Fiscal and Monetary Policy Interactions in a Model with Low Interest Rates

Online Appendix

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Appendix A. Proofs

Proof of Proposition 1. — The entrepreneur's objective is to solve the following dynamic programming problem:

$$V_t\left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right) = \max_{\{I_{jt}, D_{jt}, B_{jt}\}} C_{jt} + \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1}\left(K_{jt}, B_{jt}, D_{jt}, \varepsilon_{jt+1}\right), \tag{A1}$$

subject to

$$K_{jt} = (1-\delta) K_{jt-1} + \varepsilon_{jt} I_{jt}, \tag{A2}$$

$$B_{jt} \geq -\mu K_{jt-1}, \tag{A3}$$

$$C_{jt} + I_{jt} + B_{jt} + D_{jt} = R_{kt} K_{jt-1} + \frac{R_{t-1}}{\Pi_t} B_{jt-1} + \frac{R_{t-1}}{\Pi_t} D_{jt-1}, \tag{A4}$$

$$C_{jt} \geq 0. \tag{A5}$$

We conjecture that the value function takes the following form

$$V_t\left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right) = \phi_t^k\left(\varepsilon_{jt}\right) K_{jt-1} + \phi_t^b\left(\varepsilon_{jt}\right) B_{jt-1} + \phi_t^d\left(\varepsilon_{jt}\right) D_{jt-1}, \tag{A6}$$

where $\phi_t^i(\varepsilon_{jt})$, $i \in \{k, b, d\}$, satisfy

$$q_t^k = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^k(\varepsilon) dF(\varepsilon), \tag{A7}$$

$$1 = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^b(\varepsilon) dF(\varepsilon), \tag{A8}$$

$$1 = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^d(\varepsilon) dF(\varepsilon). \tag{A9}$$

Substituting (A2), (A4), and the above conjecture into the Bellman equation (A1), we have

$$
\begin{aligned}
&V_t\left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right) \\
&= \max_{\{I_{jt}, D_{jt}, B_{jt}\}} \left(R_{kt} + (1-\delta)\beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^k(\varepsilon) dF(\varepsilon)\right) K_{jt-1} \\
&\quad + \frac{R_{t-1}}{\Pi_t} B_{jt-1} + \frac{R_{t-1}}{\Pi_t} D_{jt-1} + \left[\beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^k(\varepsilon) dF(\varepsilon) \varepsilon_{jt} - 1\right] I_{jt} \\
&\quad + \left[\beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^b(\varepsilon) dF(\varepsilon) - 1\right] B_{jt} + \left[\beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^d(\varepsilon) dF(\varepsilon) - 1\right] D_{jt}.
\end{aligned} \tag{A10}
$$

Optimal choices of B_{jt} and D_{jt} imply that (A8) and (A9) must hold in equilibrium. Otherwise, all entrepreneurs will either save or borrow at the same time, contradicting the market-clearing conditions for bonds.

Since $I_{jt} \geq 0$ and $C_{jt} \geq 0$, it follows that $I_{jt} = 0$ if $\varepsilon_{jt} < 1/q_t^k \equiv \varepsilon_t^*$; but the firm makes as much investment as possible so that $C_{jt} = 0$ if $\varepsilon_{jt} > \varepsilon_t^*$. It follows from (A4) that when $\varepsilon_{jt} > \varepsilon_t^*$, we have

$$I_{jt} = -B_{jt} - D_{jt} + R_{kt} K_{jt-1} + \frac{R_{t-1}}{\Pi_t} B_{jt-1} + \frac{R_{t-1}}{\Pi_t} D_{jt-1}, \tag{A11}$$

$$D_{jt} = 0, \quad B_{jt} = -\mu K_{jt-1}. \tag{A12}$$

Consider first the case where $\varepsilon_{jt} < \varepsilon_t^*$ and $I_{jt} = 0$. The entrepreneurs are indifferent between borrowing and saving. Substituting the decision rules into (A10) and reorganizing yield

$$
\begin{aligned}
&V_t\left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right) \\
&= \max_{\{I_{jt}, D_{jt}, B_{jt}\}} \left(R_{kt} + (1-\delta) q_t^k\right) K_{jt-1} + \frac{R_{t-1}}{\Pi_t} B_{jt-1} + \frac{R_{t-1}}{\Pi_t} D_{jt-1}.
\end{aligned}
$$

Notice that (A8) and (A9) ensure that the indeterminacy of B_{jt} and D_{jt} does not affect the value function.

Matching the coefficients, we have

$$
\begin{aligned}
\phi_t^k(\varepsilon_{jt}) &= R_{kt} + (1-\delta) q_t^k, \\
\phi_t^b(\varepsilon_{jt}) = \phi_t^d(\varepsilon_{jt}) &= \frac{R_{t-1}}{\Pi_t}.
\end{aligned}
$$

Next we consider the case where $\varepsilon_{jt} > \varepsilon_t^*$. Substituting (A11) and (A12) into (A10) and reorganizing yield

$$\begin{aligned} & V_t\left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right) \\ = & \max_{\{I_{jt}, D_{jt}, B_{jt}\}} \left(R_{kt} + (1-\delta) q_t^k + R_{kt}\left(q_t^k \varepsilon_{jt} - 1\right) - \mu\left(1 - q_t^k \varepsilon_{jt}\right)\right) K_{jt-1} \\ & + \frac{R_{t-1}}{\Pi_t}\left(q_t^k \varepsilon_{jt}\right) B_{jt-1} + \frac{R_{t-1}}{\Pi_t}\left(q_t^k \varepsilon_{jt}\right) D_{jt-1}. \end{aligned}$$

Matching the coefficients yields

$$\begin{aligned} \phi_t^k(\varepsilon_{jt}) &= R_{kt}\left(1 + \left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1\right)\right) + (1-\delta) q_t^k + \mu\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1\right), \\ \phi_t^b(\varepsilon_{jt}) = \phi_t^d(\varepsilon_{jt}) &= \frac{R_{t-1}}{\Pi_t}\left(q_t^k \varepsilon_{jt}\right) = \frac{R_{t-1}}{\Pi_t}\left(1 + \left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1\right)\right). \end{aligned}$$

Combining the two cases above, we have

$$\begin{aligned} \phi_t^k(\varepsilon_{jt}) &= R_{kt}\left(1 + \max\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1, 0\right)\right) + (1-\delta) q_t^k + \mu \max\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1, 0\right), \\ \phi_t^b(\varepsilon_{jt}) = \phi_t^d(\varepsilon_{jt}) &= \frac{R_{t-1}}{\Pi_t}\left(1 + \max\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1, 0\right)\right), \end{aligned}$$

for $\varepsilon_{jt} \in [\varepsilon_{\min}, \varepsilon_{\max}]$. Substituting these two equations into (A7), (A8) and (A9), we obtain (13) and (14).

