Appendix for “An Efficient Ascending-Bid Auction for Multiple Objects: Comment” For Online Publication

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The following counterexample shows that sincere bidding by all bidders is not always an ex post perfect equilibrium under all rationing rules that satisfy the monotonicity property.

Example 3

Consider a case where there are two bidders, A and B, and three quantities of an object. Let u_A, u_B be the marginal value functions of the two bidders such that

$$u_A(q) = u_B(q) = \begin{cases} 5 & \text{if } q \in [0,1) \\ 1 & \text{if } q \in [1,3]. \end{cases}$$

Consider the history $h^4 = (x_A^t, x_B^t)_{t=0,1,2,3} = \big((3,3),(3,3),(3,3),(3,3)\big)$. After h^4, sincere bidding of each bidder is 1.

The result under sincere bidding $\boldsymbol{x_A^4 = 1}$

If the bidders report $x_A^4 = x_B^4 = 1$ after h^4, then the auction ends at $z^5 = (h^4, (1,1))$, yielding an assignment (x_A^*, x_B^*) such that

$$\begin{aligned} 1 \leq x_A^* &\leq 3, \\ 1 \leq x_B^* &\leq 3, \\ x_A^* + x_B^* &= 3. \end{aligned}$$

Without loss of generality, suppose that $x_A^* \geq \frac{3}{2}$. Since bidder A did not clinch at $t \leq 3$, A's payment is $y_A^* = 4x_A^*$. Therefore, A's utility is $U_A(x_A^*) - 4x_A^*$ at z^5.

The result under misreporting $\hat{x}_A^4 = 0$

If bidder A reports $\hat{x}_A^4 = 0$, and bidder B reports $x_B^4 = 1$ after h^4, then the auction ends at $\hat{z}^5 = (h^4, (0,1))$, yielding an assignment $(\hat{x}_A, \hat{x}_B)$. Since $0 = \hat{x}_A^4 < x_A^4 = 1$ and any other condition of $\hat{z}^5$ is the same as z^5, by the monotonicity property, $\hat{x}_A$ must be strictly less than x_A^*. Similarly to case with sincere bidding, A's utility at $\hat{z}^5$ is $U_A(\hat{x}_A) - 4\hat{x}_A$.

We calculate the difference between A's utilities at z^5 and $\hat{z}^5$,

$$\begin{aligned}
&\Big(U_A(x_A^*) - 4x_A^*\Big) - \Big(U_A(\hat{x}_A) - 4\hat{x}_A\Big) \\
&= \left(\int_0^{x_A^*} u_A(q)dq - 4x_A^*\right) - \left(\int_0^{\hat{x}_A} u_A(q)dq - 4\hat{x}_A\right) \\
&= \left(\int_0^{x_A^*} u_A(q)dq - \int_0^{\hat{x}_A} u_A(q)dq\right) - 4\cdot\Big(x_A^* - \hat{x}_A\Big) \\
&= \int_{\hat{x}_A}^{x_A^*} u_A(q)dq - 4\cdot\Big(x_A^* - \hat{x}_A\Big). \qquad (1)
\end{aligned}$$

Case 1: $\hat{x}_A \geq 1$. We calculate (1) such that

$$x_A^* - \hat{x}_A - 4\cdot\Big(x_A^* - \hat{x}_A\Big) = -3\cdot\Big(x_A^* - \hat{x}_A\Big) < 0.$$

Case 2: $\hat{x}_A < 1$. We calculate (1) such that

$$\begin{aligned}
&\Big(x_A^* - 1\Big) + 5\cdot\Big(1 - \hat{x}_A\Big) - 4\cdot\Big(x_A^* - \hat{x}_A\Big) \\
&= -3x_A^* - \hat{x}_A + 4 < 0 \qquad (\because x_A^* \geq \frac{3}{2}).
\end{aligned}$$

Thus, A's utility at $\hat{z}^5$ is strictly greater than that at z^5, and bidder A has an incentive to misreport after h^4. Therefore, sincere bidding by all bidder is not an ex post perfect equilibrium.

