A Prior Predictive Analysis of the Effects of Loss Aversion/Narrow Framing in a Macroeconomic Model for Asset Pricing[*†]

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Abstract

In a macroeconomic framework, I quantitatively evaluate the theory of Loss Aversion/Narrow Framing (LANF) as a resolution to the Equity Premium Puzzle (EPP). The EPP is where the neoclassical asset pricing model cannot be reconciled with the empirical fact that stocks have much higher returns than risk-free assets. The prior predictive analysis employed follows a Bayesian approach that draws realizations for preferences that describe the degree of LANF characterizing consumer's tastes. The analysis is also extended along two more dimensions: the variance of aggregate uncertainty and the elasticity of labor. The priors used are carefully defined from previous works in the literature. This Monte Carlo procedure finds that the theory is unable to jointly describe the equity premium and labor's elasticity of supply. That is, only when the labor supply elasticity is unreasonably low can LANF preferences generate any equity premiums. Alternatively, when the elasticity is more realistically high, LANF preferences fail to generate significant premiums. My analysis therefore concludes that a resolution to the EPP via a theory of LANF must be modified along the description of labor's choices. As ancillary result, the hybrid perturbation-projection method developed for this experiment is shown to be a robust technique.

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1 Introduction

The study of asset pricing is an important topic in monetary economics. As Sargent (2010) explains:

> *Important parts of modern macro are about understanding a large and interesting suite of asset pricing puzzles – puzzles about empirical failures of simple versions of efficient markets theories.*

Sargent's point is that while asset pricing theories provide logical frameworks for understanding how monetary policy is channeled through to the real economy, his interpretation is complicated by the existence of unresolved empirical anomalies. One of the most challenging of these anomolies is the Equity Premium Puzzle (EPP). The Equity Premium, the persistent excess of stock returns over the risk-free rate, is enigmatic in the sense of not being reconcilable with the predications of the neoclassical asset pricing models. For example, an early study by Mehra and Prescott (1985) estimated an equity premium of 6% which could only be reconciled with the neoclassical asset pricing model by assuming that consumers had extreme and unrealistic aversion to risk.[1] Mehra and Prescott's paper fostered a series of studies (e.g., Kocherlakota 1996, Benartzi and Thaler 1995, Mankiw and Zeldes 1991) that attempted to explain the existence of the Equity Premium by determining how households derive utility and form expectations. The present essay is predicated on the Loss-Aversion/Narrow-Framing (LANF) hypothesis of Benartzi and Thaler (1995). Their paper is one of the earliest studies to hypothesize LANF preferences as a basis for resolving the EPP paradox. Later literature (Barberis, Huang and Santos 2001, Barberis and Huang 2004 and 2008, and Grüne and Semmler 2008) extended the Benarti and Thaler findings using a similar loss-aversion framework by including LANF component into specific household's maximization problem with different utility functions.

Preferences with LANF are fundamentally different from the typical risk aversion assumption. Investors with LANF preferences have greater sensitivity to losses than to acquiring gains. A numerical example from DellaVigna (2009) based on the experimental evidence of Kahneman and Tversky (1979) illustrates the difference between the two behavioral assumptions:

[1] Mehra and Prescott (1985) observe that, over the ninety-year period 1889-1978, the average real annual yield on the Standard and Poor 500 Index is 7%. Whereas the average yield on short-term debt is less than 1%. The data they used for the risk-free rate includes asset returns on the ninety-day Treasury bill, Treasury certificate, and sixty-day to ninety-day commercial paper.

> *There are two experiments. In the first one, subjects are asked to choose between: (A) a 50% chance to gain $1,000 and a 50% chance to gain nothing; or (B) a sure gain of $500, given that they have been given $1,000. The other experiment asks to choose between: (C) a 50% chance to lose $1,000 and a 50% chance to lose nothing; or (D) a sure loss of $500, given that they have been given $2,000.*

Most individuals (84%) in the first group accepted (B) whereas most people (69%) in the second group accepted (C). Risk aversion preferences cannot explain these choices given that (B) and (D) are statistically identical lotteries. The actual mechanism that generates greater sensitivity to losses – even small losses – is achieved, in this paper, by positing a kinked utility function with the slope of the loss proportion of the function steeper than the gain proportion. Assuming a kinked preference function fundamentally alters theory's predictions about the relationship between asset demand and expected rates of return.

The works of Barberis et al. (2001), Barberis and Huang (2004 and 2008) and Grüne and Semmler (2008) have shown that LANF preferences can generate high equity premiums. Nevertheless, their works lead to a question much like the one originally faced by Mehra and Prescott (1985); namely, are the calibrated LANF preferences that generates the equity premium reasonable? To address this issue, the present paper conducts a *prior predictive analysis* and subsequent model evaluation (e.g., Geweke and Whiteman 2006, Geweke 2010). In the prior predictive analysis, prior distributions are defined along three key dimensions; the variance of aggregate uncertainty, the elasticity of labor supply, and the degree of LANF. Although a wide array of prior distributions are initially chosen (i.e., encompassing all reasonable parameters), a major contribution of the present paper is to develop a unique set of appropriate priors for a Dynamic Stochastic General Equilibrium (DSGE) model with LANF preferences. Presumably, the priors established in this paper will have further applicability in the full structural estimations.

To be more specific, the estimation method of prior prediction is an iterative Bayesian approach (Canova 1994, Geweke and Whiteman 2006, Geweke 2007) encompassing the following steps. First, the priors are defined for the parameters of interest. Second, the priors are then used to determine the model's parameters. Third, given the parameters determined in step two, the model is numerically solved for its equilibriums values. Performing these operations many times generates the statistical distributions of interest ; i.e., the equity premium, Sharpe ratios, and other volatility measures. In the final step, the actual data is compared to the distributions generated by the model. The theory of LANF is supported when estimates from the actual data fall within these distributions.

The research of Barberis et al. (2001), Barberis and Huang (2004 and 2008) and Grüne and Semmler (2008) have shown that as the margins for wealth smoothing are increased, the effects of LANF diminish. Or, stated somewhat differently, moving from the pure endowment economies proposed in Barberis et al (2001, 2004, and 2008) to the production economy hypothesized in Grüne and Semmler (2008) shows that the equity premium declines. What this means is that the extra margins of choice in the production economy (i.e., particularly precautionary savings from capital) encourage the households to smooth away from the kink in the households preference function. The model economy developed in this paper is similar to recent work of Danthine and Donaldson (2002) and Guvenen (2009) in that it adds labor choice as another margin of household smoothing. Though labor effort might reduce the equity premium, it is reasonable to believe that it may strengthen LANF's predictions for equity premium given that more realistic volatilities may drive the economy near the kink. Ultimately, however, the effects are to be discovered in the quantitative analysis.

The model in the present paper posits two types of agents: households and firms. The firm makes decisions about labor, capital investment, and borrowing. The firm owns the capital stock and employs internal and external funds to finance investment. External funds take the form of fixed rate bonds that are paid back with certainty at the end of each period. This is the risk-free asset in the model with a price that is equal to the inverse of the return - which is determined by the household's marginal rate of substitution between two periods. The household supplies labor and purchases assets produced by the firm. Equities represent claims to dividends and their value are derived from how well the firm makes investment choices. Furthermore, by assuming that the firm maximizes the expected value of investment choices, the equity values can be directly related to the value of the firm's capital stock.

Loss aversion brings discrete elements into the agents' optimization problems (decision process). This adds complexities to the model and may be one reason LANF has not been studied more thoroughly in a general equilibrium framework. However, the present paper develops a new method that is better suited to the assumption of LANF preferences. This new method, called a hybrid Perturbation-Projection method, combines perturbation and projection techniques, first introduced by Judd (1996) and recently applied by Fernández-Villaverde and Rubio-Ramirez (2006). The noteworthy aspect of this algorithm is that it exploits the beneficial properties of both methods in the presence of bifurcated functions.

The results from the application of this new method revealed that contrary to the hypothesis of Benartzi and Thaler (1995), the introduction of LANF preferences does not explain equity premium under reasonable assumptions about labor supply elasticities. Only when the labor supply elasticities are unreasonably low, by the standards of the extant literature, can LANF preferences generate an equity premium. Alternatively, when the elasticity coefficients are more realistic, LANF preferences fail to generate a premium. The conclusion from these experiments is that explaining the equity premium in terms of LANF preferences depends on the assumptions made about labor choices. Another important finding is that the hybrid perturbation-projection algorithm is a robust technique for analyzing the EPP.

The paper is organized as follows. Section 2 presents the general equilibrium model. Section 3 introduces the method used to find the dynamic properties of the model. Section 4 discusses the main findings. Section 5 concludes with suggestions for future work. The appendix details how the solution method works to simulate the results.

2 Structure of the Model

The LANF asset pricing model is comprised of two sectors: households and firms. The agents in these sectors transact in four markets: goods, stocks, bonds and labor markets.