Finally, for the entrepreneur's objective to be finite, the value function must satisfy the following condition by the Bellman equation (A1):

$$\lim_{i \to \infty} \mathbb{E}_t \frac{\beta^i \Lambda_{t+i}}{\Lambda_t} V_{t+i}\left(K_{j,t+i-1}, B_{j,t+i-1}, D_{j,t+i-1}, \varepsilon_{j,t+i}\right) = 0.$$

Using equations (A6)-(A9) we can derive the transversality condition (16). Q.E.D.

Proof of Lemma 1. — To simplify notations, we define

$$M_{t+1} = \frac{\beta \Lambda_{t+1}}{\Lambda_t},\ M_{t+1}^l = \frac{\beta \Lambda_{t+1}}{\Lambda_t}\left(1 + q_{t+1}^l\right),\ x_t = \frac{D_{t-1} R_{t-1}}{\Pi_t}. \tag{A13}$$

Then we can rewrite (34) as

$$x_t = S_t + \mathbb{E}_t M_{t+1} x_{t+1} + \mathbb{E}_t\left(M_{t+1}^l - M_{t+1}\right) x_{t+1}.$$

Leading the above equation by one period and multiplying by M_{t+1}, we obtain

$$M_{t+1}x_{t+1} = M_{t+1}S_{t+1} + \mathbb{E}_{t+1}M_{t+1}M_{t+2}x_{t+2} + \mathbb{E}_{t+1}M_{t+1}\left(M^l_{t+2} - M_{t+2}\right)x_{t+2}.$$

Following similar procedures recursively until period $t+T$, we have

$$\begin{aligned} M_{t+1}M_{t+2}...M_{t+T}x_{t+T} &= M_{t+1}M_{t+2}...M_{t+T}S_{t+T} + \mathbb{E}_{t+T}M_{t+1}M_{t+2}M_{t+T+1}x_{t+T+1} \\ &\quad + \mathbb{E}_{t+T}M_{t+1}M_{t+2}...M_{t+T+1}\left(M^l_{t+T+1} - M_{t+T+1}\right)x_{t+T+1}. \end{aligned}$$

Taking conditional expectations $\mathbb{E}_t$ on the two sides of above system of $T+1$ equations and using (A13), we obtain

(A14)

$$\frac{D_{t-1}R_{t-1}}{\Pi_t} = \mathbb{E}_t\sum_{i=0}^{T}\frac{\beta^i\Lambda_{t+i}}{\Lambda_t}S_{t+i} + \mathbb{E}_t\sum_{i=0}^{T}\frac{\beta^{i+1}\Lambda_{t+i+1}}{\Lambda_t}q^l_{t+i+1}\frac{D_{t+i}R_{t+i}}{\Pi_{t+i+1}} + \mathbb{E}_t\frac{\beta^{T+1}\Lambda_{t+1+T}}{\Lambda_t}\frac{D_{t+T}R_{t+T}}{\Pi_{T+1}}.$$

Summing over j in (16) and using the market-clearing conditions, we have

$$\lim_{i\to\infty}\mathbb{E}_t\frac{\beta^i\Lambda_{t+i}}{\Lambda_t}\left(q^k_{t+i}K_{t+i} + D_{t+i}\right) = 0.$$

Since $K_{t+i} > 0$ and $q^k_{t+i} > 0$, we have

$$\lim_{i\to\infty}\mathbb{E}_t\frac{\beta^i\Lambda_{t+i}}{\Lambda_t}D_{t+i} = 0. \tag{A15}$$

Since $q^l_{t+1+T} \geq 0$, it follows from (14) that

$$\begin{aligned} 0 \leq \mathbb{E}_t\frac{\beta^{T+1}\Lambda_{t+1+T}}{\Lambda_t}\frac{D_{t+T}R_{t+T}}{\Pi_{t+T+1}} &\leq \mathbb{E}_t\frac{\beta^T\Lambda_{t+T}}{\Lambda_t}\beta\frac{\Lambda_{t+1+T}}{\Lambda_{t+T}}(1+q^l_{t+1+T})\frac{D_{t+T}R_{t+T}}{\Pi_{t+T+1}} \\ &= \mathbb{E}_t\frac{\beta^T\Lambda_{t+T}}{\Lambda_t}\mathbb{E}_{t+T}\frac{\beta\Lambda_{t+1+T}}{\Lambda_{t+T}}(1+q^l_{t+1+T})\frac{D_{t+T}R_{t+T}}{\Pi_{t+T+1}} = \mathbb{E}_t\frac{\beta^T\Lambda_{t+T}}{\Lambda_t}D_{t+T}. \end{aligned}$$

Thus,

$$\lim_{T\to\infty}\mathbb{E}_t\frac{\beta^{T+1}\Lambda_{t+1+T}}{\Lambda_t}\frac{D_{t+T}R_{t+T}}{\Pi_{t+T+1}} = 0.$$

Taking limit in (A14) as $T\to\infty$ gives (36). Q.E.D.

Proof of Lemma 2. — Taking derivative of $R_k(\varepsilon^*)$ in (39) and reorganizing yields

$$\frac{\partial R_k(\varepsilon^*)}{\partial \varepsilon^*} = \frac{\mu \int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) - (\beta^{-1}(1+g) - 1 + \delta)F(\varepsilon^*)}{\left[\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \max(\varepsilon, \varepsilon^*)\, dF(\varepsilon)\right]^2}. \tag{A16}$$

The numerator in (A16) is strictly decreasing in ε^*, with the maximum and the minimum being $\mu\mathbb{E}[\varepsilon] \geq 0$ and $-(\beta^{-1}(1+g) - 1 + \delta) < 0$, respectively. Hence, by the intermediate value theorem, there exists a unique threshold $\varepsilon_k \in [\varepsilon_{\min}, \varepsilon_{\max}]$ such that

$$\mu \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) - (\beta^{-1}(1+g) - 1 + \delta)F(\varepsilon_k) = 0.$$

And it follows that $\partial R_k(\varepsilon^*)/\partial\varepsilon^* > 0$ if $\varepsilon^* < \varepsilon_k$; $\partial R_k(\varepsilon^*)/\partial\varepsilon^* \leq 0$ if $\varepsilon^* \geq \varepsilon_k$. Moreover, we have $\varepsilon_k = \varepsilon_k(\mu)$ strictly increasing and $\lim_{\mu\to 0} \varepsilon_k = \varepsilon_{\min}$. Q.E.D.

