Online Appendix for

"A Preferred-Habitat Model of Term Premia, Exchange Rates, and Monetary Policy Spillovers"

A. Forwards and Swaps

We show that the equilibrium with a currency forward market is equivalent to one without it but with different demands for foreign assets, home bonds and foreign bonds. The equivalence result extends to swaps because they are portfolios of forwards.

We model the demand for currency forwards as follows. Currency traders with preferences for currency forwards with maturities in $[\tau, \tau + d\tau]$ are in measure $d\tau$, and their demand, expressed in units of the home currency, is

$$Z_{et}^{(\tau)} = -(\zeta_e(\tau) + \theta_e(\tau)\gamma_t), \tag{A.1}$$

where $(\zeta_e(\tau), \theta_e(\tau))$ are functions of τ.

Since global arbitrageurs can trade costlessly foreign currency, home bonds, and foreign bonds, Covered Interest Parity (CIP) holds. Moreover, buying $Z_{et}^{(\tau)}$ of the currency forward with maturity τ is equivalent to the combination of (i) selling $Z_{et}^{(\tau)}$ of the home bond with maturity τ for home currency, (ii) selling $Z_{et}^{(\tau)}$ of home currency for foreign currency, and (iii) selling $Z_{et}^{(\tau)}$ of foreign currency for the foreign bond with maturity τ. Hence, the equilibrium with the currency forward market is equivalent to one without it but with $Z_{et}^{(\tau)}$ added to the demand (1.4) for foreign assets, $-Z_{et}^{(\tau)}$ added to the home bond demand (1.5), and $Z_{et}^{(\tau)}$ added to the foreign bond demand (1.5). The demand for foreign assets becomes

$$Z_{et} + \int_0^T Z_{et}^{(\tau)} d\tau = -\alpha_e \log(e_t) - (\zeta_{et} + \theta_e\gamma_t) - \int_0^T (\zeta_e(\tau) + \theta_e(\tau)\gamma_t) d\tau$$

instead of Z_{et}. The demand for country j bonds with maturity τ becomes

$$Z_{jt}^{(\tau)} + (-1)^{1_{\{j=H\}}} Z_{et}^{(\tau)} = -\alpha_j(\tau) \log\left(P_{jt}^{(\tau)}\right) - (\zeta_j(\tau) + \theta_j(\tau)\beta_{jt}) - (-1)^{1_{\{j=H\}}} \left(\zeta_e(\tau) + \theta_e(\tau)\gamma_t\right)$$

instead of $Z_{jt}^{(\tau)}$. Forwards induce a negative correlation between the demand for foreign assets and for home bonds, and a positive correlation between the demand for foreign assets and for foreign bonds.

B. Proofs

Proposition B.1 characterizes the equilibrium in Section II. We denote by $(\mathbf{I}_{iH}, \mathbf{I}_{iF}, \mathbf{I}_\gamma, \mathbf{I}_{\beta H}, \mathbf{I}_{\beta F})$ the five columns of the 5×5 identity matrix.

PROPOSITION B.1: *When arbitrage is global, the exchange rate e_t is given by (2.1) and bond prices $P_{jt}^{(\tau)}$ in country $j = H, F$ are given by (2.2), with $(\mathbf{A}_e, C_e)$ solving*

$$\mathbf{M}\mathbf{A}_e - \mathbf{I}_{iH} + \mathbf{I}_{iF} = 0, \tag{B.1}$$

$$-\mathbf{A}_e^\top \boldsymbol{\Gamma}\bar{\mathbf{q}} - (\pi_F - \pi_H) + \frac{1}{2}\mathbf{A}_e^\top \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \mathbf{A}_e = \mathbf{A}_e^\top \boldsymbol{\lambda}_C, \tag{B.2}$$

and $(\mathbf{A}_j(\tau), C_j(\tau))$ solving

$$\mathbf{A}_\mathbf{j}'(\tau) + \mathbf{M}\mathbf{A}_j(\tau) - \mathbf{I}_{ij} = 0, \tag{B.3}$$

$$C_j'(\tau) - \mathbf{A}_j(\tau)^\top \boldsymbol{\Gamma}\bar{\mathbf{q}} + \frac{1}{2}\mathbf{A}_j(\tau)^\top \boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \left(\mathbf{A}_j(\tau) + 2\mathbf{A}_e 1_{\{j=F\}}\right) = \mathbf{A}_j(\tau)^\top \boldsymbol{\lambda}_C, \tag{B.4}$$

with the initial conditions $\mathbf{A}_j(0) = C_j(0) = 0$, and

$$\mathbf{M} \equiv \boldsymbol{\Gamma}^\top - a\left((\theta_e \mathbf{I}_\gamma - \alpha_e \mathbf{A}_e)\mathbf{A}_e^\top + \sum_{j=H,F}\int_0^T (\theta_j(\tau)\mathbf{I}_{\beta j} - \alpha_j(\tau)\mathbf{A}_j(\tau))\mathbf{A}_j(\tau)^\top d\tau \right)\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top, \tag{B.5}$$

$$\boldsymbol{\lambda}_C \equiv a\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \left((\zeta_e - \alpha_e C_e)\mathbf{A}_e + \sum_{j=H,F}\int_0^T (\zeta_j(\tau) - \alpha_j(\tau)C_j(\tau))\mathbf{A}_j(\tau) d\tau \right). \tag{B.6}$$

Proof of Proposition B.1: Using the definitions of $(\mathbf{I}_{iH}, \mathbf{I}_{iF}, \mathbf{I}_\gamma, \mathbf{I}_{\beta H}, \mathbf{I}_{\beta F})$, we can write (2.16) as

$$\begin{aligned}
&a\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \left(\mathbf{A}_e (\theta_e \mathbf{I}_\gamma - \alpha_e \mathbf{A}_e)^\top + \sum_{j=H,F}\int_0^T \mathbf{A}_j(\tau)(\theta_j(\tau)\mathbf{I}_{\beta j} - \alpha_j(\tau)\mathbf{A}_j(\tau))^\top d\tau \right)\mathbf{q}_t \\
&+ a\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top \left((\zeta_e - \alpha_e C_e)\mathbf{A}_e + \sum_{j=H,F}\int_0^T (\zeta_j(\tau) - \alpha_j(\tau)C_j(\tau))\mathbf{A}_j(\tau) \right) \\
&= -(\mathbf{M} - \boldsymbol{\Gamma}^\top)^\top \mathbf{q}_t + \boldsymbol{\lambda}_C,
\end{aligned} \tag{B.7}$$

where the second step follows from the definitions of $(\mathbf{M}, \boldsymbol{\lambda}_C)$. We next substitute $(\mu_{et}, \{\mu_{jt}^{(\tau)}\}_{j=H,F}, \boldsymbol{\lambda}_t)$ from (2.4), (2.7), (2.8) and (B.7) into the arbitrageurs' first-order condition. Substituting into (2.10), we find an equation that is affine in $\mathbf{q}_t$.

Equation (B.1) follows by identifying the linear terms in $\mathbf{q}_t$, and (B.2) follows by identifying the constant terms. Substituting into (2.11), we find an equation that is affine in $\mathbf{q}_t$. Equation (B.3) follows by identifying the linear terms in $\mathbf{q}_t$, and (B.4) follows by identifying the constant terms. The initial conditions $\mathbf{A}_j(0) = C_j(0) = 0$ follow because the price of a bond with zero maturity is its face value, which is one.

Solving the system of (B.1)-(B.4) reduces to solving a system of 25 nonlinear scalar equations. Indeed, taking the 5×5 matrix $\mathbf{M}$ as given, we can solve the system (B.1) of five scalar equations in the elements of the 5×1 vector $\mathbf{A}_e$, the system (B.3) of five linear ODEs in the elements of the 5×1 vector $\mathbf{A}_H(\tau)$, and the same system (B.3) of five linear ODEs in the elements of the 5×1 vector $\mathbf{A}_F(\tau)$. We can then substitute back into the definition (B.5) of $\mathbf{M}$ to derive the system of 25 nonlinear scalar equations. Given a solution to that system, (B.2) determines C_e uniquely, and (B.4) determines $(C_H(\tau), C_F(\tau))$ uniquely. Since for $a = 0$, (B.5) implies $\mathbf{M} = \mathbf{\Gamma}^\top$ and (B.6) implies $\boldsymbol{\lambda}_C = 0$, (B.1) implies $\mathbf{A}_e = \left(\mathbf{\Gamma}^{-1}\right)^\top (\mathbf{I}_{iH} - \mathbf{I}_{iF})$ and (B.2) implies (2.17). ∎

Corollary B.1 specializes Proposition B.1 to the case where arbitrageurs are risk-neutral or their risk aversion goes to zero.

COROLLARY B.1: *Suppose that arbitrage is global.*

- *When arbitrageurs are risk-neutral ($a = 0$), UIP and EH hold: the expected return on foreign currency is $\mu_{et}^{UIP} \equiv i_{Ht} - i_{Ft}$, and the expected return on country-j bonds is $\mu_{jt}^{(\tau)EH} \equiv i_{jt}$. Stationarity of the real exchange rate, as per the conjecture (2.1), requires (2.17).*
- *When arbitrageurs' risk aversion goes to zero ($a \to 0$), the expected return on foreign currency goes to μ_{et}^{UIP} and the expected return on country-j bonds goes to $\mu_{jt}^{(\tau)EH}$ only when (2.17) holds. Stationarity of the real exchange rate does not require (2.17) if $\alpha_e > 0$.*

Proof of Corollary B.1: The results for $a = 0$ follow from the arguments before the corollary' statement. When a goes to zero, (B.1), (B.3) and (B.5) imply that $\mathbf{M}$ goes to $\mathbf{\Gamma}^\top$ and $(\mathbf{A}_e, \{\mathbf{A}_j(\tau)\}_{j=H,F})$ have the finite limits

$$\lim_{a\to 0} \mathbf{A}_e = \left(\mathbf{\Gamma}^{-1}\right)^\top (\mathbf{I}_{iH} - \mathbf{I}_{iF}),$$
$$\lim_{a\to 0} \mathbf{A}_j(\tau) = \left(\mathbf{\Gamma}^{-1}\right)^\top \left(I - e^{-\mathbf{\Gamma}^\top \tau}\right) \mathbf{I}_{ij}.$$

When (2.17) holds, (B.2), (B.6) (B.6) are met with $\boldsymbol{\lambda}_C$ having a zero limit and $(C_e, \{C_j(\tau)\}_{j=H,F})$ having finite limits. Equation (2.16) then implies that $\boldsymbol{\lambda}_t$ goes to zero, which means from (2.10) and (2.11) that UIP and EH hold in the limit. When instead (2.17) does not hold, (B.2) implies that $\mathbf{A}_e^\top \boldsymbol{\lambda}_C$ has a non-zero limit.

When, in addition, $\alpha_e > 0$, (B.2), (B.6) (B.6) are met with $\boldsymbol{\lambda}_C$ having a non-zero limit, $\{C_j(\tau)\}_{j=H,F}$ having finite limits, and C_e going to plus or minus infinity at the rate $\frac{1}{a}$. Equation (2.16) then implies that $\boldsymbol{\lambda}_t$ does not go to zero, which means from (2.10) and (2.11) that UIP and EH do not hold. ∎

Propositions B.2 and B.3 characterize the equilibrium in the currency and bond markets, respectively, under segmented arbitrage and the parameter restrictions assumed in Section III.

PROPOSITION B.2: *Suppose that arbitrage is segmented, the matrices* $(\boldsymbol{\Gamma}, \boldsymbol{\Sigma})$ *are diagonal, and* $\boldsymbol{\Sigma}_{3,3} = \boldsymbol{\Sigma}_{4,4} = \boldsymbol{\Sigma}_{5,5} = 0$*. The exchange rate* e_t *is given by (2.1), with* (A_{iHe}, A_{iFe}) *positive and equal to the unique solution of*

$$\kappa_{ij} A_{ije} - 1 = -a_e \alpha_e A_{ije} \left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2\right), \tag{B.8}$$

and C_e *solving*

$$\begin{aligned} & -\kappa_{iH}\bar{i}_H A_{iHe} + \kappa_{iF}\bar{i}_F A_{iFe} - (\pi_F - \pi_H) + \frac{1}{2}\sigma_{iH}^2 A_{iHe}^2 + \frac{1}{2}\sigma_{iF}^2 A_{iFe}^2 \\ & = a_e \left(\zeta_e - \alpha_e C_e\right) \left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2\right). \end{aligned} \tag{B.9}$$

Proof of Proposition B.2: The first-order condition in the currency market is

$$\mu_{et} + i_{Ft} - i_{Ht} = \mathbf{A}_e^\top \boldsymbol{\lambda}_{et}, \tag{B.10}$$

where $\mathbf{A}_e \equiv (A_{iHe}, -A_{iFe})$ and $\boldsymbol{\lambda}_{et} \equiv a_e W_{Ft}(\sigma_{iH}^2 A_{iHe}, -\sigma_{iF}^2 A_{iFe})$. It follows from (2.10) by keeping only the term $W_{Ft}\mathbf{A}_e$ in the parenthesis in (2.12), taking $\boldsymbol{\Sigma}$ to be diagonal with $\boldsymbol{\Sigma}_{3,3} = \boldsymbol{\Sigma}_{4,4} = \boldsymbol{\Sigma}_{5,5} = 0$, and replacing a by a_e. Proceeding as in the derivation of (2.16) and using $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we find $\boldsymbol{\lambda}_{et} = (\lambda_{eHt}, \lambda_{eFt})$ with

$$\lambda_{ejt} = a_e \sigma_{ij}^2 \left[\zeta_e - \alpha_e \left(A_{iHe} i_{Ht} - A_{iFe} i_{Ft} + C_e\right)\right] A_{ije} (-1)^{1_{\{j=F\}}} \tag{B.11}$$

for $j = H, F$. Since $(\boldsymbol{\Gamma}, \boldsymbol{\Sigma})$ are diagonal with $\boldsymbol{\Sigma}_{3,3} = \boldsymbol{\Sigma}_{4,4} = \boldsymbol{\Sigma}_{5,5} = 0$ and $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we can write (2.4) as

$$\mu_{et} = -A_{iHe}\kappa_{iH}(\bar{i}_H - i_{Ht}) + A_{iFe}\kappa_{iF}(\bar{i}_F - i_{Ft}) - (\pi_F - \pi_H) + \frac{1}{2}A_{iHe}^2\sigma_{iH}^2 + \frac{1}{2}A_{iFe}^2\sigma_{iF}^2. \tag{B.12}$$

Substituting $\boldsymbol{\lambda}_{et}$ from (B.11) and μ_{et} from (B.12) into (B.10), we find an equation that is affine in (i_{Ht}, i_{Ft}). Equation (B.8) follows by identifying the linear terms in (i_{Ht}, i_{Ft}), and (B.9) follows by identifying the constant terms.

When $a\alpha_e = 0$, (B.8) has the unique solution $(A_{iHe}, A_{iFe}) = (1/\kappa_{iH}, 1/\kappa_{iF})$, which is positive. Consider next the case $a\alpha_e > 0$. A solution (A_{iHe}, A_{iFe}) to

(B.8) must be positive, as can be seen by writing that equation as

$$\left[\kappa_{ij} + a_e\alpha_e\left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2\right)\right] A_{ije} = 1. \tag{B.13}$$

Since $(A_{iHe}, A_{iFe}, a\alpha_e)$ are positive, the right-hand side of (B.8) is negative. Therefore, the left-hand side is negative as well, which implies $A_{iHe} < 1/\kappa_{Hj}$ and $A_{iFe} < 1/\kappa_{Fj}$. Dividing (B.8) written for $j = H$ by (B.8) written for $j = F$, we find

$$\frac{1 - \kappa_{iH}A_{iHe}}{1 - \kappa_{iF}A_{iFe}} = \frac{A_{iHe}}{A_{iFe}} \Leftrightarrow A_{iHe} = \frac{A_{iFe}}{1 + (\kappa_{iH} - \kappa_{iF})A_{iFe}}. \tag{B.14}$$

Equation (B.14) determines A_{iHe} as an increasing function of $A_{iFe} \in [0, 1/\kappa_{iF}]$, equal to zero for $A_{iFe} = 0$, and equal to $1/\kappa_{iH}$ for $A_{iFe} = 1/\kappa_{iF}$. Substituting A_{iHe} as a function of A_{iFe} in (B.13) written for $j = F$, we find an equation in the single unknown A_{iFe}. The left-hand side of that equation is increasing in A_{iFe}, is equal to zero for $A_{iFe} = 0$, and is equal to a value larger than one for $A_{iFe} = 1/\kappa_{iF}$. Hence, that equation has a unique solution $A_{iFe} \in (0, 1/\kappa_{iF})$. Given that solution, (B.14) determines $A_{iHe} \in (0, 1/\kappa_{iH})$ uniquely. Given (A_{iHe}, A_{iFe}), (B.9) determines C_e uniquely if $\alpha_e > 0$. If $\alpha_e = 0$, then the restriction

$$\bar{i}_H - \pi_H = \bar{i}_F - \pi_F + \left(\frac{1}{2} - a_e\zeta_e\right)\left(\frac{\sigma_{iH}^2}{\kappa_{iH}^2} + \frac{\sigma_{iF}^2}{\kappa_{iF}^2}\right) \tag{B.15}$$

on model parameters must be imposed and C_e is indeterminate. ∎

PROPOSITION B.3: *Suppose that arbitrage is segmented, the matrices* $(\mathbf{\Gamma}, \mathbf{\Sigma})$ *are diagonal, and* $\mathbf{\Sigma}_{3,3} = \mathbf{\Sigma}_{4,4} = \mathbf{\Sigma}_{5,5} = 0$. *Bond prices* $P_{jt}^{(\tau)}$ *in country* $j = H, F$ *are given by (2.2), with* $A_{ij'j}(\tau)$ *equal to zero for* $j' \neq j$ *and* $(A_{ijj}(\tau), C_{ij}(\tau))$ *equal to the unique solution of the system*

$$A'_{ijj}(\tau) + \kappa_{ij}A_{ijj}(\tau) - 1 = -a_j\sigma_{ij}^2 A_{ijj}(\tau)\int_0^T \alpha_j(\tau)A_{ijj}(\tau)^2 d\tau, \tag{B.16}$$

$$C'_j(\tau) - \kappa_{ij}\bar{i}_j A_{ijj}(\tau) + \frac{1}{2}\sigma_{ij}^2 A_{ijj}(\tau)\left(A_{ijj}(\tau) - 2A_{iFe}1_{\{j=F\}}\right)$$
$$= a_j\sigma_{ij}^2 A_{ijj}(\tau)\int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau)C_j(\tau)\right] A_{ijj}(\tau)d\tau, \tag{B.17}$$

with the initial conditions $A_{ijj}(0) = C_j(0) = 0$.

Proof of Proposition B.3: The first-order condition in the country-j bond market is

$$\mu_{jt}^{(\tau)} - i_{jt} = \mathbf{A}_j(\tau)\lambda_{jt}, \tag{B.18}$$

where $\mathbf{A}_j(\tau) \equiv A_{ijj}(\tau)$ and $\lambda_{jt} \equiv a_j \sigma_{ij}^2 \int_0^T X_{jt}^{(\tau)} A_{ijj}(\tau) d\tau$. It follows from (2.11) by keeping only the term $\int_0^T X_{jt}^{(\tau)} \mathbf{A}_j(\tau) d\tau$ in the parenthesis in (2.12), taking $\mathbf{\Sigma}$ to be diagonal with $\mathbf{\Sigma}_{3,3} = \mathbf{\Sigma}_{4,4} = \mathbf{\Sigma}_{5,5} = 0$, replacing a by a_j, and conjecturing that in equilibrium $A_{ij'j}(\tau) = 0$ for $j' \neq j$. Proceeding as in the derivation of (2.16) and using $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$ and $A_{ij'j}(\tau) = 0$ for $j' \neq j$, we find

$$\lambda_{jt} = a_j \sigma_{ij}^2 \left(\int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau) \left(A_{ijj}(\tau) i_{jt} + C_j(\tau) \right) \right] A_{ijj}(\tau) d\tau \right). \tag{B.19}$$

Since $(\mathbf{\Gamma}, \mathbf{\Sigma})$ are diagonal with $\mathbf{\Sigma}_{3,3} = \mathbf{\Sigma}_{4,4} = \mathbf{\Sigma}_{5,5} = 0$, $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, and $A_{ij'j}(\tau) = 0$ for $j' \neq j$, we can write (2.7) and (2.8) as

$$\mu_{jt}^{(\tau)} = A'_{ijj}(\tau) i_{jt} + C'_j(\tau) - A_{ijj}(\tau) \kappa_{ij} (\bar{i}_j - i_{jt}) + A_{ijj}(\tau) \left(A_{ijj}(\tau) - 2 A_{iFe} 1_{\{j=F\}} \right) \sigma_{ij}^2. \tag{B.20}$$

Substituting λ_{jt} from (B.19) and μ_{jt} from (B.20) into (B.18), we find an equation that is affine in i_{jt}. Equation (B.16) follows by identifying the linear terms in i_{jt}, and (B.17) follows by identifying the constant terms. The initial conditions $A_{ijj}(0) = C_j(0) = 0$ follow because the price of a bond with zero maturity is its face value, which is one. Since the affine equation holds when (B.16) and (B.17) hold, our conjecture $A_{ij'j}(\tau) = 0$ for $j' \neq j$ is validated.

Solving (B.16) with the initial condition $A_{ijj}(0) = 0$, we find

$$A_{ijj}(\tau) = \frac{1 - e^{-\kappa_{ij}^* \tau}}{\kappa_{ij}^*}, \tag{B.21}$$

with

$$\kappa_{ij}^* \equiv \kappa_{ij} + a_j \sigma_{ij}^2 \int_0^T \alpha_j(\tau) A_{ijj}(\tau)^2 d\tau. \tag{B.22}$$

Substituting $A_{ijj}(\tau)$ from (B.21) into (B.22), we find the equation

$$\kappa_{ij}^* - \kappa_{ij} + a_j \sigma_{ij}^2 \int_0^T \alpha_j(\tau) \left(\frac{1 - e^{-\kappa_{ij}^* \tau}}{\kappa_{ij}^*} \right)^2 d\tau = 0 \tag{B.23}$$

in the single unknown κ_{ij}^*. The left-hand side of (B.23) is increasing in κ_{ij}^*, is negative for $\kappa_{ij}^* = \kappa_{ij}$, and goes to infinity when κ_{ij}^* goes to infinity. Hence, (B.23) has a unique solution $\kappa_{ij}^* > \kappa_{ij}$. Given κ_{ij}^*, (B.21) determines $A_{ijj}(\tau)$ uniquely.