2.1 Households

Infinitely-lived households enter the financial market to invest their financial wealth in both stocks and bonds. These households maximize their lifetime utility function:

$$\max_{\{c_t, l_t, s_{t+1}, B_{t+1}\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left(\begin{array}{c} \frac{\left(c_t^{\gamma}(1-l_t)^{1-\gamma}\right)^{1-\rho}}{1-\rho} + \\ \beta b_0 \gamma \bar{c}_{t+1}^{\gamma(1-\rho)-1} \left(1-\bar{l}_{t+1}\right)^{(1-\gamma)(1-\rho)} \overline{v}\left(G_{s,t+1}\right) \end{array} \right) \right\},$$

subject to:

$$c_t + p_t^f B_{t+1} + p_t^s s_{t+1} \leq B_t + s_t \left(p_t^s + d_t\right) + w_t l_t,$$

where β is the time discount rate. The term $B_t + s_t \left(p_t^s + d_t\right) + w_t l_t$ is the total wealth that the agent possesses in period t that includes: the returns from buying bonds B_t, returns from investing stocks $s_t \left(p_t^s + d_t\right)$ with the share of the stock s_t at price of p_t^s, and labor income $w_t l_t$. Expenditures that include: current consumption c_t, the purchase of bonds $p_t^f B_{t+1}$, and stock purchase $p_t^s s_{t+1}$ cannot exceed the total wealth at the end of time t. Prices for stocks and bonds at time t are p_t^s and p_t^f, respectively, while d_t is the dividend paid to the investor by the firm.

The first half of the momentary utility function, $\left(c_t^{\gamma} \left(1-l_t\right)^{1-\gamma}\right)^{1-\rho} / \left(1-\rho\right)$, follows standard neoclassical macroeconomics. This utility function, defined on consumption and labor hours l_t, has two main parameters, ρ and γ, that mutually determine the EIS (Elasticity of Intertemporal Substitution), risk aversion, and the Frisch labor supply elasticity. With this form of Cobb-Douglas (CD) utility function, risk aversion is measured by the parameter ρ (> 0), EIS is measured by $1/\rho$, and the Frisch labor supply elasticity is $((1-l)/l)\left((1-\gamma(1-\rho))/\rho\right)$. The quantitative magnitude of these parameters are shown to be important in the subsequent analysis.

The second half of the expected utility function is adapted from the framework outlined in Barberis, et al. (2001) and Barberis and Huang (2008) which includes the LANF component discounted in the period $t+1$.

$$\beta b_0 \gamma \overline{c}_{t+1}^{\gamma(1-\rho)-1}\left(1-\overline{l}_{t+1}\right)^{(1-\gamma)(1-\rho)}\left[\overline{v}\left(G_{s,t+1}\right)\right],$$

The term $\overline{v}(G_{s,t+1})$ represents the gain or loss in the value of financial wealth.

$$G_{s,t+1} = p_t^s s_{t+1}\left(\frac{p_{t+1}^s + d_{t+1}}{p_t^s} - \frac{1}{p_t^f}\right),$$

$\overline{v}(G_{s,t+1})$ governs the equity premium trajectory and $\overline{v}(x)$generates the kink in preferences - i.e., the loss-aversion element of preferences.

$$\overline{v}(x) = \begin{cases} \lambda_L x & \text{for } x \geq 0 \text{ where } \lambda_L = 1. \\ \lambda_H x & \text{for } x < 0 \text{ where } \lambda_H \geq 1. \end{cases},$$

Households value the gain or loss by the function $\overline{v}(x)$ which demonstrates that agents are more sensitive to losses ($x \leq 0$) than to gains ($x \geq 0$). The parameters λ_L and λ_H ,where λ_L $< \lambda_H$,determine how sensitive households are to gains related to losses. More specifically, if the return on stocks is less than risk-free rate, the agent's utility is reduced more than otherwise. This behavioral assumption is in accordance with prospect theory (Kahneman and Tversky, 1979) which postulates that a consumer's utility is defined over the domain of losses or gains; in this case, the losses or gains result from purchases of equities.

Additionally, prospect theory implies that households narrowly frame their choices. Previous studies (Benartzi and Thaler 1995, Barberis et al. 2001) postulate that households narrowly frame both cross-sectionally and temporally by a fraction of the marginal utility of consumption. The term $b_0 \beta \gamma \overline{c}_{t+1}^{\gamma(1-\rho)-1}\left(1-\overline{l}_{t+1}\right)^{(1-\gamma)(1-\rho)}$ represents how investors frame outcomes. Here $\left\{\overline{c}_t, \overline{l}_t\right\}$ denote the aggregate per capita consumption and labor hours for a typical participating household, b_0 is a scaling factor to signify the degree of narrow framing (Grüne and Semmler 2008, Barberis et al. 2001), and the remaining term is the marginal utility of consumption. Setting $b_0 = 0$, eliminates narrow framing, recreating the standard asset pricing model. Thus, multiplying by b_0 adjusts the function for the overall importance of utility from gains and losses in financial wealth relative to utility from consumption.

Household optimatization yields three first-order conditions:

$$
\begin{aligned}
1 &= E_t \left\{ \beta \left(\frac{1 - l_{t+1}}{1 - l_t} \right)^{(1-\gamma)(1-\rho)} \left(\frac{c_{t+1}}{c_t} \right)^{\gamma(1-\rho)-1} \left(\frac{1}{p_t^f} \right) \right\}, && (1) \\
1 &= E_t \left\{ \begin{array}{c} \beta \left(\frac{1-l_{t+1}}{1-l_t} \right)^{(1-\gamma)(1-\rho)} \left(\frac{c_{t+1}}{c_t} \right)^{\gamma(1-\rho)-1} \times \\ \left[\left(\frac{p_{t+1}^s + d_{t+1}}{p_t^s} \right) + b_0 \overline{v} \left(\frac{p_{t+1}^s + d_{t+1}}{p_t^s} - \frac{1}{p_t^f} \right) \right] \end{array} \right\}, && (2) \\
0 &= \frac{1-\gamma}{(1 - l_t)} - w_t \frac{\gamma}{c_t} && (3)
\end{aligned}
$$

Equation (1) is the inter-temporal Euler for bond purchasers, (2) is the inter-temporal Euler for stocks and (3) is the intra-temporal Euler between consumption and labor hours. The LANF preferences are embodied in equation (2). Note that if $b_0 = 0$, then equation (2) would result in the standard asset pricing model of Danthine and Donaldson (2002). Equation (2) differs from Barberis and Huang (2008) and Grüne and Semmler (2008) in its reference to a general equilibrium environment. As the households maximize their utilities with respect to both consumption and labor, our Euler has an additional labor component not included in the model of Barberis and Huang. The extension to general equilibrium also adds equation (3), the condition for intra-temporal substitution between consumption and leisure.

2.2 Firms

The firms in this economy produce the consumption good with Cobb-Douglas technology $y_t = z_t k_t^\theta l_t^{1-\theta}$ in perpetuity. The level of technology evolves according to the exogenous process

$$\log (z_{t+1}) = \eta \log (z_t) + \varepsilon_{t+1},\ \varepsilon_t \overset{iid}{\sim} N \left(0, \sigma^2\right),$$

where η represents the persistence of aggregate shock with noise having an independent and identical distribution (iid) with mean of 0 and variance of σ^2. Firm value is maximized through the distribution of dividends to the agents (owners) in the household sector. The discounted value of the firm is $\sum_{j=0}^{\infty} \beta^j \Lambda_{t+j} d_{t+j}$,where Λ_{t+j} signifies the relative price of consumption; i.e., the equity owner's marginal utility of consumption. The specific maximization problem for the firm is:

$$p_t^s = \max_{\{k_{t+j}, l_{t+j}\}} E_t \left\{ \sum_{j=0}^{\infty} \beta^j \Lambda_{t+j} d_{t+j} \right\}. \tag{4}$$

The firm owns the capital k, and funds operations internally through retained earnings and externally by selling bonds. The total supply of the bonds at price of p_t^f is constant over time and equals to $\chi\overline{k}$, where χ is a constant representing the leverage ratio. The capital stock follows a law of motion

$$i_t = k_{t+1} - (1-\delta) k_t.$$

In order to create a wedge between the return to physical capital and the return to financial capital, Danthine and Donaldson (2002) introduced a cost that adjusts the firm's capital stock from its current level in macroeconomy in their study of EPP. More specifically, Basu (1987) highlights the importance of this adjustment cost to determining such financial variables as stock prices and long term real interest rates. Existence of this cost implies diminishing returns to augmenting the quantity of capital in the economy and therefore capital stock tends to adjust in a sluggish manner with respect to any productivity shock. Following previous studies (Guvenen 2009, Danthine and Donaldson 2002), an adjustment cost of investment is persumed to be concave and given as:

$$g(k_t, i_t) = \left(\frac{\phi}{2}\right)\left(\frac{1}{k_t}\right)(i_t - \delta k_t)^2 .$$

After retaining capital for future use, paying out the net interest to bondholders, making labor payment to employees, and including the capital adjustment cost, the firm maximizes its value subject to a dividend constraint:

$$d_t = y_t - w_t l_t - i_t - (1 - p_t^f)\chi\overline{k} - g(k_t, i_t).$$

Solving the maximization problem in (4) yields the first order conditions for a typical firm

$$\begin{aligned} 0 &= E_t \left\{ \begin{array}{c} \beta\Lambda_{t+2} \left(\begin{array}{c} \theta z_{t+1} k_{t+1}^{\theta-1} l_{t+1}^{1-\theta} + (1-\delta) \\ + \frac{\phi}{2}\left(\frac{k_{t+2}-k_{t+1}}{k_{t+1}} + 1\right)^2 - \frac{\phi}{2} \end{array} \right) \\ -\Lambda_{t+1}\left(1 + \phi\frac{k_{t+1}-k_t}{k_t}\right) \end{array} \right\}, && (5) \\ w_t &= (1-\theta)\, z_t k_t^{\theta} l_t^{-\theta}. && (6) \end{aligned}$$

Equation (5) is an inter-temporal Euler equation for the firm. Since it is assumed that the households are both owners and workers, this Euler equation is equivalent to the households' inter-temporal Euler equation when they own the stock of capital and rent it to the firm. In this model, ownership by the firm enables equity to have value as it is a claim to the returns from that capital stock. Ultimately, however, the predictions in either case are the same. Equation (6) is the standard intra-temporal Euler equation that equates the marginal product of labor with the wage rate.