Proof of Lemma 3. — By Lemma 2, on $[\varepsilon_k, \varepsilon_{\max}]$, $R_k(\varepsilon^*)$ is decreasing and thus $\Phi(\varepsilon^*)$ is increasing. By (39), we compute

$$R_k(\varepsilon_k) = \frac{(1+g)/\beta - 1 + \delta - \mu \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + \mu\varepsilon_k(1 - F(\varepsilon_k))}{\varepsilon_k F(\varepsilon_k) + \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}. \tag{A17}$$

By Lemma 1, we have

$$\frac{\partial R_k(\varepsilon^*)}{\partial \varepsilon^*}|_{\varepsilon_k} = 0.$$

Thus,

$$\mu \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) - (\beta^{-1}(1+g) - 1 + \delta)F(\varepsilon_k) = 0. \tag{A18}$$

Using this equation, we can eliminate $F(\varepsilon_k)$ in (A17) to obtain

$$R_k(\varepsilon_k) = \frac{(1+g)/\beta - 1 + \delta - \mu \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}{\int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}.$$

Substituting this expression into (40) yields

$$\Phi(\varepsilon_k) = -\frac{(\beta^{-1} - 1)(1+g)}{\int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)} < 0.$$

Since $\Phi(\varepsilon_{\max}) = +\infty$ and $\Phi(\varepsilon_k) < 0$ and Φ is increasing on $[\varepsilon_k, \varepsilon_{\max}]$, it follows from the intermediate value theorem that there exists a unique value $\varepsilon_l \in (\varepsilon_k, \varepsilon_{\max})$ such that $\Phi(\varepsilon_l) = 0$.

By (39), we have

$$R_k(\varepsilon_{\min}) = \frac{(1+g)/\beta - 1 + \delta - \mu(\mathbb{E}[\varepsilon] - \varepsilon_{\min})}{\mathbb{E}[\varepsilon]}.$$

Substituting this expression into (40) yields

$$\Phi(\varepsilon_{\min}) = -\frac{(\beta^{-1}-1)(1+g) + \mu\varepsilon_{\min}}{\mathbb{E}\varepsilon} < 0.$$

When $\mu = 0$, we have

$$\Phi(\varepsilon_{\min}) = -\frac{(\beta^{-1}-1)(1+g)}{\mathbb{E}\varepsilon} < 0.$$

By (A18), ε_k is an implicit continuous function of μ and $\varepsilon_k \to \varepsilon_{\min}$ as $\mu \to 0$. By continuity, for sufficiently small $\mu \geq 0$, we have $\Phi(\varepsilon^*) < 0$ for $\varepsilon^* \in [\varepsilon_{\min}, \varepsilon_k]$. Q.E.D.

PROOF OF PROPOSITION 2. — By the assumption and Lemma 3, the investment cutoff ε^* in any steady state must satisfy $\varepsilon^* \geq \varepsilon_k$. Since $\Phi(\varepsilon_l) = 0$, by (40) and setting $\varepsilon^* = \varepsilon_l$, we have $d = 0$. Thus (41) or (43) is satisfied for $s = 0$. We deduce that $\varepsilon^* = \varepsilon_l$ is the steady-state cutoff for $s = 0$. This is the only steady state with $d = 0$ because $\Phi(\varepsilon^*)$ increases with $\varepsilon^* \in [\varepsilon_k, \varepsilon_{\max}]$ by Lemma 2.

Suppose that there is another steady state with $d > 0$ if $R^r(\varepsilon_l) > 1+g$. Then (41) implies that $R^r(\varepsilon^*) = 1+g$ for $s = 0$. Since $R^r(\varepsilon^*)$ increases with ε^* and since $R^r(\varepsilon_l) > 1+g$, we must have the steady state cutoff $\varepsilon^* < \varepsilon_l$. Since $R_k(\varepsilon^*)$ decreases with ε^* on $(\varepsilon_k, \varepsilon_l)$, it follows (40) that Φ increases with ε^* on $(\varepsilon_k, \varepsilon_l)$. Thus we have $\Phi(\varepsilon^*) < \Phi(\varepsilon_l) = 0$ for $\varepsilon^* \in (\varepsilon_k, \varepsilon_l)$, contradicting equation (40) as $d > 0$ and $R^r > 0$.

If $R^r(\varepsilon_l) < 1+g$, we show that there is another steady state with $d > 0$. It follows from (41) we must have $R^r = 1+g$. Since $R^r(\varepsilon^*)$ is a continuous and increasing function and since $R^r(\varepsilon_l) < 1+g$ and $R^r(\varepsilon_{\max}) = (1+g)/\beta > 1+g$, by the intermediate value theorem there is a unique solution $\varepsilon^* = \varepsilon_h \in (\varepsilon_l, \varepsilon_{\max})$ such that $R^r(\varepsilon^*) = 1+g$. We then have $R^r = R^r(\varepsilon_h) = 1+g$ in the steady state. It follows from (40) that $R^r d/k = \Phi(\varepsilon_h)$. Q.E.D.

PROOF OF PROPOSITION 3:. — Recall that ε_l satisfies $\Phi(\varepsilon_l^*) = 0$. Total differentiating this equation and reorganizing yield

$$\frac{\mathrm{d}\varepsilon_l}{\mathrm{d}\mu} = -\frac{\left(1 + \frac{\partial R_k(\varepsilon_l)}{\partial \mu}\right)\int_{\varepsilon_l}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}{\frac{\partial R_k(\varepsilon_l)}{\partial \varepsilon_l}\int_{\varepsilon_l}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) - (\mu + R_k(\varepsilon_l))\varepsilon_l F'(\varepsilon_l)}.$$

By (39), we have $1 + \partial R_k(\varepsilon_l)/\partial\mu > 0$ and that $\partial R_k(\varepsilon_l)/\partial \varepsilon_l < 0$. Thus we have $\mathrm{d}\varepsilon_l/\mathrm{d}\mu > 0$. By (38), $R^r(\varepsilon^*)$ increases with ε^*. It follows that both ε_l and $R^r(\varepsilon_l)$ increase with μ.

