Supplemental Appendix: "Rules versus Discretion without Common Knowledge"

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A. Extensions

Level-K thinking. In this part, we show that level-K thinking generates similar implications as higher-order doubts. Consider the case that agents are subject to level-K thinking when forming beliefs. The agents are assumed to observe the fundamental but perceive a mis-specified prior about the behavior of others: they are "level-n thinkers" for some finite integer $n \geq 0$. Level-0 agents are assumed to play $k = 0$ for all θ realizations. Level-1 agents believe that other agents are level 0. They therefore play

$$k^{(1)} = \mu\theta + \varphi k^{(0)} = \mu\theta.$$

Level-2 agents believe that other agents are level 1. They therefore choose

$$k^{(2)} = \mu\theta + \varphi k^{(1)} = \mu\theta(1 + \varphi).$$

Similarly, Level-n agents choose

$$k^{(n)} = \mu\theta(1 + \varphi + \ldots + \varphi^{n-1}).$$

The policy instrument is

$$\tau = \mu\theta + \varphi k^{(n)} = \mu\theta(1 + \varphi + \ldots + \varphi^{n}).$$

The implementability constraint is

$$y = \gamma\tau + \alpha k^{(n)} = \tau\left(\gamma + \alpha\frac{1 - \varphi^{n-1}}{1 - \varphi^{n}}\right).$$

Notice that the ratio between y and τ is different from $\alpha + \gamma$ and depends on the policy coefficient φ when the level of reasoning n is finite, which is qualitatively similar to the effect of higher-order doubts.

Noisy information. In this part, we show that the main insights also hold with noisy information. Suppose that the true signals that agents receive are

$$\begin{aligned} s &= \theta, \\ x_i &= \theta + u_i, \quad u_i \sim N(0, \sigma_x^2). \end{aligned}$$

That is, there are a public signal that is perfect and a private signal that is noisy. However, we assume that agents perceive that the public signal is also imperfect

$$\widetilde{s} = \theta + \widetilde{\varepsilon}, \quad \widetilde{\varepsilon} \sim N(0, \sigma_s^2).$$

This is a simple technical trick that accommodates public information without introducing additional aggregate shocks.

By Bayesian inference, we have

$$E_i[\theta] = \lambda_s s + \lambda_x x_i, \qquad E_i[s] = s,$$

where $\lambda_s = \frac{\widetilde{\sigma}_s^{-2}}{\sigma_\theta^{-2}+\sigma_s^{-2}+\sigma_x^{-2}}$ and $\lambda_x = \frac{\widetilde{\sigma}_x^{-2}}{\sigma_\theta^{-2}+\sigma_s^{-2}+\sigma_x^{-2}}$. The higher-order expectations are

$$\overline{E}_i[\overline{E}[\theta]] = (\lambda_s + \lambda_x\lambda_s)s + \lambda_x^2 x_i,$$
$$\overline{E}_i[\overline{E}^2[\theta]] = (\lambda_s + \lambda_x\lambda_s + \lambda_x^2\lambda_s)s + \lambda_x^3 x_i,$$

and so on. Therefore, the representation with average higher-order expectations becomes

$$k = \mu\sum_{n=0}^{\infty}\varphi^n\overline{E}^{n+1}[\theta] = \mu\frac{\lambda_x}{1-\varphi\lambda_x}\theta + \mu\frac{\lambda_s}{(1-\varphi)(1-\varphi\lambda_x)}s = \mu\left(\lambda_x + \frac{\lambda_s}{1-\varphi}\right)\frac{1}{1-\varphi\lambda_x}\theta,$$

where we have used the fact that $s = \theta$. The policy instrument can be expressed as

$$\tau = \mu\theta + \varphi k = \mu\theta + \mu\varphi\left(\lambda_x + \frac{\lambda_s}{1-\varphi}\right)\frac{1}{1-\varphi\lambda_x}\theta.$$

The implementability constraint is

$$y = \gamma\tau + \alpha k = \left(\gamma + \alpha\frac{\lambda_s + \lambda_x(1-\varphi)}{1-\varphi+\varphi\lambda_s}\right)\tau,$$

where the slope is again a function of the coefficient φ.

