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Asset Pricing with Concentrated Ownership of Capital*

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Abstract

This paper investigates how concentrated ownership of capital influences the pricing of risky assets in a production economy. The model is designed to approximate the skewed distribution of wealth and income in U.S. data. I show that concentrated ownership significantly magnifies the equity risk premium relative to an otherwise similar representative-agent economy because the capital owners' consumption is more strongly linked to volatile dividends from equity. A temporary shock to the technology for producing new capital (an "investment shock") causes dividend growth to be much more volatile than aggregate consumption growth, as in long-run U.S. data. The investment shock can also be interpreted as a depreciation shock, or more generally, a financial friction that affects the supply of new capital. Under power utility with a risk aversion coefficient of 3.5, the model can roughly match the first and second moments of key asset pricing variables in long-run U.S. data, including the historical equity risk premium. About one-half of the model equity premium is attributable to the investment shock while the other half is attributable to a standard productivity shock. On the macro side, the model performs reasonably well in matching the business cycle moments of aggregate variables, including the pro-cyclical movement of capital's share of total income in U.S. data.

Keywords: *Asset Pricing, Equity Premium, Term Premium, Investment Shocks, Real Business Cycles, Wealth Inequality.*

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1 Introduction

1.1 Overview

The distribution of wealth in the U.S economy is highly skewed. The top decile of U.S. households owns approximately 80 percent of financial wealth and about 70 percent of total wealth including real estate.[1] Shares of corporate stock are an important component of financial wealth, representing claims to the physical capital of firms. This paper investigates how concentrated ownership of capital influences the pricing of risky assets in a production economy. I show that concentrated ownership significantly magnifies the equity risk premium relative to an otherwise similar representative-agent economy because the capital owners' consumption is more strongly linked to volatile dividends from equity.

The framework for the analysis is a real business cycle model with capital adjustment costs and two types of stochastic shocks. In the baseline version of the model, the top decile of agents in the economy owns 100 percent of the productive capital stock—a setup that roughly approximates the skewed distribution of U.S. financial wealth. The consumption of the capital owners is funded from dividends and wage income. The consumption of the remaining agents (workers) is funded only from wage income. Since workers do not save, all assets (equity and bonds) are priced by the capital owners. The labor supply of capital owners and workers is inelastic, consistent with the near-zero elasticity estimates obtained by most empirical studies.[2] The ratio of the capital owners' labor supply to the total labor supply is calibrated to match the degree of income inequality in long-run U.S. data. When this ratio is equal to unity, the model collapses to a representative-agent framework. A standard "productivity shock" governs labor-enhancing technological progress and is assumed to evolve as a random walk with drift. A temporary but persistent "investment shock" impacts the technology for producing new capital. This shock is intended to capture exogenous technological changes that influence the relative contributions of new investment versus existing capital in the production of new capital goods. Empirical studies by Fischer (2006), Justiniano and Primiceri (2006), and Justiniano, et al. (2010) all suggest that shocks of this sort are an important source of macroeconomic fluctuations. The investment shock that I consider can also be interpreted as a capital quality shock or a depreciation shock that influences the economic value or obsolescence of existing capital. Liu et al. (2010) find that depreciation shocks account for up to 30 percent of output fluctuations at business cycle frequencies. Greenwood et al. (1988) were among the first to consider an investment shock in a real business cycle framework. In their model, the investment shock can influence the depreciation rate via variable capital utilization. More generally, shocks that appear in the capital accumulation equation can be interpreted as a reduced-form way of capturing financial frictions that impact the supply of new capital.[3]

[1] See Wolff (2006), Table 4.2, p. 113.

[2] For an overview of the empirical estimates, see Blundell and McCurdy (1999). Allowing for elastic labor supply on the part of workers would not change the asset pricing results because workers do not participate in financial markets.

[3] Furlanetto and Seneca (2010) explicitly distinguish between depreciation shocks, capital quality shocks,

The standard deviation of the productivity shock innovation is calibrated so that the model matches the volatility of real aggregate consumption growth in long-run U.S. data. The standard deviation of the investment shock innovation is calibrated so that the model matches the volatility of real dividend growth in the data. Figure 1 shows that dividend growth is about three times more volatile than aggregate consumption growth. While both series are less volatile in the post-World War II period, it remains true that dividend growth is about three times more volatile than consumption growth for the period 1947 to 2008. The model also captures the empirical observation that the consumption growth of stockholders is more volatile than that of non-stockholders, as documented recently by Malloy et al. (2009). Capital owners in the model demand a high equity premium because they must bear a disproportionate amount of aggregate consumption risk. In a representative-agent endowment economy with *iid* consumption growth, the equity risk premium relative to one-period bonds is given by the product of the coefficient of relative risk aversion and the variance of consumption growth.[4] The concentrated-ownership model serves to magnify the variance of the capital owners' consumption growth relative to aggregate consumption growth, thereby generating a much larger equity premium with reasonable levels of risk aversion.

Under power utility with a risk aversion coefficient of 3.5, the concentrated-ownership model can roughly match the first and second moments of key asset pricing variables in long-run U.S. data over the period 1900 to 2008. For the baseline calibration, the equity premium relative to one-period bonds is 5.6% in the model versus around 7% in the data. The equity premium relative to long-term bonds is 2.6% in the model versus around 5% in the data. Similar to Rudebusch and Swanson (2008), a long-term bond is modeled as a decaying-coupon consol with a Macauly duration of 10 years. The model's much smaller equity premium relative to long-term bonds reflects the fact that long-term bonds behave too much like equity—a result that can also occur in endowment economies.[5] The model does a good job of matching the high volatility of equity returns in the data, but somewhat overpredicts the volatility of long-term bond returns, again because these bonds behave too much like equity. When the model is calibrated to match the lower post-World War II volatilities of dividend and consumption growth, the risk aversion coefficient must be increased to 7.5 for the model to deliver an equity premium near 6% relative to one-period bonds.

Since labor supply is inelastic, capital owners must only decide the fraction of their available income to be devoted to investment, with the remaining fraction devoted to consumption. Using a power-function approximation of the true non-linear model, I derive an approximate analytical solution of the capital owner's decision rule which determines the investment-consumption ratio as a function of the existing capital stock and the two stochastic shocks. Making use of this decision rule, I derive approximate analytical expressions for the mean

and investment shocks.

[4] Specifically, we have $\log\left[E\left(R^s_{t+1}\right)/E\left(R^b_{t+1}\right)\right] = \alpha\,\mathrm{Var}[\log\left(c_{t+1}/c_t\right)]$, where R^s_{t+1} is the gross return on equity, R^b_{t+1} is the gross return on a one-period discount bond (the risk free rate), and α is the coefficient of relative risk aversion. For the derivation, see Abel (1994, p. 353).

[5] See, for example, Abel (2008), Table 2.

and variance of the equilibrium asset returns. I plot the moments of the equilibrium returns as functions of key model parameters. In simulations, the return moments generated by the non-linear model are close to those predicted by the approximate analytical solution.

In addition to the risk aversion coefficient, I investigate the impact of two other curvature parameters, namely, the elasticity of substitution between capital and labor in the production of aggregate output, and the elasticity of substitution between existing capital and new investment in the production of new capital. In both cases, lower elasticities (implying more curvature) imply higher costs of adjustment of the capital stock in response to shocks, which in turn lowers the mean return on equity as well its volatility, while holding constant the volatilities of dividend growth and aggregate consumption growth. The analytical moment expressions further reveal that about 45% of the model equity premium relative to one-period bonds is attributable to the investment shock while the remaining 55% is attributable to the productivity shock. In contrast, about 95% of the model equity premium relative to long-term bonds is attributable to the productivity shock.

On the macro side, the model performs reasonably well in matching the business cycle moments of aggregate variables, including the pro-cyclical behavior of capital's share of total income in U.S. data. In the concentrated-ownership model, *capital's* share of total income differs from the *capital owners'* share of total income to the extent that capital owners derive some income from wages. The pro-cyclical movement of capital's share in the model derives from the production technology for output, where the elasticity of capital-labor substitution is below unity, consistent with direct empirical estimates from U.S. data.

In response to a positive productivity shock, consumption, investment, dividends, and the equity price all increase relative to the no-shock trend. In contrast, a positive investment shock causes investment to increase, but at the expense of consumption and dividends which both decline. The decline in dividends leads to drop in the equity price. In simulations when both shocks are present, the growth rates of consumption, investment, dividends, and the equity price remain procyclical, consistent with data.

1.2 Related Literature

The model developed here is most closely related to Danthine and Donaldson (2002) who also employ a setup with capital owners and workers.[6] In their model, workers are not paid their marginal product but instead enter into long-term wage contracts with capital owners. The wage contract is designed to smooth workers' consumption streams by providing insurance against aggregate shocks, a mechanism they describe as "operational leverage." A persistent shock to the relative bargaining power of the two groups creates an additional source of risk that must be borne by the capital owners and contributes to a higher equity premium. Due to the insurance mechanism, capital's share of total income in the model is pro-cyclical despite the Cobb-Douglas production technology. When the bargaining power shocks are positively correlated with (temporary) productivity shocks, the model can produce an equity premium

[6] Further elaboration on the Danthine-Donaldson model can be found in Danthine et al. (2008).

relative to one-period bonds close to 6%, but the result is accompanied by too much volatility in the one-period bond return, i.e., a standard deviation in excess of 10 percent. Another drawback is the lack of independent empirical evidence that bargaining power shocks are an important source of macroeconomic fluctuations at business cycle frequencies. In contrast, there is considerable evidence to suggest the importance of investment shocks or depreciation shocks as a source of business cycle fluctuations.

Guvenen (2009) also develops a model with concentrated ownership of capital. Stockholders price equity while non-stockholders price one-period bonds. As buyers of the one-period bonds, non-stockholders have a very low elasticity of intertemporal substitution which makes them heavily dependent on the bond market to smooth their consumption, thereby producing a low equilibrium bond return, i.e., a low risk free rate. As sellers of the bonds, stockholders have a high elasticity of intertemporal substitution coupled with a relatively high risk aversion coefficient equal to 6. Stockholders must bear the risk of countercyclical interest payments to non-stockholders which amplifies the volatility of the stockholders' consumption streams, thereby raising their required rate of return on equity.[7] For the baseline model with inelastic labor supply, Guvenen's model delivers an equity premium relative to one-period bonds of about 5.5%, but he does not investigate the model's implications for long-term bonds. It is not clear how long-term bonds would be priced in Guvenen's model, since it appears that both types of agents would be willing to buy these bonds.

De Grave et al. (2010) develop a model that combines elements from both Danthine and Donaldson (2002) and Guvenen (2009). They allow for three types of agents, all with elastic labor supply: stockholders who price equity and long-term bonds, bondholders who price one-period bonds, and workers who do not save. They find that the stockholder-bondholder interaction from the Guvenen model is much less effective in generating a large equity premium when the model also includes the stockholder-worker wage bargaining shocks from the Danthine-Donaldson model. De Grave et al. assume that while one-period bonds are priced by the bondholders, long-term bonds are priced by the stockholders—a setup that seems hard to justify. An important limitation of all the foregoing models is that they abstract from long-run growth—a feature that affects the change in consumption from one period to the next. In contrast, the model developed here is calibrated to match both the mean and volatility of per capita consumption growth in long-run U.S. data.

Christiano and Fischer (2003) and Papanikolaou (2010) examine the asset pricing implications of investment specific technological change in two sector models where the "investment shock" is a geometric random walk with drift that drives growth in the investment goods-producing sector. In contrast, the investment shock in this paper is a stationary disturbance that closely resembles a depreciation shock. Finally, given the importance of the investment/depreciation shock in generating a sizeable equity premium in this paper, it is worth noting the connection with Barro (2009) who introduces two types of depreciation shocks—one representing normal fluctuations and the other representing rare disasters that destroy a signif-

[7] Guo (2004) develops a similar mechanism in the context of an endowment economy.

icant fraction of the capital stock. In this paper, a positive investment/depreciation shock can be viewed as subjecting physical capital to a kind of "mini-disaster risk" from technological obsolescence.

The remainder of the paper is organized as follows. First, I describe the model and the approximate analytical solution. I then describe the calibration procedure and investigate the model's quantitative properties. Specifically, I examine the sensitivity of the equilibrium return moments to changes in key model parameters. Next, using numerical simulations of the nonlinear model, I show that the model can match numerous quantitative features of long-run U.S. data. An appendix provides details on the model solution technique.

2 Model

The model consists of workers, capital owners, and competitive firms. There are n times more workers than capital owners, with the total number of capital owners normalized to one. The firms are owned by the capital owners. Workers and capital owners both supply labor to the firms inelastically, but in different amounts.

2.1 Workers

Workers are assumed to incur a transaction cost for saving or borrowing small amounts which prohibits their participation in financial markets. As a result, workers simply consume their labor income each period such that

$$c_t^w = w_t \, \ell_t^w,$$

where c_t^w is the individual worker's consumption, w_t is the competitive market wage, and $\ell_t^w = \ell^w$ is the constant supply of labor hours per worker.

