Supplemental Appendices: Quota Adjustment Process

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A Appendix: Proofs

A.1 Proof of Theorem 1

Proof. For a fixed $q \in Q$ and $R \in \mathcal{R}^N$, there is a unique distribution-specific student-optimal stable matching. As the set Q of quota distributions is finite, the set of distribution-specific stable matchings at all quota distributions, $\bigcup_{q \in Q} S^q(R)$, is also finite. Here, $S^q(R)$ is the set of all distribution-specific stable matchings at (R, q). Hence, there is an ESOSM because it is a maximal element of the finite set of $\bigcup_{q \in Q} S^q(R)$ in the partial order of Pareto domination. □

A.2 Proof of Proposition 1

Proof. Let μ be an ESOSM and q be the corresponding quota distribution. Suppose that at least one of the three conditions does not hold true. If (1) fails, this violates the distribution-specific stability of μ at q, which is a contradiction. If (2) fails, then $\sum_{y \in X_{k(x)}} |\mu^{-1}(y)| < \bar{q}_{k(x)}$. Thus, there is a vacant seat in some program $y \in X_{k(x)}$. Then, choose the student, denoted by j, with the highest priority among those who prefer x to the matched program in μ. Let a new matching ν be one in which only j's position changes to x from μ. Let p be a quota distribution in which $p_x = q_x + 1$ and $p_y = q_y - 1$. Then, $\mu \in \mathcal{M}$ and is distribution-specific stable at p. Clearly, ν Pareto dominates μ, which is a contradiction. If (3) fails, then new matching ν is stable and Pareto dominates μ, which is a contradiction. □

A.3 Proof of Proposition 2

Proof. By supposition, there must exist $k \in K$ such that $|X_k| \geq 2$ and $\bar{q}_k < \sum_{x \in X_k} \bar{q}_x$. Fix k.

Since $\sum_{x \in X_k} t_x \leq \bar{q}_k$, we first construct a modified target capacity t' for department k as follows.

- If $\sum t_x = \bar{q}_k$, we are done and set $t' = t$.
- Otherwise, repeatedly increase t_x by one up to $\bar{q}_x$ according to the exogenous order (returning to the first program after the last program is reached).
- Once $\sum t'_x = \bar{q}_k$, stop the procedure and we obtain t'.

Let x be a program in which $t'_x < \bar{q}_x$. Since $|X_k| \geq 2$, pick an arbitrary $y \in X_k \setminus \{x\}$. Then we can find distinct sets of students, $(U_w)_{w \in X_k}$, such that for all $w \in X_k \setminus \{y\}$, $|U_w| = t'_w$ and $|U_y| = t'_y - 1$, and for all $\ell \in N \setminus \bigcup_{w \in X_k} U_w$, for each $w \in X_k$,[20]

$$\ell' \succ_w \ell, \forall \ell' \in U_w.$$

Note that since $|N| > \bar{q}_k$, $\left|N \setminus \left(\bigcup_{w \in X_k} U_w\right)\right| \geq 2$. Let i and j be such that $i, j \in N \setminus \left(\bigcup_{w \in X_k} U_w\right)$ and $i \succ_y j$.

We claim that if students have the following preferences, then there exists an ESOSM which Pareto dominates the FDA outcome.

R_i	R_j	R_{U_w}	R_{other}
x	y	w	$\emptyset$
y	$\emptyset$	$\emptyset$	
$\emptyset$			

Since the FDA outcome does not depend on the turns of students in (2) of FDA, we consider the step after all students in U_w for all $w \in X_k$ apply to their most preferable programs. Suppose that student j is chosen in (2) of FDA. Since j prefers y the most and has not been rejected yet, she applies to y. Since there are $|X_k|$ students in the current entire applicant pool at k, j is accepted by y. Note that all programs in X_k tentatively accept exactly t'_w students.

[20]One way to construct those sets is as follows: pick any program z, and choose t'_z students in order of priority at z. Then pick any program z' from among the remaining programs and choose $t'_{z'}$ students in order of priority at z' from among the remaining students, and so on.

Now i is chosen in (2) of FDA, and applies to x. Since program x has $t'_x + 1$ applicants and all other programs have t'_w applicants, i is rejected by x in (4b) of FDA by the construction of t'. In the next step, i is chosen in (2) of FDA, and then applies to y. By the construction of t', program y must reject one student in the current applicant pool, $U_y \cup \{i, j\}$. Since j is the lowest priority among them, j is rejected at y and the FDA terminates.

In sum, the FDA outcome is

$$\begin{pmatrix} i & j & U_w & other \\ y & \emptyset & w & \emptyset \end{pmatrix},$$

which is Pareto dominated by an ESOSM

$$\begin{pmatrix} i & j & U_w & other \\ x & \emptyset & w & \emptyset \end{pmatrix},$$

as desired. □

A.4 Proof of Claim 1

Proof. ($\Rightarrow$) Suppose by contradiction that for some k, $|N| > \bar{q}_k$ and $\sum_{x \in X_k} \bar{q}_x > \bar{q}_k$. Since $\sum_{x \in X_k} \bar{q}_x > \bar{q}_k$, for any $q \in Q$ there exists $x \in X_k$ such that $q_x < \bar{q}_x$.

Fix $q \in Q$. Let x be a program such that $q_x < \bar{q}_x$. Since $|N| > \bar{q}_k$, we can find distinct sets of students, $(U_w)_{w \in X_k}$, such that for all $w \in X_k$, $|U_w| = q_w$, and for all $\ell \in N \setminus \bigcup_{w \in X_k} U_w$,

$$\ell' >_w \ell, \forall \ell' \in U_w.$$

Let i be the highest priority at x among $N \setminus \left(\bigcup_{w \in X_k} U_w\right)$, and let j be the lowest priority at y among U_y for an arbitrary $y \in X_k \setminus \{x\}$.

Consider the following preferences:

R_i	R_{U_w}	R_{other}
x	w	$\emptyset$
$\emptyset$	$\emptyset$	

Note that $\left|\bigcup_{w \in X_k} U_w\right| = \bar{q}_k$ and $|U_x \cup \{i\}| \leq \bar{q}_x$.

Then there are two ESOSMs:

$$\mu = \begin{pmatrix} i & j & U_y \setminus \{j\} & U_w & other \\ x & \emptyset & y & w & \emptyset \end{pmatrix}, \nu = \begin{pmatrix} i & j & U_y \setminus \{j\} & U_w & other \\ \emptyset & y & y & w & \emptyset \end{pmatrix},$$

a contradiction.