2.3 Equilibrium

Equations (1), (2), (3), (5), and (6) are the necessary conditions that describe optimal behavior of the agents in the general equilibrium model. Equilibrium is formally defined in Equation (7).

$$y_t - g(k_t, i_t) = c_t + i_t. \tag{7}$$

Equation (7) shows that in general equilibrium, the total consumption for the representative households plus investment cannot exceed the total production minus the adjustment cost. All variables in equilibrium are represented in aggregate quantities. The reason why we label the aggregate level in the same fashion as the individual level (i.e., c_t, i_t) is that the representative agent is assumed to be distributed uniformly over $[0, 1]$.

The clearing of the bond market requires (8):

$$B_{t+1} = \chi \overline{k}, \tag{8}$$

and the clearing of the stock market requires (9):

$$s_{t+1} = 1. \tag{9}$$

Equations (8)-(9) characterize the aggregate levels for bonds and stocks. The law of motion for bonds requires that in equilibrium, the total bonds supplied (issued) by the firms are equal to the total demand for the bonds by households. As mentioned in section 2.2, the total supply for the bonds is $\chi \overline{k}$, while the total demand for the uniformly distributed bonds is B_{t+1}. Similarly, the shares of stock are a uniformly distributed among the populace where the total supply for the stocks s_{t+1} is inelastically set to 1 throughout time. Note that the firm is not issuing new shares and therefore the price of equity is changing solely due to demand.

In equilibrium, the return to equity is

$$r_t^e = \frac{p_t^s + d_t}{p_{t-1}^s},$$

the risk-free rate is

$$r_t^f = \frac{1}{p_t^f},$$

and the average equity premium is

$$r^{ep} = \sum_{t=1}^{T} \frac{\left(r_t^e - r_t^f\right)}{T}.$$

The equity premium is approximated by:

$$r^{ep} \approx E_t[r_t^e - r_t^f].$$

3 Solution, Calibration, and Estimation Methods

3.1 Solution

A feature of all three models (defined by different calibrations) is that their steady states equity premiums are all equal to zero. That is, the risky asset and the risk-free asset will have the same returns in the economies with no uncertainty. It is known that linear solutions (certainty equivalence) do not account for uncertainty and therefore will still get simulated equity premiums of zero. In this paper, second order solutions, known as perturbations are used to account for uncertainty. Perturbation methods, first introduced by Judd (1996), solve dynamic programming problems via higher order approximations. More specifically, the central idea of perturbation is to solve for a finite set of coefficients by taking repeated derivatives of the optimality equations (1), (2), (3), (5), (6), and (7). These coefficients define the second order approximations for the allocations and prices of the model that are defined over the set of states $\{k_t, z_t, \sigma, \lambda\}$ where $\lambda = \{\lambda_L, \lambda_H\}$.

Typically, the perturbation expansion occurs around steady states defined where the model uncertainty and distortions are zero (*i.e.*, $\sigma = 0$, $\lambda = 0$). Expansions of σ and λ are taken around zero as well. This solution method is presumably only accurate for when σ and λ are near zeros. Unfortunately, in this model, for LANF preference to be important, the uncertainty parameter σ and LANF parameter λ must be nontrivial. Furthermore, λ is changing depending on the sign of the difference of the returns; it's either λ_L or λ_H.

Therefore, the perturbation solution is modified by redefining σ and λ (change of variables). This modification follows the work of Judd (1996, 2002) and Fernández-Villaverde and Rubio-Ramirez (2006) where σ and λ are estimated by a projection of the perturbation solutions back onto the optimality equations (1)-(3) and (5)-(7). The solution to this hybrid perturbation-projection method is found by minimization of the optimality equations defined over a grid of points. The grid amounts to 70 points that are intended to cross over 90 percent below and above the steady state capital. The productivity states are approximated by a 40 point grid using Tauchens procedure[2]. Details for this method are discussed in the appendix.

3.2 Calibrations

To facilitate this analysis, some parameters are calibrated based on the estimations found in other studies (Abel 1980, Danthine and Donaldson 2002, Jermann 1998, Grüne and Semmler 2008, Guvenen 2009). From these studies, three main parameterizations are defined: (i) the baseline model with Cobb-Douglas preferences (Baseline CD); (ii) the CD baseline model where the Frisch elasticity is set to zero (CD Zero Frisch), and (iii) the CD Zero Frisch model where adjustment costs of investment are zero and capital fully depreciates (CD zero Frisch $\phi = 0$, $\delta = 1$).

For all models the capital share in output is set at $\theta = 0.3$, the same value as chosen by Kydland and Prescott (1982), Jermann (1998) and Grüne and Semmler (2008). This selection comforms with the labor elasticity suggested in the data during the period studied by Mehra and Prescott (1985).The discount rate β is set to 0.99 according to a steady state return on capital of 4%. Both Danthine et al. (2002) and Guvenen (2009) assumed this rate of return in their quarterly estimates in correspondence with a quarter period. The utility power parameter ρ – the relative risk aversion – is set equal to 4 following Danthine and Donaldson (2002). For the CD model, consumption's share in utility is set to $\gamma = 0.395$. This is chosen to follow Guvenen (2009) to match the average time devoted in market activities (0.36 of discretionary time). For the Zero Frisch models, $\gamma = 1$ and labor hours l_t are set to 1. The leverage ratio χ is assumed to be 0.15 which lies in the historical range of 0.13 to 0.44 (Jermann, 1998).

Another important parameter is the cost of adjustment constant ϕ which measures the elasticity of investment. But studies that incorporate adjustment cost functions have used a

[2] See Fernández-Villaverde and Rubio-Ramirez (2006) for details

varying range for ϕ. Danthine et al. (2002) and Jermann (1998) both states that the value of ϕ is set to maximize model's ability to match a set of moments of interest; too large value of ϕ leads to low volatility of investment. For example, Abel (1980) picked ϕ in the range of $[0.27, 0.52]$, Jermann (1998) estimated it as 0.23, and Guvenen (2009) calibrated it as 0.40. The way ϕ is picked in this paper is to match the adjustment cost not "too large" (Danthine and Donaldson, 2002) and to pursue the goal of smoothing the capital stocks. Therefore, this constant is set to $0.35 \exp(k_{ss})$ to be able to replicate Tobin's Q values. All of the calibrations are detailed in Table 1.

3.3 Estimation

This paper employs a prior predictive analysis (Canova 1994, Geweke 2007) to estimate the model's volatilities and equity premiums. The analysis is conducted in four steps. First, for the unknown parameters σ, b_0, and λ_H, prior distributions are assumed. The priors are threefold: σ is an inverse-gamma distribution, b_0 is a discrete uniform distribution, and λ_H follows an uniform distribution. Second, random realizations are drawn from these priors. Third, the model is solved for the equilibrium allocations and prices. Finally, the economy is simulated for a set of allocations and prices that are meant to mimic economy's volatilities and equity premiums.

The Bayesian method described above was implemented for all of the models according to the following procedure:

1. Assume prior distributions for σ, b_0, and λ_H.

2. Draw $i = 1 : 2500$ realizations for σ, b_0, and λ_H.

3. For each draw i, solve the model for endogenous output y, consumption c, investment i, capital k, price of quity p^s, and price of bond p^f.

4. For each draw of the random variables, simulate the economy for 500 quarters and form i estimates of $\sigma(y)$, $\sigma(c)/\sigma(y)$, $\sigma(i)/\sigma(y)$, $\sigma(l)/\sigma(y)$, $E(r_t^e)$, $E(r_t^f)$, $E(r_t^{ep})$ and $E(r_t^{ep})/\sigma(r^{ep})$.

5. Plot the distributions for $\sigma(y)$, $\sigma(c)/\sigma(y)$, $\sigma(i)/\sigma(y)$, $\sigma(l)/\sigma(y)$, $E(r_t^e)$, $E(r_t^f)$, $E(r_t^{ep})$ and $E(r_t^{ep})/\sigma(r^{ep})$ with their corresponding actual data built in respectively. The data were reported in previous research (Boldrin, Christiano and Fisher 2001, Danthine and Donaldson 2002, Guvenen 2009).