General best response. In this part, we show that our results survive with more general best response function. Suppose that agents' payoff also directly depends on others' action

$$k_i = (1-\beta)E_i[\tau] + \beta E_i[k],$$

where β captures relative importance of others' action versus the policy instrument. Taking a particular policy rule $\tau = \mu\theta + \varphi k$, the effective best response becomes

$$k_i = (1-\beta)\mu E_i[\theta] + (\beta + (1-\beta)\varphi)E_i[k].$$

Without higher-order doubts, it is always the case that $k_i = k = \tau$ and the implementability constraint $y = (\alpha+\gamma)\tau$ holds.

With higher-order doubts, the aggregate action becomes

$$k = (1-\beta)\mu\theta + \lambda(\beta + (1-\beta)\varphi)k = \frac{(1-\beta)\mu}{1-\lambda(\beta+(1-\beta)\varphi)}\theta,$$

and the implementability constraint becomes

$$y = \gamma\tau + \alpha k = \left(\gamma + \alpha\frac{1-\beta}{1-\lambda\beta+(1-\lambda)(1-\beta)\varphi}\right)\tau.$$

That is, the implementability constraint now depends on the policy coefficient φ as in the main text analysis.

General shock process. In the baseline analysis, we consider the case in which the fundamental shock follows an i.i.d. process, and the degree of deviation from the full-information rationality is governed by a single parameter, λ. In this section, we extend the framework to incorporate a more general specification of the fundamental process and allow the degree of higher-order uncertainty to vary with the forecasting horizon.

Denote the fundamental process as

$$\theta_t = \sum_{k=0}^{\infty} a_k \eta_{t-k},$$

where η_{t-k} is the innovation in period $t-k$ and $\{a_k\}$ is the sequence of coefficients in the time domain. For example, an AR(1) process requires that $a_k = \rho^k$.

We allow agents to form expectations that, over time, an increasing number of agents will become informed about past innovations. Particularly, the expectations follow

$$E_{it}[E_{jt}[\eta_{t-k}]] = \lambda_k \eta_{t-k}, \quad \text{for} \quad k \geq 0. \tag{A.1}$$

We impose the restriction that $\{\lambda_k\}$ is a weakly increasing sequence with $\lambda_k \in (0, 1)$, reflecting the assumption that agents expect that a larger share of the population becomes aware of the shock the longer it has been realized.

Denote the responses of the aggregate outcome and the policy instrument as

$$y_t = \sum_{k=0}^{\infty} m_k \eta_{t-k}, \qquad \text{and} \qquad \tau_t = \sum_{k=0}^{\infty} g_k \eta_{t-k}.$$

In this setting, in order to replicate the first-best allocation, it requires that $m_k = \delta_y a_k$ and $g_k = \delta_\tau a_k$ for every k. In contrast to our baseline analysis, the number of conditions that needs to be satisfied becomes infinite. To create the necessary degree of freedom for the policymaker, we consider the following policy rule

$$\tau_t = \sum_{k=0}^{\infty} \mu_k \eta_{t-k} + \sum_{k=0}^{\infty} \varphi_k \overline{E}_t[\tau_{t+k}], \tag{A.2}$$

which allows the policymaker to respond to current and past innovations, as well as the average expectations about current and future policy instruments. This choice of policy rule is motivated by the structure of widely used survey data—such as the Survey of Professional Forecasters and Consensus Forecasts—which report expectations of aggregate variables at multiple horizons. In Supplemental Appendix B, we further demonstrate that even when the policy rule is restricted to incorporate forecasts at a single horizon, the policymaker can still achieve substantial welfare gains relative to the full-information benchmark.

Under certain conditions, the policymaker can implement the allocation $y_t = \delta_y \theta_t$ and $\tau_t = \delta_\tau \theta_t$ by suitably choosing the policy parameters.