Proof of Lemma 1

Since u_i is a weakly decreasing integer-valued function, there is a partition $\{a_0, \ldots, a_m\} \subset X_i$ with $0 = a_0 < \cdots < a_m = \lambda_i$ and values $\{b_1, \ldots, b_m\} \subset \{0, 1, \ldots, \overline{u}\}$ with $b_1 > b_2 > \cdots > b_m$ such that for each k with $1 \leq k \leq m$,

$$u_i(x_i) = b_k \text{ if } a_{k-1} < x_i < a_k.$$

Note that $m \leq T$. Consider any $x_i' \in X_i$. Let

$$k = \arg\min_{\ell}\{a_\ell : x_i' \leq a_\ell\}.$$

By the definition of Riemann Integral,

$$U_i(x_i') = \int_0^{x_i'} u_i(q)dq = \sum_{\ell=1}^{k-1} b_\ell(a_\ell - a_{\ell-1}) + b_k(x_i' - a_{k-1}). \tag{2}$$

Take any $p \in \{1, \ldots, T\}$. Define $b_0 = T + 1$. Let

$$r = \arg\min_{\ell}\{b_\ell : p - 1 < b_\ell\},$$
$$r' = \arg\min_{\ell}\{b_\ell : p \leq b_\ell\}.$$

By equation (2), we can verify that

$$a_r = \min\{\arg\max_{x_i \in X_i} U_i(x_i) - (p-1)x_i\},$$
$$a_{r'} = \max\{\arg\max_{x_i \in X_i} U_i(x_i) - px_i\}.$$

Because $b_\ell \in \mathbb{Z}$ for each ℓ,

$$\{b_\ell : p - 1 < b_\ell\} = \{b_\ell : p \leq b_\ell\}.$$

Therefore $a_r = a_r'$, that is,

$$\min\{\arg\max_{x_i \in X_i} U_i(x_i) - (p-1)x_i\} = \max\{\arg\max_{x_i \in X_i} U_i(x_i) - px_i\}.$$

To prove Lemma 2 and Proposition 1, we explain some notation and a property of the Ausubel auction.

Notation

- With full bid information, a *strategy* σ_i is a function that maps each non-terminal history $h \in H \setminus Z$ to a quantity $x_i \in X_i$, that is, $\sigma_i : H \setminus Z \to X_i$.
- For each non-terminal history $h \in H \setminus Z$, the set of histories in the subgame that follows h is given by

  $$H|_h = \{h' \in H : h' = (h, h'') \text{ for some sequence } h''\},$$

  and the set of terminal histories in the subgame is given by

  $$Z|_h = Z \cap H|_h.$$

- For each non-terminal history $h \in H \setminus Z$ and each strategy σ_i, we denote $\sigma_i|_h : H|_h \setminus Z|_h \to X_i$ the induced strategy in the subgame that follows h. For each $h' \in H_h \setminus Z_h$, $\sigma_i(h') = \sigma_i|_h(h')$.
- Let $\pi_i(\cdot)$ be the utility of bidder i at an n-tuple of strategies.

Property 1

For each $t \geq 1$, if there exists a bidder $i \in N$ such that $x_i^t = C_i^{t-1}$ and $C_i^{t-1} > 0$, then the auction ends at t, i.e., $t = L$. Therefore, if the auction does not end at t, then for each bidder $i \in N$, $x_i^t \neq C_i^{t-1}$ or $C_i^{t-1} = 0$.

Proof. Suppose that $x_i^t = C_i^{t-1}$ and $C_i^{t-1} > 0$. Then, $x_i^t = M - \sum_{j \neq i} x_j^{t-1}$. By bidding constraint for each $j \in N$, $x_j^t \leq x_j^{t-1}$. Therefore $\sum_{j \in N} x_j^t \leq M$. $\square$

Note that this property holds under all rationing rules. We use the property in proofs of Lemma 2 and Proposition 1.