Q.E.D.

PROOF OF PROPOSITION 4:. — By Lemma 2, for a sufficiently small $\mu \geq 0$, we only need to consider steady-state the investment cutoffs in $[\varepsilon_k, \varepsilon_{\max}]$. It follows from Lemma 1 that $R_k(\varepsilon^*)$ is a decreasing function of $\varepsilon^* \in [\varepsilon_k, \varepsilon_{\max}]$. Thus $\Phi(\varepsilon^*)$ increases with $\varepsilon^* \in [\varepsilon_k, \varepsilon_{\max}]$ by (40). We also know that $R^r(\varepsilon^*)$ increases with $\varepsilon^* \in [\varepsilon_{\min}, \varepsilon_{\max}]$. By (43) we have

$$\Psi(\varepsilon^*) = \left[1 - \frac{1+g}{R^r(\varepsilon^*)}\right] \frac{\alpha p_w}{R_k(\varepsilon^*)} \Phi(\varepsilon^*). \tag{A19}$$

Thus $\Psi(\varepsilon^*)$ is a product of three increasing functions on $[\varepsilon_k, \varepsilon_{\max}]$. Since $\Phi(\varepsilon_l) = 0$ and $\Phi(\varepsilon^*) < \Phi(\varepsilon_l) = 0$ for $\varepsilon^* \in [\varepsilon_k, \varepsilon_l]$, we will focus on the region $[\varepsilon_l, \varepsilon_{\max}]$ as equation (40) must hold with $R^r d \geq 0$. On this region $\Phi(\varepsilon^*) \geq 0$.

Suppose that $R^r(\varepsilon_l) > 1 + g$. Then we have

$$1 - \frac{1+g}{R^r(\varepsilon^*)} > 1 - \frac{1+g}{R^r(\varepsilon_l)} > 0$$

for $\varepsilon^* > \varepsilon_l > \varepsilon_k$. Since $\Phi(\varepsilon_l) = 0$, we have $\Phi(\varepsilon^*) > 0$ for $\varepsilon^* > \varepsilon_l$. Thus, as a product of three positive increasing functions on $[\varepsilon_l, \varepsilon_{\max}]$, $\Psi(\varepsilon^*)$ increases with $\varepsilon^* \in [\varepsilon_l, \varepsilon_{\max}]$. Since $\Psi(\varepsilon_l) = 0$ and $\Psi(\varepsilon_{\max}) = +\infty$, it follows from the intermediate value theorem that there exists a unique solution $\varepsilon_p \in (\varepsilon_l, \varepsilon_{\max})$ to equation (43). Then $R^r(\varepsilon_p) > R^r(\varepsilon_l) > 1 + g$.

Suppose that $R^r(\varepsilon_l) < 1 + g$. Then Proposition 2 shows that there exists $\varepsilon_h \in (\varepsilon_l, \varepsilon_{\max})$ such that $R^r(\varepsilon_h) = 1 + g$ and $\Psi(\varepsilon_h) = 0$. Thus $R^r(\varepsilon^*) > 1 + g$ for $\varepsilon^* \in [\varepsilon_h, \varepsilon_{\max}]$ by the monotonicity of $R^r(\varepsilon^*)$. It follows that $\Psi(\varepsilon^*)$ increases with $\varepsilon^* \in [\varepsilon_h, \varepsilon_{\max}]$ because $\Psi(\varepsilon^*)$ is a product of three positive increasing functions on $[\varepsilon_h, \varepsilon_{\max}]$. The intermediate value theorem implies that there exists a unique cutoff $\varepsilon_p \in (\varepsilon_h, \varepsilon_{\max})$ such that $\Psi(\varepsilon_p) = s/y > 0$. Then we have $R^r(\varepsilon_p) > R^r(\varepsilon_h) = 1 + g$.

For $\varepsilon^* \in (\varepsilon_l, \varepsilon_h)$, we have $R^r(\varepsilon^*) < R^r(\varepsilon_h) = 1 + g$ and thus $\Psi(\varepsilon^*) < 0$. There cannot exist a steady state with $s/y > 0$ by (43). Q.E.D.

PROOF OF PROPOSITION 5:. — As in the proof of Proposition 4, for a sufficiently small $\mu \geq 0$, we only need to consider the region $[\varepsilon_l, \varepsilon_{\max}]$ for the steady state investment cutoff. By assumption, $R^r(\varepsilon_l) < 1 + g$. By the proof of Proposition 4, $\Psi(\varepsilon^*)$ is positive and increases with $\varepsilon^* \in (\varepsilon_h, \varepsilon_{\max}]$. But $\Psi(\varepsilon^*)$ is negative for $\varepsilon^* \in (\varepsilon_l, \varepsilon_h)$. Moreover, $\Psi(\varepsilon_h) = \Psi(\varepsilon_l) = 0$. Let $\underline{s}$ be defined as in the proposition. By the intermediate value theorem, for any $s/y \in (-\underline{s}, 0)$, there exist at least two steady-state cutoffs ε_l^* and ε_h^* with $\varepsilon_l < \varepsilon_l^* < \varepsilon_h^* < \varepsilon_h$ such that (43) holds. Q.E.D.

Appendix B. Detrended Equilibrium System

The model exhibits long-run growth. To find a steady state and to study the dynamics around a steady state, we need to detrend the model around a long-run growth path. We consider transformations of $x_t = X_t/A_t$ for any variable $X_t \in \{K_t, D_t, S_t, Y_t, W_t, C_t, I_t\}$. For the marginal utility, we denote $\lambda_t = A_t\Lambda_t$. Then the detrended system can be summarized by the following 20 equations in 20 variables $\{R_{kt}, k_t, R_t, q_t^k, q_t^l, \varepsilon_t^*, d_t, \tau_t, \Pi_t, p_t^*, \Gamma_t^a, \Gamma_t^b, \Delta_t, w_t, \lambda_t, p_{wt}, N_t, y_t, c_t, i_t\}$, where $\{R_{-1}, \Delta_{-1}, d_{-1}, k_{-1}\}$ and $\{z_{mt}, z_{\tau,t}, G_{at}\}$ are given exogenously:

1) The capital return,

$$R_{kt} = \alpha (1+g)^{1-\alpha} p_{wt} k_{t-1}^{\alpha-1} N_t^{1-\alpha}. \tag{B1}$$

