Dynamic Formation of Directed Firm Networks

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Nov. 3rd, 2008

Abstract: This paper rationalizes firm's motivation to build directed links with each other and formalizes the dynamic formation process that generates the observed network structure in the citation network. Random meeting of customers and firms in the differentiated goods market are unlikely to result in perfect match between demand and supply, but the firm can earn commission fee by redirecting the customer to another firm who are able to serve the demand, only if these two firms are connected by social linkage. Therefore each firm wants to know more firms, so that they can earn more commission fee. On the other hand, every firm also desires to be known by more other firms in order to gain more redirected customers.

The model extends the dynamic formation of non-directed network model in Jackson and Rogers (2007) into directed network and results in various structure features that exhibit in actual directed networks. I then estimate the model's parameters using firm citation panel data from the NBER Patent Citation Database. Using the estimated parameters I simulate the model to generate artificial sectoral citation networks and show that the simulated network structure is similar to the network structures observed in the data.

Keyword: directed network, dynamic network formation, power-law, patent citation
JEL Classification: D85 Z13

1 Introduction

The literature on dynamic network formation has primarily been concerned with understanding the formation of non-directed networks. It has been successful in explaining many empirical characteristics observed in actual social networks.[2] However, in reality, many networks transferring goods and information flows are directed. Examples of such networks are websites connected by hyperlinks, people connected by phone call or emails, and firms connected by patent citations. In such "directed" networks, two nodes connected by one link are not symmetric but play different roles: initiator and receiver. People who send emails are not symmetric with those receiving them; patent citers are not the same as those being cited, etc. In fact, for most networks, the symmetry of nodes is at best a simplifying assumption. There has, up to now, been no model of dynamic network formation for directed networks. Any such model would have to provide insights into what leads selfish individuals to engage in network building, and to match the observed triple power-law degree distributions of networks, i.e. the in-degree, out-degree, and total-degree all follow power-law degree distribution.

This paper builds a dynamic model of directed network formation and demonstrates its application to a real-world directed network: firm citation network. The model uses profit sharing between a firm who has access to a customer and a firm who knows the technology to produce to explain individual firms' incentives to

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[2] Jackson and Rogers (2007) summarize 5 characteristics: (1) small average shortest distance between nodes, (2) positive clustering coefficients (Clustering coefficients measures how often two nodes with common friend are also friends), (3) power-law degree distribution (A quantity x obeys a power law if it is drawn from a probability distribution $p(x) \propto x^{-\alpha}$, where α is a constant parameter of the distribution known as the exponent or scaling parameter. In real world situations the scaling parameter typically lies in the range $2 < \alpha < 3$, although there are occasional exceptions, (4) positive correlation between degrees of linked nodes, and (5) negative correlation between local clustering coefficient of a node's neighborhood and the node's degree.

build directed networks. When firms and customers are randomly matched, a representative firm may not be able to produce what its customer wants, but can gain by referring this customer to another firm who knows how to produce it. The preconditions are that firms know what others in the network produce and can profit, perhaps via commission fees, for directing consumers towards them. I represent "firm i knows what firm j produces" as a directed link pointing from firm i to firm j in the network. With profit sharing as the form of commissions, firms want to know more other firms, so that they earn more commission fees, while at the same time get known by more other firms in order to obtain re-directed customers. In equilibrium, the dynamic process of network formation is decided by firm's trade off between the benefit and cost in building networks. Joharia, Mannorb and Tsitsiklis (2006) uses cost compensation to explain why private post offices build directed networks to deliver mail packages, but their model is a static one.

Standing on the shoulders of giant's, this paper extends the dynamic network formation of non-directed network in Holme and Kim (2002), Vazquez (2003), and Jackson and Rogers (2007) (JR henceforth) into directed network. [3] In non-directed network, nodes build new link either by network-based method (knowing friend's friend) or randomized method (knowing random unknown people). In directed network, there is another layer of complexity within network-based network formation method: the two directions of links. Nodes build new link via both old links in the same direction and old links in the opposite direction at different success rates. The first success rate depends on node's tendency to keep its current role as an initiator or receiver in the network; the second success rate relies on node's possibility to switch its current role to the opposite role in the network. According to Kesten (1973), when the success rates in building new links through different methods are subject to i. i. d. "popularity" shocks, the in-degree, out-degree, and total-degree distributions all converge to power-law distribution, as seen in the real directed networks. The inter-temporal causality between two types of links in the bilateral network-based network formation is the key to generate the triple power-law distribution.

The model can be extended to understand the dynamic formation of more complex networks, where there are multiple types of nodes and links. For example, exporter-market network, buyer-seller network, etc. The key to handle complex networks is modeling the inter-temporal causality between different types of links.

I then illustrate the application of the model by considering a firm citation network panel data. I construct the network data from the NBER Patent Citation Database for 42 sectors from 1985 to 1994. This is clearly a directed network: one inter-firm citation corresponds to one directed link from the citing firm[4] to the cited firm[5]. With multiple years of data, I can observe the inter-temporal change in each sectoral citation network. I can tell both the links that are newly built and the method by which the new link is built: whether the new link is introduced by an old link in the same direction, introduced by an old link in the opposite direction or by the random meeting of two previously unconnected nodes. Identifying the method through which a node builds new connections allows me to infer the success rates of building new links via the different methods for each node. By knowing the distribution of these success rates, I am able to simulate the dynamic network formation process and compare the simulated networks with the real sectoral citation networks.

The simulation allows me to test whether the simple model about degree[6] dynamic process also mimics other structure features in the real networks. The simulated network for sector s starts from a randomly generated network. Every period new links are built by a mixture of network-based and randomized network formation

[3] Network-based network formation means two unconnected nodes with common neighbor in last period connect with each other this period, or knowing friend's friend. Its opposite is randomized network formation, where randomly picked two nodes connect with each other.

[4] The firm who applies a patent and cites other existing patents.

[5] A firm whose existing patents are cited by other patent applications.

[6] A node's degree is the number of nodes that it directly connects with.

methods. In the bilateral network-based network formation, every node introduces its unconnected friends to each other. A representative node i at time t is assigned an i. i. d. "popularity draw" po_{it}^{xy} from the estimated success rate distribution of building x type new link from y type old link in sector s, where x, y $\in$ (inward, outward). A higher "popularity draw" po_{it}^{xy} means a y type old link is more likely to introduce node i an x type new link. In the randomized network formation, two unconnected nodes i and j are randomly connected by a link from i to j by possibility r_s, which is the estimated success rate of random matching in sector s. Repeat the above process for 50 periods, which is long enough for degree distributions to converge to power-law. The simulated network matches actual network not only in degree distributions, but also in clustering coefficients and other structure features.

The rest of the paper is organized as follows. The model section describes firm's motivation to build directed social network and the methods to build new links. The data section introduces the NBER Patent Citation Data and illustrates how to infer the distribution of "popularity" draws in each network-formation method. With the estimated distribution of "popularity" draws, I simulate artificial networks and compare them with the real sectoral citation networks. Lastly, the conclusion summarizes the paper.

2 The Model

2.1 Firm's Motivation

There are a continuum of firms on $[0, 1]$. Each firm plays two roles, one is the producer, and the other is the dealer. As a producer, firm i produces one unit of goods $i \in [0, 1]$ with one unit of labor. As a dealer, firm i may receive a query from a randomly matched customer j for goods $j \in [0, 1]$. The customer j only accepts the goods on $\left[j - \frac{x}{2}, j + \frac{x}{2}\right]$, and pays P dollar for it. $0 < x \ll 1$ represents customer's tolerance to substitutes. If $i \in \left[j - \frac{x}{2}, j + \frac{x}{2}\right]$, firm i serves customer j itself. Otherwise, firm i plays as a dealer and checks other producers on its contact $C_{it} = \{k | \ i \text{ knows } k\}$ at time t. If there exists one producer such that $k \in C_{it}$ and $k \in \left[j - \frac{x}{2}, j + \frac{x}{2}\right]$, dealer i introduces this customer to producer k and earns a commission fee θP. Producer k earns $(1 - \theta) P - W$. The wage rate is W normalized to 1. I assume the tolerance range x approaches 0, so that no firm knows enough producers to cover the entire product domain $[0, 1]$. Moreover, it is unlikely that any two firms in C_{it} serve overlapping markets.

I show that a firm with more connections in the producer-dealer network has higher expected profit. Denote the number of producers that i knows at time t (number of elements in C_{it}) as p_{it}. Denote the number of dealers who know i at time t (the number of firms k that $i \in C_{kt}$) as d_{it}. Assume that a firm randomly meets a customer at a constant rate L. At time t, the expected profit for firm i who knows p_{it} producers and with d_{it} dealers knowing i is

$$\pi_{it} = Lx \left[1 + \theta P p_{it} + ((1 - \theta) P - 1) d_{it}\right]. \tag{1}$$

The first part of the profit happens when the random customer who meets firm i accepts product i. The second part origins from the commission fee, when the customer does not accept product i, but takes some product produced by p_{it} other firms on firm i's contact list. The third part of the profit is from the business introduced by d_{it} dealers who know firm i, when they can not serve their customers themselves.

When I describe this dealer-producer relation by a directed link, there is a directed link from firm i to its producer. In network theory, p_{it} is called the out-degree of firm i and d_{it} is called the in-degree of firm i. Firm i's profit (1) is linearly increasing in p_{it} and d_{it}.

2.2 Methods to Build Network

Given the degree-dependent income flow in (1), a profit maximizing firm wants to expand its connections with other firms. In order to acquire information about more producers, firm i can (1a) know a new producer k from a current producer j ($j \in C_{it}$, $k \in C_{jt}$ and $k \notin C_{it}$); (2a) know a new producer k from a current dealer j ($i \in C_{jt}$ and $k \in C_{jt}$); (3a) randomly pick up a producer's advertisement on the street. In order for more dealers to know firm i, firm i can (1b) ask a current dealer j to forward firm i's information to j's producer k ($i \in C_{jt}$, $k \in C_{jt}$ and $i \notin C_{kt}$) ; (2b) ask a current producer j to forward firm i's information to a firm j's producer k($j \in C_{it}$ and $k \in C_{jt}$); (3b) send its advertisement to a random firm on the street. Methods (1a), (2a), (1b), and (2b) belong to network-based network formation, where firm i's new connections built today depend on its position in the old network yesterday. Meanwhile, methods (3a) and (3b) are called randomized network formation, in which the new connections built today are independent of last period's network topology.

The above methods to build network may involve different efficiencies. In real social network, people usually trust a friend's friend more than a random person on the street, and network-based network formation is more likely to happen through friends than the randomized method. In the firm's network here, network-based methods (1a), (2a), (1b), and (2b) are targeted and may have high success rate. While randomized methods (3a) and (3b) are aimless and may only work in very few occasions. Even among network-based methods (1a), (2a), (1b), and (2b), one method can be more efficient than the other.

