Online Appendix for "Balanced Growth Despite Uzawa"

by

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Proofs from Section I

Proof of Lemma 1

By assumption $C_t < Y_t$. Therefore, the resource constraint $Y_t = C_t + I_t/q_t$ ensures $I_t > 0$. The capital accumulation equation is $\dot{K}_t = I_t - \delta K_t$ implying

$$g_K = \frac{\dot{K}_t}{K_t} = \frac{I_t}{K_t} - \delta.$$

On a BGP g_K is constant meaning that since $I_t > 0$ the growth rates of I and K must be the same. Thus, $g_I = g_K$.

Differentiating the resource constraint and rearranging gives

$$\left(g_C - g_Y\right)\frac{C_t}{Y_t} + \left(g_I - g_q - g_Y\right)\frac{I_t/q_t}{Y_t} = 0.$$

Substituting for $\frac{I_t/q_t}{Y_t} = 1 - \frac{C_t}{Y_t}$ in this expression and using $g_I = g_K$ we have

$$\left(g_K - g_q - g_C\right)\frac{C_t}{Y_t} = g_K - g_q - g_Y.$$

If both sides of this expression equals zero we immediately obtain $g_Y = g_C = g_K - g_q$ as claimed in the lemma. Otherwise, since the growth rates are constant on a BGP it must be that C and Y grow at the same rate implying $g_Y = g_C$. But then the resource constraint implies $\frac{I_t/q_t}{Y_t} = 1 - \frac{C_t}{Y_t}$ is constant and, since $g_I = g_K$, this ensures $g_Y = g_K - g_q$. Therefore, the lemma holds.

Proof of Proposition 1

Since factors are paid their marginal products the capital share is $\theta_K = K_t F_K\left(A_t K_t, B_t L_t, s_t\right)/Y_t$. Note also that because F has constant returns to scale in its first two arguments $F_K\left(A_t K_t, B_t L_t, s_t\right) = A_t F_1\left(A_t K_t, B_t L_t, s_t\right) = A_t F_1(k_t, 1, s_t)$ where $k_t = A_t K_t/B_t L_t$.[1] Therefore, on a BGP where the capital share is positive and constant we have[2]

[1] To avoid possible confusion, note that we use $F_K(\cdot)$ and $F_L(\cdot)$ to denote the partial derivatives of $F(\cdot)$ with respect to K and L, respectively, while $F_1(\cdot)$ and $F_2(\cdot)$ denote the partial derivatives of $F(\cdot)$ with respect to its first and second arguments, respectively.

[2] Instead of assuming constant factor shares, this expression can also be obtained by assuming the rental price of capital R_t declines at rate g_q. To see this differentiate $R_t = A_t F_1\left(k_t, 1, s_t\right)$.

$$0=\frac{\dot{\theta}_K}{\theta_K}=g_A+g_K-g_Y+\frac{d\log F_1\left(k_t,1,s_t\right)}{dt}=\gamma_K+\frac{d\log F_1\left(k_t,1,s_t\right)}{dt},$$

where the final equality uses Lemma 1 and $\gamma_K=g_A+g_q$.

Taking the derivative of F_1 and using $kF_{11}+F_{12}=0$ we have

$$\gamma_K=-\frac{F_{11}\dot{k}_t+F_{1s}\dot{s}_t}{F_1}=\frac{F_{12}}{F_1}\frac{\dot{k}_t}{k_t}-\frac{F_{1s}\dot{s}_t}{F_1}=\frac{1}{\sigma_{KL}}\frac{F_2}{F}\frac{\dot{k}_t}{k_t}-\frac{F_{1s}\dot{s}_t}{F_1},$$

where the final equality uses $\sigma_{KL}=(F_1F_2)/(FF_{12})$. Since $1-\theta_K=F_2/F$ this can be rearranged to give

$$\sigma_{KL}\gamma_K=\left(1-\theta_K\right)\frac{\dot{k}_t}{k_t}-\sigma_{KL}\frac{F_{1s}\dot{s}_t}{F_1}. \tag{1}$$

To simplify (1) it will be useful to derive an expression for F_{1s}/F_1. Note that

$$\frac{\partial}{\partial K}\left[\frac{F_s(A_tK_t,B_tL_t,s_t)}{F_L(A_tK_t,B_tL_t,s_t)}\right]=\frac{F_{Ks}}{F_L}-\frac{F_{LK}F_s}{F_L^2}=\frac{F_K}{F_L}\left(\frac{F_{Ks}}{F_K}-\frac{1}{\sigma_{KL}}\frac{F_s}{F}\right). \tag{2}$$

Rearranging, we have $\frac{F_{1s}}{F_1}=\frac{F_{Ks}}{F_K}=\frac{F_L}{F_K}\frac{\partial[F_s/F_L]}{\partial K}+\frac{1}{\sigma_{KL}}\frac{F_s}{F}$. Plugging this into (1) gives

$$\sigma_{KL}\gamma_K=\left(1-\theta_K\right)\frac{\dot{k}_t}{k_t}-\sigma_{KL}\frac{F_L}{F_K}\frac{\partial\left[F_s/F_L\right]}{\partial K}\dot{s}_t-\frac{F_s\dot{s}_t}{F}. \tag{3}$$

Finally, differentiating the production function $Y_t=F(A_tK_t,B_tL_t,s_t)$ yields

$$\begin{aligned}g_Y&=\theta_K\left(g_A+g_K\right)+\left(1-\theta_K\right)\left(g_B+g_L\right)+\frac{F_s\dot{s}_t}{F},\\&=g_A+g_K-\left(1-\theta_K\right)\frac{\dot{k}_t}{k_t}+\frac{F_s\dot{s}_t}{F}.\end{aligned}$$

Using Lemma 1 and $\gamma_K=g_A+g_q$ this implies

$$\gamma_K=\left(1-\theta_K\right)\frac{\dot{k}_t}{k_t}-\frac{F_s\dot{s}_t}{F}.$$

Substituting this expression into (3) gives equation (1). This completes the proof.