2.2 Capital Owners

The capital owner's decision problem is to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left(\frac{c_t}{H_t}\right)^{1-\alpha} - 1}{1-\alpha}, \tag{1}$$

subject to the budget constraint

$$c_t \; + \; p_t^s q_{t+1}^s \; + \; p_t^b q_{t+1}^b \; + \; p_t^c q_{t+1}^c = \left(p_t^s + d_t\right) q_t^s \; + \; q_t^b + \; \left(\delta p_t^c + 1\right) q_t^c \; + \; w_t \, \ell_t^c, \tag{2}$$

where E_t represents the mathematical expectation operator, β is the subjective time discount factor, c_t is the individual capital owner's consumption, and α is the coefficient of relative risk aversion (the inverse of the intertemporal elasticity of substitution). When $\alpha = 1$, the within-period utility function can be written as $\log\left(c_t/H_t\right)$. Along the lines of Abel (1999), an individual capital owner derives utility from consumption relative to an exogenously-growing living standard index $H_t = \exp(\mu\, t)$, where μ is the economy's trend growth rate. This setup implies that capital owners today are not substantially "happier" (as measured in utility terms) than they were a hundred years ago because happiness is measured relative to an ever-improving living standard. Unlike habit formation models such as Jermann (1998) and Campbell and Cochrane (1999), the presence of H_t in the utility function here does not alter the interpretation of α as the coefficient of relative risk aversion. The net effect of H_t is to change the effective time discount factor which turns out to be useful in the calibration procedure.[8]

Capital owners derive labor income in the amount $w_t\, \ell_t^c$, where $\ell_t^c = \ell^c$ is the constant supply of labor hours per person. Capital owners may purchase the firm's equity shares in the amount q_{t+1}^s at the ex-dividend price p_t^s. Shares purchased in the previous period yield a dividend d_t. One-period discount bonds purchased in the previous period yield a single payoff of one consumption unit per bond. Capital owners may also purchase long-term bonds (consols) in the amount q_{t+1}^c at the ex-coupon price p_t^c. A long-term bond purchased in period t yields the following stream of decaying coupon payments (measured in consumption units) starting in period $t+1$: 1, δ, δ^2, ..., where δ is the decay parameter that governs the Macauly duration of the bond, i.e., the present-value weighted average maturity of the bond's cash flows.[9] When $\delta = 0$, the long-term bond collapses to a one-period bond. Equity shares are assumed to exist in unit net supply while both types of bonds exist in zero net-supply. Market clearing therefore implies $q_t^s = 1$ and $q_t^b = q_t^c = 0$ for all t.

The capital owner's first-order conditions with respect to q_{t+1}^s, q_{t+1}^b, and q_{t+1}^c are as follows:

$$p_t^s = E_t\, \beta \exp\left(-\phi\mu\right) \left[\frac{c_{t+1}}{c_t}\right]^{-\alpha} \left(p_{t+1}^s + d_{t+1}\right), \tag{3}$$

$$p_t^b = E_t\, \beta \exp\left(-\phi\mu\right) \left[\frac{c_{t+1}}{c_t}\right]^{-\alpha}, \tag{4}$$

$$p_t^c = E_t\, \beta \exp\left(-\phi\mu\right) \left[\frac{c_{t+1}}{c_t}\right]^{-\alpha} \left(1 + \delta p_{t+1}^c\right), \tag{5}$$

where $\phi \equiv 1 - \alpha$ and I have made the substitution $\left(H_{t+1}/H_t\right)^{-(1-\alpha)} = \exp\left(-\phi\mu\right)$. In equilibrium, the capital owner's budget constraint becomes $c_t = d_t + w_t\, \ell^c$, which shows that the capital owner's consumption is funded from dividends and wage income.

[8] The value of β is chosen to match the mean price-dividend ratio in long-run U.S. data. The presence of H_t allows the calibration target to be achieved with $\beta < 1$, even if risk aversion is high

[9] Rudebusch and Swanson (2008) employ a similar setup except that a long-term bond purchased in period t yields a declining coupon stream of 1, δ, δ^2... starting in period t rather than in period $t+1$.

2.3 Firms

The firm's output is produced according to the technology

$$y_t = \left\{\theta\, k_t^{\psi_y} + (1-\theta)\, \left[\left(\ell_t^c + n\, \ell_t^w\right) \exp\left(z_t\right)\right]^{\psi_y}\right\}^{\frac{1}{\psi_y}}, \quad \begin{array}{c} \theta \in (0,1) \\ \psi_y \equiv \frac{\sigma_y - 1}{\sigma_y} \\ \sigma_y \in (0,\, \infty) \end{array} \tag{6}$$

$$z_t = z_{t-1} + \mu + \varepsilon_t, \qquad \varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right), \tag{7}$$

with z_0 given. The symbol k_t is the firm's stock of physical capital and z_t is a labor-augmenting "productivity shock" that evolves as a random walk with drift. The drift parameter μ determines the trend growth rate of output. The total labor input is given by $\ell_t^c + n\, \ell_t^w$. The parameter ψ_y depends on the elasticity of substitution σ_y between capital and labor in production. When $\sigma_y = 1$ (or $\psi_y = 0$), we recover the usual Cobb-Douglas production technology. When $\sigma_y \to 0$ (or $\psi_y \to -\infty$), the production technology takes a Leontief formulation such that capital and labor become perfect compliments. When $\sigma_y \to \infty$ (or $\psi_y \to 1$), capital and labor become perfect substitutes.

Resources devoted to investment augment the firm's stock of physical capital according to the law of motion

$$k_{t+1} = B\left[(1-\lambda_t)\; k_t^{\psi_k} + \lambda_t\, i_t^{\psi_k}\right]^{\frac{1}{\psi_k}}, \quad \begin{array}{c} B > 0 \\ \psi_k \equiv \frac{\sigma_k - 1}{\sigma_k} \\ \sigma_k \in (0, \infty) \end{array} \tag{8}$$

$$\lambda_t = \lambda \exp\left(v_t\right), \qquad v_t = \rho\, v_{t-1} + u_t, \qquad u_t \sim N\left(0, \sigma_u^2\right), \tag{9}$$

with k_0 and v_0 given. The parameter ψ_k depends on the elasticity of substitution σ_k between existing capital and new investment in the production of new capital. As $\sigma_k \to 0$ (or $\psi_k \to -\infty$), the implicit cost of adjusting the capital stock from one period to the next increases.[10] A temporary but persistent "investment shock" v_t changes the relative importance of new investment versus existing capital in the production of new capital. As noted in the introduction, this shock can also be interpreted as a capital quality shock, a depreciation shock, or more generally, a financial friction that affects the supply of new capital. Starting from the above specification, we can recover the basic linear law of motion with no adjustment costs and a constant depreciation rate $\widehat{\delta}$ by imposing the following parameter settings: $\sigma_k = \infty$, $\lambda = 1/(2-\widehat{\delta})$, $B = 2 - \widehat{\delta}$, and $\sigma_u^2 = 0$.

Under the assumption that the labor market is perfectly competitive, firms take w_t as given and choose sequences of $\ell_{t+j}^c + n\, \ell_{t+j}^w$ and k_{t+1+j}, to maximize the following discounted stream of expected dividends:

$$E_0 \sum_{j=0}^{\infty} M_{t+j} \underbrace{\left[y_{t+j} - w_{t+j}\, \left(\ell_{t+j}^c + n\, \ell_{t+j}^w\right) - i_{t+j}\right]}_{d_{t+j}}, \tag{10}$$

[10] Kim (2003) shows that the intertemporal adjustment cost specification (8) can also be interpreted as a multisectoral adjustment cost that imposes a nonlinear transformation between consumption and investment in the national income identity.

subject to the production function (6) and the law of motion for capital (8). Firms act in the best interests of their owners such that dividends in period $t+j$ are discounted using the capital owner's stochastic discount factor $M_{t+j} \equiv \beta^j \exp(-\phi\mu j)(c_{t+j}/c_t)^{-\alpha}$.

The firm's first-order conditions are given by:

$$w_t = \frac{\left(1-s_t^k\right) y_t}{\ell^c + n\,\ell^w}, \tag{11}$$

$$i_t\,\mathrm{g}\left(\tfrac{k_{t+1}}{k_t},\, v_t\right) = E_t\,\beta\exp(-\phi\mu)\left[\frac{c_{t+1}}{c_t}\right]^{-\alpha}\left\{s_{t+1}^k\,y_{t+1} \;-\; i_{t+1} \;+\; i_{t+1}\,\mathrm{g}\left(\tfrac{k_{t+2}}{k_{t+1}},\, v_{t+1}\right)\right\}, \tag{12}$$

$$\text{where}\;\; \mathrm{g}\left(\tfrac{k_{t+1}}{k_t},\, v_t\right) \;\equiv\; 1 \;+\; \frac{1-\lambda\exp(v_t)}{\left(\frac{k_{t+1}}{Bk_t}\right)^{\psi_k} - 1 + \lambda\exp(v_t)},$$

$$s_t^k \;\equiv\; \frac{\theta k_t^{\psi_y}}{\theta\,k_t^{\psi_y} + (1-\theta)\,\left(\ell^c + n\,\ell^w\right)^{\psi_y}\exp\left(\psi_y\,z_t\right)},$$

which reflect the constant labor supplies ℓ^c and ℓ^w. Equation (11) shows that labor is paid its marginal product. The symbol s_t^k is used to represent capital's share of total income (or output) and $\left(1-s_t^k\right)$ represents labor's share. When $\sigma_y = 1$ (or $\psi_y = 0$), we have the Cobb-Douglas case where $s_t^k = \theta$. Comparing the first-order condition (12) to the equity pricing equation (3), we see that the ex-dividend price of an equity share is given by $p_t^s = i_t\,\mathrm{g}(k_{t+1}/k_t,\, v_t)$. The equity share is a claim to a perpetual stream of dividends $d_{t+1} = s_{t+1}^k\,y_{t+1} - i_{t+1}$ starting in period $t+1$.[11]

2.4 Approximate Analytical Solution

To facilitate a solution for the equilibrium allocations, the first-order condition (12) must be rewritten in terms of stationary variables. Since labor supply is inelastic, the combined entity of the firm and capital owner must only decide the fraction of available income to be devoted to investment, with the remaining fraction devoted to consumption. If we define the investment-consumption ratio as $x_t \equiv i_t/c_t$, then the economy's resource constraint $y_t = c_t + nc_t^w + i_t$ can

[11] After taking the derivitive of the profit function (10) with respect to k_{t+1}, I have multiplied both sides of the resulting first-order condition by k_{t+1}, which is known at time t.

be used to derive the following expressions for the equilibrium allocation ratios:

$$\frac{c_t}{y_t} = \frac{s_t^c}{1+x_t}, \tag{13}$$

$$\frac{n c_t^w}{y_t} = 1 - s_t^c, \tag{14}$$

$$\frac{i_t}{y_t} = \frac{s_t^c\, x_t}{1+x_t}, \tag{15}$$

$$\frac{d_t}{y_t} = s_t^k - \frac{s_t^c\, x_t}{1+x_t}, \tag{16}$$

where s_t^c is the capital owners' share of total income, given below. Defining the normalized capital stock as $k_{n,t} \equiv k_t / \left[\left(\ell^c + n\, \ell^w\right) \exp\left(z_t\right)\right]$, we have the following expressions:

$$\frac{y_t}{k_t} = \frac{\left[\theta\, k_{n,t}^{\psi_y} + 1 - \theta\right]^{\frac{1}{\psi_y}}}{k_{n,t}}, \tag{17}$$

$$s_t^k = \frac{\theta k_{n,t}^{\psi_y}}{\theta\, k_{n,t}^{\psi_y} + 1 - \theta}, \tag{18}$$

$$s_t^c = \frac{\theta k_{n,t}^{\psi_y} + (1-\theta)\left[\frac{\ell^c}{\ell^c + n\,\ell^w}\right]}{\theta\, k_{n,t}^{\psi_y} + 1 - \theta}, \tag{19}$$

which imply $s_t^k = s_t^c$ if capital owners do not work such that $\ell^c = 0$. Equation (18) implies $\partial s_t^k / \partial k_{n,t} < 0$ when $\sigma_y < 1$ such that $\psi_y < 0$. Capital's share of total income will therefore move in the opposite direction as the normalized capital stock $k_{n,t}$ if the elasticity of capital-labor substitution is below unity, as in the baseline calibration. A positive innovation to the productivity shock will raise z_t and thus lower $k_{n,t}$ producing pro-cyclical movement in s_t^k. A positive innovation to the investment shock will also lower $k_{n,t}$ and hence raise s_t^k because the investment shock is similar to a depreciation shock that erodes the capital stock k_t. Since labor supply is fixed, the cyclical behavior of s_t^c will be very similar to that of s_t^k.

Using the definition of $k_{n,t}$ and equation (8), the law of motion for the normalized capital stock is

$$k_{n,t+1} = B \exp\left(-z_{t+1} + z_t\right)\; k_{n,t} \left\{1 - \lambda \exp\left(v_t\right) + \lambda \exp\left(v_t\right) \left[\frac{i_t}{y_t}\, \frac{y_t}{k_t}\right]^{\psi_k}\right\}^{\frac{1}{\psi_k}}, \tag{20}$$

where the ratios i_t/y_t and y_t/k_t are given by equations (15) and (17). Similarly, the function $\mathrm{g}\left(k_{t+1}/k_t,\, v_t\right)$ that appears in the first-order condition (12) can be rewritten as follows

$$\mathrm{g}\left(\tfrac{k_{t+1}}{k_t},\, v_t\right) = \mathrm{g}_n\left(x_t,\, k_{n,t},\, v_t\right) = 1 + \frac{1 - \lambda \exp\left(v_t\right)}{\lambda \exp\left(v_t\right)} \left[\frac{i_t}{y_t}\, \frac{y_t}{k_t}\right]^{-\psi_k}. \tag{21}$$

An expression for the capital owner's consumption growth in terms of stationary variables can be obtained by combining equations (13) and (17) to yield

$$\frac{c_{t+1}}{c_t}=\left[\frac{s^c_{t+1}}{s^c_t}\right]\left[\frac{1+x_t}{1+x_{t+1}}\right]\left[\frac{y_{t+1}/k_{t+1}}{y_t/k_t}\right]\left[\frac{k_{n,t+1}}{k_{n,t}}\right]\exp\left(z_{t+1}-z_t\right) \tag{22}$$

Substituting these various expressions into equation (12) together with the capital owners' resource constraint $y_{t+1}=\left(c_{t+1}+i_{t+1}\right)/s^c_{t+1}$ yields the following transformed version of the first-order condition in terms of stationary variables:

$$\begin{aligned} x_t\,\mathrm{g}_n\left(x_t,\,k_{n,t},\,v_t\right)\left[\tfrac{\left(y_t/k_t\right)s^c_t\,k_{n,t}}{1+x_t}\right]^{\phi} \;\;=\;\; & E_t\left\{\beta\exp\left(\phi\,\varepsilon_{t+1}\right)\left[\tfrac{\left(y_{t+1}/k_{t+1}\right)s^c_{t+1}\,k_{n,t+1}}{1+x_{t+1}}\right]^{\phi}\right.\\ & \left.\times\left[\tfrac{s^k_{t+1}}{s^c_{t+1}}-x_{t+1}\left(1-\tfrac{s^k_{t+1}}{s^c_{t+1}}\right)\right]\;+\;x_{t+1}\,\mathrm{g}_n\left(x_{t+1},\,k_{n,t+1},\,v_{t+1}\right)\right\}, \end{aligned} \tag{23}$$

where I have made use of $z_{t+1}-z_t=\mu+\varepsilon_{t+1}$. Notice that the term involving $\exp\left(-\phi\mu\right)$ in the original first-order condition (12) has dropped out, leaving only β in the transformed version. There is a single decision variable x_t and two state variables, $k_{n,t}$ and v_t, with corresponding laws of motion given by equations (20) and (9).

