Misallocation and the Distribution of Global Volatility: Online Appendix on Alternative Microfoundations

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This online appendix discusses alternative microfoundations that are consistent with the paper's main results. The main results of the paper are as follows.

Result 1 (closed economy result) *at the steady state, output volatility is higher in more distorted economies.*

Result 2 (open economy result) *at the steady state, financial integration increases volatility relatively more in more distorted economies.*

In the paper, the derivation of these results builds on a strong assumption regarding the relationship between misallocation and decreasing returns. In particular, it is assumed that more efficient allocations imply aggregate production functions that have uniformly steeper marginal products. In general, models of misallocation can generate aggregate production functions that do not always satisfy this property. Despite this, it turns out that standard models of misallocation are consistent with these results, at least for some parametric restrictions, and often for the same reasons as those highlighted in the paper.

The discussion that follows will focus on models that generate steady states in which the rate of return to savers, r^{ss}, satisfies $\beta(1 + r^{ss}) = 1$ for some $\beta \in (0, 1)$. At the steady state, savings (s) fluctuate exogenously; it is assumed that $var(\ln(s))$ is the same across countries, regardless of the degree of misallocation. Finally, it will

*This note reflects my own views and not necessarily those of the World Bank, its Executive Directors or the countries they represent. Please send comments to meden@worldbank.org.

be useful to assume that the depreciation rate of capital, δ, satisfies $\delta = 1$. Nothing hinges on this assumption but it simplifies the exposition. The following lemma will be useful for establishing results 1 and 2 in the context of different models.

Lemma 1 *Let $k(r)$ denote the equilibrium capital level given r. Assume that:*

1. $\frac{\partial \ln(Y(k^{ss}))}{\partial \ln k}$ *is larger in more distorted economies, and*
2. $|\frac{\partial \ln(k(r^{ss}))}{\partial r}|$ *is larger in more distorted economies.*

Then, Results 1 and 2 hold.

Result 1 follows from the lemma's first assumption: since $\delta = 1$, it follows that, in a closed economy, $s_t = k_{t+1}$. Hence, given that $var(\ln(s))$ is the same across countries, so is $var(\ln(k))$. In the closed economy,

$$var(\ln(Y)) \approx (\frac{\partial \ln(Y)}{\partial \ln(k)})^2 var(\ln(k)) = (\frac{\partial \ln(Y)}{\partial \ln(k)})^2 var(\ln(s)) \tag{1}$$

hence, a higher elasticity of output with respect to capital implies higher output volatility in the closed economy.

To establish Result 2, note that the volatility of the capital stock in country i is approximately:

$$var(\ln(k_i)) \approx (\frac{\partial \ln(k_i(r^{ss}))}{\partial r})^2 var(r_i) \tag{2}$$

When economies are financially integrated, they face the same interest rate and $var(r_i) = var(r)$. Thus, using the second condition of the lemma, it follows that capital volatility is higher in more distorted economies. Since, under autarky, capital volatility is the same across countries, financial integration increases capital volatility relatively more in more distorted economies. The same applies to output volatility given the approximation in equation 1.

A 2x2 example. Before proceeding with the analysis of richer models of misallocation, it is useful to illustrate the conditions of Lemma 1 in a simple 2x2 example. Consider a discrete environment in which there are only two projects, $x = 1, 2$, where $A(1) > A(2)$. In this economy, there are only two possible orders of implementation: with probability $\gamma = \omega(1,1) = \omega(2,2)$, projects are implemented in the efficient order ($\pi(1) = 1$ and $\pi(2) = 2$), and with probability $1 - \gamma = \omega(1,2) = \omega(2,1)$, the order of

implementation is reverse ($\pi(1) = 2$ and $\pi(2) = 1$). Note that γ captures the probability of an efficient allocation at the micro level. Thus, the conditions of Lemma 1 hold if the discrete counterparts of $\frac{\partial \ln(Y(k^{ss}))}{\partial \ln k}$ and $|\frac{\partial \ln(k(r^{ss}))}{\partial r}|$ are decreasing in γ.

There are two possible capital levels in this economy: $k = 1$ or $k = 2$. Given a probability γ of an efficient allocation, the expected marginal product, $y(k)$, is given by:

$$y(k) = \begin{cases} \gamma A(1) + (1-\gamma)A(2) & \text{if } k = 1 \\ \gamma A(2) + (1-\gamma)A(1) & \text{if } k = 2 \end{cases} \tag{3}$$

the aggregate production function is:

$$Y(k) = \begin{cases} y(1) = \gamma A(1) + (1-\gamma)A(2) & \text{if } k = 1 \\ y(1) + y(2) = A(1) + A(2) & \text{if } k = 2 \end{cases} \tag{4}$$

In this environment, $y(1)$ is increasing in γ and $y(2)$ is decreasing in γ. The restriction $\gamma \geq 0.5$ guarantees that $y(1) \geq y(2)$.

