Supplemental Appendix for A Learning Model of Financial Instability

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This appendix contains additional calculations, extensions, and illustrations for the paper.

I. Convergence Results

Marcet and Sargent (1989), Evans and Honkapohja (2001), and others have adapted the theory of stochastic approximation to show that in the limit as the gain gets small, the evolution of beliefs can be approximated by a deterministic ordinary differential equation. Recall the belief updating equations:

$$m_{t+1} = m_t + \varepsilon(\log(Z_t) - m_t), \tag{1}$$

$$s_{t+1}^2 = s_t^2 + \varepsilon([\log(Z_t) - m_t]^2 - s_t^2). \tag{2}$$

The basic idea is that we can rewrite (1) as:

$$\frac{m_{t+1} - m_t}{\varepsilon} = \log Z(\theta_t, \theta_{t-1}, \omega_{t-1}, \chi_t) - m_t.$$

The left side of the equation can be interpreted as a finite-difference approximation of a time derivative, where ε is the notional "time" between observations. Then as $\varepsilon \to 0$ the left side of the equation converges to that time derivative. Also in this limit, more and more observations are packed into any finite interval of time, so the dynamics of outcomes are fast relative to beliefs. Thus the belief dynamics effectively average over the shock realizations and we can apply a law of large numbers.

Following Theorem 8.5.1 in Kushner and Yin (1997), as $\varepsilon \to 0$, under some regularity conditions, the beliefs converge weakly to the trajectories of the mean dynamics ODEs:

$$\dot{m} = E[\log Z(\theta, \theta, \omega, \chi)] - m, \tag{3}$$

$$\dot{s}^2 = E[\log Z(\theta, \theta, \omega, \chi) - m]^2 - s^2. \tag{4}$$

Here the expectation is with respect to the unconditional distribution of ω_t for fixed θ, and the distribution of the dividend growth shocks χ.

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The self-confirming equilibrium is clearly an equilibrium point of the mean dynamics, as seen by setting $\dot{m} = \dot{s}^2 = 0$:

$$\begin{aligned} m = \mu &= E[\log Z(\theta, \theta, \omega, \chi)] \\ s^2 = \sigma^2 &= E[(\log Z(\theta, \theta, \omega, \chi) - \mu)^2] \end{aligned}$$

I now show that the SCE is a locally stable equilibrium point of the mean dynamics, and therefore of the learning rule. As in Evans and Honkapohja (2001), the SCE is locally expectationally stable if all of the eigenvalues of the Jacobian matrix of the mean dynamics have negative real parts when evaluated at the SCE. So for small enough gain ε, and (at least) for beliefs that start near the SCE, over time beliefs will converge to the SCE. The condition on the risk aversion coefficient is sufficient but stronger than necessary, as discussed in the proof. The baseline calibration has larger risk aversion, but I verified numerically (evaluating the eigenvalues of the Jacobian with the specified parameters) that stability still holds.

THEOREM 1: *When $\gamma \leq 2$, the self-confirming equilibrium beliefs $\overline{\theta} = (\mu, \sigma)$ are a locally stable fixed point of the learning dynamics (1)-(2). Therefore as $\varepsilon \to 0$ and $t \to \infty$, the stochastic belief process $\{\theta_t\}$ will converge weakly to $\overline{\theta}$.*

PROOF:

The convergence result follows from Kushner and Yin (1997). Apart from the stability conditions established below, the regularity conditions are satisfied as the shocks are bounded, and the portfolio constraints imply that beliefs and outcomes are bounded.