Proof of Lemma 2

Consider any $t \in \{0, 1, \ldots, T\}$,

$$h^t = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq t-1} \in H^t \setminus Z^t,$$

and $(u_j)_{j \in N}$. For each $j \in N$, let σ_j^* be sincere bidding which is corresponding to u_j, and $\sigma_j^*|_{h^t}$ be induced sincere bidding in the subgame that follows h^t.

Take any $i \in N$ and $\sigma_i \in \Sigma_i|_{h^t}$. Suppose that $x_i^{t-1} \leq Q_i(p^{t-1})$. We shall show that

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Let

$$z^{L+1} = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq L}$$

be the terminal history which is reached by $(\sigma_j^*|_{h^t})_{j \in N}$, and

$$w^{L'+1} = (\hat{x}_1^s, \hat{x}_2^s, \ldots, \hat{x}_n^s)_{s \leq L'}$$

be the terminal history which is reached by $(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i})$. Denote $\{(C_j^t)_{j \in N}\}_{t=0}^{L}$ the cumulative clinches of z^{L+1}, and $\{(\hat{C}_j^t)_{j \in N}\}_{t=0}^{L'}$ the cumulative clinches of $w^{L'+1}$.

Step 1. $x_i^{L-1} \leq Q_i(p^{L-1})$.

If $L-1 = t-1$, $x_i^{L-1} = x_i^{t-1} \leq Q_i(p^{t-1}) = Q_i(p^{L-1})$. Then, let $L-1 \geq t$. By the definition of sincere bidding,

$$x_i^{L-1} = \sigma_i^*|_{h^t}\big((x_1^\ell, \ldots, x_n^\ell)_{\ell \leq L-2}\big) = \min\{x_i^{L-2}, \max\{C_i^{L-2}, Q_i(p^{L-1})\}\}.$$

By Property 1, $x_i^{L-1} \neq C_i^{L-2}$ or $C_i^{L-2} = 0$. Then, $x_i^{L-1} = \min\{x_i^{L-2}, Q_i(p^{L-1})\}$. Therefore, $x_i^{L-1} \leq Q_i(p^{L-1})$.

Step 2. For each $j \neq i$ and $s \leq \min\{L-1, L'-1\}$, $x_j^s = \hat{x}_j^s$. This implies that for each $s \leq \min\{L-1, L'-1\}$,

$$C_i^s = M - \sum_{j \neq i} x_j^s = M - \sum_{j \neq i} \hat{x}_j^s = \hat{C}_i^s.$$

For each $s \leq t-1$, obviously $x_j^s = \hat{x}_j^s$.

For the cases with $t \leq s \leq \min\{L-1, L'-1\}$, we shall show by induction. Let $s = t$. Because $x_j^t = \sigma_j^*|_{h^t}(h^t)$ and $\hat{x}_j^t = \sigma_j^*|_{h^t}(h^t)$, $x_j^t = \hat{x}_j^t$.
Let $s = k$ with $t+1 \leq k \leq \min\{L-1, L'-1\}$). Suppose that $x_j^\ell = \hat{x}_j^\ell$ for all ℓ with $t+1 \leq \ell \leq k-1$. By the definition of sincere bidding,

$$x_j^k = \sigma_j^*|_{h^t}\big((x_1^\ell, \ldots, x_n^\ell)_{\ell \leq k-1}\big) = \min\{x_j^{k-1}, \max\{C_j^{k-1}, Q_j(p^k)\}\},$$
$$\hat{x}_j^k = \sigma_j^*|_{h^t}\big((\hat{x}_1^\ell, \ldots, \hat{x}_n^\ell)_{\ell \leq k-1}\big) = \min\{\hat{x}_j^{k-1}, \max\{\hat{C}_j^{k-1}, Q_j(p^k)\}\}.$$

Since $k \leq \min\{L-1, L'-1\}$, by Property 1, $x_j^k \neq C_j^{k-1}$ or $C_j^{k-1} = 0$. Thus, $x_j^k = \min\{x_j^{k-1}, Q_j(k)\}$. Similarly, we have $\hat{x}_j^k = \min\{\hat{x}_j^{k-1}, Q_j(k)\}$. Since $x_j^{k-1} = \hat{x}_j^{k-1}$, $x_j^k = \hat{x}_j^k$.