Solving (B.17) with the initial condition $C_j(\tau) = 0$, we find

$$C_j(\tau) = \kappa_{ij}^* \bar{i}_j^* \int_0^\tau A_{ijj}(\tau) d\tau - \frac{1}{2} \sigma_{ij}^2 \int_0^\tau A_{ijj}(\tau)^2 d\tau, \tag{B.24}$$

with

(B.25) $$\kappa_{ij}^* \bar{i}_j^* \equiv \kappa_{ij}\bar{i}_j + a_j\sigma_{ij}^2 \int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau)C_j(\tau)\right] A_{ijj}(\tau)d\tau + \sigma_{ij}^2 A_{iFe}1_{\{j=F\}}.$$

Substituting $C_j(\tau)$ from (B.24) into (B.25), we find

(B.26)
$$\bar{i}_j^* = \frac{\kappa_{ij}\bar{i}_j + a_j\sigma_{ij}^2 \int_0^T \zeta_j(\tau)A_{ijj}(\tau)d\tau + \sigma_{ij}^2 A_{iFe}1_{\{j=F\}} + \frac{1}{2}a_j\sigma_{ij}^4 \int_0^T \alpha_j(\tau)\left(\int_0^\tau A_{ijj}(\tau')^2 d\tau'\right) A_{ijj}(\tau)d\tau}{\kappa_{ij}^*\left[1 + a_j\sigma_{ij}^2 \int_0^T \alpha_j(\tau)\left(\int_0^\tau A_{ijj}(\tau')d\tau'\right) A_{ijj}(\tau)d\tau\right]}.$$

Given $\bar{i}_j^*$, (B.24) determines $C_j(\tau)$ uniquely. ∎

Proof of Proposition 3.1: The property $A_{ije} > 0$ is shown in the proof of Proposition B.2. The UIP value of A_{ije} is $A_{ije}^{UIP} \equiv 1/\kappa_{ij}$, as can be seen from (B.8) by setting $a_e = 0$. When $a_e > 0$ and $\alpha_e > 0$, the proof of Proposition B.2 shows $A_{ije} < 1/\kappa_{ij}$. Differentiating (B.12) with respect to i_{Ht} and i_{Ft}, we find

$$\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ht}} = \kappa_{iH}A_{iHe} - 1 < 0,$$
$$\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ft}} = -\kappa_{iF}A_{iFe} + 1 > 0,$$

respectively. ∎

Proof of Proposition 3.2: The properties $A_{ijj}(\tau) > 0$ and $A_{ij'j} = 0$ for $j' \neq j$ are shown in the proof of Proposition B.3. The EH value of $A_{ijj}(\tau)$ is $A_{ijj}^{EH}(\tau) \equiv (1 - e^{-\kappa_{ij}\tau})/\kappa_{ij}$, as can be seen from (B.21) and (B.22) by setting $a_j = 0$. When $a_j > 0$ and $\alpha_j(\tau) > 0$, (B.22) implies $\kappa_{ij}^* > \kappa_{ij}$ and (B.21) implies $A_{ijj}(\tau) < A_{ijj}^{EH}(\tau)$. Differentiating (B.20) with respect to i_{jt}, we find

$$\frac{\partial\left(\mu_{jt}^{(\tau)} - i_{jt}\right)}{\partial i_{jt}} = A'_{ijj}(\tau) + \kappa_{ij}A_{ijj}(\tau) - 1 = (\kappa_{ij} - \kappa_{ij}^*)A_{ijj}(\tau) < 0,$$

where the second step follows from (B.21). ∎

Proof of Proposition 3.3: Consider an one-off increase in γ_t at time zero, and denote by κ_γ the rate at which γ_t reverts to its mean of zero. Equation (B.11) is modified to

(B.27)
$$\lambda_{ejt} = a_e\sigma_{ij}^2 \left[\zeta_e + \theta_e\gamma_t - \alpha_e\left(A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + A_{\gamma e}e_t + C_e\right)\right] A_{ijt}(-1)^{1_{\{j=F\}}}$$

and (B.12) is modified to

(B.28)

$$\mu_{et} = -A_{iHe}\kappa_{iH}(\bar{i}_H - i_{Ht}) + A_{iFe}\kappa_{iF}(\bar{i}_F - i_{Ft}) + A_{\gamma e}\kappa_\gamma\gamma_t - (\pi_F - \pi_H) + \frac{1}{2}A_{iHe}^2\sigma_{iH}^2 + \frac{1}{2}A_{iFe}^2\sigma_{iF}^2.$$

Substituting λ_{ejt} from (B.27) and μ_{et} from (B.28) into (B.10), we find an equation that is affine in $(i_{Ht}, i_{Ft}, \gamma_t)$. Identifying the linear terms in γ_t yields

$$\begin{aligned}&\kappa_\gamma A_{\gamma e} = a_e(\theta_e - \alpha_e A_{\gamma e})\left(A_{iHe}^2\sigma_{iH}^2 + A_{iFe}^2\sigma_{iF}^2\right)\\ &\Rightarrow A_{\gamma e} = \frac{a_e\theta_e\left(A_{iHe}^2\sigma_{iH}^2 + A_{iFe}^2\sigma_{iF}^2\right)}{\kappa_\gamma + a_e\alpha_e\left(A_{iHe}^2\sigma_{iH}^2 + A_{iFe}^2\sigma_{iF}^2\right)}.\end{aligned} \tag{B.29}$$

When $\alpha_e > 0$, (B.29) implies $A_{\gamma e} > 0$ because $\theta_e > 0$. Hence, an increase in γ_t causes the foreign currency to depreciate. Since bonds in each country are traded by a separate set of agents than those trading foreign currency, their prices do not depend on γ_t.

Consider next an one-off increase in β_{jt} at time zero, and denote by $\kappa_{\beta j}$ the rate at which β_{jt} reverts to its mean of zero. Equation (B.19) is modified to

(B.30)

$$\lambda_{jt} = a_j\sigma_{ij}^2\left(\int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau)\left(A_{ijj}(\tau)i_{jt} + A_{\beta jj}(\tau)\beta_{jt} + C_j(\tau)\right)\right]A_{ijj}(\tau)d\tau\right),$$

and (B.20) is modified to

$$\begin{aligned}\mu_{jt}^{(\tau)} =& A'_{ijj}(\tau)i_{jt} + A'_{\beta jj}(\tau)\beta_{jt} + C'_j(\tau) - A_{ijj}(\tau)\kappa_{ij}(\bar{i}_j - i_{jt}) + A_{\beta jj}(\tau)\kappa_{\beta j}\beta_{jt}\\ &+ \frac{1}{2}A_{ijj}(\tau)\left(A_{ijj}(\tau) - 2A_{iFe}1_{\{j=F\}}\right)\sigma_{ij}^2.\end{aligned} \tag{B.31}$$

Substituting λ_{jt} from (B.30) and μ_{jt} from (B.31) into (B.18), we find an equation that is affine in (i_{jt}, β_{jt}). Identifying the linear terms in β_{jt} yields

(B.32)

$$A'_{\beta jj}(\tau) + \kappa_{\beta j}A_{\beta jj}(\tau) = a_j\sigma_{ij}^2 A_{ijj}(\tau)\int_0^T \left[\theta_j(\tau) - \alpha_j(\tau)A_{\beta jj}(\tau)\right]A_{ijj}(\tau)d\tau.$$

Solving (B.32) with the initial condition $A_{\beta jj}(\tau) = 0$, we find

$$A_{\beta jj}(\tau) = \bar{\lambda}_{ij\beta j}\int_0^\tau A_{ijj}(\tau')e^{-\kappa_{\beta j}(\tau-\tau')}d\tau', \tag{B.33}$$

where

$$\bar{\lambda}_{ij\beta j} \equiv a_j\sigma_{ij}^2\int_0^T \left[\theta_j(\tau) - \alpha_j(\tau)A_{\beta jj}(\tau)\right]A_{ijj}(\tau)d\tau. \tag{B.34}$$

Substituting $A_{\beta jj}(\tau)$ from (B.33) into (B.34) and solving for $\bar{\lambda}_{ij\beta j}$, we find

$$\bar{\lambda}_{ij\beta j} = \frac{a_j \sigma_{ij}^2 \int_0^T \theta_j(\tau) A_{ijj}(\tau) d\tau}{1 + a_j \sigma_{ij}^2 \int_0^T \alpha_j(\tau) \left(\int_0^\tau A_{ijj}(\tau') e^{-\kappa_{\beta j}(\tau-\tau')} d\tau'\right) A_{ijj}(\tau) d\tau}. \tag{B.35}$$

When $a_j > 0$, (B.35) implies $\lambda_{ij\beta j} > 0$ and (B.33) implies $A_{\beta jj}(\tau) > 0$ because $(\theta_j(\tau), A_{ijj}(\tau))$ are positive. Hence, an increase in β_{jt} raises bond yields in country j. Since the foreign currency and bonds in country j' are traded by different agents than those trading bonds in country j, their prices do not depend on β_{jt}. ∎

Proposition B.4 characterizes the equilibrium under global arbitrage and the parameter restrictions assumed in Section III.

PROPOSITION B.4: *Suppose that arbitrage is global, the matrices $(\mathbf{\Gamma}, \mathbf{\Sigma})$ are diagonal, and $\mathbf{\Sigma}_{3,3} = \mathbf{\Sigma}_{4,4} = \mathbf{\Sigma}_{5,5} = 0$. The exchange rate e_t is given by (2.1) and bond prices $P_{jt}^{(\tau)}$ in country $j = H, F$ are given by (2.2), with $(\{A_{ije}\}_{j=H,F}, C_e)$ solving*

$$\kappa_{ij} A_{ije} - 1 = a\sigma_{ij}^2 \bar{\lambda}_{ijj} A_{ije} - a\sigma_{ij'}^2 \bar{\lambda}_{ij'j} A_{ij'e}, \tag{B.36}$$

$$\begin{aligned} & -\kappa_{iH}\bar{i}_H A_{iHe} + \kappa_{iF}\bar{i}_F A_{iFe} - (\pi_F - \pi_H) + \frac{1}{2}\sigma_{iH}^2 A_{iHe}^2 + \frac{1}{2}\sigma_{iF}^2 A_{iFe}^2 \\ & = a\sigma_{iH}^2 \bar{\lambda}_{iHC} A_{iHe} - a\sigma_{iF}^2 \bar{\lambda}_{iFC} A_{iFe}, \end{aligned} \tag{B.37}$$

and $(A_{ijj}(\tau), A_{ijj'}(\tau), C_j(\tau))$ solving

$$A_{ijj}'(\tau) + \kappa_{ij} A_{ijj}(\tau) - 1 = a\sigma_{ij}^2 \bar{\lambda}_{ijj} A_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'j} A_{ij'j}(\tau), \tag{B.38}$$

$$A_{ij'j}'(\tau) + \kappa_{ij'} A_{ij'j}(\tau) = a\sigma_{ij}^2 \bar{\lambda}_{ijj'} A_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'j'} A_{ij'j}(\tau), \tag{B.39}$$

$$\begin{aligned} & C_j'(\tau) - \kappa_{ij}\bar{i}_j A_{ijj}(\tau) - \kappa_{ij'}\bar{i}_{j'} A_{ij'j}(\tau) + \frac{1}{2}\sigma_{ij}^2 A_{ijj}(\tau)\left(A_{ijj}(\tau) - 2A_{iFe}1_{\{j=F\}}\right) \\ & + \frac{1}{2}\sigma_{ij'}^2 A_{ij'j}(\tau)\left(A_{ij'j}(\tau) + 2A_{iHe}1_{\{j=F\}}\right) = a\sigma_{ij}^2 \bar{\lambda}_{ijC} A_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'C} A_{ij'j}(\tau), \end{aligned} \tag{B.40}$$

with the initial conditions $A_{ijj}(0) = A_{ijj'}(0) = C_j(0) = 0$, where $j' \neq j$ and

$$\bar{\lambda}_{ijj} \equiv -\alpha_e A_{ije}^2 - \sum_{k=H,F} \int_0^T \alpha_k(\tau) A_{ijk}(\tau)^2 d\tau, \tag{B.41}$$

(B.42)
$$\bar{\lambda}_{ijj'} \equiv \alpha_e A_{ije} A_{ij'e} - \sum_{k=H,F} \int_0^T \alpha_k(\tau) A_{ijk}(\tau) A_{ij'k}(\tau) d\tau,$$

(B.43)
$$\bar{\lambda}_{ijC} \equiv (\zeta_e - \alpha_e C_e) A_{ije} (-1)^{1_{\{j=F\}}} + \sum_{k=H,F} \int_0^T (\zeta_k(\tau) - \alpha_k(\tau) C_k(\tau)) A_{ijk}(\tau) d\tau.$$

Proof of Proposition B.4: The first-order conditions are (2.10) and (2.11), with $\mathbf{A}_e \equiv (A_{iHe}, -A_{iFe})$, $\mathbf{A}_j(\tau) \equiv (A_{iHj}(\tau), A_{iFj}(\tau))$ and $\boldsymbol{\lambda}_t \equiv (\lambda_{iHt}, \lambda_{iFt})$ with

$$\text{(B.44)} \qquad \lambda_{ijt} \equiv a\sigma_{ij}^2 \left(W_{Ft} A_{ije} (-1)^{1_{\{j=F\}}} + \sum_{j'=H,F} \int_0^T X_{j't}^{(\tau)} A_{ijj'}(\tau) d\tau \right).$$

They follow from (2.10) and (2.11) by taking $\boldsymbol{\Sigma}$ to be diagonal with $\boldsymbol{\Sigma}_{3,3} = \boldsymbol{\Sigma}_{4,4} = \boldsymbol{\Sigma}_{5,5} = 0$. Proceeding as in the derivation of (2.16) and using $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we find

$$\lambda_{ijt} = a\sigma_{ij}^2 \Bigg(\left[\zeta_e - \alpha_e \left(A_{iHe} i_{Ht} - A_{iFe} i_{Ft} + C_e\right)\right] A_{ije} (-1)^{1_{\{j=F\}}}$$
$$+ \sum_{j'=H,F} \int_0^T \left[\zeta_{j'}(\tau) - \alpha_{j'}(\tau) \left(A_{iHj'}(\tau) i_{Ht} + A_{iFj'}(\tau) i_{Ft} + C_{j'}(\tau)\right)\right] A_{ijj'}(\tau) d\tau \Bigg)$$

(B.45)
$$\equiv a\sigma_{ij}^2 \left(\bar{\lambda}_{ijj} i_{jt} + \bar{\lambda}_{ijj'} i_{j't} + \bar{\lambda}_{ijC}\right).$$

Since $(\boldsymbol{\Gamma}, \boldsymbol{\Sigma})$ are diagonal with $\boldsymbol{\Sigma}_{3,3} = \boldsymbol{\Sigma}_{4,4} = \boldsymbol{\Sigma}_{5,5} = 0$ and $\beta_{Ht} = \beta_{Ft} = \gamma_t = 0$, we can write (2.7) and (2.8) as

$$\mu_{jt}^{(\tau)} \equiv A'_{iHj}(\tau) i_{Ht} + A'_{iFj}(\tau) i_{Ft} + C'_j(\tau) - A_{iHj}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht}) - A_{iFj}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft})$$

(B.46)
$$+ \frac{1}{2} A_{iHj}(\tau) \left(A_{iHj}(\tau) + 2A_{iHe} 1_{\{j=F\}}\right) \sigma_{iH}^2 + \frac{1}{2} A_{iFj}(\tau) \left(A_{iFj}(\tau) - 1_{\{j=F\}} 2A_{iFe}\right) \sigma_{iF}^2.$$

Substituting $\boldsymbol{\lambda}_t$ from (B.45) and μ_{et} from (B.12) into (2.10) (for the definitions of $(\mathbf{A}_e, \boldsymbol{\lambda}_t)$ in Section III.B), we find an equation that is affine in (i_{Ht}, i_{Ft}). Equation (B.36) follows by identifying the linear terms in (i_{Ht}, i_{Ft}), and (B.37) follows by identifying the constant terms. Substituting $\boldsymbol{\lambda}_t$ from (B.45) and $\mu_{jt}^{(\tau)}$ from (B.46) into (2.11) (for the definitions of $(\mathbf{A}_j(\tau), \boldsymbol{\lambda}_t)$ in Section III.B), we find an equation that is affine in (i_{Ht}, i_{Ft}). Equations (B.38) and (B.39) follow by identifying the linear terms in (i_{Ht}, i_{Ft}), and (B.40) follows by identifying the constant terms.

The initial conditions $A_{ijj}(0) = A_{ijj'}(0) = C_j(0) = 0$ follow because the price of a bond with zero maturity is its face value, which is one.

Solving the system of (B.36)-(B.43) reduces to solving a system of three nonlinear scalar equations. Indeed, taking $\bar{\lambda}_{iHH}$, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ and $\bar{\lambda}_{iFF}$ as given, we can solve the system (B.36) of two scalar equations in (A_{iHe}, A_{iFe}), the system (B.38) and (B.39) of two linear ODEs in $(A_{iHH}(\tau), A_{iFH}(\tau))$ (setting $(j, j') = (H, F)$), and the same system (B.38) and (B.39) of two linear ODEs in $(A_{iHF}(\tau), A_{iFF}(\tau))$ (setting $(j, j') = (F, H)$). We can then substitute back into the definitions of $\bar{\lambda}_{iHH}$, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ and $\bar{\lambda}_{iFF}$ to derive the system of three nonlinear scalar equations. Given a solution of that system, (B.40) determines $(C_H(\tau), C_F(\tau))$ uniquely, and (B.37) determines C_e uniquely if $\alpha_e > 0$. If $\alpha_e = 0$, then a parameter restriction analogous to (B.15) must be imposed and C_e is indeterminate. The results in Section III.B hold for any solution $\bar{\lambda}_{iHH}$, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ and $\bar{\lambda}_{iFF}$. ∎

Proof of Proposition 3.4: We start by proving a series of lemmas.

LEMMA B.1: *The matrix*

$$\mathbf{M}_i \equiv \begin{pmatrix} \kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH} & -a\sigma_{iF}^2 \bar{\lambda}_{iFH} \\ -a\sigma_{iH}^2 \bar{\lambda}_{iHF} & \kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF} \end{pmatrix} \tag{B.47}$$

has two positive eigenvalues.

Proof: The characteristic polynomial of $\mathbf{M}_i$ is

$$\Pi(\lambda) \equiv \left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH} - \lambda\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF} - \lambda\right) - a^2 \sigma_{iH}^2 \sigma_{iF}^2 \bar{\lambda}_{iHF} \bar{\lambda}_{iFH}. \tag{B.48}$$

For $\lambda = 0$, $\Pi(\lambda)$ takes the value

$$\begin{aligned}
\Pi(0) &= \left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}\right) - a\sigma_{iH}^2 \sigma_{iF}^2 \bar{\lambda}_{iHF} \bar{\lambda}_{iFH} \\
&> a^2 \sigma_{iH}^2 \sigma_{iH}^2 \left(\bar{\lambda}_{iHH} \bar{\lambda}_{iFF} - \bar{\lambda}_{iHF} \bar{\lambda}_{iFH}\right) \\
&= a^2 \sigma_{iH}^2 \sigma_{iH}^2 \left[\left(\alpha_e A_{iHe}^2 + \int_0^T \alpha_H(\tau) A_{iHH}(\tau)^2 d\tau + \int_0^T \alpha_F(\tau) A_{iHF}(\tau)^2 d\tau\right)\right. \\
&\quad \times \left(\alpha_e A_{iFe}^2 + \int_0^T \alpha_H(\tau) A_{iFH}(\tau)^2 d\tau + \int_0^T \alpha_F(\tau) A_{iFF}(\tau)^2 d\tau\right) \\
&\quad \left. - \left(\alpha_e A_{iHe} A_{iFe} - \int_0^T \alpha_H(\tau) A_{iHH}(\tau) A_{iFH}(\tau) d\tau + \int_0^T \alpha_F(\tau) A_{iHF}(\tau) A_{iFF}(\tau) d\tau\right)^2\right].
\end{aligned} \tag{B.49}$$

The second step in (B.49) follows because $(\kappa_{iH}, \kappa_{iF})$ are positive and because (B.41) implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are non-positive. The third step in (B.49) follows from (B.41) and (B.42). The Cauchy-Schwarz inequality associated to the scalar product

$$X \cdot Y \equiv \alpha_e xy + \int_0^T \alpha_H(\tau) X_H(\tau) Y_H(\tau) d\tau + \int_0^T \alpha_F(\tau) X_F(\tau) Y_F(\tau) d\tau$$

where $X \equiv (x, X_H(\tau), X_F(\tau))$, $Y \equiv (y, Y_H(\tau), Y_F(\tau))$, (x,y) are scalars, and $(X_H(\tau), X_F(\tau), Y_H(\tau), Y_F(\tau))$ are functions of τ, implies that (B.49) is non-negative. Hence, $\Pi(0) > 0$.

For $\lambda = \kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH}$ and $\lambda = \kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}$, $\Pi(\lambda)$ takes the value $-a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}$, which is non-positive because (B.42) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$. Since $(\kappa_{iH}, \kappa_{iF})$ are positive and $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are non-positive, $\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}$ and $\lambda = \kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}$ are positive. Since $\Pi(\lambda)$ is a quadratic function of λ, is positive for $\lambda = 0$, is non-positive for two positive values of λ, and goes to infinity when λ goes to infinity, it has two positive roots. ∎

The matrix $\mathbf{M}_i$ plays an important role in the determination of (A_{iHe}, A_{iFe}) and $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$. Equation (B.36) gives rise to the linear system

$$\mathbf{M}_i \begin{pmatrix} A_{iHe} \\ A_{iFe} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{B.50}$$

Since $\mathbf{M}_i$ has two positive eigenvalues, it is invertible, and hence (B.50) can be solved for (A_{iHe}, A_{iFe}). Equations (B.38) and (B.39) give rise to the linear system

$$\begin{pmatrix} A_{iHH}(\tau) \\ A_{iFH}(\tau) \end{pmatrix}' + \mathbf{M}_i \begin{pmatrix} A_{iHH}(\tau) \\ A_{iFH}(\tau) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{B.51}$$

for $(j, j') = (H, F)$, and to

$$\begin{pmatrix} A_{iHF}(\tau) \\ A_{iFF}(\tau) \end{pmatrix}' + \mathbf{M}_i \begin{pmatrix} A_{iHF}(\tau) \\ A_{iFF}(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{B.52}$$

for $(j, j') = (F, H)$. Since $\mathbf{M}_i$ has two positive eigenvalues, the solutions $(A_{iHH}(\tau), A_{iFH}(\tau))$ to (B.51) and $(A_{iHF}(\tau), A_{iFF}(\tau))$ to (B.52) go to finite limits when τ goes to infinity.

