Intergenerational Redistribution in the Great Recession*

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Abstract

In this paper we construct a stochastic overlapping-generations general equilibrium model in which households are subject to aggregate shocks that affect both wages and asset prices. We use a calibrated version of the model to quantify how the welfare costs of severe recessions are distributed across different household age groups. The model predicts that younger cohorts fare better than older cohorts when the equilibrium decline in asset prices is large relative to the decline in wages, as observed in the data. Asset price declines hurt the old, who rely on asset sales to finance consumption, but benefit the young, who purchase assets at depressed prices. In our preferred calibration, asset prices decline close to three times as much as wages, consistent with the experience of the US economy in the Great Recession. A model recession is almost welfare-neutral for households in the 20-29 age group, but translates into a large welfare loss of around 10% of lifetime consumption for households aged 70 and over.

Keywords: Recessions, Overlapping Generations, Asset Prices, Aggregate Risk

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1 Introduction

The current economic downturn is the most severe since the Great Depression. Household incomes have fallen significantly below trend, and the prices of real estate and stocks have plummeted. The goal of this paper is to explore the welfare consequences of a severe and long-lasting recession that results in large declines in wages and a collapse in asset prices. Our main objective is to study how these welfare costs vary across different age cohorts.

There are good reasons to believe that the welfare effects of large aggregate shocks are unevenly distributed across different generations. Young households have little financial wealth, relative to their labor income, while older households are asset-rich but have little human wealth, measured as the present discounted value of future labor income. In addition, young households who buy assets at depressed prices gain from future asset price appreciation, while older households close to the end of their life cycle cannot wait for prices to recover. Thus it is likely that a steep decline in asset prices has more serious welfare implications for older households.

In the next section, we use data from the Survey of Consumer Finances to document how the ratio of labor income to net worth varies over the life cycle. We then estimate the magnitude of the declines in net worth associated with the current recession, focussing on how these losses vary with the age of a household. To do so we decompose net worth into different types of assets and debts, and estimate losses by applying asset-class specific price deflators to age-group-specific portfolios. We find that the average household experienced a decline in net worth of $177,000 between the middle of 2007 and the trough of the asset price decline in the first quarter of 2009. These losses were heavily concentrated among older age groups: households aged 60-69 lost $312,000 on average. Since 2009, asset prices and net worth have recovered somewhat, but remain well below their 2007 values.

These empirical facts suggest that the welfare losses from large economic downturns are unevenly distributed across different age cohorts in the population. However, a more complete welfare analysis requires forecasts for the future evolution of labor income and asset prices, and an understanding of how agents will optimally adjust savings and portfolio choice behavior in response to expected future wage and price changes. In the remainder of the paper we therefore construct a stochastic general equilibrium model with overlapping generations in which households of different ages are subject to large aggregate shocks that affect both wages and asset prices. We use a realistically calibrated version of the model to assess the distributional consequences of severe recessions. One question of particular interest that we can ask within the context of this model is whether young people might actually be better off if they become economically active in the midst of a large and persistent economic downturn.

The answer to this question crucially depends on the size of the decline in equilibrium asset prices, relative to the decline in income, in response to a negative aggregate shock. If middle-aged households have a strong incentive to sell their assets in the downturn (e.g., because they strongly value smooth consumption profiles) then asset prices decline more strongly than income in equilibrium. This in turn benefits younger generations who buy these assets at low prices, compensating for the fall in earnings they experience. At the same time, we will present evidence that younger households experience disproportionately large earnings losses in recessions. Thus the overall allocation of welfare losses from a recession will depend on the quantitative importance of age differences in exposure to asset price risk relative to age differences in the direct effect of recessions on labor incomes.

To quantify the relative importance of these offsetting effects, we first compute welfare effects under the assumption that recessions are age-neutral in the sense that earnings decline proportionately for all age groups. We then extend the model to allow for the age profile of earnings to tilt during a recession, in disfavor of younger age groups. In our preferred calibration, we find that the asset price effect dominates, such that deep recessions are associated with massive welfare losses for older households and virtually no losses for the young. In an alternative version of the model that endogenizes household portfolio choice and that generates larger asset price movements, the youngest age group actually enjoys higher lifetime utility if they become economically active during a recession.

Our paper builds upon and contributes to three broad strands of the literature. First, we use a large-scale overlapping generations (OLG) model with aggregate risk to study the asset pricing and intergenerational consumption implications of large aggregate shocks. The literature that analyzes asset prices and portfolio choice in stochastic OLG economies includes Labadie (1986), Huffman (1987), Ríos-Rull (1994), Storesletten, Telmer, and Yaron (2004, 2007), Constantinides, Donaldson, and Mehra (2002), and Kubler and Schmedders (2010). Ríos-Rull (1996) investigates the properties of business cycles in this class of models, while Ball and Mankiw (2007), Smetters (2006), Krueger and Kubler (2004, 2006), Miyazaki, Saito, and Yamada (2009), and Campbell and Nosbusch (2007) analyze the allocation of aggregate consumption risk across different generations.

Second, whereas we focus on the impact of shocks to aggregate output, a number of papers study the distributional consequences across age cohorts of other types of large economy-wide shocks. Our analysis is similar in spirit to Doepke and Schneider (2006a,b)'s study of the inflationary episode of the 1970's and, to a lesser extent, to Meh, Ríos-Rull, and Terajima (2010). A number of studies employ OLG models to investigate the impact of large swings in the demographic structure of the population on factor and asset prices, as well as on the welfare of different age cohorts. Examples include Attanasio, Kitao, and Violante (2007), Krueger and Ludwig (2007), and Ríos-Rull (2001).

Finally, a recent literature estimates empirical models of aggregate consumption that allows for large declines in aggregate consumption (so-called disasters) and uses these estimates in consumption based asset pricing models. See, e.g., Barro (2006, 2009), and Nakamura, Steinsson, Barro, and Ursua (2010).[1]

The remainder of this paper is organized as follows. In Section 2 we present life cycle facts on labor income, net worth, and portfolio allocations that motivate our quantitative analysis and that we use later to calibrate the model. In Section 3 we set up our model and define a recursive competitive equilibrium. In Section 4 we analyze a sequence of simple examples that can be characterized analytically and that provide important insights into the key mechanism of the model. Section 5 is devoted to the calibration of the model, and Sections 6 and 7 report the findings from our model economies, for the cases in which aggregate shocks do not, and do, affect the shape of the age-earnings profile, respectively. Section 8 concludes. Details about the computational approach, proofs, and additional theoretical results are relegated to the Appendix.

2 Data

In this section we document the life-cycle profiles for labor income, net worth, and portfolio composition that motivate our focus on heterogeneity along the age dimension and will serve as inputs for the calibration of the model. The need for detailed data on household portfolios leads us to use the Survey of Consumer Finances (SCF). The SCF is the best source of micro data on the assets and debts of U.S. households. One advantage of the survey is that it over-samples wealthy households, using a list based on IRS data. Because the SCF weighting scheme adjusts for higher non-response rates among wealthier households, it delivers higher estimates for average income than other household surveys, such as the Current Population Survey (CPS) or Panel Study of

[1] The latter paper finds that (on average) aggregate consumption falls by 11% during a disaster period, that a large part of these consumption drops are reversed in the long run (i.e., there is only a small effect of the consumption disaster on the long-run trend of consumption), and that the average length of a disaster period is 6.5 years. Our calibration will imply a consumption process broadly consistent with these findings.

Income Dynamics (PSID). The survey is conducted every three years, with the most recent survey conducted in 2007, around the peak in asset prices.

We use the SCF to construct life-cycle profiles for labor income, total income, assets, debts, and net worth (see Table 1). These profiles are constructed by averaging (using the SCF sample weights) across households partitioned into 10-year age groups. We divide total income into an asset-type income component, and a residual non-asset-income component which we call labor income.[2] We make two adjustments relative to the SCF concept of household income. First, we add an imputation for implicit rents accruing to home owner-occupiers and attribute these rents to asset income.[3] Second, we subtract interest payments on debts, thinking of these as a negative component of asset income. We measure net worth as the value of all financial and non-financial assets, less the value of all liabilities. Our SCF-based measure of net worth excludes the present value of future pensions associated with defined benefit private pension plans and social security.

In 2007 average household income in the SCF was \$83,430, while average household net worth was \$555,660, for a net worth to income ratio of 6.66.[4] Average household assets amounted to \$659,000, with an average rate of return of 3.1%. Average household debts came to \$103,300, with an average interest rate on debts of 6.4%. The share of net asset income in total income was 0.16. Young households had negative net asset income, despite having positive net worth, reflecting the higher average interest rate paid on debts relative to the rate earned on assets.

Figure 1 plots the life-cycle profiles for labor income and net worth. Income follows the familiar humpshape over the life cycle, while net worth peaks somewhat later. For 20-29 year olds, average net worth is 1.9 times average labor income, while for households age 70 and older, the corresponding ratio is 21.1. Thus the old are much more exposed to fluctuations in asset prices than the young. We will ensure, by force of calibration, that the life-cycle patterns of labor income and net worth in our structural OLG model are identical to the empirical profiles documented here.

While Figure 1 suggests large losses for older households from a slump in asset prices, the risk

[2]Asset income is defined as interest or dividend income (minus interest payment on debts), income from capital gains and asset sales, one-third of business, farm, or self-employment income, private retirement income, and imputed rents from owner-occupied housing. Non-asset income is all other income, which includes wage and salary income, two-thirds of business, farm, or self-employment income, social security income, and a variety of public and private transfers.