Proposition A.1. *There exists a policy rule in the form of (A.2) that achieves the first-best allocation if the following*

Hankel matrix $\mathbf{H}$ *is invertible*

$$\mathbf{H} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proof. Denote the perceived law of motion for τ_t as

$$E_{it}[\tau_t] = \sum_{k=0}^{\infty} f_k \eta_{t-k}$$

This process has to be consistent with their perceived policy rule, that is

$$\begin{aligned} E_{it}[\tau_t] &= \sum_{k=0}^{\infty} \mu_k \eta_{t-k} + \sum_{k=0}^{\infty} \varphi_k E_{it}[\overline{E}_t[\tau_{t+k}]], \\ &= \sum_{k=0}^{\infty} \mu_k \eta_{t-k} + \sum_{k=0}^{\infty} \varphi_k \sum_{j=k}^{\infty} f_j \lambda_{j-k} \eta_{t-(j-k)}, \end{aligned}$$

where the last equality uses condition (A.1) and the fact that $E_{it}[\eta_{t+k}] = 0$ for $k > 0$. Matching coefficients leads to the conditions for the sequence $\{f_k\}$

$$f_k = \mu_k + \lambda_k \sum_{j=0}^{\infty} \varphi_j f_{j+k}. \tag{A.3}$$

Now consider the actual law of motion of τ_t, which is given by

$$\begin{aligned} \tau_t &= \sum_{k=0}^{\infty} \mu_k \eta_{t-k} + \sum_{k=0}^{\infty} \varphi_k \overline{E}_t[\tau_{t+k}], \\ &= \sum_{k=0}^{\infty} \mu_k \eta_{t-k} + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \varphi_j f_{j+k} \eta_{t-k}. \end{aligned}$$

Denote $\tau_t = \sum_{k=0}^{\infty} g_k \eta_{t-k}$. It follows that the sequence $\{g_k\}$ needs to satisfy

$$g_k = \mu_k + \sum_{j=0}^{\infty} \varphi_j f_{j+k}. \tag{A.4}$$

The first-best allocation requires that

$$\tau_t = \delta_\tau \theta_t, \qquad E_t[\tau_t] = \frac{\delta_y - \gamma \delta_\tau}{\alpha} \theta_t,$$

which implies that

$$g_k = \delta_\tau a_k, \qquad f_k = \frac{\delta_y - \gamma \delta_\tau}{\alpha} a_k. \tag{A.5}$$

Using condition (A.3), (A.4), and (A.5), the solution to $\{\mu_j\}$ can be obtained directly as

$$\mu_k = \frac{\delta_y - \gamma\delta_\tau - \alpha\delta_\tau\lambda_k}{\alpha}\frac{a_k}{1-\lambda_k}.$$

The sequence $\{\gamma_j\}$ needs to solve the following system of equations

$$\sum_{j=0}^{\infty} a_{k+j}\varphi_j = \frac{(\alpha+\gamma)\delta_\tau - \delta_y}{\delta_y - \gamma\delta_\tau}\frac{a_k}{1-\lambda_k}. \tag{A.6}$$

The system of equations (A.6) can be represented as

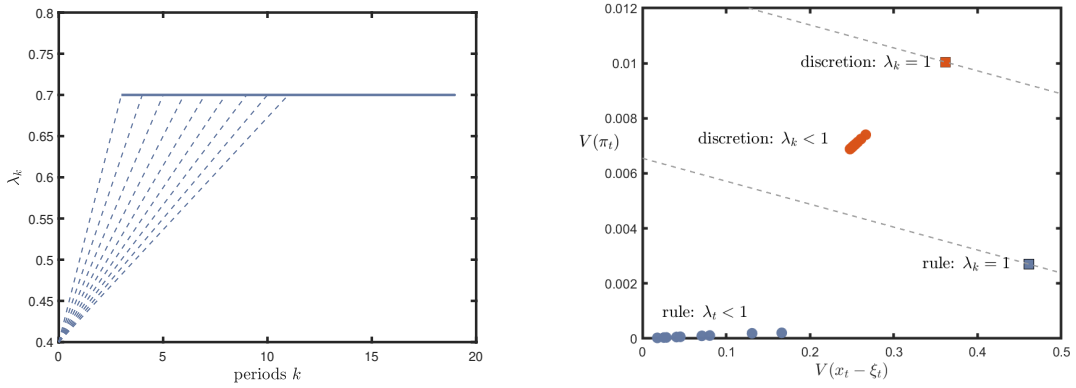
$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \vdots \end{bmatrix} = \frac{(\alpha+\gamma)\delta_\tau - \delta_y}{\delta_y - \gamma\delta_\tau} \begin{bmatrix} \frac{a_0}{1-\lambda_0} \\ \frac{a_1}{1-\lambda_1} \\ \frac{a_2}{1-\lambda_2} \\ \vdots \end{bmatrix}$$

The matrix on the left-hand side is a Hankel matrix. This system permits a solution if the Hankel matrix is invertible. □

B. Additional Results for NK Application

In Section 4, we focused on the case where agents hold symmetric higher-order doubts about current and past shocks. In this appendix, we consider an alternative scenario in which agents believe that others are more informed about shocks that occurred in the past. The expectation formation follows a specification similar to assumption A.1 in Supplemental Appendix A. Differently, we retain the same policy rule as in equation (4.10), without expanding the set of policy instruments, and assess whether it remains effective in substantially reducing social loss.