LEMMA B.2: *The normalized factor prices $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are non-negative.*

Proof: Suppose, proceeding by contradiction, that $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are negative. The solution to (B.50) is

$$A_{iHe} = \frac{\kappa_{iF} - a\sigma_{iF}^2(\bar{\lambda}_{iFF} + \bar{\lambda}_{iFH})}{\left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}}, \tag{B.53}$$

$$A_{iFe} = \frac{\kappa_{iH} - a\sigma_{iH}^2(\bar{\lambda}_{iHH} + \bar{\lambda}_{iHF})}{\left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}}. \tag{B.54}$$

The denominator in (B.53) and (B.54) is $\Pi(0) > 0$. The numerators in (B.53) and (B.54) are positive because $(\kappa_{iH}, \kappa_{iF})$ are positive and $(a\bar{\lambda}_{iHH}, a\bar{\lambda}_{iFF}, a\bar{\lambda}_{iHF}, a\bar{\lambda}_{iFH})$ are non-positive. Hence, (A_{iHe}, A_{iFe}) are positive.

When $a = 0$, (B.39) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$ for all $\tau > 0$. Since, in addition, $A_{iHe} > 0$ and $A_{iFe} > 0$, (B.42) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} \geq 0$, a contradiction.

When $a > 0$, (B.38) and (B.39) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Moreover, differentiating (B.39), we find $A''_{iFH}(0) = a\sigma^2_{iH}\bar{\lambda}_{iHF}A'_{iHH}(0) < 0$ and $A''_{iHF}(0) = a\sigma^2_{iF}\bar{\lambda}_{iFH}A'_{iFF}(0) < 0$. Hence, $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) < 0$ and $A_{iFH}(\tau) < 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A_{iHH}(\tau') > 0,\ A_{iFF}(\tau') > 0,\ A_{iHF}(\tau') < 0 \text{ and } A_{iFH}(\tau') < 0 \text{ for all } \tau' \in (0, \tau)\}.$$

If τ_0 is finite, then (i) $A_{iHH}(\tau_0) = 0$, $A'_{iHH}(\tau_0) \leq 0$, $A_{iFF}(\tau_0) \geq 0$, $A_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (ii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) = 0$, $A'_{iFF}(\tau_0) \leq 0$, $A_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (iii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$, $A'_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (iv) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) < 0$, $A_{iFH}(\tau_0) = 0$ and $A'_{iFH}(\tau_0) \geq 0$. Case (i) yields a contradiction because (B.38) for $(j, j') = (H, F)$, $A_{iHH}(\tau_0) = 0$, $A_{iFH}(\tau_0) \leq 0$ and $\bar{\lambda}_{iFH} < 0$ imply $A'_{iHH}(\tau_0) \geq 1$. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (B.39) for $(j, j') = (H, F)$, $A_{iHH}(\tau_0) > 0$, $A_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} < 0$ imply $A'_{iFH}(\tau_0) < 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) < 0$ and $A_{iFH}(\tau) < 0$ for all $\tau > 0$. Since, in addition, $A_{iHe} > 0$ and $A_{iFe} > 0$, (B.42) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} \geq 0$, a contradiction. Therefore, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are non-negative. ∎

LEMMA B.3: *The functions $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are positive for all $\tau > 0$.*

- *When $a > 0$ and $\alpha_e > 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are positive for all $\tau > 0$.*
- *When $a = 0$ or $\alpha_e = 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are zero.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. If $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$, then (B.39) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$ for all $\tau > 0$. Since, in addition, (B.53) and (B.54) imply $A_{iHe} > 0$ and $A_{iFe} > 0$, (B.42) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} > 0$, a contradiction. Hence, Lemma B.2 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} > 0$.

Equations (B.38) and (B.39) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Moreover, differentiating (B.39), we find $A''_{iFH}(0) = a\sigma^2_{iH}\bar{\lambda}_{iHF}A'_{iHH}(0) > 0$

and $A''_{iHF}(0) = a\sigma^2_{iF}\bar{\lambda}_{iFH}A'_{iFF}(0) > 0$. Hence, $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A_{iHH}(\tau') > 0,\ A_{iFF}(\tau') > 0,\ A_{iHF}(\tau') > 0 \text{ and } A_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0,\tau)\}.$$

If τ_0 is finite, then (i) $A_{iHH}(\tau_0) = 0$, $A'_{iHH}(\tau_0) \leq 0$, $A_{iFF}(\tau_0) \geq 0$, $A_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (ii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) = 0$, $A'_{iFF}(\tau_0) \leq 0$, $A_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (iii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$, $A'_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (iv) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) > 0$, $A_{iFH}(\tau_0) = 0$ and $A'_{iFH}(\tau_0) \leq 0$. Case (i) yields a contradiction because (B.38) for $(j, j') = (H, F)$, $A_{iHH}(\tau_0) = 0$, $A_{iFH}(\tau_0) \geq 0$ and $\bar{\lambda}_{iFH} > 0$ imply $A'_{iHH}(\tau_0) \geq 1$. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (B.39) for $(j, j') = (H, F)$, $A_{iHH}(\tau_0) > 0$, $A_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} > 0$ imply $A'_{iFH}(\tau_0) > 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for all $\tau > 0$.

Consider next the case $a = 0$. The properties of $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ follow because (B.38) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies $A_{iHH}(\tau) = A^{EH}_{iHH}(\tau) \equiv (1 - e^{-\kappa_{iH}\tau})/\kappa_{iH} > 0$ and $A_{iFF}(\tau) = A^{EH}_{iFF}(\tau) \equiv (1 - e^{-\kappa_{iF}\tau})/\kappa_{iF} > 0$, and (B.39) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$.

Consider finally the case $a > 0$ and $\alpha_e = 0$. Suppose, proceeding by contradiction, that $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are positive. The argument in the case $a > 0$ and $\alpha_e > 0$ implies that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$. Since $\alpha_e = 0$, (B.42) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} \leq 0$, a contradiction. Hence, Lemma B.2 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$. Since $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$, (B.39) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Since $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$, (B.38) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies that $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive for all $\tau > 0$. ∎

LEMMA B.4: *The functions $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are increasing. When $a > 0$ and $\alpha_e > 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are also increasing.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. Equations $A'_{iHH}(0) = A'_{iFF}(0) = 1$, $A'_{iHF}(0) = A'_{iFH}(0) = 0$, $A''_{iFH}(0) = a\sigma^2_{iH}\bar{\lambda}_{iHF}A'_{iHH}(0) > 0$ and $A''_{iHF}(0) = a\sigma^2_{iF}\bar{\lambda}_{iFH}A'_{iFF}(0) > 0$ imply $A'_{iHH}(\tau) > 0$, $A'_{iFF}(\tau) > 0$, $A'_{iHF}(\tau) > 0$ and $A'_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau') > 0,\ A'_{iFF}(\tau') > 0,\ A'_{iHF}(\tau') > 0 \text{ and } A'_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0,\tau)\}.$$

If τ_0 is finite, then (i) $A'_{iHH}(\tau_0) = 0$, $A''_{iHH}(\tau_0) \leq 0$, $A'_{iFF}(\tau_0) \geq 0$, $A'_{iHF}(\tau_0) \geq 0$ and $A'_{iFH}(\tau_0) \geq 0$, or (ii) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) = 0$, $A''_{iFF}(\tau_0) \leq 0$, $A'_{iHF}(\tau_0) \geq$

0 and $A_{iFH}(\tau_0)' \geq 0$, or (iii) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) > 0$, $A'_{iHF}(\tau_0) = 0$, $A''_{iHF}(\tau_0) \leq 0$ and $A'_{iFH}(\tau_0) \geq 0$, or (iv) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) > 0$, $A'_{iHF}(\tau_0) > 0$, $A'_{iFH}(\tau_0) = 0$ and $A''_{iFH}(\tau_0) \leq 0$. To analyze Cases (i)-(iv), we use

$$A''_{ijj}(\tau) + \kappa_{ij} A'_{ijj}(\tau) = a\sigma^2_{ij}\bar{\lambda}_{ijj}A'_{ijj}(\tau) + a\sigma^2_{ij'}\bar{\lambda}_{ij'j}A'_{ij'j}(\tau), \tag{B.55}$$

$$A''_{ij'j}(\tau) + \kappa_{ij'} A'_{ij'j}(\tau) = a\sigma^2_{ij}\bar{\lambda}_{ijj'}A'_{ijj}(\tau) + a\sigma^2_{ij'}\bar{\lambda}_{ij'j'}A'_{ij'j}(\tau), \tag{B.56}$$

which follow from differentiating (B.38) and (B.39), respectively.

Case (i) yields a contradiction. Indeed, if $A''_{iHH}(\tau_0) = 0$, then (B.55) for $(j, j') = (H, F)$, $A'_{iHH}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ imply $A'_{iFH}(\tau_0) = 0$. The unique solution to the linear system of ODEs (B.55) and (B.56) for $(j, j') = (H, F)$ with the initial condition $(A'_{iHH}(\tau_0), A'_{iFH}(\tau_0)) = (0, 0)$ is the function that equals (0,0) for all τ. This yields a contradiction because $(A'_{iHH}(0), A'_{iFH}(0)) = (1, 0)$. Hence, $A''_{iHH}(\tau_0) < 0$, which combined with (B.55) for $(j, j') = (H, F)$, $A'_{iHH}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ implies $A'_{iFH}(\tau_0) < 0$, again a contradiction. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (B.56) for $(j, j') = (H, F)$, $A'_{iHH}(\tau_0) > 0$, $A'_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} > 0$ imply $A''_{iFH}(\tau_0) > 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ are increasing.

In the case $a = 0$ or $\alpha_e = 0$, Lemma B.3 implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Since $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$, (B.38) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies that $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are increasing. ∎

LEMMA B.5: *The scalars A_{iHe} and A_{iFe} are positive.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. Since $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} > 0$ and $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for all $\tau > 0$ (Lemma B.3), (B.42) implies $A_{iHe}A_{iFe} > 0$. Hence, (A_{iHe}, A_{iFe}) are either both positive or both negative. Suppose, proceeding by contradiction, that they are both negative. Equations (B.53) and (B.54) imply

$$\kappa_{iH} - a\sigma^2_{iH}\bar{\lambda}_{iHH} < a\sigma^2_{iH}\bar{\lambda}_{iHF}, \tag{B.57}$$

$$\kappa_{iF} - a\sigma^2_{iF}\bar{\lambda}_{iFF} < a\sigma^2_{iF}\bar{\lambda}_{iFH}. \tag{B.58}$$

Since the left-hand side in each of (B.57) and (B.58) is positive, (B.57) and (B.58) imply

$$\Pi(0) = \left(\kappa_{iH} - a\sigma^2_{iH}\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma^2_{iF}\bar{\lambda}_{iFF}\right) - a\sigma^2_{iH}\sigma^2_{iF}\bar{\lambda}_{iHF}\bar{\lambda}_{iFH} < 0,$$

a contradiction. Hence, (A_{iHe}, A_{iFe}) are positive.

Consider next the case $a = 0$. Equation (B.36) implies $A_{iHe} = A^{UIP}_{iHe} \equiv 1/\kappa_{Hj} > 0$ and $A_{iFe} = A^{UIP}_{iFe} \equiv 1/\kappa_{Fj} > 0$. Consider finally the case $\alpha_e = 0$ and $a > 0$.

Since $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$ and $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are non-positive, (B.53) and (B.54) imply that (A_{iHe}, A_{iFe}) are positive. ∎

LEMMA B.6: *The functions $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are positive for all $\tau > 0$.*

Proof: In the case $a = 0$ or $\alpha_e = 0$, the lemma follows from Lemma B.3. To prove the lemma in the case $a > 0$ and $\alpha_e > 0$, we proceed in two steps. In Step 1, we show that $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are positive in the limit when τ goes to infinity. In Step 2, we show that $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are either increasing in τ, or increasing and then decreasing. The lemma follows by combining these properties with $A_{iHH}(0) - A_{iHF}(0) = A_{iFF}(0) - A_{iFH}(0) = 0$.

Step 1: Limit at infinity. Since the matrix $\mathbf{M}_i$ has two positive eigenvalues, the functions $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ go to finite limits when τ goes to infinity. These limits solve the system of equations

$$\kappa_{ij} A_{ijj}(\infty) - 1 = a\sigma_{ij}^2 \bar{\lambda}_{ijj} A_{ijj}(\infty) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'j} A_{ij'j}(\infty), \tag{B.59}$$

$$\kappa_{ij'} A_{ij'j}(\infty) = a\sigma_{ij}^2 \bar{\lambda}_{ijj'} A_{ijj}(\infty) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'j'} A_{ij'j}(\infty), \tag{B.60}$$

which are derived from (B.38) and (B.39) by setting the derivatives to zero. Subtracting (B.60) for $(j,j') = (F,H)$ from (B.59) for $(j,j') = (H,F)$, we find

$$\begin{aligned} &\kappa_{iH}(A_{iHH}(\infty) - A_{iHF}(\infty)) - 1 \\ &= a\sigma_{iH}^2 \bar{\lambda}_{iHH}(A_{iHH}(\infty) - A_{iHF}(\infty)) + a\sigma_{iF}^2 \bar{\lambda}_{iFH}(A_{iFH}(\infty) - A_{iFF}(\infty)). \end{aligned} \tag{B.61}$$

Subtracting (B.60) for $(j,j') = (H,F)$ from (B.59) for $(j,j') = (F,H)$, we similarly find

$$\begin{aligned} &\kappa_{iF}(A_{iFF}(\infty) - A_{iFH}(\infty)) - 1 \\ &= a\sigma_{iH}^2 \bar{\lambda}_{iHF}(A_{iHF}(\infty) - A_{iHH}(\infty)) + a\sigma_{iF}^2 \bar{\lambda}_{iFF}(A_{iFF}(\infty) - A_{iFH}(\infty)). \end{aligned} \tag{B.62}$$

The solution to the system of (B.61) and (B.62) is

$$A_{iHH}(\infty) - A_{iHF}(\infty) = \frac{\kappa_{iF} - a\sigma_{iF}^2(\bar{\lambda}_{iFF} + \bar{\lambda}_{iFH})}{\left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2 \bar{\lambda}_{iHF}\bar{\lambda}_{iFH}} = A_{iHe}, \tag{B.63}$$

$$A_{iFF}(\infty) - A_{iFH}(\infty) = \frac{\kappa_{iH} - a\sigma_{iH}^2(\bar{\lambda}_{iHH} + \bar{\lambda}_{iHF})}{\left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2 \bar{\lambda}_{iHF}\bar{\lambda}_{iFH}} = A_{iFe}, \tag{B.64}$$

where the second equality in (B.63) and (B.64) follows from (B.53) and (B.54), respectively. Since (A_{iHe}, A_{iFe}) are positive (Lemma B.5), so are $(A_{iHH}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iFH}(\infty))$.

Step 2: Monotonicity. Equations (B.38) and (B.39) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1 > 0$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Hence, $A'_{iHH}(\tau) - A'_{iHF}(\tau) > 0$ and $A'_{iFF}(\tau) - A'_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau')-A'_{iHF}(\tau') > 0 \text{ and } A'_{iFF}(\tau')-A'_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0,\tau)\}.$$

If τ_0 is infinity, then $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are increasing in τ. Suppose instead that τ_0 is finite. Then, either (i) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$, $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) \le 0$ and $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) \ge 0$, or (ii) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) > 0$, $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $A''_{iFF}(\tau_0) - A''_{iFH}(\tau_0) \le 0$. To analyze Cases (i) and (ii), we use

$$\begin{aligned}&A'_{iHH}(\tau) - A'_{iHF}(\tau) + \kappa_{iH}(A_{iHH}(\tau) - A_{iHF}(\tau)) - 1\\ &= a\sigma_{iH}^2\bar{\lambda}_{iHH}(A_{iHH}(\tau) - A_{iHF}(\tau)) + a\sigma_{iF}^2\bar{\lambda}_{iFH}(A_{iFH}(\tau) - A_{iFF}(\tau)),\end{aligned} \tag{B.65}$$

which follows by subtracting (B.39) for $(j, j') = (F, H)$ from (B.59) for $(j, j') = (H, F)$, and

$$\begin{aligned}&A'_{iFF}(\tau) - A'_{iFH}(\tau) + \kappa_{iF}(A_{iFF}(\tau) - A_{iFH}(\tau)) - 1\\ &= a\sigma_{iH}^2\bar{\lambda}_{iHF}(A_{iHF}(\tau) - A_{iHH}(\tau)) + a\sigma_{iF}^2\bar{\lambda}_{iFF}(A_{iFF}(\tau) - A_{iFH}(\tau)),\end{aligned} \tag{B.66}$$

which follows by subtracting (B.60) for $(j, j') = (H, F)$ from (B.59) for $(j, j') = (F, H)$. Differentiating (B.65) and (B.66), we find

$$\begin{aligned}&A''_{iHH}(\tau) - A''_{iHF}(\tau) + \kappa_{iH}(A'_{iHH}(\tau) - A'_{iHF}(\tau))\\ &= a\sigma_{iH}^2\bar{\lambda}_{iHH}(A'_{iHH}(\tau) - A'_{iHF}(\tau)) + a\sigma_{iF}^2\bar{\lambda}_{iFH}(A'_{iFH}(\tau) - A'_{iFF}(\tau))\end{aligned} \tag{B.67}$$

and

$$\begin{aligned}&A''_{iFF}(\tau) - A''_{iFH}(\tau) + \kappa_{iF}(A'_{iFF}(\tau) - A'_{iFH}(\tau))\\ &= a\sigma_{iH}^2\bar{\lambda}_{iHF}(A'_{iHF}(\tau) - A'_{iHH}(\tau)) + a\sigma_{iF}^2\bar{\lambda}_{iFF}(A'_{iFF}(\tau) - A'_{iFH}(\tau)),\end{aligned} \tag{B.68}$$

respectively. Equations (B.67) and (B.68) are a linear system of ODEs in the functions $(A'_{iHH}(\tau) - A'_{iHF}(\tau), A'_{iFF}(\tau) - A'_{iFH}(\tau))$.

Consider first Case (i). If $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) = 0$, then (B.67), $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ imply $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$. The unique solution to the linear system of ODEs (B.67) and (B.68) with the initial condition $(A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0), A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0)) = (0, 0)$ is the function that equals (0,0) for all τ. This yields a contradiction because $(A'_{iHH}(0) - A'_{iHF}(0), A'_{iFF}(0) -$

$A'_{iFH}(0)) = (1,1)$. Hence, $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) < 0$, which combined with (B.67), $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ implies $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) > 0$. Since $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) < 0$, $A'_{iHH}(\tau) - A'_{iHF}(\tau) < 0$ for τ larger than and close to τ_0. We define τ'_0 by

$$\tau'_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau') - A'_{iHF}(\tau') < 0 \text{ and } A'_{iFF}(\tau') - A'_{iFH}(\tau') > 0 \text{ for all } \tau' \in (\tau_0, \tau)\}.$$

If τ'_0 is finite, then either (ia) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$, $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) \geq 0$ and $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) \geq 0$, or (ib) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) < 0$, $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $A''_{iFF}(\tau_0) - A''_{iFH}(\tau_0) \leq 0$. In Case (ia), the same argument as for τ_0 implies $A''_{iHH}(\tau'_0) - A''_{iHF}(\tau'_0) > 0$, which combined with (B.67), $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ implies $A'_{iFF}(\tau'_0) - A'_{iFH}(\tau'_0) < 0$, a contradiction. In Case (ib), the same argument as for τ_0 implies $A''_{iFF}(\tau'_0) - A''_{iFH}(\tau'_0) < 0$, which combined with (B.68), $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} > 0$ implies $A'_{iHH}(\tau'_0) - A'_{iHF}(\tau'_0) > 0$, a contradiction. Therefore, τ'_0 is infinite, which means that $A_{iFF}(\tau) - A_{iFH}(\tau)$ is increasing, and $A_{iHH}(\tau) - A_{iHF}(\tau)$ is increasing in $(0, \tau_0)$ and decreasing in (τ_0, ∞).

Consider next Case (ii). A symmetric argument by switching H and F implies that $A_{iHH}(\tau) - A_{iHF}(\tau)$ is increasing, and $A_{iFF}(\tau) - A_{iFH}(\tau)$ is increasing in $(0, \tau_0)$ and decreasing in (τ_0, ∞). ∎

Using Lemmas B.1-B.6, we next prove the proposition. Since (A_{iHe}, A_{iFe}) are positive (Lemma B.5), (2.1) implies $\partial e_t/\partial i_{Ht} < 0$ and $\partial e_t/\partial i_{Ft} > 0$. When $a > 0$ and $\alpha_e > 0$, (B.41) implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative, and the proof of Lemma B.3 implies that $(\bar{\lambda}_{iHF}, \bar{\lambda}_{iFH})$ are positive. Hence,

$$a\sigma^2_{iH}\bar{\lambda}_{iHH}A_{iHe} - a\sigma^2_{iF}\bar{\lambda}_{iFH}A_{iFe} < 0, \tag{B.69}$$

$$a\sigma^2_{iF}\bar{\lambda}_{iFF}A_{iFe} - a\sigma^2_{iH}\bar{\lambda}_{iHF}A_{iHe} < 0. \tag{B.70}$$

Equations (B.69) and (B.70) also hold when $a > 0$, $\alpha_e = 0$ and $(\alpha_H(\tau), \alpha_F(\tau))$ are positive. This is because (B.41) again implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative, and the proof of Lemma B.3 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$. Combining (B.69) and (B.70) with (B.36), we find $A_{iHe} < 1/\kappa_{iH} \equiv A^{UIP}_{iHe}$ and $A_{iFe} < 1/\kappa_{iF} \equiv A^{UIP}_{iFe}$. Combining (B.69) and (B.70) with (2.10) (for the definitions of $(\mathbf{A}_e, \boldsymbol{\lambda}_t)$ in Section III.B) and (B.45), we find $\partial\left(\mu_{et} + i_{Ft} - i_{Ht}\right)/\partial i_{Ht} < 0$ and $\partial\left(\mu_{et} + i_{Ft} - i_{Ht}\right)/\partial i_{Ft} > 0$. This establishes the first bullet point of the proposition.

