Online Appendix to "The Economics of Intangible Capital"

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Abstract

This online appendix provides details on the framework discussed in Crouzet, Eberly, Eisfeldt, and Papanikolaou (2022).

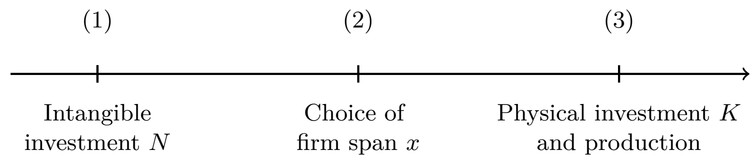

Figure 1: Timeline of the model.

1 A simple model of production with intangibles

We focus on a single firm which operates for a single period. The firm is managed by an entrepreneur, who makes operating and investment choices in order to maximize the terminal value of the profits she receives. We use the shorthand "entrepeneur" to refer to anyone who participates in the creation and dissemination of intangibles, including potentially skilled employees.

The model includes two types of capital, physical and intangible. Both types of capital can be deployed across multiple production streams, which could be different product lines, physical locations, or market segments, for example. Together, these streams determine the span, x, of the firm. To highlight the role of intangibles, we abstract from all frictions affecting physical investment: it can simply be rented on frictionless markets.

For ease of exposition, we split the model into three stages. Figure 1 sketches the timeline of decisions in the model. We next discuss the choice of physical capital, firm span, and intangibles in the reverse order in which they occur, starting with the production stage.

Stage 3: Physical capital and production

In the production stage, the firm's input N of intangible capital and its span of production x are taken as given. Profit maximization is described by:

$$
\begin{aligned}
V(N,x) \quad &= \quad \max_{\{N(s),K(s)\}_{s\in[0,x]},K} \quad \int_0^x N(s)^{1-\zeta}K(s)^{\zeta}ds - RK \\
\text{s.t.} \quad & \int_0^x K(s)\,ds \leq K \qquad (1) \\
& \left(\int_0^x N(s)^{\frac{1}{1-\rho}}\,ds\right)^{1-\rho} \leq N \qquad (2)
\end{aligned}
$$

The firm thus chooses the total amount of physical capital K, and the allocation of physical capital and intangible capital $K(s)$ to each stream of production. Each $s \in [0,x]$ indexes a different production stream. Within each stream, production uses two inputs, intangible capital, $N(s)$ and physical capital $K(s)$, and has constant returns to scale:

$$\forall s \in [0,x], \quad Y(s) = N(s)^{1-\zeta} K(s)^{\zeta}, \tag{3}$$

where $\zeta \in]0,1[$ is the elasticity of output with respect to physical capital. A production stream could represent an establishment, a product, a market segment, a geography, so long as production satisfies constant returns within that stream.

The difference between the two constraints (1) and (2) illustrates the first fundamental property of intangible capital: because intangibles are non-rival in use, they are scalable in production. The parameter ρ captures the degree of non-rivarly of the intangible input in production within the firm and across production streams.

Assumption 1 (non-rivalry within the firm)**.** $0 < \rho \leq 1$.

To see why ρ captures non-rivalry within the firm, consider two extreme cases. When $\rho = 0$, then there is no difference between physical and intangible capital: the two constraints (2) and (1) are identical and both types of capital are rival within the firm. By contrast, when ρ approaches 1, constraint (2) now becomes:

$$\lim_{\rho \to 1} \left(\int_0^x N(s)^{\frac{1}{1-\rho}} ds \right)^{1-\rho} = \max_{s \in [0,x]} N(s) \leq N. \tag{4}$$

In this case, the same intangible can be used in every production stream—that is, N becomes completely scalable, since the firm can now set $N(s) = N$ for all streams.