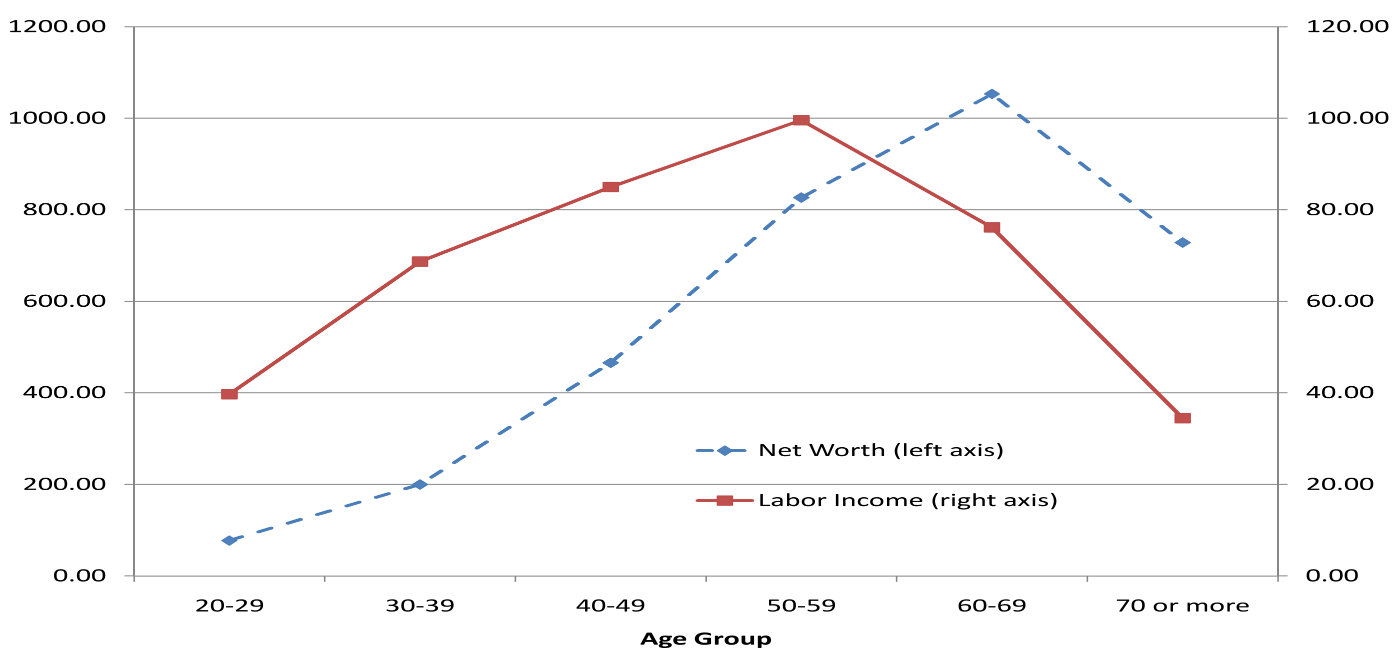
[3]We set imputed rents equal to the value of primary residence times the rate of return on all other assets. This rate of return is computed as asset income (excluding imputed rents) divided by aggregate assets (excluding the value of primary residences and the value of vehicles).

[4]Since income questions refer to the previous calendar year, while questions about wealth are contemporaneous, we adjust income measures for CPI inflation between 2006 and 2007.

Table 1: **Income and Wealth Over the Life Cycle (2007 SCF, $1,000)**

Age of Head	Total Income	Labor Income	Asset Income	Assets	Debts	Net Worth
All	83.43	70.07	13.36	659.00	103.34	555.66
20-29	38.83	39.68	-0.85	130.66	53.30	77.36
30-39	69.83	68.68	1.15	335.87	136.12	199.75
40-49	93.40	84.97	8.43	598.21	132.62	465.59
50-59	117.97	99.56	18.41	959.77	133.24	826.53
60-69	109.06	76.15	32.90	1156.96	104.10	1052.86
70+	57.56	34.46	23.11	756.76	28.48	728.28

Figure 1: **Labor Income and Net Worth by Age, SCF 2007 ($1,000)**



composition of net worth also varies quite substantially with age. To accurately estimate losses by age group, we therefore further decompose portfolios by age group and examine the patterns for relative price changes across different asset classes. In Table 2 we decompose total net worth into risky net worth and safe net worth, where we define risky net worth as the value of stocks, residential real estate, non-corporate business, and non-residential property. We define safe net worth as the value of all other assets, less all debts.[5] In aggregate, risky net worth is 93.9 of aggregate net worth. However, among 30-39 year olds, the corresponding ratio is 140.4%, while among those aged 70 or older, it is only 79.2%. These three ratios reflect three facts: (i) in aggregate, net household holdings of safe assets are very small, (ii) younger households are short in safe assets, because they tend to have substantial mortgage debt (which we classify as a riskless liability) and only small holdings of financial assets, and (iii) older households tend to have little debt and lots of assets, including a significant position in riskless financial assets.

Table 2: **Portfolio Shares as a Percentage of Net Worth**

Age of Head	(1) Stocks	(2) Res, real estate	(3) Non-corp bus.	(4) Non-res prop.	(5) RISKY N.W.	(6) Bonds + CDs	(7) Cars	(8) Other assets	(9) Debts	(10) SAFE N.W.
All	30.28	46.99	12.87	3.80	93.95	16.98	3.45	4.23	-18.60	6.05
20-29	13.20	77.67	43.31	1.28	135.46	13.66	15.26	4.51	-68.90	-35.46
30-39	26.27	96.47	12.73	4.97	140.44	13.80	9.73	4.19	-68.15	-40.44
40-49	30.41	57.62	12.55	3.81	104.38	15.17	4.44	4.49	-28.48	-4.38
50-59	32.70	42.40	13.53	3.72	92.35	17.02	2.79	3.96	-16.12	7.65
60-69	32.17	35.62	13.41	4.12	85.31	17.45	2.40	4.73	-9.89	14.69
70+	27.12	39.76	8.98	3.33	79.18	19.26	1.75	3.72	-3.91	20.82

Risky Net Worth (5) is equal to the sum of columns (1)+(2)+(3)+(4). Safe Net Worth (10) is the sum of columns (6)+(7)+(8)+(9). Total Net Worth is the sum of columns (5)+(10).

Our next task is to estimate price declines for each component of net worth. The 2007 SCF provides a snapshot of household portfolios in the middle of 2007, roughly when asset prices peaked.[6] We estimate the direct redistributive effects of dramatic changes in asset prices by using aggregate asset-class-specific price series to revalue portfolios, thereby constructing estimates for capital losses for each age group.

[5] For our purposes, stocks include stocks held directly or indirectly through mutual funds and retirement accounts and also includes closely held equity. The category "Bonds + CDs" includes bonds (directly or indirectly held), cash, transaction accounts, CDs, and the cash value of life insurance.

[6] The Federal Reserve is currently conducting a special re-survey of some of the same households, to study the effects of the financial crisis.

We assume that SCF portfolios reflect the distribution of household net worth in the second quarter of 2007.[7] We then revalue portfolios for each age group, and for each successive quarter, as follows. For all the components of safe net worth (bonds, vehicles, other assets and debts), we assume no price changes. We price stock wealth using the Wilshire 5000 price index, as of the last trading day in the quarter. We price residential real estate using the Case-Shiller National Home Price Index, which is a quarterly, repeat-sales-based index. We price non-residential property using the Moodys/REAL Commercial Property Price Index, which is a monthly repeat-sales-based index for the prices of apartments, industrial property, commercial property, and retail property. We price non-corporate business wealth using Flow of Funds data.[8] Price changes by asset type relative to 2007:2 are reported in Table 3. For stocks and residential property, values reached a low point in the first quarter of 2009 with prices respectively 46.9% and 29.5% below their 2007:2 values. The values of non-corporate business and non-residential property, by contrast, continue to decline through 2009, and are flat through 2010.[9]

Table 3: **Price Declines Relative to 2007:2 by Risky Asset Class**

	2008				2009				2010			
PRICE SERIES USED	Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4	Q1	Q2	Q3	Q4
Stocks	-12.4	-14.2	-22.0	-40.3	-46.9	-38.4	-28.6	-24.7	-20.5	-29.7	-22.1	-13.5
Res. Real Estate	-13.0	-14.9	-17.8	-23.9	-29.5	-27.3	-24.9	-25.8	-27.9	-24.5	-25.9	-28.8
Non-corp. Bus.	-4.2	-8.0	-10.6	-17.6	-22.2	-25.0	-25.4	-27.9	-26.5	-25.0	-26.0	-25.5
Non-Res. Property	-0.2	-9.6	-7.1	-14.3	-20.9	-33.9	-41.5	-39.3	-40.6	-39.9	-41.3	-40.6
Household Net Worth	-10.4	-12.7	-16.7	-26.2	-31.7	-29.0	-25.2	-24.7	-24.3	-25.3	-23.8	-22.5
ALTERNATIVE SERIES												
Stocks (Flow of Funds)	-9.9	-13.1	-22.8	-39.3	-47.7	-38.3	-27.8	-25.4	-21.6	-31.4	-22.6	-12.8
Real Estate (OFHEO)	-5.3	-5.8	-9.0	-12.6	-12.1	-10.7	-12.1	-14.3	-14.6	-12.9	-15.5	-17.2
Real Estate (Flow of Funds)	-9.1	-13.2	-16.8	-21.8	-26.3	-26.2	-25.6	-25.7	-25.0	-24.6	-27.2	-28.2

We now turn to investigating how these price changes have reduced household net worth by applying the price changes in Table 3 to the life-cycle profiles for aggregate net worth and its decomposition as outlined in Tables 1 and 2.[10] The line labelled "Household Net Worth" in Table 3 shows the implied series for aggregate household net worth. Table 4 reports changes in net worth from 2007:2 to 2009:1, in the aggregate and by age group. In the first set of columns, we

[7]Unfortunately the SCF does not provide much information about precisely when households were interviewed.

[8]In particular the Flow of Funds reports changes in market values for a variety of asset types by sector. We focus on the asset type "proprietors' investment in unincorporated business" for the household and non-profit sector.

[9]For comparison, Table 3 also reports some alternative price series. The Flow of Funds reports price changes for directly held corporate equities: this series aligns closely with the Wilshire 5000 index. The Flow of Funds also reports a price series for residential real estate, based on the Loan Performance Index from First American Corelogic. This series closely tracks the Case-Shiller series. By contrast, the house price series published by OFHEO (based on data from Fannie Mae and Freddie Mac) shows significantly smaller declines in house values.

[10]Of course, this exercise ignores the endogenous response of household portfolios to changing asset prices between 2007 and 2009.

simply report dollar losses across our risky asset types. We focus on the cumulative price changes as of the first quarter of 2009, since this was the quarter in which portfolio-weighted asset values attained their nadir (see Table 3). We then report total dollar losses, total losses as fractions of age-group specific net worth and age-group specific total income, and total losses as a fraction of average total income.

The average household saw a price-change-induced decline in net worth of $176,000 between 2007:2 and 2009:1, which amounted to 32% of 2007:2 net worth, and more than twice average annual income in 2007. Almost half of this total decline was driven by a decline in stock prices, and almost half by a decline in house prices.