Since $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive for all $\tau > 0$ (Lemma B.3), (1.1) and (2.2) imply that $(\partial y^{(\tau)}_{Ht}/\partial i_{Ht}, \partial y^{(\tau)}_{Ft}/\partial i_{Ft})$ are positive. When $a > 0$ and $\alpha_e > 0$, Lemma B.3 implies that $(A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$, and Lemma B.4 implies that $(A_{iHF}(\tau), A_{iFH}(\tau))$ are increasing. Equation (B.39) for $(j, j') = (H, F)$ implies

$$a\sigma^2_{iH}\bar{\lambda}_{iHF}A_{iHH}(\tau) + a\sigma^2_{iF}\bar{\lambda}_{iFF}A_{iFH}(\tau) > 0. \tag{B.71}$$

Multiplying both sides of (B.71) by $\bar{\lambda}_{iHH}/\bar{\lambda}_{iHF} < 0$, we find

$$a\sigma_{iH}^2\bar{\lambda}_{iHH}A_{iHH}(\tau) + a\sigma_{iF}^2\frac{\bar{\lambda}_{iHH}\bar{\lambda}_{iFF}}{\bar{\lambda}_{iHF}}A_{iFH}(\tau) < 0$$

$$\Rightarrow a\sigma_{iH}^2\bar{\lambda}_{iHH}A_{iHH}(\tau) + a\sigma_{iF}^2\bar{\lambda}_{iFH}A_{iFH}(\tau) < 0, \tag{B.72}$$

where the second step follows from $A_{iFH}(\tau) > 0$ and from the inequality $\bar{\lambda}_{iHH}\bar{\lambda}_{iFF} - \bar{\lambda}_{iHF}\bar{\lambda}_{iFH} < 0$ established in the proof of Lemma B.1. We likewise find

$$a\sigma_{iF}^2\bar{\lambda}_{iFH}A_{iFF}(\tau) + a\sigma_{iH}^2\bar{\lambda}_{iHH}A_{iHF}(\tau) > 0, \tag{B.73}$$

$$\Rightarrow a\sigma_{iF}^2\bar{\lambda}_{iFF}A_{iFF}(\tau) + a\sigma_{iH}^2\bar{\lambda}_{iHF}A_{iHF}(\tau) < 0, \tag{B.74}$$

by switching H and F. Equations (B.72) and (B.74) hold also when $a > 0$, $\alpha_e = 0$ and $(\alpha_H(\tau), \alpha_F(\tau))$ are positive. Indeed, the proof of Lemma B.3 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$, and since $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive, (B.41) implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative. Combining (B.72) and (B.74) with (B.38), we find $A_{iHH}(\tau) < (1 - e^{-\kappa_{iH}\tau})/\kappa_{iH} \equiv A^{EH}_{iHH}(\tau)$ and $A_{iFF}(\tau) < (1 - e^{-\kappa_{iF}\tau})/\kappa_{iF} \equiv A^{EH}_{iFF}(\tau)$. Combining (B.72) and (B.74) with (2.11) (for the definitions of $(\mathbf{A}_j(\tau), \boldsymbol{\lambda}_t)$ in Section III.B) and (B.45), we find $\partial\left(\mu^{(\tau)}_{Ht} - i_{Ht}\right)/\partial i_{Ht} < 0$ and $\partial\left(\mu^{(\tau)}_{Ft} - i_{Ft}\right)/\partial i_{Ft} < 0$. This establishes the second bullet point of the proposition.

When $a > 0$ and $\alpha_e > 0$, $(A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$, and hence (1.1) and (2.2) imply that $(\partial y^{(\tau)}_{Ht}/\partial i_{Ft}, \partial y^{(\tau)}_{Ft}/\partial i_{Ht})$ are positive. Moreover, combining (B.71) and (B.73) with (2.11) and (B.45), we find $\partial\left(\mu^{(\tau)}_{Ht} - i_{Ht}\right)/\partial i_{Ft} > 0$ and $\partial\left(\mu^{(\tau)}_{Ft} - i_{Ft}\right)/\partial i_{Ht} > 0$. This establishes the third bullet point of the proposition. The fourth bullet point follows from Lemma B.6, (1.1) and (2.2). ∎

Proof of Proposition 3.5: Combining (2.10) and (2.11) (for the definitions of $(\mathbf{A}_e, \mathbf{A}_j(\tau), \boldsymbol{\lambda}_t)$ in Section III.B) with (3.2), we can write the expected return of the hybrid CCT as

$$\mu^{(\tau)}_{hCCTt} \equiv \lambda_{iHt}(A_{iHe} + A_{iFH}(\tau) - A_{iHH}(\tau)) - \lambda_{iFt}(A_{iFe} + A_{iHF}(\tau) - A_{iFF}(\tau)). \tag{B.75}$$

Using (B.45), we find

$$\frac{\partial\mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}} = a\sigma_{iH}^2\bar{\lambda}_{iHH}(A_{iHe} + A_{iHF}(\tau) - A_{iHH}(\tau)) - a\sigma_{iF}^2\bar{\lambda}_{iFH}(A_{iFe} + A_{iFH}(\tau) - A_{iFF}(\tau)), \tag{B.76}$$

$$\frac{\partial\mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}} = a\sigma_{iH}^2\bar{\lambda}_{iHF}(A_{iHe} + A_{iHF}(\tau) - A_{iHH}(\tau)) - a\sigma_{iF}^2\bar{\lambda}_{iFF}(A_{iFe} + A_{iFH}(\tau) - A_{iFF}(\tau)). \tag{B.77}$$

When $a > 0$, and $\alpha_e > 0$ or $\alpha_j(\tau) > 0$, $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative. Since, in addition, $(\bar{\lambda}_{iHF}, \bar{\lambda}_{iFH})$ are non-negative, (A_{iHe}, A_{iFe}) are positive and $A_{iHH}(0) - A_{iHF}(0) = A_{iFF}(0) - A_{iFH}(0) = 0$, (B.76) and (B.77) imply that there exists a threshold $\tau^* > 0$ such that $\partial\mu^{(\tau)}_{hCCTt}/\partial i_{Ht} < 0$ and $\partial\mu^{(\tau)}_{hCCTt}/\partial i_{Ft} > 0$ for all $\tau \in (0, \tau^*)$. Since at least one of $(A_{iHH}(\tau) - A_{iHF}(\tau), A_{iFF}(\tau) - A_{iFH}(\tau))$ is increasing (proof of Lemma B.4), they are both increasing when countries are symmetric. Since, in addition, $(A_{iHH}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iFH}(\infty)) = (A_{iHe}, A_{iFe})$ (proof of Lemma B.6), (B.76) and (B.77) imply that when countries are symmetric, $\partial\mu^{(\tau)}_{hCCTt}/\partial i_{Ht} < 0$ and $\partial\mu^{(\tau)}_{hCCTt}/\partial i_{Ft} > 0$ for all $\tau > 0$, which means $\tau^* = \infty$.

Combining

$$\mu_{CCTt} \equiv \mu_{et} + i_{Ft} - i_{Ht} = \lambda_{iHt}A_{iHe} - \lambda_{iFt}A_{iFe},$$

which gives the expected return of the basic CCT and follows from (2.10) (for the definitions of $(\mathbf{A}_e, \boldsymbol{\lambda}_t)$ in Section III.B), with (B.45), (B.76) and (B.77), we find

$$\text{(B.78)}\quad \frac{\partial\left(\mu^{(\tau)}_{hCCTt} - \mu_{CCTt}\right)}{\partial i_{Ht}} = \bar{\lambda}_{iHH}(A_{iHF}(\tau) - A_{iHH}(\tau)) - \bar{\lambda}_{iHF}(A_{iFH}(\tau) - A_{iFF}(\tau)) > 0,$$

$$\text{(B.79)}\quad \frac{\partial\left(\mu^{(\tau)}_{hCCTt} - \mu_{CCTt}\right)}{\partial i_{Ft}} = \bar{\lambda}_{iFH}(A_{iHF}(\tau) - A_{iHH}(\tau)) - \bar{\lambda}_{iFF}(A_{iFH}(\tau) - A_{iFF}(\tau)) < 0,$$

where the inequalities follow because $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative, $(\bar{\lambda}_{iHF}, \bar{\lambda}_{iFH})$ are non-negative, and $(A_{iHH}(\tau) - A_{iHF}(\tau), A_{iFF}(\tau) - A_{iFH}(\tau))$ are positive for all $\tau > 0$ (Lemma B.6). Hence, the sensitivity of the hybrid CCT's expected return to (i_{Ht}, i_{Ft}) is smaller (less negative in the case of i_{Ht} and less positive in the case of i_{Ft}) than for the basic CCT. Since $(A_{iHH}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iFH}(\infty)) = (A_{iHe}, A_{iFe})$, (B.75) implies that $\mu^{(\tau)}_{hCCTt}$ goes to zero when τ goes to infinity.

Using (2.1), (2.2), (3.3) and $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we can write the return of the long-horizon CCT as

$$\begin{aligned}&A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + C_e + (\pi_F - \pi_H)t - (A_{iHe}i_{H,t+\tau} - A_{iFe}i_{F,t+\tau} + C_e + (\pi_F - \pi_H)(t+\tau))\\ &+ A_{iFF}(\tau)i_{Ft} + A_{iHF}(\tau)i_{Ht} + C_F(\tau) - (A_{iHH}(\tau)i_{Ht} + A_{iFH}(\tau)i_{Ft} + C_H(\tau)).\end{aligned}$$

Hence, (3.1) implies that the annualized expected return of the long-horizon CCT is

$$\mu^{(\tau)}_{\ell CCTt} \equiv \frac{1}{\tau}\left[A_{iHe}(1 - e^{-\kappa_{iH}\tau})(i_{Ht} - \bar{i}_H) - A_{iFe}(1 - e^{-\kappa_{iF}\tau})(i_{Ft} - \bar{i}_F) - (\pi_F - \pi_H)\tau\right.$$

$$+ A_{iFF}(\tau)i_{Ft} + A_{iHF}(\tau)i_{Ht} + C_F(\tau) - (A_{iHH}(\tau)i_{Ht} + A_{iFH}(\tau)i_{Ft} + C_H(\tau))\big], \tag{B.80}$$

and its sensitivity to (i_{Ht}, i_{Ft}) is

$$\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} = \frac{1}{\tau}\left[A_{iHe}(1 - e^{-\kappa_{iH}\tau}) + A_{iHF}(\tau) - A_{iHH}(\tau)\right], \tag{B.81}$$

$$\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} = \frac{1}{\tau}\left[-A_{iFe}(1 - e^{-\kappa_{iF}\tau}) + A_{iFF}(\tau) - A_{iFe}(\tau)\right]. \tag{B.82}$$

When $a > 0$, and $\alpha_e > 0$ or $\alpha_j(\tau) > 0$, $A_{iHe} < 1/\kappa_{iH}$ and $A_{iFe} < 1/\kappa_{iF}$. Since, in addition, $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$, the derivative of (B.81) with respect to τ at $\tau = 0$ is negative, and the derivative of (B.82) with respect to τ at $\tau = 0$ is positive. Hence, there exists a threshold $\tau^* > 0$ such that $\partial \mu^{(\tau)}_{\ell CCTt}/\partial i_{Ht} < 0$ and $\partial \mu^{(\tau)}_{\ell CCTt}/\partial i_{Ft} > 0$ for all $\tau \in (0, \tau^*)$. When countries are symmetric, we set $\kappa_r \equiv \kappa_{iH} = \kappa_{iF}$, $\sigma_r \equiv \sigma_{iH} = \sigma_{iF}$, $A_{ie} \equiv A_{iHe} = A_{iFe}$, $\Delta A(\tau) \equiv A_{iHH}(\tau) - A_{iHF}(\tau) = A_{iFF}(\tau) - A_{iFH}(\tau)$, $\Delta\bar{\lambda} \equiv \bar{\lambda}_{iHH} - \bar{\lambda}_{iHF} = \bar{\lambda}_{iFF} - \bar{\lambda}_{iFH} < 0$. Taking the difference between (B.38) and (B.39) yields

$$\Delta A'(\tau) + \kappa_r \Delta A(\tau) - 1 = a\sigma_r^2 \Delta\bar{\lambda}\Delta A(\tau),$$

which integrates to

$$\Delta A(\tau) = A_{ie}\left(1 - e^{-(\kappa_r - a\sigma_r^2\Delta\bar{\lambda})\tau}\right)$$

since $\Delta A(0) = 0$ and $\Delta A(\infty) = A_{ie}$. Substituting into (B.81) and (B.82), we find

$$\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} = -\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} = \frac{1}{\tau}A_{ie}(e^{-(\kappa_r - a\sigma_r^2\Delta\bar{\lambda})\tau} - e^{-\kappa_r\tau}) < 0. \tag{B.83}$$

Hence, $\tau^* = \infty$.

The annualized expected return of the sequence of basic CCTs is

$$\mu^{(\tau)}_{CCTt} \equiv \frac{1}{\tau}E_t\int_t^{t+\tau}(\lambda_{iHt'}A_{iHe} - \lambda_{iFt'}A_{iFe})\,dt'.$$

Using (3.1) and (B.45), we find

$$\begin{aligned}\frac{\partial \mu^{(\tau)}_{CCTt}}{\partial i_{Ht}} &= \frac{1 - e^{-\kappa_{iH}\tau}}{\kappa_{iH}\tau}\left(a\sigma_{iH}^2\bar{\lambda}_{iHH}A_{iHe} - a\sigma_{iF}^2\bar{\lambda}_{iFH}A_{iFe}\right)\\ &= \frac{1 - e^{-\kappa_{iH}\tau}}{\kappa_{iH}\tau}(\kappa_{iH}A_{iHe} - 1),\end{aligned} \tag{B.84}$$

where the second step follows from (B.36). We likewise find

$$\frac{\partial \mu^{(\tau)}_{CCTt}}{\partial i_{Ft}} = -\frac{1-e^{-\kappa_{iF}\tau}}{\kappa_{iF}\tau}(\kappa_{iF}A_{iFe}-1). \tag{B.85}$$

Combining (B.81) and (B.84), we find

$$\frac{\partial \left(\mu^{(\tau)}_{\ell CCTt} - \mu^{(\tau)}_{CCTt}\right)}{\partial i_{Ht}} = \frac{1}{\tau}\left[\frac{1-e^{-\kappa_{iH}\tau}}{\kappa_{iH}} + A_{iHF}(\tau) - A_{iHH}(\tau)\right] > 0,$$

where the inequality sign follows from (B.65) by noting that the left-hand side of (B.65) is negative. Combining (B.82) and (B.85), we likewise find

$$\frac{\partial \left(\mu^{(\tau)}_{\ell CCTt} - \mu^{(\tau)}_{CCTt}\right)}{\partial i_{Ft}} = \frac{1}{\tau}\left[-\frac{1-e^{-\kappa_{iF}\tau}}{\kappa_{iF}} + A_{iFF}(\tau) - A_{iFH}(\tau)\right] < 0.$$

Hence, the sensitivity of the long-horizon CCT's expected return to (i_{Ht}, i_{Ft}) is smaller (less negative in the case of i_{Ht} and less positive in the case of i_{Ft}) than for the corresponding sequence of basic CCTs. Since $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ go to finite limits when τ goes to infinity, (B.80) implies that $\mu^{(\tau)}_{\ell CCTt}$ goes to

$$\begin{aligned}
&\lim_{\tau\to\infty}\frac{C_F(\tau)}{\tau} - \lim_{\tau\to\infty}\frac{C_H(\tau)}{\tau} - (\pi_F - \pi_H)\\
&= \kappa_{iF}\bar{i}_F\left(A_{iFF}(\infty) - A_{iFH}(\infty)\right) - \kappa_{iH}\bar{i}_H\left(A_{iHH}(\infty) - A_{iHF}(\infty)\right)\\
&\quad - \frac{1}{2}\sigma^2_{iF}\left[A_{iFF}(\infty)\left(A_{iFF}(\infty) - 2A_{iFe}\right) - A_{iFH}(\infty)^2\right]\\
&\quad + \frac{1}{2}\sigma^2_{iH}\left[A_{iHH}(\infty)^2 - A_{iHF}(\infty)\left(A_{iHF}(\infty) + 2A_{iHe}\right)\right]\\
&\quad + a\sigma^2_{iF}\bar{\lambda}_{iFC}\left(A_{iFF}(\infty) - A_{iFH}(\infty)\right) - a\sigma^2_{iH}\bar{\lambda}_{iHC}\left(A_{iHH}(\infty) - A_{iHF}(\infty)\right) - (\pi_F - \pi_H)\\
&= \kappa_{iF}\bar{i}_F A_{iFe} - \kappa_{iH}\bar{i}_H A_{iHe} + \frac{1}{2}\sigma^2_{iF}A^2_{iFe} + \frac{1}{2}\sigma^2_{iH}A^2_{iHe}\\
&\quad + a\sigma^2_{iF}\bar{\lambda}_{iFC}A_{iFe} - a\sigma^2_{iH}\bar{\lambda}_{iHC}A_{iHe} - (\pi_F - \pi_H) = 0,
\end{aligned}$$

where the second step follows from (B.40) by noting $\lim_{\tau\to\infty} C_j(\tau)/\tau = \lim_{\tau\to\infty} C'_j(\tau)$, the third step follows from $(A_{iHH}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iFH}(\infty)) = (A_{iHe}, A_{iFe})$, and the fourth step follows from (B.37). Since

$$\lim_{\tau\to\infty}\frac{C_F(\tau)}{\tau} - \lim_{\tau\to\infty}\frac{C_H(\tau)}{\tau} - (\pi_F - \pi_H) = y^{(\infty)}_F - y^{(\infty)}_H - (\pi_F - \pi_H),$$

the difference in real yields across countries becomes zero in the limit τ goes to infinity. ∎

We next prove a lemma that we use in subsequent proofs.

LEMMA B.7: *When $a > 0$ and $\alpha_e > 0$, the functions $(A_{iFH}(\tau)/A_{iHH}(\tau), A_{iHF}(\tau)/A_{iFF}(\tau))$ are increasing.*

Proof: The functions $(A_{iHH}(\tau), A_{iFH}(\tau))$ solve the system (B.51) of linear ODEs with constant coefficients. The solution is an affine function of $(e^{-\nu_1\tau}, e^{-\nu_2\tau})$, where (ν_1, ν_2) are the eigenvalues of the matrix $\mathbf{M}_i$. Because of the initial conditions $A_{iHH}(0) = A_{iFH}(0) = 0$, we can write the solution as a linear function of $((1-e^{-\nu_1\tau})/\nu_1, (1-e^{-\nu_2\tau})/\nu_2)$. Because $(A'_{iHH}(0), A'_{iFH}(0)) = (1, 0)$, the coefficients of the linear terms sum to one for $A_{iHH}(\tau)$ and to zero for $A_{iFH}(\tau)$. Hence, we can write the solution as

$$A_{iHH}(\tau) = \frac{1-e^{-\nu_1\tau}}{\nu_1} + \phi_{HH}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \tag{B.86}$$

$$A_{iFH}(\tau) = \phi_{FH}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \tag{B.87}$$

for scalars (ϕ_{HH}, ϕ_{FH}). The eigenvalues (ν_1, ν_2) are positive (Lemma B.1), and without loss of generality we can set $\nu_1 > \nu_2$. Since $A_{iFH}(\tau)$ is positive when $a > 0$ and $\alpha_e > 0$ (Lemma B.3), $\phi_{FH} > 0$. Since

$$\frac{A_{iHH}(\tau)}{A_{iFH}(\tau)} = \frac{\frac{1-e^{-\nu_1\tau}}{\nu_1}}{\phi_{FH}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right)} + \frac{\phi_{HH}}{\phi_{FH}} = \frac{1}{\phi_{HF}\left(\frac{\nu_1}{\nu_2}\frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}} - 1\right)} + \frac{\phi_{HH}}{\phi_{FH}},$$

and the function $(\nu_1, \nu_2, \tau) \longrightarrow (1-e^{-\nu_2\tau})/(1-e^{-\nu_1\tau})$ increases in τ because its derivative has the same sign as $(e^{\nu_1\tau}-1)/\nu_1 - (e^{\nu_2\tau}-1)/\nu_2$, the function $A_{iHH}(\tau)/A_{FH}(\tau)$ is decreasing. Hence, the inverse function $A_{iFH}(\tau)/A_{HH}(\tau)$ is increasing. A similar argument using (B.52) establishes that $A_{iHF}(\tau)/A_{FF}(\tau)$ is increasing. ∎

Proof of Proposition 3.6: Consider a one-off increase in γ_t at time zero, and denote by κ_γ the rate at which γ_t reverts to its mean of zero. Equation (B.45) is modified to

$$\begin{aligned}\lambda_{ijt} = a\sigma_{ij}^2 \Bigg(& \left[\zeta_e + \theta_e\gamma_t - \alpha_e\left(A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + A_{\gamma e}\gamma_t + C_e\right)\right] A_{ije}(-1)^{1_{\{j=F\}}} \\ & + \sum_{j'=H,F}\int_0^T \left[\zeta_{j'}(\tau) - \alpha_{j'}(\tau)\left(A_{iHj'}(\tau)i_{Ht} + A_{iFj'}(\tau)i_{Ft} + A_{\gamma j'}(\tau)\gamma_t + C_{j'}(\tau)\right)\right] A_{ijj'}(\tau)d\tau \Bigg) \\ \equiv {} & a\sigma_{ij}^2\left(\bar{\lambda}_{ijj}i_{jt} + \bar{\lambda}_{ijj'}i_{j't} + \bar{\lambda}_{ij\gamma}\gamma_t + \bar{\lambda}_{ijC}\right)\end{aligned} \tag{B.88}$$

