lambda}_1}\hat{p}_0$.

4. The reasoning is identical to the preceding case.

$\square$

Lemma A.15. *There exists an equilibrium where both goods trade at multiple prices and consumer search conforms to the pattern described in Lemma 3 if and only if there exists strictly positive constants—$\lambda_0, \lambda_1, \hat{\lambda}_0, \hat{\lambda}_1$—such that*

$$\bar{v} - \underline{v} = \frac{s}{\lambda_0} + \frac{s}{\lambda_1} \tag{A.6}$$

$$\Pi_j(\sigma_0) = \Pi_j(\sigma_1) \tag{A.7}$$

$$p_1 = p_0 + \frac{s}{\lambda_0} + \underline{v} - z \tag{A.8}$$

$$1 = \lambda_0 + \lambda_1 + \hat{\lambda}_0 + \hat{\lambda}_1 \tag{A.9}$$

$$z - p_0 - \frac{s}{\lambda_0} \geq 0 \tag{A.10}$$

where

$$p_0 = \frac{1 - \hat{\lambda}_0}{1 - \hat{\lambda}_1} \cdot \frac{1 - \gamma}{\gamma} \cdot \frac{s}{\lambda_0} \qquad and \qquad p_1 = c + \frac{1 - \hat{\lambda}_1}{1 - \hat{\lambda}_0} \cdot \frac{\gamma}{1 - \gamma} \cdot \frac{s}{\lambda_1}. \tag{A.11}$$

Proof. Conditions (A.6)–(A.10) reproduce the conditions identified in Section IV.B. Thus, necessity follows from the discussion in that section. To prove sufficiency we reverse the argument. Let $\lambda_0, \lambda_1, \hat{\lambda}_0$, and $\hat{\lambda}_1$ be the strictly positive values that satisfy the conditions (A.6)–(A.10). We use these values to define four contracts, two for each good, as follows:

- Let $\sigma_0 = \langle 0, p_0 \rangle$ be a contract supplying good 0 at a price of

$$p_0 = \frac{1 - \hat{\lambda}_0}{1 - \hat{\lambda}_1} \cdot \frac{1 - \gamma}{\gamma} \cdot \frac{s}{\lambda_0}. \tag{A.12}$$

- Let $\hat{\sigma}_0 = \langle 0, \hat{p}_0 \rangle$ be a contract supplying good 0 at a price of $\hat{p}_0 = p_0 + s/\lambda_0$, where p_0 is given by (A.12).

- Let $\sigma_1 = \langle 1, p_1 \rangle$ be a contract supplying good 1 at a price of

$$p_1 = c + \frac{1 - \hat{\lambda}_1}{1 - \hat{\lambda}_0} \cdot \frac{\gamma}{1 - \gamma} \cdot \frac{s}{\lambda_1}. \tag{A.13}$$

- Let $\hat{\sigma}_1 = \langle 1, \hat{p}_1 \rangle$ be a contract supplying good 1 at a price of $\hat{p}_1 = p_1 + s/\lambda_1$, where p_1 is given by (A.13).

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Suppose each firm adopts the strategy ψ_j^* where contract σ_k ($\hat{\sigma}_k$) is offered with probability λ_k ($\hat{\lambda}_k$). Since $1 = \lambda_0 + \lambda_1 + \hat{\lambda}_0 + \hat{\lambda}_1$, all prices defined above are strictly positive and the strategy is a valid mixed strategy. To verify the equilibrium, we first characterize consumers' search behavior given ψ_j^*. Subsequently we compute the expected profits associated with each contract given the consumers' search strategy. Finally, we verify that no firm has a profitable deviation from the mixed strategy ψ_j^*.

The Optimal Search Strategy for a Type-$\underline{v}$ Consumer. Given the contracts defined above,

$$u(\sigma_0|\underline{v}) = z - p_0 \qquad\qquad u(\hat{\sigma}_0|\underline{v}) = z - \hat{p}_0 = z - p_0 - s/\lambda_0$$
$$u(\sigma_1|\underline{v}) = \underline{v} - p_1 \qquad\qquad u(\hat{\sigma}_1|\underline{v}) = \underline{v} - \hat{p}_1 = \underline{v} - p_1 - s/\lambda_1.$$

(A.10) implies that $u(\sigma_0|\underline{v}) > u(\hat{\sigma}_0|\underline{v}) \geq 0$. Moreover, by (A.8), $u(\sigma_1|\underline{v}) = \underline{v} - p_1 = \underline{v} - p_0 - s/\lambda_0 - \underline{v} + z = z - p_0 - s/\lambda_0 = u(\hat{\sigma}_0|\underline{v})$. Therefore, $u(\sigma_0|\underline{v}) > u(\sigma_1|\underline{v}) = u(\hat{\sigma}_0|\underline{v}) > u(\hat{\sigma}_1|\underline{v})$.