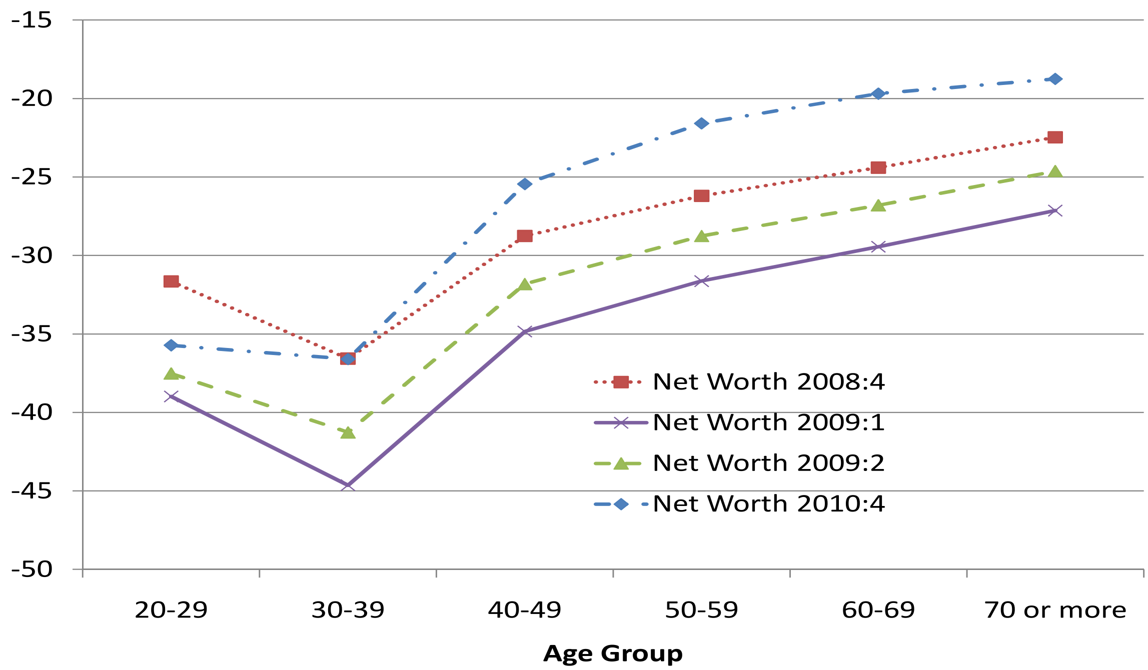
Losses varied widely by age. Younger households lost much less, while those in the 60-69 age group lost the most: $310,000 on average, or nearly four times average annual income for this age group. At the same time, differences in portfolio composition were large enough to generate substantial age variation in returns. In particular, because younger households were more leveraged, they lost more as a percentage of their net worth: 30-39 year olds lost 45% of net worth, while households older than 70 lost only 27%. In other words, absent age variation in portfolios, but given the empirical age profile for net worth, the losses experienced by younger households would have been smaller, and those experienced by older households would have been even larger.

Table 4: **Estimated Losses by Age Group as of 2009:1**

	Dollar Losses ($1,000)					Losses as Percentage of		
Age of Head	Stocks	Res. real estate	Non-corp. bus.	Non-res. property	Total Losses	Net Worth	Income	Avg. Income
All	78.99	76.97	15.90	4.42	176.27	31.7	211.3	211.3
20-29	4.79	17.71	7.44	0.21	30.16	39.0	77.7	36.1
30-39	24.64	56.80	5.65	2.08	89.16	44.6	127.7	106.9
40-49	66.47	79.07	12.98	3.71	162.24	34.8	173.7	194.5
50-59	126.91	103.29	24.85	6.43	261.47	31.6	221.6	313.4
60-69	159.01	110.53	31.36	9.08	309.97	29.4	284.2	371.5
70+	92.73	85.34	14.53	5.07	197.67	27.1	343.4	236.9

Figure 2 presents the losses as a percentage of net worth for additional dates. By the last quarter of 2010, asset prices had partially recovered, but were still 22.5% below their peak values.

Figure 2: **Percentage decline in net worth by age group relative to 2007:2**



3 The Model

The facts documented above guide our modeling choices. First, the substantial heterogeneity by household age in labor income and net worth translates into age-varying exposure to aggregate wage and asset price risk. Thus, a satisfactory quantitative analysis requires an overlapping-generations life-cycle model with aggregate shocks. Our general equilibrium approach in which asset prices respond endogenously to aggregate output fluctuations provides a theoretical link between the dynamics of income, consumption, and savings on the one hand and asset prices on the other.

Second, empirical portfolio allocations between risky and riskless assets display significant age heterogeneity, which translates into age variation in the sensitivity of net worth to aggregate shocks. This motivates us to consider models with both risky and safe assets.

Third, the direct effect of recessions on labor income varies across age groups, largely reflecting the fact that younger workers are disproportionately likely to become unemployed.[11] This fact induces us to explore a version of the model in which recessions not only change the level of the

[11] In Section 7 we will document the extent of this age variation in labor income declines in the Great Recession.

age-earnings profile, but its shape as well (see Section 7).

3.1 Stochastic Structure

Aggregate fluctuations in the model are driven by shocks to aggregate productivity z, where z has finite support Z and evolves over time according to a Markov chain with transition matrix $\Gamma_{z,z'}$.

3.2 Technology

A representative firm operates a Cobb-Douglas technology that takes as inputs a fixed factor K and labor L, and produces as output a non-storable consumption good Y. The firm's total factor productivity (TFP) varies with the aggregate productivity shock. Thus

$$Y = z\, K^{\theta}\, L^{1-\theta}$$

where $\theta \in (0,1)$ is the capital share of output. We normalize $K = 1$.

One interpretation of our assumption that capital K is in fixed supply is that K stands in for non-reproducible land or intangible capital. By making the stock of capital fixed, any changes in the demand for assets (i.e., claims to the returns to capital) must translate into movements in asset prices rather than changes in the quantity of capital. This property is important given our focus on the welfare effects of great recessions that are accompanied by large asset price declines. In the standard frictionless business cycle model, by contrast, capital and consumption are the same good, and thus this model cannot generate any movements in the relative price of capital.

3.3 Endowments

Households live for I periods and then die with certainty. Thus the economy is populated by I distinct age cohorts at any point in time. Each age cohort is composed of identical households. In each period of their lives households are endowed with one unit of time which they supply to the market inelastically. Their age- and productivity-dependent labor productivity profile is given by $\{\varepsilon_i(z)\}_{i=1}^{I}$. Indexing the productivity profile to the aggregate shock will allow us to capture heterogeneity across age groups in the impact of economic downturns on labor income (Section 7). We normalize units so that $\sum_{i=1}^{I} \varepsilon_i(z) = 1$ for all $z \in Z$. Thus the aggregate supply of labor is constant and equal to $L = 1$ at all times. This normalization also implies that aggregate output is given by $Y(z) = z$ for all $z \in Z$.

Labor markets are competitive, and therefore the economy-wide wage per labor efficiency unit supplied is equal to the marginal product of labor from the production technology:

$$w(z) = (1-\theta)z.$$

Note that because the aggregate supply of capital and labor is exogenous, and the labor share of income is constant, fluctuations in z need not be interpreted simply as neutral shocks to multi-factor productivity: they could equally well capture fluctuations in capital or labor productivity, or capital or labor utilization rates. Thus our model is consistent with a range of alternative theories regarding the fundamental sources of business cycles.

3.4 Preferences

Households have standard time-separable preferences over stochastic consumption streams $\{c_i\}_{i=1}^{I}$ that can be represented by

$$E\left[\sum_{i=1}^{I}\prod_{j=1}^{i}\beta_j\, u(c_i)\right]$$

where β_i is the time discount factor between age $i-1$ and i (we normalize $\beta_1 = 1$). Age variation in the discount factor stands in for unmodeled changes in family size and composition, age-specific mortality risk, and any other factors that generate age variation in the marginal utility of consumption. We will calibrate the profile $\{\beta_i\}_{i=1}^{I}$ so that our economy replicates the life-cycle profile for net worth documented in SCF data in Section 2.

Expectations $E(.)$ are taken with respect to the underlying stochastic process governing aggregate risk. Finally, the period utility function is of the constant relative risk aversion form

$$u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$$

where the parameter σ is the inverse of the intertemporal elasticity of substitution, and $\sigma = 1$ corresponds to log-utility.

3.5 Financial Markets

Agents trade financial assets to transfer resources over time. We consider three alternative market structures that differ in the set of assets that can be traded. In the first, households can save only by purchasing shares of the capital stock, which entitle the owner to capital income. This model allows us to highlight the main mechanism at work in the most transparent way. The second market structure features two assets: leveraged risky equity and riskless bonds. However, households' portfolio allocations are treated as exogenous parameters and calibrated to replicate the portfolio composition across risky and riskless assets by age, as observed in the SCF. With this version of the model we can assess the quantitative importance of age variation in portfolio shares for our asset pricing and welfare results. Finally we consider a third market structure in which the choice of portfolios is *endogenous*. The point of this economy is to investigate the extent to which we can rationalize observed portfolio choices over the life cycle with our model, and to document how the ability of different generations to trade risk affects equilibrium prices and the distribution of welfare losses from an adverse aggregate shock.

Given that the primary goal of the paper is to assess the distributional implications of severe recessions, we will be most interested in the quantitative results for the second economy, because in that economy we exactly replicate, by force of calibration, both the empirical life-cycle profile for net worth and the life-cycle profile for risky versus safe portfolio shares. We therefore consider the second economy our benchmark.

The one-asset economy is a special case of the two-asset economies in which the aggregate supply of riskless bonds is set to zero and each age group puts 100% of savings into risky equity. The economies with exogenous and endogenous portfolios differ only with respect to whether the division of household savings between stocks and bonds is specified exogenously or chosen optimally. We will therefore define a recursive competitive equilibrium only once, focussing on the most general model: the two-asset economy with endogenous portfolios. Also note that if the aggregate shock z can only take two values (the case we will consider in our quantitative experiments below), then the two-asset economy with endogenous portfolio choice is equivalent to an economy in which markets are sequentially complete. In Appendix A we show how to map the two-asset economy into a complete markets economy with state-contingent share purchases, and in Appendix B we discuss how we exploit this mapping in our computation of the two-asset economy with endogenous portfolio choice. Finally, note that while agents can insure against life-cycle shocks in the endogenous portfolios economy, they cannot buy insurance ex ante against the date and state in which they become economically active. Thus aggregate shocks will still have redistributive consequences between existing and newly active households.

We will now present a recursive definition of competitive equilibrium. The aggregate state of the economy is described by the current aggregate shock z and the cross-sectional distribution $A = (A_1, \ldots, A_I)$ of shares of beginning of the period total wealth, where $\sum_{i=1}^{I} A_i = 1$. Newborn households enter the economy with zero initial wealth, so that $A_1 = 0$. Individual state variables include a household's age i and its individual share of wealth, denoted by a.