The discrete counterpart of the first condition of Lemma 1 follows from equation 4:

$$\ln(Y(2)) - \ln(Y(1)) = \ln(A(1) + A(2)) - \ln(y(1)) \tag{5}$$

which is decreasing in γ because $y(1)$ is increasing in γ.

Using the relationship $\frac{\partial k}{\partial r} = 1/(\frac{\partial r}{\partial k})$, and the equilibrium condition $y = r + \delta$, the discrete counterpart of the second condition of Lemma 1 amounts to the monotonicity of $y(1) - y(2)$. This follows similarly from the fact that $y(1)$ is increasing in γ and $y(2)$ is decreasing in γ.

This benchmark highlights the intuitions underlying the relationship between misallocation and the conditions of Lemma 1. The following sections illustrate that, under certain conditions, this relationship can be obtained in standard models of misallocation. Section 1 considers limited pledgeability in the Kiyotaki and Moore [1997] setup. Section 2 considers misallocation due to adverse selection, as in Stiglitz and Weiss [1981]. Finally, section 3 explores the case of misallocation due to uncertainty, in the spirit of Asker et al. [2014].

1 Limited pledgeability

This section derives the conditions of Lemma 1 in the standard Kiyotaki and Moore [1997] framework. Both conditions can be derived with minor modifications to the original framework. While the first condition is a direct outcome of misallocation, the second condition is obtained somewhat mechanically using the assumption that constrained firms operate a constant returns technology.

In the Kiyotaki and Moore setup, the aggregate production function is an aggregation of two types of projects: "farming" projects and "gathering" projects. There is a measure 1 of farmers and a measure 1 of gatherers. Gatherers produce according to a decreasing returns technology, $G(k^g)$, where k^g is the capital stock employed by the gatherer, and the production function G satisfies $G' > 0$ and $G'' < 0$. This production technology can be thought of as a collection of "gathering" projects with productivities given by $A^G(x^g) = G'(x^g)$ for $x^g \in (0, \infty)$, and an allocation in which gatherers implement "gathering" projects in an efficient order.

Farmers have a constant returns technology given by:

$$F(k^f) = (a + c)k^f \tag{6}$$

where k^f is the capital employed by farmers, a is the pledgeable portion returns, and c is a non-pledgeable component. Similarly, this production technology can be thought of as a collection of "farming" projects with a constant productivity distribution, $A^F(x^f) = a + c$ for $x^f \in (0, \infty)$.

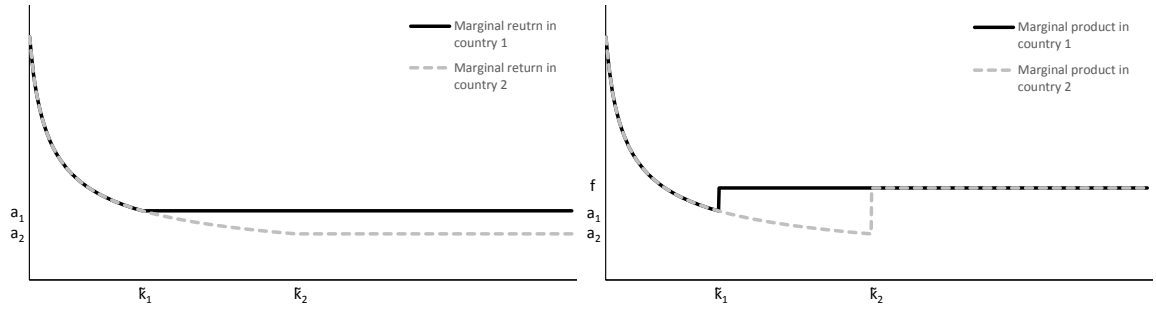
Assuming that $\lim_{k^g \to 0} G'(k_g) > a$, the equilibrium of this economy is characterized by the condition:

$$G'(k_g) \geq a \tag{7}$$

When aggregate capital is sufficiently high, $G'(k_g) = a$ and savers are indifferent between lending to gatherers and lending to farmers, taking into account that the return from lending to farmers is a rather than $a + c$.

Let $\tilde{k}$ denote the minimal capital level such that there is lending to farmers:

$$G'(\tilde{k}) = a \tag{8}$$



(a) Return to savers $R(k)$ (b) Marginal product $y(k)$

Figure 1: The return to savers and the marginal product of capital in the Kiyotaki and Moore [1997] setup, where $a_1 > a_2$ and $a_1 + c_1 = a_2 + c_2 = f$.

The aggregate production function is given by:

$$Y(k) = \begin{cases} G(k) & \text{if } k \leq \tilde{k} \\ G(\tilde{k}) + (a+c)(k-\tilde{k}) & \text{otherwise.} \end{cases} \tag{9}$$

Let $R(k)$ denote the marginal return to savers. The function $R(k)$ is given by:

$$R(k) = \begin{cases} G'(k) & \text{if } k \leq \tilde{k} \\ a & \text{otherwise.} \end{cases} \tag{10}$$

Figure 1 plots the return to savers and the marginal product curve for two different values of a (holding $a + c$ constant). While the return to savers is always decreasing in k, the marginal product is locally increasing when the rate of return reaches a and farming projects begin to be implemented.