(B.12) is modified to (B.28), and (2.7) and (2.8) are modified to

$$\mu_{jt}^{(\tau)} \equiv A'_{iHj}(\tau)i_{Ht} + A'_{iFj}(\tau)i_{Ft} + A'_{\gamma j}(\tau)\gamma_t + C'_j(\tau) - A_{iHj}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht}) - A_{iFj}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft}) + A_{\gamma j}(\tau)\kappa_\gamma\gamma_t + \frac{1}{2}A_{iHj}(\tau)\left(A_{iHj}(\tau) + 2A_{iHe}1_{\{j=F\}}\right)\sigma_{iH}^2 + \frac{1}{2}A_{iFj}(\tau)\left(A_{iFj}(\tau) - 1_{\{j=F\}}2A_{iFe}\right)\sigma_{iF}^2. \tag{B.89}$$

Substituting $\boldsymbol{\lambda}_t$ from (B.88) and μ_{et} from (B.28) into (2.10) (for the definitions of $(\mathbf{A}_e, \boldsymbol{\lambda}_t)$ in Section III.B), we find an equation that is affine in $(i_{Ht}, i_{Ft}, \gamma_t)$. Identifying the linear terms in γ_t yields

$$\kappa_\gamma A_{\gamma e} = a\sigma_{iH}^2\bar{\lambda}_{iH\gamma}A_{iHe} - a\sigma_{iF}^2\bar{\lambda}_{iF\gamma}A_{iFe}. \tag{B.90}$$

Substituting $\boldsymbol{\lambda}_t$ from (B.88) and $\mu_{jt}^{(\tau)}$ from (B.89) into (2.11) (for the definitions of $(\mathbf{A}_j(\tau), \boldsymbol{\lambda}_t)$ in Section III.B), we find an equation that is affine in $(i_{Ht}, i_{Ft}, \gamma_t)$. Identifying the linear terms in γ_t yields

$$A'_{\gamma j}(\tau) + \kappa_\gamma A_{\gamma j}(\tau) = a\sigma_{iH}^2\bar{\lambda}_{iH\gamma}A_{iHj}(\tau) + a\sigma_{iF}^2\bar{\lambda}_{iF\gamma}A_{iFj}(\tau). \tag{B.91}$$

Solving (B.91) with the initial condition $A_{\gamma j}(0) = 0$, we find

$$A_{\gamma j}(\tau) = a\sigma_{iH}^2\bar{\lambda}_{iH\gamma}\int_0^\tau A_{iHj}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' + a\sigma_{iF}^2\bar{\lambda}_{iF\gamma}\int_0^\tau A_{iFj}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau', \tag{B.92}$$

We next substitute $A_{\gamma e}$ from (B.90) and $\{A_{\gamma j}(\tau)\}_{j=H,F}$ from (B.92) into

$$\bar{\lambda}_{ij\gamma} \equiv (\theta_e - \alpha_e A_{\gamma e})A_{ije}(-1)^{1_{\{j=F\}}} - \int_0^T \alpha_H(\tau)A_{\gamma H}(\tau)A_{ijH}(\tau)d\tau - \int_0^T \alpha_F(\tau)A_{\gamma F}(\tau)A_{ijF}(\tau)d\tau, \tag{B.93}$$

which follows from the definition of $\bar{\lambda}_{ij\gamma}$ in (B.88). We find

$$(1 + a\sigma_{iH}^2 z_{\gamma HH})\bar{\lambda}_{iH\gamma} + a\sigma_{iF}^2 z_{\gamma FH}\bar{\lambda}_{iF\gamma} = \theta_e A_{iHe}, \tag{B.94}$$

$$a\sigma_{iH}^2 z_{\gamma HF}\bar{\lambda}_{iH\gamma} + (1 + a\sigma_{iF}^2 z_{\gamma FF})\bar{\lambda}_{iF\gamma} = -\theta_e A_{iFe}, \tag{B.95}$$

where

$$z_{\gamma HH} = \frac{\alpha_e}{\kappa_\gamma}A_{iHe}^2 + \int_0^T \alpha_H(\tau)A_{iHH}(\tau)\left[\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau + \int_0^T \alpha_F(\tau)A_{iHF}(\tau)\left[\int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau,$$

$$z_{\gamma FF} = \frac{\alpha_e}{\kappa_\gamma}A_{iFe}^2 + \int_0^T \alpha_H(\tau)A_{iFH}(\tau)\left[\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau$$

$$+\int_0^T \alpha_F(\tau)A_{iFF}(\tau)\left[\int_0^\tau A_{iFF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau,$$

$$\begin{aligned}
z_{\gamma HF} = &-\frac{\alpha_e}{\kappa_\gamma}A_{iHe}A_{iFe} + \int_0^T \alpha_H(\tau)A_{iFH}(\tau)\left[\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau \\
&+\int_0^T \alpha_F(\tau)A_{iFF}(\tau)\left[\int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau, \\
z_{\gamma FH} = &-\frac{\alpha_e}{\kappa_\gamma}A_{iHe}A_{iFe} + \int_0^T \alpha_H(\tau)A_{iHH}(\tau)\left[\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau \\
&+\int_0^T \alpha_F(\tau)A_{iHF}(\tau)\left[\int_0^\tau A_{iFF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau.
\end{aligned}$$

Equations (B.94) and (B.95) form a linear system of two equations in the two unknowns $(\bar{\lambda}_{iH\gamma}, \bar{\lambda}_{iF\gamma})$. Its solution is

$$\bar{\lambda}_{iH\gamma} = \frac{\theta_e}{\Delta_{z\gamma}}\left[(1+a\sigma_{iF}^2 z_{\gamma FF})A_{iHe} + a\sigma_{iF}^2 z_{\gamma FH}A_{iFe}\right] \tag{B.96}$$

$$\bar{\lambda}_{iF\gamma} = -\frac{\theta_e}{\Delta_{z\gamma}}\left[(1+a\sigma_{iH}^2 z_{\gamma HH})A_{iFe} + a\sigma_{iH}^2 z_{\gamma HF}A_{iHe}\right], \tag{B.97}$$

where

$$\Delta_{z\gamma} \equiv (1+a\sigma_{iH}^2 z_{\gamma HH})(1+a\sigma_{iF}^2 z_{\gamma FF}) - a^2\sigma_{iH}^2\sigma_{iF}^2 z_{\gamma HF}z_{\gamma FH}.$$

To complete the proof, we proceed in three steps. In Step 1, we show that $\Delta_{z\gamma}$ is positive. In Step 2, we show that $A_{\gamma e}$ is positive. This proves the first statement in the proposition. In Step 3, we show that $A_{\gamma H}(\tau)$ is positive and $A_{\gamma F}(\tau)$ is negative. This proves the second and third statements in the proposition.

Step 1: $\Delta_{z\gamma}$ is positive. Since $(z_{\gamma HH}, z_{\gamma FF})$ are non-negative, $\Delta_{z\gamma} > 0$ under the sufficient condition

$$z_{\gamma HH}z_{\gamma FF} \geq z_{\gamma HF}z_{\gamma FH}. \tag{B.98}$$

The function

$$\begin{aligned}
F(\mu) &\equiv z_{\gamma HH} + \mu(z_{\gamma HF}+z_{\gamma FH}) + \mu^2 z_{\gamma FF} \\
&= \frac{\alpha_e}{\kappa_\gamma}\left(A_{iHe}-\mu A_{iFe}\right)^2 \\
&\quad + \int_0^T \alpha_H(\tau)\left[A_{iHH}(\tau)+\mu A_{iFH}(\tau)\right]\left[\int_0^\tau \left[A_{iHH}(\tau)+\mu A_{iFH}(\tau)\right]e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau
\end{aligned}$$

$$+\int_0^T \alpha_F(\tau)\left[A_{iHF}(\tau)+\mu A_{iFF}(\tau)\right]\left[\int_0^\tau \left[A_{iHF}(\tau)+\mu A_{iFF}(\tau)\right]e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau$$

is non-negative for all μ if

$$F_0 \equiv \int_0^T \alpha(\tau)A(\tau)\left[\int_0^\tau A(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau$$

is non-negative for a non-negative and non-increasing $\alpha(\tau)$ and for a general $A(\tau)$. Since

$$F_0 = \int_0^T \phi(\tau)\Phi(\tau)\left[\int_0^\tau \Phi(\tau')d\tau'\right]d\tau,$$

where

$$\phi(\tau) \equiv \alpha(\tau)e^{-2\kappa_\gamma\tau},$$
$$\Phi(\tau) \equiv A(\tau)e^{\kappa_\gamma\tau},$$

integration by parts implies

$$F_0 = \frac{1}{2}\phi(T)\left[\int_0^T \Phi(\tau)d\tau\right]^2 - \frac{1}{2}\int_0^T \phi'(\tau)\left[\int_0^\tau \Phi(\tau')d\tau'\right]^2 d\tau. \tag{B.99}$$

The first term in the right-hand side of (B.99) is non-negative because $\alpha(\tau)$ is non-negative, and the first term is non-positive because $\alpha(\tau)$ is non-increasing. Therefore, F_0 is non-negative. Since $F(\mu)$ is quadratic in μ, its non-negativity for all μ implies

$$4z_{\gamma HH}z_{\gamma FF} \geq (z_{\gamma HF}+z_{\gamma FH})^2$$
$$\Rightarrow z_{\gamma HH}z_{\gamma FF} \geq \frac{1}{4}(z_{\gamma HF}+z_{\gamma FH})^2 = z_{\gamma HF}z_{\gamma FH} + \frac{1}{4}(z_{\gamma HF}-z_{\gamma FH})^2 \geq z_{\gamma HF}z_{\gamma FH}.$$

Therefore, (B.98) holds.

Step 2: $A_{\gamma e}(\tau)$ is positive. Substituting $(\bar{\lambda}_{iH\gamma}, \bar{\lambda}_{iF\gamma})$ from (B.96) and (B.97) into (B.90), and using the definitions of $(z_{\gamma HH}, z_{\gamma FF}, z_{\gamma HF}, z_{\gamma FH})$ and that $(\theta_e, \Delta_{z\gamma})$ are positive, we find $A_{\gamma e} > 0$ if

$$Z_{\gamma H}A_{iHe} + Z_{\gamma F}A_{iFe} > 0, \tag{B.100}$$

where

$$Z_{\gamma H} \equiv \sigma_{iH}^2(1+a\sigma_{iF}^2 z_{\gamma FF})A_{iHe} + a\sigma_{iH}^2\sigma_{iF}^2 z_{\gamma FH}A_{iFe}$$
$$= \sigma_{iH}^2 A_{iHe}$$
$$+ a\sigma_{iH}^2\sigma_{iF}^2\int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau)+A_{iFe}A_{iHH}(\tau)]\left[\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau$$

$$+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iHF}(\tau)] \left[\int_0^\tau A_{iFF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau,$$

$$\begin{aligned} Z_{\gamma F} &\equiv \sigma_{iF}^2(1 + a\sigma_{iH}^2 z_{\gamma HH})A_{iFe} + a\sigma_{iH}^2\sigma_{iF}^2 z_{\gamma HF}A_{iHe} \\ &= \sigma_{iF}^2 A_{iFe} \\ &\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)] \left[\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau \\ &\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iHF}(\tau)] \left[\int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau. \end{aligned}$$

Since $(A_{iHe}, A_{iFe}, Z_{\gamma H}, Z_{\gamma F})$ are positive, (B.100) holds.

Step 3: $A_{\gamma H}(\tau)$ is positive and $A_{\gamma F}(\tau)$ is negative. We prove that $A_{\gamma H}(\tau)$ is positive. The proof that $A_{\gamma F}(\tau)$ is negative is symmetric. Substituting $(\bar{\lambda}_{iH\gamma}, \bar{\lambda}_{iF\gamma})$ from (B.96) and (B.97) into (B.92) for $j = H$, and using the definitions of $(z_{\gamma HH}, z_{\gamma FF}, z_{\gamma HF}, z_{\gamma FH})$ and that $(\theta_e, \Delta_{z\gamma})$ are positive, we find $A_{\gamma H}(\tau) > 0$ if

$$Z_{\gamma H} \int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' - Z_{\gamma F} \int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' > 0. \tag{B.101}$$

Since $(A_{iHH}(\tau), Z_{\gamma H}, Z_{\gamma F})$ are positive, $A_{iFH}(\tau)$ is non-negative and $\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}$ is non-decreasing, (B.101) holds under the sufficient condition

$$Z_{\gamma H}A_{iHH}(\infty) - Z_{\gamma F}A_{iFH}(\infty) > 0. \tag{B.102}$$

Using the definitions of $(Z_{\gamma H}, Z_{\gamma F})$, we can write (B.102) as

$$\begin{aligned} &\sigma_{iH}^2 A_{iHe}A_{iHH}(\infty) - \sigma_{iF}^2 A_{iFe}A_{iFH}(\infty) \\ &+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)] \\ &\times \left[\int_0^\tau \left[A_{iFH}(\tau')A_{iHH}(\infty) - A_{iHH}(\tau')A_{iFH}(\infty)\right] e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau \\ &+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iHF}(\tau)] \\ &\times \left[\int_0^\tau \left[A_{iFF}(\tau')A_{iHH}(\infty) - A_{iHF}(\tau')A_{iFH}(\infty)\right] e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau > 0. \end{aligned} \tag{B.103}$$

Equation (B.39) for $(j, j') = (H, F)$ implies

$$A_{iFH}(\tau) = \frac{a\sigma_{iH}^2\bar{\lambda}_{iHF}A_{iHH}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}} - \frac{A'_{iFH}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}}, \tag{B.104}$$

which for $\tau = \infty$ becomes

$$A_{iFH}(\infty) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHH}(\infty)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}}. \tag{B.105}$$

Equation (B.38) for $(j, j') = (F, H)$ implies

$$A_{iFF}(\tau) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHF}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}} + \frac{1 - A'_{iFF}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}}. \tag{B.106}$$

Using (B.104)-(B.106) to simplify the terms in the first, second and fourth lines of (B.103), and dividing throughout by $a\sigma_{iH}^2 \sigma_{iF}^2 A_{iHH}(\infty)/(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}) > 0$, we find that (B.103) is equivalent to

$$\begin{aligned}
&\left(\frac{\kappa_{iF}}{a\sigma_{iF}^2} - \bar{\lambda}_{iFF}\right) A_{iHe} - \bar{\lambda}_{iHF} A_{iFe} \\
&- \int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)] \left[\int_0^\tau A'_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau \\
&+ \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iHF}(\tau)] \left[\int_0^\tau (1 - A'_{iFF}(\tau'))e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau > 0.
\end{aligned} \tag{B.107}$$

Equations (B.41) and (B.42) imply

$$\begin{aligned}
&- \bar{\lambda}_{iFF} A_{iHe} - \bar{\lambda}_{iHF} A_{iFe} \\
&= \int_0^T \alpha_H(\tau) A_{iFH}(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)]d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iFF}(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iHF}(\tau)]d\tau.
\end{aligned} \tag{B.108}$$

We next substitute (B.108) into (B.107). Noting that $1 - A'_{iFF}(\tau) > 0$, which follows from (B.38) for $(j, j') = (F, H)$ and (B.74), and that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHe}, A_{iFe})$ are positive and $(A_{iHF}(\tau), A_{iFH}(\tau))$ are non-negative, we find that (B.107) holds under the sufficient condition

$$\int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)] \left[A_{FH}(\tau) - \int_0^\tau A'_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau \geq 0,$$

which, in turn, holds because

$$A_{iFH}(\tau) - \int_0^\tau A'_{iFH}(\tau)e^{-\kappa_\gamma(\tau-\tau')}d\tau' \geq A_{iFH}(\tau) - \int_0^\tau A'_{iFH}(\tau')d\tau' = A_{iFH}(0) = 0.$$

■

Proof of Proposition 3.7: We prove the proposition in the case $j = H$. The proof for the case $j = F$ is symmetric. Consider a one-off increase in β_{Ht} at time zero, and denote by $\kappa_{\beta H}$ the rate at which β_{Ht} reverts to its mean of zero. The counterparts of (B.90) and (B.92) are

(B.109)
$$\kappa_{\beta H} A_{\beta He} = a\sigma_{iH}^2 \bar{\lambda}_{iH\beta} A_{iHe} - a\sigma_{iF}^2 \bar{\lambda}_{iF\beta} A_{iFe},$$
(B.110)
$$A_{\beta Hj}(\tau) = a\sigma_{iH}^2 \bar{\lambda}_{iH\beta} \int_0^\tau A_{iHj}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' + a\sigma_{iF}^2 \bar{\lambda}_{iF\beta} \int_0^\tau A_{iFj}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau,$$

where

$$\bar{\lambda}_{ij\beta} \equiv -\alpha_e A_{\gamma e} A_{ije} (-1)^{1_{\{j=F\}}}$$
(B.111)
$$+ \int_0^T \left[\theta_H(\tau) - \alpha_H(\tau) A_{\beta HH}(\tau)\right] A_{ijH}(\tau) d\tau - \int_0^T \alpha_F(\tau) A_{\beta HF}(\tau) A_{ijF}(\tau) d\tau$$

is the counterpart of (B.93). The counterparts of (B.94) and (B.95) are

$$\text{(B.112)} \qquad (1 + a\sigma_{iH}^2 z_{\beta HH}) \bar{\lambda}_{iH\beta} + a\sigma_{iF}^2 z_{\beta FH} \bar{\lambda}_{iF\beta} = \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau,$$

$$\text{(B.113)} \qquad a\sigma_{iH}^2 z_{\beta HF} \bar{\lambda}_{iH\beta} + (1 + a\sigma_{iF}^2 z_{\beta FF}) \bar{\lambda}_{iF\beta} = \int_0^T \theta_H(\tau) A_{iFH}(\tau) d\tau,$$

respectively, where

$$\begin{aligned}
z_{\beta HH} = {} & \frac{\alpha_e}{\kappa_{\beta H}} A_{iHe}^2 + \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau A_{iHH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
& + \int_0^T \alpha_F(\tau) A_{iHF}(\tau) \left[\int_0^\tau A_{iHF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau, \\
z_{\beta FF} = {} & \frac{\alpha_e}{\kappa_{\beta H}} A_{iFe}^2 + \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iFH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
& + \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iFF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau,
\end{aligned}$$

$$z_{\beta HF} = -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe} A_{iFe} + \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iHH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau$$

$$+ \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iHF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau,$$

$$z_{\beta FH} = -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe} A_{iFe} + \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau A_{iFH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau$$
$$+ \int_0^T \alpha_F(\tau) A_{iHF}(\tau) \left[\int_0^\tau A_{iFF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau.$$

The solution to the linear system of (B.94) and (B.95) is

(B.114)
$$\bar{\lambda}_{iH\beta} = \frac{1}{\Delta_{z\beta}} \left[(1 + a\sigma_{iF}^2 z_{\beta FF}) \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau - a\sigma_{iF}^2 z_{\beta FH} \int_0^T \theta_H(\tau) A_{iFH}(\tau) d\tau \right],$$
(B.115)
$$\bar{\lambda}_{iF\beta} = \frac{1}{\Delta_{z\beta}} \left[(1 + a\sigma_{iH}^2 z_{\beta HH}) \int_0^T \theta_H(\tau) A_{iFH}(\tau) d\tau - a\sigma_{iH}^2 z_{\beta HF} \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau \right],$$

where

$$\Delta_{z\beta} \equiv (1 + a\sigma_{iH}^2 z_{\beta HH})(1 + a\sigma_{iF}^2 z_{\beta FF}) - a^2 \sigma_{iH}^2 \sigma_{iF}^2 z_{\beta HF} z_{\beta FH}.$$

The same argument as in the proof of Proposition 3.6 implies $\Delta_{z\beta} > 0$.

To complete the proof, we proceed in three steps. In Step 1, we show that $(z_{\beta HF}, z_{\beta FH})$ are non-positive, and are zero when $\alpha_e = 0$. In Step 2, we show that $A_{\beta HH}(\tau)$ is positive, and that $A_{\beta HF}(\tau)$ is positive when $\alpha_e > 0$ and zero when $\alpha_e = 0$. This proves the first and second statements in the proposition. In Step 3, we show that $A_{\beta He}$ is positive. This proves the third statement in the proposition.

Step 1: $(z_{\beta HF}, z_{\beta FH})$ are non-positive, and are zero when $\alpha_e = 0$. Since Lemma B.3 implies that $A_{iFH}(\tau)$ is non-negative and $A_{iFF}(\tau)$ is positive, and Lemma B.4 implies that $A_{iHH}(\tau)$ is increasing and $A_{iHF}(\tau)$ is non-decreasing,

$$z_{\beta HF} \leq -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe} A_{iFe} + \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iHH}(\tau) e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau$$
$$+ \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iHF}(\tau) e^{-\kappa_{\beta H}(\tau-\tau')} d\tau' \right] d\tau$$
$$\leq -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe} A_{iFe} + \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \frac{A_{iHH}(\tau)}{\kappa_{\beta H}} d\tau + \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \frac{A_{iHF}(\tau)}{\kappa_{\beta H}}$$
$$= -\frac{\bar{\lambda}_{iHF}}{\kappa_{\beta H}} \leq 0,$$

where the second step follows because $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive and $(A_{iHF}(\tau), A_{iFH}(\tau))$

are non-negative, the third step follows from (B.42), and the fourth step follows from Lemma B.2. The inequality $z_{\beta FH} \leq 0$ follows similarly.

When $\alpha_e = 0$, Lemma B.3 implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Therefore, $z_{\beta HF} = z_{\beta FH} = 0$.