The representative firm issues a constant quantity B of risk-free real bonds at a price $q(z, A)$ per unit. Each bond is a promise to pay one unit of the consumption good in the next period. We treat the supply of debt B as an exogenous time- and state-invariant *parameter* of the model. Dividends for the representative firm $d(z, A)$ are then given by aggregate capital income θz plus revenue from debt issuance $q(z, A)B$ less debt repayment B :

$$d(z, A) = \theta\, z - [1 - q(z, A)]\ B \tag{1}$$

Note that returns to equity are risky, while the return to debt is safe and given by the reciprocal of the bond price. The supply of debt B determines the level of leverage in the economy: the higher is B, the more leveraged and risky are stocks. Let $p(z, A)$ denote the ex-dividend price of equity. The aggregate value of start of period wealth is the value of aggregate payments to asset holders in the period, plus the ex-dividend value of equity:

$$\begin{aligned} W(z, A) &= p(z, A) + d(z, A) + B \\ &= p(z, A) + \theta\ z + q(z, A)\ B, \end{aligned}$$

where the second equality follows from the expression for dividends in equation (1).

3.6 Household Problem

Let $y_i(z, A, a)$ and $\lambda_i(z, A, a)$ denote the optimal household policy functions for total savings and for the fraction of savings invested in leveraged equity. Let $c_i(z, A, a)$ and $a_i'(z, A, z', a)$ denote the associated policy functions for consumption and for shares of next period wealth. The dynamic programming problem of the household reads as

$$v_i(z, A, a) = \max_{c,y,\lambda,a'} \left\{ u(c) + \beta_{i+1} \sum_{z' \in Z} \Gamma_{z,z'} \; v_{i+1}(z', A', a') \right\} \qquad \text{s.t.} \tag{2}$$

$$\begin{aligned} c + y &= \varepsilon_i(z)\, w(z) + W(z, A)\, a && (3) \\ a' &= \frac{\left(\lambda \frac{[p(z',A')+d(z',A')]}{p(z,A)} + (1-\lambda) \frac{1}{q(z,A)} \right) \; y}{W(z', A')} && (4) \\ A' &= G(z, A, z'). && (5) \end{aligned}$$

The first constraint (3) is the household's budget constraint: consumption plus savings must equal labor earnings plus the household's share of start of period wealth.[12] The second constraint (4) is the law of motion for the household's share of individual wealth. This constraint merits some additional explanation. Savings in equity are given by λy, and the gross return on these savings is given by $[p(z', A') + d(z', A')] / p(z, A)$. Savings in bonds are given by $(1 - \lambda)y$, and the gross return on these savings is $1/q(z, A)$. Thus the numerator on the right-hand side of equation (4) is the gross value of the household portfolio at the beginning of next period. The household's *share* of next period wealth is this value divided by aggregate next period wealth, the denominator on the right-hand side of (4). The third constraint is the law of motion for the wealth distribution, which allows agents to forecast future prices, contingent on the sequence for future productivity. Let $G_i(z, A, z')$ denote the forecast for the share of next period wealth owned by age group i.

Definition 1. *A recursive competitive equilibrium is a value and policy functions for each age,* $v_i(z, A, a)$, $c_i(z, A, a)$, $y_i(z, A, a)$, $a_i'(z, A, z', a)$, $\lambda_i(z, A, a)$, *pricing functions* $w(z)$, $d(z, A)$, $p(z, A)$, $q(z, A)$, *and an aggregate law of motion* $G(z, A)$ *such that:*

1. *Given the pricing functions and the aggregate law of motion, the value functions* $\{v_i\}$ *solve the recursive problem of the households and* $\{c_i, y_i, a_i', \lambda_i\}$ *are the associated policy functions.*

2. *Wages and dividends satisfy*

$$w(z) = (1 - \theta)\, z \qquad \textit{and} \qquad d(z, A) = \theta\, z - [1 - q(z, A)]\, B.$$

[12] Households face the additional constraints that consumption must be non-negative, and savings at age I must be non-negative.

3. *Markets clear*

$$\begin{aligned} \sum_{i=1}^{I} c_i(z, A, A_i) &= z \\ \sum_{i=1}^{I} \lambda_i(z, A, A_i)\ y_i(z, A, A_i) &= p(z, A) \\ \sum_{i=1}^{I} [1 - \lambda_i(z, A, A_i)]\ y_i(z, A, A_i) &= B\ q(z, A). \end{aligned}$$

4. *The law of motion for the distribution of wealth is consistent with equilibrium decision rules*

$$G_1(z, A, z') = 0 \qquad \forall z' \tag{6}$$

$$G_{i+1}(z, A, z') = a_i'(z, A, z', A_i), \qquad \forall z', \ i = 1, \ldots, I-1. \tag{7}$$

4 Developing Intuition: Four Simple Example Economies

In order to develop intuition for the key mechanisms at work in our model, we now study four simple example economies: a representative agent version of our environment, and three special cases of our general OLG economy. These examples are designed to highlight a) what determines the magnitude of equilibrium asset price movements, relative to movements in output, and b) what determines how price movements translate into welfare effects that vary across different generations.

From now on, we will assume that the aggregate shock takes only two values: $Z = \{z_l, z_h\}$, where z_l stands for a severe recession. We will measure the magnitude of the decline in asset prices, relative to output (which is given by z) by

$$\xi = \frac{\ln(p_l/p_h)}{\ln(z_l/z_h)},$$

where it is understood that, in general, prices and thus the elasticity ξ, are functions of the distribution of wealth in the economy prior to the recession. An elasticity of $\xi = 3$, for example, indicates that the percentage decline in stock prices when the economy enters the recession is three times as large as the fall in output.

4.1 Example I: Representative Agent Model

Our first simple economy is the standard infinitely lived representative-agent Lucas asset pricing model (translated into our physical environment).[13] Given the existence of a representative agent, the distribution of wealth is degenerate. With Markov shocks, the only state variable is current productivity z. Furthermore, if aggregate shocks are *iid* (which we will assume in our baseline calibrations for the quantitative models), then

$$\left(\frac{p_l}{p_h}\right) = \left(\frac{z_l}{z_h}\right)^{\sigma}$$

and thus $\xi^{RA} = \sigma$. When aggregate shocks are not *iid*, the same result still obtains in two special cases: $\sigma = 1$ (unitary intertemporal elasticity of substitution) or $\beta = 1$ (no discounting). The logic for why prices become more sensitive to output as the inter-temporal elasticity of substitution $1/\sigma$ is reduced is familiar and straightforward. For the stock market to clear, stock prices (and expected returns) must adjust to output fluctuations such that the representative household neither wants to buy nor sell stocks. The less willing are households to substitute consumption over time, the more prices must fall (and expected returns increase) in order to induce households to maintain constant stock holdings in response to a decline in output and thus consumption.

In the recent Great Recession in the United States, asset prices fell roughly three times as much as output. The representative agent economy generates $\xi^{RA} = 3$ when $\sigma = 3$. Of course, although this economy is a useful benchmark, it has nothing to say about differential welfare effects across age groups, our main object of interest in this paper.

4.2 Example II: Log Utility and iid Shocks

With logarithmic utility, *iid* aggregate shocks, and an age-income profile that is independent of the aggregate shock, it is possible to characterize several features of the competitive equilibrium in closed form, even for the case in which households choose their portfolios endogenously. The following proposition is proved (by construction) in Appendix C and does not hinge on z taking only two values.

Proposition 1. *Assume that i) the period utility function is logarithmic ($\sigma = 1$), (ii) aggregate productivity shocks have no effect on relative earnings across age groups ($\varepsilon_i(z) = \varepsilon_i \ \forall z$) and (iii)*

[13] The detailed analysis is contained in Appendix D.

productivity shocks are iid over time ($\Gamma_{z,z'} = \Gamma_{z'}$). Then there exists a recursive competitive equilibrium in the two-asset economy with endogenous portfolio choice with the following properties:

1. *The distribution of wealth A is constant. Denote this distribution by $\overline{A} = (\overline{A}_1, \ldots, \overline{A}_I)$. Then:*

$$\begin{aligned} G_{i+1}(z, \overline{A}, z') &= a_i'(z, \overline{A}, z', \overline{A}_i) = \overline{A}_{i+1} \qquad \forall z, z', \ \forall i = 1, ..., I-1 \\ G_1(z, \overline{A}, z') &= 0 \qquad \forall z \end{aligned}$$

2. *Stock and bond prices are proportional to the aggregate productivity shock:*

$$\begin{aligned} q(z, \overline{A}) &= z\overline{q} \qquad \forall z \\ p(z, \overline{A}) &= z\overline{p} \qquad \forall z, \end{aligned}$$

   where $\overline{q}$ and $\overline{p}$ are constants.

3. *Portfolios are constant both over time and across age groups:*

$$\lambda_i(z, \overline{A}, \overline{A}_i) = \overline{\lambda} = \frac{\overline{p}}{\overline{p} + \overline{q}B} \qquad \forall z, \ \forall i = 1, ..., I-1.$$

4. *Consumption and savings for all age groups are proportional to the aggregate productivity shock:*

$$\begin{aligned} c_i(z, \overline{A}, \overline{A}_i) &= z\overline{c}_i = z\left[(1-\theta)\varepsilon_i + \theta\overline{A}_i + \left(\overline{A}_i - \overline{A}_{i+1}\right)(\overline{p} + \overline{q}B)\right] \\ y_i(z, \overline{A}, \overline{A}_i) &= z\overline{y}_i = z\overline{A}_{i+1}(\overline{p} + \overline{q}B) \qquad \forall z, \ \forall i = 1, ..., I-1. \end{aligned}$$

5. *The value of aggregate wealth relative to output is independent of the distribution for productivity shocks and independent of the supply of debt:*

$$\frac{p(z, \overline{A}) + q(z, \overline{A})B}{z} = \overline{p} + \overline{q}B = \Psi,$$

   where the constant Ψ depends on various model parameters of the model, but not on the value of B.

6. *The equity premium is given by*

$$\begin{aligned}
&\sum_z \Gamma_z \left\{ \sum_{z'} \Gamma_{z'} \left[\frac{p(z',\overline{A}) + \theta z' + q(z',\overline{A})B - B}{p(z,\overline{A})} \right] - \frac{1}{q(z,\overline{A})} \right\} \\
= \quad & \frac{\overline{p} + \theta + \overline{q}B}{\overline{p}} \left(\sum_z \frac{\Gamma_z}{z} \sum_z \Gamma_z z - 1 \right).
\end{aligned}$$

This proposition has various implications. First, for this particular parameterization with log utility, the model predicts that asset prices fall by exactly as much as output in the recession, $\xi = 1$. Second, portfolio allocations are counterfactual since the share invested in stocks is age-invariant in the model, but declining with age in the data. Third, since consumption of all age cohorts varies one-for-one with the aggregate shock, all households lose from a recession, including the youngest generation. Fourth, the equity premium is positive (since $E\left[\frac{1}{z}\right] \geq \frac{1}{E[z]} = 1$) and is increasing in the supply of debt B (because $\overline{p} + \overline{q}B = \Psi$ is independent of B, and $\overline{p}$ is declining in B).