Proposition 1 assumes technical change is factor augmenting, but we can generalize the proposition by relaxing this restriction. Suppose the production function is $Y = \hat{F}(K, L, s; t)$ where technical change is captured by the dependence of $\hat{F}$ on t. We can decompose technical change into a Harrod-neutral component and a non-Harrod-neutral residual. Technical change is Harrod-neutral if, holding the capital-output ratio and schooling fixed, it does not affect the marginal product of capital (Uzawa 1961). Therefore, we can define the non-Harrod-neutral component of technical change as the change in the marginal product of capital for a given capital-output ratio and schooling.

Let φ be the capital-output ratio and define $\hat{\kappa}(\varphi, s; t)$ by

$$\varphi = \frac{\hat{\kappa}(\varphi, s; t)}{\hat{F}(\hat{\kappa}(\varphi, s; t), 1, s; t)}.$$

$\hat{\kappa}(\varphi, s; t)$ is the capital-labor ratio that ensures the capital-output ratio equals φ given s and t. Differentiating this expression with respect to t while holding s and φ constant and using $\theta_K = \hat{\kappa}\hat{F}_1/\hat{F}$ implies

$$\frac{\hat{\kappa}_t}{\hat{\kappa}} = \frac{1}{1-\theta_K}\frac{\hat{F}_t}{\hat{F}}. \tag{4}$$

When technical change is Harrod-neutral $\frac{d}{dt}\log \hat{F}_1(\hat{\kappa}(\varphi, s; t), 1, s; t) = \hat{\kappa}_t \frac{\partial}{\partial \hat{\kappa}} \log \hat{F}_1 + \frac{\partial}{\partial t}\log \hat{F}_1 = 0$. Thus, we define the non-Harrod-neutral component of technical change Ψ by

$$\Psi \equiv -\sigma_{KL}\left[\hat{\kappa}_t \frac{\partial}{\partial \hat{\kappa}} \log \hat{F}_1(\hat{\kappa}(\varphi, s; t), 1, s; t) + \frac{\partial}{\partial t}\log \hat{F}_1(\hat{\kappa}(\varphi, s; t), 1, s; t)\right].$$

From this definition we have

$$\begin{aligned}\Psi &= -\sigma_{KL}\left(\frac{\hat{F}_{11}\hat{\kappa}_t}{\hat{F}_1} + \frac{\hat{F}_{1t}}{\hat{F}_1}\right), \\ &= -\sigma_{KL}\left(\frac{\hat{F}_{11}}{\hat{F}_1}\frac{\hat{\kappa}}{1-\theta_K}\frac{\hat{F}_t}{\hat{F}} + \frac{\hat{F}_{1t}}{\hat{F}_1}\right), \\ &= \frac{\hat{F}_t}{\hat{F}} - \sigma_{KL}\frac{\hat{F}_{1t}}{\hat{F}_1}, \qquad (5)\end{aligned}$$

where the second line follows from (4) and the third line uses $\hat{\kappa}\hat{F}_{11} = -\hat{F}_{12}$, the definition of σ_{KL} and $1-\theta_K = \hat{F}_2/\hat{F}$. Note that in the case where technical change is factor augmenting we have $\hat{F}(K,L,s;t) = F(A_tK, B_tL, s)$ which implies $\Psi = (1-\sigma_{KL})g_A$.

Using the expression for Ψ given in (5) we obtain the following generalization of Proposition 1.

PROPOSITION 3: *Suppose the production function is $Y = \hat{F}(K,L,s;t)$ and that investment-specific technological progress occurs at constant rate g_q. If there exists a BGP along which the income shares of capital and labor are constant and strictly positive when factors are paid their marginal products, then*

$$(1-\sigma_{KL})\,g_q + \Psi = \sigma_{KL}\frac{\hat{F}_L}{\hat{F}_K}\frac{\partial\left[\hat{F}_s/\hat{F}_L\right]}{\partial K}\dot{s}.$$

To avoid repetition, we omit the proof of Proposition 3 since it follows the same series of steps used to prove Proposition 1.

Suppose either s is constant as in Corollary 1 or the production function can be written in terms of a measure of human capital $H(L,s,t)$ implying $\frac{\partial[\hat{F}_s/\hat{F}_L]}{\partial K} = 0$ as in Corollary 2. Then Proposition 3 implies a BGP with constant and strictly positive factor shares can exist only if $(1-\sigma_{KL})g_q + \Psi = 0$. Thus, a BGP with $\sigma_{KL} \leq 1$, $g_q \geq 0$ and $\Psi \geq 0$ is possible only if technical change that affects the production function is Harrod-neutral and either the elasticity of substitution between capital and labor equals one or there is no investment-specific technological change.