More generally, the marginal rate of technical substitution of intangibles across any two production streams $N(s)$ and $N(s')$ is:

$$\nu(N(s), N(s'); \rho) = \left(\frac{N(s')}{N(s)} \right)^{\frac{\rho}{1-\rho}}.$$

This marginal rate of substitution is equal to 1 when $\rho = 0$, so that increasing $N(s')$ by a marginal unit requires reducing $N(s)$ by a marginal unit. When $\rho \to 1$, on the other hand, the marginal rate of substitution converges to:

$$\lim_{\rho \to 1} \nu(N(s), N(s'); \rho) = \begin{cases} +\infty & \text{if} \quad N(s') > N(s) \\ 0 & \text{if} \quad N(s') < N(s) \end{cases}$$

In this case, so long as $N(s') < N(s)$, increasing $N(s')$ by a marginal unit is costless: it does not require reducing $N(s)$ at all.

Given our assumption that the marginal revenue product of all streams s is the same, the optimal allocation of capital across streams is symmetric:

$$\begin{aligned}\forall s \in [0,x]\,, \quad N(s) &= x^{-(1-\rho)}\,N, \\ K(s) &= \left(\frac{A}{1-\zeta}\right)^{\frac{1}{\zeta}} N(s), \qquad \text{where} \quad A \equiv (1-\zeta)\left(\frac{\zeta}{R}\right)^{\frac{\zeta}{1-\zeta}}\end{aligned}$$

As a result, the total demand for physical capital K and the value of the firm are equal to:

$$K(N,x) = \left(\frac{A}{1-\zeta}\right)^{\frac{1}{\zeta}} N\,x^{\rho}, \tag{5}$$

and

$$V(N,x) = A\,N\,x^{\rho}. \tag{6}$$

Examining (6), we can immediately see that the quantity of the intangible N and the scope of implementation x are complements, which can allow for increasing returns to scale if the firm can increase both N and x. We next introduce a tradeoff between these two choices.

Stage 2: costs of expanding firm span

At this stage, we still take as given the initial endowment of intangible capital N. Given that initial endowment, the initial creator of the intangible asset (the entrepreneur) needs to make a choice regarding the optimal scale/span of production indexed by x.

Our key assumption is that in order to increase the span of the firm, the entrepreneur may need to give up some of the firm's surplus associated with the intangibles. We model this choice as follows. Let $0 \leq N_e \leq N$ denote the portion of intangible capital that the entrepreneur retains, and let:

$$\theta \equiv \frac{N_e}{N}.$$

Since V is linear in N, total value accruing to the entrepreneur is:

$$V(N_e, x; \theta) = \theta V(N,x).$$

We make the following assumption about the relationship between firm span x and ownership θ.

Assumption 2 (Limited excludability)**.** *The span of the firm is related to the share of intangibles retained by the entrepreneur through:*

$$x(\theta) = -\frac{1}{\delta}\log\left(\frac{\theta}{\delta}\right). \tag{7}$$

where $\delta \in (0,1]$ is a fixed parameter.

Assumption 2 states that retaining a higher ownership share of the intangible, θ, requires the entrepreneur to choose a smaller span, since $x'(\theta) = -\frac{1}{\delta\theta} < 0$. The strength of this effect is governed by the parameter δ. When δ is close to zero, the entrepreneur can increase span without giving up a large portion of her endowment of intangibles. When δ is large, on the other hand, increasing span requires forfeiting more intangibles. Thus, δ captures how easy it is for the entrepreneur to retain the surplus generated by the intangible capital as the firm grows.

Under this assumption, the entrepreneur jointly chooses span and the share of the intangible endowment to retain, θ, as follows:

$$\hat{V}_e(N) \quad = \quad \max_{\theta\in[0,1],\ x\geq 0} \quad \theta V(N,x) \quad \text{s.t.} \quad x(\theta) = -\frac{1}{\delta}\log\left(\frac{\theta}{\delta}\right). \tag{8}$$

The solution is:

$$\hat{x} \;=\; \frac{\rho}{\delta}, \quad \hat{\theta} \;=\; \delta e^{-\rho},$$

and

$$\hat{V}_e(N) \;=\; A\,N\,\delta e^{-\rho}\left(\frac{\rho}{\delta}\right)^{\rho}.$$

A high degree of non-rivalry (ρ close to 1) is associated with high firm span (high $\hat{x}$) but low retention of intangibles by the entrepreneur (low $\hat{\theta}$). A low degree of non-rivarly (ρ close to 0), or high costs of storing the intangible externally (δ high) is associated with low firm span (low $\hat{x}$) but high retention of intangibles by the entrepreneur (high $\hat{\theta}$).