Although the log utility specification delivers clean closed-form solutions, it is of limited interest from the applied perspective of assessing the welfare effects of the Great Recession, primarily because it cannot generate a large decline in asset prices, relative to the decline of output, as observed in the data.[14] We therefore now move to exploring deviations from log-utility.

4.3 Example III: Two-Period OLG Economy

Outside the log-*iid* case, closed-form solutions for OLG economies with aggregate risk are not available, to the best of our knowledge. To build intuition for this case, we begin with the simplest OLG framework, in which households live for only two periods: $I = 2$. We use this example to discuss how the curvature parameter σ affects the elasticity ξ of price changes to output changes in OLG economies. We focus on the one-asset economy ($B = 0$) and assume that households only earn labor income in the first period of life: $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$. Since young households start with zero assets, all wealth is held by old agents. Thus the wealth distribution is time invariant and degenerate in this economy. As in the representative agent model, the only state variable is the exogenous shock $z \in \{z_l, z_h\}$.

[14] However, this parameterization *is* useful as a test case for assessing the accuracy of our computational algorithm.

Consumption of young and old households is given by

$$\begin{aligned} c_1(z) &= (1-\theta)z - p(z) \\ c_2(z) &= \theta z + p(z), \end{aligned}$$

and the prices of shares are determined by the intertemporal Euler equation

$$p(z)\ \left[(1-\theta)z - p(z)\right]^{-\sigma} = \beta \sum_{z' \in \{z_L, z_h\}} \Gamma_{z,z'}\ \left[\theta z' + p(z')\right]^{-\sigma}\ \left[\theta z' + p(z')\right]. \tag{8}$$

No closed-form solution is available for the functional equation $p(z)$ that solves equation (8) outside of the special cases $\sigma = 0$ and $\sigma = 1$. However, in Appendix E we derive an approximate expression for the elasticity ξ^{2p} for this two-period ($2p$) OLG model, again assuming *iid* shocks:[15]

$$\begin{aligned} \xi^{2p} &\approx \frac{\sigma(1-\theta)}{1-\theta\frac{(R-\sigma)}{(R-1)}} \\ &= \xi^{RA} \times \frac{1-\theta}{1-\theta\frac{(R-\sigma)}{(R-1)}} \end{aligned}$$

where $R = \frac{\theta + p}{p} > 1$ is the steady state gross return on the stock.[16]

Note first that for $\sigma = 1$ this formula is exact (as shown in the previous section) and delivers $\xi^{2p} = \xi^{RA} = 1$: prices fall by exactly as much as output in a downturn. Second, ξ^{2p} is increasing in σ, and thus for $\sigma > 1$ we have $\xi^{2p} > 1$. Third, $\xi^{2p} < \xi^{RA}$. Thus, as long as the intertemporal elasticity of substitution $1/\sigma$ is smaller than one, asset prices fall by more than output in a recession, but by less than in the corresponding representative-agent economy with infinitely lived households. For example, suppose we think of a period as 30 years, take $\sigma = 3$, and set β and θ such that the model generates the same interest rate and the same wealth to income ratio as our baseline quantitative six-period model (see Section 5). These choices imply that $\beta = 0.311$ and $\theta = 0.3008$. Then, using the expression above, $\xi^{2p} = 1.97$ in the OLG economy, compared to $\xi^{RA} = 3$ in the representative agent economy.[17] The finding that stock prices are less volatile, relative to output, in OLG economies compared to the infinitely lived representative-agent economy will reappear consistently in the various economies we study. The reason for this is as follows.

[15] This approximation involves taking linear approximations to the pricing equations around the point $z_l/z_h = 1$.

[16] For θ such that $R = \beta^{-1}$ the expression simplifies to $\xi^{2p} \approx \frac{\sigma(\beta+1)}{\sigma\beta+1}$.

[17] For the calibration just described, assuming $z_l/z_h = 0.917$ and $pr(z = z_h) = 0.85$ the true elasticity is $\xi^{2p} = 1.99$, compared to the value of 1.97 from the approximate expression.

The current old generation clearly suffers from the recession since the price of the asset, the only source for old-age consumption, is lower in the bad than in the good aggregate state of the world. Moreover, for $\sigma > 1$, consumption of the old is more sensitive to aggregate shocks than consumption of the young:

$$\frac{c_1(z_h)}{c_1(z_l)} < \frac{z_h}{z_l} < \frac{c_2(z_h)}{c_2(z_l)}$$

The second inequality reflects the fact that $c_2(z_h)/c_2(z_l) = p_h/p_l > z_h/z_l$ (since $\xi^{2p} > 1$), while the first inequality follows from market clearing: $(c_1(z_h) + c_2(z_h)) / (c_1(z_l) + c_2(z_l)) = z_h/z_l$. The fact that aggregate risk is disproportionately borne by the old explains why stock prices are less volatile in this economy than in the analogous representative agent economy. Recall that stocks are effectively priced by younger agents, because the supply of stocks by the old is inelastic at any positive price. Because the old bear a disproportionate share of aggregate risk, the young's consumption fluctuates less than output. Thus smaller price changes (relative to the representative agent economy) are required to induce them to purchase the aggregate supply of equity at each date.

One might wonder whether it is possible that $c_1(z_h) < c_1(z_l)$, so that newborn households would potentially prefer to enter the economy during a recession rather than during a boom. The answer turns out to be no: while stock prices fall by more than output in the event of a recession, they never fall by enough to compensate the young for their decline in labor earnings. The logic for this result is straightforward. In a two-period OLG economy, stock prices are defined by the inter-temporal first-order condition for young households (equation 8). With *iid* shocks, the right-hand side of this condition is independent of the current value for z. Taking the ratio of the two pricing equations across states, the ratio of stock prices across states is given by

$$\frac{p_h}{p_l} = \left(\frac{c_1(z_h)}{c_1(z_l)}\right)^{\sigma}.$$

The only advantage to the young from entering the economy during a recession is that they buy stocks cheaply, $p_h/p_l > 1$. But then the optimality restriction above implies that $c_1(z_h)/c_1(z_l) > 1$, so the young must suffer relatively low consumption if they enter during a recession. Intuitively, low prices are needed to induce the young to buy stocks when the marginal utility of current consumption is high. But a high marginal utility of consumption requires low consumption for these households.

This example reveals that for the young to potentially gain from a recession, we need people to live for at least three periods, while the previous example with logarithmic preferences indicates that we also require $\sigma > 1$. We now move to a three-period example to show that the young can indeed benefit from a recession, and provide some intuition for the combination of model elements required to deliver this result.

4.4 Example IV: Three-Period OLG Economy

Now households live for three periods, $I = 3$. Households do not value consumption when young and discount the future at a constant factor $\beta_2 = \beta_3 = \beta$. They are only productive in the first period of their lives, i.e., $\varepsilon_1 = 1$ and $\varepsilon_2 = \varepsilon_3 = 0$.

By construction, young households buy as many stocks as they can afford, while the old sell all the stocks they own. Only the middle-aged make an interesting intertemporal decision, trading off current versus future consumption. In a recession, falling stock prices will have countervailing effects on the middle-aged's stock trade decision. On the one hand, low current stock prices offer an incentive to reduce stock sales to exploit higher expected stock returns (the substitution effect). On the other, consumption smoothing calls for larger stock sales, since stock sales are the only source of income for this group (the income effect).

Given that young households start their lives with zero asset holdings and that the total number of wealth shares has to sum to one, the only endogenous aggregate state variable in this simple economy is the share of assets held by old households, A_3, which we for simplicity denote by $A_3 = A$. Consequently the share of assets owned by middle-aged households is given by $A_2 = 1 - A$.

The first-order condition for middle-aged households can then be written as

$$u'\left[(1-A)\left(p(z,A)+\theta z\right) - G(z,A)p(z,A)\right] = \beta \sum_{z'} \Gamma_{z,z'} \frac{[p(z',A')+\theta z']}{p(z,A)} u'\left[G(z,A)p(z',A')\right], \tag{9}$$

where consistency requires that tomorrow's asset share of the old is equal to the number of shares purchased by the current middle-aged households: $A' = G(z, A)$. In this expression marginal utility from consumption when middle aged is equated to discounted expected marginal utility from consumption when old, adjusted by the gross return on assets $[p(z', A') + \theta z']/p(z, A)$.

The second functional equation determining the pricing and optimal policy functions states that the equilibrium demand for shares of the young, $1 - G(z, A)$, equals the number of shares that can be purchased with total labor income of the young, which is $w(z)/p(z) = (1-\theta)z/p(z)$. Thus

$$[1 - G(z, A)]\ p(z, A) = (1 - \theta)z. \tag{10}$$

Equations (9) and (10) form a pair of functional equations that jointly determine the unknown equilibrium pricing and policy functions $p(z, A)$ and $G(z, A)$. Consumption $\{c_2(z, A), c_3(z, A)\}$ and welfare $\{v_1(z, A), v_2(z, A), v_3(z, A)\}$ at all ages can easily be calculated from these equilibrium functions.[18]

In the log-case with *iid* shocks, as shown in Section 4.2, consumption of all households is proportional to the aggregate shock. It then follows directly that in this case, middle-aged and old households suffer welfare losses from the economic downturn, while newborn households (who do not value current consumption) are *exactly* indifferent between being born in normal times and in a recession. This suggests that if $\sigma > 1$, asset prices should move by more than output, which in turn should be sufficient to break the indifference of the young in favor of being born in the downturn. We briefly investigate this conjecture now.

Note that for $\sigma \neq 1$ the recursive competitive equilibrium of the model needs to be solved numerically, but this is straightforward to do with the only continuous state variable being A. We choose the same parameter values as in the previous two-period example. The capital share remains at $\theta = 0.3008$ and the time discount factor equals $\beta = (0.311)^{\frac{20}{30}} = 0.459$, resulting in the same annualized discount factor as in the two-period model.[19] The aggregate shock takes two values with $z_l/z_h = 0.917$. Thus a fall in aggregate technology leads to a decline of aggregate output in the order of 8.3%, a value we will also use below in the calibrated version of the full model. We

[18] These are given explicitly as

$$\begin{aligned}
c_3(z, A) &= A\,[p(z, A) + \theta z] && (11)\\
c_2(z, A) &= (1 - A)\,[p(z, S) + \theta z] - G(z, A)p(z, A) && (12)\\
v_3(z, A) &= u\,[c_3(z, A)] && (13)\\
v_2(z, A) &= u\,[c_2(z, A)] + \beta \sum_{z'} \Gamma_{z,z'}\ u[c_3(z', G(z, A))] && (14)\\
v_1(z, A) &= \beta \sum_{z'} \Gamma_{z,z'}\ v_2\,[z', G(z, A)]\,. && (15)
\end{aligned}$$

[19] We are assuming 30-year periods in the-two period OLG model and 20-year periods in the-three period model. Note also that it is impossible to match the empirical wealth-to-earnings ratio by appropriate choice of β in this simple model, since households only earn labor income in the first period of their life and save all of it.

assume that aggregate shocks are uncorrelated over time in this model in which a period lasts for 20 years.