If a type-$\underline{v}$ consumer believes that all firms are following the mixed strategy ψ_j^*, his optimal search strategy will be characterized by a constant cutoff value, $u^*(\underline{v})$. From Lemma A.1, $u^*(\underline{v})$ solves

$$u^*(\underline{v}) = \mathbb{E}_{\psi_j^*}\left[\max\{u^*(\underline{v}), u(\sigma|\underline{v}), 0\}\right] - s. \tag{A.14}$$

There are two possible cases:

1. Suppose that the value $u^*(\underline{v})$ that solves (A.14) is such that $u(\sigma_0|\underline{v}) \geq u^*(\underline{v}) \geq u(\sigma_1|\underline{v}) = u(\hat{\sigma}_0|\underline{v}) > u(\hat{\sigma}_1|\underline{v})$. Then, $u^*(\underline{v}) = \lambda_0(z - p_0) + (1 - \lambda_0)u^*(\underline{v}) - s \iff u^*(\underline{v}) = z - p_0 - s/\lambda_0$.

2. Suppose that the value $u^*(\underline{v})$ that solves (A.14) is such that $u(\sigma_0|\underline{v}) > u(\sigma_1|\underline{v}) = u(\hat{\sigma}_0|\underline{v}) > u^*(\underline{v}) \geq \cdots$. Then,

$$u^*(\underline{v}) = \lambda_0(z - p_0) + (\hat{\lambda}_0 + \lambda_1)\left(z - p_0 - \frac{s}{\lambda_0}\right)$$
$$+ \hat{\lambda}_1 \max\{u(\hat{\sigma}_1|\underline{v}), u^*(\underline{v}), 0\} - s$$
$$\implies u^*(\underline{v}) = (1 - \hat{\lambda}_1)(z - p_0) - (\hat{\lambda}_0 + \lambda_1)\frac{s}{\lambda_0} + \hat{\lambda}_1 u^*(\underline{v}) - s$$
$$\implies u^*(\underline{v}) = z - p_0 - \frac{\hat{\lambda}_0 + \lambda_1}{1 - \hat{\lambda}_1}\frac{s}{\lambda_0} - \frac{s}{1 - \hat{\lambda}_1}$$

By assumption, however, $z - p_0 - s/\lambda_0 > u^*(\underline{v})$. Therefore,

$$\frac{\hat{\lambda}_0 + \lambda_1}{1 - \hat{\lambda}_1}\frac{s}{\lambda_0} + \frac{s}{1 - \hat{\lambda}_1} > \frac{s}{\lambda_0} \iff \lambda_0(1 - \hat{\lambda}_1)(1 - \lambda_0 - \lambda_1 - \hat{\lambda}_0 - \hat{\lambda}_1)s < 0.$$

But since $1 - \lambda_0 - \lambda_1 - \hat{\lambda}_0 - \hat{\lambda}_1 = 0$, this final condition is a contradiction.

Since case 2 cannot apply, we conclude that under the assumed values for the λ's, $u^*(\underline{v}) = z - p_0 - s/\lambda_0$ and $u^*(\underline{v}) > u(\hat{\sigma}_1|\underline{v})$. Thus, when a type-$\underline{v}$ consumer believes all firms adopt

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the strategy ψ_j^*, he finds acceptable the contracts σ_0, $\hat{\sigma}_0$, and σ_1. If $\hat{\sigma}_1$ is the only contract available, the consumer will search in lieu of accepting $\hat{\sigma}_1$ or exiting the market.