Figure 4: Rules v.s. Discretion with Learning



(a) different paths of λ_k

(b) welfare comparison

We assume that the expectations about others' expectations follow

$$E_{it}[E_{j,t+m}[\eta_{t-k}]] = \lambda_{m+k}\eta_{t-k}, \quad \text{for} \quad k \geq 0, \tag{B.1}$$

where $\{\lambda_k\}$ is a weakly increasing sequence in k. Assumption (B.1) implies that agents believe a larger fraction of the population becomes aware of shocks that occurred further in the past, serving as a reduced-form representation of gradual learning. Moreover, when agents form expectations about others' future beliefs, an additional anticipation channel arises, captured by the term λ_{m+k}.

The left panel of Figure 4 displays the sequences of λ_k considered in our analysis. We assume that for $k < K$, λ_k is increasing, $\lambda_k = \lambda_K$ for $k \geq K$. Particularly, We impose $\lambda_0 = 0.4$, reflecting the assumption that, upon impact, agents believe only 40% of the population are aware of the shock. Over time, λ_k gradually converges to a much higher level at $\lambda_K = 0.7$. We explore a range of transition dynamics, varying the speed at which information diffuses—from a rapid adjustment over $K = 3$ periods to a more gradual transition spanning $K = 12$ periods.

We continue to employ the policy rule that

$$\pi_t = \mu\xi_t + \varphi\overline{E}_t[\pi_{t+1}].$$

Assume that agents perceive the inflation process as

$$\pi_t = \sum_{k=0}^{\infty} f_k\eta_{t-k}.$$

The sequence $\{f_k\}$ must be consistent with their perceived policy rule, which leads to

$$\sum_{k=0}^{\infty} f_k\eta_{t-k} = \mu\sum_{k=0}^{\infty}\rho^k\eta_{t-k} + \varphi\sum_{k=1}^{\infty} f_k\lambda_{k-1}\eta_{t-k+1}.$$

Matching coefficients leads to the system of equations

$$f_k = \mu\rho^k + \varphi\lambda_k f_{k+1}.$$

Note that after period K, $\lambda_k = \lambda_K$, it follows that

$$f_k = \mu\rho^k + \varphi\lambda_K f_{k+1}, \quad \text{for} \quad k > K.$$

As a result, for $k \geq K$.

$$f_k = \frac{\mu}{1 - \varphi\lambda_K\rho}.$$

And for $k < K$, they can be solved backwards recursively following $f_k = \mu\rho^k + \varphi\lambda_k f_{k+1}$.

This procedure applies for any pair of policy parameters (μ, φ). For the commitment case, we choose them to minimize the social loss

$$\mathbb{V}(\pi_t^2) + \alpha\mathbb{V}(x_t - \xi_t).$$

For discretionary policy, they are given by condition (4.9).

The right panel of Figure 4 presents a welfare comparison analogous to that in the main text. Under the simple policy rule given by equation (4.10), the exact first-best allocation under commitment is no longer achievable. Nevertheless, as indicated by the blue circles, the variances of both inflation and the output gap remain close to zero and are substantially lower than those under full information (represented by the blue square). In the absence of commitment, the lack of common knowledge also leads to welfare improvements, as shown by the red circles—consistent with the findings in our baseline analysis.