2) Evolution of capital,

$$(1+g)k_t = (1-\delta)k_{t-1} + \left((\mu + R_{kt}) k_{t-1} + \frac{R_{t-1}}{\Pi_t} d_{t-1} \right) \int_{\epsilon_t^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon). \tag{B2}$$

3) The nominal interest rate,

$$1 = \frac{\beta}{1+g} \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} \frac{R_t}{\Pi_{t+1}} \left(1 + q_{t+1}^l\right). \tag{B3}$$

4) Tobin's Q,
(B4)
$$q_t^k = \frac{\beta}{1+g} \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} R_{kt+1} \left(1 + q_{t+1}^l\right) + \frac{\beta}{1+g} \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} q_{t+1}^k (1-\delta) + \frac{\beta\mu}{1+g} \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} q_{t+1}^l.$$

5) Liquidity premium,

$$q_t^l = \int_{\varepsilon_t^*}^{\varepsilon_{\max}} \left(q_t^k \varepsilon - 1\right) dF(\varepsilon). \tag{B5}$$

6) Investment cutoff,

$$\varepsilon_t^* = 1/q_t^k. \tag{B6}$$

7) Government budget constraint,

$$\frac{R_{t-1}}{\Pi_t} \frac{d_{t-1}}{1+g} = \tau_t - G_{at} + d_t. \tag{B7}$$

8) Fiscal policy rule,

(B8)
$$(\tau_t - \tau)/y = \phi_d (d_{t-1} - d)/y + z_{\tau,t}.$$

9) Monetary policy rule,

(B9)
$$R_t = R \left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi} \exp(z_{mt}).$$

10) Pricing rule,

(B10)
$$p_t^* = \frac{\sigma}{\sigma - 1} \frac{\Gamma_t^a}{\Gamma_t^b}.$$

11) Numerator in the pricing rule,

(B11)
$$\Gamma_t^a = \lambda_t p_{wt} y_t + \beta \xi \mathbb{E}_t \left(\frac{\Pi_{t+1}}{\Pi}\right)^{\sigma} \Gamma_{t+1}^a.$$

12) Denominator in the pricing rule,

(B12)
$$\Gamma_t^b = \lambda_t y_t + \beta \xi \mathbb{E}_t \left(\frac{\Pi_{t+1}}{\Pi}\right)^{\sigma - 1} \Gamma_{t+1}^b.$$

13) Evolution of inflation,

(B13)
$$1 = \left[\xi \left(\frac{\Pi}{\Pi_t}\right)^{1-\sigma} + (1 - \xi)\, p_t^{*1-\sigma}\right]^{\frac{1}{1-\sigma}}.$$

14) Price dispersion,

(B14)
$$\Delta_t = (1 - \xi) p_t^{*-\sigma} + \xi \left(\frac{\Pi}{\Pi_t}\right)^{-\sigma} \Delta_{t-1}.$$

15) Labor demand,

(B15)
$$w_t = (1 - \alpha)(1 + g)^{-\alpha} p_{wt} k_{t-1}^{\alpha} (N_t)^{-\alpha}.$$

16) Labor supply,

(B16) $$w_t = \frac{\psi}{\lambda_t}.$$

17) Marginal utility,

(B17) $$\lambda_t = 1/c_t.$$

18) Aggregate output,

(B18) $$y_t = \Delta_t^{-1} (1+g)^{-\alpha} k_{t-1}^{\alpha} (N_t)^{1-\alpha}.$$

19) Aggregate investment,

(B19) $$(1+g) i_t = \left((\mu + R_{kt}) k_{t-1} + \frac{R_{t-1}}{\Pi_t} d_{t-1} \right) (1 - F(\varepsilon_t^*)).$$

20) Resource constraint,

(B20) $$c_t + i_t + G_{at} = y_t.$$

For the real version of our model, we set $p_{wt} = 1 - 1/\sigma$, $\Pi_t = \Delta_t = 1$, and $R_t = R_t^r$ in the above system and the detrended equilibrium system consists of 13 equations (B1)-(B7), and (B15)-(B20) in 13 variables $\{R_t,\ R_{kt},\ \lambda_t,\ \varepsilon_t^*,\ q_t^k,\ q_t^l,\ w_t,\ d_t,\ k_t,\ N_t,\ y_t,\ c_t,\ i_t\}$.

Appendix C. Steady-State System

We study the nonstochastic steady state of the detrended system with s/y and Π being exogenously given. Define real interest rate as $R^r = R/\Pi$. Let variables without time subscripts denote their steady state values. By the steady-state version of (B13), we obtain $p^* = 1$. It then follows from (B14) that $\Delta = 1$. Combining (B10), (B11), and (B12), we have $p_w = 1 - 1/\sigma$, $\Gamma^a = p_w \Gamma^b = p_w \lambda y/(1 - \beta\xi)$. With w and λ being eliminated by using (B16) and (B17), and noting that $z_\tau = z_m = 0$, we obtain a steady-state system of 11 equations in 11 variables $\{R^r,\ R_k,\ \varepsilon^*,\ q^k,\ d,\ k,\ N,\ y,\ c,\ i, q^l\}$:

1) The capital return,

(C1) $$R_k = (1 - 1/\sigma)\, \alpha\, (1+g)^{1-\alpha} k^{\alpha-1} N^{1-\alpha}.$$

2) Evolution of capital,

$$(g+\delta)k = \left((\mu + R_k)\, k + R^r d\right) \int_{\epsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon). \tag{C2}$$

3) Nominal interest rate,

$$1 = \frac{\beta}{1+g} R^r \left(1 + q^l\right). \tag{C3}$$

4) Tobin's Q,

$$q^k = \frac{\beta}{1+g} R_k \left(1 + q^l\right) + \frac{\beta}{1+g} q^k (1-\delta) + \frac{\beta}{1+g} \mu q^l. \tag{C4}$$

5) Liquidity premium,

$$q^l = \int_{\varepsilon^*}^{\varepsilon_{\max}} \left(q^k \varepsilon - 1\right) dF(\varepsilon). \tag{C5}$$

6) Investment cutoff,

$$\varepsilon^* = 1/q^k. \tag{C6}$$

7) Government budget constraint,

$$\left(\frac{R^r}{1+g} - 1\right) \frac{d}{y} = \frac{s}{y}. \tag{C7}$$

8) Labor demand,

$$\psi c = (1 - 1/\sigma)(1-\alpha)(1+g)^{-\alpha} k^\alpha N^{-\alpha}. \tag{C8}$$