The Optimal Search Strategy for a Type-$\bar{v}$ Consumer. Given the set of contracts defined above, we can compute a type-$\bar{v}$ consumer's expected utility:

$$u(\sigma_0|\bar{v}) = z - p_0 \qquad\qquad u(\hat{\sigma}_0|\bar{v}) = z - \hat{p}_0 = z - p_0 - s/\lambda_0$$
$$u(\sigma_1|\bar{v}) = \bar{v} - p_1 \qquad\qquad u(\hat{\sigma}_1|\bar{v}) = \bar{v} - \hat{p}_1 = \bar{v} - p_1 - s/\lambda_1$$

Again, $u(\sigma_0|\underline{v}) > u(\hat{\sigma}_0|\underline{v}) \geq 0$ and substituting the expression from above yields $u(\hat{\sigma}_1|\bar{v}) = \bar{v} - p_1 - s/\lambda_1 = \bar{v} - p_0 - s/\lambda_0 - \underline{v} + z - s/\lambda_1 = z - p_0$. Hence, $u(\sigma_1|\bar{v}) > u(\hat{\sigma}_1|\bar{v}) = u(\sigma_0|\bar{v}) > u(\hat{\sigma}_0|\bar{v})$. An argument analogous to the previous case establishes that under the proposed conditions, the cutoff value charactering the search strategy of a type-$\bar{v}$ consumer is $u^*(\bar{v}) = \bar{v} - p_1 - s/\lambda_1 = z - p_0$. Thus, when a type-$\bar{v}$ consumer believes all firms adopt the strategy ψ_j^*, he finds acceptable the contracts σ_0, σ_1, and $\hat{\sigma}_1$. If $\hat{\sigma}_0$ is the only contract available, the consumer will happily search in lieu of accepting $\hat{\sigma}_0$ or exiting the market.

Expected Firm Profits. Noting the consumers' search behavior we can compute the expected profits associated with each of the offered contracts. These are given by Lemma 5 above. It is straightforward to confirm that given the definition of prices and (A.7), $\Pi_j(\sigma_0) = \Pi_j(\hat{\sigma}_0) = \Pi_j(\sigma_1) = \Pi_j(\hat{\sigma}_1)$. Therefore, each contract in the support of ψ_j^* yields the same (strictly-positive) expected profit conditional on the consumers' search strategy.

No Firm has a Profitable Deviation. It remains to verify that no alternative contract can yield a greater expected profit to a typical firm given agents' search behavior and the strategy adopted by the other firms. Verification of this fact relies on the inequalities and equalities summarized by the following conditions:

$$u(\sigma_0|\underline{v}) > u(\sigma_1|\underline{v}) = u(\hat{\sigma}_0|\underline{v}) = u^*(\underline{v}) \geq \max\{u(\hat{\sigma}_1|\underline{v}), 0\}, \text{ and} \qquad (A.15)$$
$$u(\sigma_1|\bar{v}) > u(\hat{\sigma}_1|\bar{v}) = u(\sigma_0|\bar{v}) = u^*(\bar{v}) > u(\hat{\sigma}_0|\bar{v}). \qquad (A.16)$$

(A.15) and (A.16) were derived above in relation to each type of consumers' search strategy.

Suppose firm j posts the contract $\tilde{\sigma} = \langle 0, \tilde{p} \rangle$. Clearly, if $\tilde{p} \leq p_0$, the firm cannot improve its expected profits. The lower price fails to attract more consumers than p_0 and leads to strictly lower profits. If $\hat{p}_0 < \tilde{p}$, then $u^*(\bar{v}) \geq u^*(\underline{v}) = z - \hat{p}_0 > z - \tilde{p}$. Thus, the more expensive contract would not be accepted by any consumers. Finally if $p_0 < \tilde{p} < \hat{p}_0$, then $z - \tilde{p} < u^*(\bar{v})$. Hence, the contract would only be acceptable to type-$\bar{v}$ consumers. However, $\hat{\sigma}_0$ was also acceptable to such consumers and it had a higher price. Hence, firm j could do better by offering $\hat{\sigma}_0$ instead.