Step 3. $\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i})$.

We consider three cases; $L = L'$, $L > L'$ and $L < L'$.

Case 1. $L = L'$.

By step 2, for all $s \leq L-1 = L'-1$, $C_i^s = \hat{C}_i^s$. We calculate C_i^L and $\hat{C}_i^L$ for two cases with $x_i^L \geq Q_i(p^L)$ and $x_i^L < Q_i(p^L)$.

Case 1-1. $x_i^L \geq Q_i(p^L)$.

By step 1, $x_i^{L-1} \leq Q_i^{L-1}$. Thus,

$$Q_i(p^L) \leq x_i^L \leq C_i^L \leq x_i^{L-1} \leq Q_i(p^{L-1}).$$

Therefore, by lemma 1,

$$\min\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\} \leq C_i^L \leq \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\}.$$

Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Case 1-2. Let $x_i^L < Q_i(p^L)$.

We shall show that $x_i^L = x_i^{t-1}$. By the definition of sincere bidding,

$$x_i^L = \sigma_i^*|_{h^t}((x_1^s, \ldots, x_n^s)_{s \leq L-1}) = \min\{x_i^{L-1}, \max\{C_i^{L-1}, Q_i(p^L)\}\}.$$

Since $x_i^L < Q_i(p^L)$, $x_i^L = x_i^{L-1}$. If $L-1 = t-1$, $x_i^{L-1} = x_i^{t-1}$. Then, we assume $t-1 \neq L-1$. By the definition of sincere bidding,

$$x_i^{L-1} = \sigma_i^*|_{h^t}((x_1^s, \ldots, x_n^s)_{s \leq L-2}) = \min\{x_i^{L-2}, \max\{C_i^{L-2}, Q_i(p^{L-1})\}\}.$$

Since $Q_i(p^L) \leq Q_i(p^{L-1})$, $x_i^{L-1} = x_i^L < Q_i(p^{L-1})$. Hence, we have $x_i^{L-1} = x_i^{L-2}$. By repeating this procedure, $x_i^L = x_i^{L-1} = \cdots = x_i^{t-1}$. Thus, $C_i^L = x_i^{t-1}$.

Since bidder i cannot bid more quantity than x_i^{t-1} after h^t, $\hat{C}_i^L \leq x_i^{t-1}$. Then,

$$\hat{C}_i^L \leq C_i^L < Q_i(p^L).$$

Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Case 2. $L > L'$.

By step 2, for each $s \leq L'-1$, $C_i^s = \hat{C}_i^s$. Then, we calculate $\{C_i^s\}_{s=L'}^L$ and $\hat{C}_i^{L'}$. Since the auction does not end at L' in the history z^{L+1}, by Property 1 for each $j \neq i$, $x_j^{L'} \neq C_j^{L'-1}$ or $C_j^{L'-1} = 0$. Then, by the definition of sincere bidding, for each $j \neq i$,

$$x_j^{L'} = \min\{x_j^{L'-1}, Q_j(p^{L'})\}.$$

On the other hand,

$$\hat{x}_j^{L'} = \min\{\hat{x}_j^{L'-1}, \max\{\hat{C}_j^{L'-1}, Q_j(p^{L'})\}\}.$$

Since $x_j^{L'-1} = \hat{x}_j^{L'-1}$, $x_j^{L'} \leq \hat{x}_j^{L'}$. Thus,

$$\hat{C}_i^{L'} \leq M - \sum_{j \neq i} \hat{x}_j^{L'} \leq M - \sum_{j \neq i} x_j^{L'} = C_i^{L'}.$$