Tedious but straightforward calculations verify that $v(\overline{\theta}) = 1$ and $c(\overline{\theta}) = 1/(1+\delta)$. Therefore $\delta(\overline{\theta}) = \delta^*$, $q(\overline{\theta}) = 0$. In particular, using the relations above we have:

$$\mu = \log(1 + \delta^*) - \log(\delta^*) + d = -\log \beta - \log \overline{\phi} + \gamma d = \log R + \log(\phi/\overline{\phi}).$$

Therefore, defining $\psi = \phi/\overline{\phi}$, we have:

$$Z_H - R = (e^{\sigma}\psi - 1)\, R, \quad R - Z_L = \left(1 - e^{-\sigma}\psi\right) R.$$

Then we can write:

$$v(\overline{\theta}) = \frac{1 - \Gamma}{1 - \Gamma + \psi(\Gamma e^{\sigma} - e^{-\sigma})}$$

where:

$$\Gamma = \left(\frac{Z_h - R}{R - Z_L}\right)^{-\frac{1}{\gamma}} = \left(\frac{e^{(1+\gamma)\sigma} - e^{-(1-\gamma)\sigma}}{e^{(1-\gamma)\sigma} - e^{-(1+\gamma)\sigma}}\right)^{-\frac{1}{\gamma}} = e^{-2\sigma}.$$

Therefore:

$$\Gamma e^{\sigma} - e^{-\sigma} = e^{-2\sigma} e^{\sigma} - e^{-\sigma} = 0,$$

which then gives:

$$v(\overline{\theta}) = \frac{1-\Gamma}{1-\Gamma} = 1.$$

Furthermore, this implies

$$c(\overline{\theta}) = 1 - \frac{\beta^{\frac{1}{\gamma}}}{2}\left(Z_L^{1-\gamma} + Z_H^{1-\gamma}\right) = 1 - \beta^{\frac{1}{\gamma}} e^{\frac{1-\gamma}{\gamma}\mu} \overline{\phi}^{\frac{1}{\gamma}} = 1 - \beta\overline{\phi} e^{(1-\gamma)d} = \frac{1}{1+\delta^*}.$$

Thus we have $q(\overline{\theta}) = 0$ so $\omega_t \equiv 1$. This means that we only need evaluate the dynamics locally around $\omega_t = \omega_{t-1} = 1$. Recalling the definition of $\log Z_t = \log Z(\theta_t, \theta_{t-1}, \omega_{t-1}, \chi_t)$, we can evaluate:

$$\log Z(\theta, \theta, 1, \chi) = \log(1 + \delta(\theta)) - \log(\delta(\theta)) - \log \chi_t.$$

Then we have:

$$E \log Z(\overline{\theta}, \overline{\theta}, 1, \chi) = \log(1 + \delta^*) - \log(\delta^*) - d = \mu,$$

and therefore:

$$E[\left(\log Z(\overline{\theta}, \overline{\theta}, 1, \chi) - \mu\right)^2] = E(\sigma W_t)^2 = \sigma^2.$$

Thus $\overline{\theta}$ is a fixed point of the ODEs (3), and therefore of the learning dynamics.

To establish local stability, we need to evaluate the Jacobian of the mean dynamics at the SCE $\overline{\theta}$. Since the variance estimate s^2 is of second order, it does matter for the local stability. Thus it is enough to simply verify the local stability of the mean estimate. To see this explicitly, we can stack (3) into a vector, and write the dynamics as:

$$\dot{\theta} = b(\theta),$$

where I abuse notation set and $\theta = (m, s^2)$. Simple calculations establish:

$$\frac{\partial b}{\partial \theta}(\overline{\theta}) = \begin{bmatrix} \frac{\partial E \log Z}{\partial m} - 1 & \frac{\partial E \log Z}{\partial s^2} \\ 0 & -1 \end{bmatrix}.$$

So for local stability, we simply require:

$$\frac{\partial}{\partial m} E \log Z(\theta, \theta, 1, \chi)) \Big|_{\theta = \overline{\theta}} < 1.$$

From above we have:

$$E \log Z(\theta, \theta, 1, \chi) = \log(1 + \delta(\theta)) - \log(\delta(\theta)) - d,$$

so that:

$$\frac{\partial E \log Z}{\partial m} = \left(\frac{1}{1+\delta^*} - \frac{1}{\delta^*}\right)\frac{\partial \delta}{\partial m}(\overline{\theta}) = -\frac{1}{\delta^*(1+\delta^*)}\frac{\partial \delta}{\partial m}(\overline{\theta}).$$