C. Proofs of Lemma 3.1 and Proposition 3.2

Proof of Lemma 3.1. Applying Lemma 2.2 to the case of (μ^D, φ^D) yields

$$y = \left(\gamma + \frac{\alpha}{1 + \varphi^D (1-\lambda)}\right)\tau \equiv \xi(\lambda)\tau.$$

Substituting the definition of φ^D, we have

$$\xi(\lambda) = \gamma + \frac{\alpha\left[\chi\gamma^2 + (1-\chi)\right]}{\chi\gamma^2 + (1-\chi) - \chi\alpha\gamma(1-\lambda)}.$$

We differentiate $\xi(\lambda)$ with respect to λ, which leads to

$$\xi'(\lambda) = -\frac{\alpha^2\chi\gamma(\chi\gamma^2 + (1-\chi))}{(\chi\gamma^2 + (1-\chi) - \chi\alpha\gamma(1-\lambda))^2}.$$

Therefore, $\xi'(\lambda) > 0$ if $\gamma < 0$ and $\xi'(\lambda) < 0$ if $\gamma > 0$.

Proof of Proposition 3.2. With discretionary policy, the allocation needs to satisfy the first-order condition (3.1) and the implementability constraint, which yields

$$y = \xi(\lambda)\tau \quad \text{and} \quad \tau = \frac{\chi\gamma\delta_y + (1-\chi)\delta_\tau}{\chi\xi(\lambda)\gamma + (1-\chi)}\theta.$$

Define $m = \chi\gamma$ and $n = 1 - \chi$, we have

$$\tau = \frac{m\delta_y + n\delta_\tau}{m\xi(\lambda) + n}\theta, \quad y = \xi(\lambda)\frac{m\delta_y + n\delta_\tau}{m\xi(\lambda) + n}\theta.$$

Normalizing the variance of the fundamental shock to be 1, the social loss can be expressed as

$$\mathcal{L} = (\chi n^2 + (1-\chi)m^2)\left(\frac{\delta_y - \delta_\tau\xi(\lambda)}{m\xi(\lambda) + n}\right)^2.$$

Define $H(\lambda) = \frac{\delta_y - \delta_\tau\xi(\lambda)}{m\xi(\lambda)+n}$. When switching from the economy with perfect information to the one with higher-order doubts, the difference in social loss is

$$\mathcal{G}(\lambda) = (\chi n^2 + (1-\chi)m^2)\left\{\left(\frac{\delta_y - \delta_\tau\xi(\lambda)}{m\xi(\lambda) + n}\right)^2 - \left(\frac{\delta_y - \delta_\tau\xi(1)}{m\xi(1) + n}\right)^2\right\} = (\chi n^2 + (1-\chi)m^2)\left(H(\lambda)^2 - H(1)^2\right).$$

A positive $G(\lambda)$ implies that adding higher-order doubts increases the social loss.

- First consider the case where $\delta_y < \delta_\tau \xi(1)$. Given that $\xi(1) = \alpha + \gamma > 0$ and $\gamma > 0$, we have $m\xi(1) + n > 0$. By Lemma 3.1, when $\gamma > 0$, $\xi(\lambda) > \xi(1)$. It follows that $H(\lambda) < 0$ for $\lambda \in [0, 1]$. Therefore, we have

$$\text{sign}(\mathcal{G}(\lambda)) = \text{sign}(H(1) - H(\lambda)).$$

  Note that

$$H(1) - H(\lambda) = \frac{(m\delta_y + n\delta_\tau)\left[\xi(\lambda) - \xi(1)\right]}{(m\xi(\lambda) + n)(m\xi(1) + n)} > 0,$$

  and therefore the social loss increases when λ is less than 1.

- Next, consider the other case where $\delta_y > \delta_\tau \xi(1)$. First note that

$$H'(\lambda) = -\frac{(m\delta_y + n\delta_\tau)\,\xi'(\lambda)}{(m\xi(\lambda) + n)^2} > 0,$$

  as $\xi'(\lambda) < 0$ when $\gamma > 0$. Also recognize that

$$\text{sign}(\mathcal{G}(\lambda)) = \text{sign}\left(H(\lambda)^2 - H(1)^2\right).$$

  When $\gamma > 0$ and $\delta_\tau \xi(1) - \delta_y < 0$, $H(1) > 0$. The fact that $H(\lambda)$ is increasing in λ implies that $H(\lambda) < H(1)$. As a result, it is either (1) $H(\lambda)^2 < H(1)^2$ for $\lambda \in [0, 1)$ or (2) there exists $\underline{\lambda} \in (0, 1)$ such that $H(\lambda)^2 < H(1)^2$ when $\lambda \in [\underline{\lambda}, 1)$.