Suppose a firm posts $\tilde{\sigma} = \langle 1, \tilde{p} \rangle$. If $\tilde{p} < p_1$, the firm can strictly improve its expected profit by charging a slightly higher price since the contract would be accepted by all consumers. Likewise if $\tilde{p} > \hat{p}_1$, the contract is not acceptable to any consumers—even type-$\bar{v}$ consumers will search in this case. If $p_1 < \tilde{p} < \hat{p}_1$, the contract is acceptable only to type-$\underline{v}$ consumers. But $\hat{\sigma}_1$ would yield a greater profit as it would be accepted by the same number of consumers but charge them a higher price. Thus, no firm can deviate profitably from ψ^*. □

B   Increasing Search Costs

In this section we elaborate on our proposed model extension with increasing search costs. Suppose that the cost to learn the contract offer of the t-th additional firm is s_t where $0 < s_1 \leq s_2 \leq \cdots$. The sequence of search costs is common knowledge and $\lim_{t \to \infty} s_t > \bar{v}$. If the consumer searches for t periods in total, his payoff will be $u(\sigma_{i+t}|v_i) - \sum_{k=1}^{t} s_k$ if he accepts σ_{i+t} and $-\sum_{k=1}^{t} s_k$ otherwise.

Let $\sigma_0 = \langle 0, z \rangle$ and $\sigma_1 = \langle 1, \underline{v} \rangle$. Suppose firms follow a strategy (ψ^*) where σ_1 is offered with probability $\lambda_1^* \in (0, 1)$ and σ_0 is offered with probability $1 - \lambda_1^*$. A type-$\bar{v}$ consumer will search for σ_1, but only for a finite number of periods. If he searches for t^* periods at most, t^* must satisfy

$$t^* = \max\{t : \lambda_1^* u(\sigma_1|\bar{v}) + (1 - \lambda_1^*) u(\sigma_0|\bar{v}) - s_t \geq u(\sigma_0|\bar{v})\}. \tag{B.1}$$

The expected payoff from taking the final sample must exceed the payoff of accepting the available contract. We can characterize this consumer's search strategy by a decreasing sequence of cutoff values. As search costs rise over time, the consumer becomes less choosy. In period t consumer i will accept σ_{i+t} if $u(\sigma_{i+t}|\bar{v}) \geq u_t^*(\bar{v})$ where

$$u_t^*(\bar{v}) = \left(1 - (1 - \lambda_1^*)^{t^*-t}\right) u(\sigma_1|\bar{v}) + (1 - \lambda_1^*)^t u(\sigma_0|\bar{v}) - \sum_{k=0}^{t^*-t-1} s_{t^*-k} (1 - \lambda_1^*)^{t^*-t-1-k}.$$

These cutoff values can be computed via backward induction starting at t^*.

The preceding discussion has taken λ_1^* as given. However, it is determined endogenously through the equilibration of firms' profits. Given consumers' search strategy, the expected profit of a firm offering σ_0 is $\Pi_j(\sigma_0) = \left(1 - \gamma + \gamma(1 - \lambda_1^*)^{t^*}\right) z$. The term $\gamma(1 - \lambda_1^*)^{t^*}$ accounts for the small, but positive, probability that a type-$\bar{v}$ consumer accepts σ_0 after a t^*-period unsuccessful search for σ_1. Similarly,

$$\Pi_j(\sigma_1) = \left(1 + \gamma \left(\frac{1}{\lambda_1^*} - 1\right) \left(1 - (1 - \lambda_1^*)^{t^*}\right)\right) (\underline{v} - c) - \phi.$$

In equilibrium, $t^* \in \mathbb{N}$ and $\lambda_1^* \in (0, 1)$ are such that (i) t^* satisfies condition (B.1), and (ii) $\Pi_j(\sigma_0) = \Pi_j(\sigma_1)$. Additionally, firms must not have an incentive to deviate from ψ^*. Hence a collection of additional constraints, analogous to those in Theorem 3 but now accounting for the consumers' new search strategy, must be satisfied.[3] The following lemma identifies sufficient conditions ensuring that t^* and λ_1^* exist in the defined environment.

Lemma B.1. *Suppose search costs are non-decreasing. Define the functions*

$$\rho(\lambda) = \begin{cases} \max\{t : \lambda(\bar{v} - \underline{v}) - s_t \geq 0\} & \text{if } \{t : \lambda(\bar{v} - \underline{v}) - s_t \geq 0\} \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \tag{B.2}$$

$$\tau(\lambda) = \frac{\log\left(1 - \frac{1}{\gamma} \frac{\lambda(c+\phi-v+z)}{(\lambda-1)(c-v)+\lambda z}\right)}{\log(1 - \lambda)} \tag{B.3}$$

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[3]As in the case of the other extensions we analyze, we omit analyzing these conditions for brevity.