By the definition of cumulative clinches, for each $s \in \{L', \ldots, L-1\}$, $C_i^s \leq x_i^s$ and $x_i^L \leq C_i^L \leq x_i^{L-1}$. For each $s \in \{L', \ldots, L-1\}$, because $s \geq t$, x_i^s is sincere bidding. That is,

$$x_i^s = \min\{x_i^{s-1}, \max\{C_i^{s-1}, Q_i(p^s)\}\}.$$

Since the auction does not end at $s \leq L-1$ in the history z^{L+1}, by Property 1,

$$x_i^s = \min\{x_i^{s-1}, Q_i(p^s)\}.$$

Therefore, for each $s \in \{L', \dots, L-1\}$, $x_i^s \leq Q_i(p^s)$. Thus,

$$\begin{aligned} C_i^s &\leq Q_i(p^s) \qquad \forall s \in \{L', \dots, L-1\}, \\ \hat{C}_i^{L'} &\leq C_i^{L'} \leq Q_i(p^{L'}), \\ C_i^L &\leq x_i^{L-1} \leq Q_i(p^{L-1}) = \max\{\arg\max_{x_i \in X_i}(U_i(x_i) - p^L x_i)\}. \end{aligned}$$

Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

Case 3. $L < L'$.

We first show that $Q_i(p^L) \leq x_i^L$. Suppose that $Q_i(p^L) > x_i^L$. Similarly to case 1-2, we have $x_i^L = x_i^{t-1}$. By bidding constraint, $\hat{x}_i^L \leq x_i^{t-1}$. Then, $\hat{x}_i^L \leq x_i^L$. Since $L < L'$, the auction does not end at L in the history $w^{L'+1}$. Therefore, by Property 1, for each $j \neq i$, $\hat{x}_i^L \neq \hat{C}_i^L$ or $\hat{C}_i^L = 0$. By the definition of sincere bidding

$$\begin{aligned} x_j^L &= \min\{x_j^{L-1}, \max\{C_j^{L-1}, Q_j(p^L)\}\}, \\ \hat{x}_j^L &= \min\{\hat{x}_j^{L-1}, \max\{\hat{C}_j^{L-1}, Q_j(p^L)\}\} = \min\{\hat{x}_j^{L-1}, Q_j(p^L)\}. \end{aligned}$$

For each $j \neq i$, since by step 2, $x_j^{L-1} = \hat{x}_j^{L-1}$, we have $x_j^L \geq \hat{x}_j^L$. Hence for each $j \in N$, $x_j^L \geq \hat{x}_j^L$. Since the auction ends at L in the history z^{L+1}, $\sum_{j\in N} x_j^L \leq M$. Therefore, $\sum_{j\in N} \hat{x}_j^L \leq M$. This implies the auction ends at L in the history $w^{L'+1}$. This contradicts to $L < L'$. Thus, $Q_i(p^L) \leq x_i^L$.

By step 2, for each $s \leq L-1$, $C_i^s = \hat{C}_i^s$. Similarly to case 2, we have $C_i^L \leq \hat{C}_i^L$. Because $x_i^L \leq C_i^L$, $Q_i(p^L) \leq C_i^L \leq \hat{C}_i^L$. Moreover, for each $s \geq L$, $\hat{C}_i^L \leq \hat{C}_i^s$ and $Q_i(p^s) \leq Q_i(p^L)$. Thus, for each $s \geq L$, $Q_i(p^s) \leq \hat{C}_i^s$. Hence,

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$

Proof of Proposition 1

Consider any $t \in \{0, 1, \ldots, T\}$,

$$h^t = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq t-1} \in H^t \setminus Z^t,$$

and $(u_j)_{j \in N}$. For each $j \in N$, let σ_j^* be sincere bidding which is corresponding to u_j, and $\sigma_j^*|_{h^t}$ be induced sincere bidding in the subgame that follows h^t.

Take any $i \in N$ and $\sigma_i \in \Sigma_i|_{h^t}$. We shall show that

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

If $x_i^{t-1} \leq Q_i(p^{t-1})$, we can show by Lemma 2. Suppose that $x_i^{t-1} > Q_i(p^{t-1})$.