9) Aggregate output,

$$y = (1+g)^{-\alpha} k^\alpha N^{1-\alpha}. \tag{C9}$$

10) Aggregate investment,

$$(1+g)i = \left[(\mu + R_k)\, k + R^r d\right] (1 - F(\varepsilon^*)). \tag{C10}$$

11) Resource constraint,

$$c + i + G_a = y. \tag{C11}$$

As discussed in Section II, the investment cutoff ε^* can be solved for by combining (C3), (C4), (C5), (C6), and (C7). Given the inflation target Π, we obtain the nominal interest rate $R = R^r(\varepsilon^*)\Pi$. By (C6), $q^k = 1/\varepsilon^*$. By (C5), we derive q^l. With $R^r = R^r(\varepsilon^*)$, $R_k = R_k(\varepsilon^*)$ and $R^r d/k = \Phi(\varepsilon^*)$, we can determine y/k from $R_k = (1-1/\sigma)(1+g)\alpha y/k$ and $d/k = \Phi(\varepsilon^*)/R^r(\varepsilon^*)$. Noticing that equation (C10) pins down the value of i/k, we can derive $i/y = (i/k)/(y/k)$. Using the resource constraint and the exogenously given G_a/y by calibration, we obtain $c/y = 1 - G_a/y - i/y$. Dividing (C8) over (C9) and reorganizing yield the steady-state value of labor:

$$N = (1-1/\sigma)\frac{1-\alpha}{\psi}/(\frac{c}{y}).$$

Then by noting that $R_k = R_k(\varepsilon^*) = (1-1/\sigma)\alpha(1+g)^{1-\alpha}k^{\alpha-1}N^{1-\alpha}$, we can solve for k. Combining with the ratios given above, we can then determine y, d, i, c, w, and s. Finally, we have $\Gamma_a = (1-1/\sigma)\Gamma_b = (1-1/\sigma)(y/c)/(1-\beta\xi)$.

Appendix D. Linearized System

Let $\widehat{x}_t = (x_t - x)/x$ denote the log deviation from steady state for any variable x_t except for the surplus s_t and public debt d_t. For these two variables we consider level deviation relative to output, $\widetilde{d}_t = (d_t - d)/y$ and $\widetilde{\tau}_t = (\tau_t - \tau)/y$, instead of log deviation, because d may be zero and τ may be negative.

By standard linearization of the DNK model, we know the deviation of price dispersion $\widehat{\Delta}_t$ is of second-order. Thus we ignore the law of motion for the price dispersion. Moreover, the supply block can be summarized by the New-Keynesian Phillips curve. Hence, we can further eliminate $\widehat{p}^*_t$, $\widehat{\Gamma}^a_t$, and $\widehat{\Gamma}^b_t$. Then the linearized model can be summarized by a system of 16 equations in 16 variables, $\widehat{R}_{kt}$, $\widehat{k}_t$, $\widehat{R}_t$, $\widehat{q}^k_t$, $\widehat{q}^l_t$, $\widehat{\varepsilon}^*_t$, $\widetilde{d}_t$, $\widetilde{\tau}_t$, $\widehat{\Pi}_t$, $\widehat{p}_{wt}$, $\widehat{w}_t$, $\widehat{\lambda}_t$, $\widehat{N}_t$, $\widehat{y}_t$, $\widehat{c}_t$, and $\widehat{i}_t$, where $\widehat{R}_{-1}$, $\widetilde{d}_{-1}$, and $\widehat{k}_{-1}$ are predetermined, and $z_{\tau,t}$, z_{mt}, and $\widehat{G}_{at}$ are exogenous AR(1) processes:

1) The capital return,

$$\widehat{R}_{kt} = \widehat{p}_{wt} + (\alpha-1)\widehat{k}_{t-1} + (1-\alpha)\widehat{N}_t. \tag{D1}$$

2) Evolution of capital,

(D2)
$$\begin{aligned}(1+g)\widehat{k}_t =&(1-\delta)\widehat{k}_{t-1} - \left(\mu + R_k + \frac{R^r d}{k}\right)\varepsilon^{*2} f(\varepsilon^*)\widehat{\varepsilon}^*_t \\ &+ \int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)\left((\mu + R_k)\widehat{k}_{t-1} + R_k \widehat{R}_{kt} + \frac{R^r d}{k}\left(\widehat{R}_{t-1} - \widehat{\Pi}_t\right) + \frac{R^r y}{k}\widetilde{d}_{t-1}\right).\end{aligned}$$

3) Nominal interest rate,

(D3)
$$\widehat{R}_t - \mathbb{E}_t\widehat{\Pi}_{t+1} = \mathbb{E}_t\left(\widehat{\lambda}_t - \widehat{\lambda}_{t+1}\right) - \frac{q^l}{1+q^l}\mathbb{E}_t\widehat{q}^l_{t+1}.$$

4) Tobin's Q,

(D4)
$$\begin{aligned}\widehat{q}^k_t =&\mathbb{E}_t\left(\widehat{\lambda}_{t+1} - \widehat{\lambda}_t\right) + \frac{\beta}{1+g}\frac{R_k(1+q^l)}{q^k}\mathbb{E}_t\widehat{R}_{kt+1} \\ &+ \frac{\beta}{1+g}\frac{(\mu + R_k)\,q^l}{q^k}\mathbb{E}_t\widehat{q}^l_{t+1} + \frac{\beta}{1+g}(1-\delta)\mathbb{E}_t\widehat{q}^k_{t+1}.\end{aligned}$$

5) Liquidity premium,

(D5)
$$\widehat{q}^l_t = -\frac{\int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF}{q^l\varepsilon^*}\widehat{\varepsilon}^*_t.$$

6) Investment cutoff,

(D6)
$$\widehat{\varepsilon}^*_t = -\widehat{q}^k_t.$$

7) Government budget constraint,

(D7)
$$\widetilde{\tau}_t + \widetilde{d}_t = \frac{G_a}{y}\widehat{G}_{at} + \frac{R^r}{1+g}\widetilde{d}_{t-1} + \frac{R^r}{1+g}\frac{d}{y}\left(\widehat{R}_{t-1} - \widehat{\Pi}_t\right).$$

8) Fiscal policy rule,

(D8)
$$\widetilde{\tau}_t = \phi_d\widetilde{d}_{t-1} + z_{\tau,t}.$$

9) Monetary policy rule,

(D9) $$\widehat{R}_t = \phi_\pi \widehat{\Pi}_t + z_{mt}.$$

10) New-Keynesian Phillips curve,

(D10) $$\widehat{\Pi}_t = \beta \mathbb{E}_t \widehat{\Pi}_{t+1} + \kappa \widehat{p}_{wt},$$

where $\kappa = (1-\xi)(1-\beta\xi)/\xi$.