Then letting $\kappa = 1 - c$, we have:

$$\begin{aligned}\frac{\partial \delta}{\partial m}(\overline{\theta}) &= \left(\frac{\kappa}{1-\kappa} + \frac{\kappa^2}{(1-\kappa)^2}\right)\frac{\partial v}{\partial m}(\overline{\theta}) + \left(\frac{1}{1-\kappa} - 1\right)\frac{\partial \kappa}{\partial m}(\overline{\theta}) \\ &= \frac{\kappa}{(1-\kappa)^2}\frac{\partial v}{\partial m}(\overline{\theta}) + \frac{\kappa}{1-\kappa}\frac{\partial \kappa}{\partial m}(\overline{\theta}) \\ &= \delta^*\left[(1+\delta^*)\frac{\partial v}{\partial m}(\overline{\theta}) + \frac{\partial \kappa}{\partial m}(\overline{\theta})\right].\end{aligned}$$

Then since $\frac{\partial Z_H}{\partial m} = Z_H$ and $\frac{\partial Z_L}{\partial m} = Z_L$, we can write:

$$\frac{\partial v}{\partial m}(\overline{\theta}) = \frac{Z_L\left[R(R-Z_L)^{\frac{-1-\gamma}{\gamma}} + (\gamma-1)(R-Z_L)^{\frac{-1}{\gamma}}\right] + Z_H\left[R(Z_H-R)^{\frac{-1-\gamma}{\gamma}} - (\gamma-1)(Z_H-R)^{\frac{-1}{\gamma}}\right]}{\gamma\left[(R-Z_L)^{\frac{\gamma-1}{\gamma}} + (Z_H-R)^{\frac{\gamma-1}{\gamma}}\right]}.$$

But above we established that at $\overline{\theta}$:

$$(Z_H - R)^{\frac{-1}{\gamma}} = e^{-2\sigma}(R - Z_L)^{\frac{-1}{\gamma}},$$

so therefore at the SCE:

$$Z_L(R-Z_L)^{\frac{-1}{\gamma}} - Z_H(Z_H-R)^{\frac{-1}{\gamma}} = (R-Z_L)^{\frac{-1}{\gamma}}(Z_L - e^{-2\sigma}Z_H) = 0.$$

Thus we can simplify the above to:

$$\frac{\partial v}{\partial m}(\overline{\theta}) = \frac{R\left[Z_L(R-Z_L)^{\frac{-1-\gamma}{\gamma}} + Z_H(Z_H-R)^{\frac{-1-\gamma}{\gamma}}\right]}{\gamma\left[(R-Z_L)^{\frac{\gamma-1}{\gamma}} + (Z_H-R)^{\frac{\gamma-1}{\gamma}}\right]} > 0.$$

Then using the definition of $c(\theta)$ we have:

$$\frac{\partial \kappa}{\partial m}(\overline{\theta}) = (1-\gamma)\frac{\beta^{\frac{1}{\gamma}}}{2}\left(Z_L^{-\gamma}\left((Z_L - R)\frac{\partial v}{\partial m} + Z_L\right) + Z_H^{-\gamma}\left((Z_H - R)\frac{\partial v}{\partial m} + Z_H\right)\right).$$

Similar to above, at the SCE we have:

$$Z_H^{-\gamma}(Z_H - R) - Z_L^{-\gamma}(R - Z_L) = \left(Z_H^{-\gamma} e^{2\gamma\sigma} - Z_L^{-\gamma}\right)(R - Z_L) = 0,$$

so that we can simplify to:

$$\frac{\partial \kappa}{\partial m}(\bar{\theta}) = (1-\gamma)\frac{\beta^{\frac{1}{\gamma}}}{2}\left(Z_L^{1-\gamma} + Z_H^{1-\gamma}\right) = (1-\gamma)\kappa(\bar{\theta}) = (1-\gamma)\frac{\delta^*}{1+\delta^*}.$$