To facilitate an analytical solution, both sides of the transformed first-order condition are approximated as power functions around the points $\widetilde{x}=\exp\left\{E\left[\log\left(x_t\right)\right]\right\}$, $\widetilde{k}_n=\exp\left\{E\left[\log\left(k_{n,t}\right)\right]\right\}$, and $\widetilde{v}=0$ to obtain:

$$\mathrm{a}_0\left[\frac{x_t}{\widetilde{x}}\right]^{\mathrm{a}_1}\left[\frac{k_{n,t}}{\widetilde{k}_n}\right]^{\mathrm{a}_2}\exp\left[\mathrm{a}_3\,v_t\right]\;\;=\;\;E_t\,\mathrm{b}_0\left[\frac{x_{t+1}}{\widetilde{x}}\right]^{\mathrm{b}_1}\left[\frac{k_{n,t+1}}{\widetilde{k}_n}\right]^{\mathrm{b}_2}\exp\left(\mathrm{b}_3\,v_{t+1}+\phi\,\varepsilon_{t+1}\right), \tag{24}$$

where $\mathrm{a_i}$ and $\mathrm{b_i}$, $\mathrm{i}=0,1,2,3$ are Taylor series coefficients that depend on both $\widetilde{x}$ and $\widetilde{k}_n$. Similarly, the law of motion for the normalized capital stock (20) can be approximated as

$$k_{n,t+1}\;=\;\widetilde{k}_n\left[\frac{x_t}{\widetilde{x}}\right]^{\mathrm{f}_1}\left[\frac{k_{n,t}}{\widetilde{k}_n}\right]^{\mathrm{f}_2}\exp\left[\mathrm{f}_3\,v_t-\varepsilon_{t+1}\right] \tag{25}$$

where $\mathrm{f_i}$, $\mathrm{i}=1,2,3$ are Taylor series coefficients. The approximate solution is given by the following proposition.

Proposition 1. *An approximate analytical solution for the capital owner's investment-consumption ratio is given by*

$$x_t\;=\;\widetilde{x}\left[\frac{k_{n,t}}{\widetilde{k}_n}\right]^{\gamma_k}\exp\left(\gamma_v\,v_t\right),$$

where $\widetilde{x} = \exp\{E[\log(x_t)]\}$ *and* $\widetilde{k}_n = \exp\{E[\log(k_{n,t})]\}$ *are the approximation points and* γ_k *and* γ_v *are given by the solutions to*

$$(\mathrm{b_1 f_1})\,\gamma_k^2 \;+ (\mathrm{b_1 f_2} + \mathrm{b_2 f_1} - \mathrm{a_1})\,\gamma_k + \mathrm{b_2 f_2} - \mathrm{a_2} = \;0,$$

$$\gamma_v = \frac{\rho \mathrm{b_3} + \mathrm{f_3}\,(\mathrm{b_1}\gamma_k + \mathrm{b_2}) - \mathrm{a_3}}{\mathrm{a_1} - \rho \mathrm{b_1} - \mathrm{f_1}\,(\mathrm{b_1}\gamma_k + \mathrm{b_2})},$$

provided $|\mathrm{f_1}\gamma_k + \mathrm{f_2}| < 1$.
Proof: See Appendix A.

The quadratic equation for γ_k in Proposition 1 has two solutions. The condition $|\mathrm{f_1}\gamma_k + \mathrm{f_2}| < 1$ selects the stationary root. Substituting the decision rule for x_t into equation (25) yields the following reduced-form law of motion for the normalized capital stock

$$k_{n,t+1} \;=\; \widetilde{k}_n \left[\frac{k_{n,t}}{\widetilde{k}_n}\right]^{\mathrm{f_1}\gamma_k + \mathrm{f_2}} \exp\left[(\mathrm{f_1}\gamma_k + \mathrm{f_3})\; v_t - \varepsilon_{t+1}\right], \tag{26}$$

which shows that $|\mathrm{f_1}\gamma_k + \mathrm{f_2}| < 1$ is needed for stationarity. Given the stochastic properties of v_t and ε_{t+1}, the above law of motion can be used to derive an analytical expression for $Var\left[\log(k_{n,t})\right].$ The variance of the log investment-consumption ratio is then given by

$$Var\left[\log(x_t)\right] \;=\; (\gamma_k)^2\, Var\left[\log(k_{n,t})\right] + (\gamma_v)^2\, Var\,(v_t) + 2\gamma_k\gamma_v Cov\left[\log(k_{n,t})\,,\; v_t\right]. \tag{27}$$

Similarly, the capital owner's consumption growth (22) can be approximated as

$$\frac{c_{t+1}}{c_t} \;\simeq\; \exp(\mu)\left[\frac{k_{n,t}}{\widetilde{k}_n}\right]^{\mathrm{h_1}} \exp\left[\mathrm{h_2}\, v_t + \mathrm{h_3}\, u_{t+1} + \mathrm{h_4}\, \varepsilon_{t+1}\right], \tag{28}$$

where $\mathrm{h_i}$, $\mathrm{i} = 1, 2, 3, 4$ are Taylor series coefficients. The above equation can be used to derive an analytical expression for $Var\left[\log(c_{t+1}/c_t)\right].$

Later, in the model simulations, I demonstrate that the approximate analytical solution yields results which are close to those generated by an alternate solution method that preserves the model's nonlinear equilibrium conditions and employs a version of the parameterized expectation algorithm (PEA) described by Den Haan and Marcet (1990).

2.5 Asset Pricing Variables

Given the equilibrium relationships $p_t^s = i_t\, \mathrm{g}_n\,(x_t,\; k_{n,t},\; v_t)\,,\, d_t = s_t^k\, y_t - i_{t,}$ and $y_t = (c_t + i_t)\,/s_t^c$, it is straightforward to derive the following expressions for the equity price-dividend ratio and the gross equity return in terms of stationary variables:

$$\frac{p_t^s}{d_t} = \left[\frac{x_t}{s_t^k/s_t^c - \left(1 - s_t^k/s_t^c\right)\, x_t}\right] \mathrm{g}_n\left(x_t,\, k_{n,t},\, v_t\right), \tag{29}$$

$$\begin{aligned} R_{t+1}^s &= \frac{p_{t+1}^s + d_{t+1}}{p_t^s} \\ &= \frac{c_{t+1}}{c_t}\left[\frac{x_{t+1}\,\mathrm{g}_n\left(x_{t+1},\, k_{n,t+1},\, v_{t+1}\right) + s_{t+1}^k/s_{t+1}^c - \left(1 - s_{t+1}^k/s_{t+1}^c\right)\, x_{t+1}}{x_t\,\mathrm{g}_n\left(x_t,\, k_{n,t},\, v_t\right)}\right], \end{aligned} \tag{30}$$

where c_{t+1}/c_t is given by equation (22). After making the appropriate substitutions, the price-dividend ratio can be approximated as a power function of the state variables $k_{n,t}$ and v_t, while the equity return can be approximated as a power function of $k_{n,t}$, v_t, u_{t+1}, and ε_{t+1}.

The remaining asset pricing variables are the one-period bond return R_{t+1}^b (the risk free rate) and the long-term bond return R_{t+1}^c which are defined as follows:

$$R_{t+1}^b = \frac{1}{p_t^b} = \frac{1}{E_t\,\beta\exp\left(-\phi\mu\right)\left[\frac{c_{t+1}}{c_t}\right]^{-\alpha}}, \tag{31}$$

$$R_{t+1}^c = \frac{1 + \delta p_{t+1}^c}{p_t^c} = \frac{1 + \delta p_{t+1}^c}{E_t\,\beta\exp\left(-\phi\mu\right)\left[\frac{c_{t+1}}{c_t}\right]^{-\alpha}\left(1 + \delta p_{t+1}^c\right)}. \tag{32}$$

The conditional expectation in equation (31) can be computed analytically using the approximate version of c_{t+1}/c_t in equation (28). The price of the long-term bond p_t^c must computed separately as the solution to the first-order condition (5). Proceeding along the same lines as the solution for x_t, the first-order condition (5) can be approximated as

$$\begin{aligned} p_t^c &\simeq E_t\,\beta\exp\left(-\phi\mu\right)\left[\frac{c_{t+1}}{c_t}\right]^{-\alpha}\left[\frac{p_{t+1}^c}{\widetilde{p}^c}\right]^{\mathrm{b}_1^c}, \\ &\simeq \beta\exp\left(-\mu\right)\left[\frac{k_{n,t}}{\widetilde{k}_n}\right]^{-\alpha\,\mathrm{h}_1}\exp\left(-\alpha\,\mathrm{h}_2\,v_t\right)\, E_t\left[\frac{p_{t+1}^c}{\widetilde{p}^c}\right]^{\mathrm{b}_1^c}\exp\left(-\alpha\,\mathrm{h}_4\,\varepsilon_{t+1}\right), \end{aligned} \tag{33}$$

where I have substituted in the approximate expression for c_{t+1}/c_t from equation (28). The approximation point is $\widetilde{p}^c = \exp\left\{E\left[\log\left(p_t^c\right)\right]\right\}$ and $\mathrm{b}_1^c = \delta\widetilde{p}^c/\left(1 + \delta\widetilde{p}^c\right)$ is a Taylor series coefficient. The approximate analytical solution takes the form

$$p_t^c = \widetilde{p}^c\left[\frac{k_{n,t}}{\widetilde{k}_n}\right]^{\gamma_k^c}\exp\left(\gamma_v^c\, v_t\right), \tag{34}$$

where the consol pricing coefficients γ_k^c and γ_v^c depend on the investment-consumption decision rule coefficients γ_k and γ_v from Proposition 1.

Using power function approximations of the returns defined by equations (30), (31), and (32), it is straightforward to derive the following expressions for the unconditional mean log returns

$$E\left[\log\left(R^s_{t+1}\right)\right] = -\log(\beta) + \mu - \tfrac{1}{2}\left(\mathrm{b}_1\gamma_v + \mathrm{b}_3\right)^2 \sigma_u^2 - \tfrac{1}{2}\left(\phi - \mathrm{b}_1\gamma_k - \mathrm{b}_2\right)^2 \sigma_\varepsilon^2, \tag{35}$$

$$E\left[\log\left(R^b_{t+1}\right)\right] = -\log(\beta) + \mu - \tfrac{1}{2}\left(\alpha\,\mathrm{h}_3\right)^2 \sigma_u^2 - \tfrac{1}{2}\left(\alpha\,\mathrm{h}_4\right)^2 \sigma_\varepsilon^2, \tag{36}$$

$$E\left[\log\left(R^c_{t+1}\right)\right] = -\log(\beta) + \mu - \tfrac{1}{2}\left(\mathrm{b}^c_1\gamma^c_v - \alpha\,\mathrm{h}_3\right)^2 \sigma_u^2 - \tfrac{1}{2}\left(-\mathrm{b}^c_1\gamma^c_k - \alpha\,\mathrm{h}_4\right)^2 \sigma_\varepsilon^2. \tag{37}$$

Differences in the mean log returns across assets are comprised of two parts; one part depends on the volatility of the investment shock innovation while the other part depends on the volatility of the productivity shock innovation.[12] The power function approximations of the returns can also be used to derive analytical expressions for $Var\left[\log\left(R^s_{t+1}\right)\right]$, $Var\left[\log\left(R^b_{t+1}\right)\right]$, and $Var\left[\log\left(R^c_{t+1}\right)\right]$. Given the first and second moments of the log returns, the unconditional moments of R^s_{t+1}, R^b_{t+1}, and R^c_{t+1} can be computed analytically by making use of the properties of the log-normal distribution.[13]

3 Model Calibration

A time period in the model is taken to be one year. The baseline parameters are chosen simultaneously to match various empirical targets, as summarized in Table 1. The analytical moment formulas derived from the log-linear approximate solution of the model are used as starting points for the nonlinear model calibration. A process of trial and error is used to select the parameter values which are used for the nonlinear model simulations.

[12] If the exogenous living standard index H_t is omitted from the utility function (1), then the constant term μ in the mean log return expressions would be replaced by $\alpha\mu$, where α is the risk aversion coefficient. When H_t is present, the net effect is equivalent to employing a larger value of β for $\alpha > 1$.

[13] If a random variable R_t is log-normally distributed, then $E(R_t) = \exp\left\{E[\log(R_t)] + \frac{1}{2}Var[\log(R_t)]\right\}$ and $Var(R_t) = E(R_t)^2\left\{\exp\left(Var[\log(R_t)]\right) - 1\right\}$.