To compare economies with different levels of misallocation but the same distribution of projects, define $f = a+c$ as the return to farming projects, and $\lambda = a/f$ as the fraction of pledgeable returns. Note that a higher λ corresponds to a less constrained farming sector that is able to pledge a higher share of returns. As illustrated by figure 1b, the marginal product is always weakly increasing in a. It follows immediately that output (the area under the marginal product curve) is increasing in a as well, and thus efficiency is increasing in λ.

The analysis in Kiyotaki and Moore [1997] focuses on domestic reallocation of capital in response to productivity shocks, holding the aggregate capital stock fixed.

In contrast, the focus here is on fluctuations in the capital stock around the steady state of the neoclassical growth model. In this setup, there cannot be a steady state in which farming projects are implemented in two economies with different values of λ. To see this, note that a steady state requires that:

$$R(k^{ss}) + 1 - \delta = \frac{1}{\beta} \tag{11}$$

If $\frac{1}{\beta} - 1 + \delta = \lambda f$, then any economy with $\lambda' < \lambda$ will not implement any farming projects, whereas any economy with $\lambda' > \lambda$ will not converge to a steady state.

Before modifying the model to address this concern, it is instructive to consider the following lemma, that illustrates the effects of misallocation on the elasticity of output with respect to capital. The lemma implies that, at a given capital level, output is more sensitive to fluctuations in capital in more distorted economies.

Lemma 2 *Assume that $\lambda_1 > \lambda_2$, and let $Y(k, \lambda)$ denote the aggregate production function given λ. Then, for k such that $G'(k) < \lambda_2 f$, the elasticity of output with respect to capital is higher in the more constrained economy: $\frac{\partial \ln Y(k,\lambda_2)}{\partial \ln k} > \frac{\partial \ln Y(k,\lambda_1)}{\partial \ln k}$.*

To prove this lemma, note that, under the assumptions $G'(k) < \lambda_2 f$ and $\lambda_1 > \lambda_2$, the marginal unit of capital is allocated to farmers in both economies. Thus,

$$\frac{\partial \ln Y(k, \lambda_i)}{\partial \ln k} = \frac{\partial \ln(Y(k, \lambda_i))}{\partial k} \frac{\partial k}{\partial \ln(k)} = \frac{\frac{\partial Y(k,\lambda_i)}{\partial k}}{Y(k, \lambda_i)} k = \frac{f}{Y(k, \lambda_i)} k \tag{12}$$

for $i = 1, 2$. The lemma then follows from the fact that $Y(k, \lambda_2) < Y(k, \lambda_1)$, since lower pledgeable returns to farming lead to greater misallocation of capital and lower productivity in economy 2.

Note that Lemma 2 applies only under the parametric restriction $G'(k) < \lambda_2 f$, guaranteeing that the capital level is such that farming projects are implemented in both economies.[1]

Lemma 2 is, in some ways, analogous to Result 1: a change in the capital stock has a larger effect on output in economies with greater misallocation. This finding is a direct outcome of misallocation: while the productivity of marginal projects is

[1] Otherwise, the result may not apply: if $\lambda_2 f < G'(k) < \lambda_1 f$, farming projects are implemented only in the less constrained economy. It is easy to see that, in this case, the elasticity of output with respect to capital is higher in the less distorted economy.

the same in both economies, the average quality of inframarginal projects is lower in the more constrained economy. Consequently, the percent change in output induced by a percent change in capital is larger in economies with lower pledgeability. Note, however, that Lemma 2 is not equivalent to Result 1, as it offers a comparison between economies with the same capital stock, rather than a comparison of economies at their respective steady states.

To derive Results 1 and 2 in this setting, it is necessary to modify the model to allow for a steady state in which some farming projects are implemented in both economies. The model can be modified to guarantee a steady state by imposing a limit on the supply of farming projects.[2] Assume that the supply of farming projects is given by some $\bar{x}$ and that, when the economy runs out of farming projects, it implements the remaining gathering projects according to their efficient order. The aggregate production function is modified to:

$$Y(k) = \begin{cases} G(k) & \text{if } k \leq \tilde{k} \\ G(\tilde{k}) + (a+c)(k-\tilde{k}) & \text{if } \tilde{k} < k \leq \tilde{k} + \bar{x} \\ G(k-\bar{x}) + (a+c)\bar{x} & \text{if } k > \tilde{k} + \bar{x} \end{cases} \tag{13}$$