Proofs from Section II

Implications of Assumption 1

Taking the partial derivative of the production function with respect to s gives

$$F_s = -\frac{D'(s)}{D(s)}\left[bLF_L - aKF_K\right],$$

and from this we obtain

$$\frac{\partial}{\partial K}\left(\frac{F_s}{F_L}\right) = -\frac{D'(s)}{D(s)}a\left[-\frac{F_K}{F_L} - \frac{KF_{KK}}{F_L} + \frac{KF_KF_{LK}}{F_L^2}\right].$$

Since F exhibits constant returns to scale in K and L we have $F = KF_K + LF_L$ and $KF_{KK} = -LF_{LK}$. Using these results in the expression above we have

$$\frac{\partial}{\partial K}\left(\frac{F_s}{F_L}\right) = -\frac{D'(s)}{D(s)}a\frac{FF_{LK}}{F_L^2}\left(1-\sigma_{KL}\right),$$

which is strictly positive under Assumption 1 since $a > 0$, $\sigma_{KL} < 1$ and $D'(s) < 0$.

F is strictly log supermodular in K and s if and only if $F_{Ks}F - F_K F_s > 0$. Using Assumption 1 to compute these derivatives gives

$$F_{Ks}F - F_K F_s = -\frac{D'(s)}{D(s)}(a+b)LFF_{LK}\left(1 - \sigma_{KL}\right).$$

Since $a+b > 0$ and $D'(s) < 0$ it follows that under the functional form restriction in Assumption 1 the production function F is strictly log supermodular in K and s if and only if $\sigma_{KL} < 1$.

Second Order Condition of the Planner's Problem

The planner chooses z_t to maximize Y_t which is equivalent to choosing z_t to maximize $z_t^{-\theta}h(z_t)$. The first order condition is

$$-\theta z_t^{-\theta-1}h(z_t) + z_t^{-\theta}h'(z_t) = 0,$$

and the second order condition is

$$(z^*)^{-\theta-1}h(z^*)\frac{d}{dz}\mathcal{E}_h(z^*) < 0.$$

Since $\mathcal{E}_h(z)$ is strictly decreasing in z if and only if $\sigma_{KL} < 1$ it follows that the second order condition is satisfied if and only if $\sigma_{KL} < 1$.

Transition Dynamics of the Planner's Problem

After solving for optimal schooling we can write the planner's problem as

$$\max_{\{c_t\}} \int_{t_0}^{\infty} N_t e^{-\rho(t-t_0)}\frac{c_t^{1-\eta}-1}{1-\eta}dt$$

subject to

$$\dot{K}_t = q_t\left[Y_t(K_t) - N_t c_t\right] - \delta K_t.$$

where $Y_t(K_t)$ is given by (3) with $z_t = z^*$.

Solving this problem we find the planner chooses a consumption path that satisfies

(6) $$\frac{\dot{c}_t}{c_t} = -\frac{\rho+\delta+g_q}{\eta} + \frac{\theta q_t}{\eta}\frac{Y_t(K_t)}{K_t}.$$

Now let $\tilde{Y}_t = e^{-g_Y(t-t_0)}Y_t(K_t)$, $\tilde{C}_t = e^{-g_Y(t-t_0)}N_t c_t$ and $\tilde{K}_t = e^{-g_K(t-t_0)}K_t$ where g_Y is given by part (i) of Proposition 2 and $g_K = g_Y + g_q$. Using (6) and the capital accumulation equation together with the fact that q_t, A_t, B_t and N_t grow at constant rates g_q, g_A, γ_L and n, respectively, we have

$$\tilde{Y}_t = \tilde{Y}\left(\tilde{K}_t\right) = A_{t_0}^{\theta}\left(B_{t_0}N_{t_0}\right)^{1-\theta}\left(z^*\right)^{-\theta} h\left(z^*\right)\tilde{K}_t^{\theta},$$

$$\dot{\tilde{C}}_t = \left[-g_Y + n - \frac{\rho+\delta+g_q}{\eta} + \frac{\theta q_{t_0}}{\eta}\frac{\tilde{Y}(\tilde{K}_t)}{\tilde{K}_t}\right]\tilde{C}_t, \tag{7}$$

$$\dot{\tilde{K}}_t = -(g_Y + g_q + \delta)\tilde{K}_t + q_{t_0}\left[\tilde{Y}\left(\tilde{K}_t\right) - \tilde{C}_t\right]. \tag{8}$$

Since consumption and schooling can jump, K_t (or, equivalently $\tilde{K}_t$) is the economy's only state variable. The pair of differential equations (7) and (8) govern the evolution of the economy from any initial condition K_{t_0}.

Figure 3 depicts a familiar phase diagram. The vertical line labeled CC has $\tilde{K} = \tilde{K}^*$ such that

$$\frac{\tilde{Y}(\tilde{K}^*)}{\tilde{K}^*} = \frac{1}{\theta q_{t_0}}\left[\eta\left(g_Y - n\right) + \rho + \delta + g_q\right].$$

From (7), we see that $\dot{\tilde{C}}_t = 0$ along this line. The curve labeled KK has $\tilde{C} = \tilde{Y}(\tilde{K}) - (g_Y + g_q + \delta)\tilde{K}/q_{t_0}$. This curve, which from (8) depicts combinations of $\tilde{C}$ and $\tilde{K}$ such that $\dot{\tilde{K}}_t = 0$, can be upward sloping (as drawn) or hump-shaped. In either case, the two curves intersect on the upward sloping part of KK.[3] The intersection gives the unique steady-state values of $\tilde{K} = \tilde{K}^*$ and $\tilde{C} = \tilde{C}^*$, which in turn identify the unique BGP. As is clear from the figure, the BGP is reached by a unique equilibrium trajectory that is saddle-path stable.