2.3 Dynamic Network Formation Process

To exchange information with current producers and dealers, firm i spends l^p_{it} hours listening and t^p_{it} hours talking to a producer; l^d_{it} hours listening and t^d_{it} hours talking to a dealer; and l^r_{it} hours listening and t^r_{it} hours talking to a random unknown firm. While listening, firm i receives information about more producers from the talking firm; while talking, firm i broadcasts its own or other firm's information to the firms which are listening. I use $\triangle y^x_{ijt}$ to represent the number of new y gained by communicating with current x, in which x $\in$ {producer, dealer, and random firm}, $y \in$ {producer, dealer}, i is the talker, j is the listener, and t is the time period. For example, $\triangle p^p_{ijt}$ represents the number of new producers known when listening to current producer j at time t.

Since the communication is bilateral, the outcome depends on the time inputs from both sides. Suppose firm j is firm i's producer, firm k is firm i's dealer, [7]and firm r is a random unknown firm to firm i. The new links built through different methods are as follows.

$$\triangle p^p_{jit} = A_{pp}\left(t^d_{jt}\right)^\beta \left(l^p_{it}\right)^\alpha$$

$$\triangle d^p_{ijt} = A_{dp}\left(t^p_{it}\right)^\beta \left(l^d_{jt}\right)^\alpha$$

$$\triangle p^d_{kit} = A_{pd}\left(t^p_{kt}\right)^\beta \left(l^d_{it}\right)^\alpha$$

$$\triangle d^d_{ikt} = A_{dd}\left(t^d_{it}\right)^\beta \left(l^p_{kt}\right)^\alpha$$

$$\triangle p^r_{ilt} = A_{pr}\left(t^r_{rt}\right)^\beta \left(l^r_{it}\right)^\alpha$$

$$\triangle d^r_{ilt} = A_{dr}\left(t^r_{it}\right)^\beta \left(l^r_{rt}\right)^\alpha$$

A_{xy} is the technology of knowing new x through current y, in which $y \in$ {producer, dealer, and random firm}, $x \in$ {producer and dealer}. α and β measures the listener's and talker's share in the communication

[7]Notice that meanwhile firm i is also firm j's dealer and firm i is firm k's producer.

outcome respectively. I need $\alpha + \beta < 1$ to have Nash equilibrium conversation time input from both sides of communication.

The firm i maximizes firm value by choosing hour inputs l_{it}^p, t_{it}^p, l_{it}^d, t_{it}^d, l_{it}^r, and t_{it}^r.

$$
\begin{aligned}
V_t(p_{it}, d_{it}) \quad = \quad & \max_{\{l_{it}^p,\, t_{it}^p,\, l_{it}^d,\, t_{it}^d,\, l_{it}^r,\, \text{and } t_{it}^r\}} Lx\left[1 + \theta P p_{it} + ((1-\theta)P - 1) d_{it}\right] \\
& + \rho E\left[V_{it+1}(p_{it+1}, d_{it+1})\right] - p_{it}\left(l_{it}^p + t_{it}^p\right) - d_{it}\left(l_{it}^d + t_{it}^d\right) - I_t\left(l_{it}^r + t_{it}^r\right)
\end{aligned}
$$

such that

$$
\begin{aligned}
p_{it+1} \quad = \quad & (1-\delta) p_{it} + \sum_{j \in C_{it}} A_{pp} \left(t_{jt}^d\right)^\beta \left(l_{it}^p\right)^\alpha + \varepsilon_{it}^{pp} p_{it} + & (2) \\
& \sum_{i \in C_{kt}} A_{pd} \left(t_{kt}^p\right)^\beta \left(l_{it}^d\right)^\alpha + \varepsilon_{it}^{pd} d_{it} + \sum_{r \in I} A_{pr} \left(t_{rt}^r\right)^\beta \left(l_{it}^r\right)^\alpha + \varepsilon_{it}^{pr} I_t & (3)
\end{aligned}
$$

$$
\begin{aligned}
d_{it+1} \quad = \quad & (1-\delta) d_{it} + \sum_{j \in C_{it}} A_{dp} \left(t_{it}^p\right)^\beta \left(l_{jt}^d\right)^\alpha + \varepsilon_{it}^{dp} p_{it} + & (4) \\
& \sum_{i \in C_{kt}} A_{dd} \left(t_{it}^d\right)^\beta \left(l_{kt}^p\right)^\alpha + \varepsilon_{it}^{dd} d_{it} + \sum_{r \in I} A_{pr} \left(t_{it}^r\right)^\beta \left(l_{rt}^r\right)^\alpha + \varepsilon_{it}^{dr} I_t & (5)
\end{aligned}
$$

$V_t(p_{it}, d_{it})$ is the firm value a as function of its in-degree and out-degree. (2) and (4) are the network formation functions of out-degree and in-degree. ρ is the firm's discount rate. δ is the depreciation rate of contact information. In the general equilibrium, δ is set to the average speed that dealers get to know new producers, so that the average number of producers per dealer is a constant overtime. Otherwise, when δ is too small, the network degenerates to a fully connected network; or when δ is too big, the network breaks down. $\left\{\varepsilon_{it}^{pp}, \varepsilon_{it}^{pd}, \varepsilon_{it}^{pr}, \varepsilon_{it}^{dp}, \varepsilon_{it}^{dd}, \text{ and } \varepsilon_{it}^{dr}\right\}$ are the i. i. d. zero mean shocks that firm i receives in different types of social activities at time t. They capture firm i's representative's random communication productivity.

Firm i hires representatives to gain more connections, at the cost of the wage bill to representatives. In the first order conditions (6) to (11), the expected marginal profit equals the marginal cost. Every firm takes other firm's time input as given.

$$l_{it}^p = \left(V_p \alpha A_{pp}\right)^{\frac{1}{1-\alpha}} \left(t_{jt}^d\right)^{\frac{\beta}{1-\alpha}} \tag{6}$$

$$l_{it}^d = \left(V_p \alpha A_{pd}\right)^{\frac{1}{1-\alpha}} \left(t_{kt}^p\right)^{\frac{\beta}{1-\alpha}} \tag{7}$$

$$l_{it}^r = \left(V_p \alpha A_{pr}\right)^{\frac{1}{1-\alpha}} \left(t_{rt}^r\right)^{\frac{\beta}{1-\alpha}} \tag{8}$$

$$t_{it}^p = \left(V_d \beta A_{dp}\right)^{\frac{1}{1-\beta}} \left(l_{kt}^d\right)^{\frac{\alpha}{1-\beta}} \tag{9}$$

$$t_{it}^d = \left(V_d \beta A_{dd}\right)^{\frac{1}{1-\beta}} \left(l_{kt}^p\right)^{\frac{\alpha}{1-\beta}} \tag{10}$$

$$t_{it}^r = \left(V_d \beta A_{dr}\right)^{\frac{1}{1-\beta}} \left(l_{rt}^r\right)^{\frac{\alpha}{1-\beta}} \tag{11}$$

$$V_p(p_{it}, d_{it}) = PLx\theta + \rho E\left[V_p\left(p_{it+1}, d_{it+1}\right)\right](1-\delta) \tag{12}$$

$$V_d(p_{it}, d_{it}) = Lx\left(P(1-\theta) - 1\right) + \rho E\left[V_d\left(p_{it+1}, d_{it+1}\right)\right](1-\delta) \tag{13}$$

An educated guess for the firm value function is $V(p, d) = v_p p + v_d d + u$. From Bellman equations (12) and

(12), I solve for v_p and v_d.

$$v_p = \frac{PLx\theta}{1-\rho(1-\delta)} \tag{14}$$

$$v_d = \frac{Lx(P(1-\theta)-1)}{1-\rho(1-\delta)} \tag{15}$$

To firm i, a producer's (dealer's) marginal value v_p (v_d) is the discounted value of future commission fee (production profit). In first order conditions (6) to (11), firm i exerts more effort in making new producers (dealers), when its marginal value v_p (v_d) is higher.

In Nash equilibrium, every firm choose the same time input portfolio $\left\{l_{it}^{p*}, t_{it}^{d*}, l_{it}^{d*}, t_{it}^{p*}, l_{it}^{r*}, \text{ and } t_{it}^{r*}\right\}$.

$$l_{it}^{p*} = (v_p\alpha A_{pp})^{\frac{1-\beta}{1-\alpha-\beta}} (v_d\beta A_{dd})^{\frac{\beta}{1-\alpha-\beta}} \tag{16}$$

$$t_{it}^{d*} = (v_d\beta A_{dd})^{\frac{1-\alpha}{1-\alpha-\beta}} (v_p\alpha A_{pp})^{\frac{\alpha}{1-\alpha-\beta}} \tag{17}$$

$$l_{it}^{d*} = (v_p\alpha A_{pd})^{\frac{1-\beta}{1-\alpha-\beta}} (v_d\beta A_{dp})^{\frac{\beta}{1-\alpha-\beta}} \tag{18}$$

$$t_{it}^{p*} = (v_d\beta A_{dp})^{\frac{1-\alpha}{1-\alpha-\beta}} (v_p\alpha A_{pd})^{\frac{\alpha}{1-\alpha-\beta}} \tag{19}$$

$$l_{it}^{r*} = (v_p\alpha A_{pr})^{\frac{1-\beta}{1-\alpha-\beta}} (v_d\beta A_{dr})^{\frac{\beta}{1-\alpha-\beta}} \tag{20}$$

$$t_{it}^{r*} = (v_d\beta A_{dr})^{\frac{1-\alpha}{1-\alpha-\beta}} (v_p\alpha A_{pr})^{\frac{\alpha}{1-\alpha-\beta}} \tag{21}$$

Substituting Nash equilibrium time input (16) to (21) into the network formation functions (4) and (2) solves the dynamic network formation processes for firm i.

$$\begin{pmatrix} p_{it+1} \\ d_{it+1} \end{pmatrix} = F_t \begin{pmatrix} p_{it} \\ d_{it} \end{pmatrix} + R_t \tag{22}$$

$$F_t = \begin{pmatrix} F^{pp} & F^{pd} \\ F^{dp} & F^{dd} \end{pmatrix}$$

$$F_t^{pp} = 1-\delta + (v_p\alpha A_{pp})^{\frac{\alpha}{1-\alpha-\beta}} (v_d\beta A_{dd})^{\frac{\beta}{1-\alpha-\beta}} + \varepsilon_{it}^{pp}$$

$$F_t^{pd} = (v_p\alpha A_{pd})^{\frac{\alpha}{1-\alpha-\beta}} (v_d\beta A_{dp})^{\frac{\beta}{1-\alpha-\beta}} + \varepsilon_{it}^{pd}$$

$$F_t^{dp} = (v_p\alpha A_{pd})^{\frac{\alpha}{1-\alpha-\beta}} (v_d\beta A_{dp})^{\frac{\beta}{1-\alpha-\beta}} + \varepsilon_{it}^{dp}$$

$$F_t^{dd} = 1-\delta + (v_p\alpha A_{pp})^{\frac{\alpha}{1-\alpha-\beta}} (v_d\beta A_{dd})^{\frac{\beta}{1-\alpha-\beta}} + \varepsilon_{it}^{dd}$$

$$R_t = \begin{pmatrix} (v_p\alpha A_{pr})^{\frac{\alpha}{1-\alpha-\beta}} (v_d\beta A_{dr})^{\frac{\beta}{1-\alpha-\beta}} + \varepsilon_{it}^{pr} \\ (v_p\alpha A_{pr})^{\frac{\alpha}{1-\alpha-\beta}} (v_d\beta A_{dr})^{\frac{\beta}{1-\alpha-\beta}} + \varepsilon_{it}^{dr} \end{pmatrix}$$

Notice that the expectation of matrix F_t is center-symmetric. That is because every firm's inputs are the same. For example, firm i spends as much time listening to its producer j as its dealer k listens to firm i. Firm j also spends as much time talking to firm i as firm i talks to firm k. That is why the upper-left and lower-right elements of $E(F)$ are the same. Similarly, the upper and lower elements of $E(R)$ are symmetric, because firm i spends the same time listening (talking) to a random firm as a random firm spends listening (talking) to firm i.