11) Labor demand,

(D11) $$\widehat{w}_t = \widehat{p}_{wt} + \alpha \widehat{k}_{t-1} - \alpha \widehat{N}_t.$$

12) Labor supply,

(D12) $$\widehat{w}_t = -\widehat{\lambda}_t.$$

13) Marginal utility,

(D13) $$\widehat{\lambda}_t = -\widehat{c}_t.$$

14) Aggregate output,

(D14) $$\widehat{y}_t = \alpha \widehat{k}_{t-1} + (1-\alpha)\widehat{N}_t.$$

15) Aggregate investment,

(D15)
$$(1+g)\frac{i}{k}\widehat{i}_t = [1 - F(\varepsilon^*)] \left[(\mu + R_k)\widehat{k}_{t-1} + R_k \widehat{R}_{kt} + \frac{R^r d}{k}\widehat{R}_{t-1} - \frac{R^r d}{k}\widehat{\Pi}_t + \frac{R^r y}{k}\widetilde{d}_{t-1} \right] - \left(\mu + R_k + \frac{R^r d}{k} \right) f(\varepsilon^*)\varepsilon^* \widehat{\varepsilon}^*_t.$$

16) Resource constraint,

(D16) $$\frac{c}{y}\widehat{c}_t + \frac{i}{y}\widehat{i}_t + \frac{G_a}{y}\widehat{G}_{at} = \widehat{y}_t.$$

Appendix E. Additional Results

In this appendix we present some additional results not reported in the main text. First, Figure E1 shows the determinacy region for the steady state in which the interest rate is higher than the economic growth rate. We set the long-run $s/y = 4.45\%$ and fix other parameter values as in Table 1. The implied debt to GDP ratio is 120%.

Next we study welfare for different policy parameter mixes $\phi_d \in [-0.2, 0.2]$ and $\phi_\pi \in [0, 3]$ given adverse financial shocks as in Section IV. We consider parameter values in the set such that the model admits a unique equilibrium. Figures E2, E3, and E4 present the welfare losses in terms of the consumption equivalent relative to the steady state without the financial shock for the equilibria around the three steady states, respectively. We find that the welfare loss is the smallest when $\phi_d = -0.2$ and $\phi_\pi = 0$ in regime F.

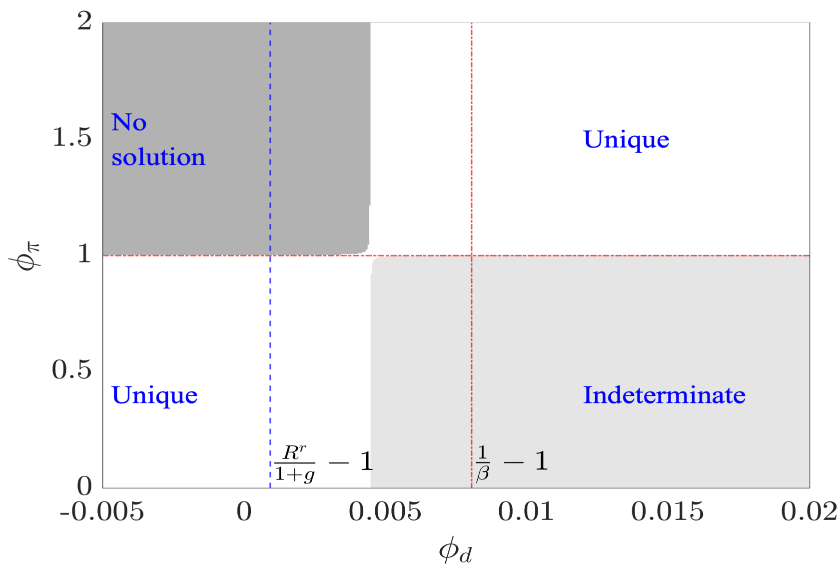

Figure E1. : Determinacy region for the steady state with $R^r > 1 + g$.

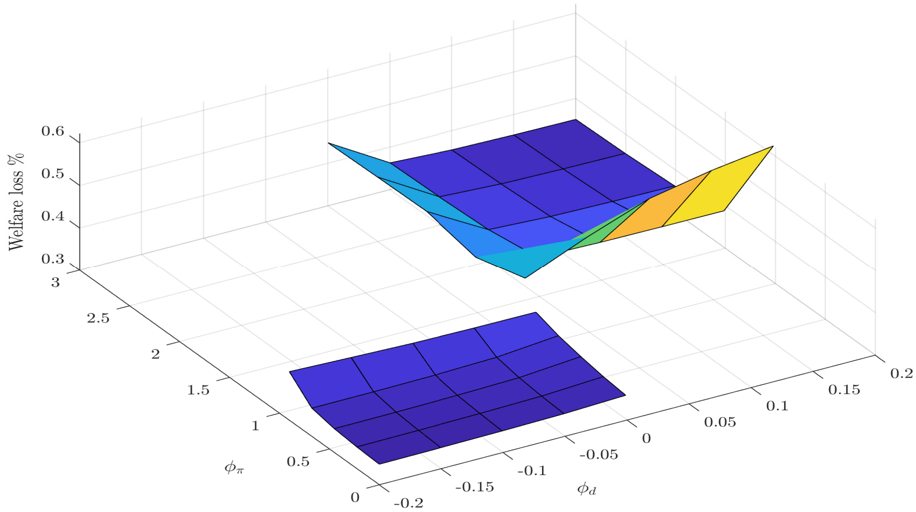

Figure E2. : Welfare loss in response to financial shocks under different policy mixes around steady state L.