Alternative Formulation of Assumption 1

Proposition 4 provides an alternative formulation of Assumption 1 that can be used whenever the marginal product of schooling is positive as guaranteed by part (i) of Assumption 2.

PROPOSITION 4: *Assumption 1 holds with $F_s(AK, BL, s) > 0$ if and only if the production function can be written as*

$$F\left(AK, BL, s\right) = (BL)^{\frac{a}{a+b}}\, G\left[AK, D(s)^{-(a+b)}BL\right]^{\frac{b}{a+b}}$$

[3]To see this, note that $\tilde{Y}'\left(\tilde{K}_t\right) = \theta\frac{\tilde{Y}(\tilde{K}_t)}{\tilde{K}_t}$. Consequently, the slope of the KK curve is $\theta\frac{\tilde{Y}(\tilde{K}_t)}{\tilde{K}_t} - \frac{g_Y+g_q+\delta}{q_{t_0}}$ which is positive when $\tilde{K} = \tilde{K}^*$ by part (iii) of Assumption 2.

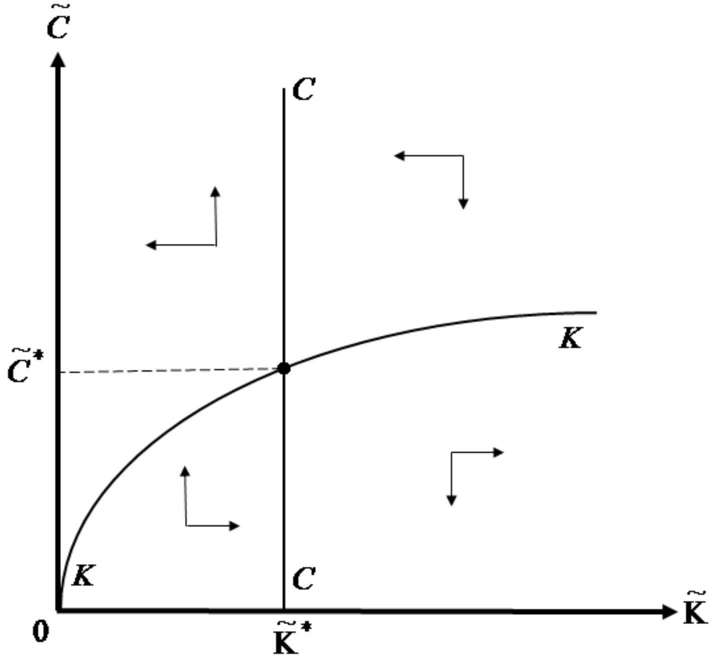

Figure 3. Transitional dynamics and stability of the balanced growth path

with $a, b > 0$, where $G(\cdot)$ is constant returns to scale, strictly increasing in both its arguments and

(i) $G(z, 1)$ is twice differentiable, and strictly concave for all z;

(ii) $\sigma^G_{KL} \equiv G_L G_K / G G_{KL} < 1$.

PROOF:

Suppose Assumption 1 holds with $F_s > 0$ and define

$$G\left[AK, D(s)^{-(a+b)}BL\right] = \left[D(s)^{-(a+b)}BL\right]^{\frac{-a}{b}} \tilde{F}\left[AK, D(s)^{-(a+b)}BL\right]^{\frac{a+b}{b}}.$$

This definition implies $G(\cdot)$ is constant returns to scale and

$$F(AK, BL, s) = \tilde{F}\left[D(s)^a AK, D(s)^{-b}BL\right] = (BL)^{\frac{a}{a+b}} G\left[AK, D(s)^{-(a+b)}BL\right]^{\frac{b}{a+b}}.$$

Differentiating $G(\cdot)$ yields

$$G_K = \left[D(s)^{-(a+b)}BL\right]^{\frac{-a}{b}} \frac{a+b}{b}\tilde{F}^{\frac{a}{b}}\tilde{F}_K > 0,$$

$$G_L = \left[D(s)^{-(a+b)}BL\right]^{\frac{-a}{b}} \frac{1}{bL}\tilde{F}^{\frac{a}{b}}\left[(a+b)L\tilde{F}_L - a\tilde{F}\right].$$

$F_s > 0$ implies $bL\tilde{F}_L - aK\tilde{F}_K > 0$. Using this result together with $\tilde{F} = K\tilde{F}_K + L\tilde{F}_L$ gives $G_L > 0$.

Next, observe that $G(z,1) = \tilde{F}(z,1)^{\frac{a+b}{b}}$. Therefore

$$G_{zz}(z,1) = \frac{a+b}{b}\tilde{F}(z,1)^{\frac{a}{b}-1}\left[\tilde{F}(z,1)\tilde{F}_{zz}(z,1) + \frac{a}{b}\tilde{F}_z(z,1)^2\right].$$

This expression is negative since $z\tilde{F}_{zz}(z,1) = -\tilde{F}_{z2}(z,1), b\tilde{F}_2(z,1) - az\tilde{F}_z(z,1) > 0$ because $F_s > 0$ and $\sigma_{KL} < 1$. It follows that $G(z,1)$ is twice differentiable, and strictly concave for all z.