Assumption 2 captures the idea that appropriating the returns that intangibles generate may be difficult for the entrepreneur. But Assumption 2 goes beyond this idea by specifying that these frictions are exacerbated by x, the span of the firm. In Crouzet et al. (2022), we discuss in more detail two potential interpretations of this assumption: one related to spillovers, and one related to imperfect pledgeability.

Under either interpretation, notice that the choice of scale by the entrepreneur is inefficient. This follows from the fact that the entrepreneur chooses x to maximize (8), rather than $V(N,x)$. The latter value function, which corresponds to either the social value of the intangible (interpretation 1) or the enterprise value of the firm (interpretation 2), can be written (given the optimal choice of the entrepreneur $\hat{x}$) as

$$\hat{V}(N) = A\,N\,\hat{x}^{\rho} = A\,N\,\left(\frac{\rho}{\delta}\right)^{\rho}.$$

Because of limited excludability, the entrepreneur always chooses a smaller span than the span that maximizes enterprise value ($x = +\infty$).

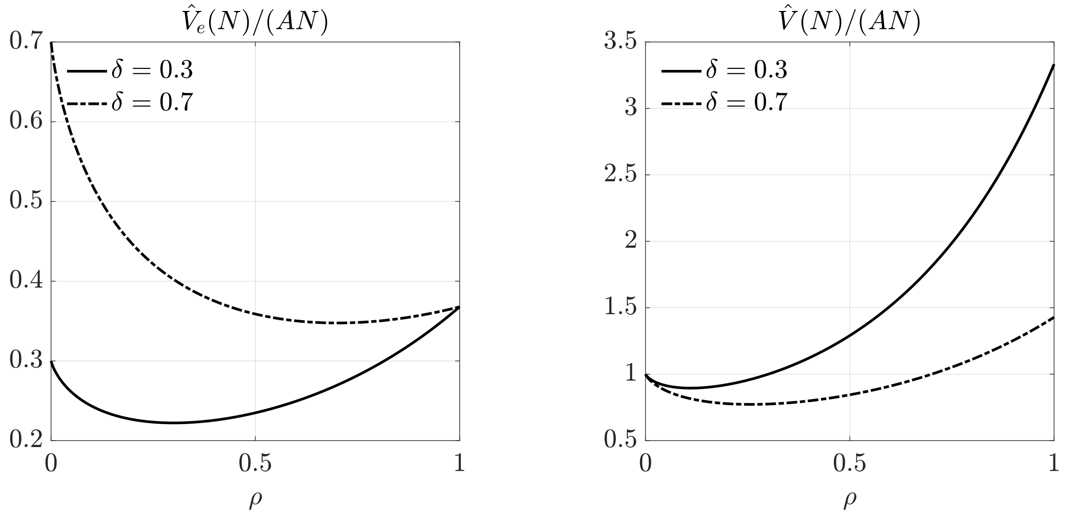

Figure 2: Comparative statics with respect to ρ and δ. The left panel reports the entrepreneur's value function, $\hat{V}_e(N)$, normalized by AB, where $A = (1-\zeta)(\zeta/R)^{\frac{\zeta}{1-\zeta}}$ and N is the intangible stock. The right panel report total enterprise value $\hat{V}(N)$, normalized by AN.

To obtain some further intuition about the interaction of non-rivalry with limited excludability, Figure 2 illustrates the comparative statics of the model with respect to ρ, non-rivalry, and δ, the limits to excludability.

The optimal span $\hat{x} = \rho/\delta$ increases with the degree of non-rivalry. However, for given limits to excludability, δ, a higher degree of non-rivalry does not necessarily make the entrepreneur better off. The left panel of Figure 2 illustrates this. When excludability is high (δ is low), given the option to adopt an intangible-intensive technology with a high non-rivalry (ρ to 1), versus using only rival capital inputs ($\rho = 0$), the entrepreneur would generally pick the former, and operate at high scale. However, when excludability is low (δ is high), that is not the case: the entrepreneur might instead pick a technology with rival capital inputs, $\rho = 0$, and focus on a single production stream ($\hat{x} = 0$).