($\Leftarrow$) Consider a matching problem without any quota adjustment, $(N, X, \bar{q}, R, \succ)$, for the given matching problem with quota adjustment. Since for all k, $|N| \leq \bar{q}_k$ or $\sum_{x \in X_k} \bar{q}_x = \bar{q}_k$, any matching of the above matching problem is in $\mathcal{M}$, the set of all feasible matchings at implementable quota distributions. It is obvious that M is a subset of all matchings of the above problem. Thus we have

$$\mathcal{M} = \mathcal{M}(\bar{q}).$$

Fix arbitrary R. Consider the DA matching of the above problem, $DA^{\bar{q}}(R)$. Note that since for any $q \in Q$, $\sum_{x \in X_k} q_x = \bar{q}_k$ for any k and $q \leq \bar{q}$, any distribution-specific student-optimal stable matching is either stable or Pareto dominated by some stable matching in the above problem. Since the DA matching is unique, and it Pareto dominates any other stable matching in the matching problem without any quota adjustment, $DA^{\bar{q}}(R)$ is a unique ESOSM of the original matching problem with quota adjustment. □

A.5 Proof of Theorem 2

We use the following two propositions to prove Theorem 2.

Proposition 5. *Suppose that a matching μ is distribution-specific stable at a quota distribution q. If there is a QASIC for μ, $\langle (x_\ell, \iota_\ell) \rangle_{\ell=0}^m$, and a matching ν and quota distribution p are constructed in a manner so that*

$$\nu(j) = \begin{cases} x_\ell & \text{if } j = \iota_\ell, \forall \ell \in \{0, \cdots, m-1\} \text{ and } \iota_\ell \text{ is a student} \\ \emptyset & \text{if } j = \iota_\ell, \forall \ell \in \{0, \cdots, m-1\} \text{ and } \iota_\ell \text{ is a dummy student} \\ \mu(j) & \text{otherwise} \end{cases}$$

and if $x_\ell \neq \mu(\iota_{\ell-1})$ for some $\ell \in \{0, \ldots, m-1\}$,

$$p_{x_\ell} = q_{x_\ell} + 1,\ p_{\mu(\iota_{\ell-1})} = q_{\mu(\iota_{\ell-1})} - 1,$$

otherwise $p_x = q_x$. Then ν is distribution-specific stable at p.

Proposition 6. *Suppose that μ and ν are distribution-specific stable matchings at q and p, respectively. If ν Pareto dominates μ, then there is a QASIC for μ.*

The propositions will be proved in the next subsections. Proposition 5 says that a QASIC preserves stability. More important is Proposition 6 which is about whether we could always reach an ESOSM. Note that Proposition 6 does not say that even if ν Pareto dominates μ, applying a QASIC for μ results in ν.

By applying the above propositions, we prove Theorem 2. Let μ be distribution-specific stable at q.

($\Rightarrow$) Suppose that there is a QASIC for μ. Then it follows from Proposition 5 that the constructed matching ν is distribution-specific stable at p, and it is obvious that ν Pareto dominates μ. Hence, μ is not an ESOSM.

($\Leftarrow$) Suppose that μ is not an ESOSM. Then there exists $p \in Q$ and a distribution-specific stable matching $\nu \in \mathcal{M}(p)$ at p such that ν Pareto dominates μ. Thus it follows from Proposition 6 that there exists a QASIC for μ.

A.5.1 Proof of Proposition 5

By the definition of QASICs, if q is implementable, then p is straightforwardly implementable. Thus it suffices to show that ν is distribution-specific stable at p. Suppose by contradiction that ν is not distribution-specific stable at p. Then there is a blocking pair (j, x) for ν at p, so $x \ P_j \ \nu(j)$. Since ν Pareto dominates μ, we have $x \ P_j \ \mu(j)$. Thus, since μ is distribution-specific stable at q, we have $|\mu^{-1}(x)| = q_x$. Suppose that x is involved in the QASIC. Then an increase, decrease, or no change of q_x is made in association with a transfer of a student, so $|\nu^{-1}(x)| = p_x$. Suppose that x is not involved in the QASIC. Then $|\nu^{-1}(x)| = |\mu^{-1}(x)| = q_x = p_x$. Hence, since (j, x) is a blocking pair for ν at p, there is $h \in \nu^{-1}(x)$ such that $j \succ_x h$. There are two cases.

(Case 1) x is a program involved in the QASIC.

If $h \in \mu^{-1}(x) \cap \nu^{-1}(x)$, then it follows from the previous paragraph that $x \ P_j \ \mu(j)$ and $j \succ_x h$, i.e., μ is not distribution-specific stable at q, a contradiction. On the other hand, if $h \notin \mu^{-1}(x) \cap \nu^{-1}(x)$, then h is a member of some unit chain (x, h) in the QASIC. Since $x \ P_j \ \mu(j)$ by the previous paragraph, $j \in \{i \mid x \ P_i \ \mu(i)\}$, and thus $h \succ_x j$, a contradiction.

(Case 2) x is a program not involved in the QASIC.

Then $\nu^{-1}(x) = \mu^{-1}(x)$ and $p_x = q_x$. Thus $h \in \mu^{-1}(x)$ and $j \succ_x h$, which means that (j, x) is a blocking pair of μ at q. This contradicts the distribution-specific stability of μ. $\square$

A.5.2 Proof of Proposition 6

We begin by introducing the following notion to prove the proposition.

Definition 6. We say that a distribution-specific stable matching ν at $p \in Q$ **minimally Pareto dominates** another distribution-specific stable matching μ at $q \in Q$ if ν Pareto dominates μ and there is no distribution-specific stable matching η at some $q' \in Q$ such that ν Pareto dominates η and η Pareto dominates μ.

Let μ and ν be distribution-specific stable matchings at $q \in Q$ and $p \in Q$, respectively. We assume, without loss of generality, that ν minimally Pareto dominates μ. We then extend matchings μ and ν to $\bar{\mu}$ and $\bar{\nu}$ in the following extended problem so that we add the set of dummy students D and treat $\varnothing$ as a real program.