Proposition 1 *According to Kesten (1973), when* $\{\varepsilon_{it}^{pp}, \varepsilon_{it}^{pd}, \varepsilon_{it}^{pr}, \varepsilon_{it}^{dp}, \varepsilon_{it}^{dd}$, *and* $\varepsilon_{it}^{dr}\}$ *are identically independent distributed cross firm and time, for 2 dimensional vector* x *with* $|x| = 1$*, as* $t \to \infty$*,* $x'\binom{p_{it}}{d_{it}}$ *follows Pareto distribution* μ_x*.*

By choosing $x = (1,0)$,$(0,1)$, and $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, I can check on the distribution of out-degree (p_{it}) in-degree (d_{it}) and total degree ($p_{it} + d_{it}$) of the network. Since the matrices F and R in (22) are symmetric, p_{it} and d_{it} have the same Pareto distribution parameter μ. Notice that Power-law distribution is the discrete time version of Pareto distribution. Since the number of links are discrete numbers, the out-degree in-degree and total degree exhibit Power-law distributions.

3 Data

3.1 Data Description

The NBER Patent Citation Database published by U. S. Patent Office reports patent applications in 42 broad SIC classifications from 1962 to 2002. With multiple years, it allows me to track the inter-temporal change of networks. With 42 sectors, it is also convenient to compare cross sectors. I use within-sector inter-firm citations made between U. S. firms from 1985 to 1995 to construct sectoral citation network for 42 sectors.

Figure 1 shows the 3-dimentional graph of a real firm citation network based on Refrigeration and Service Industry Machinery sector during 1990-1994. Each berry represents a firm; a link with arrow indicates a citation from from the citing firm to the cited firm. There are numerous layers in the network: firms with more links lie closer to the center of the network, while firms with fewer links stay in the pheriphery of the network.

The sectoral citation network is constructed as follows. Every firm is a node in the network. Every citation is a directed link pointing from the citing firm to the cited firm. At time t, an n by n adjacency matrix M_{st} summarizes sector s' citation network, where n is the total number of firms. $M_{st}(i,j) = 1$ if firm i cites firm j; otherwise $M_{st}(i,j) = 0$. Firm i's out-degree (number of outward links or producers p_{it} in the model) is the number of ones in the i^{th} row of M_{st}. Firm i's in-degree (number of inward links or dealers d_{it} in the model) is the number of ones in the i^{th} column of $M_{s,t}$. The total number of inward and outward links is called total-degree. Denote it as $t_{it} = p_{it} + d_{it}$.

3.2 Stylized Facts about Citation Network

Newman (2003) and JR summarize 5 stylized facts that socially generated networks share. Sectoral firm citation networks have all these characteristics. Notice that these 5 facts are all static, not dynamic, features of networks. I report (a), (d) and (e) in Table 1, (b) and (c) in Table 2.

(a) Average shortest distance between pairs of nodes is small.

(b) As with the other social networks, clustering coefficients[8] are larger than randomly generated networks.

(c) Power-law in-degree (d), out-degree (p), and total-degree (t) distribution (triple power-law).

(d) Positive sorting. Degrees and patent stocks of linked firms are positively correlated. Average geographic distance between the citing and the cited firms are much shorter than average distance between two random picked firms in the same sector.

(e) The clustering among the neighbors of a given node is inversely related to the node's degree.

[8] They measure how likely two nodes with a common connected node are also connected.

3.3 Degree Distribution Heterogeneity and Ratio r in JR

JR predicts that the network degree distribution is more heterogeneous and network structure is more clustered, when nodes build more new links with a friend's friend, and form fewer new links with a random node. Because a connection with a friend's friend is a type of "preferential attachment," it makes nodes with more links today get more new links tomorrow. In contrast, every node has an equal chance of building new links with random nodes in the randomized network formation, which tends to eliminate current differences in degree numbers. The patent citation network with multiple years and 42 sectors permits me to test their prediction across sectors.

For sector s at time t, I estimate the key parameter r_{st} in JR, the ratio of new links with a random node to new links with a friend's friend. On the other hand, I also estimate the power-law degree distribution parameter μ_{st} and calculate three measures of clustering coefficient C_{st}^{TT}, C_{st}, and C_{st}^{Avg} listed in JR. I give the details to estimate r_{ts}, μ_{st}^{in}, μ_{st}^{out}, μ_{st}^{total}, C_{st}^{TT}, C_{st}, and C_{st}^{Avg} in the Appendix A.

As predicted in their paper, in Figure (2a)-(2c), μ_{st}^{in}, μ_{st}^{out}, and μ_{st}^{total} are higher (degree distribution is more homogeneous), when r_{ts} is higher in sector s. In Figure (3a)-(3c), C_{st}^{TT}, C_{st}, and C_{st}^{Avg} are smaller, when r_{ts} is higher in sector s.

3.4 Simulation

To test whether the network formation process specified in model section mimics the real network formation process, I simulate a directed network for every sector and compare the simulated network with the real sectoral network. Before simulating, I need to estimate the distribution of random matrices F_s and R_s in (22) ,$G_{F_s}(F_s)$ and $G_{R_s}(R_s)$, for every sector s. I give the details for estimating $G_{F_s}(F_s)$ and $G_{R_s}(R_s)$ in the Appendix B. I then fit $\{F_{ft}\}$ and $\{R_{ft}\}$ with log-normal distribution. δ_s is set to be the average growth rate of new links. The simple simulation process is as follows.

In the simulated network for sector s, there are N nodes.

{At time $t = 0$, the adjacency matrix M_{s0} is a sparse matrix with 5% ones and 95% zeros uniformly randomly distributed.

[At time $t > 0$, ones in M_{st-1} have possibility δ_s to become zero, and possibility $1 - \delta_s$ to still be one in M_{st}.

(A node i with more than two friends plays as a matchmaker. Suppose k is i's outward friend, j is i's inward friend. With possibility F_{kt}^{11}, k wants a new outward friend from i; with possibility F_{kt}^{12}, k wants a new inward friend from i. With possibility F_{jt}^{21}, j wants a new outward friend from i; with possibility F_{jt}^{22}, j wants a new inward friend from i. F_{kt} and F_{jt} are random draws from $G_{F_s}(F_s)$. If there are m requests for an inward friend, and n requests for an outward friend, node i randomly matches $\frac{n+m}{2}$ pairs. If firm i gets more requests for inward link than requests for outward link or $m > n$ (more requests for outward link than requests for inward link or $m < n$), some node gets more than one new outward (inward) friend and some node gets zero new outward (inward) friend. However, in ex ante, every node expects to match with the same number of inward and outward friends.)[9]

Repeat the process within () for every node.

Randomly turn zeros in M_{st-1} into ones with possibility $E\left(R_s^1\right)$.[10]

Calculate the clustering coefficients $C_{st}^{TT}(t)$, $C_{st}(t)$, and $C_{st}^{Avg}(t)$ and estimate the degree distribution parameters $\mu_{st}^{in}(t)$, $\mu_{st}^{out}(t)$, and $\mu_{st}^{total}(t)$.]

Repeat the process within [] until $t = 50$. }

[9] The process in ( ) elaborates the network-based network formation. The random matrix F_s governs the likely hood that nodes in sector s build new link through existing links.

[10] This line describes the randomized method of building new link. $E(R_s)$ governs how likely two unknown nodes randomly connect with each other.

Repeat the process within {} for every sector.

In Figure 4a, 4b, 4c, 5a, 5b, and 5c, I compare the degree distribution parameters $\mu_{st}^{in}(t)$, $\mu_{st}^{out}(t)$, and $\mu_{st}^{total}(t)$ as well as the clustering coefficients $C_{st}^{TT}(t)$, $C_{st}(t)$, and $C_{st}^{Avg}(t)$ in the simulated networks with their value in real sectoral networks. Each dot represents a sector. The straight line is the 45 degree line. The $\mu_{st}^{in}(t)$, $\mu_{st}^{out}(t)$, $\mu_{st}^{total}(t)$, $C_{st}^{TT}(t)$, $C_{st}(t)$, and $C_{st}^{Avg}(t)$ reported for simulated networks are the average value of last 20 periods. The values reported for real networks are the 5 year average from 1991 to 1995.

The simulated networks mimic the real network in terms of degree distributions and clustering coefficients ($\mu_{st}^{in}(t)$, $\mu_{st}^{out}(t)$, $\mu_{st}^{total}(t)$, and $C_{st}^{TT}(t)$ in Figure 4a, 4b, 4c, and 5a. $C_{st}(t)$ and $C_{st}^{Avg}(t)$ in simulated networks deviate from their correspondents in real networks, but the rank across sectors are still retained. The more clustered sectors in real world are still more clustered in the simulated world.

In Figure 6, I compare the simulated value and real value of the correlation between the clustering among the neighbors of a given node and the node's degree for all sectors. Although most sectors still have negative correlation between the clustering among the neighbors of a given node and the node's degree, the simulated networks abandon the cross-sector rank among real networks.

Among the real networks, the more clustered sector also has more heterogeneous degree distribution (lower $\mu_{st}^{in}(t)$, $\mu_{st}^{out}(t)$, and $\mu_{st}^{total}(t)$) as displayed in Figure 7a. In Figure 7b, I show that this rank is also maintained in the simulated world.

In conclusion, the simulated sectoral networks have a structure similar as their correspondent's real networks. The cross-sector ranks in many structure measures are preserved in all but one case.

4 Conclusion

This paper extends the current literature in dynamic network formation to the directed network. The model uses profit sharing to explain firms' motivation to build directed networks. The firm that has customer access may not have the technology to produce what the consumer wants. With the directed network, the firm with customer access introduces the customer to the firm that has the technology and gets a commission fee as an incentive. The firm's trade-off between the benefit and cost of building new links through different methods determines the dynamic network formation process.