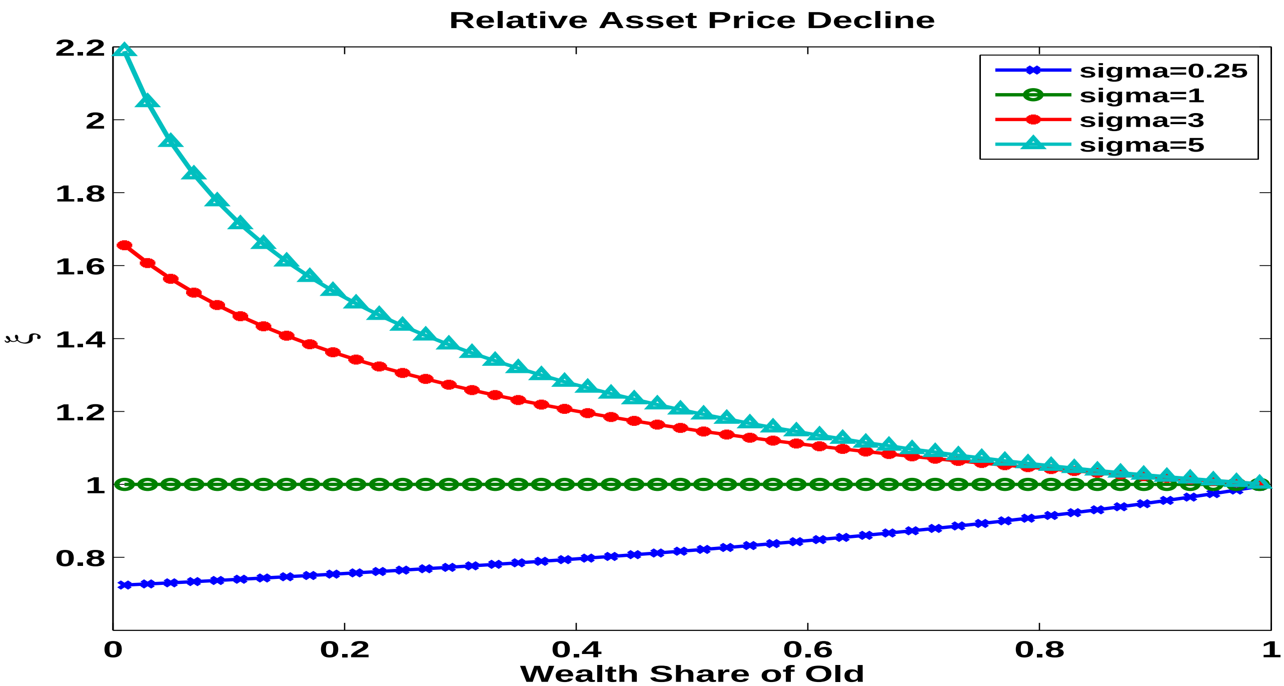


Figure 3 plots the elasticity of asset prices to output, ξ^{3p}, as a function of the share of wealth held by the old generation, for various values of the IES $1/\sigma$. Note that, as demonstrated above, for the logarithmic case $\sigma = 1$ we have $\xi^{3p} = 1$, independent of the wealth distribution A.[20]

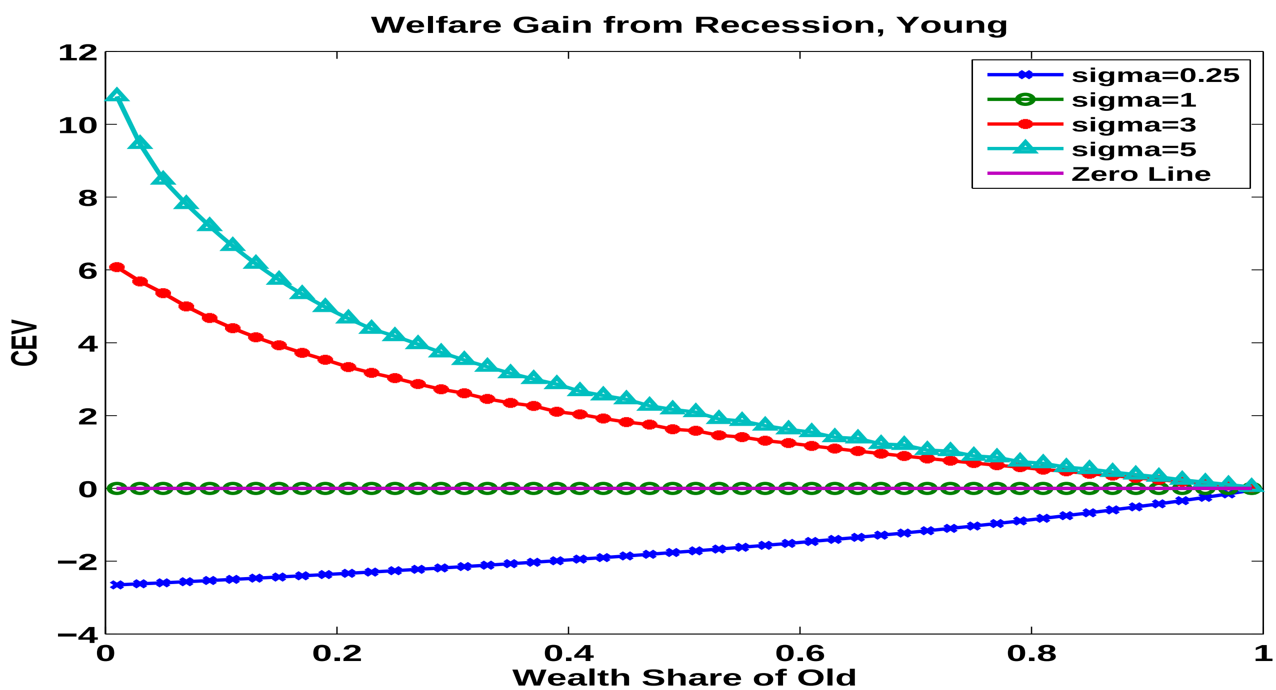
This figure displays two key findings. First, the lower is the willingness of households to intertemporally substitute consumption (the higher is σ), the larger is the fall in asset prices, relative to output, in a recession. Mechanically, when the intertemporal elasticity of substitution is less than one, the income effect dominates the substitution effect and the middle-aged sell more stocks in a recession (compared to normal times) in an attempt to smooth consumption. These extra shares must be bought by the young which, given inelastic demand, necessitates lower share prices (see equation (10)). Second, the size of asset price movements depends on the wealth distribution A when preferences are not logarithmic. The larger is the share of wealth held by the middle-aged (the smaller is A), the larger is ξ^{3p} (assuming $\sigma > 1$). Once again this is the case, since it is the net stock sales of the middle-aged that varies with the cycle. The larger the share of

[20]Given that prices move one-for-one with output, equation (10) implies that $G(z, A)$ must be independent of z, and thus that the wealth distribution must be constant, consistent with Proposition 1.

wealth in the hands of the middle-aged households relative to the old, the larger is the downward pressure on prices in response to a negative shock, since the young must buy more extra shares with the same amount of earnings.[21]

For $\sigma = 3$, the wealth share of the old generation converges to $A = 34.2\%$ if the economy experiences a long sequence of good shocks, and for $A = 0.342$ we find a price elasticity of $\xi^{3p} = 1.24$. Recall that the corresponding values for the representative-agent and two-period OLG economies were $\xi^{RA} = 3$ and $\xi^{2p} = 1.97$.

Figure 4: **Welfare Consequences of Recessions for Young Households**



The welfare consequences for young generations of starting their economic lives in a recession, relative to an expansion, are displayed in Figure 4. We measure welfare consequences as the percentage increase in consumption in all periods of a household's life, under all state contingencies, that a household born in an expansion would require to be as well off as being born in a recession, with positive numbers thus reflecting welfare gains from a recession. Again we plot these numbers as a function of the wealth distribution A and for different values for σ. We observe that the welfare consequences from recessions for the young mirror the elasticity of asset prices to output

[21] In the economies for which $\sigma \neq 1$ the wealth holdings of the old at the start of a recession need not be the same as at the start of an expansion, on average. The figure traces out the price differences between expansions and recessions, conditional on the *same* wealth distribution in the economy.

(Figure 3), confirming that this elasticity is the crucial determinant of how the welfare costs of recessions are distributed in our class of OLG economies.

The purpose of the sequence of simple examples was to develop intuition for the magnitude of asset price declines relative to output, and the resulting implications for lifetime welfare of different generations. Our last example in this section displayed welfare gains from a recession for the young as long as $\sigma > 1$. However, note that in this example we stacked the deck in several ways in favor of obtaining this result. First, young households do not value consumption today and thus are not affected directly by a decline in current aggregate consumption. Second, the middle-aged and the old have no source of income other than selling shares, which means that they bear a disproportionate share of the burden of recession. Third, aggregate shocks are purely temporary, so young households can expect asset prices to recover before they need to sell stocks later in their life cycle.

The remainder of the paper now documents the size of the asset price decline and the distribution of welfare consequences from a severe and long-lasting recession in a realistically calibrated OLG economy in which life-cycle labor income and wealth profiles match those observed in the 2007 SCF.

5 Calibration

To describe the calibration strategy, we start with the *two-asset economy with exogenous portfolios*. We use the same parameter values in the other two economies, with the exception of the life-cycle profile for portfolio shares, $\{\lambda_i\}_{i=1}^{I}$. The one-asset economy can be thought of as a special case in which portfolio shares do not vary by age. [22] In the two-asset economy with endogenous portfolios, portfolio shares are equilibrium outcomes rather than parameter choices.

We assume agents enter the economy as adults and live for $I = 6$ periods, where a period lasts for 10 years. The preference parameters to calibrate are the coefficient of relative risk aversion σ and the life-cycle profile for discount factors $\{\beta_i\}_{i=2}^{I}$. Parameters governing labor endowments over the life-cycle profile are given by $\{\varepsilon_i(z)\}_{i=1}^{I}$, and parameters governing financial markets include the supply of bonds B and the life-cycle profile for portfolio shares allocated to stocks, $\{\lambda_i\}_{i=1}^{I}$. Finally, the technology parameters are the capital share of income θ and the support and transition

[22] In other words, when all households must choose the same portfolio split λ, the choices for B and λ are irrelevant for real allocations. In Appendix F we relate asset prices in the one-asset economy to asset prices in the two-asset economy *with exogenous portfolios,* and show that in the latter one can generate equity premia of arbitrary size by appropriate choice of portfolio shares $\{\lambda_i\}$ and/or outstanding bonds B.

probability matrix Γ for the aggregate productivity shock.