The extended problem for μ is

$$\left(N \cup D, X \cup \{\varnothing\}, (R_\iota)_{\iota \in N \cup D}, (\succsim_x^{\mu})_{x \in X \cup \{\varnothing\}}, Q \times \{\bar{q}_{k_\varnothing}\}, K \cup \{k_\varnothing\}\right)$$

where

- D is a set of dummy students and D satisfies

$$|D| = \sum_{k \in K} \max\left\{\sum_{x \in k}\left(|\nu^{-1}(x)| - |\mu^{-1}(x)|\right), 0\right\},$$

- $\varnothing$ is treated as a real program with quota satisfying $q_\varnothing = |\mu^{-1}(\varnothing)|$, and which belongs to department $k_\varnothing$ with $\bar{q}_{k_\varnothing} = q_\varnothing$ which consists of only one program; that is, $X_{k_\varnothing} = \{\varnothing\}$,
- for all $x \in X$, $\succsim_x^{\mu}$ is a complete and transitive order over $N \cup D$ in which for all $i, j \in N$,

$$i \succ_x j \Rightarrow i \succ_x^{\mu} j,$$

for all $i \in N$ and $d \in D$,

$$i \succ_x^\mu d$$

for all $d, d' \in D$,

$$d \sim_x^\mu d'$$

and $\succsim_\varnothing^\mu$ is a complete and transitive order over $N \cup D$ in which for all $i \in N$ and all $d \in D$,

$$i \succ_\varnothing^\mu d,$$ [21]

and

- for all $d \in D$, R_d entails complete and transitive preferences over $X \cup \{\varnothing\}$ in which for all $x, y \in X \cup \{\varnothing\}$,

$$xI_d y.$$ [22]

In this problem, we extend μ and ν as follows:

$\bar{\mu}$: for all $i \in N$, $\bar{\mu}(i) = \mu(i)$ and each dummy student is arbitrarily matched with a program in which $|\nu^{-1}(x)| \neq |\mu^{-1}(x)|$ among departments such that $\sum_{x \in X_k} \left(|\nu^{-1}(x)| - |\mu^{-1}(x)|\right) > 0$.

$\bar{\nu}$: for all $i \in N$, $\bar{\nu}(i) = \nu(i)$ and each dummy student is arbitrarily matched with a program in which $|\nu^{-1}(x)| \neq |\mu^{-1}(x)|$ among departments such that $\sum_{x \in X_k} \left(|\nu^{-1}(x)| - |\mu^{-1}(x)|\right) < 0$ and vacant seats of program $\varnothing$ if any.

Thus all students and dummy students are matched with some program both in $\bar{\mu}$ and $\bar{\nu}$ at $(q, q_\varnothing)$ and $(p, q_\varnothing)$ respectively. Note that if for some k, $\sum_{x \in X_k} \left(|\nu^{-1}(x)| - |\mu^{-1}(x)|\right) > 0$, then since $q_k = \sum_{x \in X_k} q_x = \sum_{x \in X_k} p_x \geq \sum_{x \in X_k} |\nu^{-1}(x)| > \sum_{x \in X_k} |\mu^{-1}(x)|$, there must exist vacant seats at μ. Note also that no dummy student is matched with a program in the same department between $\bar{\nu}$ and $\bar{\mu}$.

Lemma 1. *$\bar{\mu}$ and $\bar{\nu}$ are distribution-specific stable at $(q, q_\varnothing)$ and $(p, q_\varnothing)$, respectively. Furthermore, $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$.*[23]

Proof. As noted above, $\bar{\mu}$ is feasible at $(q, q_\varnothing)$. Since no student $i \in N$ prefers $\varnothing$ to $\mu(i)$ by the individual rationality of μ, no pair of $i \in N$ and $x \in X \cup \{\varnothing\}$ blocks $\bar{\mu}$ for R at $(q, q_\varnothing)$. For all

[21] We write $i \succ_x^\mu j$ if $i \succsim_x^\mu j$ and not $j \succsim_x^\mu i$, and $i \sim_x^\mu j$ if $i \succsim_x^\mu j$ and $j \succsim_x^\mu i$, respectively.
[22] We write $xI_d y$ if $xR_d y$ and $yR_d x$.
[23] We adopt the same definition of distribution-specific stability in this extended problem as before.

$d \in D$, d finds any program indifferent. Thus $\bar{\mu}$ is distribution-specific stable at $(q, q_\varnothing)$. A similar discussion applies to $\bar{\nu}$ at $(p, q_\varnothing)$.

By construction of the preferences of dummy students, each one finds her assignment at $\bar{\mu}$ indifferent to that at $\bar{\nu}$. Furthermore, for all $x \in X$, q_x and p_x are unchanged in the extended problem from the original problem, and no student i prefers $\varnothing$ to $\mu(i)$ (thus $\nu(i)$). From these facts, since ν minimally Pareto dominates μ, $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. □

Let $N' = \{i \in N | \bar{\mu}(i) \neq \bar{\nu}(i)\}$.[24] Denote a pair of programs at $\bar{\nu}$ and $\bar{\mu}$ of student $i \in N'$ by

$$(\bar{\nu}(i), \bar{\mu}(i))$$

and a pair of programs at $\bar{\nu}$ and $\bar{\mu}$ of dummy student $d \in D$ by

$$(\bar{\nu}(d), \bar{\mu}(d)).$$

Consider a directed graph (V, E) where a set of vertices $V = K \cup \{k_\varnothing\}$ and a set of directed edges

$$E = \{(k', k'') \in [K \cup \{k_\varnothing\}] \times [K \cup \{k_\varnothing\}] \mid k(\bar{\nu}(\iota)) = k' \text{ and } k(\bar{\mu}(\iota)) = k'', \text{ for some } \iota \in N' \cup D\}.$$

That is, for all $\iota \in N' \cup D$, and for all pairs $(\bar{\nu}(\iota), \bar{\mu}(\iota))$ such that $k(\bar{\nu}(\iota)) = k'$ and $k(\bar{\mu}(\iota)) = k''$, we write a directed edge k'' to k'. We say that an ordered set of pairs $\langle(\bar{\nu}(\iota_s), \bar{\mu}(\iota_s))\rangle_{s=0}^{t}$ $(t \geq 1)$ constitutes a cycle if $\iota_0 = \iota_t$ and $k(\bar{\mu}(\iota_s)) = k(\bar{\nu}(\iota_{s+1}))$ for all $s \in \{0, 1, \ldots, t-1\}$. We denote a cycle by C which is formed visually as follows:

$$C : \bar{\nu}(\iota_0) \leftarrow \underbrace{\bar{\mu}(\iota_0), \bar{\nu}(\iota_1)}_{k} \leftarrow \underbrace{\bar{\mu}(\iota_1), \bar{\nu}(\iota_2)}_{k'} \leftarrow \ldots \leftarrow \underbrace{\bar{\mu}(\iota_{t-1}), \bar{\nu}(\iota_t)}_{k''} \leftarrow \underbrace{\bar{\mu}(\iota_t), \bar{\nu}(\iota_0)}_{k'''} \leftarrow \bar{\mu}(\iota_0).$$

Since we add dummy students in a way such that the total numbers of matched students and dummy students at each k are constant across $\bar{\mu}$ and $\bar{\nu}$, it is obvious by this construction that each $k \in K$ and $k_\varnothing$ have the same numbers (possibly none) of incoming and outgoing edges. Hence there must exist cycles $\{C^m\}$, $m \in \{1, 2, \ldots, n\}$, in which for each $\iota \in N' \cup D$, ι is involved in the exactly one cycle.