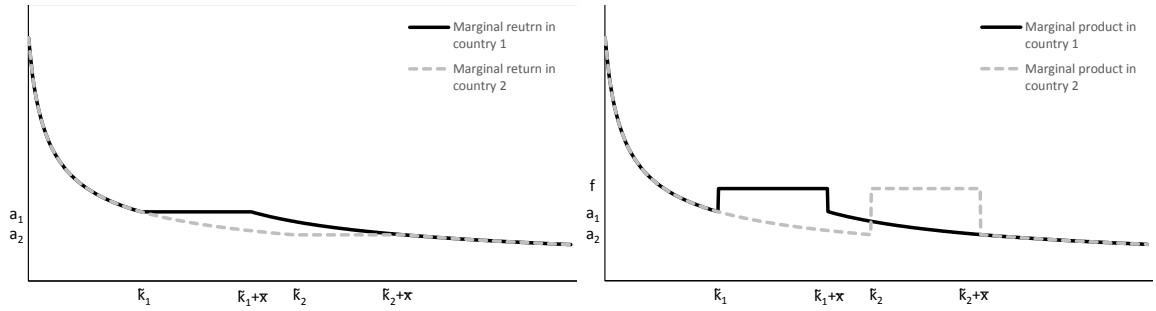
The plegeable rate of return to capital, R, is modified to:

$$R(k) = \begin{cases} G'(k) & \text{if } k \leq \tilde{k} \\ a & \text{if } \tilde{k} < k \leq \tilde{k} + \bar{x} \\ G'(k-\bar{x}) & \text{if } k > \tilde{k} + \bar{x} \end{cases} \tag{14}$$

Figure 2 plots the modified rates of return and the modified marginal product curves.

Note that, if G satisfies the Inada conditions, then this modified model always has a steady state. However, the steady state may not be unique: if $\frac{1}{\beta} - 1 + \delta = a$, then the marginal return to capital does not uniquely pin down the capital stock, as all farming projects produce the same marginal return.

[2]Note that in the Kiyotaki and Moore model, there are, implicitly, infinitely many projects, and, in equilibrium, many projects that are never implemented, regardless of the level of capital: in particular, gathering projects with $A^G(x) < a$ will never be implemented. Thus, for these projects, $\omega(x,p) = 0$ for all p, and $\int_0^\infty \omega(x,p)dp = 0$, in violation of the model's assumption that $\int_0^\infty \omega(x,p)dp = 1$ for all x.



(a) Return to savers $R(k)$ (b) Marginal product $y(k)$

Figure 2: The return to savers and the marginal product of capital in the modified Kiyotaki and Moore [1997] setup, where the supply of farming projects is limited. In the above figure, $a_1 > a_2$ and $a_1 + c_1 = a_2 + c_2 = f$.

The following lemma establishes that in this modified setup, the conditions of Lemma 1 may hold, provided that the steady state rate of return is equal to the pledgeable component of farming projects in the more constrained economy.

Lemma 3 *Assume that $\lambda_1 > \lambda_2$, and that $\lambda_2 f + 1 - \delta = \frac{1}{\beta}$. Then, there exist steady states in which the conditions of Lemma 1 hold.*

The steady state under the restriction $\lambda_2 f + 1 - \delta = \frac{1}{\beta}$ can be illustrated by figure 2. In this example, the steady state rate of return is a_2. In country 1, the steady state capital stock is $k_1^{ss} = \tilde{k}_2 + \bar{x}$, which is the unique capital level consistent with that return. In country 2, the steady state capital stock may take any value $k_2^{ss} \in [\tilde{k}_2, \tilde{k}_2 + \bar{x}]$. The marginal product in country 1 is equalized with the marginal return ($R_1(k^{ss}) = y_1(k^{ss}) = a_2$). The marginal product in country 2 is greater than the marginal return, and is given by f.

To establish the first condition of Lemma 1, note that the maximum level of steady state capital in country 2 is $\tilde{k}_2 + \bar{x} = k_1^{ss}$. By equation 12, for a steady state in which $k_1^{ss} = k_2^{ss}$, the elasticity of output is higher in the more distort economy:

$$\frac{\partial \ln(Y_2(k_2^{ss}))}{\partial \ln(k)} = \frac{\partial \ln(Y_2(k_1^{ss}))}{\partial \ln(k)} = \frac{f k_1^{ss}}{Y_2(k_1^{ss})} > \frac{\lambda_2 f k_1^{ss}}{Y_1(k_1^{ss})} = \frac{\partial \ln(Y_1(k_1^{ss}))}{\partial \ln(k)} \tag{15}$$

where the inequality follows from $Y_1(k_1^{ss}) > Y_2(k_1^{ss})$ and $\lambda_2 < 1$. By continuity, this property holds for any $k_2^{ss} = k_1^{ss} - \epsilon$, provided that $\epsilon > 0$ is sufficiently small. Note that while the lemma guarantees the existence of steady states which satisfy this

property, additional assumptions are needed in order to guarantee that any steady state satisfies this property.[3]

It is useful to clarify the components of the proof that build on the mechanisms highlighted in the paper, and those that do not. The proof builds on three features of the modified Kiyotaki and Moore environment: (a) both economies can have similar steady state capital levels; (b) the marginal product of capital at the steady state is higher in the more distorted economy and (c) steady state output is lower in the more distorted economy.

The model in the paper relies on features (b) and (c), but does not require feature (a). In particular, misallocation increases the return to marginal units of capital relative to inframarginal units. Given the assumption that allocations can be ranked according to the steepness of their marginal product curves, the relationship between volatility and misallocation does not depend on the relative values of steady state capital stocks. However, since the Kiyotaki and Moore framework does not imply a uniform ranking of allocations in terms of decreasing returns, the result requires (a) as well.