We first calibrate the preference and life-cycle parameters using a non-stochastic version of the economy, in which the productivity shock is set to its average value $\bar{z} = 1$. We then specify the stochastic process for productivity z.

Let r_e denote the net return on equity in the non-stochastic economy, and let r_b be the net return on bonds. Our calibration strategy can be summarized as follows:

1. Fix risk aversion, σ, to a benchmark value of 3. We will conduct sensitivity analysis with respect to this parameter.
2. Set the life-cycle labor endowment profile $\{(1-\theta)\varepsilon_i(\bar{z})\}_{i=1}^{I}$ equal to the empirical 2007 SCF life-cycle profile for labor income, and the portfolio shares $\{\lambda_i\}_{i=1}^{I}$ equal to age-group-specific shares of risky assets in net worth from the SCF.
3. Set the capital share θ and the supply of bonds B so that the model generates realistic returns to risky and safe assets, r_e and r_f.
4. Set the life-cycle profile $\{\beta_i\}_{i=2}^{I}$ so that the model generates the 2007 SCF life-cycle profile for net worth documented in Section 2, given the other determinants of life-cycle saving: risk aversion σ, the profile for earnings $\{(1-\theta)\varepsilon_i(\bar{z})\}_{i=1}^{I}$, the exogenous portfolio shares $\{\lambda_i\}_{i=1}^{I}$, and the returns to risky and safe assets, r_e and r_b.

We now describe this calibration procedure in more detail. It delivers realistic (as measured by the SCF 2007) joint life-cycle profiles for earnings, net worth, and portfolio composition. These are the necessary ingredients for our calibrated OLG model to serve as a suitable laboratory for exploring the distributional impact of aggregate shocks.

Returns Following Piazzesi, Schneider, and Tuzel (2007), we target annual real returns on safe and risky assets of 0.75% and 4.75% per annum, where the latter is the return on an equally weighted portfolio of stocks returning 6.94% and housing returning 2.52%. Given that our period length is 10 years, this implies

$$1 + r_b = \frac{1}{q} = 1.0075^{10}. \tag{16}$$

and

$$1 + r_e = \frac{\theta - (1-q)B + p}{p} = 1.0475^{10}. \tag{17}$$

We now describe in more detail how we calibrate the life-cycle profiles, before discussing how to compute the pair (θ, B) that delivers the target returns r_e and r_b.

Life-cycle profiles As described in Section 2 we generate empirical life-cycle profiles by computing age-group-specific means in the SCF for holdings of safe assets, net worth, and non-asset income. The youngest age group corresponds to households aged $20-29$, and the sixth and oldest age group corresponds to households aged 70 and above.

The budget constraints of households in the deterministic version of the model can be written as

$$\begin{aligned} c_i &= (1-\theta)\varepsilon_i + R_i y_{i-1} - y_i \text{ for } i = 1, \ldots, N-1 \\ c_N &= (1-\theta)\varepsilon_N + R_N y_{N-1}, \end{aligned}$$

where y_i is total savings for age group i (net worth for age group $i+1$) and where the gross return on savings between age i and $i+1$ is given by

$$R_{i+1} = \lambda_i(1 + r_e) + (1 - \lambda_i) r_b.$$

We measure $\{(1-\theta)\varepsilon_i\}_2^I$ as ten times average annual labor income (as defined in Section 2) of age group i, $\{y_i\}_1^{I-1}$ as the average net worth of age group $i+1$, and $\{\lambda_i\}_1^{I-1}$ as the fraction of risky assets in aggregate net worth for age group $i+1$. Note that returns vary by age because λ_i is age-varying and $r_e > r_b$. Because agents in our model enter the economy with zero initial wealth, we re-categorize asset income for the youngest group in the SCF as labor income: thus we set $R_1 y_0 = 0$ and set $(1-\theta)\varepsilon_1$ equal to ten times average annual labor income for the youngest group plus the data value for $R_1 y_0$. We also set $y_I = 0$, since the oldest group does not save in our model.

Given the sequences $\{(1-\theta)\varepsilon_i\}$, $\{y_i\}$ and $\{\lambda_i\}$, the budget constraints imply a life-cycle consumption profile, $\{c_i\}$. This consumption profile can be used to back out the sequence of time discount factors that supports the age-varying profile for returns. In particular, in the non-stochastic version of the model, the household's intertemporal first-order condition implies that

$$\beta_{i+1} = \left(\frac{c_{i+1}}{c_i}\right)^{\sigma} \frac{1}{R_{i+1}}.$$

Note that the consumption profile is derived directly from household budget constraints and is pinned down by the *data* on labor income, net worth, and returns. Thus, the consumption profile is independent of preference parameters, and in particular independent of the choice for risk aversion, σ. However, supporting this consumption profile as an equilibrium outcome requires a discount factor profile $\{\beta_i\}$ that does depend on σ. For example, suppose that $\lambda_i = \lambda$ and thus $R_i = R$ for all i. Then, the larger is σ, the more sensitive is the implied profile for β_i to age variation in consumption.

Figure 5: **Life-Cycle Profiles for Consumption, Net Worth, and Labor Income**

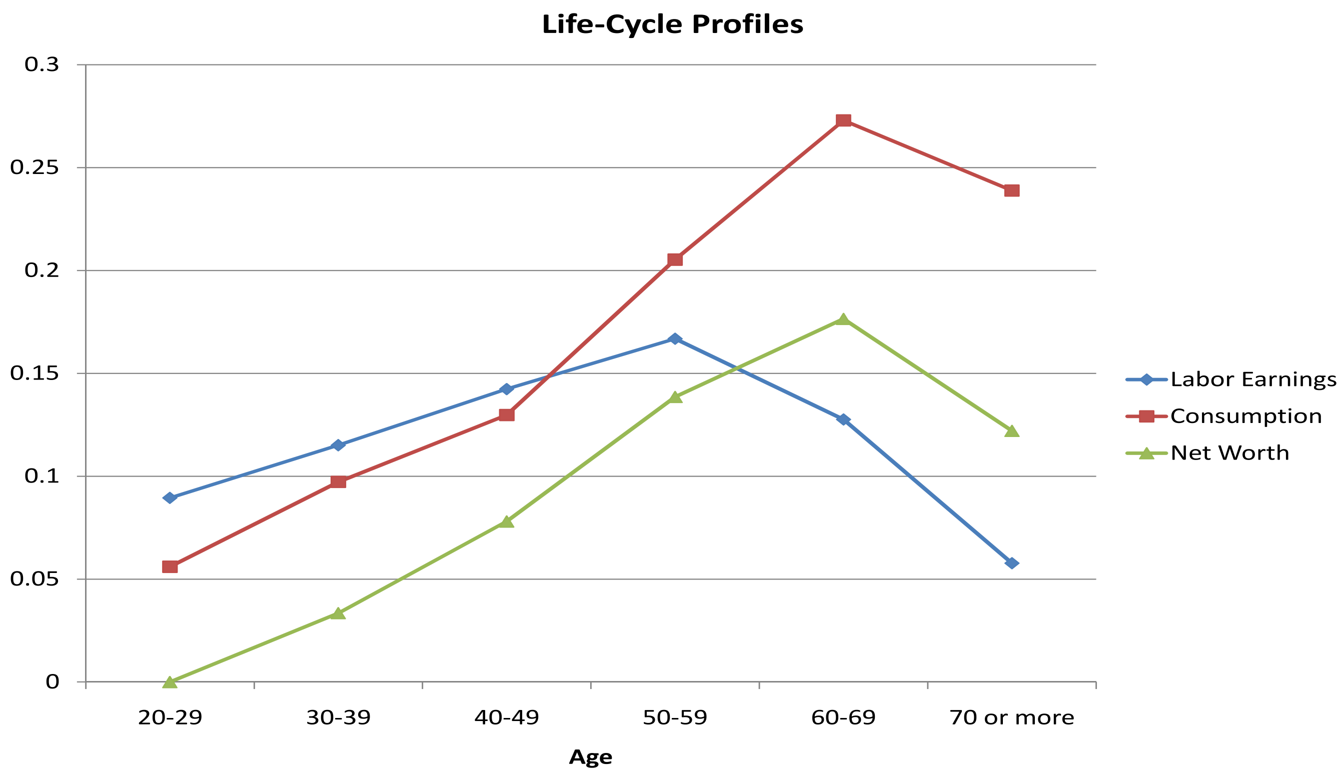
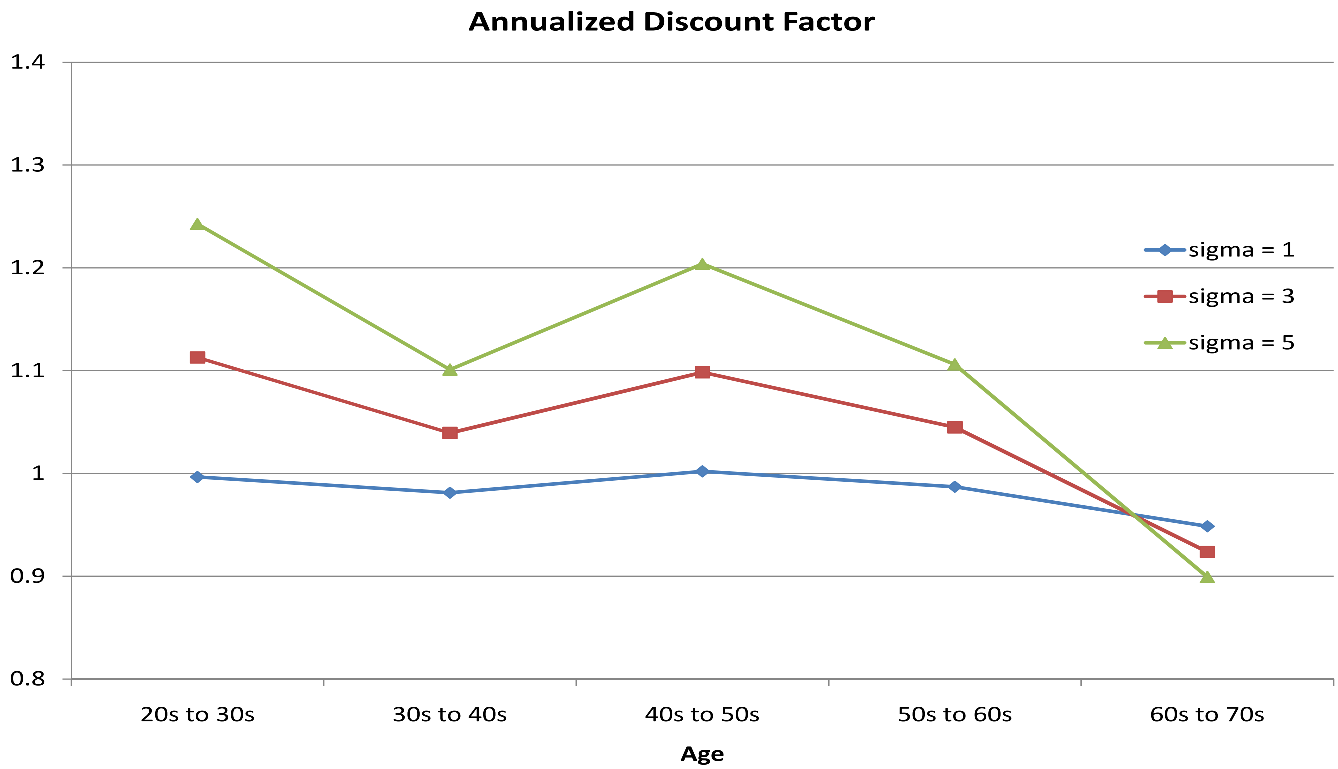