Finally, we have

$$G_{KL} = \left[D(s)^{-(a+b)}BL\right]^{\frac{-a}{b}} \frac{a+b}{b}\tilde{F}^{\frac{a}{b}}\left[\tilde{F}_{KL} + \frac{a}{b}\frac{\tilde{F}_K\tilde{F}_L}{\tilde{F}} - \frac{a}{b}\frac{\tilde{F}_K}{L}\right],$$

meaning

$$\sigma^G_{KL} = \frac{\tilde{F}_K\tilde{F}_L + \frac{a}{b}\tilde{F}_K\tilde{F}_L - \frac{a}{b}\frac{\tilde{F}\tilde{F}_K}{L}}{\tilde{F}\tilde{F}_{KL} + \frac{a}{b}\tilde{F}_K\tilde{F}_L - \frac{a}{b}\frac{\tilde{F}\tilde{F}_K}{L}},$$

which is less than one since $\sigma_{KL} < 1$.

The converse can be proved in the same manner after defining

$$\tilde{F}\left[D(s)^a AK, D(s)^{-b}BL\right] = \left[D(s)^{-b}BL\right]^{\frac{a}{a+b}} G\left[D(s)^a AK, D(s)^{-b}BL\right]^{\frac{b}{a+b}}.$$

This completes the proof.

Necessity of Functional Form

Consider an economy that satisfies the assumptions required for Lemma 1 to hold and has production function $F(K, L, s; t)$ which is constant returns to scale in its first two arguments. Suppose factors are paid their marginal products and schooling is chosen to satisfy

$$s_t = \arg\max_s F(K_t, L_t, s; t) \text{ subject to } L_t = D(s) N_t.$$

We assume this optimization problem has a unique interior maximum.

Suppose the economy is on a BGP from time T onwards with a constant capital share $\theta_K \in (0,1)$. With a slight abuse of notation define $\tilde{F}$ by

$$\tilde{F}(K, L, s; t) = \tilde{F}\left[A_t K D(s)^a, B_t L D(s)^{-b}\right] \equiv F\left[A_t K D(s)^a, B_t L D(s)^{-b}, s_T; T\right],$$

where $b = 1 + a\theta_K / (1 - \theta_K)$, while A_t and B_t are defined by

$$A_t \equiv e^{g_Y(t-T)} D(s_t)^{-a} \frac{K_T}{K_t},$$
$$B_t \equiv e^{g_Y(t-T)} D(s_t)^{b} \frac{L_T}{L_t}.$$

Since a and b jointly satisfy a single restriction, $\tilde{F}$ defines a one dimensional family of functions.

Differentiating the definitions of A_t and B_t together with the constraint $L_t = D(s_t)N_t$ and using Lemma 1 we obtain

$$\gamma_K \equiv \frac{\dot{A}_t}{A_t} + g_q = a(n - g_L),$$
$$\gamma_L \equiv \frac{\dot{B}_t}{B_t} = g_Y - n - \frac{\theta_K}{1 - \theta_K}\gamma_K.$$

γ_K is the total rate of capital-augmenting technical change, while γ_L is the rate of labor-augmenting technical change. When both n and the labor force growth rate g_L are constant then γ_K and γ_L are also constant. Also, provided schooling is increasing over time $n > g_L$ implying that $a > 0$ if and only if γ_K is strictly positive.

We can now prove the following proposition. Part (i) shows that on the BGP F has a one dimensional family of representations of the form $\tilde{F}\left[A_t K D(s)^a, B_t L D(s)^{-b}\right]$. From the expressions for γ_K and γ_L above we see that each member of this family has a different combination of capital-augmenting and labor-augmenting technical change. When we say the production function can be represented by $\tilde{F}$ we mean that the equilibrium allocation and the marginal products of capital, labor and schooling on the BGP are the same under $\tilde{F}$ as under F. However, this does not imply that counterfactual experiments using $\tilde{F}$ will necessarily coincide with counterfactuals under F. The first order impact of some policy changes (e.g., schooling subsidies, capital taxation) depends on σ_{KL} and $\sigma_{Ks} \equiv (F_K F_s)/(F_{Ks} F)$. Therefore, in part (ii) of the proposition we show that if σ_{KL} is constant on the BGP then $\sigma_{KL} = \tilde{\sigma}_{KL} \equiv (\tilde{F}_K \tilde{F}_L)/(\tilde{F}_{KL}\tilde{F})$ and that $\tilde{\sigma}_{Ks} \equiv (\tilde{F}_K \tilde{F}_s)/(\tilde{F}_{Ks}\tilde{F})$ can be written as a function of $\tilde{\sigma}_{KL}$, a and b. Consequently, if σ_{KL} and σ_{Ks} are constant on the BGP then there exist unique values of a and b such that $\tilde{\sigma}_{KL} = \sigma_{KL}$ and $\tilde{\sigma}_{Ks} = \sigma_{Ks}$. Thus, knowing σ_{KL} and σ_{Ks} is sufficient to separate the roles played by capital-augmenting and labor-augmenting technical change. Moreover, when a and b are chosen appropriately counterfactual analysis using $\tilde{F}$ instead of F will, to a first order, give accurate predictions.