The model extends the network-based network formation method in Jackson and Rogers (2007) by modeling the inter-temporal causality between two types of links. A current link in one direction may introduce new links in both directions. The inter-temporal causality between links in two directions is the key to generating triple power-law degree distribution of in-degree, out-degree, and total-degree, as observed in real directed networks.

The empirical part of the paper constructs the sectoral firm citation network from the NBER Patent Citation Database and estimates the model parameters from the panel network data. The simulated networks have a structure similar to the real ones. Meanwhile, it also proves the predictions in Jackson and Rogers (2007).

In the future research, the extended model can understand the dynamic formation of more complex networks with multiple types of nodes and links.

5 Appendix A

To fit in RJ2007's non-directed network environment, I ignore the direction in the citation networks and treat them as non-directed networks. For sector s at time t, the adjacency matrix becomes $\hat{M}_{s,t}(i,j) = \max\left(M_{s,t}(i,j), M_{s,t}(j,i)\right)$. Firm i and firm j are called "old friend" at time t, if i and j are connected at both t and $t-1$ ($\hat{M}_{s,t-1}(i,j)=1$ and $\hat{M}_{s,t}(i,j)=1$). Firm i and j are called "new friend" if firm i does not connect with firm j at time $t-1$, but i connects with j at time t ($\hat{M}_{s,t-1}(i,j)=0$ and $\hat{M}_{s,t}(i,j)=1$).

Conditional on firm i and j are new friends, they are called "network-based new friend" or "friend's friend" to each other, if there exists at least one firm $k \neq i,j$ such that k is connected with both i and j at time $t-1$ ($\hat{M}_{s,t-1}(i,:)*\hat{M}_{s,t-1}(:,j) \geq 1$, where $M_{s,t-1}(i,:)$ is the i^{th} row of matrix $\hat{M}_{s,t-1}$ and $\hat{M}_{s,t-1}(:,j)$ is the j^{th} column of matrix $\hat{M}_{s,t-1}$). Conditional on firm i and j are new friends, they are called "random new friend" to each other, if there is no such firm k that is connected with both i and j at time $t-1$ ($\hat{M}_{s,t-1}(i,:)*\hat{M}_{s,t-1}(:,j)=0$).

Mathematically $r_{s,t}$ is calculated as:

$$r_{s,t}=\frac{nnz\left[not\left(\hat{M}_{s,t-1}*\hat{M}_{s,t-1}\right).*not\left(\hat{M}_{s,t-1}\right).*\hat{M}_{s,t}\right]}{nnz\left[\hat{M}_{s,t-1}*\hat{M}_{s,t-1}.*not\left(\hat{M}_{s,t-1}\right).*\hat{M}_{s,t}\right]}. \tag{23}$$

The numerator (denominator) in (23) is the number of new random (friend's) friends made in sector s at time t. nnz counts the number of none zero elements. The (i,j) element in $\hat{M}_{s,t-1}*\hat{M}_{s,t-1}$ is the number of common friends that firm i and j share at time $t-1$. $not(M)$ replaces positive elements in M with zeros, and replaces zeros with ones. Therefore the (i,j) element in $not\left(\hat{M}_{s,t-1}*\hat{M}_{s,t-1}\right)$ is one, if firm i and firm j has no common friend at time $t-1$, it is zero otherwise. The (i,j) element in $not\left(\hat{M}_{s,t-1}\right).*\hat{M}_{s,t}$ is one, if firm i and j are new friends to each other, it is zero otherwise. All together, the (i,j) element in $not\left(\hat{M}_{s,t-1}*\hat{M}_{s,t-1}\right).*not\left(\hat{M}_{s,t-1}\right).*\hat{M}_{s,t}$ is positive, if firm i and j are "random new friends", it is zero otherwise. Similarly, the (i,j) element in $\hat{M}_{s,t-1}*\hat{M}_{s,t-1}.*not\left(\hat{M}_{s,t-1}\right).*\hat{M}_{s,t}$ is positive, if firm i and j are "network-based new friends", it is zero otherwise.

Newman (2003) gives several ways to measure network clustering. Jackson and Rogers (2007) examines three commonly used clustering coefficients in the literature. They are:

$$C^{TT}(M_{s,t})=\frac{\sum_{i;j\neq i;k\neq i,j}M_{s,t}(i,j)\,M_{s,t}(j,k)\,M_{s,t}(k,i)}{\sum_{i;j\neq i;k\neq i,j}M_{s,t}(i,j)\,M_{s,t}(j,k)},$$

$$C(M_{s,t})=\frac{\sum_{i;j\neq i;k\neq i,j}\hat{M}_{s,t}(i,j)\,\hat{M}_{s,t}(j,k)\,\hat{M}_{s,t}(k,i)}{\sum_{i;j\neq i;k\neq i,j}\hat{M}_{s,t}(i,j)\,\hat{M}_{s,t}(j,k)},$$

and

$$C^{Avg}(M_{s,t})=\frac{1}{n}\sum_{i}\frac{\sum_{j\neq i;k\neq i,j}\hat{M}_{s,t}(i,j)\,\hat{M}_{s,t}(j,k)\,\hat{M}_{s,t}(k,i)}{\sum_{j\neq i;k\neq i,j}\hat{M}_{s,t}(i,j)\,\hat{M}_{s,t}(j,k)}.$$

They all measure the likelyhood that two nodes are connected, conditional on these two nodes are connected with a common node. The first two definitions are the same when the network is non-directed. The third definition gives an equal weight to every node; while the first two definitions give a bigger weight to the node with more links.

To estimate $\mu^{x}_{s,t}$ in sector s at time t, I run OLS regress of $ln(1-F_{s,t}\left(d^{x}_{f,t}\right))$ on $ln(d^{x}_{f,t})$, where x={in,out and total}, $F_{s,t}(d^{x})$ is the c. d. f. of x-degree distribution in sector s at time t. $\hat{\mu}^{x}_{s,t}$ is equal to the absolute value of the OLS coefficient before $ln(d^{x}_{f,t})$. Note that standard deviation of $\left\{ln(d^{x}_{f,t})\right\}$ is equal to $\frac{1}{\hat{\mu}^{x}_{s,t}}$, which measures the heterogeneity of x-degree distribution.

6 Appendix B

In sector s' firm citation network, I identify the realization of $F_{f,t}$ and $R_{f,t}$ for node f at time t with the following steps.

(1) Identify new friend.

If $M_{s,t}(j,i) = 1$ and $M_{s,t-1}(j,i) = 0$, node j is node i's new inward friend at time t. If $M_{s,t}(i,j) = 1$ and $M_{s,t-1}(i,j) = 0$, node j is node i's new outward friend at time t. If $M_{s,t}(j,i) = 1$ and $M_{s,t-1}(j,i) = 1$, node j is node i's old inward friend at time t.

(2) Identify the source of new friend.

Suppose node j is node i's new inward friend at time t.

If there exists a node k, such that node k is an inward friend of node i, and inward or outward friend of node j at time $t-1$; then node j is node i's new inward friend introduced by inward friend k. The total number of such node j introduced by inward friend are denoted as $\Delta d^d_{f,t}$.

If there exists a node k, such that node k is an outward friend of node i, and k is inward or outward friend of node j at time $t-1$ or $j = k$; then node j is node i's new inward friend introduced by outward friend k. The total number of such node j is denoted as $\Delta d^p_{f,t}$.

If node i has both inward and outward common friends who are friend of j, then I attribute half to $\Delta d^p_{f,t}$ and half to $\Delta d^d_{f,t}$.

If node j and i do not have any type of common friend, then node j is node i's random new friend and belongs to $\Delta d^r_{f,t}$.

Similarly, new outward friend introduced by inward friend, outward friend, and random new friend $\Delta p^d_{f,t}$, $\Delta p^p_{f,t}$, and $\Delta p^r_{f,t}$ are identified.

(3) Estimate δ, the possibility to drop an old link. If $M_{s,t}(j,i) = 0$ and $M_{s,t-1}(j,i) = 1$, then the old link between i and j is dropped at time t. Denote $drop_{f,t}$ as the number of inward link dropped by firm f at time t. In the entire network, the possibility to drop an old link is $\delta = \frac{\sum_f drop_{f,t}}{\sum_f d_{f,t-1}}$.

(4) Infer the elements in random matrix $F_{f,t}$ and $R_{f,t}$. N is the total number of nodes in the network.

$$F^{11}_{f,t} = 1 - \delta + \frac{\Delta p^p_{f,t}}{p_{f,t}},$$

$$F^{12}_{f,t} = \frac{\Delta p^d_{f,t}}{d_{f,t}},$$

$$F^{21}_{f,t} = \frac{\Delta d^p_{f,t}}{p_{f,t}},$$

$$F^{22}_{f,t} = 1 - \delta + \frac{\Delta d^d_{f,t}}{d_{f,t}},$$

$$R^1_{f,t} = \frac{\Delta p^r_{f,t}}{N},$$

$$R^2_{ft} = \frac{\Delta d^r_{ft}}{N}.$$

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1	2	3	4	5	6	7	8
Patent Category	SIC87 Code	Average Shortest Distance between Two Nodes	Correlation in Total Degree	Correlation in Patent Stock	Great Circle Distance between Random Nodes (Kilometer)	Great Circle Distance between Linked Nodes (Kilometer)	Correlation between Local Clustering and Total Degree
1	20	3.568	0.256	0.174	6941.487	1061.766	-0.059
2	22	4.161	0.335	0.274	6991.833	793.612	-0.057
6	281	4.347	0.223	0.259	6487.991	988.637	-0.052
7	286	4.897	0.334	0.309	7159.114	724.757	-0.062
8	282	3.833	0.262	0.219	6724.499	724.359	-0.048
9	287	4.065	0.326	0.317	6835.288	839.865	-0.059
11	284	3.311	0.318	0.285	6250.190	804.956	-0.051
12	285	4.130	0.189	0.222	6735.441	662.364	-0.048
13	289	5.133	0.374	0.216	6643.541	844.491	-0.056
14	283	3.888	0.198	0.143	6366.191	1498.119	-0.058
15	1329	3.249	0.175	0.276	6566.865	894.687	-0.051
16	30	3.311	0.250	0.367	6950.680	994.404	-0.057
17	32	5.817	0.336	0.428	6890.830	918.441	-0.061
19	331+	2.903	0.215	0.127	6534.431	606.400	-0.036
20	333+	3.484	0.267	0.197	6769.626	963.375	-0.021
21	34-	5.574	0.259	0.308	6766.991	1064.213	-0.058
23	351	3.580	0.159	0.370	6766.586	939.101	-0.063
24	352	5.963	0.270	0.285	7044.228	1033.380	-0.036
25	353	5.962	0.197	0.383	6777.036	918.523	0.000
26	354	5.424	0.180	0.296	6983.690	941.041	-0.048
27	357	3.580	0.167	0.191	6687.346	1618.286	-0.062
29	355	6.037	0.201	0.328	6648.464	891.189	-0.058
30	356	5.947	0.180	0.315	6695.607	929.471	-0.060
31	358	4.035	0.223	0.214	7215.740	1116.276	-0.025
32	359	1.675	0.294	0.243	6127.337	1262.372	-0.045
35	361+	3.626	0.105	0.211	6974.893	1305.647	-0.050
36	362	3.978	0.152	0.246	7042.136	1016.173	-0.057
38	363	2.123	0.191	0.289	6359.496	670.735	-0.017
39	364	4.759	0.285	0.378	6888.798	1174.515	-0.057
40	369	4.059	0.233	0.299	6863.550	1238.117	-0.055
42	365	3.315	0.144	0.183	7250.125	1236.902	-0.061
43	366+	4.280	0.170	0.251	6953.626	1325.210	-0.059
46	371	3.764	0.215	0.457	7060.468	944.429	-0.056
47	376	2.404	-0.061	0.249	7256.875	1975.178	0.212
49	373	1.967	0.268	0.102	6917.394	1721.915	0.056
50	374	3.891	0.115	0.273	5475.854	775.826	-0.045
51	375	2.238	0.194	0.199	6505.385	1770.709	0.191
52	379-	1.589	0.044	0.065	5743.242	1252.619	NaN
53	348+	4.113	0.257	0.271	6884.717	1379.926	-0.018
54	372	2.872	-0.002	0.336	5551.671	1193.121	-0.053
55	38-	6.257	0.193	0.412	6945.959	1379.452	-0.058
56	99	6.860	0.277	0.270	6904.848	1336.209	-0.054