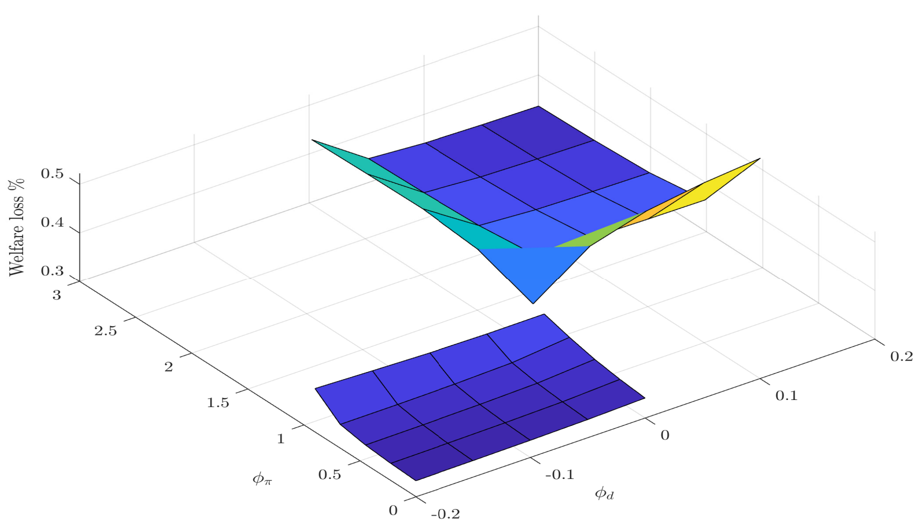

Figure E3. : Welfare loss in response to financial shocks under different policy mixes around steady state H.

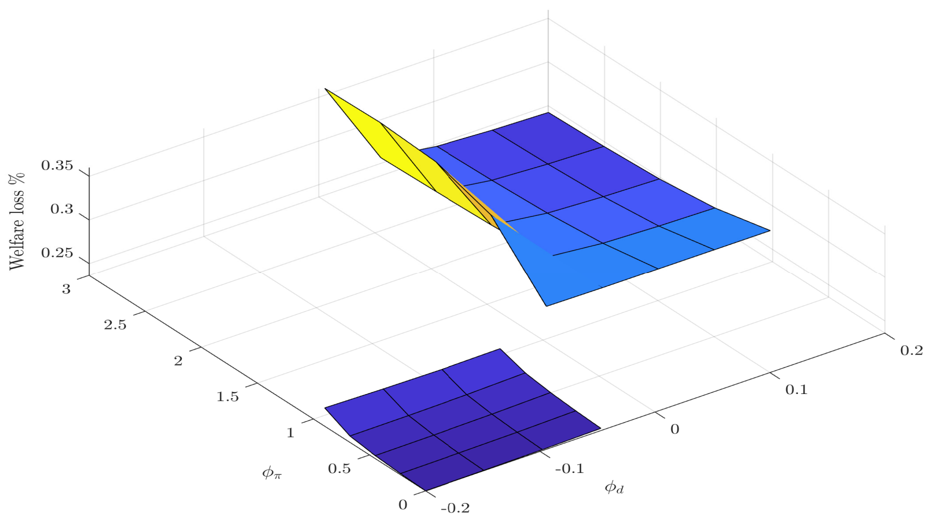

Figure E4. : Welfare loss in response to financial shocks under different policy mixes around the steady state with $R^r > 1 + g$.

Appendix F. The Standard New Keynesian Block

Retailers are monopolistically competitive. Their role is to introduce nominal price rigidities. In each period t they buy intermediate goods from entrepreneurs at the real price p_{wt} and sell good j at the nominal price P_{jt}. Intermediate goods are transformed into final goods according to the CES aggregator

$$Y_t = \left[\int_0^1 Y_{jt}^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1.$$

Thus retailers face demand given by

$$Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\sigma} Y_t, \tag{F1}$$

where the price index is given by

$$P_t \equiv \left[\int_0^1 P_{jt}^{1-\sigma} dj\right]^{\frac{1}{1-\sigma}}. \tag{F2}$$

Aggregating equation (F1) yields aggregate output equation (31).

To introduce price stickiness, we assume that each retailer is free to change its price in any period only with probability $1-\xi$, following Calvo (1983). To introduce trend inflation, we follow Erceg, Henderson and Levin (2000) and assume that whenever the retailer is not allowed to reset its price, its price is automatically increased at the steady-state inflation rate. The retailer selling good j chooses the nominal price P_{jt}^* in period t to maximize the discounted present value of real profits

$$\max_{P_t^*} \sum_{k=0}^{\infty} \xi^k \mathbb{E}_t \left[\frac{\beta^k \Lambda_{t+k}}{\Lambda_t} \left(\frac{\Pi^k P_{jt}^*}{P_{t+k}} - p_{w,t+k}\right) Y_{jt+k}^*\right],$$

subject to the demand curve

$$Y_{jt+k}^* = \left(\frac{\Pi^k P_{jt}^*}{P_{t+k}}\right)^{-\sigma} Y_{t+k}, \ k \geq 0,$$

where Π denotes the steady-state inflation target. We use the household intertemporal marginal rate of substitution as the stochastic discount factor because retailers must hand in all profits to households who are the shareholders.

The first-order condition gives the pricing rule

$$P_{jt}^* = P_t^* \equiv \frac{\sigma}{\sigma-1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\xi)^k \Lambda_{t+k} p_{w,t+k} P_{t+k}^{\sigma} Y_{t+k} (\Pi^k)^{-\sigma}}{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\xi)^k \Lambda_{t+k} P_{t+k}^{\sigma-1} (\Pi^k)^{1-\sigma} Y_{t+k}}$$

for all j. Let $p_t^* = P_t^*/P_t$. We can then write the pricing rule in a recursive form as follows

$$p_t^* = \frac{\sigma}{\sigma-1} \frac{\Gamma_t^a}{\Gamma_t^b},$$

where

$$\begin{aligned} \Gamma_t^a &= \Lambda_t p_{wt} Y_t + \beta\xi \mathbb{E}_t \left(\frac{\Pi_{t+1}}{\Pi}\right)^{\sigma} \Gamma_{t+1}^a, \\ \Gamma_t^b &= \Lambda_t Y_t + \beta\xi \mathbb{E}_t \left(\frac{\Pi_{t+1}}{\Pi}\right)^{\sigma-1} \Gamma_{t+1}^b. \end{aligned}$$

It follows from (F2) and Calvo price setting that

$$1 = \left[\xi \left(\frac{\Pi}{\Pi_t}\right)^{1-\sigma} + (1-\xi)\, p_t^{*1-\sigma}\right]^{\frac{1}{1-\sigma}}.$$