Let $\bar{\lambda} = \frac{\gamma(\underline{v}-c)}{\phi-(1-\gamma)(\underline{v}-c-z)}$ and suppose there exists $\hat{\lambda} \in (0, \bar{\lambda})$ such that $\rho(\hat{\lambda}) > \tau(\hat{\lambda})$. Then there exists $t^* \in \mathbb{N}$ and $\lambda_1^* \in (0,1)$ such that

1. $t^* = \max\{t : \lambda_1^* u(\sigma_1|\bar{v}) + (1-\lambda_1^*)u(\sigma_0|\bar{v}) - s_t \geq u(\sigma_0|\bar{v})\}$; and,

2. $\left(1 + \gamma \left(\frac{1}{\lambda_1^*} - 1\right)\left(1 - (1-\lambda_1^*)^{t^*}\right)\right)(\underline{v} - c) - \phi = \left(1 - \gamma + \gamma(1-\lambda_1^*)^{t^*}\right)z.$

Proof. When $\sigma_0 = \langle 0, z\rangle$ and $\sigma_1 = \langle 1, \underline{v}\rangle$, $u(\sigma_1|\bar{v}) = \bar{v} - \underline{v}$ and $u(\sigma_0|\bar{v}) = 0$. Now, $\rho(\lambda)$ is a non-decreasing step function taking on integer values. It is the blue curve is Figure B.1.

The function $\tau(\lambda)$ is defined as the value of τ that solves

$$\left(1 + \gamma \left(\frac{1}{\lambda} - 1\right)\left(1 - (1-\lambda)^\tau\right)\right)(\underline{v} - c) - \phi = \left(1 - \gamma + \gamma(1-\lambda)^\tau\right)z \qquad \text{(B.4)}$$

When a solution exists, it is given by (B.3). $\tau(\lambda)$ is strictly increasing and continuous. Also, $\lim_{\lambda \to 0^+} \tau(\lambda) = \frac{z-\underline{v}+\phi+c}{\gamma(\underline{v}-c)}$ and $\tau(\lambda)$ asymptotes to infinity as $\lambda \to \bar{\lambda}^- = \frac{\gamma(\underline{v}-c)}{\phi-(1-\gamma)(\underline{v}-c-z)}$. $\tau(\lambda)$ is the red curve in Figure B.1.

If $\rho(\hat{\lambda}) > \tau(\hat{\lambda})$ there will exist some $\lambda_1^* \in (\hat{\lambda}, \bar{\lambda})$ where $t^* = \tau(\lambda_1^*)$ and (λ_1^*, t^*) satisfies the lemma's conditions. $\square$
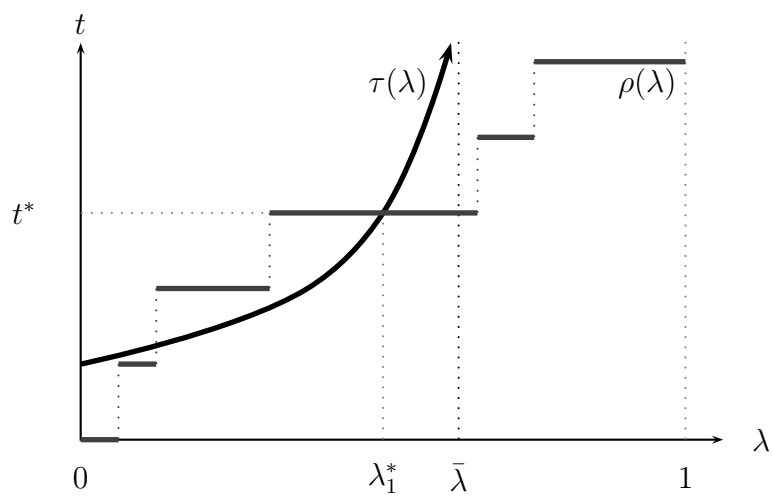
Figure B.1: An illustration of the situation in Lemma B.1.

References

DeGroot, M. H. (1970). *Optimal Statistical Decisions*. McGraw-Hill Book Company, New York.