To obtain the most accurate results, the prior distributions are carefully picked. That is, the prior distributions should not only cover a large range of possibilities but also be consistent with economic intuition by having the correct signs. Barberis and Huang. (2008) adopt separate values for b_0 within the range of $[0, 0.1]$. Grüne and Semmler (2008) indicated that the degree of narrow framing can vary from 0 to 3. This paper assumes a prior for b_0 of a discrete uniform distribution from the set $[0.3, 1, 3, 10, 50, 100]$. This choice is designed to encompass the ranges studied in Barberis and Huang (2008) and Grüne and Semmler (2008). Following other asset pricing works (i.e., Jacquier, Polson and Rossi, 1994), this paper assumes σ in an inverse-gamma distribution. This special distribution allows the uncertainty centering above zero. The Baseline CD model assumes a shape parameter of 2.75 and a scale parameter $1.75 * 0.0045$ (this roughly matches the volatility of output). Previous studies also allow λ_H to have different values. To determine how loss aversion can impact the equity premium, this study assumes λ_H takes the form of a uniform distribution $[1, 200]$. The choice is designed to encompass the range studied in Barberis and Huang (2008) and Grüne and Semmler (2008) where λ_H falls in the range of $[1, 10]$ or the set of $[3, 5, 10, 20]$ respectively.

Following the five step procedure listed above will reveal how well the hypothetical economy captures the actual distributions of the macroeconomic volatilities and equity premiums. If the hypothesis is supported by the analysis; i.e., that agent have LANF preferences, then the data should fall within the predictive densities of the model. These predictive densities are accomplished, for each model estimate, by a non-parametric kernel smoother. The inverse cumulative distribution values for the actual data are derived from the predictive densities.

4 Results

This section analyzes how the LANF preferences can generate an equity premium and related economic volatilities. The results are represented by comparing the three models (Baseline CD model, CD Zero Frisch model and Zero Frisch $\phi = 0, \delta = 1$ model) with the actual data[3] and with the performances from the extant literature (Guvenen, BCF[4] and DD[5]). This comparison is shown in Table 2. To demonstrate how the three models capture the actual data via these volatilities and equity premium, Figure 1, 3 and 4 indicate the prior

[3]Data source: Boldrin, Christiano, and Fisher 2001, Danthine and Donaldson 2002, Guvenen 2009.
[4]Boldrin, Christiano, and Fisher (2001).
[5]Danthine and Donaldson (2002).

predictive densities for these three models respectively. The blue curves depict the prior predictive densities (estimated by a non-parametric kernel) and the red dashed line represents the actual data. Note that if the red line is within the distribution of the volatility, the model performs satisfactorily. On the other hand, if the red line falls outside the distribution, the model fails to describe the specific statistic. Table 3 maps these features into the prior cumulative distribution functions (C.D.F.) for the three models together. The conclusions are based on a two-sided 95% confidence level.

4.1 Baseline CD Model

Table 2 shows that Baseline CD model generates satisfactory volatility of output; 1.84 compared to 1.89 found in the data. The model also accurately predicts the relative volatility of consumption to output; 0.65 compared with 0.7. The value $\sigma(i)/\sigma(y)$ in the table shows, a good match for the baseline model and the actual data. In fact, the Baseline CD model performs as well as any other model tested (Guvenen, BCF[6] and DD[7]).

Even though the baseline model is able to predict volatilities that match the actual data, it is unable to explain the following statistics: relative volatility of labor to output, equity premium, and Sharpe ratio. As shown in Table 2, the volatility of labor to output $\sigma(l)/\sigma(y)$ for the baseline model is 0.26 which compares poorly to the 0.8 found in the actual data. The baseline model also fails to explain the main focus of this study – the equity premium. The baseline model's equity premium is 1.46 percent compared to 6.17 percent for the actual data. The Sharpe ratio displays the households' return for their investment expressed as the ratio of equity premium to its standard deviation. The Baseline CD value of 4.62 is much higher than the 0.32 found in the actual data.

Figure 1 and Table 3 jointly describe how the Baseline CD model performs in explaining the equity premium puzzle and other economic volatilities. Evidently, distributions of volatilities of output, consumption and investment fully encompass the actual data whereas the actual data lines for labor, equity premium and Sharpe ratio either fall outside of or lies peripheral to the corresponding distributions at 95% level. More precisely, the inverse C.D.F. does not fall within the range of $[0.025, 0.975]$. Column two shows the performance for Baseline CD model.

Even though the Baseline CD cannot explain the high equity premium, Figure 2 offers insights into the influences of the prior predictive distributions of technology shock σ, loss aversion parameter λ_H and the narrow framing parameter b_0 on the equity premium. The

[6] Boldrin, Christiano, and Fisher (2001).
[7] Danthine and Donaldson (2002).

uppermost panel shows that the technology shock parameter influences the equity premium to some extent. The LANF components, loss aversion λ_H and narrow framing b_0, were analyzed separately. Neither parameter displayed a close relationship with the equity premium in the baseline model. These weak results provide little support for the Benartzi and Thaler hypothesis in models with labor supply elasticities in the baseline ranges. The implication of these findings is that inclusion of a labor choice in the LANF preference model does not guarantee the high equity premium observed in actual data. Moreover, this model fails to explain the Sharpe ratio. For the measures the actual data falls outside of the predictive densities. Hence, a conclusion drawn from the Baseline CD model is that LANF preferences alone cannot resolve the equity premium puzzle. Additionally, LANF parameters have no effect on the equity premium; it is mainly determined by the variance of the technology.

4.2 Inelasticity of Labor

The Baseline CD model's results are in sharp contrast to the previous research of Barberis et al. (2001), Barberis and Huang (2008), and Grüne and Semmler (2008). A fundamental reason for this difference is that these studies assumed some combination: (i) inelastic labor; (ii) zero investment costs; and (iii) full depreciation. Presumably, as noted in Grüne and Semmler (2008), the elimination of smoothing margins and/or increased volatilities makes fluctuations in asset prices more costly thereby generating a higher equity premium. Two additional experiments are conducted to test this hypothesis. The first test is denoted as the CD Zero Frisch model in Table 2. In this model $\gamma = 1$ implying that households cannot smooth consumption by substituting labor for leisure. The second experiment sets $\delta = 1$ and $\phi = 0$ in order to study the effects of full depreciation and the inclusion of investments costs. This is denoted as Zero Frisch $\phi = 0, \delta = 1$ model.

Column four in Table 2 presents the result for the CD Zero Frisch model. The volatilities generated from this model are close to the actual data as well as those in the Baseline CD model. In terms of equity premium, changing the assumptions about the elasticity of labor alters the results significantly. Specifically, the equity premium is now 2.59 compared to 1.46 in Baseline CD model, a 72.6% increase whereas the Sharpe ratio remains high. The prior predictive densities are illustrated in Figure 3. The densities in the first three panel do not change notably, but there is a slight improvement in the estimated magnitude of the equity premium shown in the bottom-left panel. However, because the labor is supplied perfectly inelastic, the predictive distribution in the fourth panel of Figure 3 for the volatility of $\sigma(l)/\sigma(y)$ poorly describes the actual data (the straight blue line). The inverse C.D.F.s for this presented in Table 3, show a similar story to Figure 3. First, the volatilities of output and consumption capture the actual data at 95% level. Alternatively, the volatility of $\sigma(l)/\sigma(y)$ and the Sharpe ratio cannot be explained by this model. Secondly, the distribution of investment shifts to the right. An inverse C.D.F. of 0.033 in Zero Frisch model compared to 0.185 in CD Baseline, a change not evident in Figure 3. Finally, the improvement of equity premiums is illustrated by the change in the inverse C.D.F from 0.989 in Baseline CD model to 0.954 in this model.

Danthine and Donaldson (2002) consider adjustment cost as a necessary component in a production economy since these costs drive a wedge between the return to physical capital and financial capital. They also argue that without this cost, the marginal value of capital equals the price of the investment good, a fact not supported by the data. An example offered by Huffman and Wynne (1999) illustrates the importance of this adjustment cost. Inputs used to produce computers cannot easily and swiftly be converted into the physical capital such as equipment or skilled labor that are needed to produce heavy industrial equipment. Therefore, the adjustment cost is designed to feature the difficulty of reorienting the production of new capital goods from one specific sector to another. Another parameter, the depreciation rate δ, reduces the return to investment.

Column five of Table 2 presents the results of a simulation of Zero Frisch $\phi = 0, \delta = 1$ model. The result shows that with zero adjustment costs and full depreciation the volatility of output and relative volatility of investment decrease drastically, but that the relative volatility of consumption increases as well as the volatility of output. This is inconsistent with theory. Nevertheless, this specification improves the estimates of the equity premium when compared to the Baseline CD and Zero-Frisch models. The Sharpe ratio decreases compared to the previous two models, but stays higher than the actual data. Moreover, Figure 4 shows that the actual data lie within the long-tail of the simulated densities. Unfortunately, we see the positive results of LANF are generated at the expense of the macroeconomic performance of the model. Combining LANF preferences with the assumption of a perfectly inelastic labor supply (Grüne and Semmler 2008, Barberis et al. 2001, and Barberis and Huang 2008) generates a sizeable equity premium. However, assuming $\delta = 1$ distorts the predictions of other volatilities, such as the consumption/output ratio and investment/output ratio.

