When will the Social Security Trust Fund Run Out?: Simulation by Linearization about the Current State

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Abstract

We estimate the time until the Social Security trust fund runs out by simulating an overlapping generations model with stochastic life spans, immigration, aggregate shocks, and a tax and transfer policy calibrated to the U.S. economy. This class of fiscal policy problems also highlights the need for a solution method that can accommodate unstable steady states and nonstationarity. We detail such a solution method in which we linearize the model around the current state, updating the approximated characterizing equations each period. Our baseline simulation is calibrated to match the forecast of the Social Security Trustees report. We find that the major source of uncertainty is not economic fluctuations, but uncertainty about the fundamental processes driving the economy and Social Security system. We show that changes in policy must be quite large to avoid trust fund insolvency within the next three decades.

1. Introduction and Literature Review

A large amount of current research is focused on the effects of changing fiscal policy in the United States, both with regard to countercyclical policy (see Christiano, et al (2010), Kumhof, et al (2010), and Zubiary (2010)) and to reducing the national debt (see Gomes, et al (2010) and Traum and Yang (2010)). Regarding questions about reducing the national debt, the two main contributors to U.S. deficit spending now and long into the future are the Social Security system and the government health care benefits of Medicare and Medicaid (see CBO (2010)). Because Social Security policy (as well as Medicare and Medicaid policy) affect age cohorts differently, overlapping generations (OLG) dynamic stochastic general equilibrium (DSGE) models are the theoretical tool of choice for these studies.

In this paper, we calibrate an OLG model with stochastic life spans, immigration, aggregate shocks, and a tax and transfer program similar to Social Security to the United States. We simulate this model in order to estimate the time until the Social Security trust fund runs out, as well as 95 percent confidence intervals around that point estimate. Our simulations imply that the Social Security trust fund is likely to run out in 26 years. However, 95 percent confidence intervals suggest that the trust fund could run out anytime between 24 and 30 years from now.

An additional contribution of this paper is that we detail a solution method for the broad class of DSGE models that have unstable steady states and are characterized by nonstationarity. Recent official projections have noted that the current state of U.S. tax policy is not sustainable (see CBO (2010) and GAO (2007)) and, therefore, is not a steady-state. However, current DSGE solution methods rely on the models exhibiting long-run stationarity. Our solution method accommodates nonstationarity by linearizing the characterizing equations of the model around the current state each period and updating those approximations each successive period. This solution technique is similar to the adaptive control method of approximating around some path of central tendency (see Kendrick, 2002). However, our method lets the path around which the approximation is based be determined each period by the approximation forecast.

The paper proceeds as follows. Section 2 lays out the model. Section 3 presents a stationary version of the model. Section 4 presents the unstable steady state and the calibration. Section 5 details the updating linearization around the current state solution method and a simulation of the trust fund. Section 6 presents how some policy experiments change the simulated time path of the trust fund balance, and Section 7 concludes.

2. The Model

Demographics

Households live for a maximum of S periods. Each period a new cohort of households is born and some portion of existing households of all ages die. In addition, each period new households of various ages immigrate into the economy. The populations of households of various ages evolve according to the following laws of motion.

$$N'_{s+1} = N_s(\rho_{s+1} + \iota_{s+1}) \text{ for } 1 \leq s \leq S-1 \tag{2.1}$$

Where N_s is the population aged s, ρ_{s+1} is the probability the household lives to age $s+1$ given it has already lived to age s, ι_{s+1} is the immigration rate for households as a faction of the current age s population. A prime on a variable ($'$) denotes its value in the following period.

Age 1 households arrive via birth after all immigration has occurred and agents are one period older.

$$N'_1 = \sum_{s=1}^{S} f_s N_s \tag{2.2}$$

Where f_s is the fertility rate for households of age s.

Households

The objective of existing households is to maximize the expected value of utility over their lifetime. All households are endowed with the same amount of labor at a given age. We assume they do not work when young, prior to age E, and cease working at an exogenously given retirement age of R.

Households accumulate capital over time by saving a portion of their wage income. They also receive a transfer payment (denoted T) each period which are the proceeds from liquidating the capital of the households that die at the end of the previous period. Finally, households participate in a public social security system by paying a portion of their wage income in taxes up to age $R-1$, and receiving a benefit payment (denoted b) each period thereafter until death.

For ease of analysis we choose to set up the households' problems as dynamic programs and write them using Bellman equations. For individuals in a generic cohort, aged s this is:

$$V_s(\Omega) = \max_{k_{s+1}'} u\{c_s\} + \beta\rho_{s+1}E\{V_{s+1}(\Omega')\}$$

Where Ω is the information set, $u\{.\}$ is the within-period utility function, and β is the household's subjective discount factor. Note that because households do not live forever, their value functions vary by age.

Household consumption is defined by the following budget constraint.

$$c_s = w\bar{\ell}_s(1-\tau) + (1+r-\delta)k_s - k_{s+1}' + b_s + T \quad (2.3)$$

for $1 \leq s \leq S$

Where w is the wage rate, $\bar{\ell}_s$ is the household endowment of labor ate age s, τ is the tax rate on labor income, r is the return on capital, δ is the rate of capital depreciation, k_s is the household's holdings of bonds coming into the period, b_s is the pension benefit payment received, and T is a lump-sum transfer.

The solution gives the following Euler equation:

$$u^c\{c_s\} = \beta\rho_{s+1}E\{u^c\{c_s'\}(1+r'-\delta)\} \quad (2.4)$$

Where $u^c\{.\}$ denotes the marginal utility of consumption.

We use versions of equation (2.4) for $1 \leq s \leq S-1$.

In order to solve for its own transition function, $k_{s+1}' = k_s(\Omega)$, the household needs know the value functions for ages s and s+1 and it needs to form an expectation of the aggregate capital stock, K'. This means it also needs to know the transition functions of all the other households and their arguments. The transition functions for the oldest cohort are trivial. Since $V_{S+1}(\Omega') = 0$, the household will choose $k_{S+1}' = 0$. Transition functions for other cohorts will be found using numerical techniques explained below.

Firms

Firms hire labor and capital to produce final goods which are either consumed or invested as new capital goods. They use a simple Cobb-Douglas production technology. The representative firm's problem is:

$$\max_{K,L} K^{\alpha}(e^{gt+z}L)^{1-\alpha} - rK - wL$$

Where K is the capital hired by the firm, L is the amount of labor it hires, g is the exogenous growth rate of labor-augmenting technology, and z is a stochastic technology shock.

The solution is characterized by the following three equations.

$$r = \alpha Y/K \quad (2.5)$$

$$w = (1-\alpha)Y/L \quad (2.6)$$

$$Y = K^{\alpha}(e^{gt+z}L)^{1-\alpha} \quad (2.7)$$

Technology is assumed to evolve over time according to the following law of motion.

$$z' = \psi_z z + e_z{}'; \; e_z{}' \sim iid(0, \sigma_z^2) \quad (2.8)$$

Government

Each period the government collects revenues and makes payments on two separate accounts. The first is a redistribution of the capital of deceased households over the current population. We assume an equal share for each household regardless of age. Since this is a pure redistribution scheme, the account must balance each period.

$$T' = \frac{\sum_{s=1}^{S} N_s(1-\rho_s)k_s}{\sum_{s=1}^{S} N_s'} \quad (2.9)$$

The second is the social security system, which accumulates a balance over time on a trust fund, denoted H, as illustrated below.

$$H' = (1+r)H + \sum_{s=E}^{R-1} N_s \tau w \bar{\ell}_s - \sum_{s=R}^{S} N_s b_s \quad (2.10)$$

Benefits are assigned when a household retires at age R and are a function of the average index of monthly earnings (AIME) at retirement. The index is inflated each year by the percent growth in wages. For simplicity we use the trend growth of the economy which is very similar in our model to the growth of wages. We assume that the benefit is some fraction, θ, of this value.

$$b_R = \theta a_R \quad (2.11)$$

For any individual AIME evolves as a running average over ages E to R according to:

$a'_{s+1} = \frac{s-E-1}{s-E} a_s(1+g) + \frac{1}{s-E} w\bar{\ell}_s$ for $E \le s \le R-1$

However, since there is immigration in our model, new individuals who have zero past earnings for purposes of AIME calculations are continually entering the cohort. We take the weighted average of the surviving domestic workers' AIME and zero for immigrant workers when calculating the cohort's new value next year.

$$a'_{s+1} = \frac{\rho_{s+1}}{\rho_{s+1}+\iota_{s+1}} \left[\frac{s-E-1}{s-E} a_s(1+g) + \frac{1}{s-E} w\bar{\ell}_s\right] \text{ for } E \le s \le R-1 \quad (2.12)$$

Once set at retirement benefits remain constant until death, however immigration averaging applies in this case also. New immigrants of retirement age or older receive no benefits.

$$b'_{s+1} = \frac{\rho_{s+1}}{\rho_{s+1}+\iota_{s+1}} b_s \text{ for } s > R \quad (2.13)$$

In order to assure the government does not violate a transversality condition we assume that the tax (τ) and replacement (θ) parameters are state dependent and adjust as the size of the trust fund nears some upper or lower bounds (H_{min} & H_{max}).

$$\tau = \begin{cases} \tau_1(H) & H_{min} \le H \le H_{low} \\ \bar{\tau} & H_{low} \le H \le H_{up} \\ \tau_2(H) & H_{up} \le H \le H_{max} \end{cases}$$

$$\tau'_1(H) < 0, \tau_1(H_{min}) = 1, \tau_1(H_{low}) = \bar{\tau}$$

$$\tau'_2(H) < 0, \tau_2(H_{up}) = \bar{\tau}, \tau_1(H_{max}) = 0 \quad (2.14)$$

$$\theta = \begin{cases} \theta_1(H) & H_{min} \le H \le H_{low} \\ \bar{\tau} & H_{low} \le H \le H_{up} \\ \theta_2(H) & H_{up} \le H \le H_{max} \end{cases}$$

$$\theta'_1(H) > 0, \theta_1(H_{min}) = 0, \theta_1(H_{low}) = \bar{\theta}$$

$$\theta'_2(H) > 0, \ \theta_2(H_{up}) = \bar{\theta}, \theta_1(H_{max}) = \infty \quad (2.15)$$

Market-clearing and Aggregation

The capital and labor market clearing conditions are given by:

$$K = \sum_{s=1}^{S} N_s k_s + H \quad (2.16)$$

$$L = \sum_{s=1}^{S} N_s \bar{\ell}_s \quad (2.17)$$

There is also a goods market clearing condition, $Y + (1-\delta)K = \sum_{s=1}^{S} c_s + K'$, but it is redundant by Walras Law.

Equations (2.1) through (2.18) define the model. There are $S+1$ exogenous state variables: the cohort populations, $\{N_s\}_{s=1}^{S}$, and the technology shock, z. Since capital prior to age E is assumed to be zero, there are $2(S-E)+1$ endogenous state variables: the bond holdings for each cohort, $\{k_s\}_{s=E+1}^{S}$, AIME for each cohort from labor force entry until retirement, $\{a_s\}_{s=E}^{R-1}$, benefits for every cohort thereafter, $\{b_s\}_{s=R}^{S}$, and the balance on the social security trust fund, H.

3. Stationarizing the Model

Our model as written is non-stationary. Technology has a trend rate of growth, g, and the population may also be growing over time. We can write equations (2.1) & (2.2) in matrix notation.

$$\mathbf{N}' = \mathbf{\Gamma N} \; ; \mathbf{\Gamma} = \begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_{S-1} & f_S \\ \rho_2+\iota_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \rho_3+\iota_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \rho_S+\iota_S & 0 \end{bmatrix}$$

Where $\mathbf{N}$ is the $S \times 1$ vector of cohort populations. We define the total population as $= \mathbf{1}_{1\times S}\mathbf{N}$. The growth rate of the population comes from $N' = (1+n')N$ and by substitution this is $n' = \frac{\mathbf{1}_{1\times S}\mathbf{\Gamma N}}{\mathbf{1}_{1\times S}\mathbf{N}} - 1$

In order to solve our model using the numerical techniques we propose, it is necessary to transform the non-stationary variables to stationary ones. Some per capita variables, such as consumption and wages, will grow at the long-run rate of g. We transform these variables by defining a stationary version that removes this growth. We denote these transformed variables with a carat (^). $\hat{x} \equiv x/e^{gt}$ for $x \in (\{k_s\}_{s=2}^{S}, \{a_s\}_{s=E}^{R}, \{b_s\}_{s=R}^{S}, \{c_s\}_{s=1}^{S}, w)$

To transform the cohort populations we need to remove a unit root, which we do by dividing by the total population, N. $\hat{x} \equiv x/N$ for $y \in (\{N_s\}_{s=1}^{S}, L)$

Finally some aggregate variable grow at the rate g and also have a unit root. $\hat{x} \equiv x/(Ne^{gt})$ for $y \in (Y, K, H)$

If we assume a within-period utility function, $u(c) = \frac{1}{1-\gamma}(c^{1-\gamma} - 1)$, the transformed equations that define the stationary model are:

$$z' = \psi_z z + e_z{}'; \; e_z{}' \sim iid(0, \sigma_z^2) \tag{3.1}$$

$$n' = \frac{\mathbf{1}_{1\times S}(\bar{\mathbf{\Gamma}}+\hat{\mathbf{\Gamma}})\hat{\mathbf{N}}}{\mathbf{1}_{1\times S}\hat{\mathbf{N}}} - 1 \tag{3.2}$$

$$\hat{N}_{s+1}'(1+n') = \hat{N}_s(\rho_s + \iota_s + \varepsilon_{s+1}') \tag{3.3}$$

for $1 \leq s \leq S-1$

$$\hat{N}_1'(1+n') = \sum_{s=1}^{S}(f_s + \nu_1')\hat{N}_s \tag{3.4}$$

$$\hat{c}_s = \hat{w}\bar{\ell}_s(1-\tau) + (1+r-\delta)\hat{k}_s - (1+g)\hat{k}_{s+1}' + \hat{b}_s + \hat{T} \tag{3.5}$$

for $1 \leq s \leq S$

$$\hat{c}_s{}^{-\gamma} = \beta E\{[\hat{c}_s'(1+g)]^{-\gamma}(1+r'-\delta)\} \tag{3.6}$$

for $1 \leq s \leq S-1$

$$r = \alpha\hat{Y}/\hat{K} \tag{3.7}$$

$$\hat{w} = (1-\alpha)\hat{Y}/\hat{L} \tag{3.8}$$

$$\hat{Y} = \hat{K}^\alpha(e^z\hat{L})^{1-\alpha} \tag{3.9}$$

$$\hat{T}' = \frac{\sum_{s=1}^{S}\hat{N}_s(1-\rho_s)\hat{k}_s}{(1+n')\sum_{s=1}^{S}\hat{N}_s'} \tag{3.10}$$

$$\hat{H}'(1+g) = \hat{H} + \sum_{s=E}^{R-1}\hat{N}_s\tau\hat{w}\bar{\ell}_s - \sum_{s=R}^{S}\hat{N}_s\hat{b}_s \tag{3.11}$$

$$\hat{a}_{s+1}'(1+g) = \frac{\rho_{s+1}}{\rho_{s+1}+\iota_{s+1}}\left[\frac{s-E-1}{s-E}\hat{a}_s + \frac{1}{s-E}\hat{w}\bar{\ell}_s\right] \tag{3.12}$$

for $E \leq s \leq R-1$

$$\hat{b}_R = \theta\hat{a}_R \tag{3.13}$$

$$\hat{b}_{s+1}'(1+g) = \frac{\rho_{s+1}}{\rho_{s+1}+\iota_{s+1}}\hat{b}_s \tag{3.14}$$

for $s > R$

$$\hat{K} = \sum_{s=1}^{S}\hat{N}_s\hat{k}_s + \hat{H} \tag{3.15}$$

$$\hat{L} = \sum_{s=1}^{S}\hat{N}_s\bar{\ell}_s \tag{3.16}$$

4. Calibration and Steady States

Table 1 lists the parameters of the model which will need to be given specific numerical values.

