Online Appendix to
"Hybrid All-Pay and Winner-Pay Contests"

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1. Introduction

In this online appendix, I provide proofs that were omitted from Lagerlöf (2020). In addition, I show the calculations that were used for Figures 2, 4, and 5 of that paper. For convenience, in the next section I restate some of the equations (and an assumption) from Lagerlöf (2020). The numbering of those equations is thus the same as in that paper.

2. Restatement of Expressions from Lagerlöf (2020)

$$x_i = y_i h\left(\frac{1}{p_i}\right). \tag{5}$$

$$Y(s_i, p_i) = \left[\frac{s_i}{f(h(1/p_i), 1)}\right]^{\frac{1}{t}}, \qquad X(s_i, p_i) = Y(s_i, p_i)\, h\left(\frac{1}{p_i}\right). \tag{6}$$

$$C[s_i, p_i(\mathbf{s})] \stackrel{\text{def}}{=} p_i(\mathbf{s})\, Y[s_i, p_i(\mathbf{s})] + X[s_i, p_i(\mathbf{s})]. \tag{8}$$

Assumption 1. *The production function and the CSF satisfy at least one of the following three sets of conditions:*

(i) $t \leq 1$ *and* $\varepsilon_i(\mathbf{s})\,\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \leq 2$ *(for all* i*,* p_i*, and* $\mathbf{s}$*);*

(ii) $tr \leq 1$*,* $r\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \leq 2$*, and*

$$p_i(\mathbf{s}) = \frac{w_i s_i^r}{\sum_{j=1}^n w_j s_j^r} \quad \textit{(for all } i, p_i, \textit{ and } \mathbf{s} \neq \mathbf{0}\textit{)}, \tag{9}$$

where $r > 0$ *and* $w_i > 0$ *are parameters;*

(iii) $p_i(\mathbf{s})$ *is given by (9),* $f(x_i, y_i) = x_i^\alpha y_i^\beta$ *(with* $\alpha > 0$ *and* $\beta > 0$*), and* $\alpha r \leq 1$ *(for all* i*).*

$$[v_i - Y(s_i, p_i(\mathbf{s}))]\frac{\partial p_i(\mathbf{s})}{\partial s_i} \leq C_1(s_i, p_i). \tag{10}$$

$$(v - y^*)\frac{\widehat{\varepsilon}(n)}{ns^*} = C_1\left[s^*, \frac{1}{n}\right] \Leftrightarrow (v - y^*)\, t\widehat{\varepsilon}(n) = y^* + nx^*. \tag{11}$$

$$y^* = \frac{t\widehat{\varepsilon}(n)v}{1 + nh(n) + t\widehat{\varepsilon}(n)}. \tag{12}$$

$$\widehat{\varepsilon}(n) = \frac{r(n-1)}{n}. \tag{14}$$

$$\frac{\partial x^*}{\partial n} < 0 \Leftrightarrow \sigma(n) > -\frac{n(n-2)h(n) - 1}{(n-1)\left[1 + \frac{rt(n-1)}{n}\right]}, \qquad \frac{\partial y^*}{\partial n} > 0 \Leftrightarrow \sigma(n) > \frac{n(n-2)h(n) - 1}{(n-1)nh(n)}. \tag{16}$$

$$R^A = t\widehat{\varepsilon}(n)v. \tag{18}$$

$$R^H = \left[1 - \frac{y^*}{v}\right] R^A = \left[\frac{1}{v[1 + nh(n)]} + \frac{1}{R^A}\right]^{-1}. \tag{19}$$

$$y_i^* = \frac{rtp_i^*(1 - p_i^*)v_i}{rtp_i^*(1 - p_i^*) + p_i^* + h\left(\frac{1}{p_i^*}\right)}, \qquad \text{for } i = 1, 2. \tag{23}$$

$$R^H = rtp_1(1 - p_1)\frac{r\beta[p_1 v_1 + (1 - p_1)v_2] + v_1 + v_2}{[r\beta(1 - p_1) + 1](r\beta p_1 + 1)}. \tag{24}$$

$$w_1 = w_2\left(\frac{p_1}{1 - p_1}\right)^{1 + r\beta}\left(\frac{r\beta(1 - p_1) + 1}{r\beta p_1 + 1}\frac{v_2}{v_1}\right)^{rt}. \tag{25}$$

3. Proofs of Propositions 1-5 and 7-10

3.1. Proof of Proposition 1

To prove the proposition, we can invoke Theorem 3.1 in Reny (1999), which guarantees the existence of a pure strategy equilibrium under the conditions that the strategy sets are compact, contestant i's payoff function is quasiconcave in s_i, and the game is better-reply secure. The first condition is readily taken care of by, without loss of generality, imposing a constraint $s_i \leq \bar{s}$, where $\bar{s}$ is some finite and sufficiently large constant; this ensures that each player's strategy set $[0, \bar{s}] \stackrel{\text{def}}{=} S$ is closed and bounded and thus compact. The requirement that the payoff functions are quasiconcave will be investigated at the end of this proof. To show that the game is better-reply secure, we can rely on Proposition 1 in Bagh and Jofre (2006). This says that a game is better-reply secure if it is *payoff secure* and *weakly reciprocal upper semicontinuous (wrusc)*.[1] We know that, in the hybrid contest, each player's payoff function is continuous everywhere, except possibly at the origin. This means that the potentially problematic issue with showing the two properties is what happens at the point $\mathbf{s} = \mathbf{0}$.

In order to prove that the game is payoff secure at $\mathbf{s} = \mathbf{0}$, we must show that each player can, for every $\epsilon > 0$, secure a payoff of $p_i(\mathbf{0})\, v_i - \epsilon$ at $\mathbf{s} = \mathbf{0}$. A player is said to be able to secure a payoff of $p_i(\mathbf{0})\, v_i - \epsilon$ at $\mathbf{s} = \mathbf{0}$ if there exists $\widetilde{s}_i$ such that $\pi_i\left(\widetilde{s}_i, \mathbf{s}'_{-\mathbf{i}}\right) \geq p_i(\mathbf{0})\, v_i - \epsilon$ for all $\mathbf{s}'_{-\mathbf{i}}$ in some open neighborhood of $\mathbf{0}_{-\mathbf{i}}$. The hybrid contest is indeed payoff secure at $\mathbf{s} = \mathbf{0}$. To see this, note that there exists $\widetilde{s}_i > 0$ such that

$$\pi_i\left(\widetilde{s}_i, \mathbf{0}_{-\mathbf{i}}\right) = p_i\left(\widetilde{s}_i, \mathbf{0}_{-\mathbf{i}}\right) v_i - C\left[\widetilde{s}_i, p_i\left(\widetilde{s}_i, \mathbf{0}_{-\mathbf{i}}\right)\right] = v_i - C\left[\widetilde{s}_i, 1\right] > p_i(\mathbf{0})\, v_i. \tag{A1}$$

The second equality in (A1) follows from the assumption that, for any $\widetilde{s}_i > 0$, $p_i\left(\widetilde{s}_i, \mathbf{0}_{-\mathbf{i}}\right) = 1$; the inequality in (A1) follows from (i) the assumption that $p_i(\mathbf{0}) < 1$ and (ii) the fact that $C\left[\widetilde{s}_i, 1\right]$ can be made arbitrarily small by choosing a $\widetilde{s}_i$ close enough to zero. Moreover, π_i is continuous at $\left(\widetilde{s}_i, \mathbf{0}_{-\mathbf{i}}\right)$. Therefore, (A1) implies that for every $\epsilon > 0$ and for all $\mathbf{s}'_{-\mathbf{i}}$ in some open neighborhood of $\mathbf{0}_{-\mathbf{i}}$, we have $\pi_i\left(\widetilde{s}_i, \mathbf{s}'_{-\mathbf{i}}\right) \geq p_i(\mathbf{0})\, v_i - \epsilon$.

The *graph* of the game is defined as $\Gamma = \{(\mathbf{s}, \pi_1, \cdots, \pi_n) \in S^n \times \mathbb{R}^n \mid \pi_i(\mathbf{s}) = \pi_i, \forall i\}$. The closure of Γ is denoted by $\overline{\Gamma}$. The *frontier* of Γ, denoted by $\mathrm{Fr}\, \Gamma$, is defined as the set of points that are in $\overline{\Gamma}$ but not in Γ. In order to prove that the game is wrusc, we must show that for any $(\mathbf{s}, \beta_1, \ldots, \beta_n)$ in the frontier of the game, there is a player i and $\widetilde{s}_i$ such that $\pi_i\left(\widetilde{s}_i, \mathbf{s}'_{-\mathbf{i}}\right) > \beta_i$. The game is indeed wrusc. To verify this, first note that, since the origin is the only point of discontinuity, any point in $\mathrm{Fr}\, \Gamma$ must be of the form $(\mathbf{0}, \gamma_1 v_1, \cdots, \gamma_n v_n)$, where for some $\mathbf{s}^\tau \to \mathbf{0}$ and every i, we have $\lim_{\tau\to\infty} \pi_i(\mathbf{s}^\tau) = \gamma_i v_i$. We must also have $\sum_{i=1}^n \gamma_i = 1$. Hence, for some i, $\gamma_i < 1$. Suppose, without loss of generality, that $\gamma_1 < 1$. Because $\lim_{s_1\to 0} \pi_i(s_1, \mathbf{0}_{-\mathbf{1}}) = v_1$, there exists $\widetilde{s}_i > 0$ such that $\pi_i\left(\widetilde{s}_i, \mathbf{0}_{-\mathbf{1}}\right) > \gamma_1 v_1$.