Let

$$z^{L+1} = (x_1^s, x_2^s, \ldots, x_n^s)_{s \leq L}$$

be the terminal history which is reached by $(\sigma_j^*|_{h^t})_{j \in N}$, and

$$w^{L'+1} = (\hat{x}_1^s, \hat{x}_2^s, \ldots, \hat{x}_n^s)_{s \leq L'}$$

be the terminal history which is reached by $(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i})$. Denote $\{(C_j^t)_{j \in N}\}_{t=0}^{L}$ the cumulative clinches of z^{L+1}, and $\{(\hat{C}_j^t)_{j \in N}\}_{t=0}^{L'}$ the cumulative clinches of $w^{L'+1}$.

We consider three cases; $L > t$, $L' > L = t$ and $L' = L = t$.

Case 1. $L > t$.

Since $L - 1 \geq t$, by the definition of sincere bidding,

$$x_i^{L-1} = \sigma_i^*|_{h^t}\big((x_1^\ell, \ldots, x_n^\ell)_{\ell \leq L-2}\big) = \min\{x_i^{L-2}, \max\{C_i^{L-2}, Q_i(p^{L-1})\}\}.$$

By Property 1, $x_i^{L-1} \neq C_i^{L-2}$ or $C_i^{L-2} = 0$. Then, $x_i^{L-1} = \min\{x_i^{L-2}, Q_i(p^{L-1})\}$. Therefore, $x_i^{L-1} \leq Q_i(p^{L-1})$, which is the same argument as step 1 of Lemma 2. Note that we only use the assumption $x_i^{t-1} \leq Q_i(p^{t-1})$ in step 1 of Lemma 2. Thus, we can prove this case similarly to Lemma 2.

Case 2. $L' > L = t$.

For each $j \in N$ and each $s \leq t - 1$, obviously $x_j^s = \hat{x}_j^s$. Therefore, for each

$s \le t-1 = L-1$, $C_i^s = \hat{C}_i^s$. We will calculate C_i^L and $\{\hat{C}_i^s\}_{s=L}^{L'}$.

We first show that $Q_i(p^L) \le C_i^L$. By the definition of sincere bidding,

$$x_i^L = \min\{x_i^{L-1}, \max\{C_i^{L-1}, Q_i(p^L)\}\}.$$

Since $x_i^{L-1} \ge C_i^{L-1}$ and $x_i^{L-1} > Q_i(p^{L-1}) \ge Q_i(p^L)$,

$$x_i^L = \max\{C_i^{L-1}, Q_i(p^L)\}.$$

Therefore, $Q_i(p^L) \le x_i^L$. Because $x_i^L \le C_i^L \le x_i^{L-1}$, $Q_i(p^L) \le C_i^L$.

Next we show that $C_i^L \le \hat{C}_i^L$. For each $j \ne i$, because $t = L$, $x_j^L = \sigma_j^*|_{h^t}(h^t)$ and $\hat{x}_j^L = \sigma_j^*|_{h^t}(h^t)$. Therefore, for each $j \ne i$, $x_j^L = \hat{x}_j^L$. Since the auction does not end at L in the history $w^{L'+1}$,

$$\hat{C}_i^L = M - \sum_{j \ne i} \hat{x}_j^L = M - \sum_{j \ne i} x_j^L.$$

On the other hand, since the auction ends at L in the history z^{L+1},

$$C_i^L \le M - \sum_{j \ne i} x_j^L.$$

Therefore, $C_i^L \le \hat{C}_i^L$.

Hence, $Q_i(p^L) \le C_i^L \le \hat{C}_i^L$. Furthermore, for all $s \ge L+1$, $Q_i(p^s) \le \hat{C}_i^s$, because $Q_i(p^s) \le Q_i(p^L)$ and $\hat{C}_i^L \le \hat{C}_i^s$. Thus,

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \ge \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \ne i}).$$

Case 3. $L' = L = t$.

For each $j \in N$ and each $s \le t-1$, obviously $x_j^s = \hat{x}_j^s$. Furthermore, for each $j \ne i$, $x_j^L = \sigma_j^*|_{h^t}(h^t) = \hat{x}_j^L$. Since for each $s \le L-1$, $C_i^s = \hat{C}_i^s$, we calculate C_i^L and $\hat{C}_i^L$.