Table 1 Stylized Facts (a), (d) and (e) for Firm Citation Networks

1	2	3	4	5	6	7	8
Patent Category	SIC87 Code	mu-in	mu-out	mu-total	CTT	C	Cavg
1	20	0.666	0.592	0.639	0.045	0.043	0.139
2	22	0.817	0.724	0.791	0.050	0.049	0.178
6	281	0.744	0.675	0.717	0.056	0.054	0.151
7	286	0.791	0.714	0.756	0.055	0.052	0.229
8	282	0.656	0.576	0.638	0.056	0.055	0.173
9	287	0.794	0.723	0.768	0.055	0.053	0.235
11	284	0.612	0.521	0.603	0.049	0.047	0.135
12	285	0.876	0.753	0.825	0.053	0.052	0.150
13	289	0.692	0.653	0.680	0.049	0.048	0.161
14	283	0.925	0.905	0.910	0.046	0.046	0.188
15	1329	0.568	0.508	0.551	0.041	0.040	0.122
16	30	0.888	0.849	0.866	0.054	0.051	0.243
17	32	0.867	0.749	0.829	0.050	0.050	0.174
19	331+	0.885	0.720	0.837	0.023	0.022	0.055
20	333+	0.850	0.714	0.804	0.038	0.041	0.085
21	34-	1.035	0.961	1.003	0.054	0.052	0.223
23	351	0.567	0.539	0.571	0.051	0.050	0.178
24	352	1.043	0.933	1.010	0.040	0.039	0.104
25	353	0.907	0.852	0.891	0.053	0.049	0.161
26	354	1.146	1.010	1.091	0.042	0.042	0.133
27	357	0.572	0.566	0.580	0.056	0.053	0.298
29	355	0.983	0.936	0.967	0.054	0.050	0.198
30	356	0.958	0.859	0.920	0.053	0.051	0.234
31	358	1.204	1.018	1.155	0.042	0.039	0.074
32	359	1.494	1.021	1.300	0.008	0.008	0.038
35	361+	0.826	0.803	0.817	0.059	0.057	0.177
36	362	0.896	0.836	0.890	0.048	0.048	0.179
38	363	0.780	0.566	0.744	0.017	0.012	0.028
39	364	0.953	0.858	0.907	0.047	0.044	0.139
40	369	0.750	0.659	0.722	0.050	0.050	0.166
42	365	0.727	0.709	0.731	0.048	0.045	0.173
43	366+	0.626	0.614	0.622	0.056	0.052	0.295
46	371	0.649	0.608	0.649	0.055	0.054	0.172
47	376	0.963	0.742	0.911	0.010	0.009	0.008
49	373	1.259	0.915	1.122	0.015	0.013	0.022
50	374	0.797	0.625	0.743	0.018	0.017	0.063
51	375	1.540	0.986	1.345	0.012	0.007	0.014
52	379-	1.157	0.683	1.005	0.000	0.000	0.000
53	348+	0.867	0.741	0.844	0.034	0.036	0.069
54	372	0.809	0.689	0.782	0.015	0.015	0.059
55	38-	0.681	0.696	0.692	0.056	0.051	0.282
56	99	0.931	0.869	0.911	0.049	0.047	0.185