Table 1: Baseline Parameter Values

Parameter	Log-linear Model	Nonlinear Model	Description/Target
n	9	9	Capital owners = top income decile
$\frac{\ell^c}{\ell^c+n\ell^w}$	0.063	0.061	Mean $s_t^c = 0.4$, income share of top decile
θ	0.836	0.801	Mean $s_t^k = 0.36$, capital's share of income
α	3.5	3.5	Mean equity premium $\simeq$ 6 %
σ_y	0.55	0.55	Empirical estimates: 0.4 - 0.6
σ_k	0.45	0.45	Std. dev. equity return $\simeq$ 20 %
B	1.071	1.078	Mean $k_t/y_t = 2.8$
λ	0.0029	0.0032	Mean $i_t/y_t = 0.22$
μ	0.0203	0.0203	Mean consumption growth = 2.03 %
σ_ε	0.0558	0.0564	Std. dev. consumption growth = 3.51 %
σ_u	0.2909	0.2584	Std. dev. dividend growth = 11.7 %
ρ	0.9	0.9	Corr. $\left(p_t^s/d_t,\ p_{t-1}^s/d_{t-1}\right) \simeq 0.9$
β	0.9518	0.9519	Mean $p_t^s/d_t = 26.6$
δ	0.9650	0.9648	Consol duration = 10 years

The number of workers per capital owner is set to $n = 9$ so that capital owners represent the top income decile of households in the model economy. At the baseline calibration, capital owners supply 6 percent of the total labor input so that the top income decile in the model earns 40 percent of total income on average, consistent with the long-run average income share measured by Piketty and Saez (2003). I investigate the sensitivity of the results to changing the trend value of s_t^c, which is adjusted by changing the relative magnitudes of ℓ^c and ℓ^w. When $\ell^w = 0$, we have $s_t^c = 1$ for all t and the model collapses to a representative agent framework. When $\ell^c = 0$, we have $s_t^c = s_t^k$ for all t and we have the basic capitalist-worker framework employed by Judd (1985), Lansing (1999), and others. The production function parameter θ is chosen so that the average value of capital's share of total income in the model matches the corresponding U.S. average.[14] Table 2 compares the model distribution for income and wealth to the corresponding distribution in the U.S. economy. The U.S. financial wealth distribution data are from Wolff (2006), covering the period 1983 to 2001. The Gini coefficient data for income are from Heathcote et al. (2010) using the Current Population Survey for the period 1967 to 2005.

[14] Capital's share of total income is defined as 1− labor's share, where labor's share for the period 1947 to 2008 is obtained from <www.bls.gov/data>, series ID PRS85006173.

Table 2: Income and Wealth Distribution: Data versus Model

Statistic	U.S. Data	Model
Top decile share Income	40%[a]	40%
Top decile share Financial wealth	80%[b]	100%
Gini coefficient Income	0.32 - 0.42[c]	0.30
Gini coefficient Financial wealth	0.89 - 0.93[b]	0.90

Sources: a = Piketty and Saez (2003), b = Wolff (2006), c = Heathcote et al. (2010).

The parameters α, σ_y, and σ_k each govern an aspect of curvature in the model. The baseline risk aversion coefficient $\alpha = 3.5$ is chosen to achieve an equity premium relative to one-period bonds close to 6 percent. The baseline value of the capital-labor substitution elasticity is $\sigma_y = 0.55$. Chirinko (2008) reviews the many studies that have attempted to estimate this parameter using various econometric methods. He concludes that "the weight of the evidence suggests a value of [the elasticity parameter] in the range of $0.40 - 0.60$." The baseline value of the capital–investment substitution elasticity is $\sigma_k = 0.45$. In conjunction with the other parameters, this value delivers an empirically plausible volatility for the model's equity return. I examine the sensitivity of the results to changes in α, σ_y, and σ_k.

The volatility of the productivity shock innovation σ_ε is chosen so that the model matches the standard deviation of real per capita consumption growth in long-run annual U.S. data. The volatility of the investment shock innovation σ_u is chosen so that the model matches the standard deviation of dividend growth for the S&P 500 stock index.[15] I examine the sensitivity of the results to changes in the magnitude of both σ_ε and σ_u. I also examine the implications of calibrating the model to match the post-World War II volatilities of dividend and consumption growth.[16]

The parameter δ is set so that the Macauly duration of the long-term bond is $D = 10$ years. The Macauly duration is the present-value-weighted average maturity of the bond's cash flows, computed as follows:

$$D = \frac{\sum_{t=0}^{\infty} \left(\widetilde{M}\,\delta\right)^t (t+1)}{\sum_{t=0}^{\infty} \left(\widetilde{M}\,\delta\right)^t} = \frac{1}{1 - \widetilde{M}\,\delta}, \tag{38}$$

where $\widetilde{M}$ is the trend stochastic discount factor defined as $\widetilde{M} = \exp\left[E \log\left(M_{t+1}\right)\right] = \beta \exp\left(-\mu\right).$

[15] The series for real stock prices, real dividends, and real per capita consumption employed in the paper are from Robert Shiller's website <http://www.econ.yale.edu/~shiller/>. The price-dividend ratio in year t is defined as the value of the S&P 500 stock index at the beginning of year $t+1$, divided by the accumulated dividend over year t.

[16] For the period 1947 to 2008, the standard deviation of real dividend growth is 5.4% while the standard deviation of real per capita consumption growth is 1.75%.

4 Quantitative Results

4.1 Impulse Response Functions

Figure 2 plots the model response to a one standard error innovation of the productivity shock (blue line) and the investment shock (red line). The responses are computed using the solution of the nonlinear model which is outlined in Appendix B. In both cases, the figure shows the percentage deviation from the no-shock trend. The effects of the productivity shock innovation are permanent due to the unit root in the law of motion (7), whereas the effects of the investment shock are temporary, but very persistent.

An important distinction between the two shocks is that a positive productivity shock expands the amount of available output that can be used to increase both consumption and investment. In contrast, a positive investment shock serves to increase investment at the expense of consumption. The investment shock is very similar to a depreciation shock, as discussed in more detail later. A positive investment shock temporarily erodes the capital stock relative to the no-shock trend which in turn reduces output relative to no-shock trend. The capital owner's consumption recovers more quickly than the worker's consumption because a positive investment shock temporarily boosts the capital owner's share of total income s_t^c. This is so because both s_t^c and s_t^k move in the opposite direction as the normalized capital stock $k_{n,t}$ when $\sigma_y < 1$ such that $\psi_y < 0$. Despite the drops in capital and total output, the capital owners' share of that output rises, which serves to accelerate the recovery of the capital owners' consumption relative to the workers' consumption.

The effect of the two shocks on asset prices is also very different. A positive productivity shock allows for a permanent increase in dividends which permanently raises the equity price. Bond prices also increase to satisfy the no-arbitrage condition across the different asset classes. In contrast, a positive investment shock stimulates investment temporarily, but since output is reduced (due to the erosion of the capital stock), there are now less resources to pay dividends, so dividends must be reduced for a time. The reduction in dividends temporarily lowers the equity price. Bond prices also decline to satisfy the no-arbitrage condition. This feature of the model is consistent with empirical evidence that the stock market reacts negatively to technology innovations that accelerate the obsolescence of existing capital (see Hobijn and Jovanovic, 2001).

As the investment shock dissipates, the level of investment returns to the no shock-trend while both dividends and the equity price recover upwards, but then slightly overshoot the no-shock trend. The overshooting occurs because a positive investment shock boosts capital's share of total income s_t^k in a persistent manner, thus providing some additional resources from which to pay dividends. The fact that a temporary investment shock can induce a large move in the equity price helps the model to match the volatility of equity returns in U.S. data. However, as we shall see in the simulations, the volatility of the model price-dividend ratio is still below the volatility observed in the data.

4.2 Sensitivity of Return Moments to Key Parameters

Figures 3 and 4 plot the mean and standard deviation of the asset returns as key parameters are varied. A vertical line in each panel marks the baseline value for each parameter being examined. The return moments are computed using the approximate analytical solution of the model. The approximate log-linear solution employs a slightly different baseline calibration for the parameters θ, B, λ, σ_ε, and σ_u, as shown in Table 1. This is done so that the approximate solution matches the same empirical targets as the nonlinear model.

For the first four cases, when a given parameter is changed, the remaining non-curvature parameters are adjusted to maintain the same empirical targets. The three curvature parameters α, σ_y, and σ_k are maintained at their baseline values except when they are the subject of a particular sensitivity experiment. In the final two cases, the standard deviation of a shock innovation is being varied. In these instances, when σ_ε is being varied, I hold σ_u constant at its baseline value and vice versa when σ_u is being varied. Hence for these two plots only, the model does not match the volatilities of U.S. consumption and dividend growth growth except at the baseline calibration.

The top two panels in Figure 3 show the effect of changing the trend value of the capital owners' share of total income, i.e., $\widetilde{s}^c = \exp\left\{E\left[\log\left(s_t^c\right)\right]\right\}$. At the extreme right we have $\widetilde{s}^c = 1$ which is achieved by setting $\ell^w = 0$ so that the model collapses to a representative-agent framework. At the extreme left, we have $\widetilde{s}^c = \widetilde{s}^k = 0.36$ which is achieved by setting $\ell^c = 0$ so that the model coincides with a basic capitalist-worker framework. Intermediate values of $\widetilde{s}^c$ are obtained by varying the ratio $\ell^c / \left(\ell^c + n\,\ell^w\right)$.[17] Starting from $\widetilde{s}^c = 1$ at the extreme right, we see that the representative-agent version of the model yields a small equity premium and a low volatility of equity returns. Papanikolaou (2010) also obtains a small equity premium in a representative-agent model with nonstationary investment shocks.

As $\widetilde{s}^c$ declines towards the lower bound of $\widetilde{s}^k = 0.36$, the equity premium relative to the one-period bond increases dramatically and the return volatilities for all assets increase. The intuition is straightforward: a decline in $\widetilde{s}^c$ implies that a higher proportion of the capital owners' consumption is funded from dividends rather than wage income. Since dividend growth is about three times more volatile than aggregate consumption growth (in both the model and the data), the capital owners demand a higher rate of return on equity to compensate for the risk of linking their consumption stream to volatile dividends. The return on the one-period bond actually declines with $\widetilde{s}^c$ due to the capital owners' precautionary saving motive which causes them to bid up the price of the bond. At the baseline calibration with $\widetilde{s}^k = 0.40$ and $\widetilde{s}^k = 0.36$, the model produces an equity premium relative to one-period bonds that is close to 6 percent and an equity return volatility of about 20 percent. Both figures are close to those in the data. But as noted in the introduction, the equity premium relative to the consol bond is much smaller, only around 3 percent, since these bonds behave too much like equity in this framework.

[17] Specifically, I vary ℓ^c between 0 and 1 with $\ell^w = 1 - \ell^c$.

The middle panel of Figure 3 shows the effect of increasing the risk aversion coefficient α. At the extreme left when $\alpha = 0$, capital owners are risk neutral and the equity premium relative to both types of bonds is zero. Moreover, since the stochastic discount factor is constant when $\alpha = 0$, the return volatility of the bonds is also zero. As risk aversion increases, the mean return on equity increases rapidly while the mean return on one-period bonds actually declines, again due to the capital owners' precautionary saving motive. The mean return on the consol initially declines a bit with risk aversion (due to the precautionary saving motive) but then starts increasing with risk aversion but at a slower rate than the equity return. The return volatilities all increase with risk aversion because the stochastic discount factor becomes more variable, thus increasing the volatility of the equilibrium asset prices.

The bottom panel of Figure 3 shows the effect of changing the substitution elasticity between capital and labor in production. At the extreme right when $\sigma_y = 1$, the production function is Cobb-Douglas such that $s_t^k = \theta$ for all t. Empirical estimates for σ_y are in the range of 0.4 to 0.6. As σ_y declines, the curvature of the production technology increases, while holding fixed the volatilities of dividend growth and aggregate consumption growth. The figure shows that changes in σ_y have only a mild effect on return moments over most of the range examined. However, at the extreme left when σ_y approaches a value of 0.5, the effect on return moments is more pronounced. In this region of the parameter space, more curvature in the production function serves to lower the mean and volatility of the equity return, with the effect of shrinking the equity premium relative to one-period bonds. A smaller value of σ_y effectively imposes a higher cost of adjusting the capital stock in response to shocks, which makes equity appear less risky relative to the one-period bond.

The top panel of Figure 4 shows the effect of changing the substitution elasticity between existing capital and new investment in the production of new capital. At the extreme right when $\sigma_k = 1$, the capital law of motion is Cobb-Douglas and the equity price can be represented simply as $p_t^s = i_t/\lambda_t$. A smaller value of σ_k implies more curvature in the capital law of motion while holding fixed the volatilities of dividend growth and aggregate consumption growth. Similar to the effect of changing σ_y, a smaller value of σ_k reduces the mean and volatility of the equity return and shrinks the equity premium relative to the one-period bond.

The middle panel of Figure 4 shows the effect of changing the standard deviation of the productivity shock innovation σ_ε while holding σ_u constant at the baseline value. Higher values of σ_ε raises the equity premium relative to both types of bonds. In particular, higher values of σ_ε stimulate precautionary saving which serves to reduce the required rate of return on the one-period bond.

The bottom panel of Figure 4 shows the effect of changing the volatility of the investment shock innovation σ_u while holding σ_ε constant at its baseline value. As noted in the introduction, the investment shock can be viewed as subjecting capital owners to a kind of "mini-disaster risk" that boosts the required return on equity. Higher values of σ_u raise the equity premium relative to the one-period bond, but the premium relative the consol bond is little changed. As seen previously with the impulse response functions plotted in Figure 2, the consol bond responds to the investment shock in much the same way as equity.