We set the number of periods in our model and interpret the period so that S periods corresponds to 100 years. We assume agents become financially independent at and enter the labor force at age 16, which gives $E = \text{round}(\frac{16}{100}S)$. We assume retirement occurs at age 67 so that $R = \text{round}(\frac{67}{100}S)$. The depreciation rate is set to correspond to an annual rate of 5%; $\delta = 1 - (1 - 0.05)^{100/S}$. Similarly, β is chosen to yield an average annual rate of time preference of approximately 2% when coupled with the age-specific mortality hazard. And g is chosen to yield an annual growth rate of technology of 1.0%, $g = (1 + 0.01)^{100/S}$.

The capital share in GDP (α) is set to 0.35. γ is the intertemporal elasticity of substitution and we set this to 1.0, which yields logarithmic utility. The benefit to AIME ratio (θ), or replacement rate is set to .40. The payroll tax rate (τ) is chosen to make total social security benefits and taxes equal in the steady state.

Effective labor supply, fertility rates, survival rates and immigration rates by age are estimated using data from a variety of sources. Data on effective labor comes from the Bureau of Labor Statistics' Current Population Survey. Data on immigration rates come from the US Census Bureau. Fertility rates come from Nishiyama & Smetters (2007). Cumulative survival rates come from the Center for Disease Control's (CDC) mortality tables. We fit polynomials to the data by age. For fertility and immigration we fit the number of births or immigrants of a certain age as a percent of the population of that age.

Data for effective labor supply comes from quarterly earnings data for 2001 through 2010. We use earnings because our effective labor includes both hours worked and the productivity of the worker. Since wage rates should be proportional to productivity, we can simply use earnings which is hours worked times the wage rate per hour. We normalize so that the average earnings over the ages reported is one. We then fit earnings by age to the average age of the cohort using a 6th-order polynomial in the age. Since these polynomials are ill-behaved at the ends, we interpolate exponentially to get better fit there. Figure 1 shows the data and the fitted curve. When we simulate we choose the size of a period in years, and use this fitted curve to get effective labor for each cohort.

Data for immigration is available from 2005 detailing the number of those who immigrated between 2000 and 2005. Immigrants are grouped into cohorts of five years.

We calculate the number of immigrants as a percentage of the US population in 2000. We then fit this percentage by age to the average age of the cohort using a 6th-order polynomial in the age. We interpolate linearly at the ends. Figure 2 shows the data and the fitted curve. When we simulate we choose the size of a period in years, and use this fitted curve to get immigration rates for each cohort.

For fertility rates the data are available in 5-year cohorts as well. Fertility rates below age 15 and above age 50 are effectively zero. We proceed as above and fit this data with a 3rd-order polynomial in age. Again we interpolate, but only on the upper end. The data and fitted curve are shown in Figure 3.

For survival rates, we fit data on the cumulative probability of surviving to a particular age. We infer the conditional survival rates from this fitted polynomial. Data are available for 10-year cohorts. We fit this with a third-order polynomial and interpolate on the upper end so that mortality reaches 100% at age 100. The data and fitted curve are shown in Figure 4.

For both fertility and survival rates we introduce exogenous changes over time that are consistent with the social security trustees' assumptions. The trustees assume a life expectancy of 22.9 and 24.8 years beyond age 65 for males and females in the year 2085. We treat calibrate our survival rates over time so that the average life expectancy for all agents in the model at age 65 converges to the average of the trustees assumptions (23.6) for males and females by 2085. The initial life expectancy at the beginning of the simulation is 19.6 years.

For fertility the trustees' intermediate assumption is that total fertility rate will drop from its current value of 2.06 to 2.00 by the year 2035. We estimate an AR(1) model for total fertility using the trustees' report data since 1997 and find an autocorrelation parameter of .889. We assume that fertility rates across all ages decay from their current values at this rate per year converge to a total fertility rate of 2.00 in the long-run.

The threshold levels for adjusting policy from equations (2.14) & (2.15) are set such that taxes and transfers begin to change when the size of trust fund reaches 50% of the capital stock, either positive or negative. The second threshold is when the size is 100%.

With the model calibrated we can easily solve for the steady state. We do so using numerical techniques. The steady state is summarized in Table 2 and in Figure 5.

5. Solution and Simulation

We propose solving and simulating our model in the same way that many dynamic stochastic general equilibrium (DSGE) models with infinitely-lived agents are solved and simulated by linear approximation. To see the parallels we first outline the methodology for the infinitely-lived representative agent case.

Consider a simple infinitely-lived agent's problem.

$V(k; \mathrm{z}) = \max_{k'} u(c) + \beta E\{V(k'; z)\}$

With $c = w\bar{\ell} + (1 + r - \delta)k - k'$, $y = y(k, z)$, $r = y_k(k, z)$ & $w = y(k, z) - y_k(k, z)k$.

The Euler equation in this case is:

$$u^c(c) = \beta E\{u^c(c')(1 + r' - \delta)\} \tag{5.1}$$

The single endogenous state variable is k and we have assumed there is a single technology shock, z. To solve this model we first log-linearize our Euler equation about the model's steady state. We can write this in the form below, where the tildes (~) denote log-deviations from steady state values.

$$E_t\{T + F\tilde{k}_{t+1} + G\tilde{k}_t + H\tilde{k}_{t-1} + L\tilde{z}_{t+1} + M\tilde{z}_t\} = 0 \tag{5.2}$$

Where F, G, H, L & M are coefficients that are functions of parameters and steady state values. When linearizing about the steady state, T will be zero.

Assuming a log linear law of motion for z, $\tilde{z}_{t+1} = (1 - N)\bar{z} + N\tilde{z}_t + e_{t+1}$, and assuming that the transition function, $k_{t+1} = \phi(k_t, z_{t+1})$, can also be written in log-linear form we can find its coefficient values.

$$\tilde{k}_{t+1} = P\tilde{k}_t + Q\tilde{z}_{t+1} + U \tag{5.3}$$

The techniques for finding the numerical values of P & Q are well-known and involve solving a quadratic in P.[1] Solution techniques for U are less commonly used, but easy to derive. They can be shown to yield a U equal to zero when linearizing about the steady state.

[1] See Uhlig (1999) or Christiano (2002).

Next, consider an OLG model with a similar setup. An age s agent solves the following problem.

$$V_s(k_s, \mathrm{z}) = \max_{k_{s+1}{}'} u(c_s) + \beta E\{V_{s+1}(k_{s+1}{}', \mathrm{z}')\}$$

With $c_s = w\bar{\ell}_s + (1 + r - \delta)k_s - k'_{s+1}$, $K = \sum_{i=1}^{J} N_s k_s$, $y = f(k; \mathrm{z})$, $r = f_k(k; \mathrm{z})$, $w = f(k; \mathrm{z}) - f_k(k; \mathrm{z})k$

The Euler equation in this case is:

$$u^c(c_s) = \beta E\{u^c(c'_{s+1})(1 + r' - \delta)\}$$

If we set up and solve each agent's problem and then stack the variables for each agent such that $\mathbf{x} \equiv \begin{bmatrix} x_1 \\ \vdots \\ x_S \end{bmatrix}$, we get the following matrix representation of the model[2], where bold variables indicate matrices.

$$\mathbf{V}(\mathbf{k}, z) = \max_{\mathbf{k}'} \mathbf{u}(\mathbf{c}) + \beta\boldsymbol{\Delta}E\{\mathbf{V}(\mathbf{k}', z')\}$$

with $\mathbf{c} = w\bar{\boldsymbol{\ell}}(1 - \tau) + (1 + r - \delta)\mathbf{k} - \boldsymbol{\Delta}\mathbf{k}'$, $K = \mathbf{1}_{1\times S} \cdot (\mathbf{N} \circ \mathbf{k})$, $y = y(K; \mathrm{z})$, $r = y_k(K; \mathrm{z})$, $w = y(K; \mathrm{z}) - y_k(K; \mathrm{z})K$, $\boldsymbol{\Delta} \equiv \begin{bmatrix} \mathbf{0}_{(S-1)\times 1} & \mathbf{I}_{(S-1)\times(S-1)} \\ 0 & \mathbf{0}_{1\times(S-1)} \end{bmatrix}$. $\mathbf{V}(\mathbf{k}, z)$ and $\mathbf{u}(\mathbf{c})$ are S×1 vector-valued functions.

The stacked Euler equations are:

$$\mathbf{u}^c(\mathbf{c}) = \beta\boldsymbol{\Delta}E\{\mathbf{u}^c(\mathbf{c}')(1 + r' - \delta)\} \tag{5.3}$$

where $\mathbf{u}^c$ is an S×1 vector of the derivatives of $\mathbf{u}(\mathbf{c})$ with respect to the s^{th} element of $\mathbf{c}$. Note that the final S^{th} row is dropped since the S aged agent has no Euler equation.

We can solve and simulate this model just as we do the DSGE model above.

We write the log-linearized versions of the Euler equations in the following form:

$$E_t\{\mathbf{T} + \mathbf{F}\tilde{\mathbf{k}}_{t+1} + \mathbf{G}\tilde{\mathbf{k}}_t + \mathbf{H}\tilde{\mathbf{k}}_{t-1} + \mathbf{L}\tilde{z}_{t+1} + \mathbf{M}\tilde{z}_t\} = 0 \tag{5.4}$$

We use the same numerical techniques as above to solve for the matrices $\mathbf{P}$ & $\mathbf{Q}$ in the log-linearized transition functions.

$$\tilde{\mathbf{k}}_{t+1} = \mathbf{P}\tilde{\mathbf{k}}_t + \mathbf{Q}\tilde{z}_{t+1} + \mathbf{U} \tag{5.5}$$

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[2] Note that a $'$ always denotes next period, not a transpose. A transpose is denoted with a **T** superscript, instead.

To simulate our particular model we use the linearized transition functions for our stationary model laid out in section 3.[3]

$$\tilde{\mathbf{X}}_{t+1} = \mathbf{P}\tilde{\mathbf{X}}_t + \mathbf{Q}\tilde{\mathbf{Z}}_{t+1} + \mathbf{U} \tag{5.6}$$

Where $\mathbf{X}_t = [\{k_{s,t+1}\}_{s=2}^{S} \ \{a_{s,t+1}\}_{s=E+1}^{R} \ \{b_{s,t+1}\}_{s=R}^{S}]$ and $\mathbf{Z}_t = [z_t \ \{f_{s,t} \ \iota_{s,t} \ \rho_{s,t}\}_{s=1}^{S}]$.

Along with the exogenous laws of motion defined by equations (3.1) – (3.4) which we rewrite collectively as:

$$\tilde{\mathbf{Z}}_{t+1} = \mathbf{N}\tilde{\mathbf{Z}}_t + \mathbf{e}_{t+1} \tag{5.7}$$

Our technique works only as an approximation, since the **P** and **Q** coefficients in (5.6) will generally vary from period to period. We are therefore introducing an additional source of approximation error. However, the benefit is a reduction the approximation due to the Taylor-series expansion of our behavioral equations. Since these are normally taken about the steady state, and our simulations are only rarely in the steady state, the first approximation error is much smaller than the second. We illustrate this below.

We begin our simulation with initial conditions for the log-deviations of the state variables, $\mathbf{X}_t$ & $\mathbf{Z}_t$, from their steady state values. We also draw a series of random shocks for the values of $\mathbf{e}_t$ in each period. Equations (5.5) & (5.6) allow us to generate a time series for the log-deviations of our state variables from their steady states.

We can reconstruct the stationary versions of state variables by treating them as percent deviations using $\hat{x}_t = \bar{x}e^{\tilde{x}_t}$. We can also construct the total population using an initial value, the formula $N_{t+1} = (1 + n_{t+1})N_t$, and by noting that n_{t+1} is a function of our stationary state variable by equation (3.5).

Finally, we can construct non-stationary variables by putting in the appropriate trend and/or unit root, $x_t = e^{gt}\hat{x}_t$, $x_t = N_t\hat{x}_t$ or $x_t = N_t e^{gt}\hat{x}_t$. Once we have a time-series for all state variables in the non-stationary model, we can find the value of any other variable of interest by using the appropriate structural equation(s) from section 2.

[3] We use equations (3.9), (3.15), (3.16) & (3.17) as the dynamic equations which are linearized. Equations (3.1) – (3.4) define the exogenous laws of motion. The remaining equations are used as definitions.

The methodology above works well for simulations where the state of the economy deviates only in a neighborhood about the steady state. However, our model is dynamically unstable. This means that even if we start out at the model's steady state values, the stochastic shocks will drive us away from that point and the model will explode thereafter. We need a simulation technique that will be accurate when we are far from the steady state.

One technique that fits the bill is to linearize about the current state of the economy rather than around the steady state. We can use equations (5.4), (5.5) & (5.7), but we reinterpret the tilde as the deviation of the variable from its value now, rather than its value in the steady state. We rewrite these equations noting that the coefficients will now be time dependent since we linearize about a different point each period.

$$E_t\{\mathbf{T}_t + \mathbf{F}_t\widetilde{\mathbf{X}}_{t+1} + \mathbf{G}_t\widetilde{\mathbf{X}}_t + \mathbf{H}_t\widetilde{\mathbf{X}}_{t-1} + \mathbf{L}_t\widetilde{\mathbf{Z}}_{t+1} + \mathbf{M}_t\widetilde{\mathbf{Z}}_t\} = 0 \tag{5.8}$$

$$\widetilde{\mathbf{Z}}_{t+1} = \mathbf{N}\widetilde{\mathbf{Z}}_t + (\mathbf{N} - \mathbf{I})(\mathbf{Z}_t - \bar{\mathbf{Z}}) + \mathbf{e}_{t+1} \tag{5.9}$$

$$\widetilde{\mathbf{X}}_{t+1} = \mathbf{P}_t\widetilde{\mathbf{X}}_t + \mathbf{Q}_t\widetilde{\mathbf{Z}}_{t+1} + \mathbf{U}_t \tag{5.10}$$

In this case the matrices $\mathbf{T}_t$ and $\mathbf{U}_t$ will generally not be zero. Since the current state is $(\mathbf{X}_t, \mathbf{Z}_t)$ when we move to next period this becomes $(\mathbf{X}_{t-1}, \mathbf{Z}_{t-1})$. $\mathbf{Z}_t$ is found immediately by using (5.5). So we linearize about the point $(\mathbf{X}_{t-1}, \mathbf{Z}_t)$. This means (5.9) can be rewritten as:

$$\widetilde{\mathbf{X}}_{t+1} = \mathbf{U}_t \tag{5.11}$$

$\mathbf{U}_t$ can be shown to be:

$$\mathbf{U}_t = -(\mathbf{F}_t + \mathbf{F}_t\mathbf{P}_t + \mathbf{G}_t)^{-1}[\mathbf{T}_t + (\mathbf{F}_t\mathbf{Q}_t + \mathbf{L}_t)(\mathbf{N} - \mathbf{I})(\mathbf{Z}_t - \bar{\mathbf{Z}})] \tag{5.12}$$

Simulation proceeds by first setting the values of the initial state. As one simulates each period sequentially:

- (5.9) gives the next value for $\mathbf{Z}_t$.
- One solves for the values of $\mathbf{P}_t$, $\mathbf{Q}_t$, & $\mathbf{U}_t$ by linearizing about $(\mathbf{X}_{t-1}, \mathbf{Z}_t)$.
- (5.10) gives the next value for $\widetilde{\mathbf{X}}_t$
- (5.11) gives $\mathbf{X}_t = \mathbf{X}_t + \mathbf{U}_t$.
- One then proceeds to the next period.