To establish the second condition, note that at any interior steady state in which $k_2^{ss} \in (\tilde{k}_2, \tilde{k}_2 + \bar{x})$,

$$\frac{\partial k_2(r^{ss})}{\partial r} = -\infty \tag{16}$$

as any small change in the interest rate leads to a measurable change in the capital stock in country 2. It thus follows that $|\frac{\partial \ln(k_2(r^{ss}))}{\partial r}| > |\frac{\partial \ln(k_1(r^{ss}))}{\partial r}|$, consistent with the second condition of Lemma 1.

While the modified Kiyotaki and Moore setup generates comparative statics that are consistent with the second condition of Lemma 1, it does so for reasons that are largely unrelated to misallocation. In the modified Kioytaki and Moore setup, this result builds heavily on the technological assumption of constant returns to farming projects, which, in this model, are not an endogenous outcome of misallocation. This

[3]Under the conditions of Lemma 3, the added assumption $G(\tilde{k}_2) < \tilde{k}_2 f$ guarantees that the elasticity of output with respect to capital is higher in country 2 in any steady state. To see this, note that, under this assumption, the average product of capital in country 2 is increasing in k_2 for $k_2 \in [\tilde{k}_2, \tilde{k}_2 + \bar{x}]$: $Y_2/k_2 = (G(\tilde{k}_2) + (k_2 - \tilde{k}_2)f)/k_2 = (G(\tilde{k}_2) - \tilde{k}_2 f)/k_2 + f$. It follows that, under this condition, the steady state elasticity of output with respect to capital in country 2 (which is given by fk_2^{ss}/Y_2^{ss}) is decreasing in k_2^{ss}. Since $k_2^{ss} = k_1^{ss}$ is the maximum steady state capital level in country 2, it follows that the elasticity of output with respect to capital is higher in country 2 at any steady state.

is in contrast to Result 1 which is closely tied to misallocation and the mechanisms highlighted in the paper.

2 Adverse selection

Stiglitz and Weiss [1981] study optimal lending behavior in an environment in which projects are heterogeneous in their risk. Their model highlights the possibility of credit rationing whenever there is asymmetric information between borrowers and lenders regarding the riskiness of projects. This section illustrates that credit rationing generates comparative statics consistent with Results 1 and 2, and for similar reasons.

While Stiglitz and Weiss [1981] restrict attention to environments in which projects have the same mean returns, it will be useful to modify their setup to allow for heterogeneous returns. Consider an economy with two types of projects: safe projects indexed $x \in [0, \bar{x}]$, and risky projects indexed $v \in [0, \bar{x}]$. The safe project x delivers a certain return of $B = A(x)$, where $A(\cdot)$ is decreasing and $A(x) \geq 0$. The risky project v delivers a return $B = 2A(v)$ with probability 0.5, and $B = 0$ otherwise. Thus, if $x = v$, the mean returns of projects x and v are the same, but v is more risky.

Borrowers make positive profits only when their projects' realized returns exceed their debt obligations. Otherwise, they default and realize 0 profits. Given an interest rate of $\tilde{r}$, the profits associated with a project that realizes a return of B are:

$$\Pi(B, \tilde{r}) = \max\{B - (1 + \tilde{r}), 0\} \tag{17}$$

In an event of default, lenders seize the project's returns, B. The return to lenders is therefore:

$$\rho(B, \tilde{r}) = \min\{1 + \tilde{r}, B\} \tag{18}$$

When applying for a loan, borrowers know the distribution of their projects' returns. It is assumed that borrowers apply for loans only when doing so is associated with strictly positive expected profits ($E_B(\Pi(B, \tilde{r})) > 0$).

I will consider two economies: in economy 1, lenders are able to differentiate between risky and safe projects. In economy 2, lenders are unable to distinguish between safe and risky projects. In both economies, it is assumed that lenders do not observe projects' mean returns.

I begin by characterizing the equilibrium in economy 1. In this environment, a risk neutral lender sets two interest rates: a safe interest rate, r^*, and a risky interest rate, $\tilde{r}$. Only safe borrowers can access loans at the safe interest rate. Note that owners of safe projects will find it optimal to apply for loans only when x is such that:

$$A(x) > 1 + r^* \tag{19}$$

thus, there will be no default associated with lending to safe projects.

In contrast, risky projects will borrow whenever their returns exceed $1 + \tilde{r}$ *in the good state*:

$$2A(v) > 1 + \tilde{r} \tag{20}$$

In the bad state, risky projects will default. Thus, the expected return from lending to risky projects is $0.5(1+\tilde{r})$. In equilibrium, lenders are indifferent between safe and risky projects. Thus,

$$1 + r^* = 0.5(1 + \tilde{r}) \tag{21}$$

It follows that the equilibrium implements the efficient allocation. To see this, note that the set of safe projects that are implemented is characterized by the condition $A(x) > 1 + r^*$, and the set of risky projects that are implemented is characterized by the condition $2A(v) > 2(1 + r^*)$, or $A(v) > 1 + r^*$. It follows that there is a unique cutoff $\underline{v} = \underline{x}$, such that projects are implemented if and only if their expected returns exceed $A(\underline{v}) = A(\underline{x})$.