[24]Since $\bar{\nu}$ Pareto dominates $\bar{\mu}$, this set is equivalent to $\{i \in N | \bar{\nu}(i) P_i \bar{\mu}(i)\}$.

Note that there may be multiple cycles at this stage, but it is important that all students and dummy students are involved in exactly one cycle. For the sake of exposition, for each C^m, the set of programs which are involved in $\bar{\nu}$ and $\bar{\mu}$ are denoted by $C^m_{\leftarrow}$ and $C^m_{\rightarrow}$, respectively.

There are potentially multiple ways to construct such cycles, but the following lemma indicates that we can find cycles in which if $\bar{\mu}(\iota_s) \neq \bar{\nu}(\iota_{s+1})$ then $|\bar{\mu}^{-1}(\bar{\nu}(\iota_{s+1}))| < \bar{q}_{\bar{\nu}(\iota_{s+1})}$. Note that no cycle is constituted only of dummy students since in each k if there are outgoing edges by dummy students then there are no incoming edges by dummy students and vice versa.

Lemma 2. *Among the sets of cycles constructed above, there must exist the set of cycles* $\{C^m\}$, $m \in \{1, 2, \ldots, n\}$, *such that if* $\bar{\mu}(\iota_s) \neq \bar{\nu}(\iota_{s+1})$ *then* $|\bar{\mu}^{-1}(\bar{\nu}(\iota_{s+1}))| < \bar{q}_{\bar{\nu}(\iota_{s+1})}$.

Proof. Since all of $N' \cup D$ are involved in exactly one cycle, if all $\iota \in N' \cup D$ exchange their matched programs at $\bar{\mu}$ along with $(\bar{\nu}(\iota), \bar{\mu}(\iota))$ then we obtain $\bar{\nu}$. The fact that $\bar{\nu}$ is feasible at $(p, q_\varnothing)$ implies that the desired cycles exist. □

We then claim that the set of cycles satisfying the above condition consists of exactly one cycle.

Lemma 3. *The set of cycles found in Lemma 2 is constituted of exactly one cycle.*

Proof. Suppose not. If there is more than one distinct cycle, then we construct new matchings for each cycle as follows: for each $m \in \{1, 2, \ldots, n\}$,

$$\eta^{C^m}(\iota) = \begin{cases} \bar{\nu}(\iota) & \iota \in C^m \\ \bar{\mu}(\iota) & otherwise \end{cases}$$

and p^m updates q sequentially (p^m_ν then p^m_μ),

$$\begin{aligned} p^m_{\nu(\iota)} &= q_{\nu(\iota)} + 1 && \iota \in C^m \\ p^m_{\mu(\iota)} &= q_{\mu(\iota)} - 1 && \iota \in C^m \\ p^m_x &= q_x \ (x \in X \cup \{\varnothing\}) && otherwise. \end{aligned}$$

Since exchanges along with all the cycles at $\bar{\mu}$ preserve feasibility and those cycles are distinct, $\{\eta^{C^m}\}$ are all feasible at $(p^m, q_\varnothing)$, respectively. If there exists some m such that η^{C^m} is distribution-specific stable at $(p^m, q_\varnothing)$, then it contradicts the fact that $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. Thus we assume that all $\{\eta^{C^m}\}$ are not distribution-specific stable. That is, since η^{C^m} is individually rational, for each η^{C^m} there exists a pair of j^m and y^m that blocks η^{C^m}. Since $\bar{\mu}$ is distribution-specific stable at $(q, q_\emptyset)$, $y^m \in C^m_{\leftarrow}$.

For all $y^m \in C^m_{\leftarrow}$, let j^{y^m} be of the highest priority among those who desire y^m at $\bar{\mu}$ if such a student exists, and be a dummy student involved in $(y^m = \bar{\nu}(d), \bar{\mu}(d))$ if there is no student who desires y^m at $\bar{\mu}$. Denote them by $\{j^{y^m_0}, \ldots, j^{y^m_{t-1}}\}$, and possibly $j^{y^m_s} = j^{y^m_{s'}}$ for some s, s' with $s \neq s'$. For any C^m, if j^{y^m} is not a dummy student, then $j^{y^m} \in N'$, for otherwise $j^{y^m} \in N \setminus N'$ whose matched program does not change among $\bar{\mu}, \eta^{C^1}, \ldots, \eta^{C^n}$, and $\bar{\nu}$, which contradicts the distribution-specific stability of $\bar{\nu}$. Since $\bar{\nu}$ is distribution-specific stable, for all m,

$$\bar{\nu}(j^m) R_{j^m} y^m P_{j^m} \bar{\mu}(j^m).$$

For each m, if $j^{y^m_0}, \ldots, j^{y^m_{t-1}}$ are all involved in C^m, we find a shorter cycle by adding $(y^m_s, \bar{\mu}(j^{y^m_s}))$ and replacing the original $(y^m_s, \bar{\mu}(\iota))$ with $(y^m_s, \bar{\mu}(j^{y^m_s}))$ in a sequential manner. That is,