Step 2: $A_{\beta HH}(\tau)$ is positive, and $A_{\beta HF}(\tau)$ is positive when $\alpha_e > 0$ and zero when $\alpha_e = 0$. Since $(\Delta_{z\beta}, \theta_H(\tau), A_{iHH}(\tau))$ are positive, $(A_{iFH}(\tau), z_{\beta FF})$ are non-negative, and $z_{\beta FH} \leq 0$, (B.114) implies $\bar{\lambda}_{iH\beta} > 0$. When $\alpha_e > 0$, $A_{iFH}(\tau) > 0$. Since, in addition, $z_{\beta HH} \geq 0$ and $z_{\beta FH} \leq 0$, (B.115) implies $\bar{\lambda}_{iF\beta} > 0$. When $\alpha_e = 0$, (B.115) and $A_{iFH}(\tau) = z_{\beta HF} = 0$ imply $\bar{\lambda}_{iF\beta} = 0$.

Since $(\bar{\lambda}_{iH\beta}, A_{iHH}(\tau))$ are positive and $(\bar{\lambda}_{iF\beta}, A_{iFH}(\tau))$ are non-negative, (B.110) implies $A_{\beta HH}(\tau) > 0$. When $\alpha_e > 0$, $A_{iHF}(\tau) > 0$. Since, in addition, $(\bar{\lambda}_{iH\beta}, \bar{\lambda}_{iF\beta}, A_{iFF}(\tau))$ are positive, (B.110) implies $A_{\beta HF}(\tau) > 0$. When $\alpha_e = 0$, (B.110) and $A_{iHF}(\tau) = \bar{\lambda}_{iF\beta} = 0$ imply $A_{\beta HF}(\tau) = 0$.

Step 3: $A_{\beta He}$ is positive. Substituting $(\bar{\lambda}_{\beta HH}, \bar{\lambda}_{\beta HF})$ from (B.114) and (B.115) into (B.109), and using the definitions of $(z_{\beta HH}, z_{\beta HF}, z_{\beta FH}, z_{\beta FF})$, we find $A_{\beta He} > 0$ if

$$Z_{\beta H} \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau - Z_{\beta F} \int_0^T \theta_H(\tau) A_{iFH}(\tau) d\tau > 0, \tag{B.116}$$

where

$$\begin{aligned}
Z_{\beta H} &\equiv \sigma_{iH}^2 (1 + a\sigma_{iF}^2 z_{\beta FF}) A_{iHe} + a\sigma_{iH}^2 \sigma_{iF}^2 z_{\beta HF} A_{iFe} \\
&= \sigma_{iH}^2 A_{iHe} \\
&\quad + a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \left[\int_0^\tau [A_{iHe} A_{iFH}(\tau') + A_{iFe} A_{iHH}(\tau')] e^{-\kappa_{\beta H}(\tau - \tau')} d\tau' \right] d\tau \\
&\quad + a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau [A_{iHe} A_{iFF}(\tau') + A_{iFe} A_{iHF}(\tau')] e^{-\kappa_{\beta H}(\tau - \tau')} d\tau' \right] d\tau, \\
Z_{\beta F} &\equiv \sigma_{iF}^2 (1 + a\sigma_{iH}^2 z_{HH}) A_{iFe} + a\sigma_{iH}^2 \sigma_{iF}^2 z_{HF} A_{iHe} \\
&= \sigma_{iF}^2 A_{iFe} \\
&\quad + a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau [A_{iHe} A_{iFH}(\tau') + A_{iFe} A_{iHH}(\tau')] e^{-\kappa_{\beta H}(\tau - \tau')} d\tau' \right] d\tau \\
&\quad + a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_F(\tau) A_{iHF}(\tau) \left[\int_0^\tau [A_{iHe} A_{iFF}(\tau') + A_{iFe} A_{iHF}(\tau')] e^{-\kappa_{\beta H}(\tau - \tau')} d\tau' \right] d\tau.
\end{aligned}$$

Since $(\theta_H(\tau), A_{iHH}(\tau))$ are positive, $A_{iFH}(\tau)$ is non-negative, and $A_{iFH}(\tau)/A_{iHH}(\tau)$ is non-decreasing (increasing when $a > 0$ and $\alpha_e > 0$ from Lemma B.7, and zero when $a = 0$ or $\alpha_e = 0$), the ratio $\int_0^T \theta_H(\tau) A_{iFH}(\tau) d\tau / \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau$ is bounded above by $A_{iFH}(\infty)/A_{iHH}(\infty)$. Since, in addition $(Z_{\beta H}, Z_{\beta F})$ are positive, (B.116) holds for all positive functions $\theta_H(\tau)$ under the sufficient condition

$$Z_{\beta H}A_{iHH}(\infty) - Z_{\beta F}A_{iFH}(\infty) > 0. \tag{B.117}$$

Using the definitions of $(Z_{\beta H}, Z_{\beta F})$, we can write (B.117) as

$$\begin{aligned}
&\sigma^2_{iH}A_{iHe}A_{iHH}(\infty) - \sigma^2_{iF}A_{iFe}A_{iFH}(\infty)\\
&+ a\sigma^2_{iH}\sigma^2_{iF}\int_0^T \alpha_H(\tau)\left[A_{iFH}(\tau)A_{iHH}(\infty) - A_{iHH}(\tau)A_{iFH}(\infty)\right]\\
&\times\left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau\\
&+ a\sigma^2_{iH}\sigma^2_{iF}\int_0^T \alpha_F(\tau)\left[A_{iFF}(\tau)A_{iHH}(\infty) - A_{iHF}(\tau)A_{iFH}(\infty)\right]\\
&\times\left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iHF}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau > 0.
\end{aligned} \tag{B.118}$$

Using (B.104)-(B.106) to simplify the terms in the first, second and fourth lines of (B.118), and dividing throughout by $a\sigma^2_{iH}\sigma^2_{iF}A_{iHH}(\infty)/(\kappa_{iF} - a\sigma^2_{iF}\bar{\lambda}_{iFF}) > 0$, we find that (B.118) is equivalent to

$$\begin{aligned}
&\left(\frac{\kappa_{iF}}{a\sigma^2_{iF}} - \bar{\lambda}_{iFF}\right)A_{iHe} - \bar{\lambda}_{iHF}A_{iFe}\\
&- \int_0^T \alpha_H(\tau)A'_{iFH}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau\\
&+ \int_0^T \alpha_F(\tau)(1 - A'_{iFF}(\tau))\left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iHF}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau > 0.
\end{aligned} \tag{B.119}$$

We next substitute (B.108) into (B.119). Noting that $1 - A'_{iFF}(\tau) > 0$ and that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHe}, A_{iFe})$ are positive and $(A_{iHF}(\tau), A_{iFH}(\tau))$ are non-negative, we find that (B.119) holds under the sufficient condition

$$\begin{aligned}
&\int_0^T \alpha_H(\tau)\Bigg\{A_{iFH}(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)]d\tau\\
&-A'_{iFH}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]\Bigg\}d\tau \geq 0,
\end{aligned}$$

which, in turn, holds under the sufficient condition

$$\int_0^T \alpha_H(\tau)\Bigg\{A_{iFH}(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)]d\tau$$

$$-A'_{iFH}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFH}(\tau')+A_{iFe}A_{iHH}(\tau')]d\tau'\right]\Bigg\}d\tau \geq 0. \tag{B.120}$$

Equation (B.120) holds under the sufficient condition that the function

$$G(\tau) \equiv \frac{A_{iFH}(\tau)}{\int_0^\tau [A_{iHe}A_{iFH}(\tau')+A_{iFe}A_{iHH}(\tau')]d\tau'}$$

is non-increasing because the term in curly brackets in (B.120) is the negative of the numerator of $G'(\tau)$. The function $G'(\tau)$ is non-increasing under the sufficient condition that the function

$$G_1(\tau) \equiv \frac{A'_{iFH}(\tau)}{A_{iHe}A_{iFH}(\tau)+A_{iFe}A_{iHH}(\tau)}$$

is non-increasing. Equation (B.39) for $(j,j')=(H,F)$ implies

$$\begin{aligned}
G_1(\tau) &= \frac{a\sigma_{iH}^2\bar{\lambda}_{iHF}A_{iHH}(\tau)+(a\sigma_{iF}^2\bar{\lambda}_{iFF}-\kappa_{iF})A_{iFH}(\tau)}{A_{iHe}A_{iFH}(\tau)+A_{iFe}A_{iHH}(\tau)} \\
&= \frac{a\sigma_{iH}^2\bar{\lambda}_{iHF}+(a\sigma_{iF}^2\bar{\lambda}_{iFF}-\kappa_{iF})\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}}{A_{iHe}\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}+A_{iFe}}.
\end{aligned}$$

Since $\bar{\lambda}_{iFH} \geq 0$, $\bar{\lambda}_{iFF} \leq 0$ and $A_{iFH}(\tau)/A_{iHH}(\tau)$ is non-decreasing, $G_1(\tau)$ is non-increasing. ∎

C. Model Estimation

1. *Numerical Solution Method*

We derive a system of twenty-five nonlinear scalar equations in the elements of the 5×5 matrix $\mathbf{M}$. We adopt the exponential specification (4.6) and (4.7) for the functions $\{(\alpha_j(\tau),\theta_j(\tau))\}_{j=H,F}$, and set $T=\infty$. Using the exponential specification and $T=\infty$, we can compute the integrals involving $\{\mathbf{A}_j(\tau)\}_{j=H,F}$ in the definition (B.5) of $\mathbf{M}$ by treating them as Laplace transforms of $\{\mathbf{A}_j(\tau)\}_{j=H,F}$. These Laplace transforms can be computed by solving systems of linear equations with coefficients that are linear in the elements of $\mathbf{M}$. The computation of the Laplace transforms does not require solving the ODE system (B.3) for $\{\mathbf{A}_j(\tau)\}_{j=H,F}$, which would entail computing eigenvalues and eigenvectors of $\mathbf{M}$.

We define the Laplace transform

$$\boldsymbol{\mathcal{A}}_j(s) \equiv \int_0^\infty \mathbf{A}_j(\tau)e^{-s\tau}d\tau$$

of $\mathbf{A}_j(\tau)$, and

$$\boldsymbol{\mathfrak{X}}_j(s) \equiv \int_0^\infty \mathbf{X}_j(\tau) e^{-s\tau} d\tau$$

of $\mathbf{X}_j(\tau) \equiv \mathbf{A}_j(\tau)\mathbf{A}_j(\tau)^\top$. Multiplying (B.3) by $e^{-s\tau}$, taking integrals of both sides from zero to infinity, and using the property that the Laplace transform of $\mathbf{A}_j'(\tau)$ is s times that of $\mathbf{A}_j(\tau)$ (this property follows from integration by parts), we find

$$(s\mathbf{I}+\mathbf{M})\boldsymbol{\mathcal{A}}_j(s) = \frac{1}{s}\mathbf{I}_{ij} \to \boldsymbol{\mathcal{A}}_j(s) = \frac{1}{s}(s\mathbf{I}+\mathbf{M})^{-1}\mathbf{I}_{ij}, \tag{C.1}$$

where $\mathbf{I}$ denotes the 5×5 identity matrix. Multiplying (B.3) from the right by $\mathbf{A}_j(\tau)^\top$, and adding to the resulting equation its transpose, we find

$$\mathbf{A}_j'(\tau)\mathbf{A}_j(\tau)^\top + \mathbf{A}_j(\tau)\mathbf{A}_j'(\tau)^\top + \mathbf{M}\mathbf{A}_j(\tau)\mathbf{A}_j(\tau)^\top + \mathbf{A}_j(\tau)\mathbf{A}_j(\tau)^\top\mathbf{M}^\top - \mathbf{I}_{ij}\mathbf{A}_j(\tau)^\top - \mathbf{A}_j(\tau)\mathbf{I}_{ij}^\top = 0. \tag{C.2}$$

Multiplying (C.2) by $e^{-s\tau}$, taking integrals of both sides from zero to infinity, and using the definition of $\mathbf{X}_j(\tau)$ and the property that the Laplace transform of $\mathbf{X}_j'(\tau)$ is s times that of $\mathbf{X}_j(\tau)$, we find

$$\left(\frac{s}{2}\mathbf{I}+\mathbf{M}\right)\boldsymbol{\mathfrak{X}}_j(s) + \boldsymbol{\mathfrak{X}}_j(s)\left(\frac{s}{2}\mathbf{I}+\mathbf{M}\right)^\top = \mathbf{I}_{ij}\boldsymbol{\mathcal{A}}_j(s)^\top + \boldsymbol{\mathcal{A}}_j(\tau)\mathbf{I}_{ij}^\top. \tag{C.3}$$

The solutions to (C.1) and (C.3) can be computed by solving systems of linear equations with coefficients that are linear in the elements of $\mathbf{M}$. There are twenty-five scalar equations in the system for (C.1). There are fifteen scalar equations in the system for (C.3) because $\boldsymbol{\mathfrak{X}}_j(s)$ is a symmetric matrix. Equation (C.1) has a unique solution $\boldsymbol{\mathcal{A}}_j(s)$ if $s\mathbf{I}+\mathbf{M}$ is invertible. Equation (C.3) is a Lyapunov equation and has a unique solution $\boldsymbol{\mathfrak{X}}_j(s)$ under the sufficient condition that the eigenvalues of $\frac{s}{2}\mathbf{I}+\mathbf{M}$ have positive real parts.

Using the Laplace transforms $(\boldsymbol{\mathcal{A}}_j(s), \boldsymbol{\mathfrak{X}}_j(s))$ and the exponential specifications (4.6) and (4.7), we can compute the integrals involving $\mathbf{A}_j(\tau)$ in the definition of $\mathbf{M}$ as

$$\int_0^\infty \theta_j(\tau)\mathbf{I}_{\beta j}\mathbf{A}_j(\tau)^\top d\tau = -\theta_{j0}\mathbf{I}_{\beta j}\boldsymbol{\mathcal{A}}_j'(\theta_{j1})^\top, \tag{C.4}$$

$$\int_0^\infty \alpha_j(\tau)\mathbf{A}_j(\tau)\mathbf{A}_j(\tau)^\top d\tau = \alpha_{j0}\boldsymbol{\mathfrak{X}}_j(\alpha_{j1}). \tag{C.5}$$

Deriving (C.4) requires using the property that the Laplace transform of $\tau\mathbf{A}_j(\tau)$ is minus the derivative of the Laplace transform of $\mathbf{A}_j(\tau)$. The derivative $\boldsymbol{\mathcal{A}}_j'(s)$ can be computed as a function of $\boldsymbol{\mathcal{A}}_j(s)$ by differentiating (C.1):

$$\boldsymbol{\mathcal{A}}_j(s) + (s\mathbf{I}+\mathbf{M})\boldsymbol{\mathcal{A}}_j'(s) = -\frac{1}{s^2}\mathbf{I}_{ij} \Rightarrow \boldsymbol{\mathcal{A}}_j'(s) = -(s\mathbf{I}+\mathbf{M})^{-1}\left(\boldsymbol{\mathcal{A}}_j(s) + \frac{1}{s^2}\mathbf{I}_{ij}\right). \tag{C.6}$$

Using (C.4) and (C.5), together with

$$\mathbf{A}_e = \mathbf{M}^{-1}\left(\mathbf{I}_{iH} - \mathbf{I}_{iF}\right), \tag{C.7}$$

which follows from (B.1), we can write (B.5) as

$$\mathbf{M} \equiv \boldsymbol{\Gamma}^\top - a\Bigg[\Big(\theta_e \mathbf{I}_\gamma - \alpha_e \mathbf{M}^{-1}\left(\mathbf{I}_{iH} - \mathbf{I}_{iF}\right)\Big)\left(\mathbf{I}_{iH} - \mathbf{I}_{iF}\right)^\top \left(\mathbf{M}^{-1}\right)^\top$$

$$- \sum_{j=H,F}\Big(\theta_{j0}\mathbf{I}_{\beta j}\boldsymbol{\mathcal{A}}_j'(\theta_{j1})^\top + \alpha_{j0}\boldsymbol{\mathfrak{X}}_j(\alpha_{j1})\Big)\Bigg]\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\top. \tag{C.8}$$

The right-hand side of (C.8) is a function of $\mathbf{M}$, derived from (C.1), (C.3) and (C.6). Therefore, (C.8) forms a system of twenty-five nonlinear scalar equations in the twenty-five elements of $\mathbf{M}$. Given $\mathbf{M}$, we derive $\mathbf{A}_j(\tau)$ by solving the ODE system (B.3), and we obtain $\mathbf{A}_e$ from (C.7). Given $\mathbf{A}_j(\tau)$ and $\mathbf{A}_e$, we solve for $C_j(\tau)$ and C_e from (B.2), (B.4) and (B.6).

We solve the system of twenty-five nonlinear scalar equations using a continuation algorithm.

- Step 0 of the algorithm solves the system for zero risk aversion $a^{(0)} = 0$. The solution is $\mathbf{M} = \boldsymbol{\Gamma}^\top$.
- Step $i+1$ of the algorithm solves the system for risk aversion $a^{(i+1)} = a^{(i)} + s^{(i+1)}$, where $a^{(i)}$ is risk aversion for step i and $s^{(i+1)}$ is a small step size. The solution $\mathbf{M}^{(i)}$ in step i is used as initial condition for solving the system in step $i+1$. This ensures that the solution in step $i+1$ is found quickly and is close to the solution in step i.
- The algorithm ends when $a^{(i+1)} = a$.

If there are multiple solutions for $\mathbf{M}$, the continuation algorithm picks the solution that converges to the unique solution $\mathbf{M} = \boldsymbol{\Gamma}^\top$ when risk aversion goes to zero.

2. *MLE for Alternative Parametrizations*

Table C.1 reports parameter estimates and standard errors when innovations to the currency demand factor are allowed to correlate with innovations to the bond demand factors. The correlations are small (correlation $\sigma_{\gamma,\beta H}/\sqrt{\sigma^2_{\gamma,\beta H} + \sigma^2_{\gamma,\beta F} + \sigma^2_\gamma} = -0.020$ between innovations to the currency demand factor and the home bond demand factor, and correlation $\sigma_{\gamma,\beta F}/\sqrt{\sigma^2_{\gamma,\beta H} + \sigma^2_{\gamma,\beta F} + \sigma^2_\gamma} = -0.185$ between innovations to the currency demand factor and the foreign bond demand factor). The estimates for the remaining parameters are similar to those in Table 1. The

TABLE C.1—ESTIMATED MODEL PARAMETERS FOR CORRELATED CURRENCY AND BOND DEMAND.

Parameter	Value	Standard Error
σ_{iH}	1.144	0.077
σ_{iF}	0.867	0.056
$\sigma_{iH,iF}$	0.309	0.075
κ_{iH}	0.102	0.061
κ_{iF}	0.127	0.049
κ_{β}	0.059	0.058
κ_{γ}	0.041	0.116
$a\theta_e\kappa_{\gamma,iH}$	-142.1	109.5
$a\theta_e\kappa_{\gamma,iF}$	148.1	116.7
$a\theta_0\sigma_{\beta}$	887.9	173.6
$a\theta_e\sigma_{\gamma}$	760.9	395.2
$a\alpha_0$	5.977	3.143
$a\alpha_e$	71.76	34.59
$a\theta_e\sigma_{\gamma,\beta H}$	-15.23	109.8
$a\theta_e\sigma_{\gamma,\beta F}$	-143.4	123.2

counterparts of the figures in Sections IV.C and IV.D (not reported here) are similar as well.

Table C.2 reports parameter estimates and standard errors for $(\alpha_1, \theta_1) = (0.25, 0.5)$ and Table C.3 does the same for $(\alpha_1, \theta_1) = (0.3, 0.75)$. The estimates of all parameters except for $(a\theta_0\sigma_\beta, a\alpha_0)$ are similar to those in Table 1. The estimates for $(a\theta_0\sigma_\beta, a\alpha_0)$ become larger to compensate for the faster convergence of the functions $\{(\alpha_j(\tau), \theta_j(\tau))\}_{j=H,F}$ to zero. The counterparts of the figures in Sections IV.C and IV.D (not reported here) are similar.

Table C.4 reports parameter estimates and standard errors for the case where $K = 21$ and $\mathbf{p}_t$ includes US and German yields with maturities from one to ten years and the Dollar-Euro log exchange rate. The parameter estimates are similar to those in Table 1. The counterparts of the figures in Sections IV.C and IV.D (not reported here) are similar as well.

3. Sensitivity Analysis

Table C.5 shows the derivatives of the elements of the covariance matrix $\mathbf{A}\hat{\mathbf{\Sigma}}_{\Delta t}\mathbf{A}^\top$ of innovations to the vector $\mathbf{p}_t$ in the MLE estimation with respect to the thirteen model parameters $(\sigma_{iH}, \sigma_{iF}, \sigma_{iH,iF}, \kappa_{iH}, \kappa_{iF}, \kappa_\beta, \kappa_\gamma, \theta_e\kappa_{\gamma,iH}, \theta_e\kappa_{\gamma,iF}, a\theta_0\sigma_\beta, a\theta_e\sigma_\gamma, a\alpha_0, a\alpha_e)$. The rows correspond to matrix elements and the columns to model parameters. Since $\mathbf{A}\hat{\mathbf{\Sigma}}_{\Delta t}\mathbf{A}^\top$ is a 5×5 symmetric matrix, Table C.5 is 15×13. We denote elements of $\mathbf{A}\hat{\mathbf{\Sigma}}_{\Delta t}\mathbf{A}^\top$ based on the corresponding observables. Consider, for example, the element in the third row and second column of $\mathbf{A}\hat{\mathbf{\Sigma}}_{\Delta t}\mathbf{A}^\top$ (which is the

TABLE C.2—ESTIMATED MODEL PARAMETERS FOR $(\alpha_1, \theta_1) = (0.25, 0.5)$.

Parameter	Value	Standard Error
σ_{iH}	1.163	0.076
σ_{iF}	0.882	0.058
$\sigma_{iH,iF}$	0.326	0.083
κ_{iH}	0.143	0.059
κ_{iF}	0.136	0.046
κ_{β}	0.047	0.058
κ_{γ}	0.159	0.102
$a\theta_e\kappa_{\gamma,iH}$	-159.5	123.4
$a\theta_e\kappa_{\gamma,iF}$	193.6	135.2
$a\theta_0\sigma_{\beta}$	1079.0	222.3
$a\theta_e\sigma_{\gamma}$	983.2	477.4
$a\alpha_0$	14.68	6.659
$a\alpha_e$	80.47	38.74

same as the element in the second row and third column). That element is the covariance between innovations to the home ten-year yield and the foreign one-year yield, and is denoted by $y_H^{(10)} y_F^{(1)}$. We express the derivatives as elasticities by dividing by the matrix element and multiplying by the model parameter. We highlight in red the elasticities that are within 80% of the maximum elasticity in absolute value, and in yellow the elasticities that are within 50-80%. Table C.5 indicates which elements of $\mathbf{A}\hat{\mathbf{\Sigma}}_{\Delta t}\mathbf{A}^\top$ are the main determinants of which parameters.