This property demonstrates the complementarity we emphasized in Crouzet et al. (2022) between non-rivalry and excludability: non-rivalry may increase the returns to intangibles, but the entrepreneur will only value the associated intangible asset to the extent that the benefits are appropriable. To see this, note that the cross-partial derivative of the entrepreneur's value function is equal to:

$$\frac{\partial^2 \log(\hat{V}_e(N))}{\partial \rho\, \partial \delta} = -1/\delta < 0,$$

so that non-rivalry (high ρ) and excludability (low δ) are complements.

Finally, the comparison of the left and right panels of Figure 2 further illustrate the fact that the entrepreneur's scale choices may be inefficient. In the right panel, for all values of δ, a non-rival technology (ρ close to 1) yields higher social value (under the interpretation of δ relating to positive spillovers to other firms) or enterprise value (under the interpretation of δ as relating to imperfect pledgeability). This conflicts with the preferences of the entrepreneur, who would rather choose a technology with non-scalable inputs when δ is sufficiently high.

Stage 1: Intangible investment/creation

The last step is to determine the initial amount of the intangible asset N. To do so, we need to take the perspective of the entrepreneur who invests in producing new intangibles. More specifically, the entrepreneur exerts effort ι subject to a convex cost $c(\iota)$ to generate a new intangible asset. The process of generating new intangibles can be risky: exerting effort ι yields intangible capital $N \sim f(N;\iota)$. Exerting higher effort yields ex-ante better outcomes: we assume that if $\iota' > \iota$ then $f(N;\iota') \overset{\text{fosd}}{>} f(N;\iota)$. Given these assumptions, the entrepreneur solves

$$\max_{\iota} \int \hat{V}_e(N) f(N;\iota) dN - c(\iota) \tag{9}$$

which after substituting for (9) yields the optimality condition

$$A \left[\delta e^{-\rho} \left(\frac{\rho}{\delta} \right)^{\rho} \right] \frac{\partial}{\partial e} E[N;\hat{\iota}] = \frac{\partial}{\partial \iota} c(\hat{\iota}). \tag{10}$$

Examining (10), we see that the dependence of the entrepreneur's optimal effort choice on ρ and δ is determined by how the term in brackets depends on ρ and δ.

We can immediately see that $\partial\hat{\iota}/\partial\delta < 0$; if the entrepreneur needs to give up more value in order to scale her firm (δ is higher) then her marginal valuation of the intangible N is lower, which leads to lower ex-ante investment in generating intangibles. By contrast, the comparative statics with respect to the non-rivalry parameter ρ are more subtle. It turns out that the term in brackets is increasing in ρ if $\rho > \delta$, and decreasing otherwise. This is again related to the complementarity between non-rivalry and excludability. Scalability may generate value, but the entrepreneur will only value the associated intangible asset to the extent that the benefits are sufficiently appropriable. Otherwise, if the entrepreneur cannot appropriate a significant share of the rents (δ is high enough) she will exert less effort in generating new intangibles for local increases in non-rivalry ρ.

Additionally, the model features underinvestment in innovation since the entrepreneur's effort choice depends on her private value of the intangible, which in general is lower than the social value. This is similar to models of endogenous growth with spillovers. Perhaps surprisingly, however, we see that the degree of under-investment can be greater for intangibles that are highly scalable (higher ρ) if excludability is low enough (δ is high).

Last, it is important to emphasize the distinction between ex-post rents to the entrepreneur

$$A\left[\delta e^{-\rho}\left(\frac{\rho}{\delta}\right)^{\rho}\right]N - c(\hat{\imath}), \tag{11}$$

with ex-ante rents

$$A\left[\delta e^{-\rho}\left(\frac{\rho}{\delta}\right)^{\rho}\right]E[N;\hat{\imath}] - c(\hat{\imath}). \tag{12}$$

If there is selection on which entrepreneurs enter the market (or equivalently if failure $N = 0$ is a feasible outcome despite the amount of effort involved) then focusing on ex-post compensation to entrepreneurs (11) will overstate their payoff. Put simply, ex-post rents (11) can be positive even if rents are zero ex-ante (12) due to free entry of entrepreneurs.