Using the analytical expressions for the mean log returns given by equations (35) through (37), it is possible to decompose the equity premium into two parts, each attributable to one of the two shock innovations.[18] At the baseline calibration, 45 percent of the equity premium relative to one-period bonds is attributable to the temporary investment shock, while 55 percent is attributable to the permanent productivity shock. In contrast, only 6 percent of the equity premium relative to consols is attributable to the investment shock while 95 percent is attributable to the productivity shock. The investment shock accounts for the high volatility of the dividend stream which is a significant source of risk relative to the one-period bond. The capital owner's stochastic discount factor is strongly linked to dividend growth and hence is strongly influenced by the investment shock. The consol bond comes with its own stream of payments, the value of which is influenced by the variability of the stochastic discount factor. The productivity shock is the main source of equity risk relative to the consol bond because this shock affects the stochastic growth rate of the dividend stream. In contrast, the coupon payments from the consol do not grow over time but rather decay at a constant rate.

4.3 Nonlinear Model Simulations

Figure 5 shows that there is close agreement between the log-linear approximate solution of the model and the solution of the nonlinear model that employs the parameterized expectation algorithm. Lansing (2010) demonstrates the accuracy of a very similar approximate solution method by comparison to the exact solution in an endowment economy with autocorrelated dividend growth.

Figure 6 demonstrates the similarity of the investment shock to a depreciation shock of the sort considered by Liu et al. (2009). The figure shows a scatterplot of gross capital growth k_{t+1}/k_t versus the investment-capital ratio i_t/k_t generated by a long simulation of the nonlinear model. The top panel plots the mean relationship (in blue) between the two ratios by inserting the mean value $E(\lambda_t)$ from the simulation into the capital law of motion (8). The dashed lines show the corresponding shifts in the mean relationship from adding or subtracting one standard deviation of λ_t from its mean value. The upward-sloping straight line (in red) is the hypothetical relationship implied by a linear law of motion for capital with no adjustment costs and a constant annual depreciation rate, i.e., $k_{t+1}/k_t = 1-\widehat{\delta}+i_t/k_t$. The hypothetical constant depreciation rate is computed from the simulation as $\widehat{\delta} = 1 + E(i_t/k_t) - E(k_{t+1}/k_t) = 0.067$. The vertical intercept of the hypothetical relationship is $1 - \widehat{\delta}$ so that a model with stochastic depreciation would imply a shifting vertical intercept of the straight line. Comparing the slope of the straight line (equal to 1.0) to the slope of the mean relationship in the model (equal to 0.82) shows that capital adjustment costs are relatively small on average, i.e., when $\lambda_t = E(\lambda_t)$ and $i_t/k_t = E(i_t/k_t)$. The investment shock shifts the value of λ_t upwards or

[18] Although the decomposition is computed using the expressions for the mean log returns in equations (35) through (37), one can assume that a roughly similar decomposition holds for the mean returns which are plotted in Figures 3 and 4.

downwards in a persistent manner, thus shifting the relationship between k_{t+1}/k_t and i_t/k_t so as to generate the cloud of points shown in the lower panel of Figure 6. A roughly similar cloud of points could be generated by a model with no adjustment costs and stochastic variation in the depreciation rate $\widehat{\delta}$.

Table 3 presents unconditional moments of the model's asset pricing variables computed from a long simulation of the nonlinear model using the baseline parameter values shown in Table 1. The table also shows the corresponding statistics from U.S. data.[19] Figures 7 and 8 provide a visual comparison between the model and the data for selected variables.

Table 3: Unconditional Asset Pricing Moments

Variable	Dates	Statistic	U.S. Data	Model
p_t^s/d_t	1871-2008	Mean	26.6	27.0
		Std. Dev.	13.8	5.24
		Corr. Lag 1	0.93	0.86
$R_{t+1}^s - 1$	1900-2008	Mean	8.0%	8.0%
		Std. Dev.	20.4%	21.7%
		Corr. Lag 1	0.00	-0.04
$R_{t+1}^b - 1$	1900-2008	Mean	1.1%	2.4%
		Std. Dev.	4.7%	6.0%
		Corr. Lag 1	0.62	0.87
$R_{t+1}^c - 1$	1900-2008	Mean	2.6%	5.4%
		Std. Dev.	10.0%	15.5%
		Corr. Lag 1	0.11	-0.04
Sharpe Ratio	1900-2008	$\frac{\text{Mean}\left(R_{t+1}^s - R_{t+1}^b\right)}{\text{Std. Dev.}\left(R_{t+1}^s - R_{t+1}^b\right)}$	0.345	0.243
Sharpe Ratio	1900-2008	$\frac{\text{Mean}\left(R_{t+1}^s - R_{t+1}^c\right)}{\text{Std. Dev.}\left(R_{t+1}^s - R_{t+1}^c\right)}$	0.270	0.113

Note: Model results computed from a 20,000 period simulation.

Table 3 and the top panel of Figure 7 show that the model underpredicts the volatility of the U.S. price-dividend ratio. The model standard deviation is about 5% versus almost 14% in the data. The volatility of the U.S. price-dividend ratio is influenced by a dramatic bubble-like run-up starting in the mid-1990s that is mostly unwound by the end of the data sample in 2008. A large literature finds evidence that real-world stock prices exhibit "excess volatility" when compared to the discounted stream of ex post realized dividends.[20] If findings of excess volatility in the data are genuine, then one would not expect a fully rational model like this one to be able to match the volatility of the U.S. price dividend ratio. An extension of the present model that allows for boundedly-rational behavior on the part of capital owners could potentially magnify the volatility of the price-dividend ratio, providing a better match with the data.[21]

[19] The U.S. real return data shown in Table 3 are for equity, long-term bonds, and short term bills, from Dimson, et al. (2002), updated through 2008.

[20] Lansing and LeRoy (2010) provide a recent update on this literature.

[21] For an example along these lines, see Lansing (2011).

Despite underpredicting the volatility of the price-dividend ratio, the model provides a good match with mean and volatility of the U.S. equity return, which are around 8% and 20%, respectively. Recall that the later statistic is matched by construction due to the choice of the curvature parameter σ_k in the capital law of motion (8). The model somewhat overpredicts the mean and volatility of the U.S. short-term bond return, although it should be noted that the return data constructed by Dimson, et al. (2002, updated) pertain to a 3-month "bill" whereas the short-term bond in the model has a one-year maturity. As noted previously, the model's long-term bond behaves too much like equity so the mean and volatility of the consol are too high relative to the mean and volatility of the U.S. long-term bond return. This deficiency in the model is well-summarized by the Sharpe ratio comparison shown at the bottom of Table 3. Finally, the model does capture the fact that returns on equity and long-terms bonds exhibit zero or weak autocorrelation in the data while returns on short-term bonds exhibit strong positive autocorrelation.

The bottom panel of Figure 7 shows that the model equity premium relative to one-period bonds is procyclical, exhibiting a correlation coefficient with output growth of 0.46 versus a value of 0.19 in the data. Table 4 below shows that the model equity premium relative to consol bonds exhibits a correlation coefficient with output growth of 0.88. This result is consistent with the finding reported earlier that about 95 percent of the equity premium relative to consol bonds is attributable to the productivity shock. The introduction of additional stochastic disturbances that affect equity and bonds in a differential manner would help to reduce the overly-procyclical nature of the model equity premium. One such example would be to introduce stochastic variation in the parameter δ that governs the decay rate of the consol coupon payments. Moreover, if movements in δ were countercyclical, this feature would make the consol less risky relative to equity, thus helping to magnify the associated equity premium.

Figure 8 shows that asset returns in both the data and the model exhibit time-varying means and volatilities. The time-varying behavior in the data suggests the presence of nonlinearities. The time-varying behavior in the model is endogenous, owing to the nonlinear nature of the various functional forms and equilibrium conditions. In contrast, Bansal and Yaron (2004) introduce exogenous time-varying volatility in the stochastic process for consumption growth within an endowment economy.

Tables 4, 5, and 6 show that the model performs reasonably well in matching the business cycle moments of aggregate macro variables.[22] In Table 4, the model variables all exhibit strong correlations with output growth—a typical feature of productivity-shock driven real business cycle models.

[22] Data on per capita real GDP from 1870-2008 are from <www.globalfinancialdata.com>. Data on real business fixed investment from 1929-2008 are from the U.S. Bureau of Economic Analysis/Haver Analytics.

Table 4: Correlations with Output Growth

Variable	Dates	U.S. Data Corr. w/ $\Delta \log (y_t)$	Model Corr. w/ $\Delta \log (y_t)$
$\Delta \log (y_t)$	1871-2008	1.00	1.00
$\Delta \log (c_t^a)$	1890-2008	0.53	0.94
$\Delta \log (d_t)$	1872-2008	0.22	0.73
$\Delta \log (i_t)$	1930-2008	0.23	0.79
$\Delta \log (p_t^s)$	1872-2008	0.14	0.48
$R_{t+1}^s - R_{t+1}^b$	1900-2008	0.19	0.49
$R_{t+1}^s - R_{t+1}^c$	1900-2008	0.21	0.89

Note: Model results computed from a 20,000 period simulation.

In Table 5, the model underpredicts the volatility of output growth relative to aggregate consumption growth given by $c_t^a = c_t + nc_t^w$. This feature is attributable to the model's underprediction of investment growth volatility—about 7% in the model versus about 16% in the data. Due to capital adjustment costs, investment growth in the model is only about 1.8 times more volatile than output growth, whereas investment growth in the data is about 3 times more volatile than output growth. Barlevy (2004, p. 983) notes the difficulty of generating sufficient investment volatility in real business cycle models with capital adjustment costs. However, the model does a good job of predicting the volatility of equity price growth—about 20% in the model versus about 18% in the data.

Table 5: Volatility of Macro Variables

Variable	Dates	U.S. Data Std. Dev.	Model Std. Dev.
$\Delta \log (y_t)$	1871-2008	5.28%	3.70%
$\Delta \log (c_t^a)$	1890-2008	3.51%	3.50%
$\Delta \log (d_t)$	1872-2008	11.7%	11.5%
$\Delta \log (c_t)$	–	–	9.03%
$\Delta \log (c_t^w)$	–	–	2.52%
$\Delta \log (i_t)$	1930-2008	16.2%	6.45%
$\Delta \log (p_t^s)$	1872-2008	17.9%	20.0%
$\frac{\text{Std. Dev. } [\Delta \log(c_t)]}{\text{Std. Dev. } [\Delta \log(c_t^w)]}$	1982-2004	1.63	3.58

Note: Model results computed from a 20,000 period simulation.

The bottom row of Table 5 shows that the capital owners' consumption growth is 3.58 times more volatile than the workers' consumption growth. The source of the extra volatility for capital owners is their heavy reliance on volatile dividends to fund consumption. The procyclical behavior of capital's share of total income s_t^k (discussed below) implies that labor's share is countercyclical, which helps to smooth the consumption of the workers relative to that of capital owners. In the model of Danthine and Donaldson (2002), the source of extra volatility for capital owners is the wage contract which smoothes workers' consumption at the expense of larger fluctuations in capital owners' consumption. In the version of their model that delivers an equity premium approaching 6%, the capital owners' consumption growth is 10 times more volatile than aggregate consumption growth.[23] By comparison, Table 5 shows that the capital owners' consumption growth in the present model is only 2.6 times more volatile than aggregate consumption growth. In the model of Guvenen (2009), the source of extra volatility for stockholders is the bond market; stockholders make interest payments to bondholders which smooths the bondholders' consumption but magnifies the volatility of stockholders' consumption. Guvenen's model delivers a consumption growth volatility ratio for stockholders relative to non-stockholders of 2.4. Citing several empirical studies, he argues that measured volatility ratios in the range of 1.5 to 2.0 are likely to represent a lower bound for the true ratio.

Malloy, et al. (2009) study consumption growth data for stockholders and non-stockholders for the period 1982 to 2004.[24] Using their data, the consumption growth volatility ratio for the two groups is 1.63, as shown in bottom row of Table 5. The corresponding volatility ratio in the model is more than twice as large at 3.58. The sample period 1982 to 2004 employed in the study by Malloy, et al. falls within the so-called "Great Moderation" era which is characterized by relatively mild macroeconomic fluctuations. In contrast, the model is calibrated to match the volatility of observed dividend growth for the period 1872 to 2008, which includes the Great Depression and other significant bear markets. These events likely magnified the volatility of stockholders' consumption relative to non-stockholders' consumption in the data.

Dividends are much less volatile in the post-World War II sample period as can be clearly seen from Figure 1. It is still the case, however, that dividend growth is about three times more volatile than aggregate consumption growth for the period 1947 to 2008. When the model is calibrated to match the lower post-World War II volatilities of dividend and consumption growth, the capital owner's risk aversion parameter must be increased to $\alpha = 7.5$ for the model to deliver an equity premium near 6% relative to one-period bonds. The value $\alpha = 7.5$ remains within the plausible range of 0 to 10 considered by Mehra and Prescott (1985). There is no theoretical reason to think that stock market investors would ignore the pre-World War II data. The persistent memory of the pre-World War II data in the minds of investors could serve as a key determinant of today's equity premium. Along these lines, Cogley and Sargent (2008) develop a model where agents' persistent beliefs about dividend and consumption growth formed during the Great Depression contribute to a large equity premium.

Finally, Table 6 shows that the model captures the procyclical movement of capital's share of total income in U.S. data. However, capital's share in the model is significantly more volatile than the corresponding U.S. value for the period 1947 to 2008. Again, expanding the sample period to include the Great Depression and other bear markets would likely magnify the volatility of capital's share in the data. As noted earlier, the procyclical movement of capital's share in the model derives from the production technology for output, where the

[23] See Table 6, Panel A (p. 62) in Danthine and Donaldson (2002). They do not report the volatility of consumption growth for workers.

[24] The data are available from <www.kellogg.northwestern.edu/faculty/vissing/htm/research1.htm>

capital-labor substitution elasticity σ_y is below unity. Intuitively, when $\sigma_y < 1$, the capital stock and the effective labor input $(\ell^c + n\,\ell^w)\exp(z_t)$ are compliments. This complementarity allows capital to derive proportionally more benefits from a positive realization of a labor–enhancing productivity shock. In contrast, when $\sigma_y = 1$, the benefits of a positive productivity shock are shared proportionally between inputs so that income shares remain constant.