Table 2 Stylized Facts (b) and (c) for Firm Citation Networks

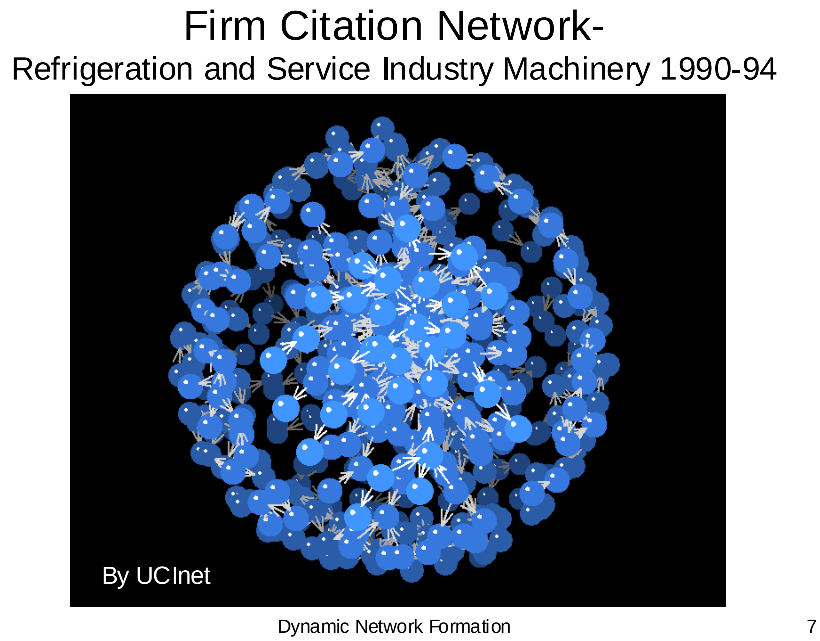

Figure 1

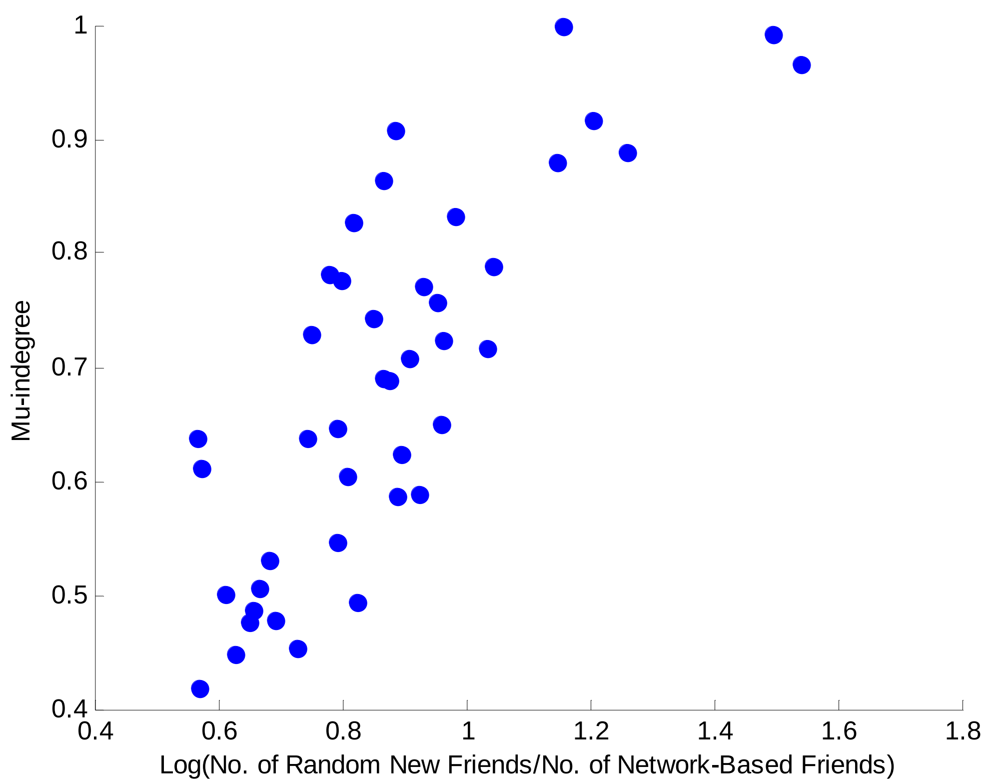

Figure 2a

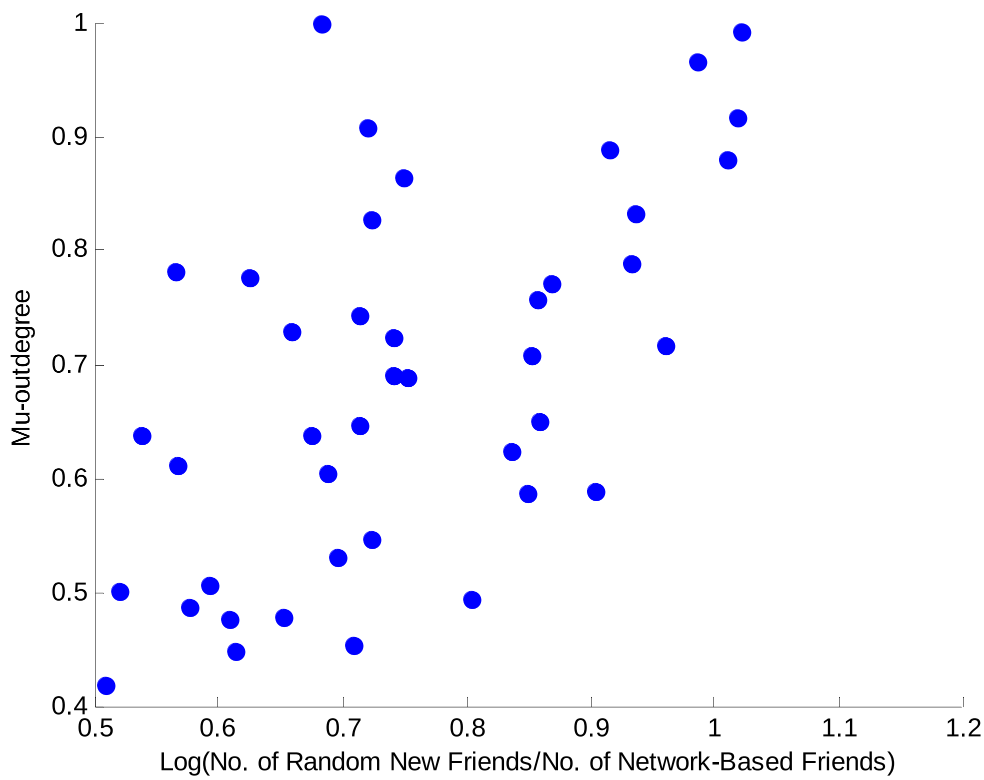

Figure 2b

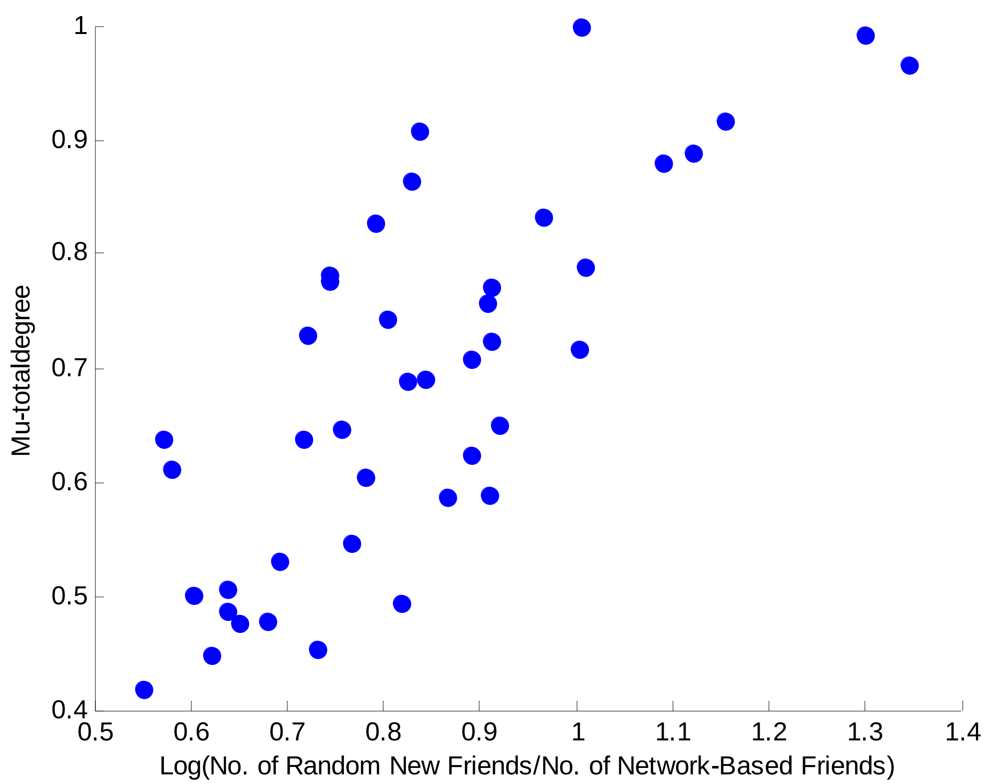

Figure 2c

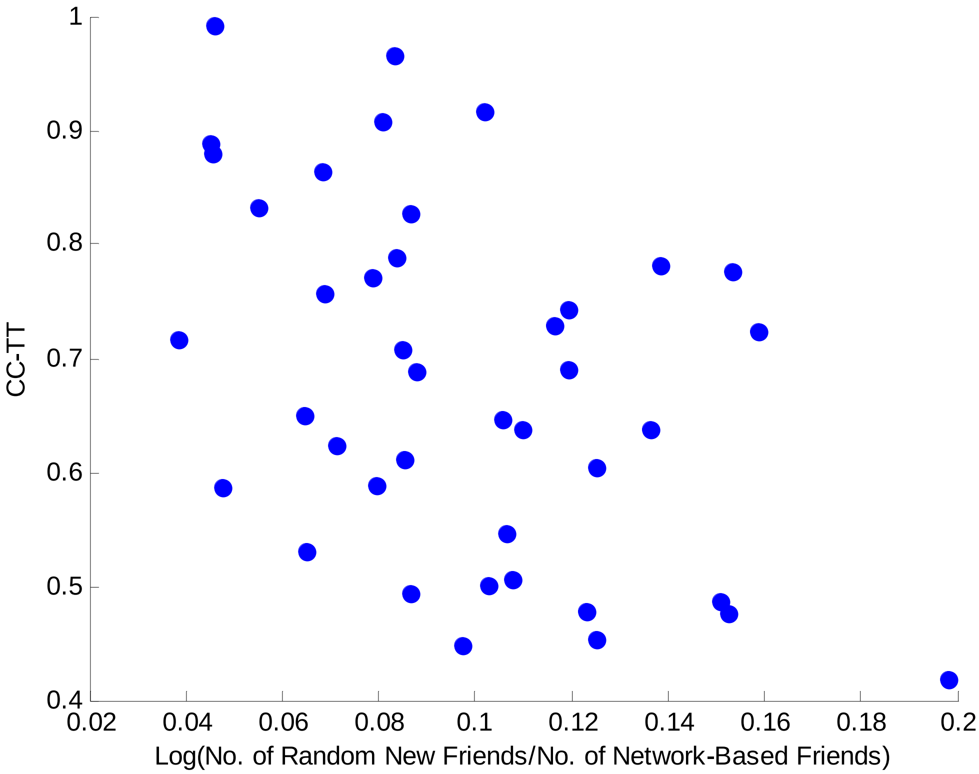