2 Economic implications of intangible capital

In this section, we provide some simple formal implications of the model outlined in the previous section regarding the four issues discussed in Crouzet et al. (2022): productivity growth; factor income shares; Tobin's Q and investment; market structure.

2.1 Aggregate Productivity and Factor Shares

Consider the model in the previous section. If we assume that there is no heterogeneity in ρ and δ and then clear the market for physical capital to determine the equilibrium interest rate R, we can write aggregate output Y with some abuse of notation as

$$Y = \left(\frac{\rho}{\delta}\right)^{\rho\,(1-\zeta)} N^{1-\zeta}\, K^{\zeta}. \tag{13}$$

As we examine (13), it is useful to keep in mind that the simple model from the previous section has constant total factor productivity. As such, aggregate output is a function of the economies' stock of physical capital K, the quantity of intangibles N, with an adjustment for the fact that intangibles are non-rival (the first term). Here, for simplicity, we ignore the effort cost e to generate new intangibles when constructing output (consistent with the data). Consider the extreme case where the only measurable input to production is physical capital, that is, aggregate statistics can only measure Y_t and K_t. Taking logs of (13), we can define measured TFP (in logs) as

$$tfp \equiv \log Y - \zeta \log K = \rho\,(1-\zeta)\ (\log\rho - \log\delta) + (1-\zeta)\log N. \tag{14}$$

Examining equation (14), we see that measured productivity depends not only on the 'stock' of intangible capital N but also on their degree of non-rivalry and appropriability (i.e. ρ and δ). Specifically, the non-rivalrous nature of intangibles can imply that once an intangible asset is developed, output can increase rapidly as the intangible capital is applied in many locations or applications simultaneously (which depends on ρ). Similarly, the optimal scale of deployment is a function of appropriability (determined by δ).

Equation (14) can then be interpreted from two perspectives: either intangibles account for the entirety of the Solow residual; or they pose a measurement challenge for which Solow residual measures must adjust. Crouzet et al. (2022) discuss these two perspectives in more detail.

Factor Income Shares

The fact that intangibles are typically hard to measure implies that factor shares are also mismeasured. Depending on the implicit assumptions researchers make, the share of output that would accrue to intangibles is can be allocated to either physical capital, labor, or 'rents', where the latter is defined as monopoly profits. As an illustration, let us now re-interpret the fixed factor K in the model as a composite input good consisting of physical capital M and labor L, both in fixed supply,

$$K \equiv M^{\alpha} L^{1-\alpha}. \tag{15}$$

In this case, the share of output that accrues to 'intangibles' (i.e. not to K) is equal to

$$\frac{Y - R\,K}{Y} = 1 - \zeta, \tag{16}$$

while the factor share of physical capital and labor is $\alpha\,\zeta$ and $(1-\alpha)\,\zeta$, respectively.

Now, the question is whether the factor share of intangibles (16) should be allocated to labor or capital. If we view δ as reflecting frictions between the entrepreneur and outside financiers, then we can

$$\frac{N_e}{N} = \delta e^{-\rho} \qquad \text{and} \qquad \frac{N - N_e}{N} = 1 - \delta e^{-\rho} \tag{17}$$

as the share of intangibles that accrues to the entrepreneur, and to outside investors, respectively. The greater the degree of non-rivalry (higher ρ) the smaller is the share that accrues to the entrepreneur, as she chooses to give up a larger fraction of her surplus to achieve a higher scale. The stronger the limits to excludability (higher δ), the more of the surplus the entrepreneur chooses to retain — as limited pleadgeability make expansion too costly for her.

Equation (17) gives some guidance on how the factor shares of intangibles should be allocated between capital and labor. The entrepreneur's share N_e/N should likely be considered labor income if it is the case that human capital is the key input in the production of new intangibles. The residual part $1 - N_e/N$, however, is the part that outside investors have a claim to, which could (though it need not be) be part of capital income.