Next, I derive the equilibrium in economy 2. In economy 2, lenders are restricted to one interest rate, $\tilde{r}$, that applies to both risky and safe borrowers. I follow Stiglitz and Weiss [1981] and define the average dollar return to loans as a function of the interest rate, $\bar{\rho}(\tilde{r})$. If $1+\tilde{r} > A(0)$, only risky projects borrow, and the average return is $\bar{\rho} = 0.5(1 + \tilde{r})$. If $1 + \tilde{r} < A(0)$, both safe and risky borrowers apply for loans. An interior solution is characterized by the thresholds $\underline{v}(\tilde{r})$ and $\underline{x}(\tilde{r})$, respectively:

$$A(\underline{x}(\tilde{r})) = 2A(\underline{v}(\tilde{r})) = 1 + \tilde{r} \tag{22}$$

The average return to the lender is then:

$$\bar{\rho}(\tilde{r}) = \begin{cases} \frac{0.5(1+\tilde{r})\underline{v}(\tilde{r})+(1+\tilde{r})\underline{x}(\tilde{r})}{\underline{v}(\tilde{r})+\underline{x}(\tilde{r})} & \text{if } 1 + \tilde{r} \in (0, A(0)) \\ 0.5(1 + \tilde{r}) & \text{if } 1 + \tilde{r} > A(0) \end{cases} \tag{23}$$

Note that the restriction $r^* = \tilde{r}$ implies that the allocation is inefficient, since there are too many risky projects being implemented. To see this, note that in the efficient allocation, the set of implemented projects consists of equal measures of safe and risky projects ($\underline{v} = \underline{x}$), whereas in economy 2 the measure of risky projects is always larger than the measure of safe projects ($\underline{v} \geq \underline{x}$).

It is possible to construct examples in which the function $\bar{\rho}(\tilde{r})$ is non-monotone and realizes an interior maximum in the region $1 + \tilde{r} \in (0, A(0))$, in which both risky and safe projects borrow.[4] Let r^o denote this interior maximum. As illustrated in Stiglitz and Weiss [1981], there are multiple levels of credit supply that yield the optimality of r^o.

Lemma 4 *[Stiglitz and Weiss] Assume that the market clearing interest rate, r^{mc}, satisfies $r^{mc} > r^o$ and $\bar{\rho}(r^{mc}) < \bar{\rho}(r^o)$. Then, the equilibrium interest rate is $\tilde{r} = r^o$.*

The proof is immediate: rather than setting the market clearing interest rate, the lenders realize higher returns by setting the lower interest rate r^o and rationing credit.

To embed this framework in a neoclassical growth model, it will be useful to assume that $A(x)$ is the gross return (or that $\delta = 1$) and that the representative household is the representative lender, facing the return $\bar{\rho}(\tilde{r})$. The following lemma states that if there is credit rationing at the steady state, then the conditions of Lemma 1 apply.

Lemma 5 *Assume that $\beta\bar{\rho}(r^o) = 1$. Then, there exist steady states in which: $\frac{\partial \ln(Y_2^{ss})}{\partial \ln(k_2)} = 1 > \frac{\partial \ln(Y_1^{ss})}{\partial \ln(k_1)}$ and $\frac{\partial \ln(k_2(\rho^{ss}))}{\partial \rho} = -\infty < \frac{\partial \ln(k_1(\rho^{ss}))}{\partial \rho}$.*

As illustrated by Lemma 4, there are multiple capital levels that imply an equilibrium return of $\bar{\rho}(r^o)$. I will restrict attention to k_2^{ss} in which there is credit rationing. Using $r^{mc}(k)$ to denote the market clearing interest rate given k, it will be assumed that $r^{mc}(k_2^{ss}) > r^o$ and $\bar{\rho}(r^{mc}(k_2^{ss})) < \bar{\rho}(r^o)$.

When credit is rationed, a marginal expansion in credit supply does not lead to a change in the lending rate. This is a generic feature of credit rationing (see Corollary 1 in Stiglitz and Weiss [1981]). Thus, since the supply of funding in country 2 adjusts through the financing of credit-rationed projects, the resulting increase in output is proportional to the increase in the capital stock. In contrast, in country 1, output

[4] To provide a concrete example, I confirm this numerically for the productivity distribution $A(x) = \bar{x} - x$.

increases less than proportionately since the projects funded at the margin are less productive than the inframarginal projects. Essentially, the dynamics implied by this model are, at least locally, the same as in the random allocation example.