- We begin with y^m_s, and replace $(y^m_s, \bar{\mu}(\iota))$ with $(y^m_s, \bar{\mu}(j^{y^m_s}))$.
- Then we consider pair $(y^m_{s'} = \nu(\iota'), \mu(\iota'))$ next to pair $(\bar{\nu}(j^{y^m_s}), \bar{\mu}(j^{y^m_s}))$ in C^m.
  - If ι' is a dummy student or is of the highest priority among those who desire $\nu(\iota')$ at $\bar{\mu}$, namely $\iota' = j^{y^m_{s'}}$, then we proceed to the pair next to $(\bar{\nu}(\iota'), \bar{\mu}(\iota'))$. Note that $\iota' \neq j^{y^m_s}$.
  - Otherwise $\iota' \neq j^{y^m_{s'}}$ and we replace $(y^m_{s'}, \mu(\iota'))$ with $(y^m_{s'}, \bar{\mu}(j^{y^m_{s'}}))$. Note that $j^{y^m_{s'}} \neq j^{y^m_s}$.
- Then we consider pair $(y^m_{s''} = \nu(\iota''), \mu(\iota''))$ next to pair $(\bar{\nu}(j^{y^m_{s'}}), \bar{\mu}(j^{y^m_{s'}}))$ in C^m.
  - If ι'' is a dummy student or is of the highest priority among those who desire $\nu(\iota'')$ at $\bar{\mu}$, namely $\iota'' = j^{y^m_{s''}}$, then we proceed to the pair next to $(\bar{\nu}(\iota''), \bar{\mu}(\iota''))$. Note that $\iota'' \neq j^{y^m_{s''}}$.
  - If $\iota'' \neq j^{y^m_{s''}}$ and $j^{y^m_{s''}} = j^{y^m_s}$, then we find a shorter cycle such that

$$y^m_{s'} \leftarrow \underbrace{\bar{\mu}(j^{y^m_{s'}}), y^m_{s''}}_{k'} \leftarrow \underbrace{\bar{\mu}(j^{y^m_s}), y^m_{s'}}_{k''} \leftarrow \bar{\mu}(j^{y^m_{s'}}).$$

  - Otherwise $\iota'' \neq j^{y^m_{s''}}$ and $\iota'' \neq j^{y^m_s}$, and we replace $(y^m_{s''}, \mu(\iota''))$ by $(y^m_{s''}, \bar{\mu}(j^{y^m_{s''}}))$. Note that $j^{y^m_{s''}} \neq j^{y^m_{s'}}$.

Since all of $j^{y_1^m}, \ldots, j^{y_t^m}$ are involved in C^m and $C^m_{\leftarrow}$ is finite, this process finds a shorter cycle than C^m. Then a matching induced by exchanging programs at $\bar{\mu}$ along with the above shorter cycle is feasible, distribution-specific stable and Pareto dominates $\bar{\mu}$, which contradicts the fact that $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. Hence, for each m, there exists $y_s^m \in C^m_{\leftarrow}$ and $j^{y_s^m}$ who desires y_s^m at $\bar{\mu}$, is of the highest priority among those who desire y_s^m at $\bar{\mu}$ and is not involved in C^m.

Now we consider $y_s^m \in C^m_{\leftarrow}$ for all m, s. As above, for all m there must exist j_s^m involved in C^m who is of the highest priority among those who desire $y_{s'}^{m'} \in C^{m'}_{\leftarrow}$ for some $m' \neq m$. Then we find a new cycle such that students who are of the highest priority at $\bar{\mu}$ are involved in different cycles. By construction, exchanges along with this new cycle induces a matching which is not equal to $\bar{\nu}$ that is feasible, distribution-specific stable, and Pareto dominates $\bar{\mu}$, which violates the fact that $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. Therefore, the set of cycles found in Lemma 2 is constituted by exactly one cycle. $\square$

From the above lemmata, we denote C by the cycle found above.

Lemma 4. *No $x \in C_{\leftarrow} \cap X$ appears more than once.*

Proof. Let C be $\langle(\bar{\nu}(\iota_s), \bar{\mu}(\iota_s))\rangle_{s=0}^t$. Suppose that there exists $x \in X$ which appears more than once in the cycle. That is, there are at least two distinct students $\iota_{s'}$ and $\iota_{s''}$ (we assume without loss of generality that $s' < s''$). We know by Lemma 2 that C satisfies $k(\bar{\mu}(\iota_s)) = k(\bar{\nu}(\iota_{s+1}))$, $\forall s \in \{0, 1, \ldots, t-1\}$ and if $\bar{\mu}(\iota_s) \neq \bar{\nu}(\iota_{s+1})$ then $\bar{\mu}^{-1}(\bar{\nu}(\iota_{s+1})) < \bar{q}_{\bar{\nu}(\iota_{s+1})}$.

We next decompose C into two cycles, C^1 and C^2, as follows:

$$C^1 : x = \bar{\nu}(\iota_{s'}) \leftarrow \underbrace{\bar{\mu}(\iota_{s'}), \bar{\nu}(\iota_{s'+1})} \leftarrow \bar{\mu}(\iota_{s'+1}), \cdots, \bar{\nu}(\iota_{s''-1}) \leftarrow \underbrace{\bar{\mu}(\iota_{s''-1}), x = \bar{\nu}(\iota_{s'})}_{k'} \leftarrow \bar{\mu}(\iota_{s'})$$

and

$$C^2 : \bar{\nu}(\iota_0) \leftarrow \bar{\mu}(\iota_0), \cdots, \bar{\nu}(\iota_{s'-1}) \leftarrow \underbrace{\bar{\mu}(\iota_{s'-1}), x = \bar{\nu}(\iota_{s''})}_{k'} \leftarrow \bar{\mu}(\iota_{s''})), \cdots, \bar{\nu}(\iota_0) \leftarrow \bar{\mu}(\iota_0).$$

Then, since the above two cycles C^1 and C^2 satisfy $k(\bar{\mu}(\iota_{s''-1})) = k(\bar{\nu}(\iota_{s''})) = k(x) = k(\bar{\nu}(\iota_{s'}))$ and $k(\bar{\mu}(\iota_{s'-1})) = k(\bar{\nu}(\iota_{s'})) = k(x) = k(\bar{\nu}(\iota_{s''}))$, respectively, C^1 and C^2 are the set of cycles found in Lemma 2, which contradicts Lemma 3. Therefore, $x \in C_{\leftarrow} \cap X$ appears just once. $\square$

Lemmata 3 and 4 tell us that there exists a unique cycle and all programs in $C_{\leftarrow}$ but $\varnothing$ are distinct. We further see the students' priority in N'.

Lemma 5. *For each* $x = \bar{\nu}(i_s) \in \nu(N')$*,* $i_s \succsim_x^{\mu} j$*,* $\forall j \in \{j \in N | x P_j \bar{\mu}(j)\}$*.*

Proof. As before, all students in N' are involved in C; that is, $\bar{\nu}(N') = C_{\leftarrow}$. Suppose not. Then there exists $x \in C_{\leftarrow}$ and j such that $j \succsim_x^{\mu} i_s$. Obviously, $j \notin D$. We know that $j \notin N \setminus N'$, otherwise violating the distribution-specific stability of $\bar{\nu}$. Thus $j \in N'$. We then find a shorter cycle similar to the process in Lemma 3, and exchanges along with the shorter cycle at $\bar{\mu}$ induce a matching which is not equal to $\bar{\nu}$ that is feasible, distribution-specific stable and that Pareto dominates $\bar{\mu}$, which contradicts the fact that $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. $\square$

The next lemma allows us to consider at most one dummy student at each $k \in K$.