Table 6: Capital Share of Total Income

Statistic	U.S. Data 1947-2008	Model
Mean	0.362	0.362
Std. Dev.	0.015	0.056
Corr. Lag 1	0.83	0.98
Corr $\left[s_t^k\ , \Delta \log(y_t)\right]$	0.31	0.22

Note: Model results computed from a 20,000 period simulation.

5 Conclusion

A long history of research since Mehra and Prescott (1985) has sought to develop models that can account for the high mean and high volatility of observed equity returns relative to bond returns. One branch of this research has focused on investigating modifications to agents' preferences that govern attitudes towards risk or intertemporal substitution. Another branch has focused on investigating changes to the structure of the cash flows that are priced by agents in the model. This paper falls into the second category. The basic intuition for the results is that capital owners demand a high equity premium to compensate for the risk of linking their consumption to a highly volatile dividend stream. Dividend growth in U.S. data is about three times more volatile than aggregate consumption growth. Since ownership of financial wealth in the U.S. economy is highly concentrated at the extreme upper end, the owners of financial wealth must bear a disproportionate share of the risk from fluctuating dividends.

In the model, investment shocks, which are similar to depreciation shocks, influence the volatility of dividend growth and thus contribute to both a large equity premium and the high volatility of equity returns. The model can match many quantitative features of U.S. data under rational expectations, but it notably underpredicts the volatility of the price-dividend ratio and the volatility of investment growth. These deficiencies could potentially be addressed by a richer model that allows for non-fundamental asset price movements which empirical evidence suggests are present in real-world stock market data.

A Appendix: Approximate Solution (Proposition 1)

Taking logarithms of both sides of the transformed first-order condition (23) and then applying a first-order Taylor series approximation to each side yields equation (24). The Taylor-series coefficients are themselves functions of the approximation points $\widetilde{x}$, $\widetilde{k}_n$, and $\widetilde{v} = 0$.

The conjectured form of the solution $x_{t+1} = \widetilde{x}\left(k_{n,t+1}/\widetilde{k}_n\right)^{\gamma_k} \exp\left(\gamma_v\, v_{t+1}\right)$ is substituted into the right-side of equation (24) together with the approximate law of motion (25) that governs $k_{n,t+1}$ and $v_{t+1} = \rho v_t + u_{t+1}$. After evaluating the conditional expectation and then collecting terms, we have:

$$x_t = \underbrace{\widetilde{x}\left[\frac{\mathrm{a}_0}{\mathrm{b}_0}\right]^{\frac{1}{\mathrm{a}_1 - \mathrm{f}_1(\mathrm{b}_1\gamma_k + \mathrm{b}_2)}} \exp\left[\frac{\frac{1}{2}\left(\mathrm{b}_1\gamma_v + \mathrm{b}_3\right)^2\sigma_u^2 + \frac{1}{2}\left(\phi - \mathrm{b}_1\gamma_k - \mathrm{b}_2\right)^2\sigma_\varepsilon^2}{\mathrm{a}_1 - \mathrm{f}_1\left(\mathrm{b}_1\gamma_k + \mathrm{b}_2\right)}\right]}_{\widetilde{x}}$$

$$\times\left[\frac{k_{n,t}}{\widetilde{k}_n}\right]^{\overbrace{\frac{\mathrm{f}_2\left(\mathrm{b}_1\gamma_k + \mathrm{b}_2\right) - \mathrm{a}_2}{\mathrm{a}_1 - \mathrm{f}_1\left(\mathrm{b}_1\gamma_k + \mathrm{b}_2\right)}}^{\gamma_k}} \exp[\underbrace{\frac{\rho\left(\mathrm{b}_1\gamma_v + \mathrm{b}_3\right) + \mathrm{f}_3\left(\mathrm{b}_1\gamma_k + \mathrm{b}_2\right) - \mathrm{a}_3}{\mathrm{a}_1 - \mathrm{f}_1\left(\mathrm{b}_1\gamma_k + \mathrm{b}_2\right)}}_{\gamma_v}\, v_t] \quad \text{(A.1)}$$

which shows that the conjectured form is correct. Solving for the undetermined coefficients γ_k and γ_v yields the expressions shown in Proposition 1.

The undetermined coefficients $\widetilde{x}$ and $\widetilde{k}_n$ solve the following system of nonlinear equations

$$\widetilde{x} = \frac{\beta\left(\widetilde{s}^k/\widetilde{s}^c\right) Q \exp\left[\frac{1}{2}\left(\mathrm{b}_1\gamma_v + \mathrm{b}_3\right)^2\sigma_u^2 + \frac{1}{2}\left(\phi - \mathrm{b}_1\gamma_k - \mathrm{b}_2\right)^2\sigma_\varepsilon^2\right]}{1 - \beta\left[1 - \left(1 - \widetilde{s}^k/\widetilde{s}^c\right) Q\right] \exp\left[\frac{1}{2}\left(\mathrm{b}_1\gamma_v + \mathrm{b}_3\right)^2\sigma_u^2 + \frac{1}{2}\left(\phi - \mathrm{b}_1\gamma_k - \mathrm{b}_2\right)^2\sigma_\varepsilon^2\right]}, \quad \text{(A.2)}$$

$$1 = B\exp(-\mu)\left\{1 - \lambda + \lambda\left[\frac{\widetilde{s}^c\,\widetilde{x}}{1 + \widetilde{x}}\,\widetilde{y/k}\right]^{\psi_k}\right\}^{\frac{1}{\psi_k}}, \quad \text{(A.3)}$$

$$\text{where} \quad Q \equiv 1 - \frac{(1-\lambda)\,B^{\psi_k}}{\exp\left(\psi_k\,\mu\right)}, \qquad \widetilde{y/k} = \frac{\left[\theta\,\widetilde{k}_n^{\psi_y} + (1-\theta)\right]^{\frac{1}{\psi_y}}}{\widetilde{k}_n},$$

$$\widetilde{s}^k = \frac{\theta\widetilde{k}_n^{\psi_y}}{\theta\,\widetilde{k}_n^{\psi_y} + 1 - \theta}, \qquad \widetilde{s}^c = \frac{\theta\widetilde{k}_n^{\psi_y} + (1-\theta)\left[\frac{\ell^c}{\ell^c + n\,\ell^w}\right]}{\theta\,\widetilde{k}_n^{\psi_y} + 1 - \theta}.$$

Equation (A.2) is derived from equation (A.1) after substituting in the expressions for the Taylor series coefficients a_0 and b_0 and then canceling terms. Equation (A.3) is the law of motion for the normalized capital stock (20) evaluated at the approximation point.

B Appendix: Nonlinear Model Solution

The impulse response functions and quantitative simulations are generated using the solution method outlined below that preserves the model's nonlinear equilibrium conditions. The method employs a version of the parameterized expectation algorithm (PEA) described by Den Haan and Marcet (1990).

The transformed first-order condition (23) can be represented as:

$$\mathrm{f}\left(x_t,\, k_{n,t},\, v_t\right) \;=\; E_t\, \mathrm{h}\left(x_{t+1},\, k_{n,t+1},\, v_{t+1,}\, \varepsilon_{t+1}\right) \tag{B.1}$$

where $\mathrm{h}(\cdot)$ is the nonlinear object to be forecasted. For purposes of constructing the conditional expectation, the function $\mathrm{h}(\cdot)$ is approximated as

$$\mathrm{h}\left(\cdot\right) \;\simeq\; d_0\; \left[k_{n,t}\right]^{d_k} \exp\left[d_v\, v_t + d_u\, u_{t+1} + d_\varepsilon\, \varepsilon_{t+1}\right], \tag{B.2}$$

where d_0, d_k, d_v, d_u, and d_ε are regression coefficients that are obtained by projecting the nonlinear function $\mathrm{h}(\cdot)$ onto the form (B.2) during repeated simulations of the model, as described below. The initial guesses for d_0 through d_ε are determined analytically using the approximate decision rule from Proposition 1, together with the power function approximations (24) and (25).

Given a set of initial guesses for d_0 through d_ε, a simulation is run where the conditional expectation on the right side of (B.1) is constructed each period as

$$E_t\, \mathrm{h}\left(\cdot\right) \;=\; d_0\; \left[k_{n,t}\right]^{d_k} \exp\left[d_v\, v_t + \tfrac{1}{2}\left(d_u\, \sigma_u\right)^2 + \tfrac{1}{2}\left(d_\varepsilon\, \sigma_\varepsilon\right)^2\right]. \tag{B.3}$$

Given the forecast $E_t\, \mathrm{h}\left(\cdot\right)$, the nonlinear function (B.1) is solved each period for x_t using a nonlinear equation solver. The state variables $k_{n,t}$ and v_t evolve according to the exact laws of motion (20) and (9). During the simulation, realized values of the nonlinear function $\mathrm{h}\left(\cdot\right)$ are constructed. At the end of the simulation, the realized values of $\mathrm{h}\left(\cdot\right)$ are projected onto the form (B.2) to obtain new guesses for d_0 through d_ε. The simulation is then repeated using the new guesses for d_0 through d_ε with the same sequence of draws for the shock innovations u_{t+1} and ε_{t+1}. The procedure is stopped when the guesses for d_0 through d_ε do not change from one simulation to the next.

An analogous procedure is used to construct the conditional expectations in the bond pricing equations (4) and (5) to solve for p_t^b and p_t^c each period. Specifically, the nonlinear objects to be forecasted are approximated by power functions of the state variables and shock innovations as follows:

$$
\begin{aligned}
p_t^b &= E_t \beta \exp(-\phi\mu) \left[\frac{c_{t+1}}{c_t}\right]^{-\alpha} \\
&= E_t\, M_{t+1},
\end{aligned}
$$

where
$$
M_{t+1} \simeq d_0^b\, [k_{n,t}]^{d_k^b} \exp\left[d_v^b\, v_t + d_u^b\, u_{t+1} + d_\varepsilon^b\, \varepsilon_{t+1}\right], \tag{B.4}
$$

$$
\begin{aligned}
p_t^c &= E_t \beta \exp(-\phi\mu) \left[\frac{c_{t+1}}{c_t}\right]^{-\alpha} \left(1 + \delta p_{t+1}^c\right) \\
&= E_t\, M_{t+1} + E_t\, \delta M_{t+1}\, p_{t+1}^c,
\end{aligned}
$$

where
$$
\delta M_{t+1}\, p_{t+1}^c \simeq d_0^c\, [k_{n,t}]^{d_k^c} \exp\left[d_v^c\, v_t + d_u^c\, u_{t+1} + d_\varepsilon^c\, \varepsilon_{t+1}\right]. \tag{B.5}
$$

The initial guesses for the regression coefficients d_0^b through d_ε^b and d_0^c through d_ε^c are determined analytically using the approximate solution of the model. After each simulation, new guesses for the regression coefficients are obtained by projecting the realized values of the nonlinear functions M_{t+1} and $\delta M_{t+1}\, p_{t+1}^c$ onto the forms shown in (B.4) and (B.5) until convergence is achieved.

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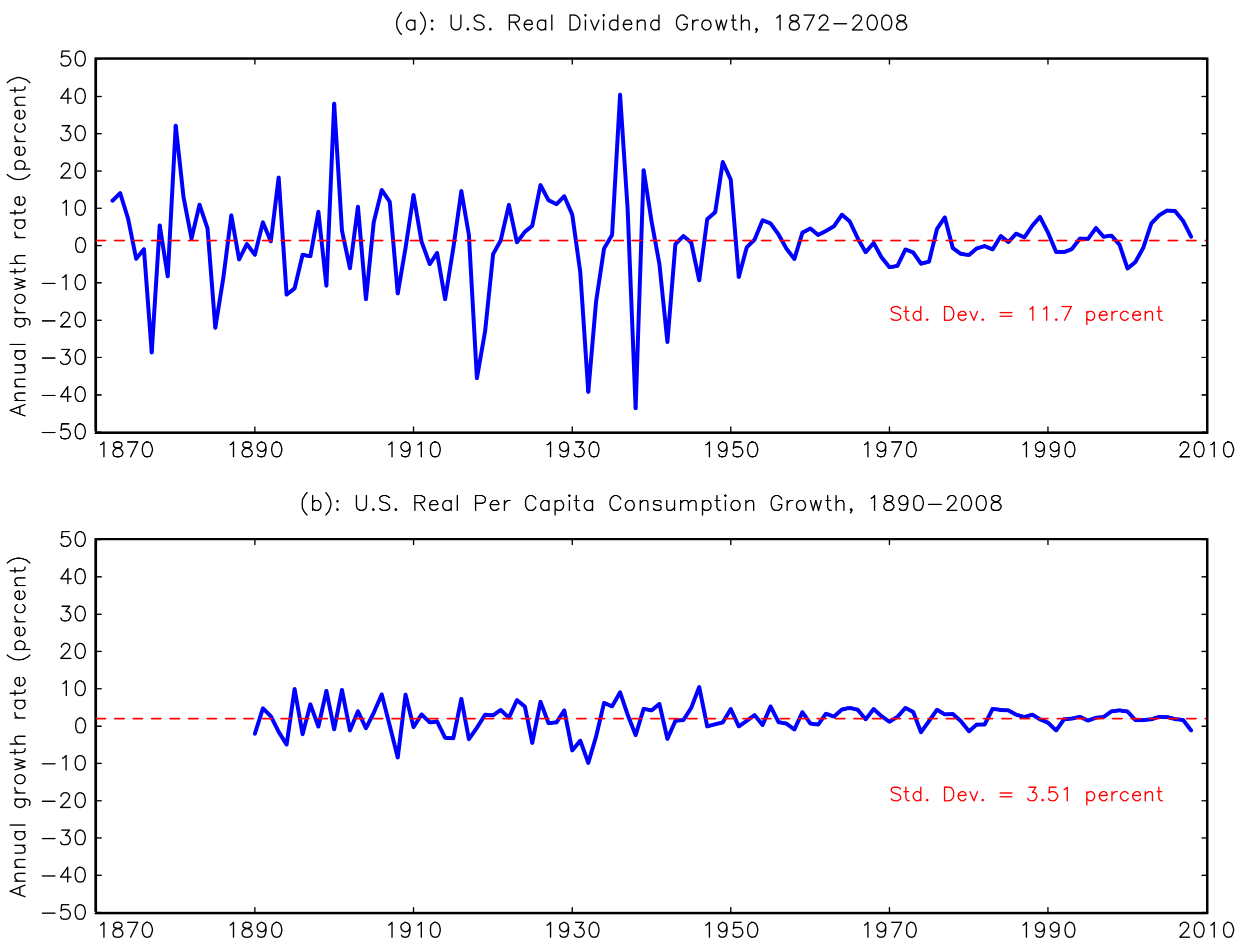

Figure 1: Dividend growth is about three times more volatile than consumption growth.