5 Conclusion

Benartzi and Thaler (1995) suggested LANF preferences as a possible explanation for the equity premium puzzle. Barberis et al. (2001), Barberis and Huang (2004 and 2008) and

Grüne and Semmler (2008) included LANF preferences in a partial equilibrium model in an attempt to clarify the EPP. Their findings supported the Benartzi and Thaler hypothesis. Alternatively, Danthine and Donaldson (2002) and Guvenen (2009) included labor choice in their general equilibrium models by assuming that households obtain utilities not only from consumption as in partial equilibrium, but also from leisure/labor component (DSGE model). The present paper tests the Benartzi and Thaler hypothesis in the context of a DSGE model. Conducting the tests entailed two steps: (i) solving the DSGE model with an uncertainty component, and (ii) estimating the model with non-smooth elements – the LANF preference function. Step (i) was completed by using the hybrid perturbation-projection method. To overcome the difficulties of implementing step (ii), this paper used prior predictive analysis; an estimation method that employs an iterative Bayesian approach.

The fundamental finding of the paper is that LANF preferences cannot explain the equity premium under reasonable assumptions about the standard deviations of important macroeconomic variables. When the model accurately predicts these macroeconomic volatilities, it does not produce an equity premium commensurate with past empirical findings. Only by including both unrealistic labor elasticities and depreciation rates could the model generate a reasonable equity premium.

Other studies have used alternative (non-LANF) assumptions to explain the EPP. For example, in column 7 of Table 2, Boldrin et al. (2001) generated an equity premium, 6.63% under the assumption of habit persistent preferences, which actually exceeds most empirical estimates of the EPP. Danthine and Donaldson (2002) and Guvenen (2009) generate equity premiums of 5.23% and 4.21%, respectively, by using imperfect risk sharing mechanisms. However, the results of these models are also undermined by their estimates of the the standard deviation of labor which are not reconcilable with actual empirical observations. Thus, the paradox is that models with LANF preferences fail to improve the predictions of the equity premium, just like all other theories, because the equilibrium volatility for labor hours is unreasonable.

From these results it appears that the Benartzi and Thaler hypothesis needs to be modified to include other dimensions about utility function. One possible modification is to extend the work of Guvenen (2009) who utilizes GHH preferences (Greenwood, Hercowitz and Huffman, 1998). In this case, the utility function allows for separation of risk aversion and labor supply elasticity. Another avenue is the work of Cho and Cooley (1994) which focuses on the intensity of hours worked and the elasticity of labor supply.

References

[1] Abel, A.B. 1980. Empirical Investment Equations: An Intergrative Framework, Carnegie-Rochester Conf. Ser. on Public Policy 12, 39-91.

[2] Barberis, N., M. Huang, and T. Santos. 2001. Prospect Theory and Asset Prices. The Quarterly Journal of Economics 116, 1-53.

[3] Barberis, N., and M. Huang. 2004. Preferences with Frames: A New Utility Specification That Allows for The Framing of Risks. Working paper, Yale University.

[4] Barberis, N., and M. Huang. 2008. The Loss Aversion/Narrow Framing Approach to The Equity Premium Puzzle. In: Rajnish Mehra, Editor, 2008. The Handbook of the Equity Risk Premium, North-Holland, Amsterdam.

[5] Basu, P. 1987. An Adjustment Cost Model of Asset Pricing. International Economic Review 28, 609-621.

[6] Benartzi, S., and R. H. Thaler. 1995. Myopic Loss Aversion and The Equity Premium Puzzle. The Quarterly Journal of Economics 110, 73-92.

[7] Boldrin, M., L. Christiano, and J. Fisher. 2001. Habit Persistence, Asset Returns, and the Business Cycle. American Economic Review 91, 149-166.

[8] Canova, F. 1994. Statistical Inference in Calibrated Models. Journal of Applied Econometrics 9, S123-S144.

[9] Cho, J., and T.F. Cooley. 1994. Employment and Hours over The Business Cycle. Journal of Economic Dynamics and Control 18, 411-432.

[10] Danthine, J., and J. Donaldson. 2002. Labor Relations and Asset Returns. The Review of Economic Studies 69, 41-64.

[11] DellaViglla, S. 2009. Psychology and Economics: Evidence from the Field. Journal of Economics Literature 47, 315-372.

[12] Drèze, J.H. 1981. Inferring Risk Tolerance from Deductibles in Insurance Contracts. Geneva Papers on Risk and Insurance 20, 48-52.

[13] Fernández-Villaverde, J., and J. Rubio-Ramirez. 2006. Solving DSGE Models with Perturbation Methods and A Change of Variables. Journal of Economic Dynamic and Control 30, 2509-2531.

[14] Geweke, J., and C. Whiteman. 2006. Bayesian Forecasting. In G. Elliott, C., W. J. Granger and A., Timmerman, The Handbook of Economic Forecasting. Amsterdam: North-Holland.

[15] Geweke, J. 2007. Models, Computational Experiments and Reality. Working paper, University of Iowa.

[16] Geweke, J. 2010. Complete and Incomplete Econometric Models. Princeton University Press. Princeton, New Jersey.

[17] Greenwood, J. Z. Hercowitz and G. W., Huffman. 1998. Investment, Capacity Utilization, and the Real Business Cycles. American Economic Review 78, 402-417.

[18] Grüne, L., and W. Semmler. 2008. Asset Pricing with Loss Aversion. Journal of Economic Dynamics & Control 32, 3253-3274.

[19] Guvenen, F. 2009. A Parsimonious Macroeconomic Model for Asset Pricing. Econometrica 77, 1711-1750.

[20] Huffman, G.W. and M.A.Wynne. 1999. The Role of Intratemporal Adjustment Costs in A Multisector Economy. Journal of Monetary Economics 43, 317-350.

[21] Jermann, U. 1998. Asset Pricing in Production Economies. Journal of Monetary Economics 41, 257-275.

[22] Jacquier, E., N. G. Polson, and P.E. Rossi. 1994. Bayesian Analysis of Stochastic Volatility Models. Journal of Business & Economic Statistics 12, 371-389.

[23] Judd, K.L. 1996. Approximation, Perturbation, and Projection Solution Methods in Economics. In: Amman, H., Kendrick, D. and Rust, J., Editors, 1996. Handbook of Computational Economics, North-Holland, Amsterdam.

[24] Judd, K.L., and S.Guu. 1997.Asymptotic Methods for Aggregate Growth Models. Journal of Economic Dynamics and Control 21,1025-1042.

[25] Judd, K.L. 2002. Perturbation Methods with Nonlinear Changes of Variables. Working paper, Hoover Institution, Stanford University.

[26] Kahneman, D., and A. Tversky. 1979. Prospect Theory: An Analysis of Decision under Risk. Econometrica 47, 263-292.

[27] Kocherlakota, N.R. 1996. The Equity Premium: It's Still A Puzzle. Journal of Economic Literature 34, 42-71.

[28] Kydland, F.E. and E.C. Prescott. 1982. Time to Build and Aggregate Flutuations. Econometrica 50, 1345-1370.

[29] Ljungqvist, L., and T. Sargent. 2004. Recursive Macroeconomic Theory. The MIT Press. Cambridge, Massachusetts.

[30] Mankiw, N.G., and S.P. Zeldes. 1991. The Consumption of Stockholders and Nonstockholders. Journal of Financial Economics 29, 97-112.

[31] Masulis, R.W. 1988. The Debt/Equity Choice. Ballinger, Cambridge, MA.

[32] Mehra, R., E. Prescott. 1985. The Equity Premium. Journal of Monetary Economics 15, 145-161.

[33] Sargent, T. 2010. Interview with Thomas Sargent. The Region September, 27-39.

Tables

Table 1: Parameterizations for CD Baseline

θ	Capital Share	0.30
δ	Depreciation Rate	0.02
β	Time Discount Factor	0.99
χ	Leverage Ratio	0.15
γ	Consumption Share	0.395
ρ	Relative Risk Aversion	4
ϕ	Elasticity of Investment	$0.35 \times \exp(k_{ss})$
η	Persistence of Aggregate Shock	0.95

Table 2: Model Performance

	Data	Model			Guvenen	BCF	DD
		CD Baseline	CD Zero Frisch	CD Zero Frisch $\phi = 0$, $\delta = 1$			
$\sigma(y)$	1.89	1.84	1.74	2.05	1.95	1.97	1.77
$\sigma(c)/\sigma(y)$	0.70	0.65	0.59	0.96	0.78	0.69	0.82
$\sigma(i)/\sigma(y)$	2.39	2.57	2.80	1.10	1.76	1.67	1.72
$\sigma(l)/\sigma(y)$	0.80	0.26	0	0	0.50	0.51	–
$E(r^{ep})$	6.17	1.46	2.59	2.82	4.21	6.63	5.23
$E(r^{ep})/\sigma(r^{ep})$	0.32	4.62	4.81	2.67	0.24	0.36	0.21

Table 3: Inverse Prior Cumulative Distribution Function and Features

Feature	Inverse C.D.F. at Data: CD Baseline	CD Zero Frisch	CD Zero Frisch $\phi = 0$, $\delta = 1$
$\sigma(y)$	0.683	0.714	0.636
$\sigma(c)/\sigma(y)$	0.797	0.969	0.000
$\sigma(i)/\sigma(y)$	0.185	0.033	1.000
$\sigma(l)/\sigma(y)$	1	1	0
$E(r^{ep})$	0.989	0.954	0.980
$E(r^{ep})/\sigma(r^{ep})$	0.005	0.007	0.016

Figures

Figure 1: Prior Predicitive Densities for CD Baseline Model (dashed line is actual data).