Lemma 6. *For each* k *involved in* C*, there is at most one* d *in* X_k *at* $\bar{\mu}$*.*

Proof. Suppose that there are more than one dummy student who are matched with some program at k in C, namely,

$$(\bar{\nu}(d), \bar{\mu}(d)), \dots, (\bar{\nu}(d'), \bar{\mu}(d')).$$

Since $\bar{\mu}(d), \bar{\mu}(d') \in k$, we have the shorter cycle which replaces pairs between $(\nu(d), \mu(d))$ and $(\nu(d'), \mu(d'))$ in C with $(\nu(d), \mu(d'))$.

Since there is only one cycle involved with distinct programs and each student i involving $(\bar{\nu}(i), \bar{\mu}(i))$ is of the highest priority among those who desire $\bar{\nu}(i)$ at $\bar{\mu}$ by Lemma 5, a matching obtained by exchanges along with the shorter cycle above preserves feasibility and distribution-specific stability, and Pareto dominates $\bar{\mu}$, which violates the fact that $\bar{\nu}$ minimally Pareto dominates $\bar{\mu}$. $\square$

Eventually, we obtain cycle $\langle(\bar{\nu}(\iota_s), \bar{\mu}(\iota_s))\rangle_{s=0}^{t}$ such that

- $\iota_0, \cdots, \iota_t$ are distinct, and at least one ι_s is a student,
- $\iota_s \in N' \Rightarrow \iota_s \succsim_{\bar{\nu}(\iota_s)}^{\mu} j$, $\forall j \in \{j \in N | \bar{\nu}(\iota_s) P_j \bar{\mu}(j)\}$,
- $\bar{\mu}(\iota_{s-1}) \neq \emptyset \Rightarrow \bar{\mu}(\iota_{s-1}) \in X_{k(\bar{\nu}(\iota_s))}$ and $\bar{\mu}(\iota_{s-1}) = \emptyset \Rightarrow \bar{\mu}(\iota_{s-1}) = \bar{\nu}(\iota_s)$,

  for the latter part, since $X_{k_\emptyset} = \{\emptyset\}$ and $\bar{\mu}(\iota_{s-1}) \in X_{k(\bar{\nu}(\iota_s))}$, $\bar{\mu}(\iota_{s-1}) = \bar{\nu}(\iota_s)$.
- $\bar{\mu}(\iota_{s-1}) \neq \bar{\nu}(\iota_s)$ implies $q_{\bar{\nu}(\iota_s)} < \bar{q}_{\bar{\nu}(\iota_s)}$.

Now we would like to show that the cycle above is indeed QASIC. For the original problem, the cycle we obtain can be represented as $\langle (x_s, \iota_s) \rangle_{s=0}^t$ in which $x_s = \bar{\nu}(\iota_s)$:

$$(x_0, \iota_0) \leftarrow (x_1, \iota_1) \leftarrow \cdots \leftarrow (x_t, \iota_t) \leftarrow (x_0, \iota_0)$$

such that

- students, dummy students and programs involved are distinct, and at least one student is involved,
- if $\iota_s \in N$, then $x_s P_{i_s} \mu(i_s)$ and $i_s \succ_{x_s} j$ for all $j \in \{j \in N \backslash \{i_s\} | x P_j \mu(j)\}$,
- if $\iota_s \notin N$, then (x_s, ι_s) is a unit chain by a dummy student,
- for all s, $\bar{\mu}(\iota_{s-1}) \neq \varnothing \Rightarrow \bar{\mu}(\iota_{s-1}) \in X_{k(x_s)}$ and $\bar{\mu}(\iota_{s-1}) = \varnothing \Rightarrow \bar{\mu}(\iota_{s-1}) = x_s$,
- $\bar{\mu}(\iota_{s-1}) \neq x_s$ implies $q_{x_s} < \bar{q}_{x_s}$,

as is desired. □

A.6 Proof of Proposition 3

Proof. Let f and g be the QAP mechanisms under Q and Q', respectively. Note that the QAP starts from the DA matchings in both distributions. If there is a matching, then $\mu \in f(R)$ such that μ Pareto dominates ν for some $\nu \in g(R)$. Let the corresponding quota distribution of μ be p. As $p \in Q$ and $Q \subset Q'$, μ is either an ESOSM or is Pareto dominated by some $\eta \in g(R)$ under Q'. The former implies $\mu \in g(R)$ while the latter implies that η Pareto dominates ν. Thus, in both cases, $\nu \notin g(R)$, a contradiction. Therefore, all the students' welfare unanimously improves as flexibility increases. □

A.7 Proof of Theorem 3

Proof. Let f be the ex-post student-optimal stable correspondence. Suppose not. Then there exists $i \in N$, R, R'_i, $\nu \in f(R'_i, R_{-i})$ such that

$$\nu(i) P_i \mu(i), \forall \mu \in f(R).$$

Let the corresponding quota distribution for ν be p. Since $\nu \in f(R'_i, R_{-i})$,

$$\nu = DA^p(R'_i, R_{-i}).$$

Since the DA is strategy-proof, we know that

$$DA^p_i(R)\ R_i\ DA^p_i(R'_i, R_{-i}).$$

Since $DA^p(R)$ is a distribution-specific student-optimal stable matching, there must exist $\eta \in f(R)$ such that

$$\eta(i) R_i DA^p_i(R),$$

a contradiction. $\square$

A.8 Proof of Theorem 4

Proof. By counterexample, suppose that $N = \{i_1, i_2, i_3, i_4\}$, $X = \{x_1, x_2, x_3\}$, and $(\bar{q}_{x_1}, \bar{q}_{x_2}, \bar{q}_{x_3}) = (2, 2, 1)$. Moreover, $K = \{k, k'\}$ such that $k(x_1) = k(x_2) = k$, $k(x_3) = k'$, $(\bar{q}_k, \bar{q}_{k'}) = (3, 1)$ so that $Q = \{(1, 2, 1), (2, 1, 1)\}$. Preferences R and priorities $\succ$ are as follows:

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	$\succ_{x_1}$	$\succ_{x_2}$	$\succ_{x_3}$
x_2	x_1	x_1	x_2	i_1	i_3	
x_1	x_3	x_2	x_3	i_4	i_2	any .
				i_2	i_4	
				i_3	i_1	

$$DA^{(1,2,1)}(R) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_1 & x_3 & x_2 & x_2 \end{pmatrix}, \quad DA^{(2,1,1)}(R) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_1 & x_1 & x_2 & x_3 \end{pmatrix}$$

are ESOSMs. Let f be an ex-post student-optimal stable and strategy-proof mechanism. Suppose the mechanism selects $DA^{(2,1,1)}(R)$ for R. If i_4 declares

$$R'_{i_4} : x_2\ x_1\ x_3,$$

the DA matchings for that preference profile and two quota distributions are

$$DA^{(1,2,1)}(R'_{i_4}, R_{-i_4}) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_1 & x_3 & x_2 & x_2 \end{pmatrix}, \quad DA^{(2,1,1)}(R'_{i_4}, R_{-i_4}) = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 \\ x_1 & x_3 & x_2 & x_1 \end{pmatrix}$$

respectively. As $DA^{(2,1,1)}(R'_{i_4}, R_{-i_4})$ is Pareto dominated by $DA^{(1,2,1)}(R'_{i_4}, R_{-i_4})$ at (R'_{i_4}, R_{-i_4}), $DA^{(1,2,1)}(R'_{i_4}, R_{-i_4})$ is an ESOSM at (R'_{i_4}, R_{-i_4}). Then, the mechanism must select

$$f(R'_{i_4}, R_{-i_4}) = DA^{(1,2,1)}(R'_{i_4}, R_{-i_4}).$$

Obviously, i_4 has an incentive to misstate his/her preference, because

$$x_2 = f_{i_4}(R'_{i_4}, R_{-i_4}) \; P_{i_4} \; f_{i_4}(R) = x_3.$$

As i_2 and i_4 play a symmetric role, if $f(R)$ is $DA^{(1,2,1)}(R)$ then i_2 has an incentive to misreport his/her preferences. Therefore, no single-valued function is ex-post student-optimal stable and strategy-proof. □

A.9 Proof of Proposition 4

Proof. We employ the method created by Hafalir et al. (2022). Consider diversity choice rule C^d in Hafalir et al. (2022) with a diversity index of 1 for all feasible distributions and 0 for all other distributions.[25] Then the diversity index function is ordinal concave and monotone, so by Theorem 2 and Proposition 2 of Hafalir et al. (2022), C^d is path independent and satisfies the law of aggregate demand. By Hatfield and Milgrom (2005), the student-proposing deferred acceptance mechanism is strategy-proof and distribution-specific student-optimal stable.

Furthermore, by the construction of C^d, the algorithm proceeds as if it is a sequential dictatorship. Hence the outcome is also efficient. Together with the above, the mechanism is strategy-proof and ex-post student optimal stable. □

[25]To apply Hafalir et al. (2022), there must be a merit ranking over contracts. In our setting, contracts are just pairs of a student and a program. Since all programs have common priorities, a merit ranking ranks contracts with a student of the highest priority first, then ranks contracts with a student of the second highest priority, and so on. In addition, here the domain of distributions is the set of distributions of a matching, not quota distributions.

A.10 Proof of Claim 2

Proof. Let μ be the ESOSM and q be the corresponding quota distribution. Suppose that it is not weakly stable. Then, because μ is feasible and individually rational, and if $yP_j\mu(j)$, then for all $\ell \in \mu^{-1}(y)$ with $\ell \succ_y j$, there exists a pair (i, x) such that $xP_i\mu(i)$ and $g(w(\mu) + e_x) = 1$.

As $g(w(\mu) + e_x) = 1$, $\sum_{y \in k(x)} |\mu^{-1}(y)| < q_k$, which implies that there is a vacant seat in some program in department $k(x)$. Let the program be z. Then, quota distribution q' such that $q'_x = q_x + 1$ and $q'_z = q_z - 1$ while holding the other quotas unchanged is also feasible ($q' \in Q$).

Thus, it is clear that a QASIC must exist. As i prefers x to $\mu(i)$, at least one student wants x. No matter who is of the highest priority at x among students who prefer x to their matched program at μ, because z has a vacant seat, we can find a QASIC. That is, starting from the unit chain (x, j), if there are unit chains involving students connected to (x, j), we are done. Otherwise a dummy student who is assigned z connects to (x, j). This result contradicts the fact that μ is an ESOSM. □

B Appendix: Simulation results

Each trial was conducted 1000 times. The unconstrained deferred acceptance mechanism, denoted by UDA, corresponds to the implementation of the DA algorithm under the assumption that there is no upper bound on the number of students each department may admit. In this setting, the capacity of each program is given by $\bar{q}_x$.

		$\gamma = 0.1$				$\gamma = 0.2$			
$(\alpha, \beta) = (0, 0)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	762.33	762.33	763.46	781.66	1416.38	762.33	764.50	782.79	1597.71
2nd ranked	483.32	483.32	482.98	487.69	449.79	483.32	482.99	487.94	366.06
Average rank	2.69	2.69	2.69	2.62	1.45	2.69	2.68	2.62	1.29
Better off than DA	–	0.00	60.88	49.11	951.38	0.00	83.99	51.82	1077.84
Worse off than DA	–	0.00	59.70	0.00	0.00	0.00	80.83	0.00	0.00

		$\gamma = 0.3$				$\gamma = 0.4$			
$(\alpha, \beta) = (0, 0)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	762.33	762.33	765.34	782.97	1710.80	762.33	765.60	782.99	1781.25
2nd ranked	483.32	483.32	483.29	487.97	297.09	483.32	483.19	487.99	247.51
Average rank	2.69	2.69	2.68	2.62	1.20	2.69	2.68	2.62	1.16
Better off than DA	–	0.00	106.66	52.23	1143.01	0.00	117.72	52.29	1179.87
Worse off than DA	–	0.00	102.08	0.00	0.00	0.00	112.85	0.00	0.00

		$\gamma = 0.5$			
$(\alpha, \beta) = (0, 0)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	762.33	762.33	765.55	782.99	1839.61
2nd ranked	483.32	483.32	483.10	487.99	202.74
Average rank	2.69	2.69	2.68	2.62	1.12
Better off than DA	–	0.00	122.30	52.29	1208.16
Worse off than DA	–	0.00	117.79	0.00	0.00