2.2 Tangible Investment and Tobin's Q

The model outlined in the previous section can shed light on the role that intangibles may have played in creating a wedge between average and marginal q. In order to make the connection to Q-theory clearer, we assume that the total stock of physical capital is fixed, instead of rented on markets at a fixed marginal cost R. A firm with installed physical capital K, intangibles N, and span x has value:

$$\begin{aligned} W(K,N,x) &= \max_{\{N(s),K(s)\}_{s\in[0,x]}} \int_0^x N(s)^{1-\zeta} K(s)^{\zeta} ds && (18) \\ \text{s.t.} \quad [q_K] & \int_0^x K(s)\, ds \leq K && (19) \\ [q_N] & \left(\int_0^x N(s)^{\frac{1}{1-\rho}}\, ds \right)^{1-\rho} \leq N, && (20) \end{aligned}$$

where q_K and q_N are the Lagrange multipliers on the physical and intangible capital allocation constraints, that is, the marginal increase in firm value associated with a marginal increase in K or in N:

$$\frac{\partial W}{\partial K} = q_K, \quad \frac{\partial W}{\partial N} = q_N.$$

Following similar steps as before, the solution for enterprise value and these shadow values is:

$$\begin{aligned} W(K,N,x) &= N^{1-\zeta} K^{\zeta} x^{\rho(1-\zeta)}, && (21) \\ q_K &= \zeta \left(\frac{N}{K}\right)^{1-\zeta} x^{\rho(1-\zeta)}, && (22) \\ q_N &= (1-\zeta) \left(\frac{K}{N}\right)^{\zeta} x^{\rho(1-\zeta)}. && (23) \end{aligned}$$

Note that, if we impose $q_K = R$, we obtain:

$$K = \left(\frac{\zeta}{R}\right)^{\frac{1}{1-\zeta}} N x^{\rho} \quad \text{and} \quad V(N,x) = W(K,N,x) - RK = (1-\zeta)\left(\frac{\zeta}{R}\right)^{\frac{\zeta}{1-\zeta}} N x^{\rho},$$

that is, the same solution as in Section 3. Thus the model of Section 3 can be thought of as a particular case of the fixed-capital model, if q_K is set fixed to $q_K = R$.

The solution in Equations (21)-(23) implies the following firm value decomposition:

$$W(K,N,x) = q_K K + q_N N. \tag{24}$$

This decomposition offers some insight into the growing disconnect between average Q_K and marginal q_K for physical capital. Rewriting Equation (24) as:

$$\frac{W}{K} \equiv Q_K = q_K + q_N \frac{N}{K}, \tag{25}$$

we see that intangibles introduce a wedge between average Q_K and marginal q_K. The wedge, $Q_K - q_K = \frac{N}{K} q_N$, depends positively on the ratio of intangible to physical capital, N/K, since:

$$Q_K - q_K = q_N \frac{N}{K} = (1-\zeta)\left(\frac{N}{K}\right)^{1-\zeta}\left(\frac{\rho}{\delta}\right)^{\rho(1-\zeta)}. \tag{26}$$

In a dynamic extension of this model, physical investment rates would be an increasing function of the marginal value of physical capital q_K. Thus, the growing disconnect between average Q_K and physical investment rates (a function of marginal q_k could be explained by an rising ratio of intangible to physical capital N/K.

2.3 Rents and Market structure

To illustrate the role that market power could play, consider again the fixed-capital model described in Equations (18)-(20), but assume that total sales are given by:

$$\left(\int_0^x N(s)^{1-\zeta} K(s)^{\zeta}\right)^{\frac{1}{\mu}},$$

where $\mu > 1$ is a fixed parameter that creates a wedge between the average and marginal revenue product of both K and N. The wedge is a simple way to capture the rents associated with production, and could be microfounded, for instance, as a markup of output prices over marginal cost in a monopolistic setting.[1] In this case, firm value W, and marginal q_K and q_N are given by:

$$W(K,N,x) \quad = \quad x^{\frac{(1-\zeta)\rho}{\mu}} N^{\frac{1-\zeta}{\mu}} K^{\frac{\zeta}{\mu}} \tag{27}$$

[1] We choose to introduce this wedge to the sum of all production across streams. If, instead, it applied individually to each stream, i.e. $Y(s) = \left(N(s)^{1-\zeta} K(s)^{\zeta}\right)^{1/\mu}$, then the firm would have a motive to increase its span even when $\rho = 0$, because a higher span would counterbalance decreasing returns at the stream level. We leave this mechanism out of the model, since it is independent of the non-rivalry of intangibles, but it is possible to include it in the model. It generally leads to a higher choice of span, all other things equal.