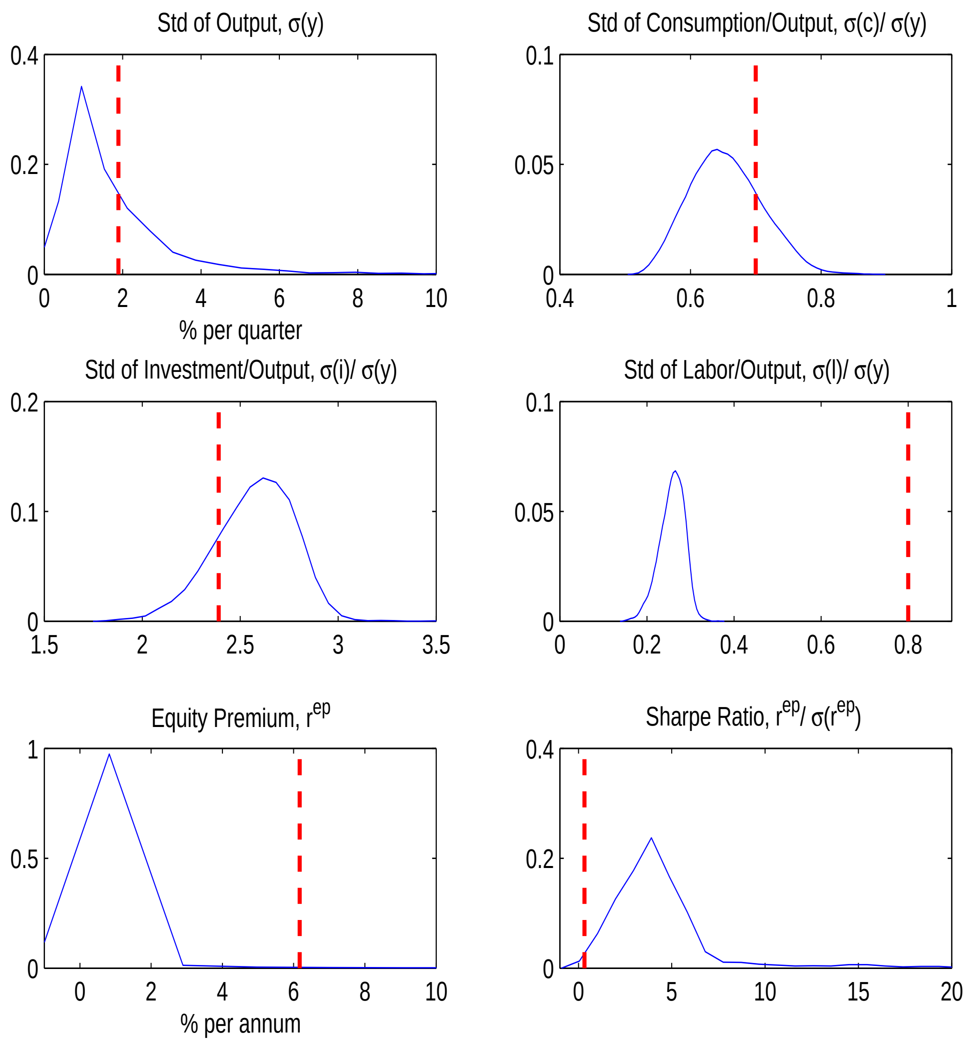


Figure 2: Relationship Between Equity Premium and Variable Estimates (CD Baseline Model).

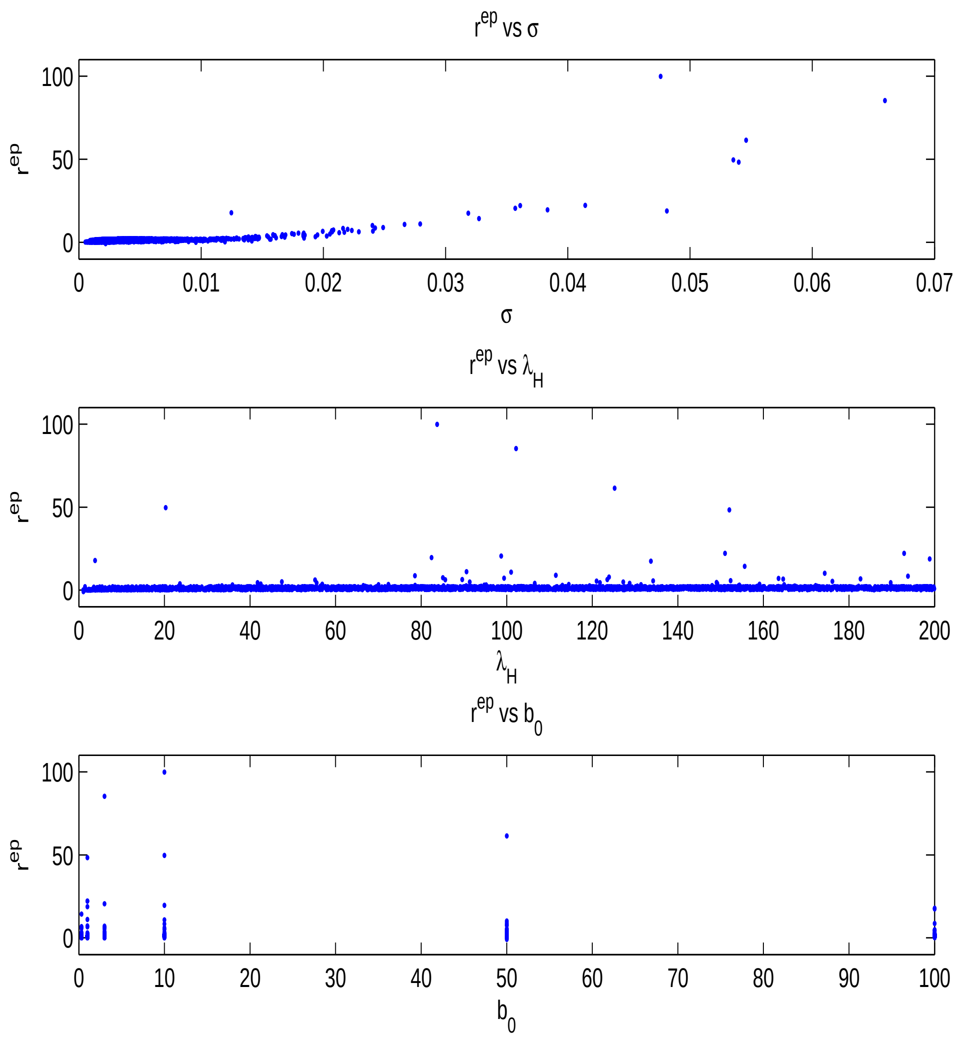


Figure 3: Prior Predicitve Densities for CD Zero Frisch Model(dashed line is actual data).