When the number of state variables is large, as it is in our model, computation of the $\mathbf{P}_t$, & $\mathbf{Q}_t$ matrices can be very burdensome. Equation (5.11) shows that these matrices are not necessary for calculating next period's state, but equation (5.12) shows that they are needed to accurately calculate $\mathbf{U}_t$.

It is possible to approximate $\mathbf{P}_t$, & $\mathbf{Q}_t$, however, and avoid calculating them each period of the simulation. This can be done by calculating the values for some carefully chosen states only once – perhaps the steady state, or other states near which the simulation will need to be performed – and using these fixed values every period in (5.12). Since this will introduce inaccuracies, we run a simulation to gauge how important they are in our model. Figure 6 plots the time paths of the trust fund and the Social Security surplus for the baseline parameterization of our model using S=50, which corresponds to 2-year periods. The exact simulation illustrated is the one where $\mathbf{P}_t$, & $\mathbf{Q}_t$ are computed for each period in the simulation. The approximate simulation uses the steady state values of $\mathbf{P}$ & $\mathbf{Q}$, but calculates a new value of $\mathbf{U}_t$ each period. Finally, for comparison purposes we also show the results of linearizing about the steady state and using the same linearized transition function each period. The approximation method works quite well, matching the exact method very closely. It generates mean absolute deviations of .0071 and .0010 for the trust fund and surplus, respectively. The corresponding values for linearizing about the steady state are .1883 and .0141. We use this approximation when simulating for the remainder of the paper.

6. Policy Experiments

With the basic methodology in place, we are now ready to proceed with simulation of the model. We first simulate a baseline model where we calibrate the initial state of the economy to match the current situation in the US. We focus on the time-path of the Social Security trust fund, H_t. We then consider a policy change, resimulate the model, and compare the resulting time-path with the baseline.

As we have noted, however, the model is unstable and will rarely generate the values reported in Table 2 and Figure 5. In order to simulate the model we need to first choose a starting state. The state is defined by S values for each population cohort, 1 value for the aggregate productivity, S-E values for asset holdings of each cohort, R-E-1 values for the average index of monthly earnings (AIME) for each cohort of workers, S-R+1 values for the benefits paid to each cohort of retirees, and 1 value for the trust fund. The values we have chosen are shown in Tables 7 – 10.

We fit the initial distribution of the population to match that of the US population for 2010. This fitted polynomial is shown in Table 7 along with the steady state distribution of the population.

The distribution of asset holdings by age is calibrated to wealth data by age groups from 2007 reported by the US Census Bureau. Table 8 shows the data, the fitted polynomial and steady state distribution for asset holdings.

The initial distribution of Average Index of Monthly Earnings (AIME) by age comes from a random sample of 1% of social security participants available on the Social Security Administration's website. We took the history of wage earnings for individuals in the sample and applied the wage indexation adjustments published by the Social Security Administration to get an AIME value for each individual. We then sorted individuals by age and calculated the average value for each age group. This is the data which is plotted in Figure 9 along with the fitted values and steady state values for ages 16 to 67.

The initial value of benefits are from the Social Security Trust Fund annual report and use data for the average monthly benefit by age as of December 2006. This data is reported in Figure 10.

While these calibration exercises pin down the relatives sizes of the variables across cohorts, they give to guidance as to how large the values are relative to the steady state. We therefore assume that variables have the same average value across cohorts as the steady state. The one exception is the overall capital stock, which we assume is actually above its steady state value by 25%. This gives us short-run dynamics consistent with the current economic slowdown. For similar reasons we also assume the current technology level is one standard deviation below the trend value. Finally, we choose an initial trust fund balance that yields a ratio of taxes collected to that balance of 23.32%, which matches the numbers from the Social Security Trustee's report.

The baseline benefits replacement rate is assumed to be 40%, and the baseline payroll tax rate is .72 times the steady state value, which would yield a long-run trust fund balance of zero. This value is chosen to match the Trustee's report forecast that the trust fund will go to zero in 2037.

We run 1,000 Monte Carlo simulations of the economy from this starting point, which we set as the year 2011. All of our simulations use S=50, corresponding to 2-year periods. Figure 11 plots the time path of a zero shock simulation along with 90% confidence bands from the Monte Carlos. The simulations show that the trust fund rises gradually until it peaks in 2019 and then falls explosively in a negative direction. The trust fund turns negative for the first time in 2037. The social security surplus (taxes minus benefits) starts off negative, rises almost to a break-even point in 2015 and then explodes negatively. The sudden upswing at the end of the sample in figure 11 for the Social Security surplus is a result of the forced change in taxes and benefits.

We compare this baseline time path with the following broad sets of policies:

- Reductions in benefits.
- Increases in the payroll tax.
- Changes in immigration policy.
- Increases in the growth rate of technology.

One way to restore long-run balance to the system would be to lower the level of benefits to retirees. To explore this option we consider a change in the replacement rates. Holding current benefit levels constant we reduce the percent of calculated AIME paid as benefits to retirees starting in the current year. The drop in the rate is immediate and permanent, but since it does not affect the benefits of older cohorts its effects on the trust fund are gradual and build up only slowly over time. Figure 12 plot the average and 90% confidence bands over 1000 simulations for the baseline and three different reductions in the replacement rate. Reducing the replacement rate by 25% is sufficient to restore fiscal balance in the long-run, but a reduction of 10% is not.

Figure 13 shows the effects of increasing the payroll tax. As with benefits reductions, small changes are insufficient. A tax increase of 25%, reverses the tendency for the trust fund to fall in the long-run, but a 10% increase is not large enough.

Figure 14 shows the effects of simultaneously raising taxes and lowering benefits by the same amount. Here a change of 10% is almost sufficient to restore fiscal balance, and a change of 25% is more than adequate.

For the case of increased immigration we double the halve immigration rates for all cohorts we exacerbate the insolvency problem; the trust fund turns negative about 4 years earlier as do changes in fiscal policy. By comparison, doubling immigration rates postpones these problems, but does not eliminate them. The trust fund turns negative in 2075, rather than 2037.

Increased growth is argued to be one way to resolve Social Security insolvency. To test this we simulate our model with growth rates of technology of 3% per year and 5% per year. The time paths are plotted in figure 16. Technical progress clearly improves the fiscal situation. With 2% growth the trust fund still turns negative, albeit 15 years later than the baseline. 3% growth is sufficient to eliminate Social Security insolvency, even in the long run.

The conditions necessary to ensure the trust fund does not go negative for the above simulations seem unlikely. Benefits must fall by between 10% and 25% or taxes must rise roughly the same amount. Growth must be three times higher than is currently the case to eliminate the eventual insolvency of the system. We consider finally a more balanced policy approach. We increase taxes by 10%, lower the replacement rate by 10%, increase immigration by 50%, and assume that technology growth rises from 1% to 1.5%. The time path for this scenario is plotted in figure 17. The figure shows that the trust never falls into deficit.

7. Conclusions

This paper has presented an OLG model with relatively short periods. Rather than solve the model exactly, we have linearized it and have done so about the current state of the economy, rather than about the steady state. This allows us to solve and simulate models with much greater dimensionality that we could by solving exactly using either analytical or numerical methods. In addition, by linearizing about the current state rather than about the steady state, we are able to more accurately simulate our model which is locally unstable near the steady state. This accuracy comes at a cost of greater computing time, but for relatively simple models like ours, this cost is not overwhelming.

Our model still suffers from the curse of dimensionality, however. For example as the size of the periods in the model get smaller, the number of cohorts rises. The number of state variables in the model is $3S - 2E + 1$. So as the number of cohorts rises, so does the state space. With large enough state spaces the computation of the linear coefficients **P** & **Q** in the transition function becomes computationally burdensome.

A model with idiosyncratic shocks to members of cohorts would be intractable with our solution method. For example, a ten-period-lived-agent model with only 2 values for an idiosyncratic shock each period would give $2^{10} - 1 = 1023$ different agents of various ages, whereas our current model with hundred-period-lived agents has 100 different agents.

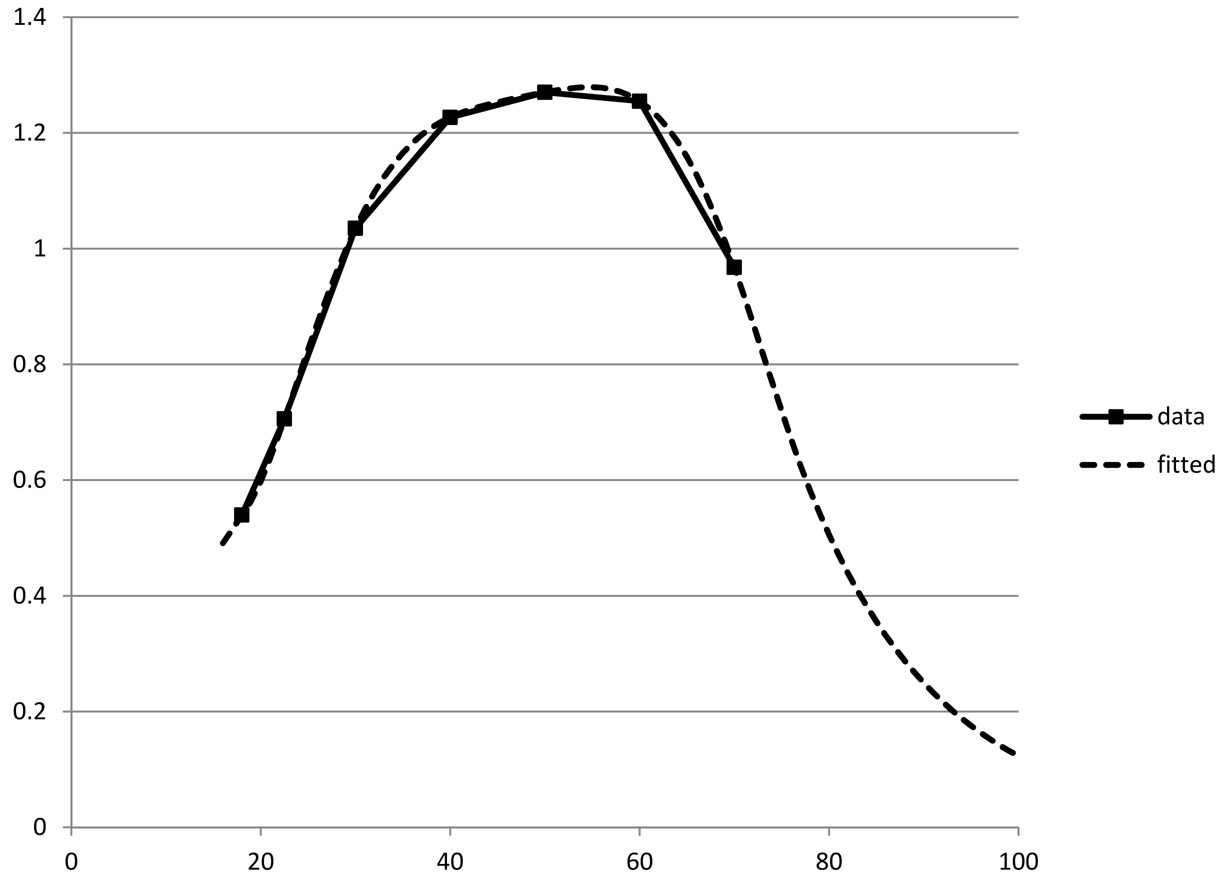
Despite its reliance on a representative agent for each cohort our model does yield some useful results. First, no single policy on its own is sufficient to resolve the looming insolvency of Social Security. Taxes must be raised by roughly 25% in order to do so. Or benefits could be cut by roughly the same amount. Neither of these fixes seems feasible in the current political climate. Easing immigration restrictions moves things in the right direction, but has insufficiently large effects. Finally, steady state growth needs to be implausibly high to solve the fiscal crisis on its own.

While none of these policies is sufficient on its own to solve the problem, a mix of policies looks promising. Increasing taxes and lowering benefits for new retirees by 10%, while eliminating wage indexing and allowing a 50% increase in immigration is sufficient to solve the problem if long run growth can also be raised by 50%, as shown in our final simulation. Similar mixes of policy would also be sufficient. However any solution requires the implementation of a number of policy changes.

The fundamental instability of the Social Security system makes long-run predictions very imprecise. Any Social Security system that defines fixed benefits while relying on stochastic tax revenues will be subject to this instability. A more appropriate arrangement would be for benefits to be somehow dependent on the state of the economy. This seems like a fruitful area for future research.