Thus we have
$$\frac{\partial E \log Z}{\partial m}(\bar{\theta}) = -\frac{\partial v}{\partial m}(\bar{\theta}) - (1-\gamma)\frac{\delta^*}{(1+\delta^*)^2}.$$

When $\gamma < 1$ this is clearly negative. More generally, we require:

$$\frac{\partial v}{\partial m}(\bar{\theta}) > (\gamma - 1)\frac{\delta^*}{(1+\delta^*)^2} - 1.$$

For $\gamma \leq 2$, since $\delta^* > 0$ the right side of the above inequality is negative. Since $\frac{\partial v}{\partial m}(\bar{\theta}) > 0$ the condition is guaranteed to hold.

These conditions are stronger than necessary, but the relatively complicated formulas make weaker conditions difficult to establish. For example, with $\gamma = 3$ a sufficient condition would be $\frac{\partial v}{\partial m}(\bar{\theta}) > 1$. Such a condition is easy to verify with given parameters, as in the calibrated model. But it is harder to provide conditions on primitives. In the Merton continuous time analogue of the model, the portfolio decision rule is $v = (m - r)/(\gamma s^2)$, so $\frac{\partial v}{\partial m}(\bar{\theta}) = \frac{1}{\gamma \sigma^2}$. Then as long as $\sigma^2 < 1/\gamma$ we know that $\frac{\partial v}{\partial m}(\bar{\theta}) < 1$.

II. Averaging and the Bifurcation Theorem

The paper shows that for small shocks, we can approximate the dynamics of the mean of the stochastic system by the averaged dynamics. Figure 1 shows the averaged dynamics for m_t and s_t, along with quantiles of the belief distribution from 1,000 simulations. In both cases the time paths are initialized near the self-confirming equilibrium values. The figure clearly illustrates that the averaging in the small noise approximation provides a good characterization of the belief distribution for the baseline parameterization.

I now show how to verify the conditions in the bifurcation theorem. The key is to calculate the Jacobian $J(\varepsilon)$, whose eigenvalues determine the stability of the SCE $\bar{X}$. The higher-order derivatives needed for the higher-order approximations B and C follow similarly. Explicitly, we have $X_t = [m_t, s_t^2, m_{t-1}, s_{t-1}^2, \omega_{t-1}]'$ so

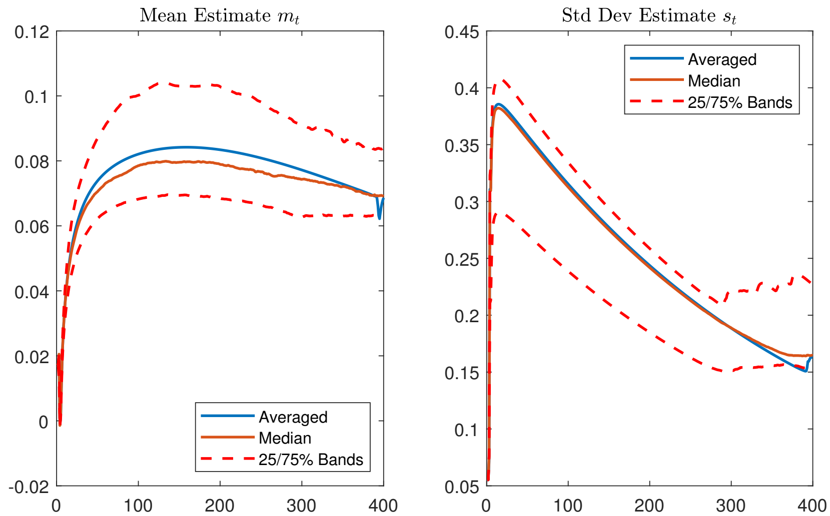

FIGURE 1. BELIEFS m_t AND s_t FROM THE AVERAGED DYNAMICS.