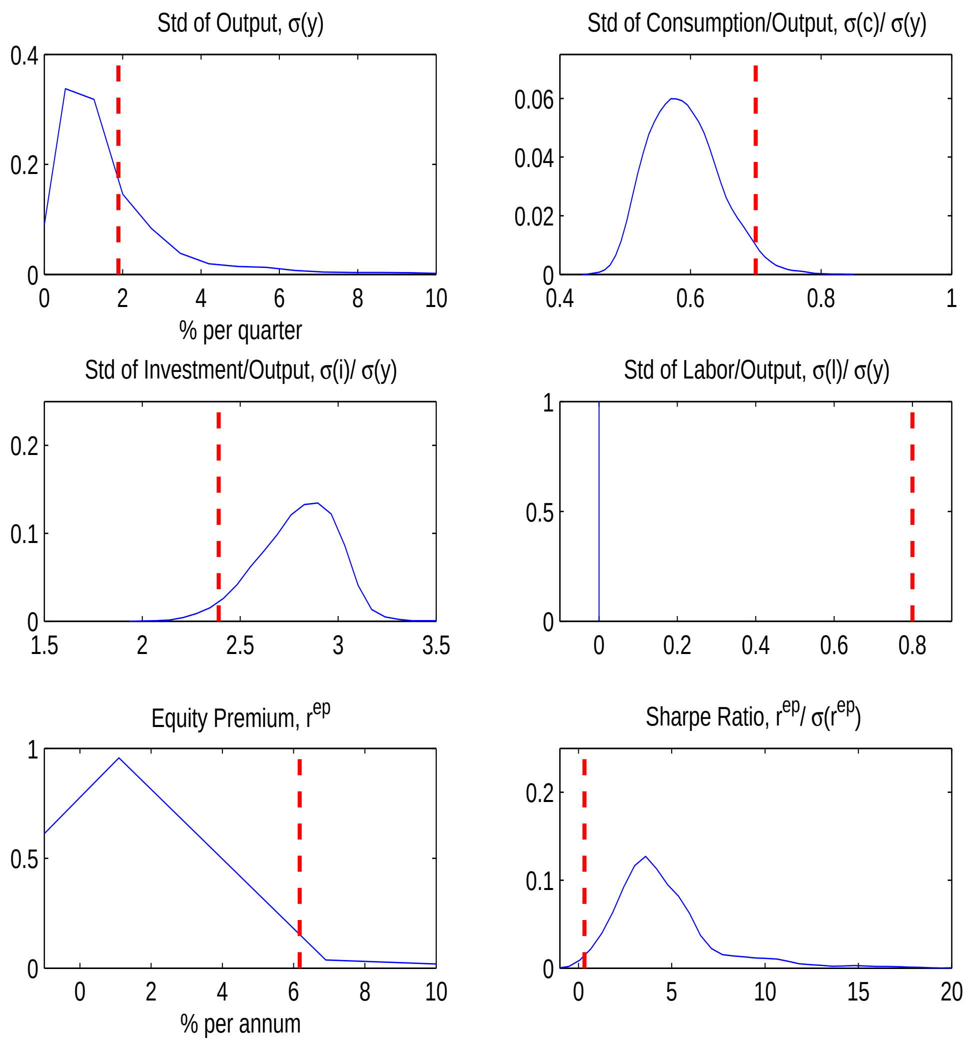
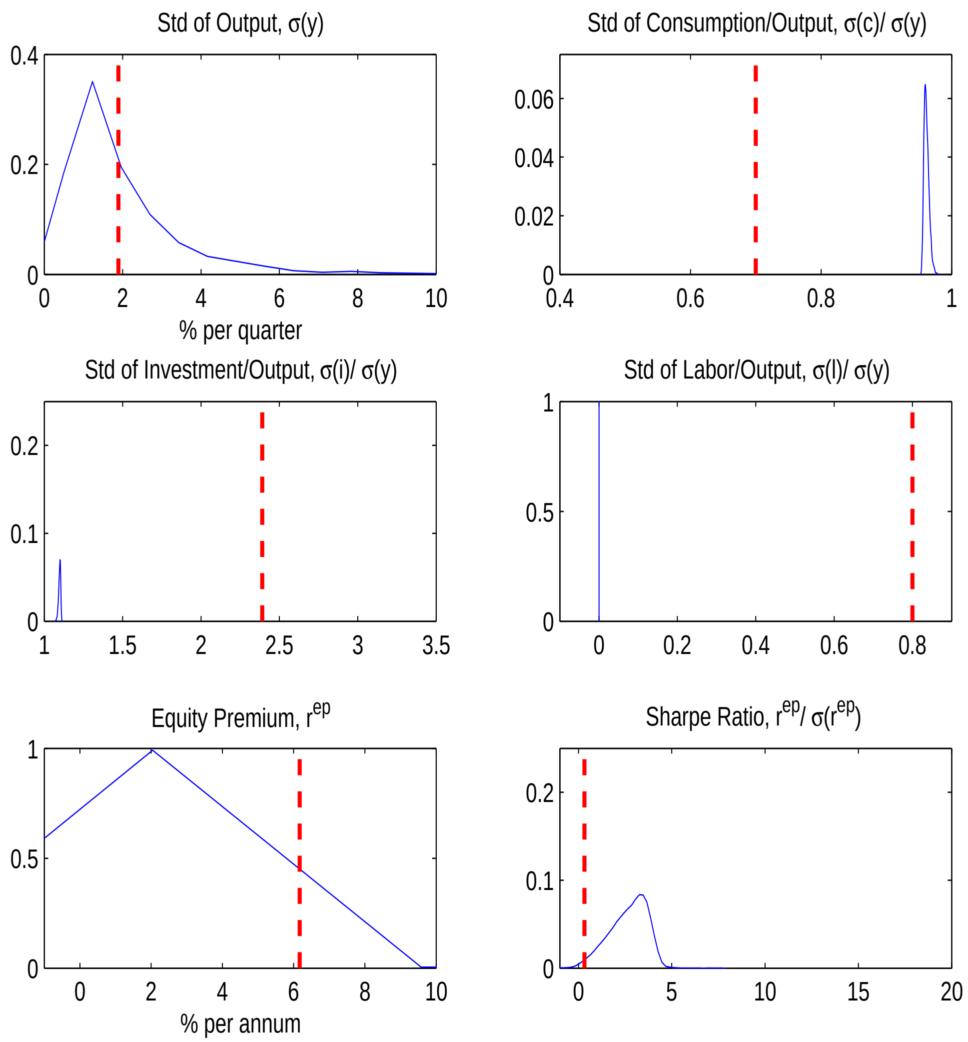


Figure 4: Prior Predicitve Densities for CD $\phi = 0$, $\delta = 1$ Model(dashed line is actual data).



Appendix

Details for Hybrid Perturbation-Projection Method

Perturbation with COV

The solution method here makes use of Taylor series expansion with changes of variables (Judd 1996, 2002; Fernández-Villaverde and Rubio-Ramirez 2006). Every policy function (i.e.,
l_t, k_{t+1}, etc.) is first approximated by a perturbation solution:

$$f(z,k,\sigma,\lambda) \approx \sum_{i,j,m,n} \frac{1}{(i+j+m+n)!} \left. \frac{\partial^{i+j+m+n} f(z,k,\sigma,\lambda)}{\partial z^i \partial k^j \partial \sigma^m \partial \lambda^n} \right|_{\{0,kss,0,0\}} z^i (k-kss)^j \sigma^m \lambda^n \quad (10)$$

where $\lambda_L = \lambda_H = \lambda = 0$ (i.e., no LANF). The solution in (10) is, presumably, accurate around $\{z,k,\sigma,\lambda\} = \{0,kss,0,0\}$. Then, λ and σ are replaced by a change of variables (COV) defined by either a constant or a polynomial in the states. More specifically, let the COV be;

$$\begin{aligned} y_{3,t} &= \tau_0 \sigma, \\ y_{4,t} &= \alpha_0 \lambda + \alpha_1 z_t + \alpha_2 (k_t - kss). \end{aligned}$$

To get a better understanding of how COV works, consider a simple example from Judd (2002). At first, the researcher has a basic second order Taylor series expansion of a function $f(x)$:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 \quad (11)$$

where x has been expanded around a. The COV is then defined by $y = Y(x)$ with an inverse function existing as $x = X(y)$. The COV finds $g(y) = f(X(y))$ at $y = b = Y(a)$. Note that $g(y)$ can be approximated with Chain Rule at second order by:

$$\begin{aligned} g(y) &= f(X(y)) \\ &\approx f(X(b)) + f'(X(b))X'(b)(y-b) + \frac{1}{2}(X'(b)^2 f''(X(b)) + f'(X(b))X''(b))(y-b)^2 \\ &= f(a) + f'(a)X'(b)(y-b) + \frac{1}{2}(X'(b)^2 f''(a) + f'(a)X''(b))(y-b)^2. \end{aligned}$$

More concretely, suppose $a = 1$ and the COV is $y = Y(x) = \log(x)$. Then, we immediately see that $b = 0$ and the inverse function is $x = X(y) = \exp(y)$. As a result, $X'(b) = 1$ and

$X''(b) = 1$. The COV expansion is thus:

$$f(a) + f'(a)\log(x) + \frac{1}{2}(f''(a) + f'(a))\log(x)^2,$$

where $\{f(a), f'(a), f''(a)\}$ are presumed to be known from (11).