PROPOSITION 5: *Suppose for all $t \geq T$ the economy's equilibrium trajectory $\{Y_t, K_t, L_t, s_t\}$ is a BGP with constant and strictly positive factor shares. On the BGP*

(i) The production function F can be represented by $\tilde{F}$ in the sense that for all $t \geq T$

$$\begin{aligned}
\tilde{F}(K_t, L_t, s_t; t) &= F(K_t, L_t, s_t; t), \\
\tilde{F}_K(K_t, L_t, s_t; t) &= F_K(K_t, L_t, s_t; t), \\
\tilde{F}_L(K_t, L_t, s_t; t) &= F_L(K_t, L_t, s_t; t), \\
\tilde{F}_s(K_t, L_t, s_t; t) &= F_s(K_t, L_t, s_t; t);
\end{aligned}$$

(ii) $\tilde{\sigma}_{KL}$ and $\tilde{\sigma}_{Ks}$ satisfy

$$\frac{1}{\tilde{\sigma}_{Ks}} - 1 = (a+b)\left(\frac{1}{\tilde{\sigma}_{KL}} - 1\right),$$

and if σ_{KL} is constant then $\tilde{\sigma}_{KL} = \sigma_{KL}$.

PROOF:

Without loss of generality let $T = 0$. Output at $t \geq 0$ is given by

$$\begin{aligned}
F(K_t, L_t, s_t; t) = Y_t = e^{g_Y t} Y_0 = e^{g_Y t} F(K_0, L_0, s_0; 0) &= F\left(e^{g_Y t} K_0, e^{g_Y t} L_0, s_0; 0\right), \\
&= F\left(A_t K_t D(s_t)^a, B_t L_t D(s_t)^{-b}, s_0; 0\right), \\
&= \tilde{F}(K_t, L_t, s_t; t).
\end{aligned}$$

To show the marginal products of capital are equal, we use the facts that the capital share is constant over time and capital is paid its marginal product. Therefore

$$\begin{aligned}\frac{K_t F_K\left(K_t, L_t, s_t; t\right)}{Y_t} = \theta_K = \frac{K_0 F_1\left(K_0, L_0, s_0; 0\right)}{Y_0} &= \frac{e^{g_Y t} K_0 F_1\left(e^{g_Y t} K_0, e^{g_Y t} L_0, s_0; 0\right)}{e^{g_Y t} Y_0}, \\ &= \frac{A_t K_t D\left(s_t\right)^a F_1\left(A_t K_t D\left(s_t\right)^a, B_t L_t D\left(s_t\right)^{-b}, s_0; 0\right)}{Y_t}, \\ &= \frac{K_t \tilde{F}_K\left(K_t, L_t, s_t; t\right)}{Y_t}.\end{aligned}$$

Dividing each side by K_t/Y_t gives $F_K\left(K_t, L_t, s_t; t\right) = \tilde{F}_K\left(K_t, L_t, s_t; t\right)$. Identical logic using the labor share gives $F_L\left(K_t, L_t, s_t; t\right) = \tilde{F}_L\left(K_t, L_t, s_t; t\right)$.

To complete the proof of part (i) we show equality of the marginal products of schooling. Optimal schooling choice implies

$$\frac{D'\left(s_t\right) L_t}{D\left(s_t\right)} = -\frac{F_s\left(K_t, L_t, s_t; t\right)}{F_L\left(K_t, L_t, s_t; t\right)}.$$

This means the ratio of the marginal product of schooling to output can be written as

$$\frac{F_s\left(K_t, L_t, s_t; t\right)}{Y_t} = -\left(1-\theta_K\right) \frac{D'\left(s_t\right)}{D\left(s_t\right)}.$$

We now show that same equation holds for $\tilde{F}$. Differentiating $\tilde{F}$ with respect to s and dividing by output gives

$$\begin{aligned}\frac{\tilde{F}_s\left(K_t, L_t, s_t; t\right)}{Y_t} &= \frac{1}{Y_t}\frac{D'\left(s_t\right)}{D\left(s_t\right)}\left[a A_t K_t D\left(s_t\right)^a F_1\left(A_t K_t D\left(s_t\right)^a, B_t L_t D\left(s\right)^{-b}, s_0; 0\right)\right. \\ &\quad \left. - b B_t L_t D\left(s_t\right)^{-b} F_2\left(A_t K_t D\left(s_t\right)^a, B_t L_t D\left(s_t\right)^{-b}, s_0; 0\right)\right], \\ &= \left[a\theta_K - b\left(1-\theta_K\right)\right]\frac{D'\left(s_t\right)}{D\left(s_t\right)}, \\ &= -(1-\theta_K)\frac{D'\left(s_t\right)}{D\left(s_t\right)}.\end{aligned}$$

To prove part (ii) we start by noting that when σ_{KL} is constant on the BGP, the homogeneity of F implies

$$
\begin{aligned}
\sigma_{KL} &= \frac{F_1\left(K_0, L_0, s_0; 0\right) F_2\left(K_0, L_0, s_0; 0\right)}{F_{12}\left(K_0, L_0, s_0; 0\right) F\left(K_0, L_0, s_0; 0\right)}, \\
&= \frac{F_1\left(e^{g_Y t} K_0, e^{g_Y t} L_0, s_0; 0\right) F_2\left(e^{g_Y t} K_0, e^{g_Y t} L_0, s_0; 0\right)}{F_{12}\left(e^{g_Y t} K_0, e^{g_Y t} L_0, s_0; 0\right) F\left(e^{g_Y t} K_0, e^{g_Y t} L_0, s_0; 0\right)}, \\
&= \frac{F_1\left(A_t K_t D\left(s_t\right)^a, B_t L_t D\left(s_t\right)^{-b}, s_0; 0\right) F_2\left(A_t K_t D\left(s_t\right)^a, B_t L_t D\left(s_t\right)^{-b}, s_0; 0\right)}{F_{12}\left(A_t K_t D\left(s_t\right)^a, B_t L_t D\left(s_t\right)^{-b}, s_0; 0\right) F\left(A_t K_t D\left(s_t\right)^a, B_t L_t D\left(s_t\right)^{-b}, s_0; 0\right)}, \\
&= \frac{\tilde{F}_K\left(K_t, L_t, s_t; t\right) \tilde{F}_L\left(K_t, L_t, s_t; t\right)}{\tilde{F}_{KL}\left(K_t, L_t, s_t; t\right) \tilde{F}\left(K_t, L_t, s_t; t\right)}, \\
&= \tilde{\sigma}_{KL}.
\end{aligned}
$$