Similarly, as the equilibrium is locally equivalent to the random allocation example, a marginal change in k_2 leaves $\bar{\rho}$ unchanged, whereas a marginal change in k_1 leads to a decline in the rate of return. Thus, a small increase in the required rate of return, ρ, can be accommodated in country 1 by small adjustments to r^* and $\tilde{r}$, which lead to a small adjustment in the amount of credit. However, in country 2, $\bar{\rho}(r^0)$ is the maximum level of the expected return function $\bar{\rho}(\cdot)$. Thus, an increase in the required expected return will lead to a dry-up of credit in country 2, and hence $\frac{\partial k_2(\rho^{ss})}{\partial \rho} = -\infty$.

This analysis illustrates that the stark results obtained from the random allocation example can be generated as equilibrium outcomes in a much richer environment. In particular, note that the only difference between countries 1 and 2 is in the ability of lenders to assess the riskiness of projects. The ability to discriminate based on risk is sufficient for guaranteeing the efficient allocation, even when lenders are unable to observe projects' expected returns. In contrast, absent the ability to discern between risky and safe projects, the equilibrium may exhibit credit rationing. By definition, credit rationing implies a condition in which there are some projects that are denied financing while financing is granted to other projects with identical characteristics. This generically implies a local "flatness" of the marginal product curve, as marginal increases in funding can adjust through the financing of credit-rationed projects rather than through a decline in the rate of return.

3 Uncertainty

This section generates the conditions of Lemma 1 in an environment in which ex-post misallocation is an outcome of ex-ante uncertainty regarding projects' returns. Asker et al. [2014] illustrate that, when it is costly for firms to adjust capital in response to idiosyncratic productivity shocks, there will be some misallocation of capital ex-post. In what follows, I consider a simplified version of this model, in which the degree of uncertainty determines the degree of ex-post misallocation.

Consider the following setup. The ex-post distribution of projects is log Normal with mean 0 and standard deviation of 1 ($\ln(A(x)) \sim N(0,1)$). At the time of

borrowing, project owners receive a noisy signal $\zeta(x)$, which is correlated with their productivity.[5] In particular, the ex-post returns to project x are:

$$\ln(A(x)) = \sqrt{1-\sigma^2}\zeta(x) + \sigma\epsilon(x) \tag{24}$$

where $\sigma \in [0,1]$, and $\zeta(x)$ and $\epsilon(x)$ are independently distributed Normal variables: $\zeta(x), \epsilon(x) \sim N(0,1)$. Note that the ex-post distribution of $\ln(A(x))$ is given by the standard Normal distribution, and does not depend on σ.[6] At the time of borrowing, project owners observe $\zeta(x)$ but not $\epsilon(x)$.

In this model, σ determines the degree of uncertainty. When $\sigma = 0$, $\ln(A(x)) = \zeta(x)$, and there is no ex-ante uncertainty; when $\sigma = 1$, project owners do not have any information regarding the expected returns to their projects.

Project owners are risk neutral and borrow whenever expected returns exceed the interest rate:

$$E(A(x)|\zeta(x)) = E(\exp(\sqrt{1-\sigma^2}\zeta(x) + \sigma\epsilon(x))|\zeta(x)) \tag{25}$$

$$= \exp(\sqrt{1-\sigma^2}\zeta(x))E(\exp(\sigma\epsilon(x))) \geq 1 + r$$

Taking logs yields the following cutoff, $\bar{\zeta}(r)$, such that projects are implemented if and only if $\zeta(x) > \bar{\zeta}(r)$:

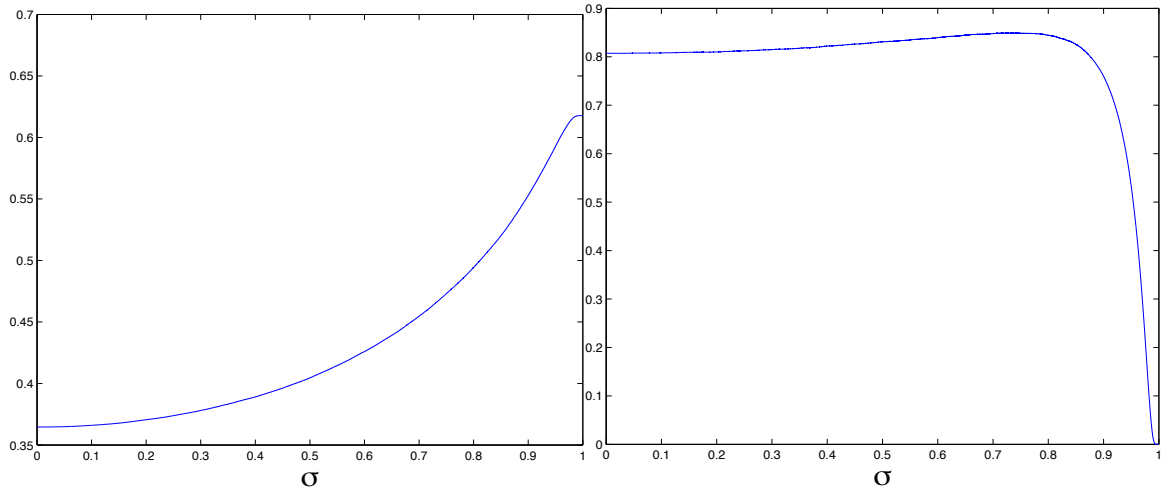
$$\sqrt{1-\sigma^2}\bar{\zeta}(r) = \ln(1+r) - \ln(E(\exp(\sigma\epsilon(x)))) \tag{26}$$