- The variance $y_H^{(1)} y_H^{(1)}$ of innovations to the home one-year yield determines σ_{iH} because it is sensitive to only σ_{iH}. Indeed, in the row corresponding to $y_H^{(1)} y_H^{(1)}$, there is a red in the column corresponding to σ_{iH} and no other reds or yellows.

- The variance $y_F^{(1)} y_F^{(1)}$ of innovations to the foreign one-year yield determines σ_{iF} because it is sensitive to only σ_{iF}. Indeed, in the row corresponding to $y_F^{(1)} y_F^{(1)}$, there is a yellow in the column corresponding to σ_{iF} and no other reds or yellows.

- The covariance $y_H^{(1)} y_F^{(1)}$ between innovations to the home and the foreign one-year yield determines $\sigma_{iH,iF}$ because it is sensitive to only $\sigma_{iH,iF}$. Indeed, in the row corresponding to $y_H^{(1)} y_F^{(1)}$, there is a red in the column corresponding to $\sigma_{iH,iF}$ and no other reds or yellows.

- The covariance $y_H^{(1)} y_H^{(10)}$ between innovations to the home one- and ten-year

TABLE C.3—ESTIMATED MODEL PARAMETERS FOR $(\alpha_1, \theta_1) = (0.3, 0.75)$.

Parameter	Value	Standard Error
σ_{iH}	1.165	0.076
σ_{iF}	0.888	0.058
$\sigma_{iH,iF}$	0.322	0.082
κ_{iH}	0.149	0.061
κ_{iF}	0.138	0.046
κ_{β}	0.055	0.058
κ_{γ}	0.158	0.104
$a\theta_e\kappa_{\gamma,iH}$	-162.9	124.5
$a\theta_e\kappa_{\gamma,iF}$	196.9	135.7
$a\theta_0\sigma_{\beta}$	1472.0	307.7
$a\theta_e\sigma_{\gamma}$	989.4	476.9
$a\alpha_0$	20.36	8.964
$a\alpha_e$	81.21	38.73

TABLE C.4—ESTIMATED MODEL PARAMETERS USING ALL YIELDS FROM ONE TO TEN YEARS.

Parameter	Value	Standard Error
σ_{iH}	1.180	0.075
σ_{iF}	0.940	0.065
$\sigma_{iH,iF}$	0.348	0.084
κ_{iH}	0.094	0.054
κ_{iF}	0.115	0.045
κ_{β}	0.033	0.050
κ_{γ}	0.129	0.088
$a\theta_e\kappa_{\gamma,iH}$	-145.4	120.6
$a\theta_e\kappa_{\gamma,iF}$	178.7	134.9
$a\theta_0\sigma_{\beta}$	729.1	155.7
$a\theta_e\sigma_{\gamma}$	946.4	472.2
$a\alpha_0$	4.569	2.737
$a\alpha_e$	80.12	39.55

TABLE C.5—ELASTICITIES OF THE ELEMENTS OF $A\hat{\mathbf{\Sigma}}_{\Delta t}A^{\top}$ WITH RESPECT TO THE MODEL PARAMETERS.

Element	σ_{iH}	σ_{iF}	$\sigma_{iH,iF}$	κ_{iH}	κ_{iF}	κ_{β}	κ_{γ}	$a\theta_e\kappa_{\gamma,iH}$	$a\theta_e\kappa_{\gamma,iF}$	$a\theta_0\sigma_{\beta}$	$a\theta_e\sigma_{\gamma}$	$a\alpha_0$	$a\alpha_e$
$y_H^{(1)}$ $y_H^{(1)}$	1.911	0.003	0.020	-0.156	-0.002	-0.000	-0.006	-0.041	0.017	0.035	0.074	-0.031	-0.034
$y_H^{(1)}$ $y_H^{(10)}$	1.513	0.021	0.067	-0.418	-0.016	-0.016	-0.042	-0.152	0.076	0.282	0.309	-0.264	-0.154
$y_H^{(1)}$ $y_F^{(1)}$	1.013	0.122	0.910	-0.082	-0.082	-0.000	0.014	0.025	0.035	0.059	-0.171	-0.053	0.072
$y_H^{(1)}$ $y_H^{(10)}$	0.906	0.357	0.657	-0.068	-0.404	-0.023	0.080	0.033	0.133	0.513	-0.587	-0.445	0.276
$y_H^{(1)}$ e	2.347	-0.096	-0.470	-0.260	0.072	-0.002	0.178	0.907	-0.404	0.059	-1.371	-0.046	0.089
$y_H^{(10)}$ $y_H^{(10)}$	1.792	0.077	0.182	-0.738	-0.055	-0.097	-0.087	-0.303	0.171	1.937	0.580	-0.815	-0.285
$y_H^{(10)}$ $y_F^{(1)}$	0.918	0.347	0.655	-0.375	-0.101	-0.024	0.070	0.101	0.073	0.507	-0.607	-0.443	0.279
$y_H^{(10)}$ $y_H^{(10)}$	1.179	0.661	0.582	-0.406	-0.411	-0.074	0.100	0.066	0.124	2.090	-0.722	-0.991	0.328
$y_H^{(10)}$ e	2.022	-0.313	-0.384	-0.498	0.100	-0.015	0.560	0.737	-0.487	0.436	-1.246	-0.261	0.079
$y_F^{(1)}$ $y_F^{(1)}$	0.019	1.660	0.266	-0.002	-0.153	-0.000	-0.005	0.015	-0.037	0.030	0.059	-0.026	-0.023
$y_F^{(1)}$ $y_H^{(10)}$	0.068	1.348	0.255	-0.009	-0.442	-0.013	-0.026	0.064	-0.141	0.239	0.238	-0.224	-0.102
$y_F^{(1)}$ e	-0.675	2.639	-0.166	0.087	-0.271	-0.002	0.164	-0.453	1.054	0.032	-1.350	-0.023	-0.021
$y_H^{(10)}$ $y_H^{(10)}$	0.256	1.531	0.410	-0.055	-0.771	-0.103	-0.044	0.154	-0.267	1.852	0.368	-0.747	-0.151
$y_H^{(10)}$ e	-0.814	2.290	-0.092	0.113	-0.523	-0.006	0.521	-0.541	0.889	0.191	-0.964	-0.105	-0.165
e e	0.115	0.136	-0.039	-0.015	-0.012	-0.001	-0.364	0.076	0.068	0.021	2.082	-0.014	-2.077

yield determines κ_{iH}. Indeed, in the row corresponding to $y_H^{(1)}y_H^{(10)}$, there is a yellow in the column corresponding to κ_{iH} and no other reds or yellows, except for a yellow in the column corresponding to σ_{iH}, which is determined above.

- The covariance $y_F^{(1)}y_F^{(10)}$ between innovations to the foreign one- and ten-year yield determines κ_{iF}. Indeed, in the row corresponding to $y_F^{(1)}y_F^{(10)}$, there is a yellow in the column corresponding to κ_{iH} and no other reds or yellows, except for a yellow in the column corresponding to σ_{iF}, which is determined above.
- The covariance $y_H^{(1)}e$ between innovations to the home one-year yield and the exchange rate determines $a\theta_e\kappa_{\gamma,iH}$. Indeed, in the row corresponding to $y_H^{(1)}e$, there is a red in the column corresponding to $a\theta_e\kappa_{\gamma,iH}$ and no other reds or yellows except in columns corresponding to parameters already determined and except for a yellow in the column corresponding to $a\theta_e\sigma_\gamma$.
- The covariance $y_F^{(1)}e$ between innovations to the foreign one-year yield and the exchange rate determines $a\theta_e\kappa_{\gamma,iF}$. Indeed, in the row corresponding to $y_F^{(1)}e$, there is a red in the column corresponding to $a\theta_e\kappa_{\gamma,iF}$ and no other reds or yellows except in columns corresponding to parameters already determined and except for a yellow in the column corresponding to $a\theta_e\sigma_\gamma$.
- The covariance $y_H^{(10)}e$ between innovations to the home ten-year yield and the exchange rate, and its foreign counterpart $y_F^{(10)}e$, determine κ_γ. Indeed, in the rows corresponding to $y_H^{(10)}e$ and $y_F^{(10)}e$, there is a red in the column corresponding to κ_γ and no other reds or yellows except in columns corresponding to parameters already determined and except for a yellow in the column corresponding to $a\theta_e\sigma_\gamma$.
- The variances $y_H^{(10)}y_H^{(10)}$ and $y_F^{(10)}y_F^{(10)}$ of innovations to the home and foreign ten-year yields, and the covariance $y_H^{(10)}y_F^{(10)}$ of innovations between these yields, determine κ_β, $a\theta_0\sigma_\beta$ and $a\alpha_0$. Indeed, in the rows corresponding to $y_H^{(10)}y_H^{(10)}$, $y_F^{(10)}y_F^{(10)}$ and $y_H^{(10)}y_F^{(10)}$, there are reds or yellows in the columns corresponding to κ_β, $a\theta_0\sigma_\beta$ and $a\alpha_0$ and no other reds or yellows except in columns corresponding to parameters already determined.
- The variance ee of innovations to the exchange rate, and the covariances between innovations to yields and the exchange rate, determine $a\theta_e\sigma_\gamma$ and $a\alpha_e$. Indeed, in the row corresponding to ee, there is a red in the columns corresponding to $a\theta_e\sigma_\gamma$ and ee and no other reds or yellows except in columns corresponding to parameters already determined. Moreover, in the column corresponding to $a\theta_e\sigma_\gamma$ there are several other yellows, while in the column corresponding to α_e there are no other reds or yellows.

The intuition for the determination of $a\alpha_e$ and $a\theta_e\sigma_\gamma$ is as follows. The effect of $a\alpha_e$ on the volatility of the exchange rate is particularly pronounced because higher demand elasticity of currency traders causes both larger under-reaction of the exchange rate to short rates and larger attenuation of currency demand shocks. Both effects lower the volatility of the exchange rate. By contrast, the effect of $a\theta_e\sigma_\gamma$ on the volatility of the exchange rate is less pronounced and more comparable to the effect on the covariance between the exchange rate and bond yields. This is because higher volatility of currency demand shocks causes larger under-reaction of the exchange rate to short rates, which tempers the effect of the demand shocks' higher volatility on the volatility of the exchange rate.

4. GMM

We use four sets of moments for GMM. A first set of moments concern one-year yields. They are the standard deviation of one-year yields $y_{jt}^{(1)}$ and of their annual change $\Delta y_{jt}^{(1)} \equiv y_{j,t+1}^{(1)} - y_{jt}^{(1)}$, and the standard deviation of the one-year yield differential $y_{Ht}^{(1)} - y_{Ft}^{(1)}$ between home and foreign.

A second set of moments concern the exchange rate. They are the standard deviation of the annual (log) exchange rate change $\Delta \log e_t \equiv \log e_{t+1} - \log e_t$; the correlation between $\Delta \log e_t$ and the one-year yield differential $y_{Ht}^{(1)} - y_{Ft}^{(1)}$; the correlation between $\Delta \log e_t$ and the annual change $\Delta y_{jt}^{(1)}$ in the home and foreign one-year yield; and the correlation between the five-year change in the exchange rate $\Delta^{(5)} \log e_t \equiv \log e_{t+5} - \log e_t$ and the five-year yield differential $y_{Ht}^{(5)} - y_{Ft}^{(5)}$.

A third set of moments concern yields across all maturities up to twenty years. They are the standard deviation of yields $y_{jt}^{(\tau)}$ and of their annual change $\Delta y_{jt}^{(\tau)} \equiv y_{j,t+1}^{(\tau)} - y_{jt}^{(\tau)}$; the correlation between the annual changes $\Delta y_{jt}^{(1)}$ in one-year yields and $\Delta y_{jt}^{(\tau)}$ in all other yields; and the standard deviation of yield differentials $y_{Ht}^{(\tau)} - y_{Ft}^{(\tau)}$ for all maturities.

A final set of moments concern trading volume. They are the trading volume of US government bonds with maturities between zero and three years, and with maturities between eleven and thirty years, as a fraction of total US government bond trading volume (denoted by $\tilde{V}_H(0 \leq \tau \leq 3)$ and $\tilde{V}_H(11 \leq \tau \leq 30)$, respectively).

The total number of target moments is $N = 12 + 7(\mathcal{N}_T - 1)$, where $\mathcal{N}_T$ is the number of bond maturities (we subtract one to not double-count the one-year maturity). With maturities going from one to twenty years in annual increments, $\mathcal{N}_T$ equals twenty and the number of target moments is 145 ($=12 + 7 \times 19$). We refer to the twelve moments that do not depend on maturity as scalar.

We estimate the model by choosing the model parameters that minimize

$$\mathcal{L} \equiv \sum_{n=1}^{N} w_n(\hat{m}_n - m_n)^2, \tag{C.9}$$

the weighted sum of squared differences between the empirical moments $\{\hat{m}_n\}_{n=1,..,N}$ and their model-implied counterparts $\{m_n\}_{n=1,..,N}$. We set the weights w_n to one for scalar moments and to $\frac{1}{\mathbb{N}_T}$ for moments that depend on maturity, so that each type of moment receives the same weight (for moments corresponding to the one-year maturity, we use $1 + \frac{1}{\mathbb{N}_T}$).

To compute the model-implied moments of exchange rates and bond yields, we first compute the unconditional covariance and autocovariance of the state vector $\mathbf{q}_t$. Integrating the dynamics (1.7) of $\mathbf{q}_t$ between $-\infty$ and t, we find

$$\mathbf{q}_t - \bar{\mathbf{q}} = \int_{-\infty}^{t} e^{-\mathbf{\Gamma}(t-s)}\mathbf{\Sigma} d\mathbf{B}_s. \tag{C.10}$$

Equation (C.10) implies that the unconditional covariance matrix of $\mathbf{q}_t$ is

$$\mathbb{C}\text{ov}(\mathbf{q}_t, \mathbf{q}_t^\top) = \int_{-\infty}^{t} e^{-\mathbf{\Gamma}(t-s)}\mathbf{\Sigma}\mathbf{\Sigma}^\top e^{-\mathbf{\Gamma}^\top(t-s)} ds \equiv \hat{\mathbf{\Sigma}}. \tag{C.11}$$

Differentiating (C.11) with respect to t and noting that the derivative is zero, we find

$$\mathbf{\Gamma}\hat{\mathbf{\Sigma}} + \hat{\mathbf{\Sigma}}\mathbf{\Gamma}^\top = \mathbf{\Sigma}\mathbf{\Sigma}^\top, \tag{C.12}$$

which is a Lyapunov equation and has a unique solution $\hat{\mathbf{\Sigma}}$ because the eigenvalues of $\mathbf{\Gamma}$ have positive real parts. The unconditional autocovariance matrix of $\mathbf{q}_t$ is

$$\begin{aligned} \mathbb{C}\text{ov}(\mathbf{q}_t, \mathbf{q}_{t'}^\top) &= \int_{-\infty}^{t} e^{-\mathbf{\Gamma}(t-s)}\mathbf{\Sigma}\mathbf{\Sigma}^\top e^{-\mathbf{\Gamma}^\top(t'-s)} ds \\ &= \left[\int_{-\infty}^{t} e^{-\mathbf{\Gamma}(t-s)}\mathbf{\Sigma}\mathbf{\Sigma}^\top e^{-\mathbf{\Gamma}^\top(t-s)} ds\right] e^{-\mathbf{\Gamma}^\top(t'-t)} \\ &= \hat{\mathbf{\Sigma}} e^{-\mathbf{\Gamma}^\top(t'-t)}, \end{aligned} \tag{C.13}$$

for $t' > t$, where the last step in (C.13) follows from (C.11).

Bond yields and log exchange rates in the model are affine functions of the state vector $\mathbf{q}_t$. The covariance between two such affine functions $\mathbf{X}\mathbf{q}_t + X_0$ and $\mathbf{Y}\mathbf{q}_{t'} + Y_0$ for 1×5 constant vectors $(\mathbf{X}, \mathbf{Y})$, scalars (X_0, Y_0), and $t' > t$ is

$$\mathbb{C}\text{ov}(\mathbf{X}\mathbf{q}_t + X_0, \mathbf{Y}\mathbf{q}_{t'} + Y_0) = \mathbf{X}\mathbb{C}\text{ov}(\mathbf{q}_t, q_{t'}^\top)\mathbf{Y}^\top. \tag{C.14}$$

Table C.4 reports parameter estimates and standard errors for GMM. The GMM point estimates are similar to the MLE ones in Table 1. GMM also delivers estimates for (a_1, θ_1), which are similar to the values $(0.15, 0.3)$ that we use in Section IV. The counterparts of the figures in Sections IV.C and IV.D (not reported here) are similar except for the confidence intervals for the UIP regressions and the exchange-rate responses to monetary policy. These are significantly wider under GMM because of the higher standard errors of the estimates for $(\theta_e\kappa_{\gamma,iH}, \theta_e\kappa_{\gamma,iF}, a\theta_e\sigma_\gamma, a\alpha_e)$.

Table C.6—Estimated Model Parameters using GMM.

Parameter	Value	Standard Error
σ_{iH}	1.429	0.148
σ_{iF}	0.751	0.140
$\sigma_{iH,iF}$	1.054	0.083
κ_{iH}	0.126	0.030
κ_{iF}	0.090	0.020
κ_β	0.050	0.009
κ_γ	0.134	0.102
$a\theta_e\kappa_{\gamma,iH}$	-267.1	550.4
$a\theta_e\kappa_{\gamma,iF}$	252.1	527.7
$a\theta_0\sigma_\beta$	648.9	80.27
$a\theta_e\sigma_\gamma$	762.7	1067.0
$a\alpha_0$	4.740	3.302
$a\alpha_e$	73.38	106.3
α_1	0.144	0.031
θ_1	0.374	0.014

5. *Variance Decomposition*

To compute the variance decomposition, we must account for the correlation between the instantaneous innovations to the home short rate i_{Ht} and the foreign short rate i_{Ft}, which arises because of the off-diagonal element $\sigma_{iH,iF}$ of the diffusion matrix $\mathbf{\Sigma}$. Attributing all the return variance that arises because of the Brownian motion B_{iHt} to i_{Ht} amounts to attributing to i_{Ht} all the variance that arises because of its correlated part with i_{Ft}. To attribute the return variance due to the correlated part symmetrically across i_{Ht} and i_{Ft}, we rotate the Brownian motions (B_{iHt}, B_{iFt}) to new versions (B_{RiHt}, B_{RiFt}) that render the diffusion matrix symmetric. We then compute the fraction of return variance that arises because of each Brownian motion in the rotated vector $\mathbf{B}_{Rt} \equiv (B_{RiHt}, B_{RiFt}, B_{\gamma t}, B_{\beta Ht}, B_{\beta Ft})^\top$ and attribute it to the corresponding element of $\mathbf{q}_t \equiv (i_{Ht}, i_{Ft}, \gamma_t, \beta_{Ht}, \beta_{Ft})^\top$.

We denote the rotated diffusion matrix by $\mathbf{\Sigma}_R$. Since $\mathbf{\Sigma}_R$ is symmetric and the product of $\mathbf{\Sigma}$ times a rotation matrix $\mathbf{R}$,

$$\mathbf{\Sigma}_R\mathbf{\Sigma}_R = \mathbf{\Sigma}_R\mathbf{\Sigma}_R^\top = (\mathbf{\Sigma}\mathbf{R})(\mathbf{\Sigma}\mathbf{R})^\top = \mathbf{\Sigma}\mathbf{R}\mathbf{R}^\top\mathbf{\Sigma}^\top = \mathbf{\Sigma}\mathbf{\Sigma}^\top.$$

Therefore, $\mathbf{\Sigma}_R = (\mathbf{\Sigma}\mathbf{\Sigma}^\top)^{\frac{1}{2}}$.

Since the log exchange rate and bond yields are affine in $\mathbf{q}_t$, the same calculation as in (4.1) implies that the surprise component of a currency or bond log return over a horizon Δt is $\mathbf{A}_r \int_t^{t+\Delta t} e^{-\mathbf{\Gamma}(t+\Delta t-s)}\mathbf{\Sigma}_R d\mathbf{B}_{Rs}$, where the 1×5 vector $\mathbf{A}_r$ is endogenously determined from the model parameters. The variance of the return is $\mathbf{A}_r\hat{\mathbf{\Sigma}}_{\Delta t}\mathbf{A}_r^\top$, where

$$\hat{\mathbf{\Sigma}}_{\Delta t} \equiv \int_t^{t+\Delta t} e^{-\mathbf{\Gamma}(t+\Delta t-s)}\mathbf{\Sigma}_R\mathbf{\Sigma}_R^\top e^{-\mathbf{\Gamma}^\top(t+\Delta t-s)}ds = \int_t^{t+\Delta t} e^{-\mathbf{\Gamma}(t+\Delta t-s)}\mathbf{\Sigma}\mathbf{\Sigma}^\top e^{-\mathbf{\Gamma}^\top(t+\Delta t-s)}ds$$

is the covariance matrix (computed in Section IV.A) of the innovations to $\mathbf{q}_t$ over horizon Δt. The return variance that arises because of the j'th element of the vector $\mathbf{B}_{Rt}$ is $\mathbf{A}_r\hat{\mathbf{\Sigma}}_{\Delta t,j}\mathbf{A}_r^\top$, where

$$\hat{\mathbf{\Sigma}}_{\Delta t,j} \equiv \int_t^{t+\Delta t} e^{-\mathbf{\Gamma}(t+\Delta t-s)}\mathbf{\Sigma}_R\mathfrak{J}_j\mathbf{\Sigma}_R^\top e^{-\mathbf{\Gamma}^\top(t+\Delta t-s)}ds$$

and $\mathfrak{J}_j$ is a matrix whose terms are zero except for the term (j,j), which is one. The matrix $\hat{\mathbf{\Sigma}}_{\Delta t,j}$ satisfies the Lyapunov equation

$$\mathbf{\Gamma}\hat{\mathbf{\Sigma}}_{\Delta t,j} + \hat{\mathbf{\Sigma}}_{\Delta t,j}\mathbf{\Gamma}^\top = \mathbf{\Sigma}_R\mathfrak{J}_j\mathbf{\Sigma}_R^\top - e^{-\mathbf{\Gamma}\Delta t}\mathbf{\Sigma}_R\mathfrak{J}_j\mathbf{\Sigma}_R^\top e^{-\mathbf{\Gamma}^\top\Delta t}.$$

The fraction of return variance generated by the j'th element of the vector $\mathbf{B}_{Rt}$ is

$$\frac{\mathbf{A}_r\hat{\mathbf{\Sigma}}_{\Delta t,j}\mathbf{A}_r^\top}{\mathbf{A}_r\hat{\mathbf{\Sigma}}_{\Delta t}\mathbf{A}_r^\top}.$$

6. Correlations

The top left panel of Figure C.1 shows the correlation between changes to the one-year and τ-year home bond yields as function of τ. The top right panel shows the same correlation for foreign bond yields. The bottom left panel shows the correlation between changes to the log exchange rate and the τ-year home and foreign bond yields as function of τ. The bottom right panel shows the correlation between changes to the τ-year home and foreign bond yields as function of τ. All changes are quarterly. The empirical correlations are the red circles or brown diamonds. The model-implied correlations are the blue solid or purple dashed lines. The black dashed or black dotted lines show the model-implied correlations when arbitrageurs are risk-neutral. The model-implied correlations can be computed as described in (C.14).