Next define $\hat{h}(z) \equiv F(z, 1, s_0; 0)$. Then we have

$$
\tilde{F}\left(K, L, s; t\right) = B_t L D(s)^{-b} \hat{h}\left[\frac{A_t K}{B_t L} D(s)^{a+b}\right].
$$

Taking derivatives of this expression implies

$$
\begin{aligned}
\tilde{\sigma}_{KL} &= \frac{\mathcal{E}_{\hat{h}}\left[\frac{A_t K}{B_t L} D(s)^{a+b}\right] - 1}{\mathcal{E}_{\hat{h}'}\left[\frac{A_t K}{B_t L} D(s)^{a+b}\right]}, \\
\tilde{\sigma}_{Ks} &= \frac{\frac{b}{a+b} - \mathcal{E}_{\hat{h}}\left[\frac{A_t K}{B_t L} D(s)^{a+b}\right]}{\frac{b}{a+b} - 1 - \mathcal{E}_{\hat{h}'}\left[\frac{A_t K}{B_t L} D(s)^{a+b}\right]}.
\end{aligned}
$$

On the BGP we also have

$$
\theta_K = \frac{K_t \tilde{F}_K\left(K_t, L_t, s_t; t\right)}{Y_t} = \mathcal{E}_{\hat{h}}\left[\frac{A_t K_t}{B_t L_t} D(s_t)^{a+b}\right].
$$

Combining these expressions and using $b = 1 + a\theta_K / \left(1 - \theta_K\right)$ we have that on the BGP

$$
\frac{1}{\tilde{\sigma}_{Ks}} - 1 = (a+b)\left(\frac{1}{\tilde{\sigma}_{KL}} - 1\right).
$$

This completes the proof.

"Time-in-School" Model

A firm that employs K_t units of physical capital and hires L_t time units from workers with schooling s_t at time t produces

$$F\left(A_tK_t, B_tL_t, s_t\right) = \tilde{F}\left[A_tK_t\left(1-s_t\right)^a, B_tL_t\left(1-s_t\right)^{-b}\right]$$

units of output. The production technology satisfies Assumption 1 and the parameter restrictions in Assumption 2 also apply. Aggregate output is simply the sum of the outputs produced by all firms.

Since $F(\cdot)$ has constant returns to scale in its first two arguments we can define the intensive form production function by $f(k,s) \equiv F(k,1,s)$ where $f(\cdot)$ is output per effective unit of labor and $k = A_tK/B_tL$ is the ratio of effective capital to effective labor. Using Assumption 1 the intensive form production function can be written as $f(k,s) = (1-s)^{-b}h\left[k(1-s)^{a+b}\right]$.

The competitive firms take the rental rate per unit of capital, R_t, and the wage schedule per unit of time, $W_t(s)$, as given. A firm that hires workers with education s_t chooses L_t and k_t to maximize $B_tL_t\left[f\left(k_t,s_t\right) - r_tk_t - w_t\left(s_t\right)\right]$, where $r_t \equiv R_t/A_t$ is the rental rate per effective unit of capital and $w_t\left(s_t\right) \equiv W_t\left(s_t\right)/B_t$ is the wage per effective unit of labor. Profit maximization implies, as usual, that

$$f_k\left(k_t, s_t\right) = r_t \tag{9}$$

and[4]

$$f\left(k_t, s_t\right) - r_tk_t = w_t\left(s_t\right). \tag{10}$$

We define the functions $\kappa(s,r)$ and $\omega(s,r)$ such that $f_k\left[\kappa(s,r), s\right] \equiv r$ and $\omega(s,r) \equiv f\left[\kappa(s,r), s\right] - r\kappa(s,r)$. Then, in equilibrium, $k_t = \kappa\left(s_t, r_t\right)$ and $w_t\left(s_t\right) = \omega\left(s_t, r_t\right)$.

An individual alive at time t who seeks to maximize dynastic utility should choose s to maximize her own wage income, $B_t(1-s)\,\omega\left(s, r_t\right)$, taking the rental rate per unit of effective capital as given. The rental rate will determine, via (9), how much capital the individual will be allocated by her employer as a reflection of her schooling choice. The individual's education decision is separable from her choice of consumption. The first-order condition for income maximization at time t requires

$$\left(1-s_t\right)\omega_s\left(s_t, r_t\right) = \omega\left(s_t, r_t\right).$$

But using $\omega\left(s, r_t\right) \equiv f\left[\kappa\left(s, r_t\right), s\right] - r_t\kappa\left(s, r_t\right)$ and noting (9), we have $\omega_s\left(s_t, r_t\right) = f_s\left[\kappa\left(s_t, r_t\right), s_t\right]$. In other words, the marginal effect of schooling on the wage reflects only the direct effect of schooling on per capita output; the extra output that comes from a greater capital allocation to more highly educated workers,

[4]Equation (10) is the zero-profit condition, which is implied by the optimal choice of L_t in an equilibrium with positive output.