Figure 1

Data and Fitted Curve for Effective Labor by Age[4]

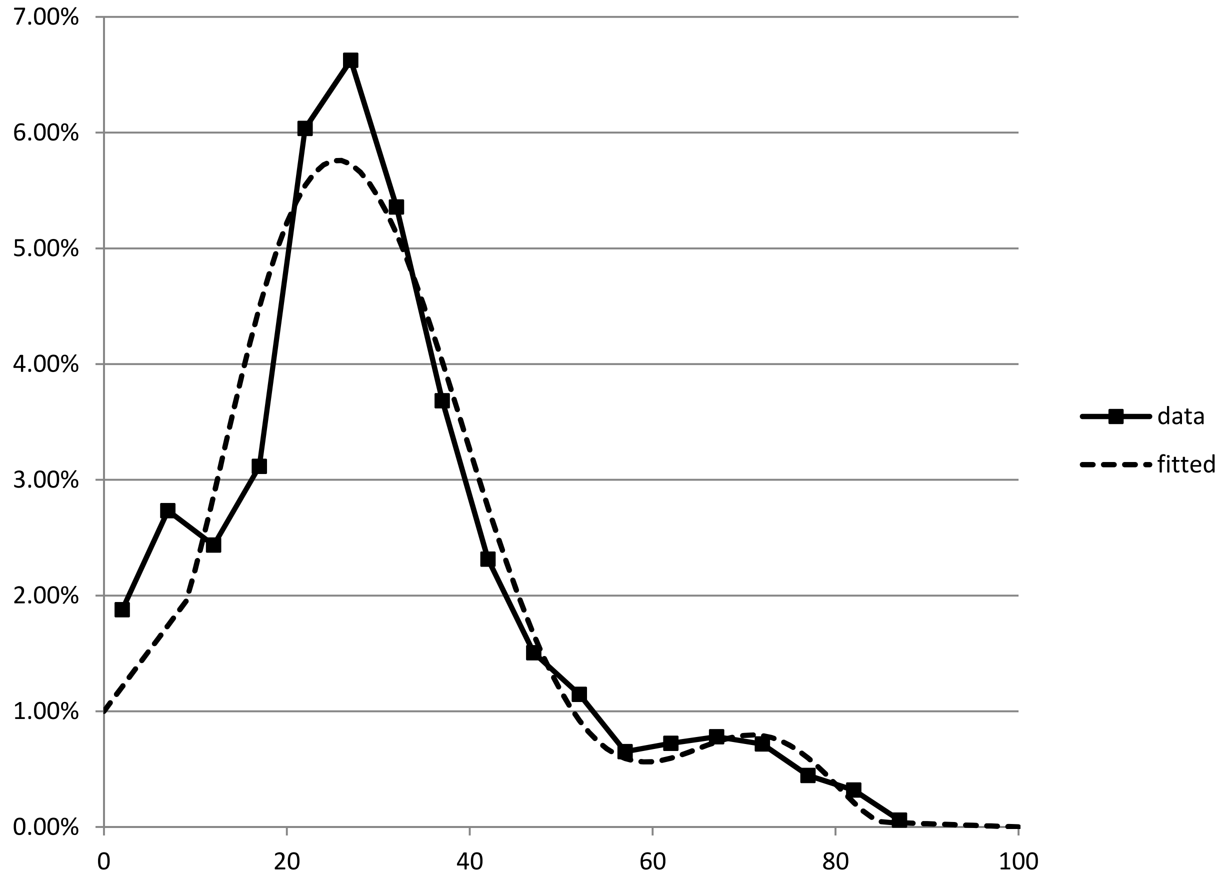


[4] Data are from the US Bureau of Labor Statistics' Current Population Survey.

Figure 2

Data and Fitted Curve for Immigration Rates by Age[5]

(immigration rates are over a 5-year period)

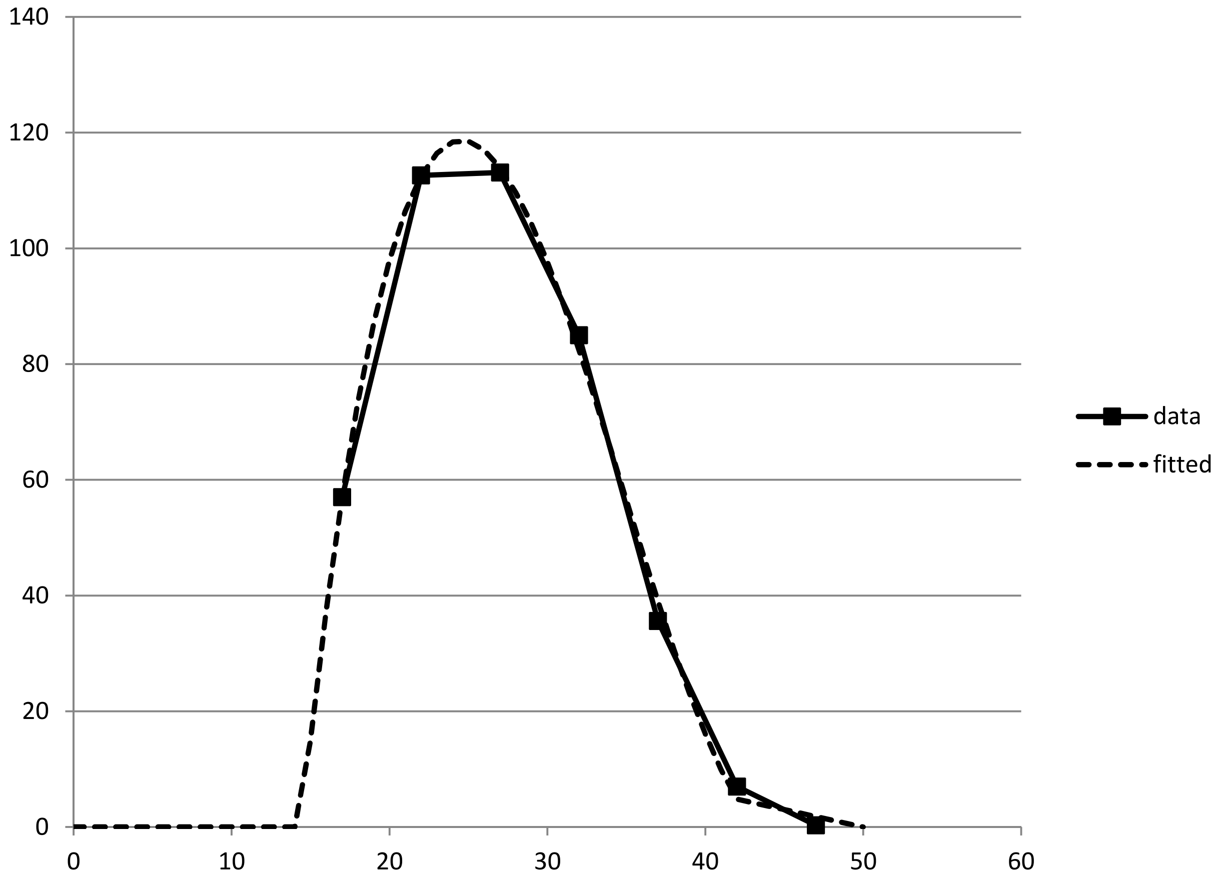


[5] Data are from the US Census Bureau.

Figure 3

Data and Fitted Curve for Fertility Rates by Age[6]

(births per 1000 for females of indicated age per year)

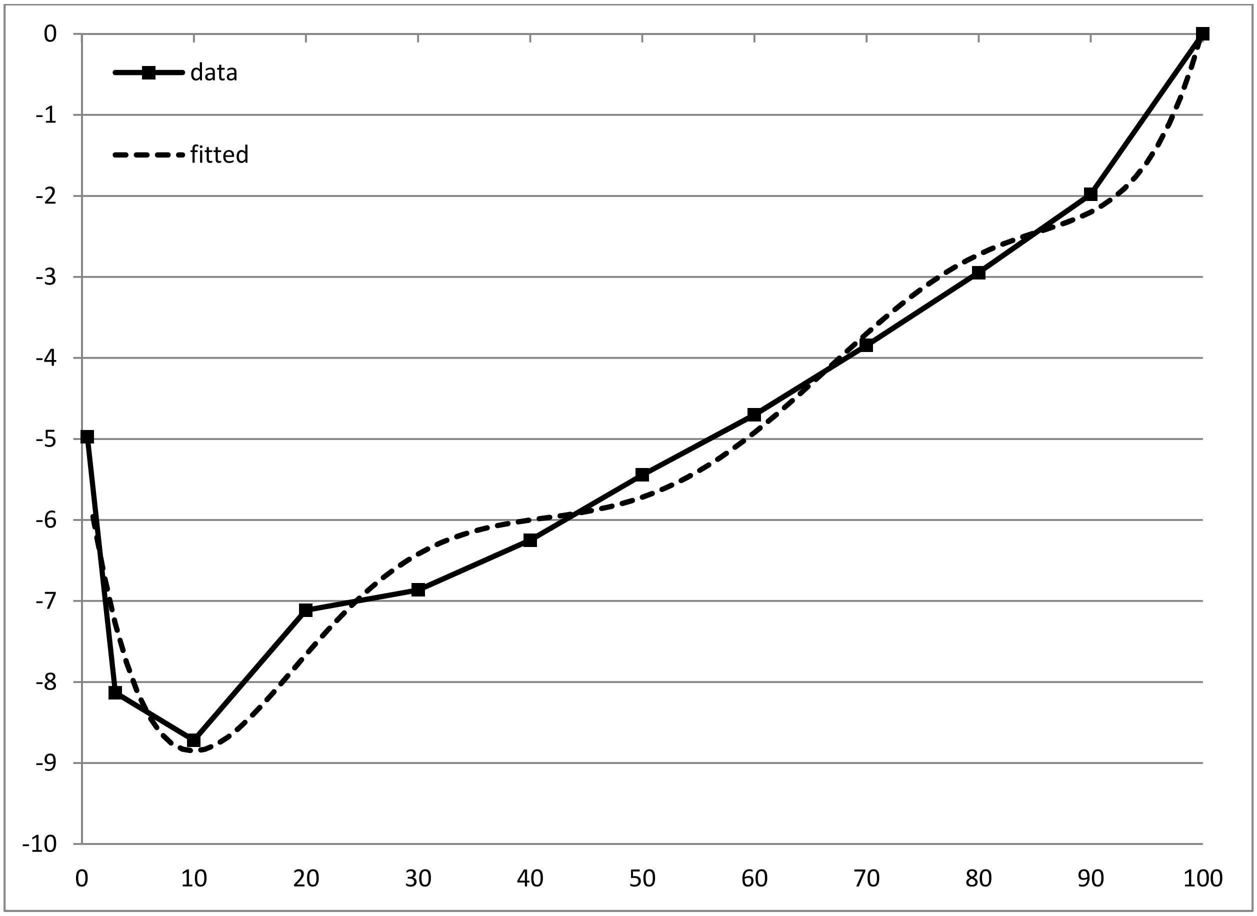


[6] Data are from Nishiyama (2004).

Figure 4

Data and Fitted Curve for Mortality Hazard Rates by Age[7]

(natural logarithms of probabilities)



[7] Data are from the US Center for Disease Control's mortality tables.

Figure 5

Steady State Values of Selected Variables by Age

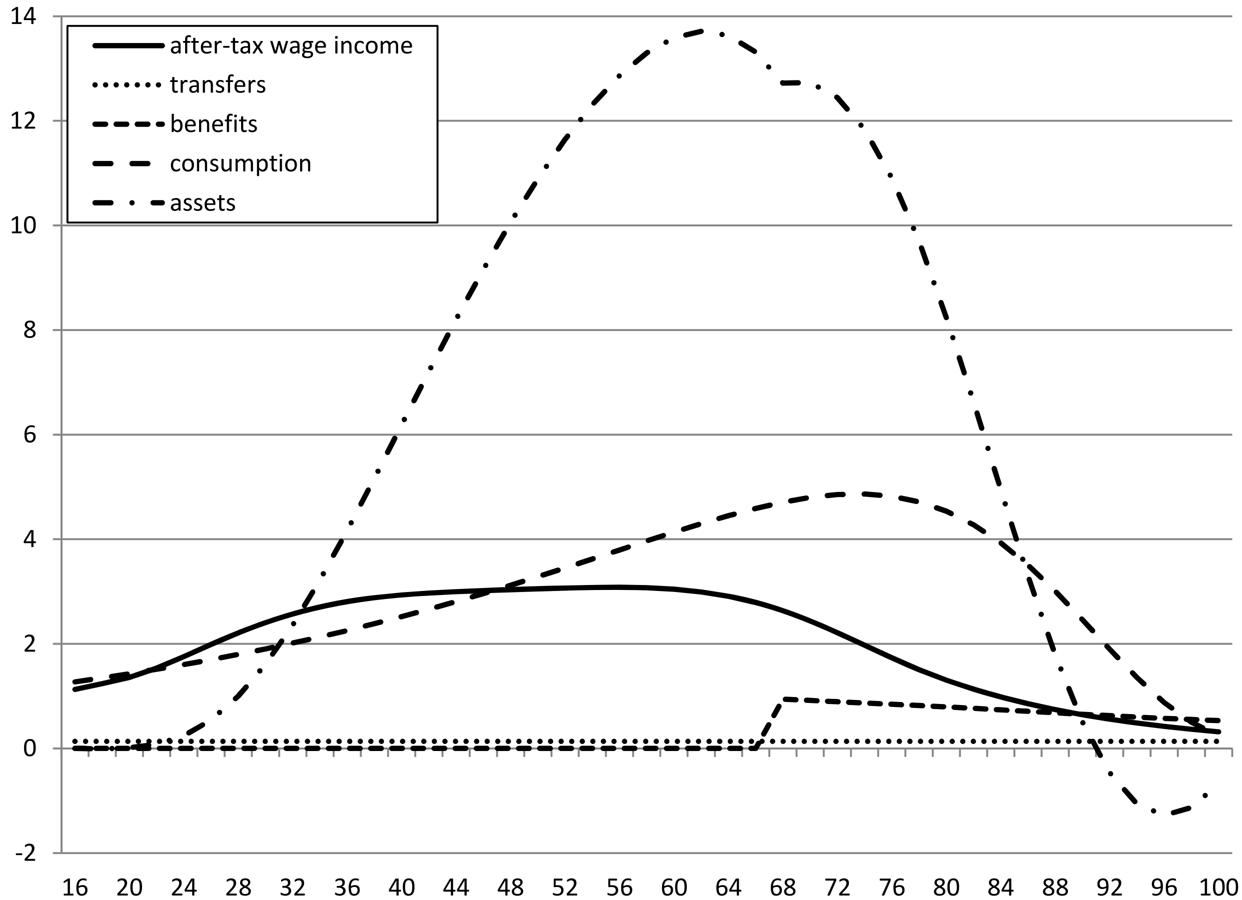
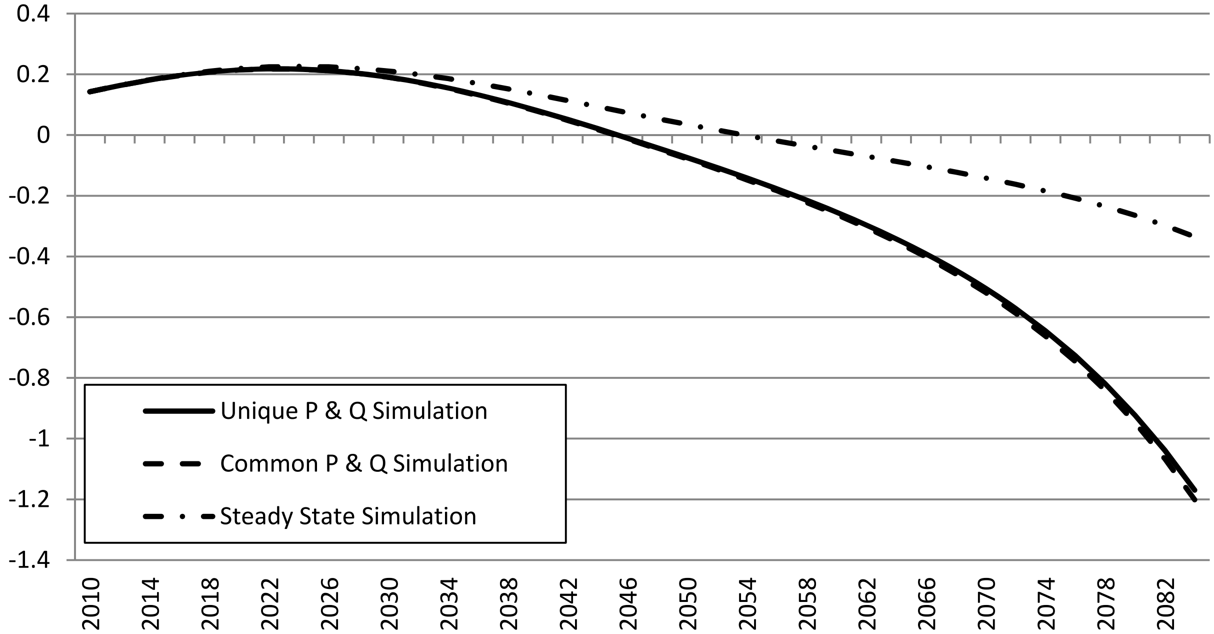