To prove the proposition, it remains to show that, under the conditions stated there, player i's payoff function is quasiconcave in s_i. I will do this by showing that $\frac{\partial^2 \pi_i}{\partial s_i^2} < 0$ at any point where $\frac{\partial \pi_i}{\partial s_i} = 0$. From the analysis in the main text, it follows that we can write the derivative of contestant i's payoff function with respect to s_i as $\frac{\partial \pi_i}{\partial s_i} = [v_i - Y(s_i, p_i(\mathbf{s}))] \frac{\partial p_i}{\partial s_i} - C_1(s_i, p_i)$. Differentiating again yields

$$\frac{\partial^2 \pi_i}{\partial s_i^2} = -\left[Y_1(s_i, p_i) + Y_2(s_i, p_i) \frac{\partial p_i}{\partial s_i}\right] \frac{\partial p_i}{\partial s_i} + [v_i - Y(s_i, p_i)] \frac{\partial^2 p_i}{\partial s_i^2} - C_{11}(s_i, p_i) - C_{12}(s_i, p_i) \frac{\partial p_i}{\partial s_i}.$$

[1]The proof below that the hybrid contest has those two properties will follow the proof in Example 3 of Bagh and Jofre (2006) fairly closely.

Now note that $C_{12}(s_i, p_i) = C_{21}(s_i, p_i) = Y_1(s_i, p_i)$. For a value of s_i for which $\frac{\partial \pi_i}{\partial s_i} = 0$ holds, we also have $v_i - Y(s_i, p_i) = \frac{C_1(s_i,p_i)}{\partial p_i/\partial s_i}$. Moreover, $C_1(s_i, p_i) = \left[p_i + h\left(\frac{1}{p_i}\right)\right] Y_1(s_i, p_i)$ and

$$C_{11}(s_i, p_i) = \left[p_i + h\left(\frac{1}{p_i}\right)\right] Y_{11}(s_i, p_i) = \frac{1-t}{ts_i}\left[p_i + h\left(\frac{1}{p_i}\right)\right] Y_1(s_i, p_i)$$

(cf. (6) and (8)). Therefore, evaluated at a value of s_i where $\frac{\partial \pi_i}{\partial s_i} = 0$, the second-derivative can be written

$$\frac{\partial^2 \pi_i}{\partial s_i^2}\Big|_{\frac{\partial \pi_i}{\partial s_i}=0} = -\left[2Y_1(s_i, p_i) + Y_2(s_i, p_i)\frac{\partial p_i}{\partial s_i}\right]\frac{\partial p_i}{\partial s_i} + \left[\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i} - \frac{1-t}{ts_i}\right]\left[p_i + h\left(\frac{1}{p_i}\right)\right] Y_1(s_i, p_i). \quad \text{(A2)}$$

The expression in (A2) is strictly negative if and only if

$$\left[2\frac{Y_1(s_i, p_i)\, s_i}{Y(s_i, p_i)} + \frac{Y_2(s_i, p_i)\, p_i}{Y(s_i, p_i)}\frac{\partial p_i}{\partial s_i}\frac{s_i}{p_i}\right]\frac{\partial p_i}{\partial s_i} > \left[\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i} - \frac{1-t}{ts_i}\right]\left[p_i + h\left(\frac{1}{p_i}\right)\right]\frac{Y_1(s_i, p_i)\, s_i}{Y(s_i, p_i)}. \quad \text{(A3)}$$

Now note that $\frac{Y_1(s_i,p_i)s_i}{Y(s_i,p_i)} = \frac{1}{t}$ and

$$\begin{aligned}\frac{Y_2(s_i, p_i)\, p_i}{Y(s_i, p_i)} &= -\frac{1}{t}(s_i)^{\frac{1}{t}}\left[f\left(h\left(\frac{1}{p_i}\right), 1\right)\right]^{-\frac{1}{t}-1} f_1\left[h\left(\frac{1}{p_i}\right), 1\right] h'\left(\frac{1}{p_i}\right)\left(\frac{-1}{p_i^2}\right) \times p_i\left[\frac{s_i}{f(h(1/p_i), 1)}\right]^{-\frac{1}{t}} \\ &= \frac{1}{t}\frac{f_1\left[h\left(\frac{1}{p_i}\right), 1\right] h\left(\frac{1}{p_i}\right)}{f\left(h\left(\frac{1}{p_i}\right), 1\right)} \times \frac{h'\left(\frac{1}{p_i}\right)\frac{1}{p_i}}{h\left(\frac{1}{p_i}\right)} = -\frac{\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)}{t}.\end{aligned}$$

Inequality (A3) can therefore be written as

$$\left[\frac{2}{t} - \frac{\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)\varepsilon_i(\mathbf{s})}{t}\right]\frac{\partial p_i}{\partial s_i} > \left[\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i} - \frac{1-t}{ts_i}\right][p_i + h(1/p_i)]\frac{1}{t}$$

or, equivalently, as

$$\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)\varepsilon_i(\mathbf{s}) < 2 - \left[\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i} - \frac{1-t}{ts_i}\right]\frac{\left[p_i + h\left(\frac{1}{p_i}\right)\right]}{\partial p_i/\partial s_i}. \quad \text{(A4)}$$

The last term in the above inequality is strictly negative for all $t \le 1$. Therefore, a sufficient condition for (A4) to hold is that $\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)\varepsilon_i(\mathbf{s}) \le 2$. This proves the claim for part (i) of Assumption 1. In order to prove the claim for part (ii), note that the derivative of the CSF in (9) can be written as $\frac{\partial p_i}{\partial s_i} = rp_i(1-p_i)/s_i$, and the second-derivative is given by $\frac{\partial^2 p_i}{\partial s_i^2} = rp_i(1-p_i)[r(1-2p_i)-1]/s_i^2$. Thus, the term in square brackets in (A4) becomes

$$\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i} - \frac{1-t}{ts_i} = \frac{r(1-2p_i)-1}{s_i} - \frac{1-t}{ts_i} = \frac{tr(1-2p_i)-1}{ts_i},$$

which is non-positive for all p_i if $tr \le 1$. Moreover, $\varepsilon_i(\mathbf{s}) = r(1-p_i) \le r$. Hence the result follows. Finally consider part (iii). The additional Cobb-Douglas assumption means that we can write the last term in (A4) as

$$\left[\frac{\partial^2 p_i/\partial s_i^2}{\partial p_i/\partial s_i} - \frac{1-t}{ts_i}\right]\left[\frac{p_i + h\left(\frac{1}{p_i}\right)}{\partial p_i/\partial s_i}\right] = \left[\frac{tr(1-2p_i)-1}{ts_i}\right]\left[\frac{p_i + \frac{\alpha}{\beta}p_i}{rp_i(1-p_i)/s_i}\right] = \frac{tr(1-2p_i)-1}{r\beta(1-p_i)}.$$

Moreover, the left-hand side of (A4) simplifies to $\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)\varepsilon_i(\mathbf{s}) = \alpha r(1-p_i)$. Inequality (A4) therefore becomes

$$\alpha r(1-p_i) < 2 - \frac{tr(1-2p_i)-1}{r\beta(1-p_i)} \Leftrightarrow \alpha\beta r^2(1-p_i)^2 < 2r\beta(1-p_i) - tr(1-2p_i) + 1.$$

This inequality is most stringent at $p_i = 0$ (and it is strictly less stringent for higher values of p_i). It therefore suffices if the inequality holds weakly when evaluated at $p_i = 0$:

$$\alpha\beta r^2 \leq 2r\beta - tr + 1 = r\beta - \alpha r + 1 \Leftrightarrow 0 \leq r\beta(1-\alpha r) + 1 - \alpha r \Leftrightarrow \alpha r \leq 1,$$

which gives us the result. □

3.2. Proof of Proposition 2

First consider the claim in the last sentence of the proposition. To verify that $\mathbf{s} = \mathbf{0}$ cannot be a Nash equilibrium, note that $\pi_i(\mathbf{0}) = p_i(\mathbf{0})\,v_i < v_i$. Moreover, by assumption we have $p_i(s_i, \mathbf{0}_{-\mathbf{i}}) = 1$ for any $s_i > 0$. Therefore, if contestant i were to deviate from $s_i = 0$ to some $s_i > 0$, her payoff would equal $\pi_i(s_i, \mathbf{0}_{-\mathbf{i}}) = v_i - C[s_i, 1]$. But $C[s_i, 1]$ can be made arbitrarily small by choosing an s_i close enough to zero and, hence, for such an s_i the deviation is profitable.