$f_k\kappa_s$, just offsets the extra part of revenue that the firm must pay for that capital, $r\kappa_s$. Consequently, we can rewrite the first-order condition as

$$(1-s_t)\, f_s\left[\kappa\left(s_t,r_t\right),s_t\right] = f\left[\kappa\left(s_t,r_t\right),s\right] - f_k\left[\kappa\left(s_t,r_t\right),s_t\right]\kappa\left(s_t,r_t\right) \ .$$

Now replace $f\left(k,s\right)$ by $(1-s)^{-b}h\left[k(1-s)^{a+b}\right]$ and use this representation to calculate $f_s\left(\cdot\right)$ and $f_k\left(\cdot\right)$. After rearranging terms, this yields

$$\begin{aligned}(b-1)\,h\left[\kappa\left(s_t,r_t\right)\left(1-s_t\right)^{a+b}\right] &= (a+b-1)\,h'\left[\kappa\left(s_t,r_t\right)\left(1-s_t\right)^{a+b}\right] \\ &\quad \times \kappa\left(s_t,r_t\right)\left(1-s_t\right)^{a+b}\end{aligned}$$

or

$$\mathcal{E}_h\left[\kappa\left(s_t,r_t\right)\left(1-s_t\right)^{a+b}\right] = \frac{b-1}{a+b-1} \ .$$

Since $\kappa(s_t,r_t)=k_t=A_tK_t/B_tL_t$ and $L_t=N_t(1-s_t)$ this expression is identical to the first order condition for optimal schooling choice given in the paper.

Dynasties' intertemporal optimization decisions yield the same consumption and savings choices as in the planner's problem. To see this, start from the no arbitrage condition $\iota_t = R_t/p_t + g_p - \delta$ where ι_t denotes the real interest rate and $p_t = 1/q_t$ is the equilibrium price of a unit of capital.[5] Combining this with $r_t = R_t/A_t$ gives

$$r_t = \frac{1}{q_tA_t}\left(\iota_t + g_q + \delta\right). \tag{11}$$

Individuals' optimal schooling choices imply $\kappa(s_t,r_t)(1-s_t)^{a+b} = z^*$ for all $t \geq t_0$ where z^* takes the same value as in the planner's problem. Therefore, aggregate output is given by (3) with $z_t = z^*$, just as in the planner's problem.

Using $f\left(k,s\right) = (1-s)^{-b}h\left[k(1-s)^{a+b}\right]$ the first order condition for profit maximization (9) yields

$$r_t = (1-s_t)^a h'(z^*).$$

Substituting this expression into the capital market clearing condition $k_t = \kappa(s_t,r_t)$ and using (11) implies the real interest rate satisfies

$$\iota_t = -g_q - \delta + q_tA_t^{\theta}\left(\frac{B_tN_t}{K_t}z^*\right)^{1-\theta} h'(z^*).$$

[5] The no-arbitrage condition states that the real interest rate on a short-term bond equals the dividend rate on a unit of physical capital plus the rate of capital gain on capital equipment (positive or negative), minus depreciation.

Combining this equation with the representative dynasty's Euler equation $\dot{c}_t/c_t = (\iota_t - \rho)/\eta$ and using $\mathcal{E}_h(z^*) = \theta$ and (3) gives

$$\frac{\dot{c}_t}{c_t} = -\frac{\rho + \delta + g_q}{\eta} + \frac{\theta q_t}{\eta}\frac{Y_t(K_t)}{K_t}.$$

Noting that this equation is identical to equation (6) we see that consumption per capita satisfies the same differential equation as in the planner's problem. Since the capital accumulation equation is also the same in both cases we conclude that consumption and the aggregate capital stock follow the same equilibrium trajectory as in the planner's problem.

Schooling Choice in the "Manager-Worker" Model

Recall that the production function can be written as

$$\tilde{F}\left[A_t K D(s)^a, B_t L D(s)^{-b}\right] = B_t L D(s)^{-b} h\left[k D(s)^{a+b}\right]$$

where $s = M/L$, $k = A_t K/B_t L$ and $D(s) = [1 + s/(1-m)]^{-1}$. Since $W_{Mt} = \tilde{F}_M$ and $W_{Lt} = \tilde{F}_L$, differentiating yields

$$\begin{aligned} W_{Mt} &= (a+b) B_t D(s_t)^{-b} \frac{D'(s_t)}{D(s_t)} h\left[k_t D(s_t)^{a+b}\right]\left\{-\frac{b}{a+b} + \mathcal{E}_h\left[k_t D(s_t)^{a+b}\right]\right\}, \\ W_{Lt} &= B_t D(s_t)^{-b} h\left[k_t D(s_t)^{a+b}\right] \\ &\quad \left(1 - \mathcal{E}_h\left[k_t D(s_t)^{a+b}\right] + (a+b)\frac{s_t D'(s_t)}{D(s_t)}\left\{\frac{b}{a+b} - \mathcal{E}_h\left[k_t D(s_t)^{a+b}\right]\right\}\right). \end{aligned}$$

Substituting these expressions into $(1-m)W_{Mt} = W_{Lt}$ and using $D'(s) = -D(s)^2/(1-m)$ implies that, in equilibrium

$$\mathcal{E}_h\left[\left(1 + \frac{s_t}{1-m}\right)^{-(a+b)} k_t\right] = \frac{b-1}{a+b-1}.$$

The fact that $\mathcal{E}_h(z)$ is declining in z ensures stability of the equilibrium.