Figure 6

Comparison of Simulation Methods for the Trust Fund and Social Security Surplus



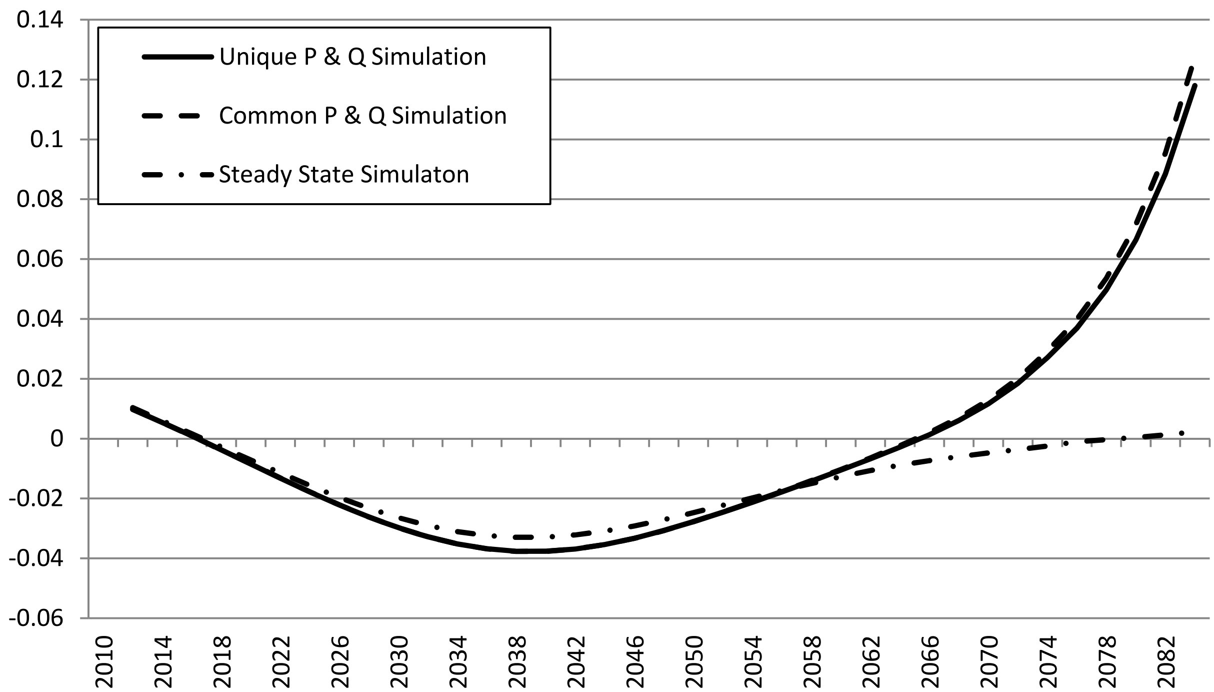


Figure 7

Starting and Steady State Distributions of the Population by Age

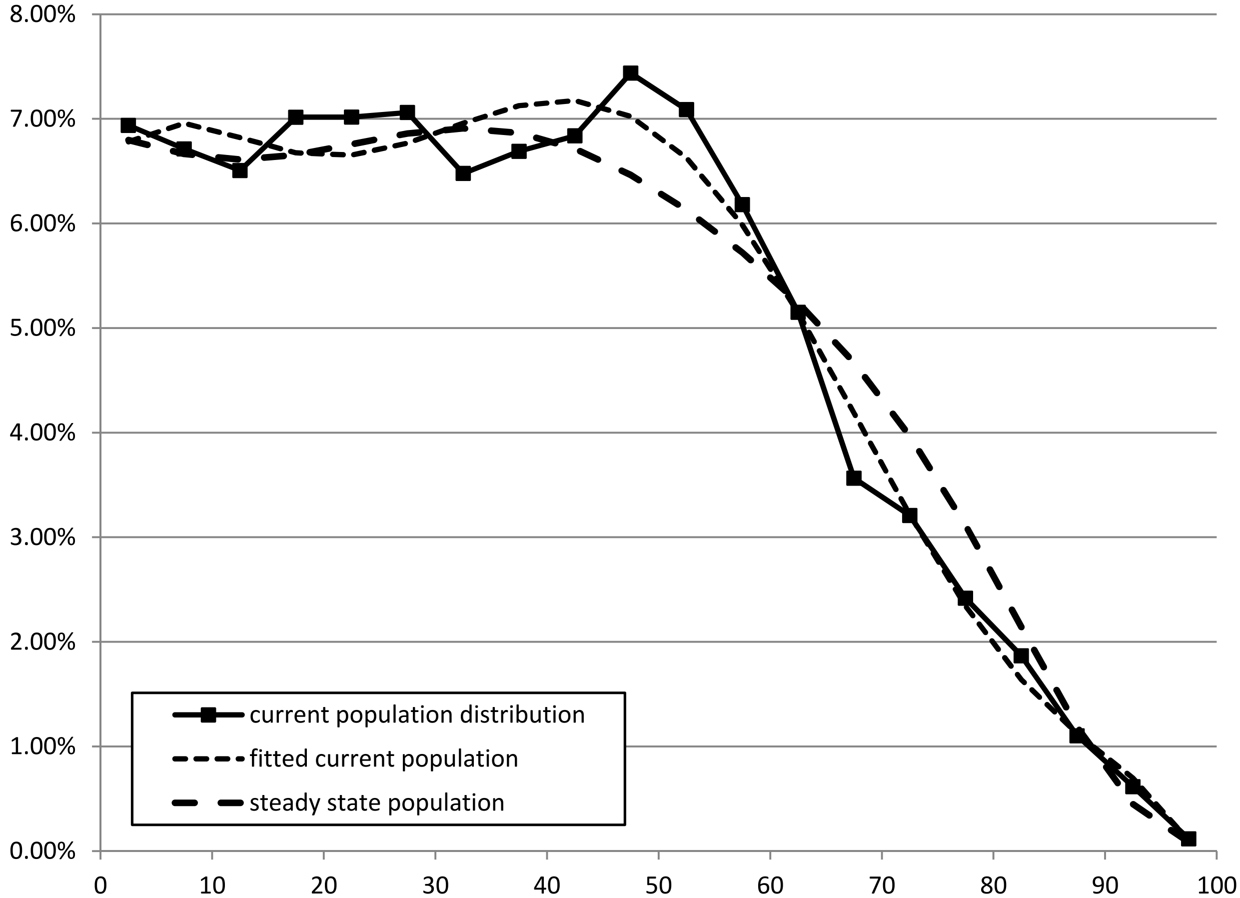


Figure 8

Starting and Steady State Distributions of Capital by Age

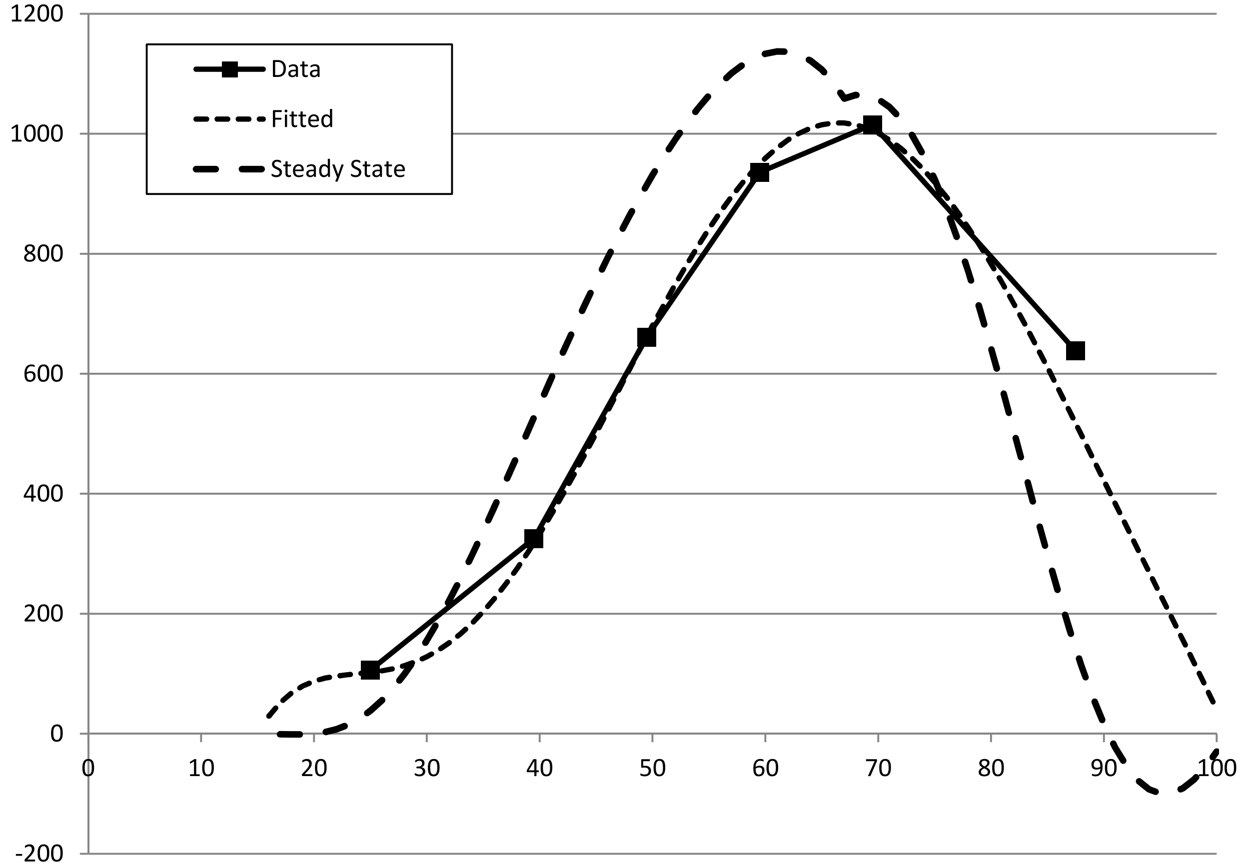


Figure 9

Starting and Steady State Distributions of AIME by Age