Figure 3a

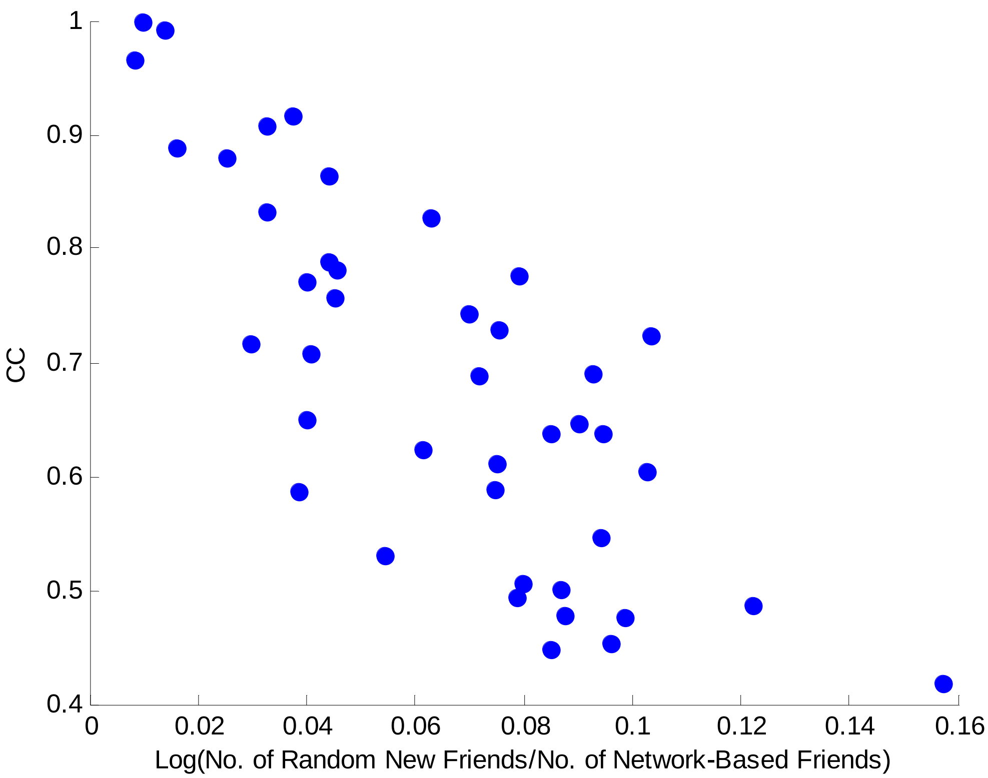

Figure 3b

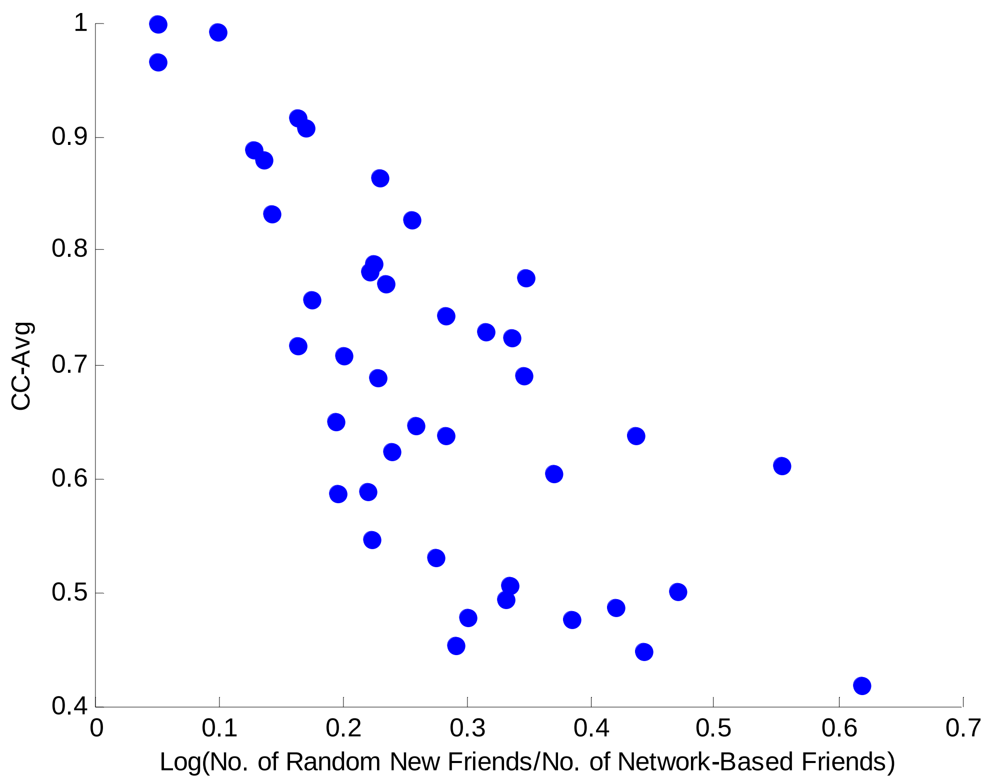

Figure 3c

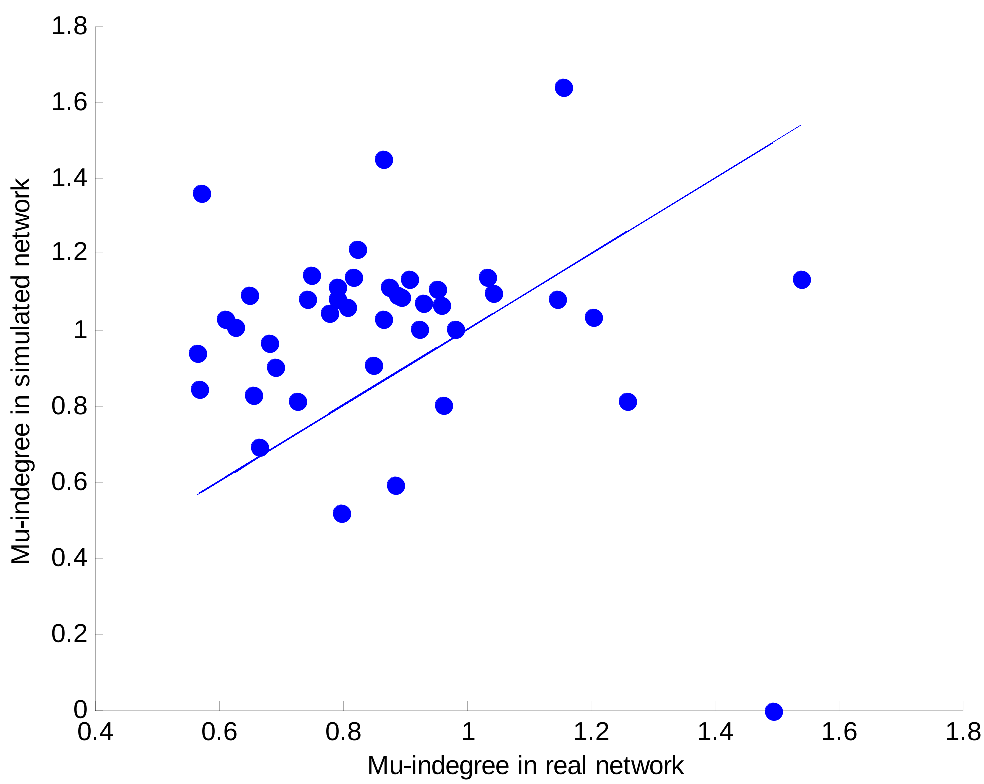

Figure 4a

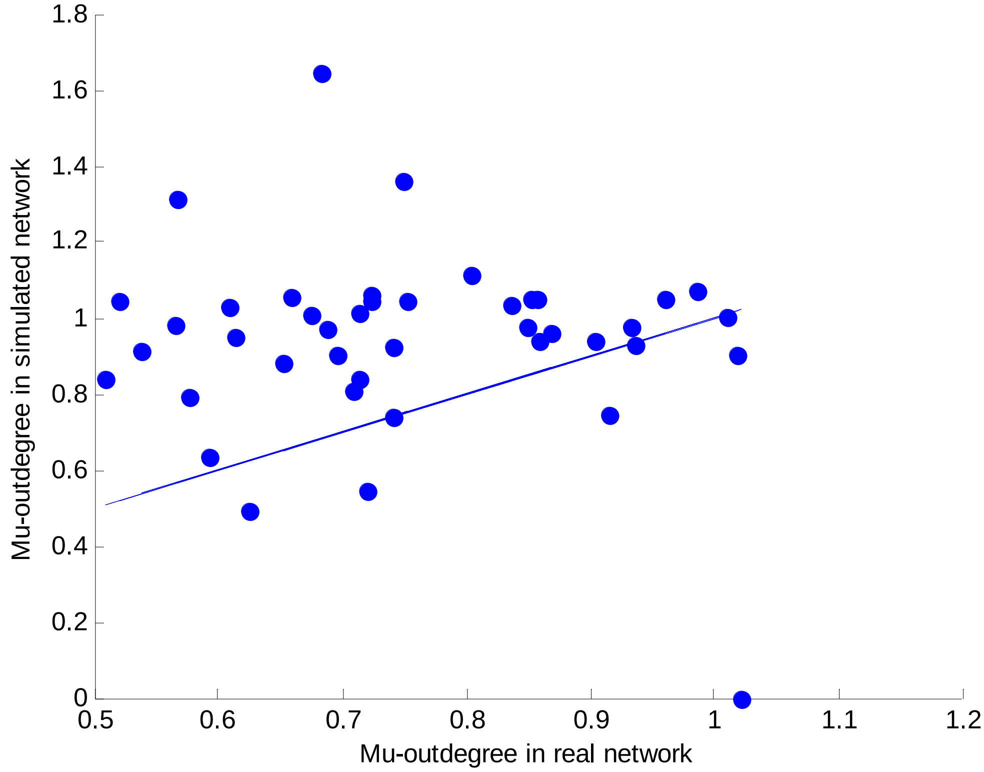

Figure 4b

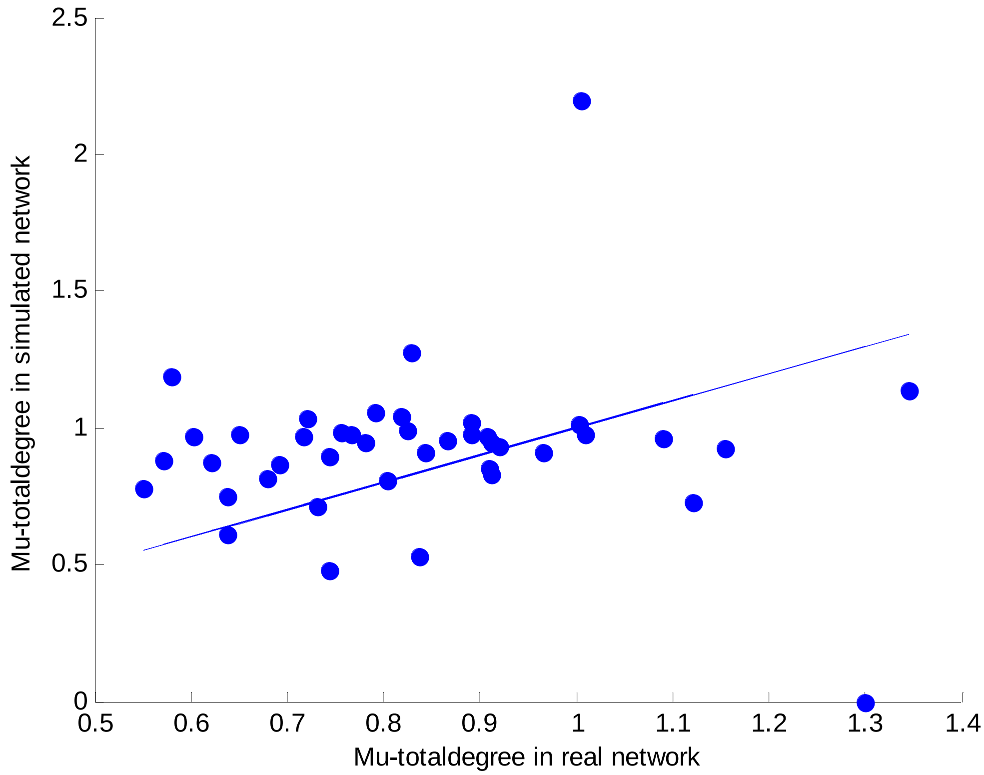

Figure 4c