Note: The solid and dashed lines also show quantiles from the simulated distribution.

that the averaged dynamics are:

$$\bar{G}(X_t) = \left[\begin{array}{c} m_t + \varepsilon(E \log Z(\theta_t, \theta_{t-1}, \omega_{t-1}, \chi_t) - m_t) \\ s_t^2 + \varepsilon(E(\log Z(\theta_t, \theta_{t-1}, \omega_{t-1}, \chi_t) - m_t)^2 - s_t^2) \\ m_t \\ s_t^2 \\ 1 + q(\theta_{t-1})\omega_{t-1} E(\chi_t) \end{array} \right].$$

In the following, all expressions and functions are evaluated at the SCE, so I suppress the arguments. First, since Z_t is binomial, we can write $E \log Z = 0.5(\log Z_L + \log Z_H)$ where Z_L is associated with shock realization $\chi_L = \exp(\mu - \sigma)$ and Z_H with $\chi_H = \exp(\mu + \sigma)$. Similarly, $E(\log Z - m)^2 = .5(\log Z_L - m)^2 + .5(\log Z_H - m)^2$. Therefore for any $x \in X$, $\frac{\partial E \log Z}{\partial x} = 0.5(\frac{\partial \log Z_L}{\partial x} + \frac{\partial \log Z_H}{\partial x})$ and so on. Also let $\bar{\chi} = E\chi_t = .5(\chi_L + \chi_H)$. Then we have:

$$\begin{aligned} J_1(\varepsilon) &= \frac{\partial \bar{G}_1(\bar{X})}{\partial X} \\ &= \varepsilon \left[\frac{1}{\varepsilon} + \frac{\partial E \log Z}{\partial m_t} - 1, \frac{\partial E \log Z}{\partial s_t^2}, \frac{\partial E \log Z}{\partial m_{t-1}}, \frac{\partial E \log Z}{\partial s_{t-1}^2}, \frac{\partial E \log Z}{\partial \omega_{t-1}} \right] \\ &= \varepsilon \left[\frac{1}{\varepsilon} + \frac{1}{1+\delta} \frac{\partial \delta}{\partial m}, \frac{1}{1+\delta} \frac{\partial \delta}{\partial s^2}, \bar{\chi} \frac{\delta}{1+\delta} \frac{\partial q}{\partial m} - \frac{1}{\delta} \frac{\partial \delta}{\partial m}, \bar{\chi} \frac{\delta}{1+\delta} \frac{\partial q}{\partial s^2} - \frac{1}{\delta} \frac{\partial \delta}{\partial s^2}, -1 \right]. \end{aligned}$$

Then we can also write:

$$J_2(\varepsilon) = \varepsilon \begin{bmatrix} (\log Z_L - m)\left(\frac{\partial \log Z_L}{\partial m_t} - 1\right) + (\log Z_H - m)\left(\frac{\partial \log Z_H}{\partial m_t} - 1\right) \\ \frac{1}{\varepsilon} + (\log Z_L - m)\frac{\partial \log Z_L}{\partial s_t^2} + (\log Z_H - m)\frac{\partial \log Z_H}{\partial s_t^2} - 1 \\ (\log Z_L - m)\frac{\partial \log Z_L}{\partial m_{t-1}} + (\log Z_H - m)\frac{\partial \log Z_H}{\partial m_{t-1}} \\ (\log Z_L - m)\frac{\partial \log Z_L}{\partial s_{t-1}^2} + (\log Z_H - m)\frac{\partial \log Z_H}{\partial s_{t-1}^2} \\ (\log Z_L - m)\frac{\partial \log Z_L}{\partial \omega_{t-1}} + (\log Z_H - m)\frac{\partial \log Z_H}{\partial \omega_{t-1}} \end{bmatrix}'$$

where the $\log Z$ derivative expressions can be evaluated as in $J_1(\varepsilon)$. Similarly, we have:

$$J_3(\varepsilon) = [1, 0, 0, 0, 0], \quad J_4(\varepsilon) = [0, 1, 0, 0, 0],$$

and finally:

$$J_5(\varepsilon) = \left[0, 0, \bar{\chi}\frac{\partial q}{\partial m}, \bar{\chi}\frac{\partial q}{\partial s^2}, 0\right].$$