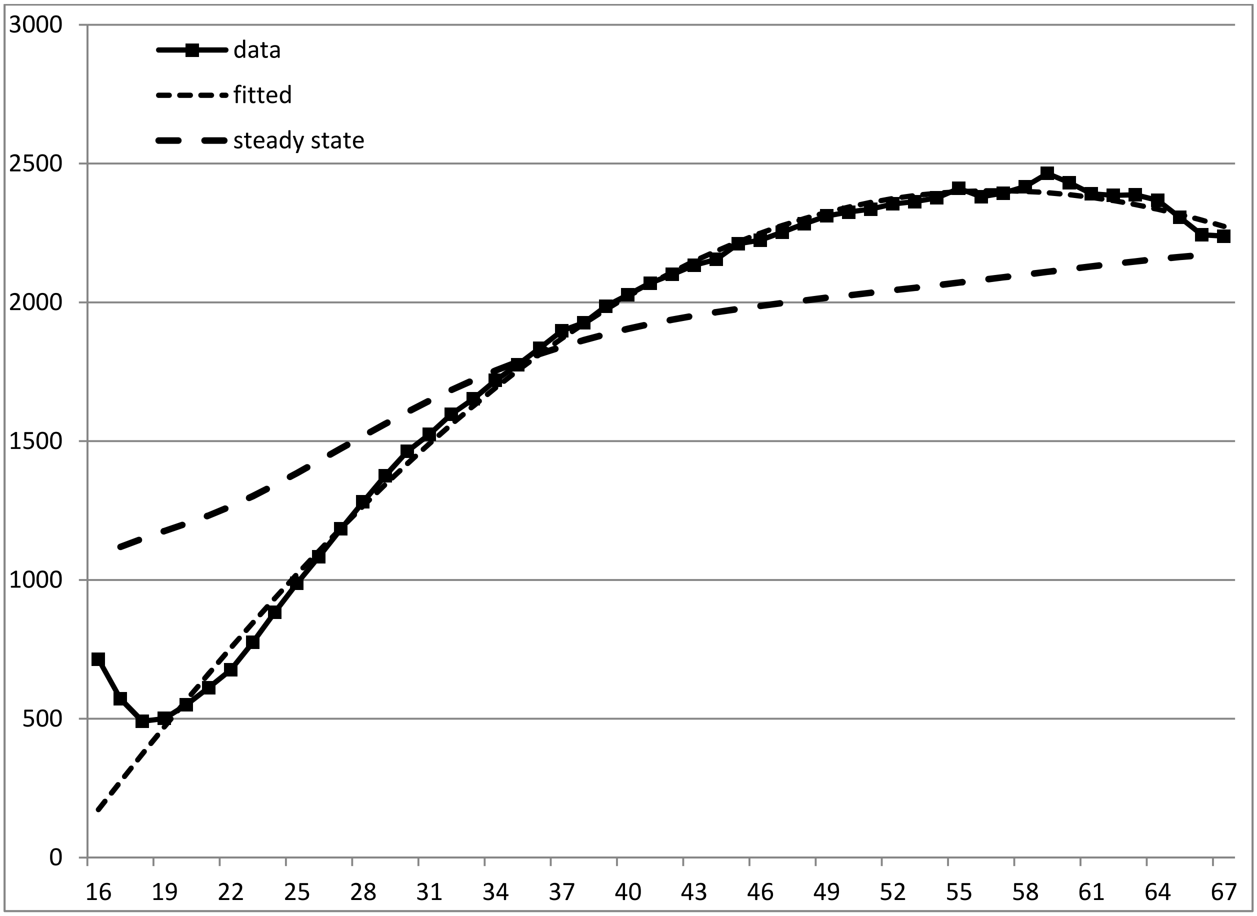


Figure 10

Starting and Steady State Distributions of Benefits by Age

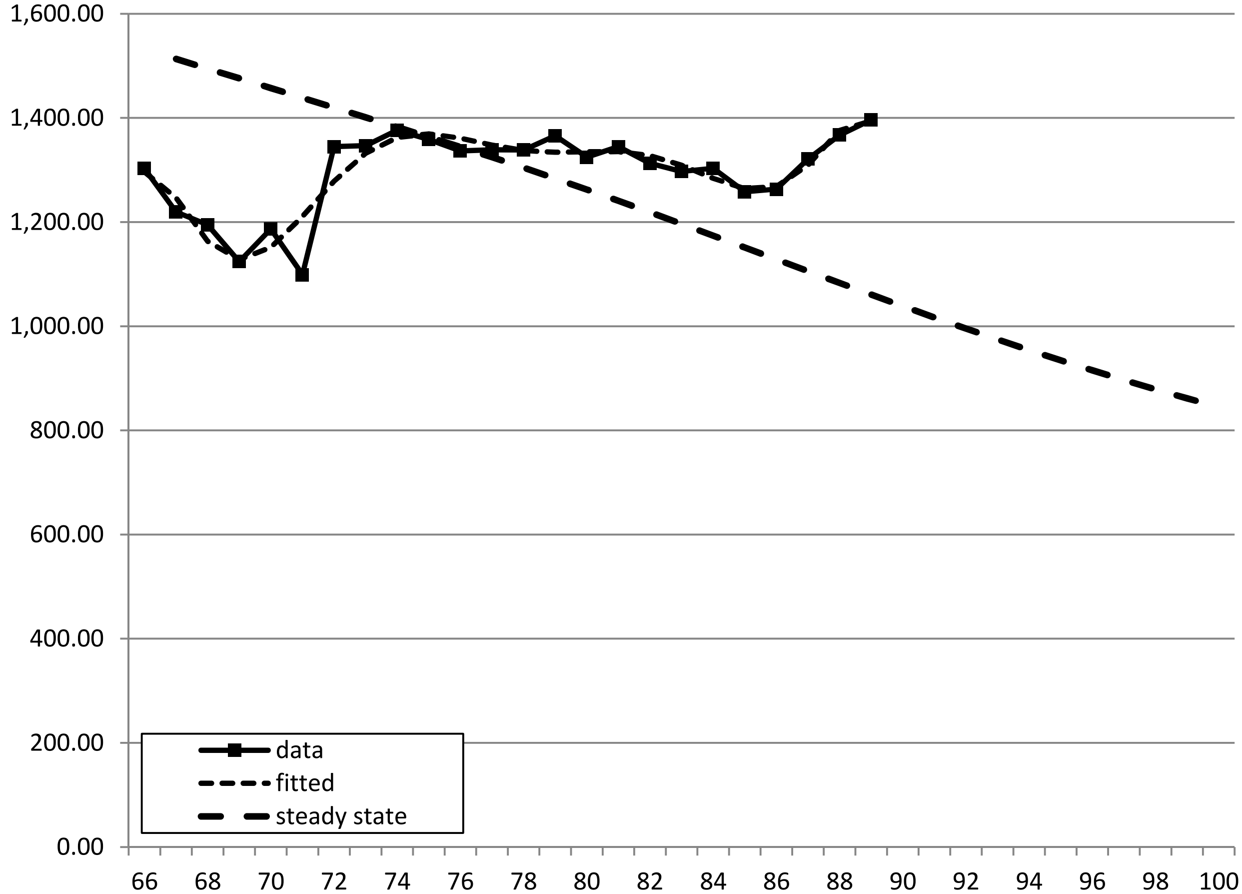
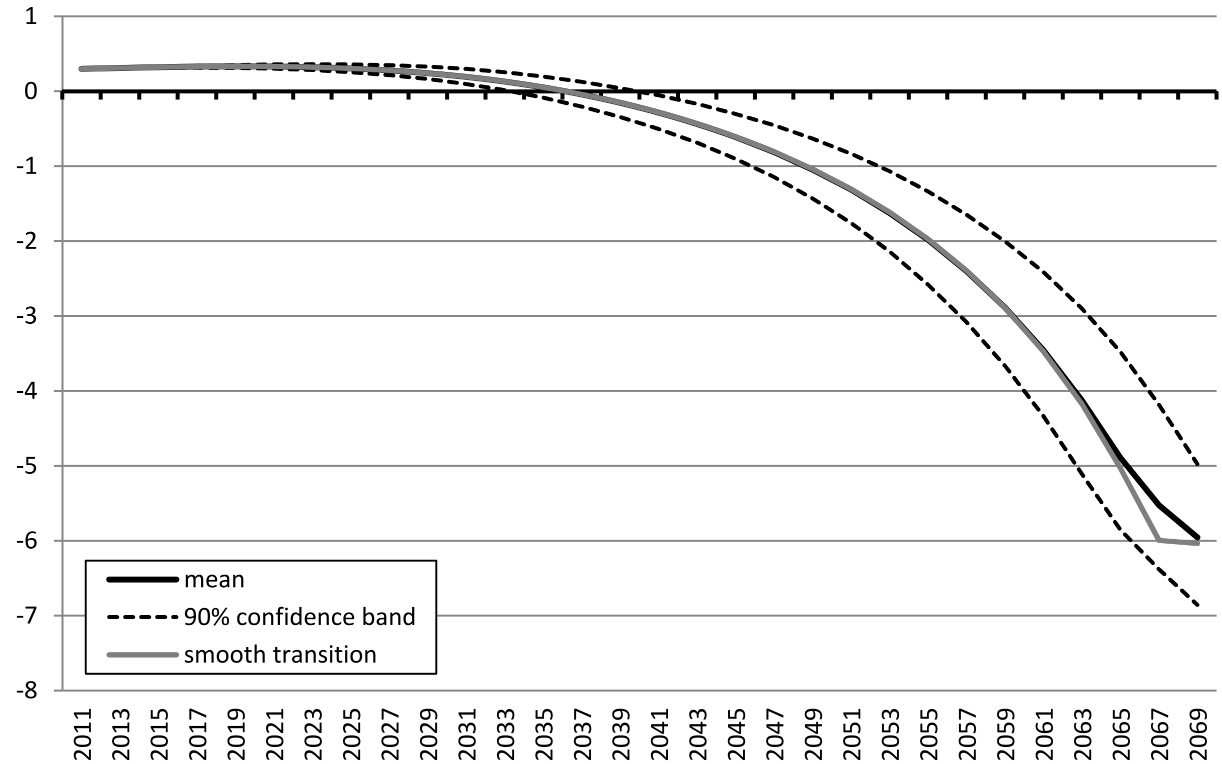


Figure 11

Time Paths for the Trust Fund & Social Security Surplus in the Baseline Case



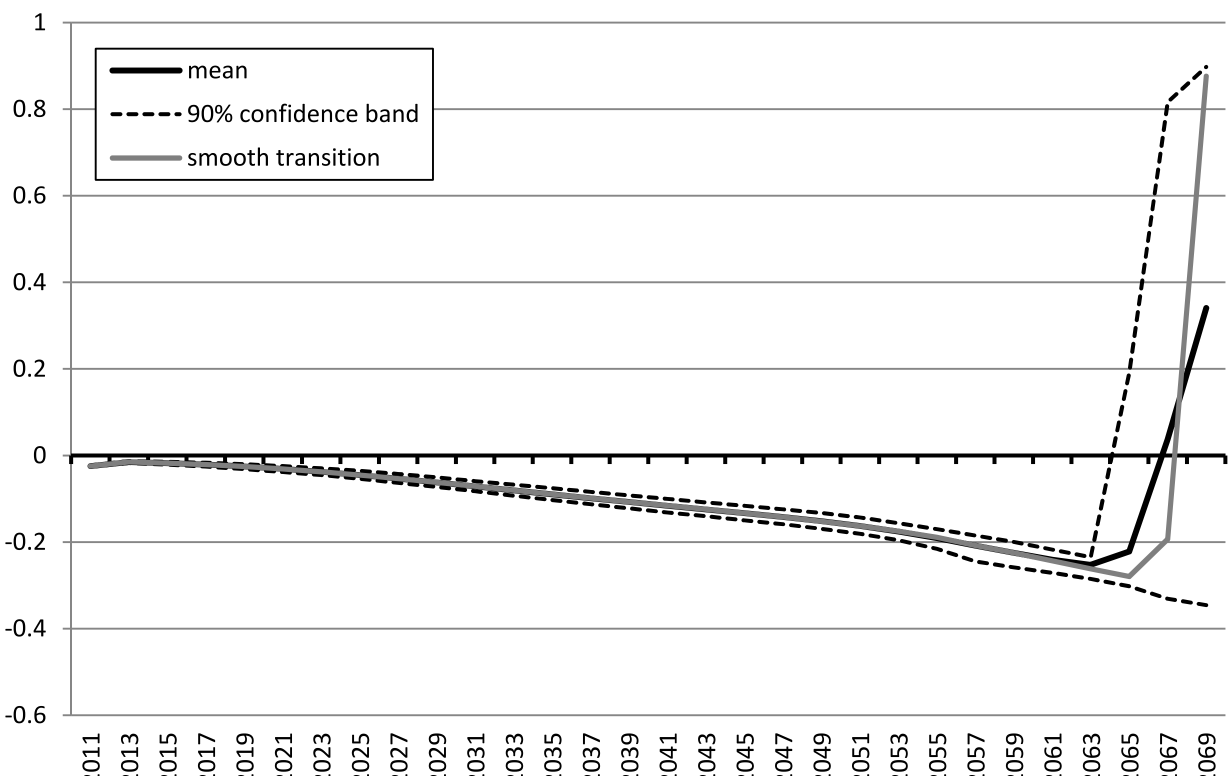
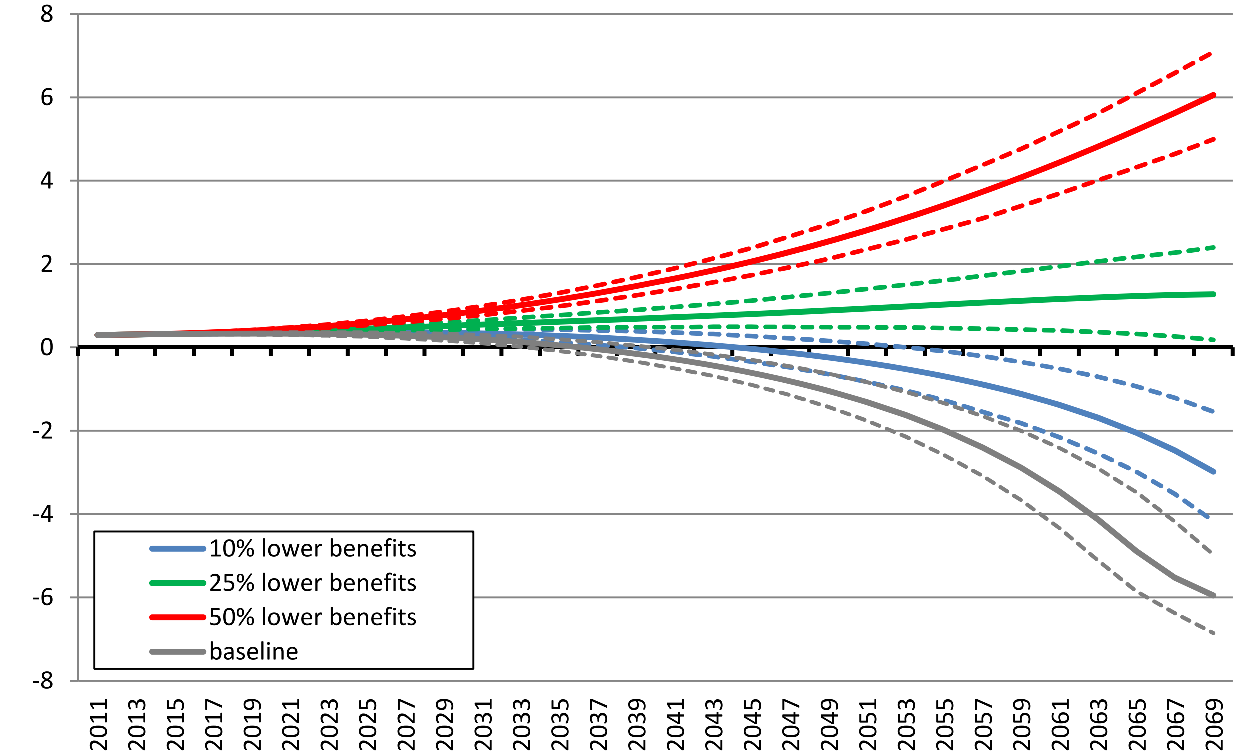


Figure 12
Time Paths for the Trust Fund & Social Security Surplus with Reduced Benefits