We can thus conclude that in any equilibrium, $\mathbf{s} \neq \mathbf{0}$. Moreover, we know that each contestant's payoff function is continuous and differentiable for all $\mathbf{s} \neq \mathbf{0}$. In addition, Assumption 1 takes care of the second-order condition. It follows that the analysis in the text that precedes the first-order condition (10) is valid and that this first-order condition indeed characterizes the equilibria of the model. □

3.3. Proof of Proposition 3

Under symmetry, the expression in (5) can be written as $x^* = h(n)\,y^*$. Plugging this into (11) and then solving for y^* yields (12). The solution to this linear equation system is unique, and so the model has a unique equilibrium within the family of symmetric equilibria. The expression for s^* is obtained by plugging $h(1/p_i) = h(n)$ and $y_i = y^*$ into the equality $s_i = y_i^t f[h(1/p_i), 1]$, which was derived in footnote 13 in Lagerlöf (2020). □

3.4. Proof of Proposition 4

The claims about v, t, and α are straightforward to verify, so the calculations are omitted. Consider the condition for y^* to be strictly increasing in n. Differentiating the expression for y^* in (12), we have

$$\frac{\partial y^*}{\partial n} = \frac{\widehat{\varepsilon}\,'(n)\left[1 + nh(n) + t\widehat{\varepsilon}(n)\right] - \widehat{\varepsilon}(n)\left[h(n) + nh'(n) + t\widehat{\varepsilon}\,'(n)\right]}{(tv)^{-1}\left[1 + nh(n) + t\widehat{\varepsilon}(n)\right]^2} > 0$$
$$\Leftrightarrow \widehat{\varepsilon}\,'(n)\left[1 + nh(n)\right] > \widehat{\varepsilon}(n)\left[h(n) + nh'(n)\right].$$

Differentiating (14), we obtain $\widehat{\varepsilon}\,'(n) = r/n^2$. Using this and (14) in the second inequality above yields $1 + nh(n) > n(n-1)\left[h(n) + nh'(n)\right] = n(n-1)\,h(n)\left[1 - \sigma(n)\right]$, which simplifies to the condition in (16). Next consider to the condition for x^* to be strictly decreasing in n. We have $x^* = h(n)\,y^*$, where y^* is given by (12). Differentiating yields

$$\frac{\partial x^*}{\partial n} = \frac{\left[\widehat{\varepsilon}\,'(n)\,h(n) + \widehat{\varepsilon}(n)\,h'(n)\right]\left[1 + nh(n) + t\widehat{\varepsilon}(n)\right] - \widehat{\varepsilon}(n)\,h(n)\left[h(n) + nh'(n) + t\widehat{\varepsilon}\,'(n)\right]}{(tv)^{-1}\left[1 + nh(n) + t\widehat{\varepsilon}(n)\right]^2} < 0 \Leftrightarrow$$

$$\left[\widehat{\varepsilon}'(n)\,h(n)+\widehat{\varepsilon}(n)\,h'(n)\right][1+nh(n)]+t\left[\widehat{\varepsilon}(n)\right]^2 h'(n)<\widehat{\varepsilon}(n)\,h(n)\left[h(n)+nh'(n)\right].$$

Dividing through by $\widehat{\varepsilon}(n)$ and using $\widehat{\varepsilon}'(n)/\widehat{\varepsilon}(n)=1/[n(n-1)]$, the inequality simplifies to

$$\left[\frac{h(n)}{n(n-1)}+h'(n)\right][1+nh(n)]+t\widehat{\varepsilon}(n)\,h'(n)<h(n)\left[h(n)+nh'(n)\right]$$

or, equivalently, $h(n)[1-(n-1)\sigma(n)][1+nh(n)]-t\widehat{\varepsilon}(n)(n-1)h(n)\sigma(n)<n(n-1)[h(n)]^2[1-\sigma(n)]$, which simplifies to the condition in (16). Finally consider the claim that $\sigma(n)\geq 1$ is sufficient for both conditions in (16) to hold. Substituting $\frac{n-2}{n-1}$ (which is smaller than unity) for $\sigma(n)$ in the condition for $\frac{\partial y^*}{\partial n}$ in (16) yields

$$\frac{n-2}{n-1}>\frac{n(n-2)h(n)-1}{n(n-1)h(n)}\Leftrightarrow(n-2)nh(n)>n(n-2)h(n)-1\Leftrightarrow 1>0,$$

which always holds. And substituting 1 for $\sigma(n)$ in the condition for $\frac{\partial x^*}{\partial n}$ in (16) yields

$$1>-\frac{n(n-2)h(n)-1}{(n-1)[1+t\widehat{\varepsilon}(n)]}\Leftrightarrow(n-1)[1+t\widehat{\varepsilon}(n)]>-n(n-2)h(n)+1$$
$$\Leftrightarrow n-2+t\widehat{\varepsilon}(n)(n-1)>-n(n-2)h(n),$$

which again always holds. □

3.5. Proof of Proposition 5

The first equality in (19) follows immediately from (11) and (18), since $nC\left[s^*,\frac{1}{n}\right]=y^*+nx^*$. To verify the second equality, note that

$$\left(1-\frac{y^*}{v}\right)R^A=\left(1-\frac{R^A/v}{1+nh(n)+R^A/v}\right)R^A=\frac{R^A[1+nh(n)]v}{[1+nh(n)]v+R^A}=\left[\frac{1}{[1+nh(n)]v}+\frac{1}{R^A}\right]^{-1},$$

where the first equality uses (12) and (18). The claim that $R^H<R^A$ follows immediately from (19) and $y^*>0$. The claims about v, t, and α are straightforward to verify, so the calculations are omitted. Consider the condition for R^{H} to be weakly increasing in n. By differentiating the right-most expression for R^{H} in (19), we have

$$\frac{\partial R^H}{\partial n}=-\left[\frac{1}{v[1+nh(n)]}+\frac{1}{R^A}\right]^{-2}\left[-\frac{h(n)+nh'(n)}{v[1+nh(n)]^2}-\frac{\partial R^A/\partial n}{(R^A)^2}\right]\geq 0\Leftrightarrow\frac{\partial R^A/\partial n}{(R^A)^2}\geq-\frac{h(n)[1-\sigma(n)]}{v[1+nh(n)]^2}.$$

By differentiating the expression in (18) (also using (14)), we obtain $\partial R^A/\partial n=tvr/n^2$. By plugging this and the expression for R^{A} in (18) (combined with (14)) into the above inequality and then rewriting, we have

$$rt(n-1)^2[\sigma(n)-1]h(n)\leq[1+nh(n)]^2=1+2nh(n)+n^2h(n)^2\Leftrightarrow h(n)^2-Kh(n)\geq-\frac{1}{n^2},\quad\text{(A5)}$$

where K is defined in Proposition 5 in Lagerlöf (2020). Since $h(n)>0$, this inequality always holds if $K\leq 0$. Suppose $K>0$. Then the left-hand side is negative for all $h(n)<K$, and it is minimized at $h(n)=K/2$. Evaluating inequality (A5) at $h(n)=K/2$ yields

$$-\frac{K^2}{4}\geq-\frac{1}{n^2}\Leftrightarrow K\leq\frac{2}{n}\Leftrightarrow\sigma(n)\leq 1+\frac{4n}{tr(n-1)^2}.\quad\text{(A6)}$$

Thus if (A6) holds, then (A5) is always satisfied. If (A6) is violated, then also (A5) is violated for values of $h(n)$ between the two roots of (A5). Solving for these roots (by completing the square), we have:

$$h(n)^2 - Kh(n) = -\frac{1}{n^2} \Leftrightarrow \left[h(n) - \frac{K}{2}\right]^2 = \frac{n^2K^2}{4n^2} - \frac{4}{4n^2} \Leftrightarrow h(n) = \frac{K}{2} \pm \frac{1}{2n}\sqrt{n^2K^2 - 4}.$$

Thus, total expenditures are increasing in n if and only if (i) inequality (A6) holds or (ii) inequality (A6) is violated and $h(n) \notin (\Xi_L, \Xi_H)$, where Ξ_L and Ξ_H are defined in Proposition 5 in Lagerlöf (2020). $\square$