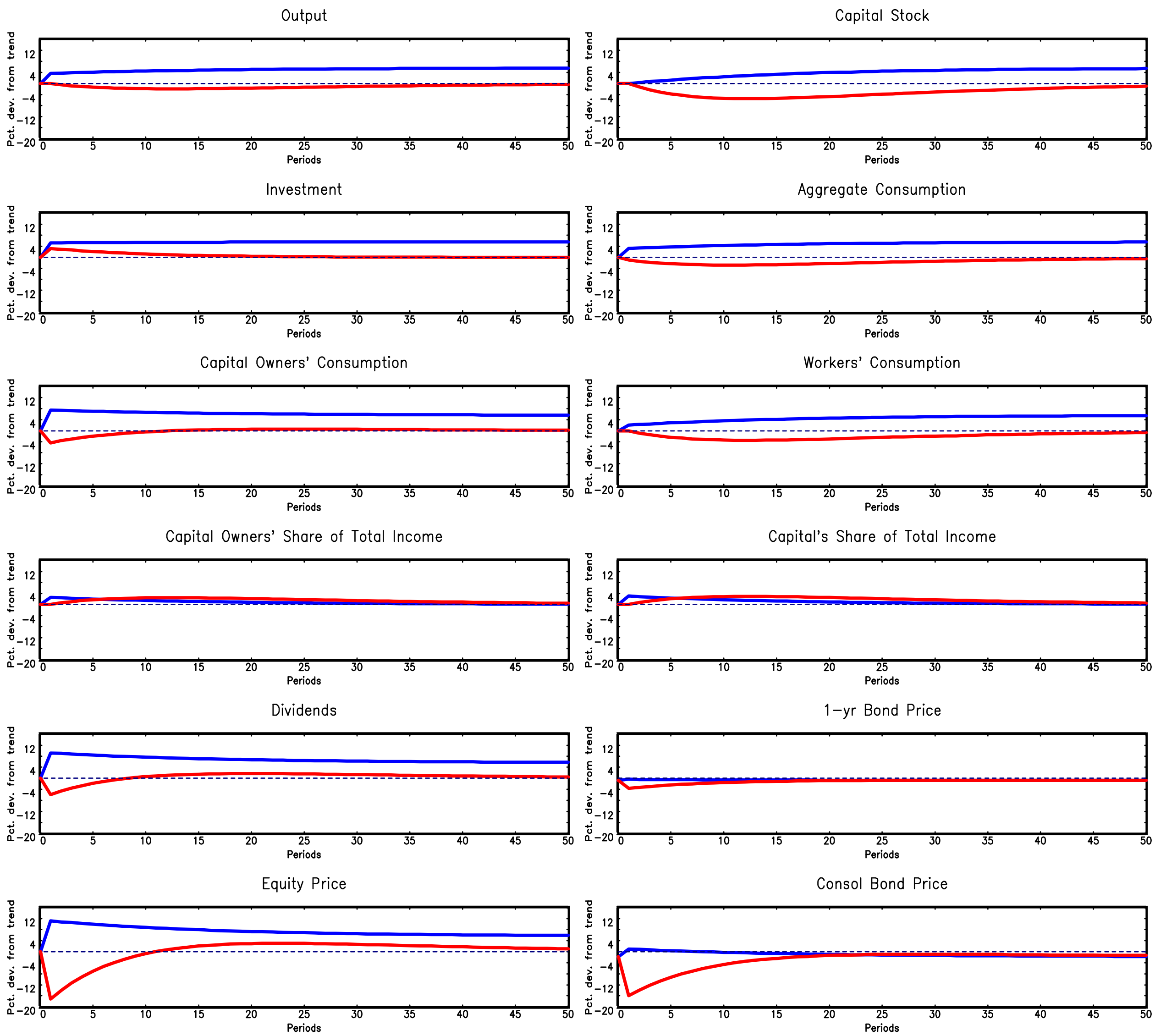

Figure 2: Impulse responses to one standard error innovations of the productivity shock (blue line) and the investment shock (red line).

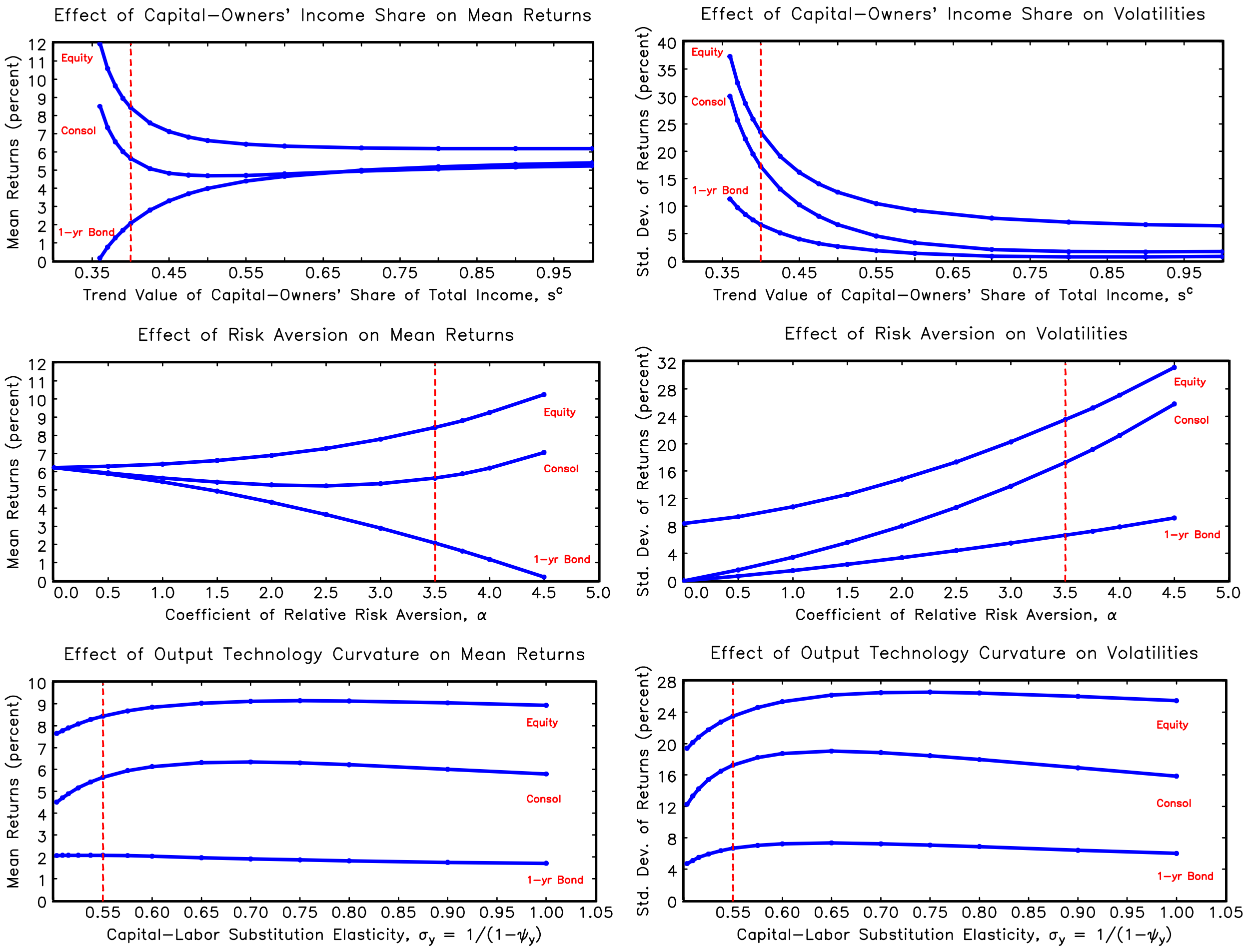

Figure 3: Effect on return moments of changing $\widetilde{s}^c$, α, and σ_y.

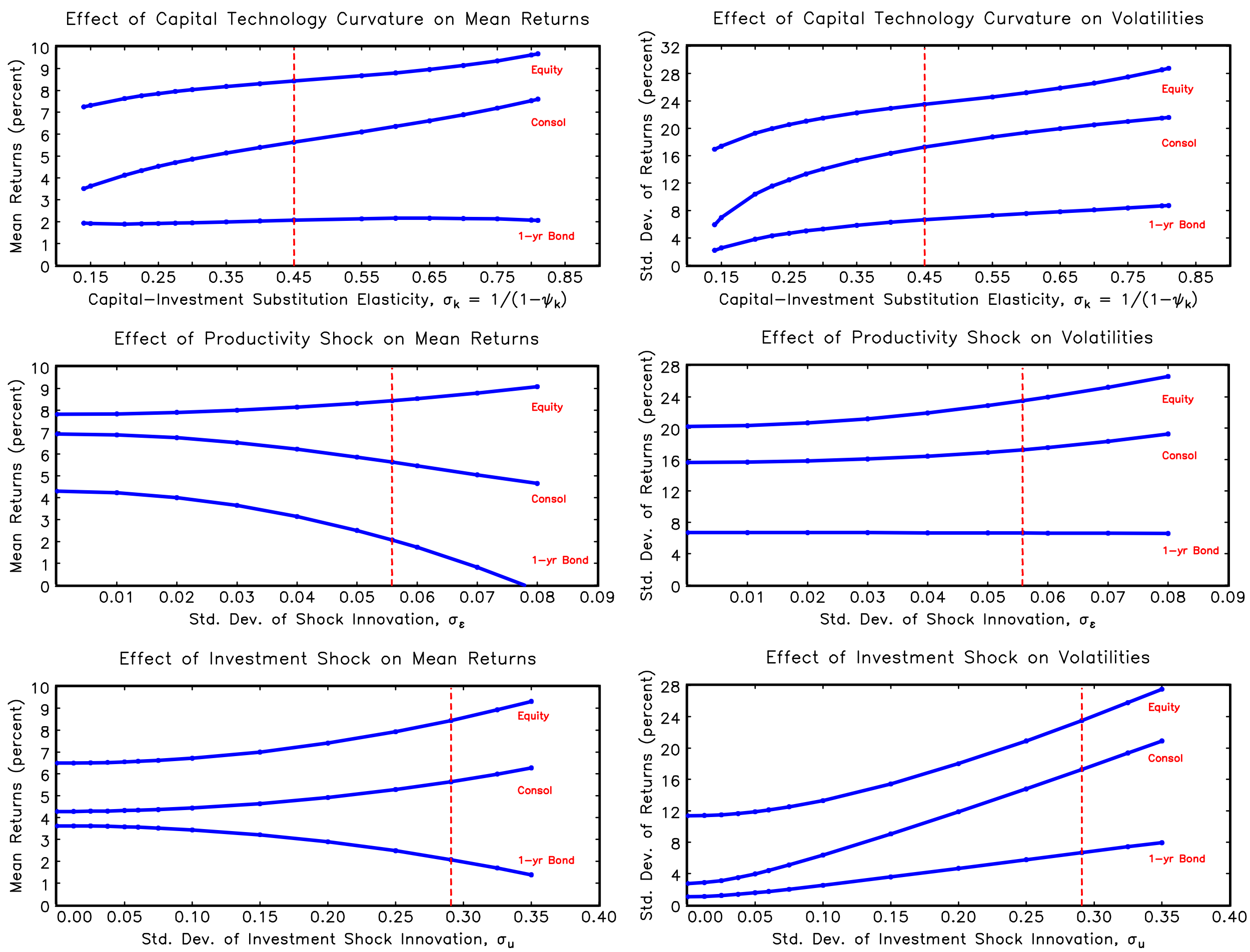

Figure 4: Effect on return moments of changing σ_k, σ_ε, .and σ_u.