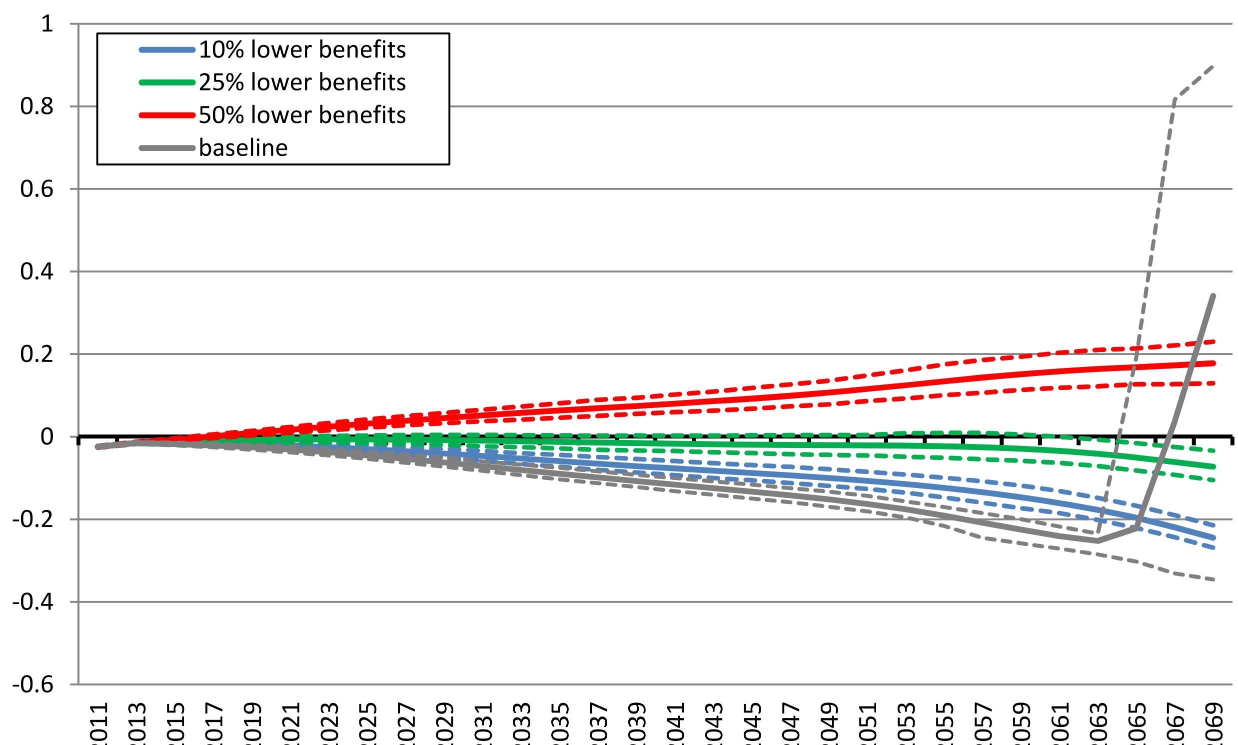
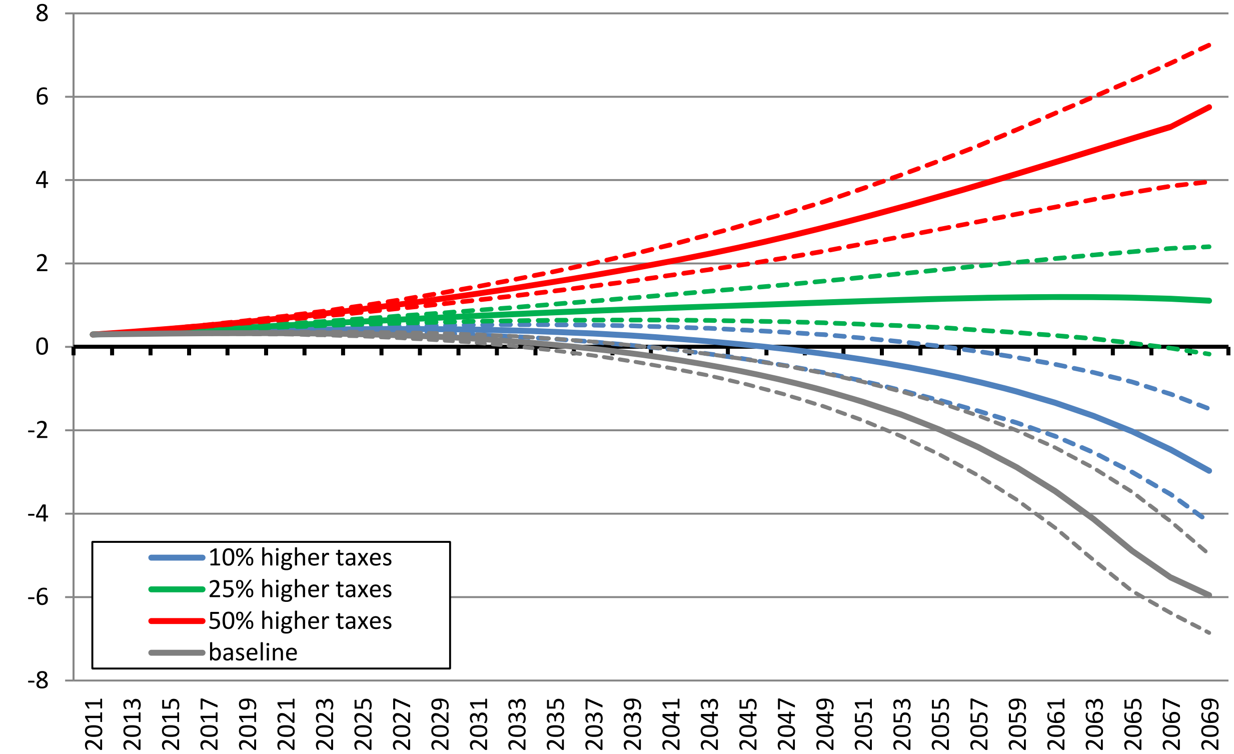


Figure 13
Time Paths for the Trust Fund & Social Security Surplus with Higher Taxes



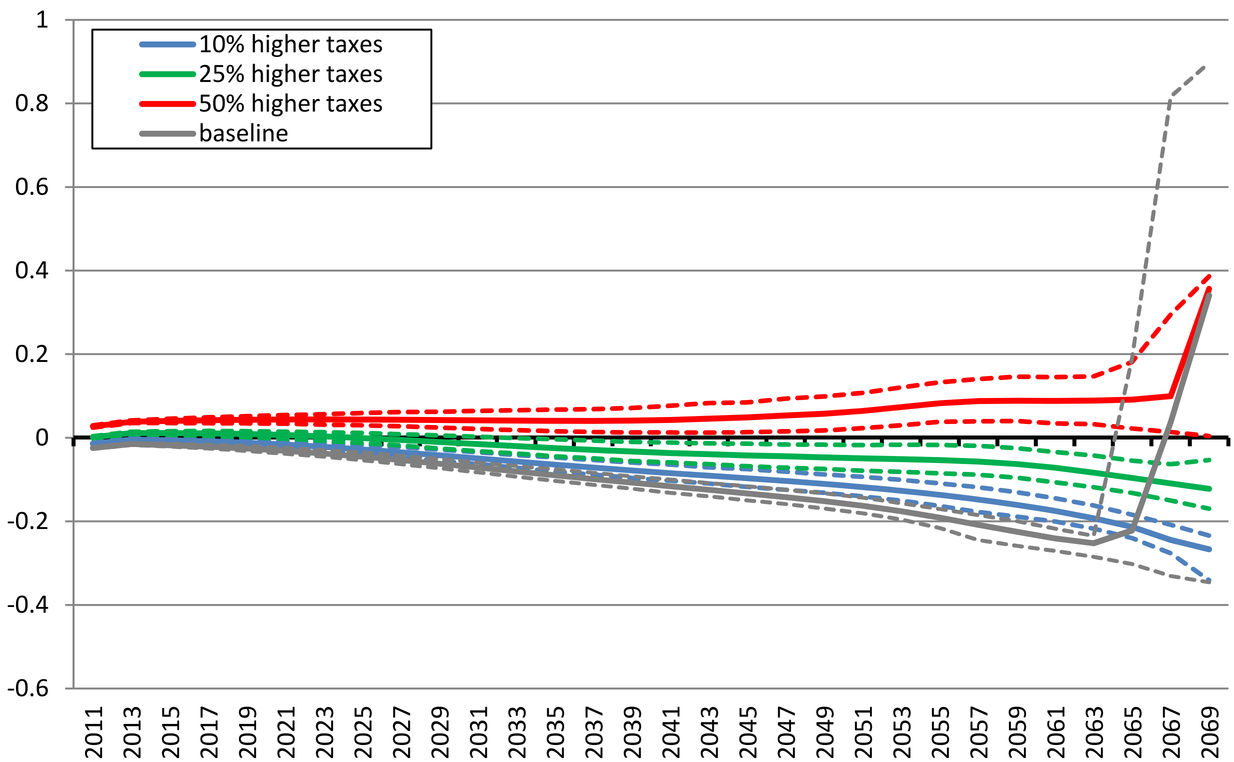
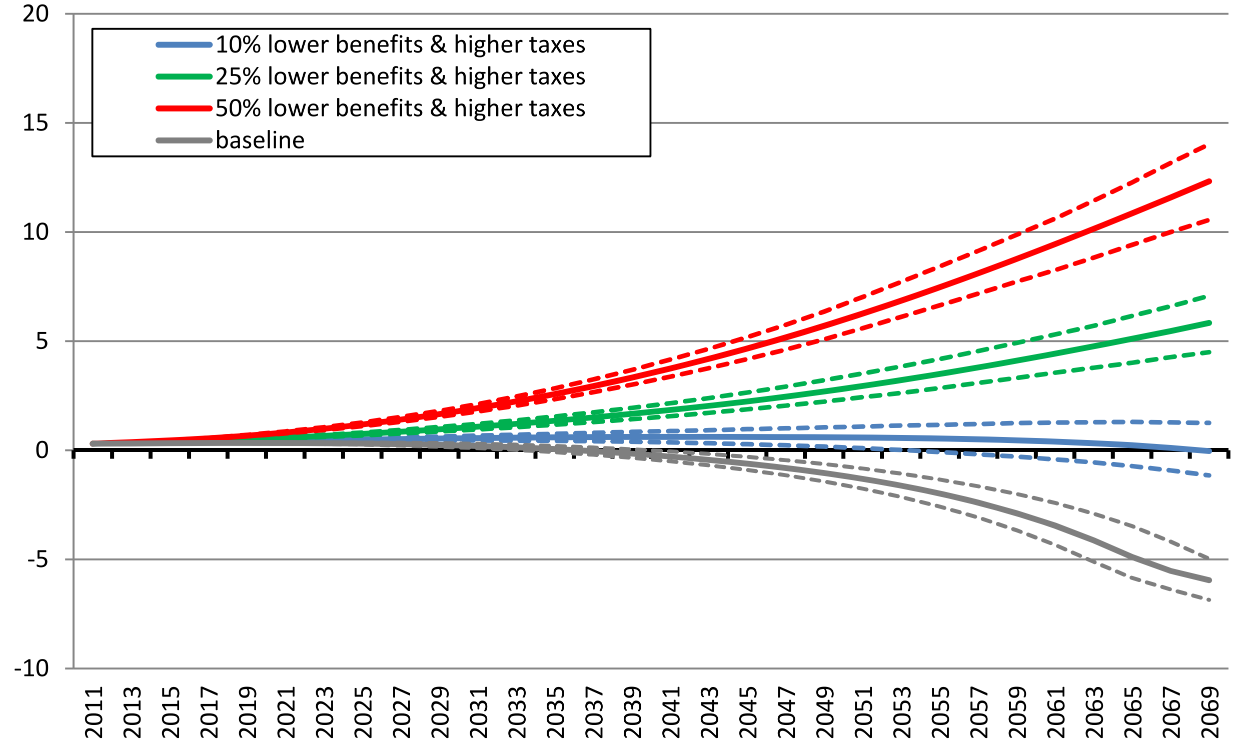


Figure 14
Time Paths for the Trust Fund & Social Security Surplus with Lower Benefits and Higher Taxes



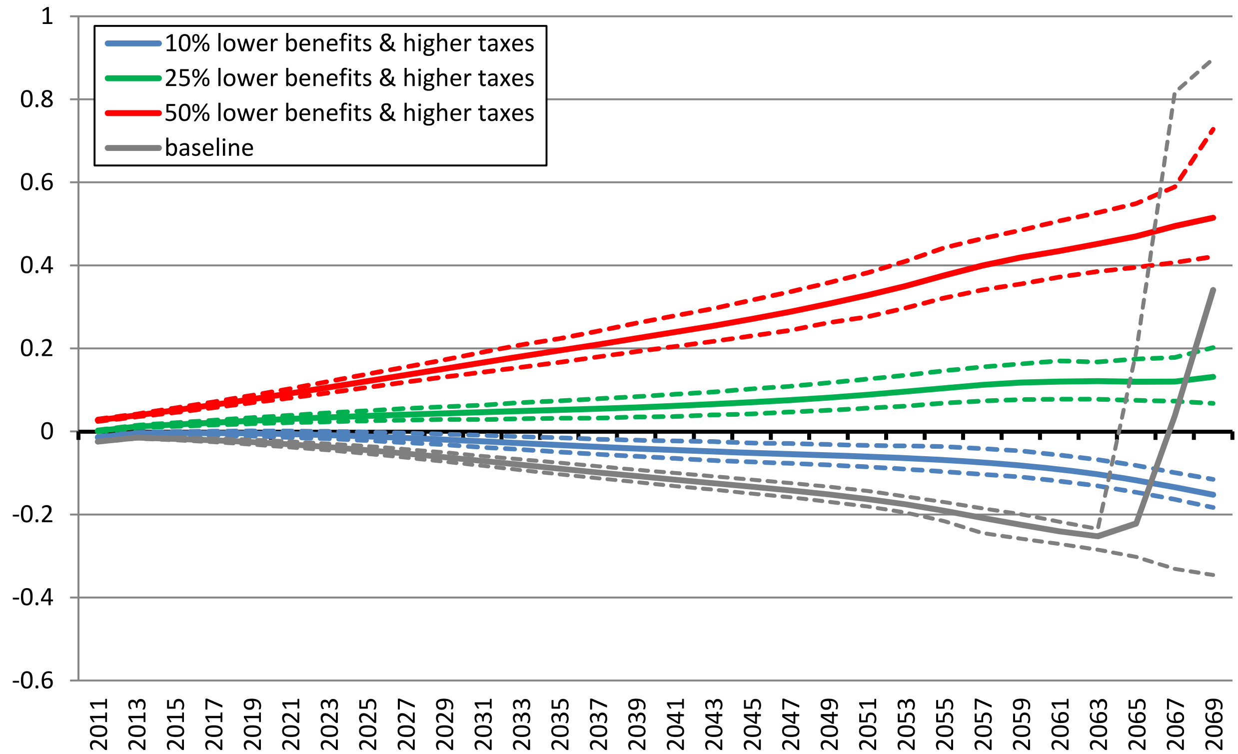
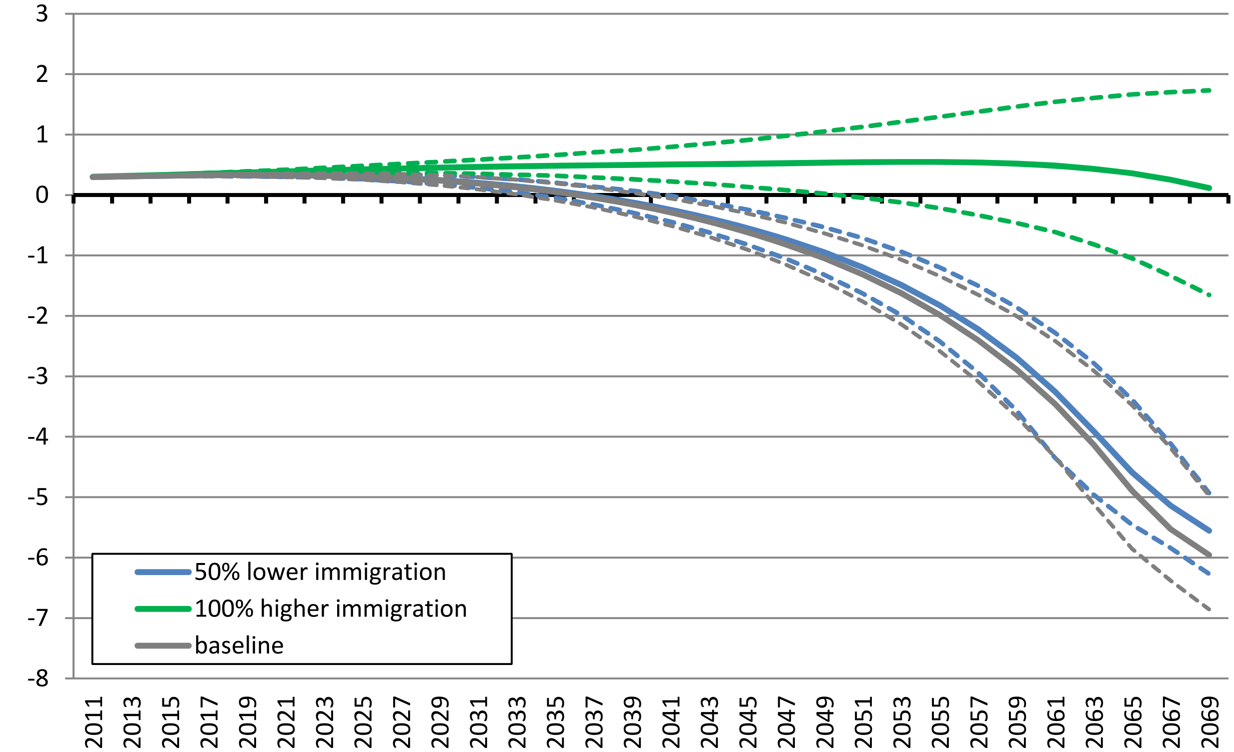