3.6. Proof of Proposition 7

The first-order condition in (10) can be written as

$$(v_i - y_i^*)\frac{rp_i^*(1-p_i^*)}{s_i^*} = \frac{1}{ts_i^*}C(s_i^*, p_i^*) \Leftrightarrow rt(v_i - y_i^*)p_i^*(1-p_i^*) = \left[p_i^* + h\left(\frac{1}{p_i^*}\right)\right]y_i^*, \quad \text{(A7)}$$

where the relationships $C_1(s_i^*, p_i^*) = \frac{1}{ts_i^*}C(s_i^*, p_i^*)$ and $C(s_i^*, p_i^*) = \left[p_i^* + h\left(\frac{1}{p_i^*}\right)\right]y_i^*$ were used. By solving (A7) for y_i^*, we obtain (23). The remaining parts of the characterization claim are either shown in the main text or straightforward. It remains to prove the uniqueness claim. Note that the equilibrium is defined recursively: The only endogenous variable in the equality $\Upsilon(p_1) = 0$ is p_1; moreover, given a value of p_1^*, the winner-pay investments y_1^* and y_2^* are uniquely determined by (23). To prove the claim, it thus suffices to show that if $r\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right) \leq 1$ for all $p_i \in [0,1]$, then the equation $\Upsilon(p_1) = 0$ has a unique root. A sufficient condition for this, in turn, is that $\Upsilon(p_1)$ is strictly increasing (by Proposition 1 in Lagerlöf (2020), we know that the equation has at least one root). The equation $\Upsilon(p_1) = 0$ can equivalently be written as $\widehat{\Upsilon}(p_1) = 0$, where

$$\begin{aligned}\widehat{\Upsilon}(p_1) = {} & \ln\left[\frac{w_2 v_2^{rt}}{w_1 v_1^{rt}}\right] + \ln p_1 + r\ln f\left[h\left(\frac{1}{1-p_1}\right), 1\right] + rt\ln\left[rtp_1(1-p_1) + p_1 + h\left(\frac{1}{p_1}\right)\right] \\ & - \ln(1-p_1) - r\ln f\left[h\left(\frac{1}{p_1}\right), 1\right] - rt\ln\left[rtp_1(1-p_1) + 1 - p_1 + h\left(\frac{1}{1-p_1}\right)\right].\end{aligned}$$

Differentiating with respect to p_1 yields

$$\begin{aligned}\widehat{\Upsilon}'(p_1) = {} & \frac{1}{p_1} + \frac{rf_1\left[h\left(\frac{1}{1-p_1}\right),1\right]h'\left(\frac{1}{1-p_1}\right)\frac{1}{(1-p_1)^2}}{f\left[h\left(\frac{1}{1-p_1}\right),1\right]} + \frac{rt\left[rt(1-2p_1) + 1 - h'\left(\frac{1}{p_1}\right)\frac{1}{p_1^2}\right]}{rtp_1(1-p_1) + p_1 + h\left(\frac{1}{p_1}\right)} \\ & + \frac{1}{1-p_1} + \frac{rf_1\left[h\left(\frac{1}{p_1}\right),1\right]h'\left(\frac{1}{p_1}\right)\frac{1}{p_1^2}}{f\left[h\left(\frac{1}{p_1}\right),1\right]} - \frac{rt\left[rt(1-2p_1) - 1 + h'\left(\frac{1}{1-p_1}\right)\frac{1}{(1-p_1)^2}\right]}{rtp_1(1-p_1) + 1 - p_1 + h\left(\frac{1}{1-p_1}\right)} \\ = {} & \frac{1}{p_1(1-p_1)} - \frac{r\eta\left(\frac{1}{1-p_1}\right)\sigma\left(\frac{1}{1-p_1}\right)}{1-p_1} - \frac{r\eta\left(\frac{1}{p_1}\right)\sigma\left(\frac{1}{p_1}\right)}{p_1} \\ & + \frac{rt\left[rt(1-2p_1) + 1 - h'\left(\frac{1}{p_1}\right)\frac{1}{p_1^2}\right]}{rtp_1(1-p_1) + p_1 + h\left(\frac{1}{p_1}\right)} - \frac{rt\left[rt(1-2p_1) - 1 + h'\left(\frac{1}{1-p_1}\right)\frac{1}{(1-p_1)^2}\right]}{rtp_1(1-p_1) + 1 - p_1 + h\left(\frac{1}{1-p_1}\right)}. \quad \text{(A8)}\end{aligned}$$

Under the assumption that $r\eta\left(\frac{1}{p_i}\right)\sigma\left(\frac{1}{p_i}\right)\leq 1$ for all p_i, the first line of (A8) is non-negative. The second line of (A8) is strictly positive if

$$\frac{rt\left[rt\left(1-2p_1\right)\right]}{rtp_1\left(1-p_1\right)+p_1+h\left(\frac{1}{p_1}\right)}-\frac{rt\left[rt\left(1-2p_1\right)\right]}{rtp_1\left(1-p_1\right)+1-p_1+h\left(\frac{1}{1-p_1}\right)}\geq 0\Leftrightarrow$$

$$\left(1-2p_1\right)\left[1-p_1+h\left(\frac{1}{1-p_1}\right)-p_1-h\left(\frac{1}{p_1}\right)\right]=\left(1-2p_1\right)^2+\left(1-2p_1\right)\int_{\frac{1}{p_1}}^{\frac{1}{1-p_1}}h'\left(z\right)dz\geq 0.$$

But, since $h'<0$, the last inequality holds for all $p_1\in[0,1]$ (with equality if, and only if, $p_1=0.5$). □

3.7. Proof of Proposition 8

Under the assumption that $v_1=v_2$, (A7) simplifies to $rt\left(v-y_i^*\right)p_i^*\left(1-p_i^*\right)=\left[p_i^*+h\left(\frac{1}{p_i^*}\right)\right]y_i^*$. Since the expression in square brackets is strictly increasing in p_i^* and since $p_1^*\left(1-p_1^*\right)=p_2^*\left(1-p_2^*\right)$, the equality implies that $p_1^*>p_2^*\Leftrightarrow y_1^*<y_2^*$. Moreover, since $\left[p_i^*+h\left(\frac{1}{p_i^*}\right)\right]y_i^*=C\left(s_i^*,p_i^*\right)$, it also implies that $y_1^*<y_2^*\Leftrightarrow C\left(s_1^*,p_1^*\right)>C\left(s_2^*,p_2^*\right)$. This proves part (i). Next turn to part (ii). By taking logs of the three equations (23) and $\Upsilon(p_1^*)=0$, we have

$$\ln r+\ln t+\ln\left(v_1-y_1^*\right)+\ln p_1^*+\ln\left(1-p_1^*\right)=\ln\left[p_1^*+h\left(\frac{1}{p_1^*}\right)\right]+\ln y_1^*, \tag{A9}$$

$$\ln r+\ln t+\ln\left(v_2-y_2^*\right)+\ln p_1^*+\ln\left(1-p_1^*\right)=\ln\left[1-p_1^*+h\left(\frac{1}{1-p_1^*}\right)\right]+\ln y_2^*, \tag{A10}$$

$$\ln p_1^*+\ln w_2+r\ln f\left[h\left(\frac{1}{1-p_1^*}\right),1\right]+rt\ln y_2^*=\ln\left(1-p_1^*\right)+\ln w_1+r\ln f\left[h\left(\frac{1}{p_1^*}\right),1\right]+rt\ln y_1^*. \tag{A11}$$

Now set $v_1=v_2=v$ in (A9) and (A10). Then differentiate (A9) with respect to w_1:

$$-\frac{1}{v-y_1^*}\frac{\partial y_1^*}{\partial w_1}+\left[\frac{1}{p_1^*}-\frac{1}{1-p_1^*}\right]\frac{\partial p_1^*}{\partial w_1}=\frac{1-h'\left(\frac{1}{p_1^*}\right)\frac{1}{\left(p_1^*\right)^2}}{p_1^*+h\left(\frac{1}{p_1^*}\right)}\frac{\partial p_1^*}{\partial w_1}+\frac{1}{y_1^*}\frac{\partial y_1^*}{\partial w_1}\Leftrightarrow$$

$$\left[\frac{1-2p_1^*}{p_1^*\left(1-p_1^*\right)}\right]\frac{\partial p_1^*}{\partial w_1}=\frac{\frac{1}{p_1^*}\left[p_1^*+\sigma\left(\frac{1}{p_1^*}\right)h\left(\frac{1}{p_1^*}\right)\right]}{p_1^*+h\left(\frac{1}{p_1^*}\right)}\frac{\partial p_1^*}{\partial w_1}+\frac{v}{y_1^*\left(v-y_1^*\right)}\frac{\partial y_1^*}{\partial w_1}\Leftrightarrow$$

$$\left[\frac{1-2p_1^*}{1-p_1^*}\right]\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*}=\frac{p_1^*+\sigma\left(\frac{1}{p_1^*}\right)h\left(\frac{1}{p_1^*}\right)}{p_1^*+h\left(\frac{1}{p_1^*}\right)}\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*}+\frac{v}{v-y_1^*}\frac{\partial y_1^*}{\partial w_1}\frac{w_1}{y_1^*}\Leftrightarrow$$