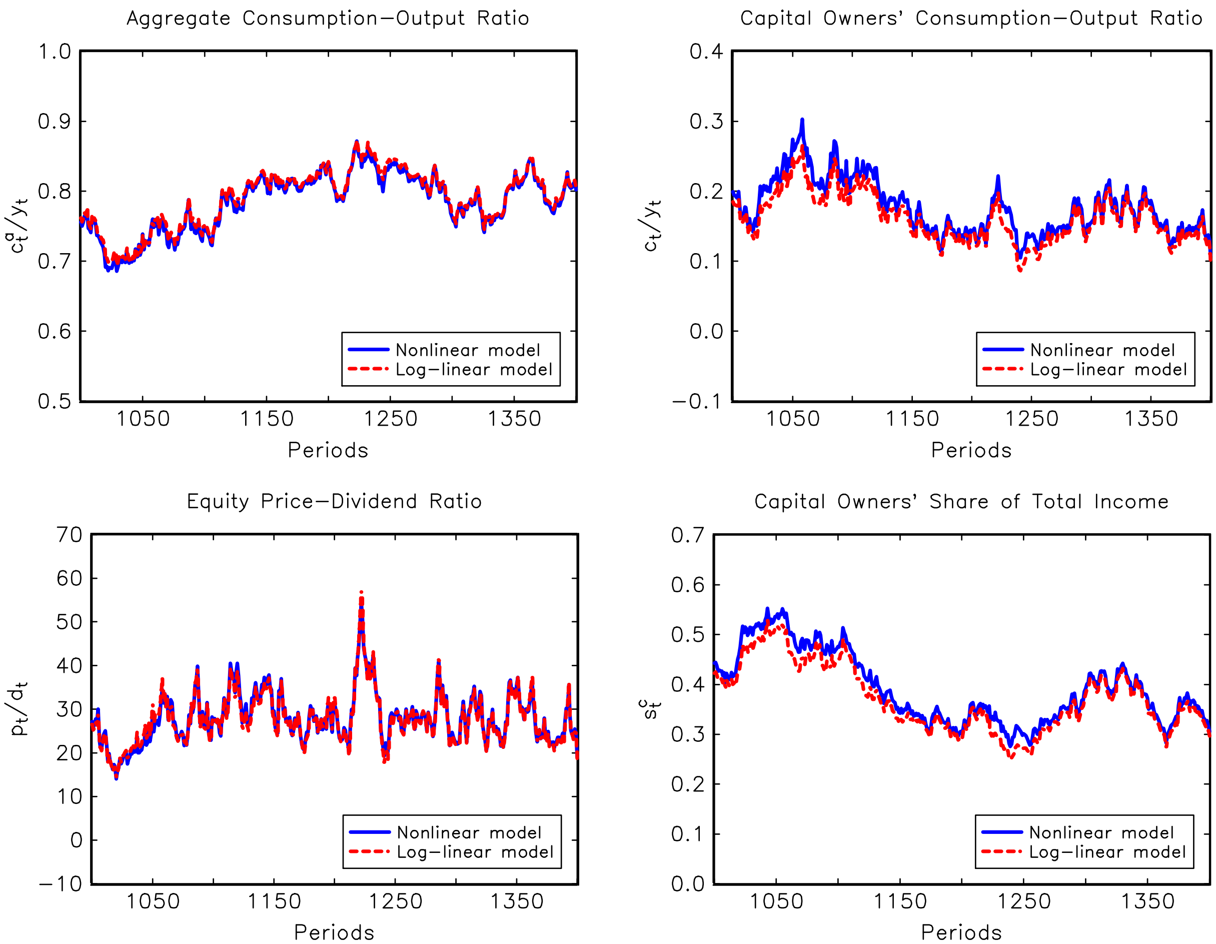

Figure 5: Nonlinear model solved using PEA versus log-linear approximate solution.

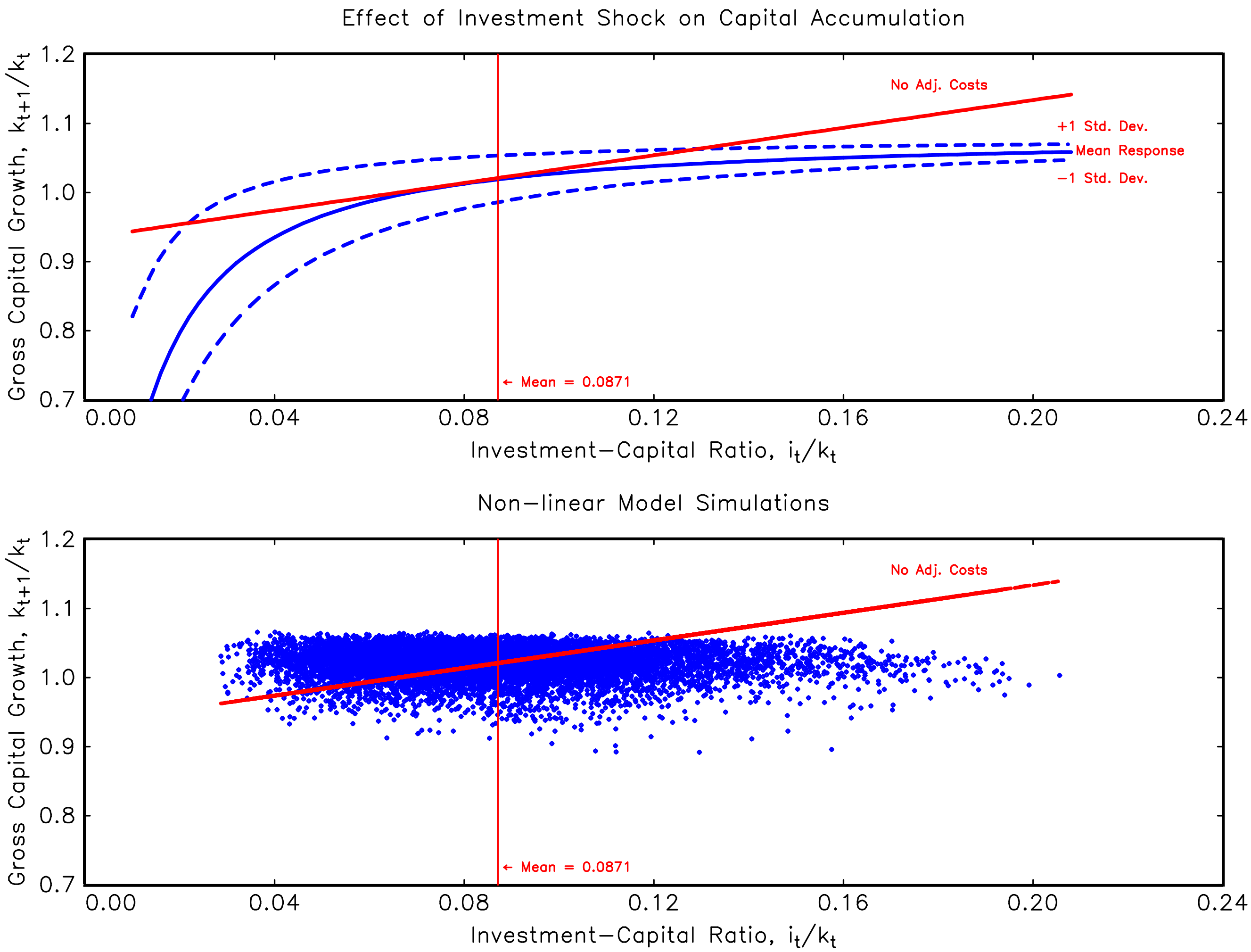

Figure 6: The effect of the investment shock is similar to stochastic depreciation.

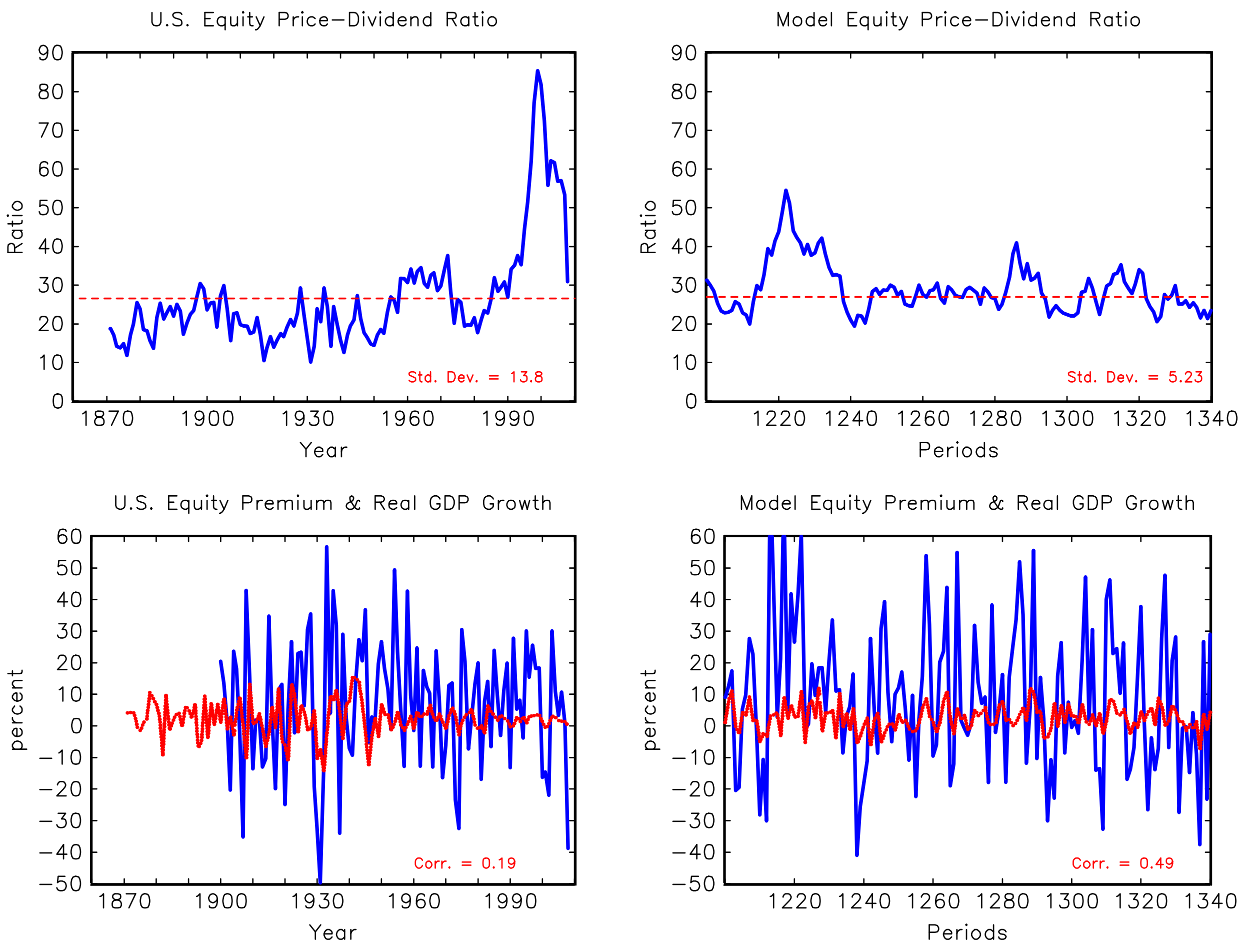

Figure 7: Asset pricing variables: Data versus model.

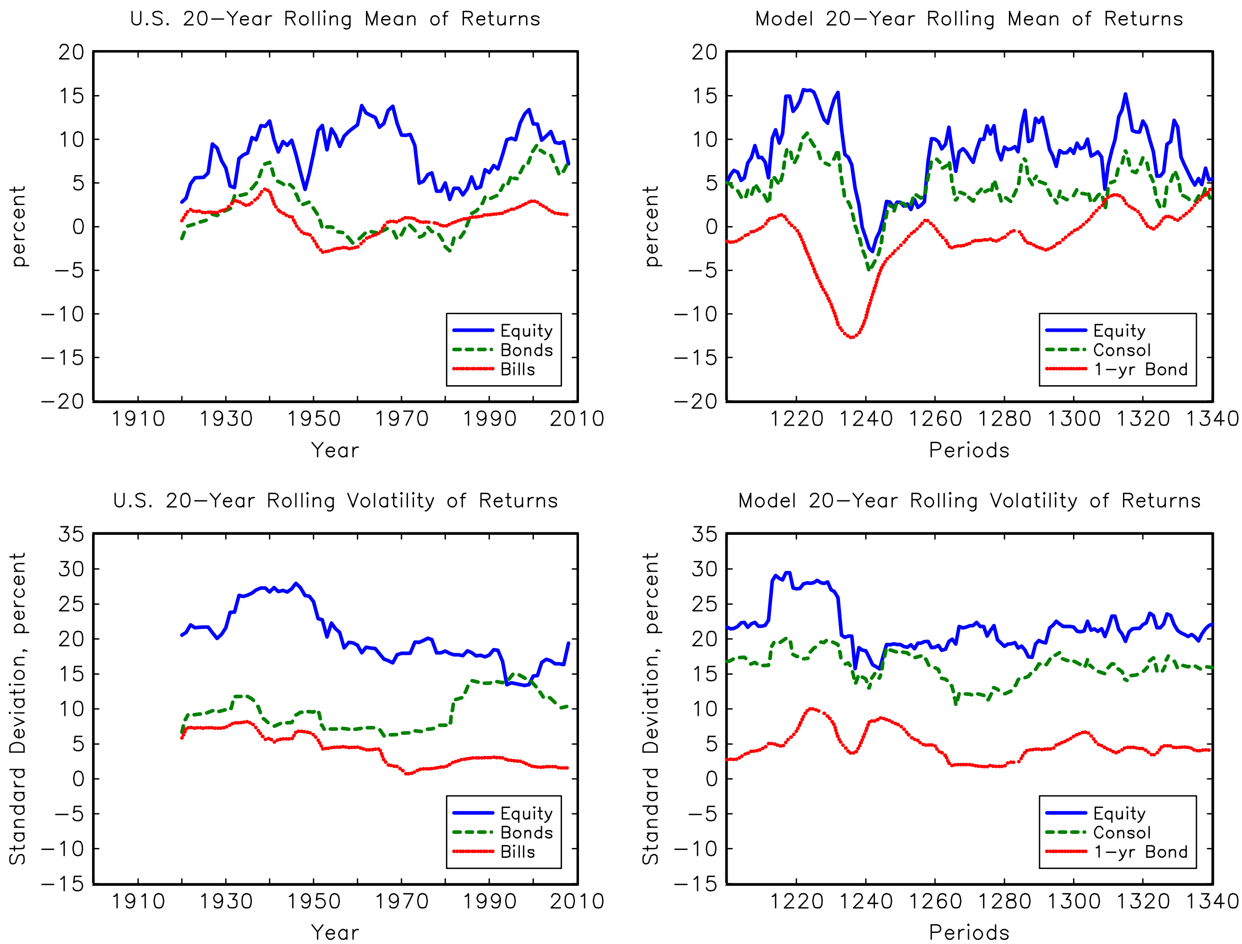

Figure 8: Asset returns exhibit time-varying means and volatilities.