$$q_K = \frac{\zeta}{\mu}\left(\frac{N}{K}\right)^{1-\zeta}\left(\frac{K^\zeta}{N^{1-\zeta}}\right)^{1-\frac{1}{\mu}} x^{\frac{\rho(1-\zeta)}{\mu}} \tag{28}$$

$$q_N = \frac{1-\zeta}{\mu}\left(\frac{K}{N}\right)^{\zeta}\left(\frac{K^\zeta}{N^{1-\zeta}}\right)^{1-\frac{1}{\mu}} x^{\frac{\rho(1-\zeta)}{\mu}} \tag{29}$$

This solution implies a more general version of the firm value decomposition in Equation (24):

$$W(K,N,x) = \overbrace{q_K K + q_N N}^{(a)} + \underbrace{(\mu-1)q_K K + (\mu-1)q_N}_{(b)}. \tag{30}$$

This decomposition highlights two broad sources of firm value. Term (a) is the value of installed capital, which makes up all of firm value in the absence of rents ($\mu = 1$), as was the case in Equation (24). Term (b) is the net value of rents, which is positive only when $\mu > 1$. Additionally, each of these two terms (value of installed assets, and rents) can be decomposed between a contribution of physical and a contribution of intangible capital. Consistent with the idea that the intangible asset itself is not the rent, this decomposition assumes no specific relationship between μ (a fixed parameter) and the stock of intangibles (N, which we take as given in this application of the model).

To see the effect of non-rivalry and limited excludability on rents, define total rents per unit of physical capital:

$$\Gamma(K,N,x) \equiv (\mu-1)\left(q_K + q_N\frac{N}{K}\right), \tag{31}$$

and it is straightforward to see that:

$$\Gamma(K,N,x) = \frac{\mu-1}{\mu}\left(\frac{N}{K}\right)^{1-\zeta}\left(\frac{K^\zeta}{N^{1-\zeta}}\right)^{1-\frac{1}{\mu}} x^{\frac{\rho(1-\zeta)}{\mu}}. \tag{32}$$

Of these rents, we assume that only $\Gamma^e(K,N,x)$ are appropriable by the entrepreneur, where:

$$\Gamma^e(K,N,x) \equiv \delta e^{-\frac{\delta(1-\zeta)x}{\mu}}\Gamma(K,N,x). \tag{33}$$

Note that here, for simplicity, we assumed that span x and retention θ are related through:

$$x(\theta) = -\frac{1-\zeta}{\mu\delta}\log\left(\frac{\theta}{\delta}\right),$$

which helps clarify the parallels between this model and the model with variable capital and no rents. Indeed, as before, we assume that the entrepreneur chooses the span x to maximize her claim, which, in this model, is proportional to the value of rents she can appropriate. The optimal span is the same as before, namely $\hat{x} = \frac{\rho}{\delta}$, implying:

$$\hat{\Gamma}(K, N) = \frac{\mu - 1}{\mu} \left(\frac{N}{K}\right)^{1-\zeta} \left(\frac{K^{\zeta}}{N^{1-\zeta}}\right)^{1-\frac{1}{\mu}} \left(\frac{\rho}{\delta}\right)^{\frac{(1-\zeta)\rho}{\mu}}, \tag{34}$$

$$\hat{\Gamma}^{e}(K, N) = \delta e^{-\frac{(1-\zeta)\rho}{\mu}} \hat{\Gamma}(K, N). \tag{35}$$

Relative to a model with $\rho = 0$ and $\delta = 0$, this expression shows that both limits to excludability and non-rivalry affect both the total size of rents. Higher non-rivalry (ρ closer to 1) generally leads to a lower share of rents accruing to the entrepreneur, while lower excludability (higher δ) generally implies that the entrepreneur retains a higher share of total rents.

References

Nicolas Crouzet, Janice Eberly, Andrea Eisfeldt, and Dimitris Papanikolaou. The economics of intangible capital. The Journal of Economic Perspectives, forthcoming, 2022.