It remains to evaluate the additional derivatives in the expressions $\frac{\partial q}{\partial \theta_i}$ and $\frac{\partial \delta}{\partial \theta_i}$, for $\theta_i = m, s^2$. For δ and the case of $\theta_i = m$, these explicit calculations are given in the proof of Theorem 2, where embedded in these expressions are further derivatives $\frac{\partial v}{\partial m}$ and $\frac{\partial \kappa}{\partial m}$. The derivatives for $\theta_i = s^2$ follow similarly. For q, the derivatives at the SCE are given by:

$$\frac{\partial q}{\partial \theta_i} = -R\kappa\delta\frac{\partial v}{\partial \theta_i},$$

which can be evaluated given the expressions above.

III. Additional Quantitative Results

A. *Predictability of Returns*

In this section, I discuss the time series predictability of returns. In the model, returns are strongly positively autocorrelated over one- and two-year horizons. In addition, a strong positive return in a current period makes a crash more likely several periods in the future. In the data, there is little or no evidence of autocorrelations in annual market returns, and weak evidence of time series reversals.[1] In Table 1, I report time regressions of the form:

$$R_{t+f-1,t+f} = a_f + b_f R_{t-1,t} + \epsilon_{t+f-1,t+f}$$

where t indexes the current date and $f \in \{1, 2, 3, 4, 5\}$ is the number of years ahead. The table reports the results from the Shiller (2023) data, along with

[1]These facts are distinct from the well-documented cross-sectional momentum and reversal.

simulations of my baseline model. The simulated return data use the same 1,000 simulated time series as in the main text, and I report the mean coefficient and t-statistic from the regressions in each simulation run.

Table 1— Results from return predictability regressions

	Data		Simulations	
Years	Coefficient	t-Stat	Coefficient	t-Stat
1	0.0181	0.22	0.6509	27.75
2	-0.1889	-2.32	0.1345	4.32
3	0.0934	1.13	-0.1309	-4.19
4	-0.0613	-0.73	-0.2086	-6.75
5	-0.1302	-1.56	-0.2056	-6.65

Note: Coefficients from return predictability regressions in the data along with average regression coefficients and t-statistics from 1000 simulations of the calibrated model.

The model generates return predictability from persistent changes in beliefs θ_t and net asset positions ω_t. Moreover, the dynamics of the cycles in my model mean that a market boom today predicts a crash in the future. Both the autocorrelation and the reversal are much stronger in the model than in the data. The data shows some evidence of reversal at two years and very weak evidence of longer-term reversal.

B. Intermediate Between Open and Closed Economies

The baseline small open economy model has very different properties from a standard closed economy version. Here I report some simulations from alternative intermediate specifications, where interest rates adjust each period but enough to ensure that there is zero debt. I find that the qualitative features of the closed economy model persist in many intermediate cases. This suggests that the debt dynamics and asset allocations features of the open economy are more crucial for asset pricing implications than the lack of interest rate variation.

Using the results above we can write the demand for bonds (scaled by dividends) as:

$$\frac{B_t}{D_t} = (1+\delta(R))(1-c(R))(1-v(R))\left(1 + R\frac{B_{t-1}}{D_{t-1}}\chi_t\right),$$

where I emphasize the dependence on the interest rate R. In the small open economy case R is a fixed constant, while in the closed economy case R adjusts to level, say R_t^*, so that $B_t = 0$. For an intermediate case, I suppose that bonds are supplied with an interest elasticity which is finite (unlike the small open economy) and positive (unlike the open economy).