Case 3-1. $C_i^L = x_i^L$.

By the definition of sincere bidding,

$$x_i^L = \min\{x_i^{L-1}, \max\{C_i^{L-1}, Q_i(p^L)\}\}.$$

Since $x_i^{L-1} \geq C_i^{L-1}$ and $x_i^{L-1} > Q_i(p^{L-1}) \geq Q_i(p^L)$,

$$x_i^L = \max\{C_i^{L-1}, Q_i(p^L)\}.$$

If $x_i^L = Q_i(p^L)$, then $C_i^L = Q_i(p^L)$ and we have

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Suppose that $x_i^L = C_i^{L-1}$. Then, $C_i^{L-1} \geq Q_i(p^L)$ and $C_i^L = C_i^{L-1}$. Since $\hat{C}_i^L \geq \hat{C}_i^{L-1}$ and $C_i^{L-1} = \hat{C}_i^{L-1}$, $\hat{C}_i^L \geq C_i^L$. Therefore, $\hat{C}_i^L \geq C_i^L \geq Q_i(p^L)$. Hence

$$\pi_i((\sigma_j^*|_{h^t})_{j \in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j \neq i}).$$

Case 3-2. $C_i^L > x_i^L$.

First, we show that for each $j \in \{1, \ldots, i-1\}$, $C_j^L = x_j^{L-1}$. Suppose that there exists $j \in \{1, \ldots, i-1\}$ such that $C_j^L \neq x_j^{L-1}$. By the definition of our rationing rule,

$$\begin{aligned} C_j^L &= \min\{x_j^{L-1}, x_j^L + M - \sum_{k=j}^{n} x_k^L - \sum_{k=1}^{j-1} C_k^L\} \\ &= x_j^L + M - \sum_{k=j}^{n} x_k^L - \sum_{k=1}^{j-1} C_k^L. \end{aligned}$$

Therefore,

$$M = \sum_{k=j+1}^{n} x_k^L - \sum_{k=1}^{j} C_k^L.$$

Since for each $k \in N$, $x_k^L \leq C_k^L$, and $\sum_{k \in N} C_k^L = M$,

$$M = \sum_{k=j+1}^{n} x_k^L - \sum_{k=1}^{j} C_k^L \leq \sum_{k \in N} C_k^L = M.$$

Therefore, for each $k \geq j+1$, $x_k^L = C_k^L$. Because $i \geq j+1$, this contradicts to $C_i^L > x_i^L$.

Next, we show that $C_i^L \leq \hat{C}_i^L$. By the definition of our rationing rule,

$$C_i^L = \min\{x_i^{L-1}, x_i^L + M - \sum_{j=i}^{n} x_j^L - \sum_{j=1}^{i-1} C_j^L\} \qquad = \min\{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} C_j^L\},$$

$$\hat{C}_i^L = \min\{x_i^{L-1}, \hat{x}_i^L + M - \hat{x}_j^L - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} \hat{C}_j^L\} \quad = \min\{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} \hat{C}_j^L\}.$$

For each $j \leq i-1$, since $x_j^{L-1} = C_j^L$ and $x_j^{L-1} \geq \hat{C}_j^L$,

$$C_j^L \geq \hat{C}_j^L.$$

Therefore,

$$\min\{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} C_j^L\} \leq \min\{x_i^{L-1}, M - \sum_{j=i+1}^{n} x_j^L - \sum_{j=1}^{i-1} \hat{C}_j^L\}.$$

Hence, $C_i^L \leq \hat{C}_i^L$.

Similarly to case 2, we can show that $Q_i(p^L) \leq C_i^L$. Therefore, $Q_i(p^L) \leq C_i^L \leq \hat{C}_i^L$. Thus,

$$\pi_i((\sigma_j^*|_{h^t})_{j\in N}) \geq \pi_i(\sigma_i, (\sigma_j^*|_{h^t})_{j\neq i}).$$