To solve for aggregate quantities, note that the measure of projects that realize $\zeta(x) > \bar{\zeta}(r)$ must be equal to the capital stock. Using ϕ to denote the probability density function of the standard Normal distribution, k is given by:

$$k(r) = \int_{\bar{\zeta}(r)}^{\infty} \phi(\zeta)d\zeta \tag{27}$$

[5] In Asker et al. [2014], firms invest dynamically and current productivity is an informative signal of future productivity. In this framework, $\zeta(x)$ can be interpreted as last period's productivity and $\epsilon(x)$ as an idiosyncratic productivity shock.

[6] The standard deviations of $\sqrt{1-\sigma^2}\zeta(x)$ and $\sigma\epsilon(x)$ are $\sqrt{1-\sigma^2}$ and σ, respectively, and hence the variance of $\ln(A(x))$ is the sum $(\sqrt{1-\sigma^2})^2 + \sigma^2 = 1$.



(a) $\frac{\partial \ln Y(k^{ss})}{\partial \ln k}$ as a function of σ (b) $|\frac{\partial \ln(k(r^{ss})}{\partial r}|$ as a function of σ

Figure 3: The conditions of Lemma 1 in the uncertainty model. The first condition requires an increasing relationship between σ and $\frac{\partial \ln Y(k^{ss})}{\partial \ln k}$. The second condition requires an increasing relationship between σ and $|\frac{\partial \ln(k(r^{ss}))}{\partial r}|$. The figure illustrates that both conditions are satisfied for $\sigma \in [0, 0.8]$.

Aggregate output is the expected returns of all implemented projects:

$$Y(r) = \int_{\bar{\zeta}(r)}^{\infty} \int_{-\infty}^{\infty} \phi(\epsilon)\phi(\zeta) \exp(\sqrt{1-\sigma^2}\zeta + \sigma\epsilon) d\epsilon d\zeta \tag{28}$$

$$= E(\exp(\sigma\epsilon)) \int_{\bar{\zeta}(r)}^{\infty} \phi(\zeta) \exp(\sqrt{1-\sigma^2}\zeta) d\zeta$$

To assess the conditions of Lemma 1, note that the elasticity of output with respect to capital is:

$$\frac{\partial \ln(Y)}{\partial \ln(k)} = \frac{\partial \ln(Y)}{\partial k} k = \frac{\frac{\partial Y}{\partial k}}{Y} k = \frac{1+r}{Y} k \tag{29}$$

And, using equations 26 and 27, the derivative of capital with respect to $\ln(1+r)$ is:

$$|\frac{\partial k}{\partial \ln(1+r)}| = \frac{\phi(\bar{\zeta}(r))}{\sqrt{1-\sigma^2}} \tag{30}$$

and hence:

$$|\frac{\partial \ln(k)}{\partial \ln(1+r)}| = \frac{\phi(\bar{\zeta}(r))}{\sqrt{1-\sigma^2} \int_{\bar{\zeta}(r)}^{\infty} \phi(\zeta) d\zeta} \tag{31}$$

Equations 29 and 30 can be assessed numerically for different values of σ at their corresponding steady states. Figure 3 illustrates the results under the assumption that $r^{ss} = 0.02$.

The first condition of Lemma 1, requiring that $\frac{\partial \ln Y(k^{ss})}{\partial \ln k}$ is increasing in σ, appears to be a robust feature of this model. This condition guarantees Result 1, but not Result 2. The second condition of Lemma 1, requiring that $|\frac{\partial \ln(k(r^{ss}))}{\partial r}|$ is increasing in σ, is satisfied for $\sigma \in [0, 0.8]$ but not for the entire range $\sigma \in [0, 1]$. To understand why higher values of σ violate this condition, note that, in the extreme case of $\sigma = 1$, all projects are implemented if and only if the mean value of the log Normal distribution exceeds $1 + r$. This implies that, close to this limit, the measure of projects implemented at the margin, $\phi(\bar{\zeta}(r))$, will be small (unless $1 + r$ happens to equal to the mean of the log Normal distribution). From equation 30, this will generate an equilibrium capital stock that is relatively insensitive to changes in the interest rate.

To summarize, this online appendix establishes that the main insights of the paper can be generated by three off-the-shelf models of misallocation, at least under certain parametric restrictions. It is useful to keep in mind that the models considered here were not constructed with the purpose of studying the effects of misallocation on the shape of the aggregate production function. In light of this, Result 1 appears to be relatively more robust than Result 2, as the latter relies on a local property of the marginal product curve around the steady state, whereas the former relates to the ratio of the marginal return and the average return. However, Result 2 appears to hold as well under some parametric restrictions.

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