Figure 15
Time Paths for the Trust Fund & Social Security Surplus with Different Immigration Schemes



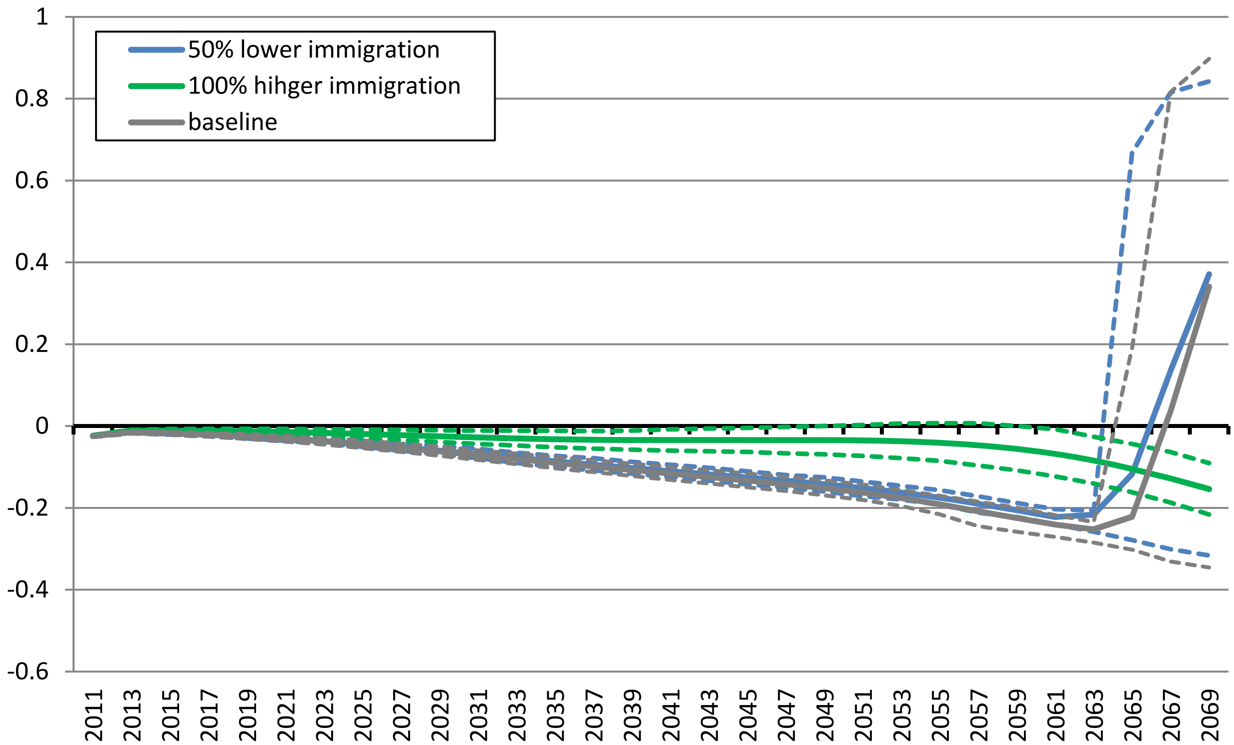
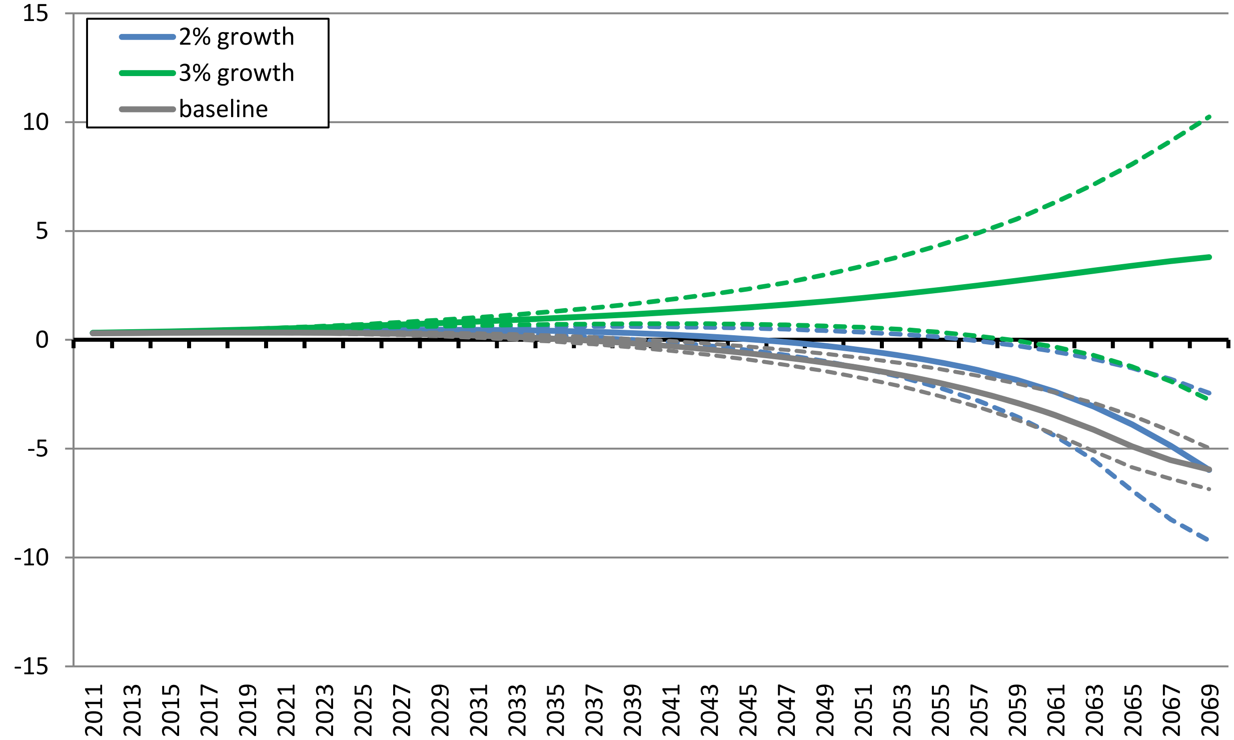


Figure 16
Time Paths for the Trust Fund & Social Security Surplus with Higher Growth Rates for Technology



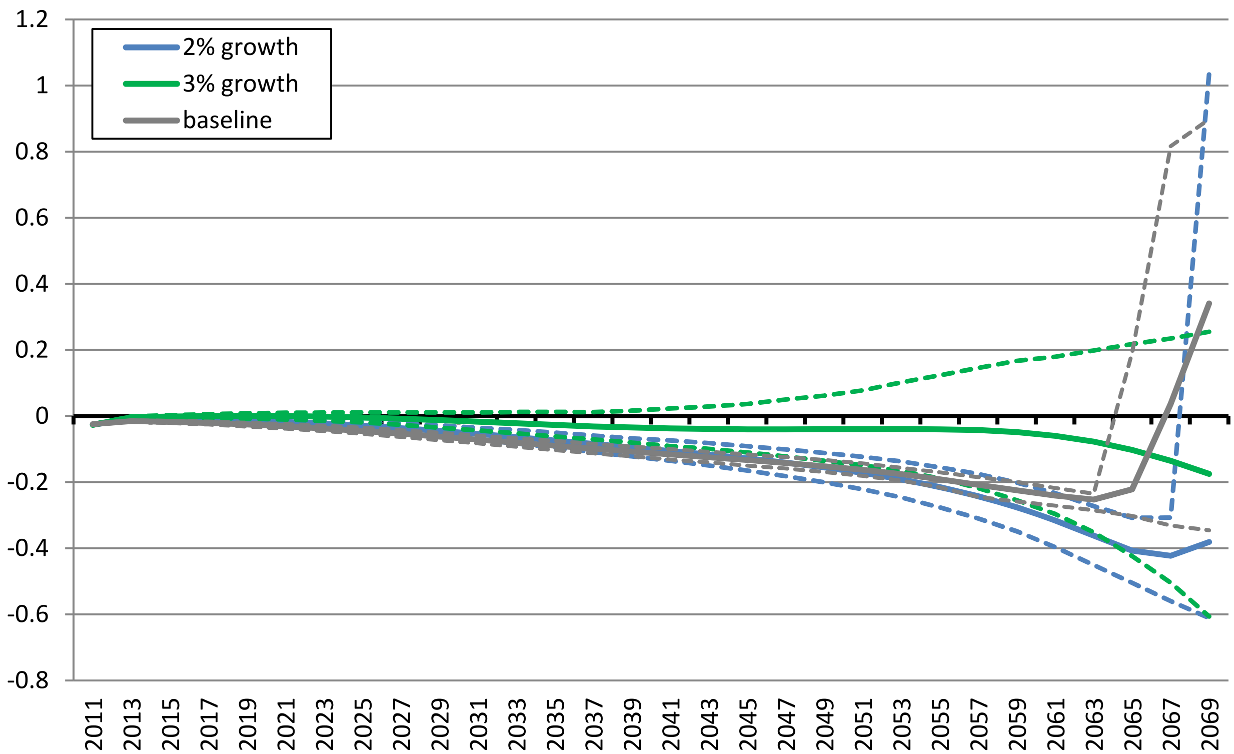


Figure 17

Time Paths for the Trust Fund & Social Security Surplus under a Balanced Reform Scenario (preliminary based on S=25, 100 Monte Carlos)

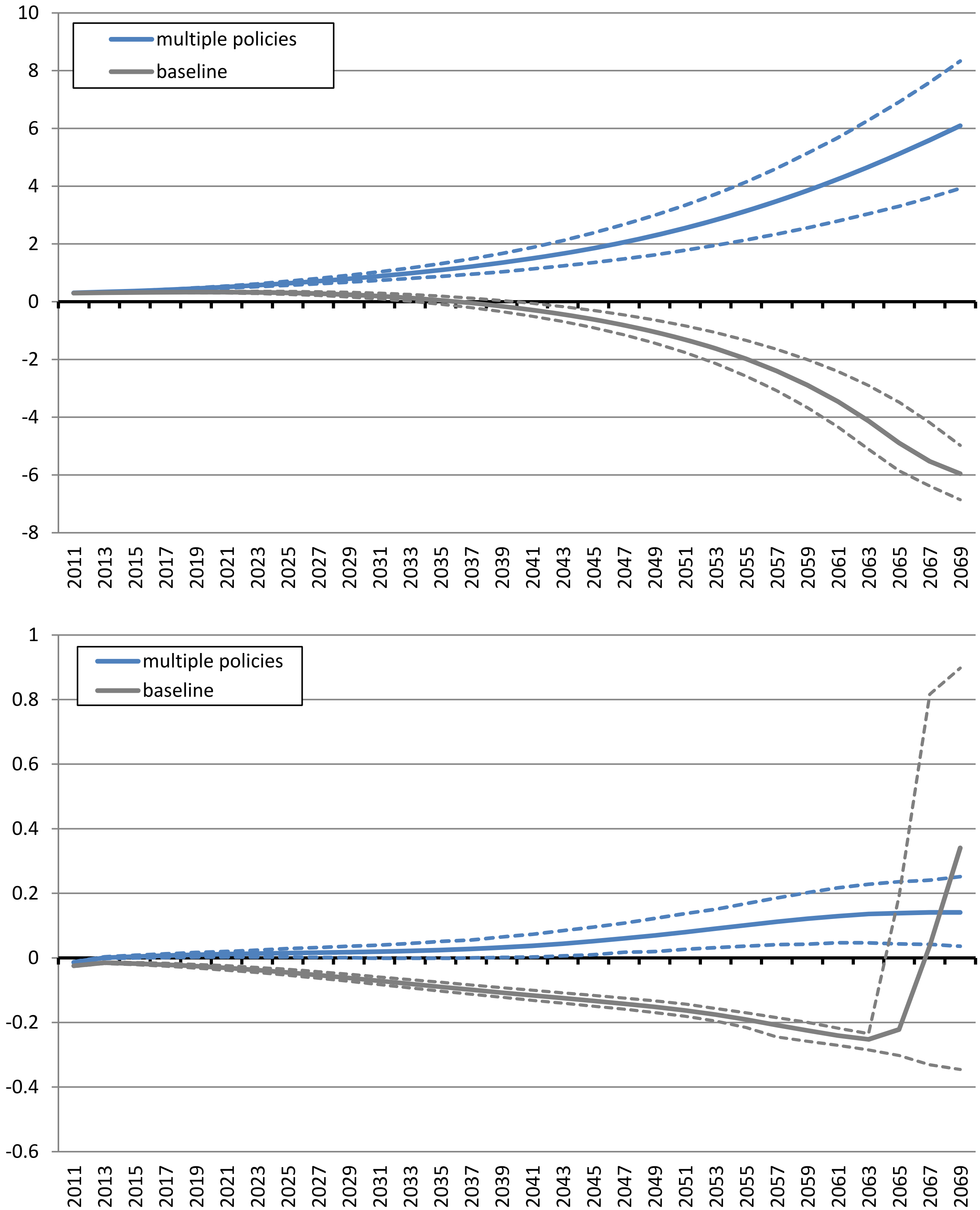


Table 1
List of Model Parameters

S	maximum age in periods
E	period workers enter the labor force
R	period workers retire
$\{\bar{f}_s\}_{s=1}^{S}$	average fertility rates by age
$\{\bar{\imath}_s\}_{s=2}^{S}$	average immigration rates by age
$\{\bar{\rho}_s\}_{s=2}^{S}$	average survival rates by age
$\{\bar{\ell}_s\}_{s=1}^{S}$	effective labor endowment by age
τ	payroll tax rate
δ	capital depreciation rate
β	subjective discount factor
g	growth rate of technology
γ	coefficient of relative risk aversion
α	capital share in GDP
θ	pension benefits as percent of AIME

In addition we have parameters governing the stochastic processes.

ψ_z	autocorrelation
σ_z^2	variance

Table 2
Baseline Calibration & Selected Steady State Values

α	0.35	$\bar{K}$	5.6705
γ	1	$\bar{H}$	0.0000
$\bar{g}^*$	0.01	$\bar{Y}$	3.1975
δ^*	0.05	$\bar{C}$	2.4695
β^*	0.992	$\bar{I}$	0.8241
θ	0.4	$\bar{L}$	0.8086
S	50	$\bar{r}^*$	0.0943
		$\bar{w}$	2.5705
		$\bar{T}$	0.1348
		$\bar{B}$	0.1316
		$\bar{n}^*$	0.0068
		τ	0.0456

* values are quoted in per annum terms

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