In this study, the proposed transformation is:

$$Y(x_t) = \begin{bmatrix} y_{1,t} \\ y_{2,t} \\ y_{3,t} \\ y_{4,t} \end{bmatrix} = \begin{bmatrix} z_t \\ k_t - kss \\ \tau_0\sigma \\ \alpha_0\lambda + \alpha_1 z_t + \alpha_2 (k_t - kss) \end{bmatrix},$$

where $x_t = [z_t, k_t, \sigma, \lambda]'$. The inverse function is thus:

$$X(y_t) = \begin{bmatrix} Z(y_{1,t}) \\ K(y_{2,t}) \\ \Sigma(y_{3,t}) \\ \Lambda(y_{1,t}, y_{2,t}, y_{4,t}) \end{bmatrix} = \begin{bmatrix} y_{1,t} \\ y_{2,t} + kss \\ y_{3,t}/\tau_0 \\ (y_{4,t} - \alpha_1 y_{1,t} - \alpha_2 y_{2,t})/\alpha_0 \end{bmatrix}.$$

And, suppose that an initial second order perturbation gave equation (10) for k_{t+1} of:

$$\begin{aligned}
k_{t+1} &= \mathcal{K}(x_t) \\
&\approx \mathcal{K}_{<0,0,0,0>} + \mathcal{K}_{<1,0,0,0>} z_t + \mathcal{K}_{<0,1,0,0>}(k_t - kss) + \mathcal{K}_{<0,0,1,0>}\sigma + \mathcal{K}_{<0,0,0,1>}\lambda + \ldots \\
&\quad \mathcal{K}_{<2,0,0,0>} {z_t}^2 + \mathcal{K}_{<0,2,0,0>}(k_t - kss)^2 + \mathcal{K}_{<0,0,2,0>}\sigma^2 + \mathcal{K}_{<0,0,0,2>}\lambda^2 + \ldots \\
&\quad 2\mathcal{K}_{<1,1,0,0>} z_t(k_t - kss) + 2\mathcal{K}_{<1,0,1,0>} z_t\sigma + 2\mathcal{K}_{<1,0,0,1>} z_t\lambda + \ldots \\
&\quad 2\mathcal{K}_{<0,1,1,0>}(k_t - kss)\sigma + 2\mathcal{K}_{<0,1,0,1>}(k_t - kss)\lambda + \ldots \\
&\quad 2\mathcal{K}_{<0,0,1,1>}\sigma\lambda,
\end{aligned}$$

where

$$\mathcal{K}_{<i,j,m,n>} = \frac{1}{(i+j+m+n)!} \left. \frac{\partial^{i+j+m+n}\mathcal{K}(z, k, \sigma, \lambda)}{\partial z^i \partial k^j \partial \sigma^m \partial \lambda^n} \right|_{\{0,kss,0,0\}}.$$

Applying the COV gives:

$$
\begin{aligned}
k_{t+1} &= \mathcal{K}\left(Z(y_{1,t}), K(y_{2,t}), \Sigma(y_{3,t}), \Lambda(y_{1,t}, y_{2,t}, y_{4,t})\right) \\
&\approx \mathcal{K}_{<0,0,0,0>} + \left(\mathcal{K}_{<1,0,0,0>} + \mathcal{K}_{<0,0,0,1>}\Lambda_{<1,0,0>}\right) y_{1,t} + \ldots \\
&\quad \left(\mathcal{K}_{<0,1,0,0>} + \mathcal{K}_{<0,0,0,1>}\Lambda_{<0,1,0>}\right) y_{2,t} + \ldots \\
&\quad \left(\mathcal{K}_{<0,0,1,0>}\Sigma_{<1>}\right) y_{3,t} + \ldots \\
&\quad \left(\mathcal{K}_{<0,0,0,1>}\Lambda_{<0,0,1>}\right) y_{4,t} + \ldots \\
&\quad \left(\mathcal{K}_{<2,0,0,0>} + 2\mathcal{K}_{<1,0,0,1>}\Lambda_{<1,0,0>} + \mathcal{K}_{<0,0,0,2>}\Lambda^2_{<1,0,0>}\right) y^2_{1,t} + \ldots \\
&\quad \left(\mathcal{K}_{<0,2,0,0>} + 2\mathcal{K}_{<0,1,0,1>}\Lambda_{<0,1,0>} + \mathcal{K}_{<0,0,0,2>}\Lambda^2_{<0,1,0>}\right) y^2_{2,t} + \ldots \\
&\quad \left(\mathcal{K}_{<0,0,2,0>}\Sigma^2_{<1>} + \mathcal{K}_{<0,0,1,0>}\Sigma_{<2>}\right) y^2_{3,t} + \ldots \\
&\quad \left(\mathcal{K}_{<0,0,0,2>}\Lambda^2_{<0,0,1>}\right) y^2_{4,t} + \ldots \\
&\quad 2\left(\mathcal{K}_{<1,1,0,0>} + \mathcal{K}_{<1,0,0,1>}\Lambda_{<0,1,0>} + \mathcal{K}_{<0,1,0,1>}\Lambda_{<1,0,0>} + \mathcal{K}_{<0,0,0,2>}\Lambda_{<1,0,0>}\Lambda_{<0,1,0>}\right) y_{1,t}y_{2,t} + \ldots \\
&\quad 2\left(\mathcal{K}_{<1,0,1,0>}\Sigma_{<1>} + \mathcal{K}_{<0,0,1,1>}\Sigma_{<1>}\Lambda_{<1,0,0>}\right) y_{1,t}y_{3,t} + \ldots \\
&\quad 2\left(\mathcal{K}_{<1,0,0,1>}\Lambda_{<0,0,1>} + \mathcal{K}_{<0,0,0,2>}\Lambda_{<1,0,0>}\Lambda_{<0,0,1>}\right) y_{1,t}y_{4,t} + \ldots \\
&\quad 2\left(\mathcal{K}_{<0,1,1,0>}\Sigma_{<1>} + \mathcal{K}_{<0,0,1,1>}\Sigma_{<1>}\Lambda_{<0,1,0>}\right) y_{2,t}y_{3,t} + \ldots \\
&\quad 2\left(\mathcal{K}_{<0,1,0,1>}\Lambda_{<0,0,1>} + \mathcal{K}_{<0,0,0,2>}\Lambda_{<0,1,0>}\Lambda_{<0,0,1>}\right) y_{2,t}y_{4,t} + \ldots \\
&\quad 2\left(\mathcal{K}_{<0,0,1,1>}\Sigma_{<1>}\Lambda_{<0,0,1>}\right) y_{3,t}y_{4,t},
\end{aligned}
$$

where

$$
\begin{aligned}
\Lambda_{<i,j,n>} &= \left.\frac{\partial^{i+j+n}\Lambda\left(y_1, y_2, y_4\right)}{\partial y_1^i \partial y_2^j \partial y_4^n}\right|_{\{\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{y}_4\}}, \\
\Sigma_{<i>} &= \left.\frac{\partial^i \Sigma\left(y_3\right)}{\partial y_3^i}\right|_{\{\bar{y}_1,\bar{y}_2,\bar{y}_3,\bar{y}_4\}}
\end{aligned}
$$

My choice of COV gives $\Sigma_{<1>} = 1/\tau_0$, $\Sigma_{<2>} = 0$, $\Lambda_{<1,0,0>}{=}{-\alpha_1/\alpha_0}$, $\Lambda_{<0,1,0>}{=}{-\alpha_2/\alpha_0}$, and $\Lambda_{<0,0,1>}$ $=1/\alpha_0$ for example.

Projection Methods

Given the COV transformations for the policy solution set: $\{c_t, l_t, k_{t+1}, p_t^s, p_t^f\}$, the next step in the solution method quantifies the unknown parameters of the COV; $\{\lambda, \tau_0, \alpha_0, \alpha_1, \alpha_2\}$, by examing of the Euler Equation Errors (EER). Following Fernández-Villaverde and Rubio-Ramirez (2006) and Judd (2002), the COV solutions are projected onto the EER and minimized by choice of parameters. To reduce the dimension of the estimation set, α_0 and τ_0, normalized to one leaving the set $\{\sigma, \lambda, \alpha_1, \alpha_2\}$ to be found.

In the next step, using (6), the optimality equations (1), (2), (3), (5), and (7) are evaluated using the COV perturbation solutions for any given set of states and unknown parameters: $\{k_t, z_t, \sigma, \lambda, \alpha_1, \alpha_2\}$. These equations are stacked into a vector that is denoted $EER\left(k_t, z_t, \sigma, \lambda, \alpha_1, \alpha_2\right)$. By summing up EER by element and across sets of values for

$\{k, z\}$, the minimization problem is:

$$\min_{\{\sigma,\lambda,\alpha_1,\alpha_2\}} \sum_{j,k_i,z_i} \left| EER_j(k_i, z_i, \sigma, \lambda, \alpha_1, \alpha_2) \right|. \tag{12}$$

Following Fernández-Villaverde and Rubio-Ramirez (2006) a grid for k is made by a grid of 70 points intended so that $\{k_i\}_{i=1}^{70}$ crosses over 90 percent below and above the steady state capital. A grid of 40 productivity points for z is found by employing Tauchens procedure given a calibrations for η and a drawn σ from the prior. The grid $\{z_i\}_{i=1}^{40}$ has a Markov transition martrix that is used to compute the expectations in equation (12).

5.0.1 Evaluation of Solution Method

If the Hybrid Perturbation-Projection method is an improvement, then the Euler equation errors (EERs) evaluated at the solutions should be in magnitude smaller than the EERs evaluated at the regular perturbation solutions represented in (10). Figure (5) shows the relationship between the relative EERs (the ratio of the EERs under the hybrid method to the EERs under the regular perturbation method) and realized variables σ, λ_H, and b_0. We see that the relative EERs are all less than one implying that the Hybrid Perturbation-Projection method reduces the computational modeling error. Also, as expected, the relative EERs are related to σ, λ_H, and b_0. Low realizations for σ, λ_H, and b_0 reduce the importance of LANF preferences and technology shocks. This economy, with small distortions, is described well by the regular perturbation method. In total, the evidence suggests successful minimizations of the EERs using the proposed Hybrid Perturbation-Projection method.

Figure 5: Relationship Between Euler Errors and Variable Estimates (CD Baseline Model).