Rather than specifying an explicit supply curve, I analyze cases where the equilibrium interest rate R_t is a weighted average of the open and closed economy interest rates:

$$R_t = \alpha R + (1-\alpha) R_t^*.$$

Varying the weight α between zero and one effectively traces out the implications of different supply elasticities.

TABLE 2— SUMMARY STATISTICS FOR VARIATIONS ON THE MODEL.

Statistic	Data	$\alpha = 1$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$	$\alpha = 0$	$\alpha = 0.5$ (Opt)
$E(\log P/D)$	3.25	3.16	3.31	3.46	3.59	3.68	3.70	3.20
$\sigma(\log P/D)$	0.46	0.46	0.40	0.33	0.24	0.15	0.15	0.38
$100 \times E(\log Z)$	6.51	6.44	5.56	4.91	4.40	4.11	4.10	5.95
$100 \times \sigma(\log Z)$	16.90	16.93	15.30	13.97	12.71	11.70	11.17	15.47
Crash Frequency	25.25	38.30	41.16	46.30	59.82	266.18	–	36.19

Note: Statistics from data and simulations of the baseline model ($\alpha = 1$), the closed economy version with the baseline parameters ($\alpha = 0$), and a range of intermediate cases ($0 < \alpha < 1$).

The results from the simulations for different α values are reported in Table 2. The setting is the same as in Table 1 in the paper, where I fix the baseline parameters and consider the same shock realizations, iterations, and sample length as above. Thus $\alpha = 1$ repeats the data from Table 1 for the baseline case while $\alpha = 0$ is the closed economy with the baseline parameterization. As expected, the statistics for the asset pricing data vary continuously with the weighting α, with the mean P/D ratio increasing and its volatility falling with α. However the crash frequency is relatively insensitive to α, increasing only slightly until the weight on the closed economy interest rate drops to 0.2. The last column shows the results for the case of $\alpha = 0.5$ where now I re-optimize over the parameter vector (which yielded a larger gain of 0.01 and slightly larger risk aversion of nearly 4). The results come close to the baseline small open economy parameterization both in terms of the fit to the asset price statistics (the error is 0.204) and roughly match the timing of market crashes.

IV. Beliefs and Survey Expectations

As discussed in the paper, several papers have shown that survey measures of expected returns are positively correlated with price-dividend ratios. The top row of Table 3 reports a such a finding, using the subjective expectations data series from Nagel and Xu (2022).[2] Consistent with the previous literature, the correlation between the survey expected excess return and the empirical price-dividend ratio is positive, at 0.476.

[2]This series combines and imputes subjective survey expectations from multiple sources, and is available quarterly from 1987:2 to 2021:4.

TABLE 3— CORRELATIONS BETWEEN SUBJECTIVE BELIEFS AND THE LOG PRICE-DIVIDEND RATIO.

Data Sources	Correlation
Nagel-Xu (2022): excess return, Shiller $\log P/D$	0.476
Baseline model: m, $\log P/D$	-0.027
Baseline model: $(m - \log R)/s$, $\log P/D$	0.472
Model with $(\varepsilon_m, \varepsilon_s)$: m, $\log P/D$	0.257

Note: The first line uses the Nagel and Xu (2022) subjective expected excess returns and the price-dividend ratio from the Shiller data set. The others use simulated data from 1,000 simulations of the baseline model or variations of it.

However in my model, the subjective excess return is essentially uncorrelated with the price-dividend ratio. Table 3 reports the correlations from 1,000 simulations of 2,000 periods, discarding the first 1,000 periods. For the baseline calibration of the model from above, the correlation between expected returns m_t and the log price-dividend ratio is -0.027. Thus the baseline model misses the positive correlation between survey expected returns and the price dividend ratio. What really matters for agents in the model is the subjective risk-return tradeoff, which determines their portfolio position, and hence drives asset flows and returns. The third row of Table 3 reports the correlation between the subjective Sharpe ratio and the log-price dividend ratio in the model. The value of 0.472 nearly matches the procyclical correlation observed in the survey expectations.