$$\left[\frac{1-2p_1^*-A_1\left(1-p_1^*\right)}{1-p_1^*}\right]\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*}=\frac{v}{v-y_1^*}\frac{\partial y_1^*}{\partial w_1}\frac{w_1}{y_1^*}, \tag{A12}$$

where $A_1\stackrel{\text{def}}{=}\left[p_1^*+\sigma\left(\frac{1}{p_1^*}\right)h\left(\frac{1}{p_1^*}\right)\right]/\left[p_1^*+h\left(\frac{1}{p_1^*}\right)\right]$. Similarly, by differentiating (A10) with respect to w_1 and then rewriting, we obtain the following equality (the derivation is very similar to the one above):

$$\left[\frac{1-2p_1^*+A_2p_1^*}{1-p_1^*}\right]\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*}=\frac{v}{v-y_2^*}\frac{\partial y_2^*}{\partial w_1}\frac{w_1}{y_2^*}, \tag{A13}$$

where $A_2 \overset{\text{def}}{=} \left[1 - p_1^* + \sigma\left(\frac{1}{1-p_1^*}\right) h\left(\frac{1}{1-p_1^*}\right)\right] / \left[1 - p_1^* + h\left(\frac{1}{1-p_1^*}\right)\right]$. Finally differentiate (A11) with respect to w_1:

$$\frac{1}{p_1^*}\frac{\partial p_1^*}{\partial w_1} + \frac{rf_1\left[h\left(\frac{1}{1-p_1^*}\right),1\right] h'\left(\frac{1}{1-p_1^*}\right)\frac{1}{(1-p_1^*)^2}}{f\left[h\left(\frac{1}{1-p_1^*}\right),1\right]}\frac{\partial p_1^*}{\partial w_1} + rt\frac{1}{y_2^*}\frac{\partial y_2^*}{\partial w_1}$$

$$= -\frac{1}{1-p_1^*}\frac{\partial p_1^*}{\partial w_1} + \frac{1}{w_1} - \frac{rf_1\left[h\left(\frac{1}{p_1^*}\right),1\right] h'\left(\frac{1}{p_1^*}\right)\frac{1}{(p_1^*)^2}}{f\left[h\left(\frac{1}{p_1^*}\right),1\right]}\frac{\partial p_1^*}{\partial w_1} + rt\frac{1}{y_1^*}\frac{\partial y_1^*}{\partial w_1} \Leftrightarrow$$

$$\frac{1}{p_1^*(1-p_1^*)}\frac{\partial p_1^*}{\partial w_1} - \frac{r\eta\left(\frac{1}{1-p_1^*}\right)\sigma\left(\frac{1}{1-p_1^*}\right)}{1-p_1^*}\frac{\partial p_1^*}{\partial w_1} + rt\frac{1}{y_2^*}\frac{\partial y_2^*}{\partial w_1} = \frac{1}{w_1} + \frac{r\eta\left(\frac{1}{p_1^*}\right)\sigma\left(\frac{1}{p_1^*}\right)}{p_1^*}\frac{\partial p_1^*}{\partial w_1} + rt\frac{1}{y_1^*}\frac{\partial y_1^*}{\partial w_1} \Leftrightarrow$$

$$\left[\frac{1 - r\eta\left(\frac{1}{1-p_1^*}\right)\sigma\left(\frac{1}{1-p_1^*}\right)p_1^* - r\eta\left(\frac{1}{p_1^*}\right)\sigma\left(\frac{1}{p_1^*}\right)(1-p_1^*)}{p_1^*(1-p_1^*)}\right]\frac{\partial p_1^*}{\partial w_1} + rt\frac{1}{y_2^*}\frac{\partial y_2^*}{\partial w_1} = \frac{1}{w_1} + rt\frac{1}{y_1^*}\frac{\partial y_1^*}{\partial w_1} \Leftrightarrow$$

$$\frac{1-B}{1-p_1^*}\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*} + rt\frac{\partial y_2^*}{\partial w_1}\frac{w_1}{y_2^*} = 1 + rt\frac{\partial y_1^*}{\partial w_1}\frac{w_1}{y_1^*}, \tag{A14}$$

where

$$B \overset{\text{def}}{=} r\eta\left(\frac{1}{1-p_1^*}\right)\sigma\left(\frac{1}{1-p_1^*}\right)p_1^* + r\eta\left(\frac{1}{p_1^*}\right)\sigma\left(\frac{1}{p_1^*}\right)(1-p_1^*).$$

Now evaluate the three equations (A12)-(A14) above at $w_1 = w_2$ (all expressions below, until the end of the proof, are evaluated at symmetry, even though this is not everywhere explicitly indicated):

$$-A\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*} = \frac{v}{v-y^*}\frac{\partial y_1^*}{\partial w_1}\frac{w_1}{y^*}, \quad A\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*} = \frac{v}{v-y^*}\frac{\partial y_2^*}{\partial w_1}\frac{w_1}{y^*}, \quad 2(1-B)\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*} + rt\frac{\partial y_2^*}{\partial w_1}\frac{w_1}{y^*} = 1 + rt\frac{\partial y_1^*}{\partial w_1}\frac{w_1}{y^*},$$

where $A \overset{\text{def}}{=} [1 + 2\sigma(2)h(2)] / [1 + 2h(2)]$, $B \overset{\text{def}}{=} r\eta(2)\sigma(2)$, and y^* is the common value of y_1^* and y_2^* when evaluated at symmetry. Solving this equation system yields

$$\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*} = \frac{1}{2\left[1 - B + rtA\frac{v-y^*}{v}\right]}, \quad \frac{\partial y_1^*}{\partial w_1}\frac{w_1}{y^*} = -\frac{A\frac{v-y^*}{v}}{2\left[1 - B + rtA\frac{v-y^*}{v}\right]}, \quad \frac{\partial y_2^*}{\partial w_1}\frac{w_1}{y^*} = \frac{A\frac{v-y^*}{v}}{2\left[1 - B + rtA\frac{v-y^*}{v}\right]}. \tag{A15}$$

From these results, most of the comparative statics claims follow. To prove the only remaining claim, the one about the all-pay investments, note that the relationship $x_1^* = h\left(\frac{1}{p_1^*}\right)y_1^*$ implies that (at symmetry)

$$\frac{\partial x_1^*}{\partial w_1}\frac{w_1}{x^*} = \sigma(2)\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*} + \frac{\partial y_1^*}{\partial w_1}\frac{w_1}{y^*} = \frac{\sigma(2) - A\frac{v-y^*}{v}}{2\left[1 - B + rtA\frac{v-y^*}{v}\right]}, \tag{A16}$$

where x^* is the common value of x_1^* and x_2^* when evaluated at symmetry. Thus,

$$\frac{\partial x_1^*}{\partial w_1} > 0 \Leftrightarrow \sigma(2) > A\frac{v-y^*}{v} = \frac{1 + 2\sigma(2)h(2)}{1 + 2h(2) + \frac{tr}{2}} \Leftrightarrow \sigma(2) > \frac{1}{1 + \frac{tr}{2}} = \frac{2}{2 + tr},$$

where the first equality is obtained by using (23). Similarly, from the relationship $x_2^* = h\left(\frac{1}{1-p_1^*}\right)y_2^*$ we have (at symmetry)

$$\frac{\partial x_2^*}{\partial w_1}\frac{w_1}{x^*} = -\sigma(2)\frac{\partial p_1^*}{\partial w_1}\frac{w_1}{p_1^*} + \frac{\partial y_2^*}{\partial w_1}\frac{w_1}{y^*} = \frac{-\sigma(2) + A\frac{v-y^*}{v}}{2\left[1 - B + rtA\frac{v-y^*}{v}\right]}, \tag{A17}$$

which has the opposite sign to (A16). □

3.8. Proof of Proposition 9

Equation (A7) can be restated as $rt\left(v_i - y_i^*\right) p_i^* \left(1 - p_i^*\right) = C\left(s_i^*, p_i^*\right)$. Since $p_1^*\left(1 - p_1^*\right) = p_2^*\left(1 - p_2^*\right)$, the equality implies that $v_1 - y_1^* > v_2 - y_2^* \Leftrightarrow C\left(s_1^*, p_1^*\right) > C\left(s_2^*, p_2^*\right)$. We can also write (A7) as

$$rt\left(\frac{v_i}{y_i^*} - 1\right) p_i^*\left(1 - p_i^*\right) = p_i^* + h\left(\frac{1}{p_i^*}\right).$$

Since the right-hand side is strictly increasing in p_i^* and since $p_1^*\left(1 - p_1^*\right) = p_2^*\left(1 - p_2^*\right)$, the equality implies that $p_1^* > p_2^* \Leftrightarrow \frac{y_1^*}{v_1} < \frac{y_2^*}{v_2}$. □