		$\gamma = 0.1$				$\gamma = 0.2$			
$(\alpha, \beta) = (0.3, 0.3)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	425.80	462.03	462.72	473.31	734.54	494.57	495.40	507.93	932.27
2nd ranked	361.18	382.07	381.96	387.66	506.11	398.85	399.24	405.06	546.58
Average rank	4.39	4.11	4.11	4.04	2.64	3.90	3.90	3.83	2.11
Better off than DA	–	150.39	178.19	190.66	871.71	254.85	288.74	296.11	1121.92
Worse off than DA	–	0.00	25.92	0.00	0.00	0.00	31.21	0.00	0.00

		$\gamma = 0.3$				$\gamma = 0.4$			
$(\alpha, \beta) = (0.3, 0.3)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	425.80	520.81	521.31	534.76	1083.40	540.11	540.95	555.00	1204.99
2nd ranked	361.18	411.33	411.47	417.05	547.65	419.69	420.03	425.05	531.96
Average rank	4.39	3.76	3.76	3.69	1.84	3.67	3.67	3.60	1.66
Better off than DA	–	323.71	365.06	362.29	1250.48	368.38	415.26	406.68	1330.54
Worse off than DA	–	0.00	39.22	0.00	0.00	0.00	43.39	0.00	0.00

		$\gamma = 0.5$			
$(\alpha, \beta) = (0.3, 0.3)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	425.80	555.91	556.76	571.76	1306.92
2nd ranked	361.18	425.68	425.92	430.86	506.44
Average rank	4.39	3.60	3.60	3.53	1.54
Better off than DA	–	400.28	449.25	438.82	1386.58
Worse off than DA	–	0.00	45.09	0.00	0.00

		$\gamma = 0.1$				$\gamma = 0.2$			
$(\alpha, \beta) = (0.5, 0.5)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	159.53	165.71	166.05	171.39	228.18	172.82	173.21	182.03	297.07
2nd ranked	171.00	178.35	178.60	183.28	249.60	186.44	186.85	194.12	319.15
Average rank	8.11	7.81	7.81	7.69	5.74	7.51	7.50	7.36	4.62
Better off than DA	–	120.19	148.94	164.33	811.55	230.17	261.79	285.13	1120.25
Worse off than DA	–	0.00	25.85	0.00	0.00	0.00	27.45	0.00	0.00

		$\gamma = 0.3$				$\gamma = 0.4$			
$(\alpha, \beta) = (0.5, 0.5)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	159.53	181.09	181.71	193.36	366.48	187.99	188.59	202.13	433.22
2nd ranked	171.00	195.76	196.43	205.68	380.45	203.40	204.14	215.23	432.15
Average rank	8.11	7.21	7.21	7.04	3.93	6.99	6.98	6.80	3.46
Better off than DA	–	329.77	366.45	390.35	1294.53	404.15	441.02	468.11	1405.38
Worse off than DA	–	0.00	31.62	0.00	0.00	0.00	32.11	0.00	0.00

		$\gamma = 0.5$			
$(\alpha, \beta) = (0.5, 0.5)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	159.53	195.39	195.88	212.06	498.83
2nd ranked	171.00	211.56	212.20	223.92	476.63
Average rank	8.11	6.79	6.78	6.60	3.11
Better off than DA	–	469.43	504.66	530.80	1483.52
Worse off than DA	–	0.00	31.02	0.00	0.00

		$\gamma = 0.1$				$\gamma = 0.2$			
$(\alpha, \beta) = (0.8, 0.8)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	89.79	90.75	90.88	95.38	99.12	91.77	92.02	100.22	108.54
2nd ranked	91.15	92.26	92.19	96.52	101.11	93.31	93.15	100.84	110.88
Average rank	12.22	12.07	12.06	11.84	10.86	11.92	11.92	11.53	9.78
Better off than DA	–	77.26	113.02	166.89	576.66	147.34	191.97	292.29	871.74
Worse off than DA	–	0.00	30.16	0.00	0.00	0.00	36.77	0.00	0.00

		$\gamma = 0.3$				$\gamma = 0.4$			
$(\alpha, \beta) = (0.8, 0.8)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	89.79	92.97	93.33	104.64	118.11	93.97	94.30	108.90	127.71
2nd ranked	91.15	94.52	94.29	104.97	120.92	95.44	95.29	109.06	131.26
Average rank	12.22	11.76	11.75	11.24	8.90	11.62	11.61	10.98	8.17
Better off than DA	–	220.09	272.91	403.47	1061.15	275.34	330.62	493.44	1192.12
Worse off than DA	–	0.00	43.80	0.00	0.00	0.00	45.62	0.00	0.00

		$\gamma = 0.5$			
$(\alpha, \beta) = (0.8, 0.8)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	89.79	95.41	95.75	112.84	137.52
2nd ranked	91.15	96.81	96.68	112.90	141.98
Average rank	12.22	11.44	11.43	10.73	7.56
Better off than DA	–	341.61	394.94	573.42	1289.81
Worse off than DA	–	0.00	43.91	0.00	0.00

		$\gamma = 0.1$				$\gamma = 0.2$			
$(\alpha, \beta) = (1.0, 1.0)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	82.32	83.16	83.34	88.14	90.61	83.97	84.22	93.21	98.79
2nd ranked	82.84	83.78	83.48	88.27	91.18	84.73	84.35	93.02	99.41
Average rank	13.00	12.88	12.87	12.61	11.87	12.76	12.76	12.28	10.94
Better off than DA	–	78.75	113.45	227.36	594.27	148.22	191.17	405.66	904.32
Worse off than DA	–	0.00	33.98	0.00	0.00	0.00	42.37	0.00	0.00

		$\gamma = 0.3$				$\gamma = 0.4$			
$(\alpha, \beta) = (1.0, 1.0)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	82.32	85.16	85.38	98.25	107.03	85.80	86.00	103.09	115.25
2nd ranked	82.84	85.88	85.37	97.67	107.71	86.48	85.92	102.07	115.98
Average rank	13.00	12.61	12.60	11.96	10.15	12.53	12.52	11.69	9.48
Better off than DA	–	228.52	279.27	559.53	1098.88	268.49	323.14	679.53	1230.05
Worse off than DA	–	0.00	50.48	0.00	0.00	0.00	54.27	0.00	0.00

		$\gamma = 0.5$			
$(\alpha, \beta) = (1.0, 1.0)$	DA	$FDA^{\tau=q}$	$FDA^{\tau=\frac{q}{2}}$	QAP	UDA
1st ranked	82.32	87.54	87.75	107.76	123.54
2nd ranked	82.84	88.03	87.50	106.59	124.32
Average rank	13.00	12.34	12.34	11.42	8.89
Better off than DA	–	352.94	403.19	782.47	1324.95
Worse off than DA	–	0.00	50.50	0.00	0.00

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