In fact, many of the surveys ask for positive or negative outlooks rather than quantitative predictions, and so could reflect investors' subjective risk-adjusted assessments rather than simply their expected return forecast. For example, four of the six sources of survey beliefs analyzed by Greenwood and Shleifer (2014) report subjective measures of stock market optimism, rather than quantitative predictions of market returns. In addition, one of the main Gallup surveys asks respondents about expected returns on their own portfolios, which would combine the endogenous portfolio weight (v_t in the model) with beliefs.

The largest movements in beliefs in the model come as a consequence of the collapse of prices during a market crash. In the simulations, the largest realized returns occur in the immediate aftermath of a crash, as the stock price first falls to a very low level during the crash and then rebounds to a higher (but still historically low) value shortly thereafter. The volatility in returns before and after the crash leads to a spike in the estimated volatility s_t, while the large return after the collapse leads a large upward revision in both the expected return m_t. Both of these changes in beliefs take a long time to fade away. A variation in the model that would dampen the collapse of prices in a crash, such as having pool of outside investors willing to buy at low prices, would lead to less dramatic changes in beliefs after a crash and increase the correlation between expected returns and prices.

Variations in the belief specification alter the model dynamics and hence its predictions. In the baseline model, the gain ε governs the speed of adjustment of both the expected return m_t and the subjective variance s_t^2. While this is a useful benchmark, the belief components could drift at different rates, with gain ε_m for the mean and ε_s for the variance.[3] The fourth row of Table 3 considers a version of the model that separates $(\varepsilon_m, \varepsilon_s)$, where the calibration includes the previous asset pricing moments but also aims to match the correlation between m_t and $\log P_t/D_t$, with a target of 0.476. The other statistics and the parameter values for this specification are given in the last column of Table 4. For comparison, the table repeats the columns for the data and baseline model from Table 1 in the paper. With this minor change in the belief dynamics, the model is able to generate a larger correlation of 0.257, and the duration between crashes (which was un-targeted) decreases to 27.3 years, which is closer to the empirical value. But the fit of the model on the asset pricing moments worsens, with the relative error increasing from 0.031 to 0.234. This model variation that separates learning speeds improves along some dimensions, but is not a complete fix.

Table 4— Summary statistics from data and simulations.

Statistic	Data	Baseline	Two Gains
$E(\log P/D)$	3.25	3.16	3.43
$\sigma(\log P/D)$	0.46	0.46	0.50
$100 \times E(\log Z)$	6.51	6.44	5.32
$100 \times \sigma(\log Z)$	16.90	16.93	18.43
Error	–	0.031	0.234
β	–	0.979	0.975
γ	–	3.259	1.174
ε	–	0.0051	0.0010 0.0103
Crash Frequency	25.25	38.30	27.30
$100 \times E(\log \Delta C)$	0.86	1.57	1.58
$100 \times \sigma(\log \Delta C)$	1.27	16.99	17.43

Note: US data and simulations of the baseline model and a version with two learning gains.

My main focus in the paper was to analyze the dynamics of a learning model developed as an interpretation of Minsky's theory. As such, it was kept simple in order to aid the analysis. If we interpret survey measures as capturing risk-adjusted subjective beliefs, then my model also fits with this survey evidence. But to directly fit the survey measures of expected returns, variations on the baseline model seem necessary. The more general belief specification gets part of the way there, but other sources of shocks or belief specifications may be required.

[3]The limit theory would go with a simple scale factor adjustment. For example, we could specify $\varepsilon_m = \varepsilon$ and $\varepsilon_s = k_s\varepsilon$ and let $\varepsilon \to 0$ as in the theory. Then the limit dynamics for s_t^2 would be scaled by k_s.

For example, Adam, Marcet and Beutel (2017) introduce labor income shocks and subjective beliefs with persistent and transitory components in order to fit moments of asset prices and survey expectations.

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