3.9. Proof of Proposition 10

The Cobb-Douglas specification (Assumption 5 in Lagerlöf, 2020) implies $h(m) = \frac{\alpha}{\beta} m^{-1}$. By using this in (23), we get

$$v_1 - y_1^* = \frac{v_1\left(p_1 + \frac{\alpha}{\beta} p_1\right)}{rtp_1\left(1 - p_1\right) + p_1 + \frac{\alpha}{\beta} p_1} = \frac{v_1 \frac{t}{\beta}}{rt\left(1 - p_1\right) + \frac{t}{\beta}} = \frac{v_1}{r\beta\left(1 - p_1\right) + 1}, \tag{A18}$$

$$v_2 - y_2^* = \frac{v_2\left[1 - p_1 + \frac{\alpha}{\beta}\left(1 - p_1\right)\right]}{rtp\left(1 - p_1\right) + 1 - p_1 + \frac{\alpha}{\beta}\left(1 - p_1\right)} = \frac{v_2 \frac{t}{\beta}}{rtp_1 + \frac{t}{\beta}} = \frac{v_2}{r\beta p_1 + 1}. \tag{A19}$$

Moreover, it follows from (A7) that the expected total equilibrium expenditures can be written as $R^H = rtp_1\left(1 - p_1\right) \times \left[\left(v_1 - y_1^*\right) + \left(v_2 - y_2^*\right)\right]$. Plugging (A18) and (A19) into this expression yields the expression for R^H stated in (24). Next, taking logs of both sides of (24), we can write

$$\begin{aligned}\ln R^H \quad = \quad & \ln rt + \ln p_1 + \ln\left(1 - p_1\right) + \ln\left\{r\beta\left[p_1 v_1 + \left(1 - p_1\right) v_2\right] + v_1 + v_2\right\} \\ & - \ln\left[r\beta\left(1 - p_1\right) + 1\right] - \ln\left(r\beta p_1 + 1\right)\end{aligned}$$

Differentiating yields:

$$\begin{aligned}\frac{\partial \ln R^H}{\partial p_1} &= \frac{1}{p_1} - \frac{1}{1 - p_1} + \frac{r\beta\left(v_1 - v_2\right)}{r\beta\left[p_1 v_1 + \left(1 - p_1\right) v_2\right] + v_1 + v_2} + \frac{r\beta}{r\beta\left(1 - p_1\right) + 1} - \frac{r\beta}{r\beta p_1 + 1} \\ &= \frac{1 - 2p_1}{p_1\left(1 - p_1\right)} + \frac{r\beta\left(v_1 - v_2\right)}{r\beta\left[p_1 v_1 + \left(1 - p_1\right) v_2\right] + v_1 + v_2} + \frac{(r\beta)^2\left(2p_1 - 1\right)}{(r\beta)^2 p_1\left(1 - p_1\right) + r\beta + 1} \\ &= \frac{\left(1 - 2p_1\right)\left(r\beta + 1\right)}{p_1\left(1 - p_1\right)\left[(r\beta)^2 p_1\left(1 - p_1\right) + r\beta + 1\right]} + \frac{r\beta\left(v_1 - v_2\right)}{r\beta\left[p_1 v_1 + \left(1 - p_1\right) v_2\right] + v_1 + v_2} \stackrel{\text{def}}{=} F\left(p_1\right).\end{aligned} \tag{A20}$$

First consider the case $v_1 = v_2$. Then it is clear from inspection that (A20) is positive for $p_1 < \frac{1}{2}$ and negative for $p_1 > \frac{1}{2}$. Hence, $\widehat{p}_1 = \frac{1}{2}$. Next consider the case $v_1 > v_2$. The derivative w.r.t. p_1 of the first term in (A20) is strictly negative:

$$\frac{\partial T\left(p_1\right)}{\partial p_1} = \left(r\beta + 1\right) \frac{-2p_1\left(1 - p_1\right)\left[(r\beta)^2 p_1\left(1 - p_1\right) + r\beta + 1\right] - \left(1 - 2p_1\right)^2\left[2(r\beta)^2 p_1\left(1 - p_1\right) + r\beta + 1\right]}{p_1^2\left(1 - p_1\right)^2\left[(r\beta)^2 p_1\left(1 - p_1\right) + r\beta + 1\right]^2} < 0, \tag{A21}$$

where T is short-hand notation for the first term in (A20). Moreover, by inspection, the second term in (A20) is strictly decreasing in p_1. Therefore, $\partial^2 \ln R^H / \partial p_1^2 < 0$. Moreover, evaluated at $p_1 = \frac{1}{2}$, the expression in (A20) is strictly positive, whereas it approaches $-\infty$ as $p_1 \to 1$. It follows that $\widehat{p}_1 \in \left(\frac{1}{2}, 1\right)$. In particular, for any $v_1 \geq v_2$, $\widehat{p}_1$ is characterized by $F(\widehat{p}_1) = 0$.

One can verify that $F(p_1)$ is strictly increasing in v_1 and strictly decreasing in v_2. Hence, $\partial \widehat{p}_1 / \partial v_1 > 0$ and $\partial \widehat{p}_1 / \partial v_2 < 0$ (the former result will also follow from computations shown below). In order to do comparative statics w.r.t. $r\beta$, differentiate the first term of $F(p_1)$ w.r.t. $r\beta$:

$$\frac{(1-2p_1)}{p_1(1-p_1)} \frac{(r\beta)^2 p_1(1-p_1) + r\beta + 1 - (r\beta+1)\left[2r\beta p_1(1-p_1)+1\right]}{\left[(r\beta)^2 p_1(1-p_1) + r\beta + 1\right]^2}$$
$$= \frac{-(1-2p_1)}{p_1(1-p_1)} \frac{p_1(1-p_1) r\beta \left[2(r\beta+1) - r\beta\right]}{\left[(r\beta)^2 p_1(1-p_1) + r\beta + 1\right]^2} = -\frac{(1-2p_1) r\beta(r\beta+2)}{\left[(r\beta)^2 p_1(1-p_1) + r\beta + 1\right]^2}.$$

Then differentiate the second term of $F(p_1)$ w.r.t. $r\beta$:

$$(v_1 - v_2) \frac{r\beta\left[p_1 v_1 + (1-p_1) v_2\right] + v_1 + v_2 - r\beta\left[p_1 v_1 + (1-p_1) v_2\right]}{\left\{r\beta\left[p_1 v_1 + (1-p_1) v_2\right] + v_1 + v_2\right\}^2} = \frac{(v_1 - v_2)(v_1 + v_2)}{\left\{r\beta\left[p_1 v_1 + (1-p_1) v_2\right] + v_1 + v_2\right\}^2}.$$

Thus, if $v_1 = v_2$, then $F(p_1)$ is constant w.r.t. $r\beta$ and $\partial \widehat{p}_1 / \partial(r\beta) = 0$. And if $v_1 > v_2$, then $F(p_1)$ is strictly increasing in v_1 and $\partial \widehat{p}_1 / \partial(r\beta) > 0$.

Given the Cobb-Douglas specification in Assumption 5 in Lagerlöf (2020), the equation $\Upsilon(p_1^*) = 0$, which defines the equilibrium value of p_1, becomes

$$\frac{\frac{w_2 v_2^{rt}}{w_1 v_1^{rt}} p_1 \left[\left(\frac{\alpha}{\beta}\right)^\alpha (1-p_1)^\alpha\right]^r}{\left[rt p_1(1-p_1) + 1 - p_1 + \frac{\alpha}{\beta}(1-p_1)\right]^{rt}} = \frac{(1-p_1)\left[\left(\frac{\alpha}{\beta}\right)^\alpha p_1^\alpha\right]^r}{\left[rt p_1(1-p_1) + p_1 + \frac{\alpha}{\beta} p_1\right]^{rt}} \Leftrightarrow$$

$$\frac{\frac{w_2 v_2^{rt}}{w_1 v_1^{rt}} p_1 (1-p_1)^{r\alpha}}{(1-p_1)^{rt}\left(rt p_1 + 1 + \frac{\alpha}{\beta}\right)^{rt}} = \frac{p_1^{r\alpha}(1-p_1)}{p_1^{rt}\left[rt(1-p_1) + 1 + \frac{\alpha}{\beta}\right]^{rt}} \Leftrightarrow \frac{\frac{w_2 v_2^{rt}}{w_1 v_1^{rt}} p_1^{1+r\beta}}{\left(rt p_1 + \frac{t}{\beta}\right)^{rt}} = \frac{(1-p_1)^{1+r\beta}}{\left[rt(1-p_1) + \frac{t}{\beta}\right]^{rt}} \Leftrightarrow$$

$$w_1 = w_2 \left[\frac{p_1}{1-p_1}\right]^{1+r\beta} \left[\frac{r(1-p_1) + \frac{1}{\beta}}{r p_1 + \frac{1}{\beta}} \frac{v_2}{v_1}\right]^{rt} = w_2 \left[\frac{p_1}{1-p_1}\right]^{1+r\beta} \left[\frac{r\beta(1-p_1) + 1}{r\beta p_1 + 1} \frac{v_2}{v_1}\right]^{rt},$$

which gives us (25). The result that $\lim_{v_1 \to \infty} \widehat{p}_1 < 1$ follows from inspection of (A20): $F(\widehat{p}_1) = 0$ is inconsistent with $\lim_{v_1 \to \infty} \widehat{p}_1 = 1$. Similarly, the result that $\lim_{v_1 \to \infty} \widehat{w}_1 = 0$ follows from (25) and the fact that $\lim_{v_1 \to \infty} \widehat{p}_1 < 1$.