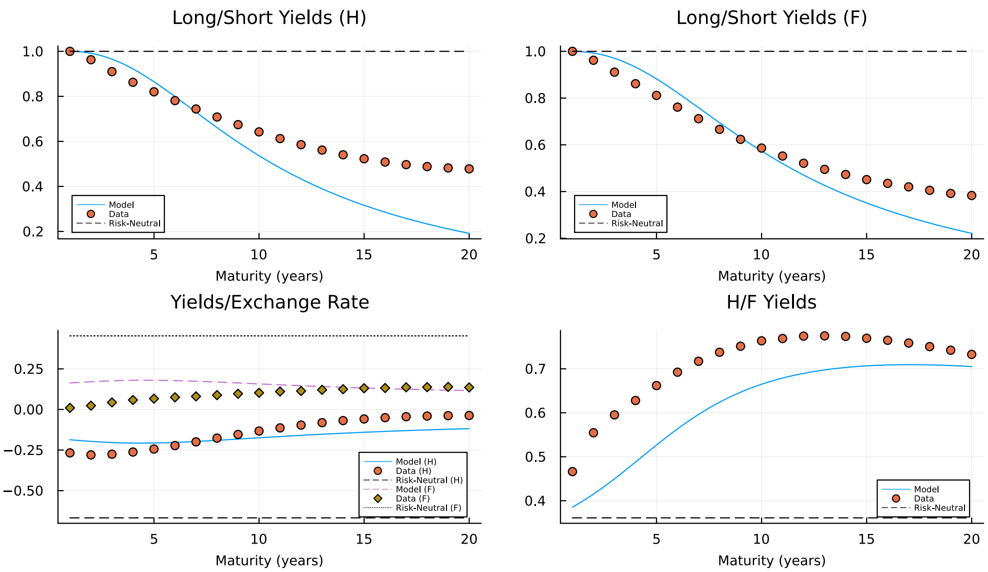

Figure C.1. Correlations between the exchange rate and bond yields.

The empirical correlations are close to the model-implied ones. The correlations between long-maturity (10- to 20-year) home and foreign bond yields are around 70%. They are higher than the correlations when arbitrageurs are risk-neutral, which are around 35%. When arbitrageurs are risk-neutral, a country's bond yields move only because of that country's short rate. Therefore, the correlation between bond yields across countries is equal to that between short rates. When arbitrageurs are risk-averse, the correlation between bond yields is influenced also by the bond demand factors, which generate positive comovement for the reasons explained in Section IV.C.

The correlations between changes to the exchange rate and long-maturity (10- to 20-year) bond yields are positive for foreign yields, negative for home yields, and smaller than 20% in absolute value. They are significantly smaller in absolute value than the correlations between home and foreign bond yields for the reasons explained in Section IV.C. They are smaller in absolute value than the correlations when arbitrageurs are risk-neutral because the bond demand factors become a significant driver of long-maturity bond yields but do not generate comovement between yields and the exchange rate.

The correlations between short- and long-maturity bond yields are equal to one when arbitrageurs are risk-neutral because a country's bond yields move only because of that country's short rate. When arbitrageurs are risk-averse, the correlations become significantly smaller than one because the bond demand factors become a significant driver of long-maturity bond yields while generating only small comovement between short- and long-maturity yields.

7. *Predictive Regressions*

Bilson (1981) and Fama (1984) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{e_t}{e_{t+\Delta\tau}}\right) = a_{\text{UIP}} + b_{\text{UIP}}\left(y_{Ft}^{(\Delta\tau)} - y_{Ht}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the rate of foreign currency depreciation over horizon $\Delta\tau$. The independent variable is the foreign-minus-home $\Delta\tau$-year yield differential. Bilson (1981) and Fama (1984) assume that the horizon $\Delta\tau$ is short (monthly). **?** perform the same regression for longer horizons. The coefficient b_{UIP} of this regression depends on second moments of bond yields and log exchange rates, and can be computed as described in (C.14).

Lustig, Stathopoulos and Verdelhan (2019) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{P_{F,t+\Delta\tau}^{(\tau-\Delta\tau)}e_{t+\Delta\tau}}{P_{Ft}^{(\tau)}e_t}\right) - \frac{1}{\Delta\tau}\log\left(\frac{P_{H,t+\Delta\tau}^{(\tau-\Delta\tau)}}{P_{Ht}^{(\tau)}}\right) = a_{\text{LSV}} + b_{\text{LSV}}\left(y_{Ft}^{(\Delta\tau)} - y_{Ht}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the return over horizon $\Delta\tau$ of the hybrid CCT constructed using bonds with maturity τ. The independent variable is the foreign-minus-home $\Delta\tau$-year yield differential. Since log bond prices are affine functions of the state vector $\mathbf{q}_t$, the coefficient b_{LSV} of this regression can be computed as described in (C.14).

Lloyd and Marin (2020) and Chernov and Creal (2021) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{e_t}{e_{t+\Delta\tau}}\right) = a_{\text{UIP}\ell s} + b_{\text{UIP}\ell}\left(y_{Ft}^{(\Delta\tau)} - y_{Ht}^{(\Delta\tau)}\right) + b_{\text{UIPs}}\left[\left(y_{Ft}^{(\tau_2)} - y_{Ft}^{(\tau_1)}\right) - \left(y_{Ht}^{(\tau_2)} - y_{Ht}^{(\tau_1)}\right)\right] + e_{t+\Delta\tau}.$$

The dependent variable is the rate of foreign currency depreciation over horizon $\Delta\tau$. The independent variables are the foreign-minus-home $\Delta\tau$-year yield differential and the foreign-minus-home slope differential between years τ_1 and τ_2. The coefficients b_{UIPs} and $b_{\text{UIP}\ell}$ of this regression can be computed as described in (C.14).

Fama and Bliss (1987) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{P_{j,t+\Delta\tau}^{(\tau-\Delta\tau)}}{P_{jt}^{(\tau)}}\right) - y_{jt}^{(\Delta\tau)} = a_{\text{FB}} + b_{\text{FB}}\left(f_{jt}^{(\tau-\Delta\tau,\tau)} - y_{jt}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the log return over horizon $\Delta\tau$ of the country-j bond with maturity τ in excess of the $\Delta\tau$-year spot rate (yield). The independent variable is the slope of the country-j term structure as measured by the difference between the forward rate between maturities $\tau - \Delta\tau$ and τ, and the $\Delta\tau$-year spot rate. Since log bond prices are affine functions of the state vector $\mathbf{q}_t$, and the forward rate is

$$f_{jt}^{(\tau-\Delta\tau,\tau)} = -\frac{\log\left(\frac{P_{jt}^{(\tau)}}{P_{jt}^{(\tau-\Delta\tau)}}\right)}{\Delta\tau},$$

the coefficient b_{FB} of this regression can be computed as described in (C.14).

Campbell and Shiller (1991) perform the regression

$$y_{j,t+\Delta\tau}^{(\tau-\Delta\tau)} - y_{jt}^{(\tau)} = a_{\text{CS}} + b_{\text{CS}}\frac{\Delta\tau}{\tau-\Delta\tau}\left(y_{jt}^{(\tau)} - y_{jt}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the change over horizon $\Delta\tau$ in the yield of a country-j bond with initial maturity τ. The independent variable is the difference between the country-j spot rates for maturities τ and $\Delta\tau$, normalized so that b_{CS} is equal to one under the EH. The coefficient b_{CS} of this regression can be computed as described in (C.14).

8. *Monetary Policy Transmission*

The top left and top right panels of Figure C.2 show, respectively, how the cut to the home short rate described in Section IV.D affects arbitrageur bond holdings at time zero as function of maturity, and how it affects their currency holdings over time. The bottom left and right panels show the same for the cut to the foreign short rate. Holdings of home bonds are shown in blue and of foreign bonds in red. Holdings are expressed as fraction of US GDP.

Following the cut to the home short rate, aggregate holdings of home bonds by arbitrageurs increase by 0.205% of GDP, as can be derived by integrating the blue line over maturities. The sharpest increase occurs for maturities around three years. Arbitrageur holdings of foreign bonds remain essentially unchanged—they decrease by 0.001% of GDP. Arbitrageur currency holdings increase by 0.751% of GDP following the cut, and then decline gradually to their pre-cut value. The responses to the foreign short rate's cut have similar magnitudes (with currency holdings decreasing following the cut).

The top left and top right panels of Figure C.3 show, respectively, how the QE purchases in the home country described in Section IV.D affect arbitrageur bond holdings at time zero as function of maturity, and how they affect their currency holdings over time. The bottom left and right panels show the same for the QE purchases in the foreign country. The coloring and units are as in Figure C.2.

Following the purchases of home bonds, aggregate holdings of home bonds by arbitrageurs decrease by 7.160% of GDP. Arbitrageurs thus sell to the central bank bonds worth 71.6% (=7.16/10) of all the bonds that the central bank purchases

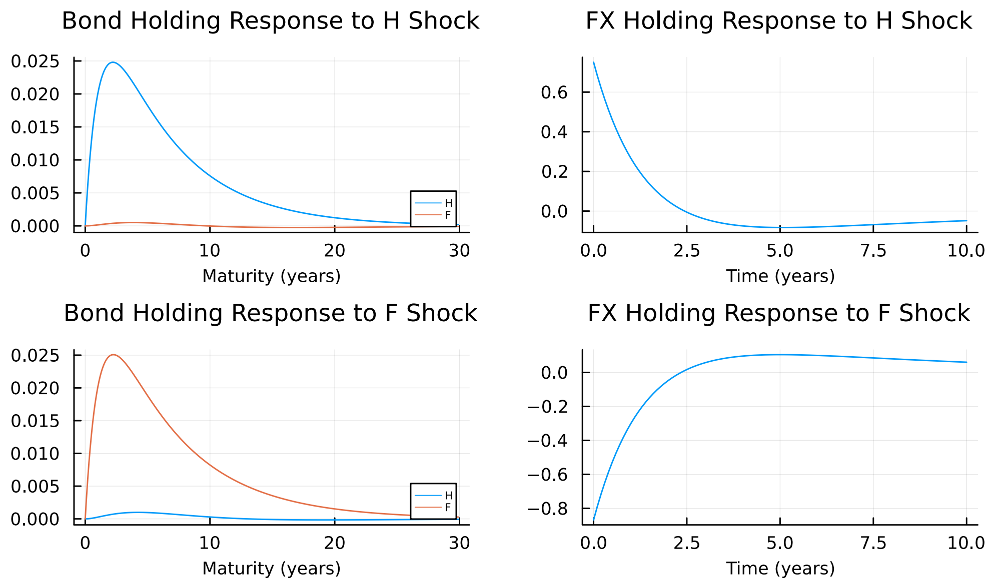

Figure C.2. Conventional monetary policy and arbitrageur holdings – Short rate cut.

(which are assumed to be worth 10% of GDP). The maturities that arbitrageurs sell the most are around five years. Arbitrageur holdings of foreign bonds increase by 1.105% of GDP, as they seek to partly replace the home bonds that they sell to the central bank by foreign bonds. Arbitrageur currency holdings increase by 0.844% of GDP following the cut, and then decline gradually to their pre-cut value. The responses to the purchases of foreign bonds have similar magnitudes (with currency holdings decreasing following the cut).

Figures C.4 and C.5 are the counterparts of Figures 4 and 5 for subsamples. For each of Figures C.4 and C.5, the left two columns correspond to the subsample 06/1986-12/2007 and the right two columns to the subsample 01/1999-04/2021. Our results are reasonably stable when viewed in conjunction with the confidence intervals in Figures 4 and 5.

Figure C.6 shows additional comparative statics of the effects of QE. We vary the slope parameter α_0 of bond demand, the slope α_e of currency demand, and the correlation parameter $\sigma_{iH,iF}$ between the home and the foreign short rate. All parameters except for the one that varies are set to their estimated values, except in the case of $\sigma_{iH,iF}$ where we also vary σ_{iF} to keep the standard deviation $\sqrt{\sigma_{iH,iF}^2 + \sigma_{iF}^2}$ of innovations to the foreign short rate constant. The estimated value of the parameter that varies corresponds to one in the x-axis by normalizing the units. The figure shows the ten-year yield and the exchange rate. Results

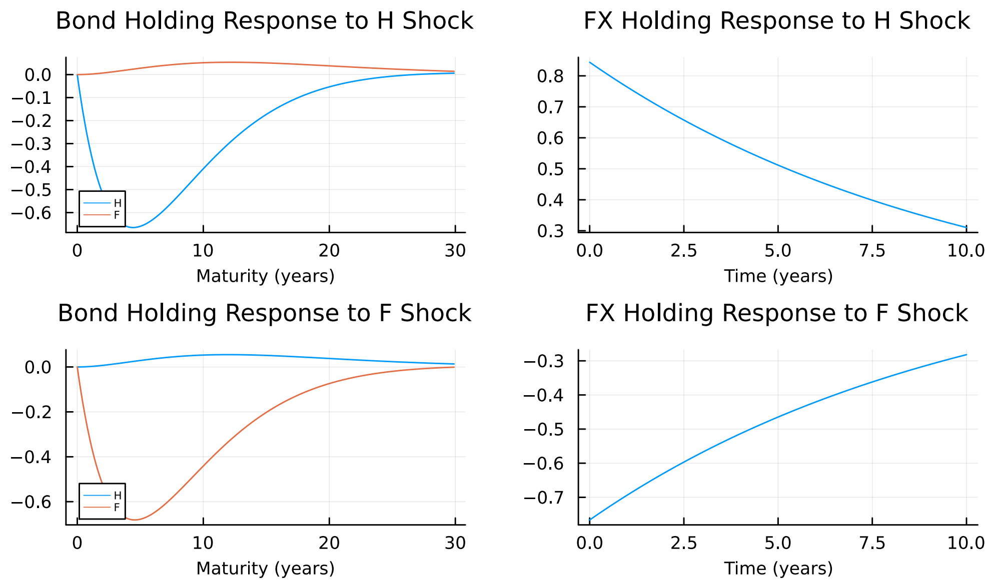

Figure C.3. Unconventional monetary policy and arbitrageur holdings – Bond purchases.

in the top row are for α_0, in the middle row for α_e, and in the bottom row for $\sigma_{iH,iF}$.

When α_0 increases, QE has weaker effects on domestic bond yields. This is because bond investors require a smaller price increase to sell domestic bonds to the central bank. As a result, arbitrageurs sell fewer domestic bonds to the central bank, and adjust less their hedges in currency and foreign bonds. Because of the arbitrageurs' reduced trading, QE has weaker effects on the exchange rate and foreign bond yields.

When α_e increases, QE has weaker effects on domestic bond yields. This is because arbitrageurs can adjust their hedge in currency with smaller price impact, and thus become more willing to sell domestic bonds to the central bank. Because arbitrageurs adjust more their currency hedge, they also adjust more their foreign-bond hedge. As a result, QE has stronger effects on foreign bond yields. The effects of QE on the exchange rate are non-monotone because on the one hand the adjustment to the currency hedge is larger but on the other hand price impact in the currency market is smaller.

When $\sigma_{iH,iF}$ increases, QE has stronger effects on domestic and foreign bond yields, and weaker effects on the exchange rate. The intuition for foreign bond yields is that arbitrageurs are better able to use foreign bonds to replace the domestic bonds that they sell to the central bank, causing foreign bond yields to decrease by more. The intuition for the exchange rate is that it becomes less

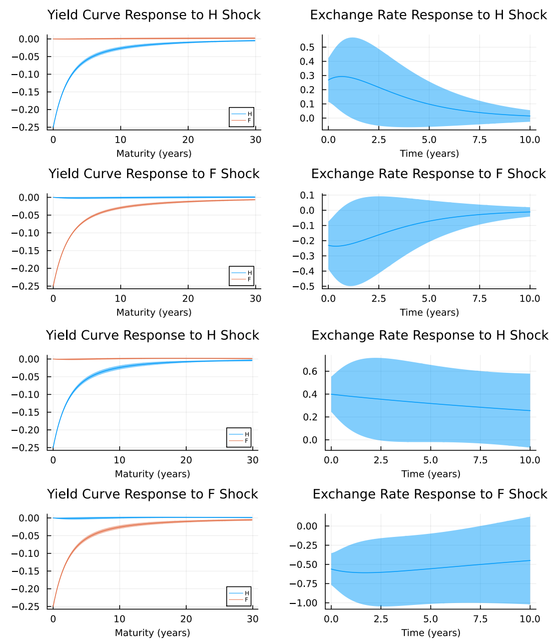

Figure C.4. Conventional monetary policy – Subsamples.

responsive to short-rate shocks. (In the limit of perfectly correlated short rates, the exchange rate is fixed.) Therefore, currency becomes less valuable as a hedge, and the exchange rate responds less to QE. The intuition for domestic yields is as follows. On the one hand, domestic yields should decrease by less because arbitrageurs are better able to use foreign bonds to replace the domestic bonds that they sell to the central bank. On the other hand, domestic yields should decrease by more because currency becomes less valuable as a hedge. Either effect can dominate, and the second one does under our estimated parameter values.

*

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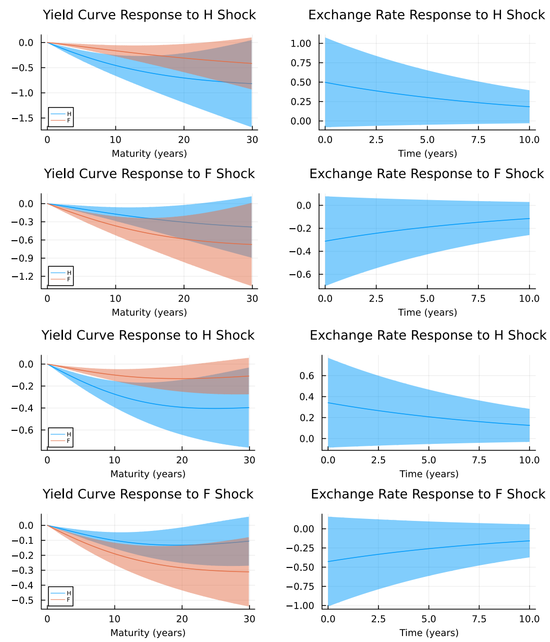

Figure C.5. Unconventional monetary policy – Subsamples.

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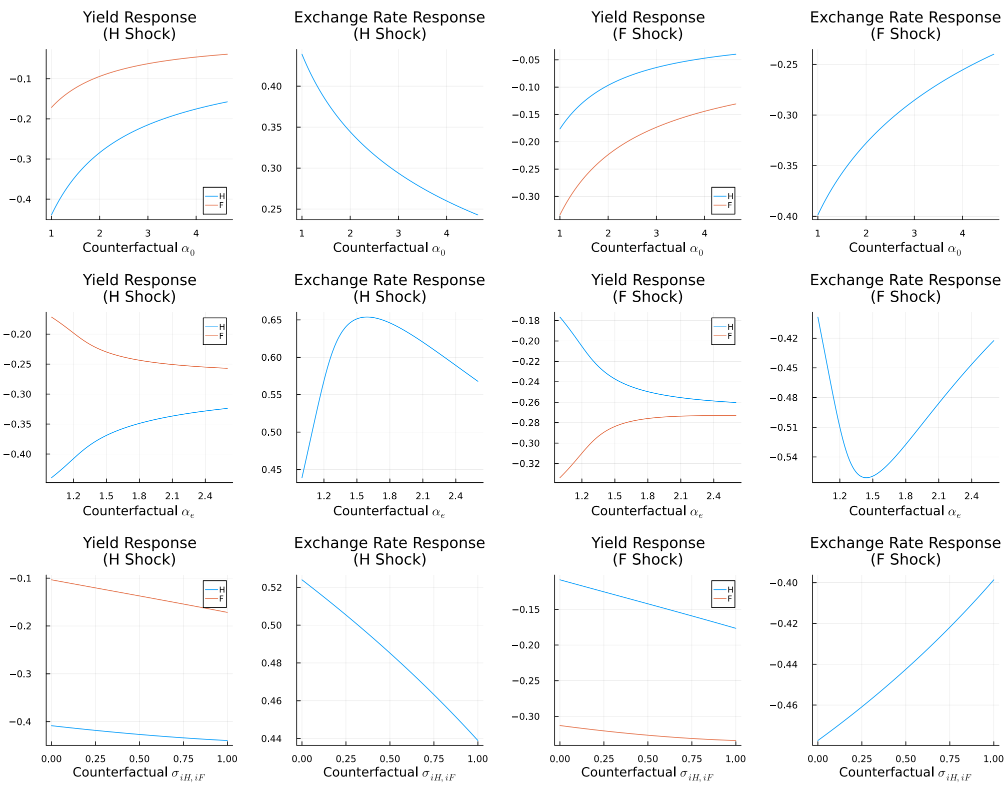

Figure C.6. Additional comparative statics of effects of QE.