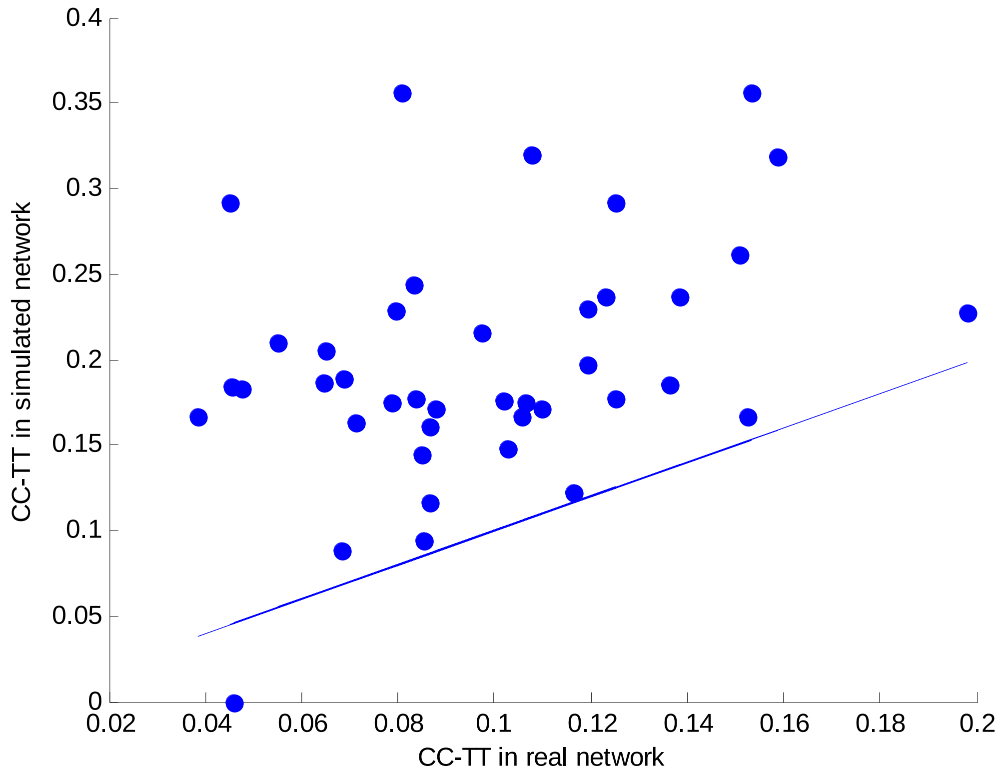

Figure 5a

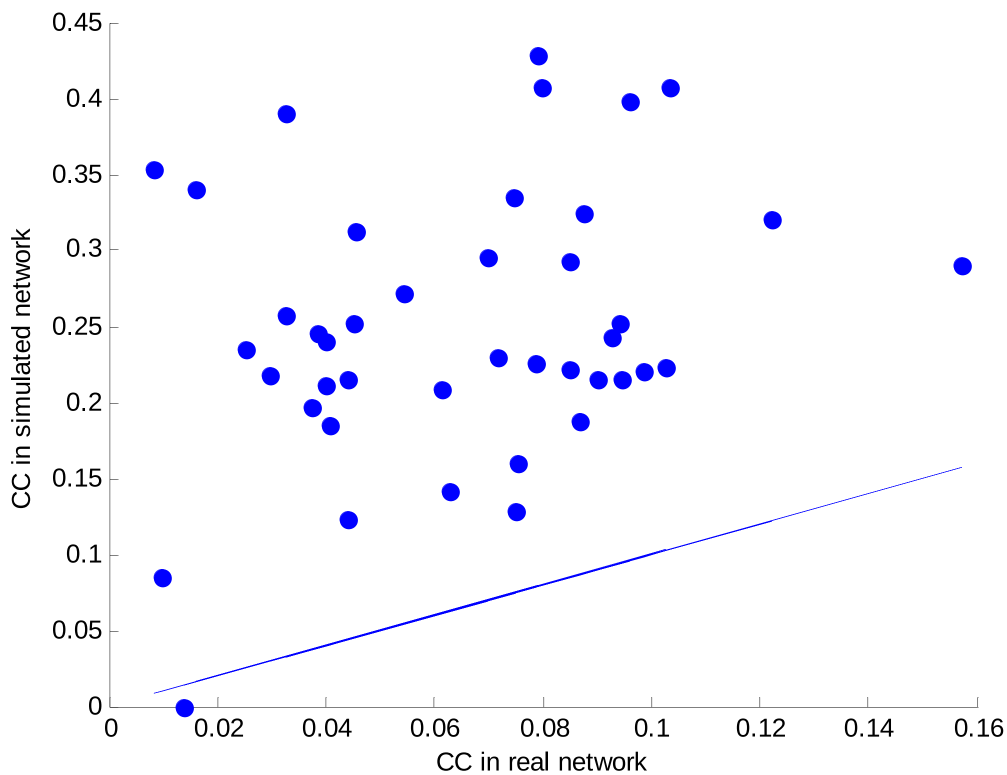

Figure 5b

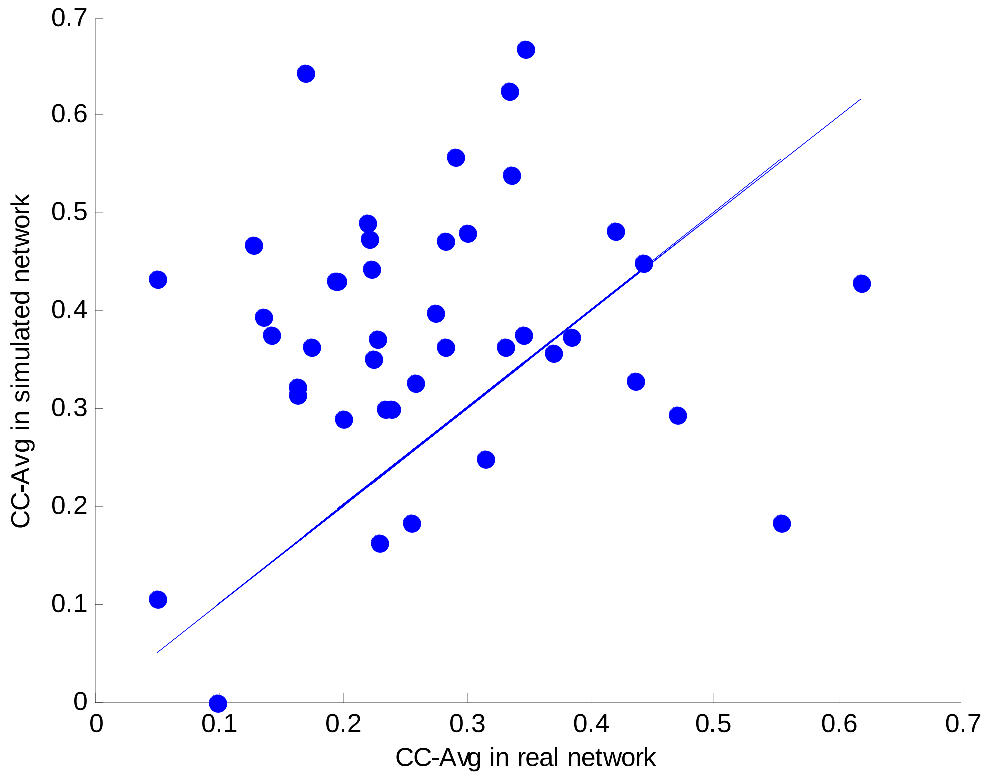

Figure 5c

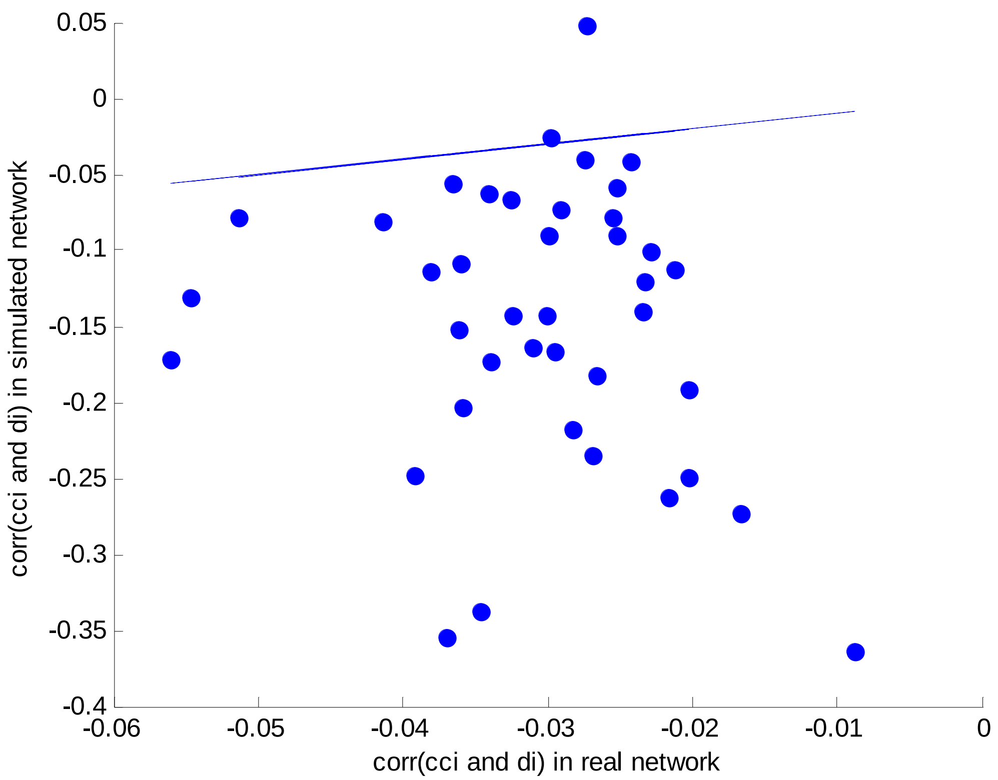

Figure 6

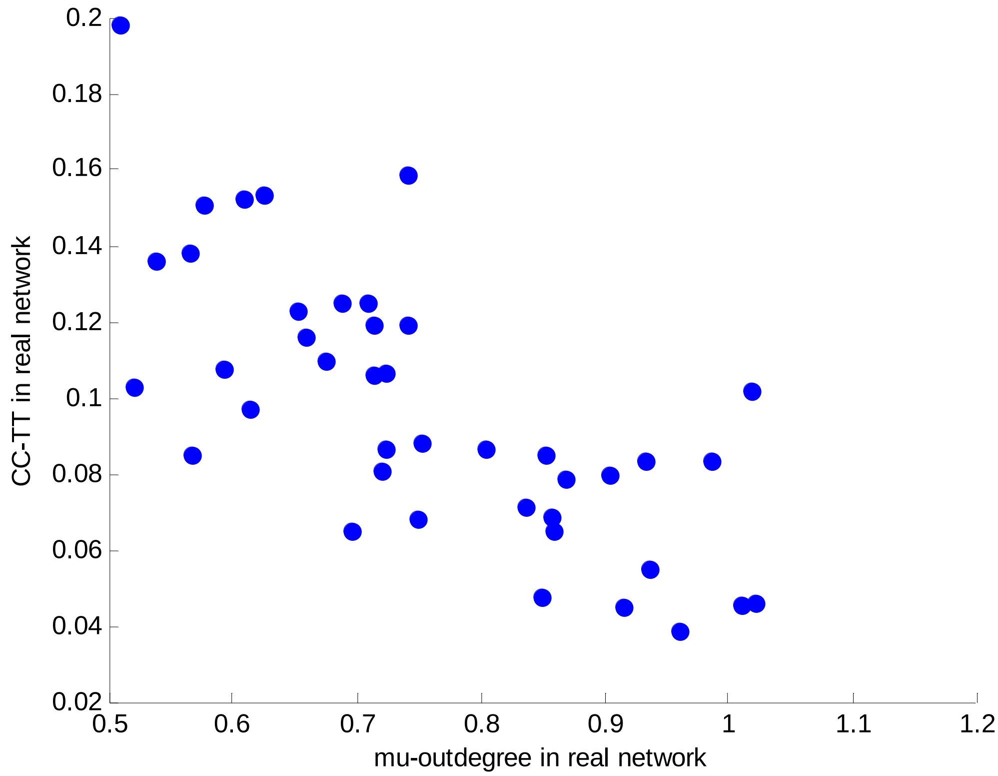

Figure 7a

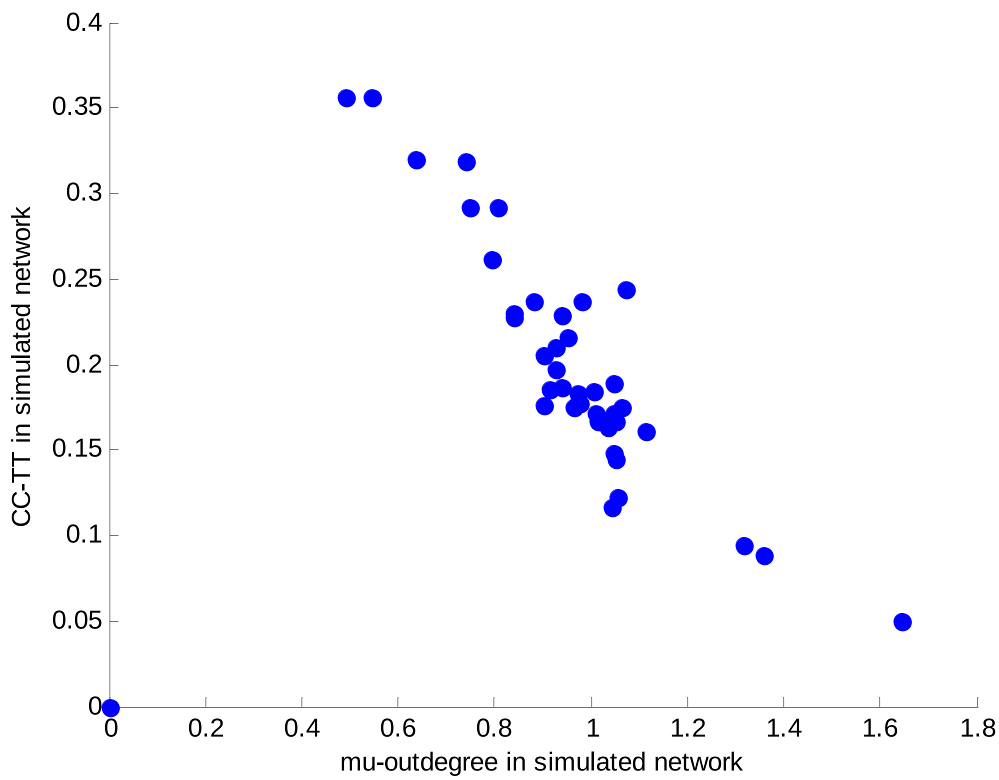

Figure 7b