It remains to prove the last limit result stated in the proposition. In order to do that, we must first derive the value of $\lim_{v_1 \to v_2} \partial \widehat{p}_1 / \partial v_1$. To this end, differentiate both sides of $F(\widehat{p}_1) = 0$, to obtain:

$$\begin{aligned} &\frac{\partial T(\widehat{p}_1)}{\partial p_1} \frac{\partial \widehat{p}_1}{\partial v_1} - \frac{(r\beta)^2 (v_1 - v_2)^2}{\left[r\beta\left[\widehat{p}_1 v_1 + (1-\widehat{p}_1) v_2\right] + v_1 + v_2\right]^2} \frac{\partial \widehat{p}_1}{\partial v_1} \\ &+ \frac{r\beta\left\{r\beta\left[\widehat{p}_1 v_1 + (1-\widehat{p}_1) v_2\right] + v_1 + v_2 - (v_1 - v_2)(r\beta \widehat{p}_1 + 1)\right\}}{\left[r\beta\left[\widehat{p}_1 v_1 + (1-\widehat{p}_1) v_2\right] + v_1 + v_2\right]^2} = 0. \end{aligned} \tag{A22}$$

The numerator of the last term simplifies to $r\beta(r\beta+2) v_2 > 0$. Since we also know, from above, that $\partial T(\widehat{p}_1) / \partial p_1 < 0$, it follows that $\partial \widehat{p}_1 / \partial v_1 > 0$. Next, take the limit $v_1 \to v_2$ of both sides of (A22):

$$\left[\lim_{v_1 \to v_2} \frac{\partial T(\widehat{p}_1)}{\partial p_1}\right] \left[\lim_{v_1 \to v_2} \frac{\partial \widehat{p}_1}{\partial v_1}\right] + \frac{r\beta(r\beta+2) v_2}{\left[r\beta v_2 + v_2 + v_2\right]^2} = 0.$$

From (A21) we also have

$$\lim_{v_1\to v_2}\frac{\partial T(\widehat{p}_1)}{\partial p_1}=(r\beta+1)\frac{-8\left[\left(\frac{r\beta}{2}\right)^2+r\beta+1\right]}{\left[\left(\frac{r\beta}{2}\right)^2+r\beta+1\right]^2}=-\frac{8(r\beta+1)}{\left(\frac{r\beta}{2}\right)^2+r\beta+1}=-\frac{32(r\beta+1)}{(r\beta+2)^2}.$$

Thus, $\lim_{v_1\to v_2}\frac{\partial\widehat{p}_1}{\partial v_1}=\left[-\frac{r\beta(r\beta+2)v_2}{(r\beta+2)^2v_2^2}\right]/\left[-\frac{32(r\beta+1)}{(r\beta+2)^2}\right]=\frac{r\beta(r\beta+2)}{32(r\beta+1)v_2}$. We can now prove the last limit result stated in the proposition. Take logs of (25) and evaluate at $p=\widehat{p}_1$:

$$\ln\widehat{w}_1=\ln w_2-(1+r\beta)\ln(1-\widehat{p}_1)+(1+r\beta)\ln\widehat{p}_1-rt\ln(r\beta\widehat{p}_1+1)+rt\ln[r\beta(1-\widehat{p}_1)+1]-rt\ln v_1+rt\ln v_2.$$

Differentiate both sides w.r.t. v_1:

$$\begin{aligned}\frac{1}{\widehat{w}_1}\frac{\partial\widehat{w}_1}{\partial v_1} &= \left[\frac{1+r\beta}{1-\widehat{p}_1}+\frac{1+r\beta}{\widehat{p}_1}-\frac{tr^2\beta}{r\beta\widehat{p}_1+1}-\frac{tr^2\beta}{r\beta(1-\widehat{p}_1)+1}\right]\frac{\partial\widehat{p}_1}{\partial v_1}-\frac{rt}{v_1}\\ &= \left[\frac{1+r\beta}{(1-\widehat{p}_1)\widehat{p}_1}-\frac{tr^2\beta(r\beta+2)}{(r\beta\widehat{p}_1+1)[r\beta(1-\widehat{p}_1)+1]}\right]\frac{\partial\widehat{p}_1}{\partial v_1}-\frac{rt}{v_1}.\end{aligned}$$

Next take the limit $v_1\to v_2$ of both sides:

$$\lim_{v_1\to v_2}\left[\frac{1}{\widehat{w}_1}\right]\left[\lim_{v_1\to v_2}\frac{\partial\widehat{w}_1}{\partial v_1}\right]=\left[4(1+r\beta)-\frac{tr^2\beta(r\beta+2)}{\left(\frac{r\beta}{2}+1\right)\left(\frac{r\beta}{2}+1\right)}\right]\left[\lim_{v_1\to v_2}\frac{\partial\widehat{p}_1}{\partial v_1}\right]-\frac{rt}{v_2}\Leftrightarrow$$

$$\begin{aligned}\frac{1}{w_2}\lim_{v_1\to v_2}\frac{\partial\widehat{w}_1}{\partial v_1} &= 4\left[(1+r\beta)-\frac{tr^2\beta(r\beta+2)}{(r\beta+2)^2}\right]\frac{r\beta(r\beta+2)}{32(r\beta+1)v_2}-\frac{rt}{v_2}\\ &= \frac{r\beta(r\beta+2)}{8v_2}-\frac{rt}{v_2}-\frac{tr^3\beta^2}{8(r\beta+1)v_2}=\frac{r\beta(r\beta+2)}{8v_2}-\frac{rt\left[8(r\beta+1)+(r\beta)^2\right]}{8(r\beta+1)v_2}.\end{aligned}$$

Thus,

$$\begin{aligned}&\lim_{v_1\to v_2}\frac{\partial\widehat{w}_1}{\partial v_1}<0\Leftrightarrow\frac{r\beta(r\beta+2)}{8v_2}<\frac{rt\left[8(r\beta+1)+(r\beta)^2\right]}{8(r\beta+1)v_2}\\ &\Leftrightarrow\frac{\beta}{\alpha+\beta}<\frac{8(r\beta+1)+(r\beta)^2}{(r\beta+2)(r\beta+1)}=\frac{5r\beta+6}{(r\beta+2)(r\beta+1)}+1,\end{aligned}$$

which always holds. □

4. Calculations Used for Figures 2, 4, and 5 in Lagerlöf (2020)

4.1. Calculations Used for Figure 2

Assume a CES production function, a CSF of the generalized Tullock form (as in eq. (9) in Lagerlöf, 2020), and that $t=1$ and $r\leq 1$. Under these assumptions, condition (i) in Assumption 1 is satisfied for all $\sigma\leq 1$. Thus suppose that $\sigma>1$. Table 1 in Lagerlöf (2020) tells us that, under the stated assumptions, $\eta\left(\frac{1}{p_i}\right)=\left(\frac{\alpha}{1-\alpha}\right)^{\sigma}p_i^{\sigma-1}/\left[\left(\frac{\alpha}{1-\alpha}\right)^{\sigma}p_i^{\sigma-1}+1\right]$. For $\sigma>1$, this expression is strictly increasing in p_i. Therefore, since

$p_i \leq 1$, an upper bound on $\eta\left(\frac{1}{p_i}\right)$ is given by $\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} / \left[\left(\frac{\alpha}{1-\alpha}\right)^{\sigma}+1\right]$. It follows that condition (i) in Assumption 1 (i.e., $r\eta\left(\frac{1}{p_i}\right)\sigma \leq 2$) is satisfied for all $p_i \in [0,1]$ if

$$r\frac{\left(\frac{\alpha}{1-\alpha}\right)^{\sigma}}{\left(\frac{\alpha}{1-\alpha}\right)^{\sigma}+1}\sigma \leq 2 \Leftrightarrow (r\sigma-2)\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} \leq 2.$$

This inequality is satisfied for all $\sigma \leq 2/r$. Suppose $\sigma > 2/r$. Then the inequality can be rewritten as

$$\alpha \leq \frac{\left(\frac{2}{r\sigma-2}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{2}{r\sigma-2}\right)^{\frac{1}{\sigma}}} \stackrel{\text{def}}{=} \Theta(\sigma,r).$$

This is the function that is graphed in Figure 2 in Lagerlöf (2020). Note that the derivative of $\Theta(\sigma,r)$ has the same sign as the derivative of $\frac{1}{\sigma}[\ln 2-\ln(r\sigma-2)]$. Differentiating the latter expression with respect to σ yields

$$\frac{\ln(r\sigma-2)-\ln 2-\frac{r\sigma}{r\sigma-2}}{\sigma^2}, \tag{A23}$$

which clearly is negative for all $r\sigma \leq 4$. Moreover, the numerator in (A23) is increasing in σ and for sufficiently large values of σ the numerator is positive. Thus, for all $\sigma \leq 4/r$, $\Theta(\sigma,r)$ is downward-sloping and there is a unique σ, such that $\sigma > 4/r$, for which $\Theta(\sigma,r)$ is minimized. This value of σ, which I denote by σ^*, is characterized by $\ln(r\sigma^*-2)-\ln 2-\frac{r\sigma^*}{r\sigma^*-2}=0$. The values of σ^* shown in the table in Figure 2 in Lagerlöf (2020) are obtained by solving this equation numerically for different r values (using Maple). The table also shows the associated minimized values values of $\Theta(\sigma,r)$, denoted by $\alpha^*=\Theta(\sigma^*,r)$. □

4.2. Calculations Used for Figure 4

In Figure 4 in Lagerlöf (2020) there are two graphs that indicate the part of the parameter space where R^{H} is decreasing in n (at $n=10$). I here describe how these graphs were obtained. By assuming a CES production function (which implies $h(n)=\left(\frac{\alpha}{(1-\alpha)n}\right)^{\sigma}$) and by setting $t=r=1$, we can write

$$\begin{aligned}
h(n) &> \Xi_L \Leftrightarrow \left(\frac{\alpha}{1-\alpha}\right)^{\sigma} n^{-\sigma} > \frac{(n-1)^2(\sigma-1)-2n}{2n^2}-\frac{1}{2n}\sqrt{\frac{\left[(n-1)^2(\sigma-1)-2n\right]^2}{n^2}-4} \Leftrightarrow \\
\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} &> \frac{(n-1)^2(\sigma-1)-2n-\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}}{2n^{2-\sigma}} \\
&= \frac{\left[(n-1)^2(\sigma-1)-2n\right]^2-\left[\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2\right]}{2n^{2-\sigma}\left[(n-1)^2(\sigma-1)-2n+\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}\right]} \\
&= \frac{2n^{\sigma}}{(n-1)^2(\sigma-1)-2n+\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}} \Leftrightarrow
\end{aligned}$$

$$
\begin{aligned}
\frac{\alpha}{1-\alpha} &> \frac{2^{\frac{1}{\sigma}} n}{\left[(n-1)^2(\sigma-1)-2n+\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}\right]^{\frac{1}{\sigma}}} \Leftrightarrow \\
\alpha &> \frac{2^{\frac{1}{\sigma}} n}{2^{\frac{1}{\sigma}} n+\left[(n-1)^2(\sigma-1)-2n+\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}\right]^{\frac{1}{\sigma}}}. \qquad \text{(A24)}
\end{aligned}
$$

Similarly we can write

$$
\begin{aligned}
h(n) &< \Xi_H \Leftrightarrow \left(\frac{\alpha}{1-\alpha}\right)^{\sigma} n^{-\sigma} < \frac{(n-1)^2(\sigma-1)-2n}{2n^2}+\frac{1}{2n}\sqrt{\frac{\left[(n-1)^2(\sigma-1)-2n\right]^2}{n^2}-4} \Leftrightarrow \\
\left(\frac{\alpha}{1-\alpha}\right)^{\sigma} &< \frac{(n-1)^2(\sigma-1)-2n+\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}}{2n^{2-\sigma}} \\
&= \frac{\left[(n-1)^2(\sigma-1)-2n\right]^2-\left[\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2\right]}{2n^{2-\sigma}\left[(n-1)^2(\sigma-1)-2n-\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}\right]} \\
&= \frac{2n^{\sigma}}{(n-1)^2(\sigma-1)-2n-\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}} \Leftrightarrow
\end{aligned}
$$

$$
\begin{aligned}
\frac{\alpha}{1-\alpha} &< \frac{2^{\frac{1}{\sigma}} n}{\left[(n-1)^2(\sigma-1)-2n-\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}\right]^{\frac{1}{\sigma}}} \Leftrightarrow \\
\alpha &< \frac{2^{\frac{1}{\sigma}} n}{2^{\frac{1}{\sigma}} n+\left[(n-1)^2(\sigma-1)-2n-\sqrt{\left[(n-1)^2(\sigma-1)-2n\right]^2-4n^2}\right]^{\frac{1}{\sigma}}}. \qquad \text{(A25)}
\end{aligned}
$$

The expressions in (A24) and (A25) are then evaluated at $n = 10$. The resulting expressions can then, in principle, be plotted with the help of some appropriate software. However, I have instead computed values of the right-hand sides of (A24) and (A25), evaluated at $n = 10$ and different σ's. Then I plotted the associated pairs of (σ, α) using the LaTeX package TikZ. □

4.3. Calculations Used for Figure 5

Recall from the proof of Proposition 10 in Lagerlöf (2020) that $\widehat{p}$ is characterized by $\digamma(\widehat{p}_1) = 0$, where

$$
\digamma(p_1) = \frac{(1-2p_1)(r\beta+1)}{p_1(1-p_1)\left[(r\beta)^2 p_1(1-p_1)+r\beta+1\right]} + \frac{r\beta(v_1-v_2)}{r\beta\left[p_1 v_1+(1-p_1)v_2\right]+v_1+v_2}.
$$

Also recall that $\widehat{w}_1$ is given by

$$
\widehat{w}_1 = w_2\left(\frac{\widehat{p}_1}{1-\widehat{p}_1}\right)^{1+r\beta}\left(\frac{r\beta(1-\widehat{p}_1)+1}{r\beta\widehat{p}_1+1}\frac{v_2}{v_1}\right)^{rt}.
$$

	v_1	1	1.5	2	3	4	5	6	7	8	9	10	20	50	100	∞
$\alpha = .1$	$\widehat{p}_1$	.500	.517	.528	.542	.550	.555	.559	.562	.564	.565	.567	.573	.577	.578	.580
	$\widehat{w}_1$	1	.743	.599	.436	.344	.285	.243	.212	.188	.169	.153	.080	.033	.017	0
$\alpha = .5$	$\widehat{p}_1$	.500	.510	.517	.526	.531	.534	.537	0.538	.540	0.541	.542	.546	.549	.550	.551
	$\widehat{w}_1$	1	.704	.547	0.381	.294	.239	.202	.174	.153	.137	.124	.064	.026	.013	0
$\alpha = .9$	$\widehat{p}_1$	.500	.502	.504	.506	.507	.508	.509	.509	.509	.510	.510	.511	.511	.512	.512
	$\widehat{w}_1$	1	.673	.508	.342	.258	.207	.173	.148	.130	.116	.104	.052	.021	.011	0

Table 1: Computed values of $\widehat{p}_1$ and $\widehat{w}_1$ used in Figure 5 of Lagerlöf (2020).

Now set $r = t = v_2 = 1$. Moreover, to start with, assume $\alpha = \beta = \frac{1}{2}$. We then get

$$F(p_1) = \frac{(1-2p_1)\left(\frac{1}{2}+1\right)}{p_1(1-p_1)\left(\left(\frac{1}{2}\right)^2 p_1(1-p_1)+\frac{1}{2}+1\right)} + \frac{\frac{1}{2}(v_1-1)}{\frac{1}{2}(p_1 v_1 + 1 - p_1) + v_1 + 1} \tag{A26}$$

and

$$\widehat{w}_1 = w_2\left(\frac{\widehat{p}_1}{1-\widehat{p}_1}\right)^{\frac{3}{2}} \frac{\frac{1}{2}(1-\widehat{p}_1)+1}{\frac{1}{2}\widehat{p}_1+1}\frac{1}{v_1} = \frac{w_2}{v_1}\left(\frac{\widehat{p}_1}{1-\widehat{p}_1}\right)^{\frac{3}{2}} \frac{3-\widehat{p}_1}{2+\widehat{p}_1}. \tag{A27}$$

By using Maple and the expression in (A26), the equality $F(\widehat{p}_1) = 0$ can be solved for $\widehat{p}_1$, given various values of v_1. Thereafter, by plugging $\widehat{p}_1$ into (A27), we can compute $\widehat{w}_1$. Doing this yields the numbers in rows 3 and 4 (i.e., the ones for $\alpha = 0.5$) of Table 1 in the present document. The numbers for $\alpha = 0.1$ and $\alpha = 0.9$ are obtained similarly. □

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