Figure 5 shows life-cycle profiles for consumption, net worth, and labor income from the non-stochastic version of our model, and Figure 6 displays the implied calibrated profile $\{\beta_i\}_1^I$, with β_1

normalized to 1. Note that β_i is generally larger than one. This reflects the fact that the data indicate strong growth in income and net worth over the life cycle between the 20-29 age group and the 50-59 age group. However, β_i should not be interpreted solely as capturing pure time preference: it also incorporates the effects of age variation in family size and composition on the marginal utility on consumption.

Figure 6: **Implied Discount Factors for Various Elasticities of Substitution**



Technology Because the model is calibrated to replicate observed life-cycle profiles for earnings, net worth, and portfolio composition, it will also replicate the ratio of aggregate safe assets to aggregate net worth, and the ratio of aggregate net worth to aggregate (10 year) labor income.[23] These ratios in the SCF are, respectively:

$$\frac{qB}{p+qB} = 0.082 \qquad \text{and} \qquad \frac{p+qB}{1-\theta} = \frac{1}{10} \times 7.84.$$

[23]In our model, each age group is assumed to be of equal size. Thus, given that we replicate SCF portfolios for each age group, the appropriate aggregate targets are simple unweighted averages across age groups. Because these age groups are not of exactly identical size in the SCF, these aggregate targets also do not correspond exactly to SCF population averages, but the differences turn out to be quite small.

The expressions for returns in equations (16-17) define q and p as functions of θ and B. Substituting in these functions, the two ratios above can be used to solve for θ and B. The solutions are $\theta = 0.3008$ and $B = 0.048$.

Aggregate risk We assume that the aggregate shock z takes one of two values, $z \in Z = \{z_l, z_h\}$. We set the ratio $\frac{z_l}{z_h} = 0.917$ so that in a recession output falls by 8.3%. This corresponds to the size of the gap that opened up between actual real GDP per capita and trend real GDP per capita between the NBER start and end dates for the recession (which we take to be 2007:4 and 2009:2 respectively).[24] We assume that z is *iid* over time, with the probability of $z = z_h$ equal to 0.85. Given that a period is 10 years, this implies that the expected duration of periods of high productivity is $10/0.15 = 66.7$ years, while the expected duration of periods of low productivity is $10/0.85 = 11.8$ years. Thus in our calibration, a great recession involves a very large and quite long-lasting decline in output, but is a fairly rare event. Finally, we normalize the z's so that average output is equal to one.

We think of the age profile for labor income calibrated to 2007 SCF data as corresponding to the age profile for earnings in normal times: thus $\varepsilon_i(z_h) = \varepsilon_i(\bar{z})$. We will consider two alternative models for how this age profile changes in a recession. In Section 6 we start by assuming $\varepsilon_i(z_l) = \varepsilon_i(z_h) = \varepsilon_i(\bar{z})$. In this case, recessions reduce labor income of all groups proportionately. In Section 7 we then consider an alternative calibration in which recessions also change the slope of the age profile, reducing labor income relatively more for the young and relatively less for the old. We will use micro data from the CPS to quantify this change in the life-cycle profile induced by a change in the aggregate state of the economy.

6 Results with Acyclical Age-Earnings Profiles

We now document the asset price and welfare implications of a large recession. We start with the economy in which labor income profiles $\{\varepsilon_i\}$ do not vary with z, since our results from the simple models in Section 4 provide us with clear guidance of what to expect, qualitatively, in this economy. Our baseline value for the intertemporal elasticity of substitution is one-third: $\sigma = 3$. We begin by describing the results for the one-asset economy in which only stocks are traded, and then move to the two other economies, the two-asset economy with exogenous portfolios and the model in which portfolios are endogenous.

[24]We define trend growth to be the average growth rate in GDP per capita between 1947:1 and 2007:4, which was 2.1% per year.

One-asset economy In order to display the dynamics of asset prices, we simulate each model economy, assuming that the sequence for aggregate productivity involves a long period of normal times, a recession in period zero, and a return to normal times in subsequent periods. Recall that the shock involves a 8.3% decline in productivity and that agents assign a 15% probability to a recession occurring in any period.

Figure 7 displays the implied realized paths for asset prices. In the one-asset economy, the stock price falls by 17% in the period of a recession. The finding that stock prices decline by more than output is consistent with the intuition developed in Section 4 that for an elasticity of substitution smaller than one, asset prices are more volatile than the underlying shocks. Prices decline by 17%, and the elasticity of prices to output is 2.06 (see Table 5).[25] Quantitatively, the decline in the model stock price is smaller than the change in U.S. household net worth from asset price peak (2007:2) to trough (2009:1), which was 32%. However, recall that the period length in our model is 10 years. In this light, we note that asset prices in the United States have partially recovered since early 2009, and by the end of 2010 net worth was only 22% below its mid-2007 peak (see Table 3).

In the recovery period after the shock, the stock price over-shoots, rising above its long-run value. This reflects the endogenous dynamics of the wealth distribution in the model. When the shock hits, older households, and especially those in the 60–69 year-old age group, sell additional equity to fund consumption. Thus, in the period after the recession, a larger share of aggregate wealth is held by younger cohorts, who are net buyers of the asset, while less is held by older cohorts, who are net sellers. This translates into higher net demand for equity in the period after the recession and thus a higher stock price.

Table 5 shows that the size of the asset price decline is declining in the intertemporal elasticity of substitution. With log consumption, prices move one-for-one with output, as in the simpler economies described in Section 4. As in the two-period economy, the price elasticity is a concave function of the coefficient of relative risk aversion: the magnitude of the additional price response as σ increases from 3 to 5 is much smaller than when σ goes from 1 to 3. Recall that in an analogous representative-agent economy with *iid* shocks, the elasticity is linear in (and in fact equal to) σ.

Table 6 displays the welfare consequences of a one-period model recession by age group, with

[25] Recall that the corresponding value for the two-period OLG economy described in Section 4 was very similar, at 1.97.

Figure 7: **Equilibrium Asset Prices for All Calibrated Economies**

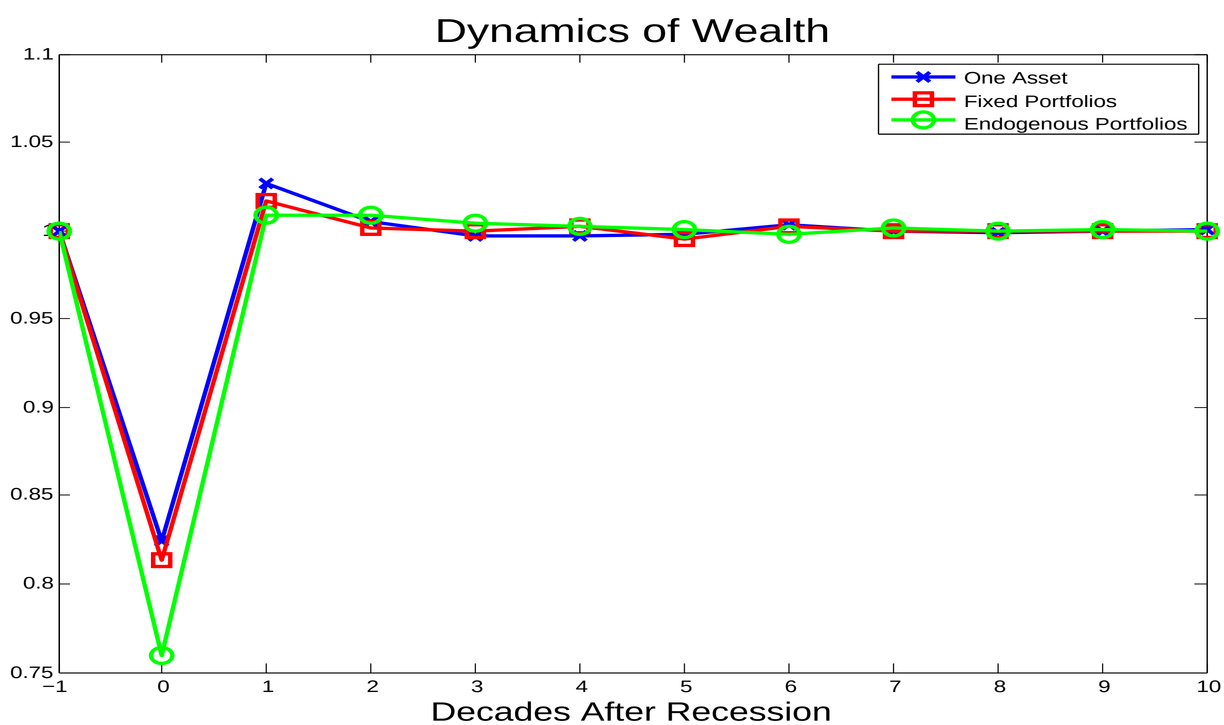


Table 5: **Relative Price Decline** $\xi = \left(\frac{\ln(p_0/p_{-1})}{\ln(z_0/z_{-1})}\right)$ **for Each Economy**

Economy	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
Single Asset	1.00	2.06	2.65
Fixed Portfolios			
–Stock	1.04	2.19	2.86
–Bond	1.01	2.55	3.49
–Wealth	1.04	2.22	2.91
Endogenous Portfolios			
–Stock	1.00	2.89	4.90
–Bond	1.00	2.94	4.95
–Wealth	1.00	2.90	4.90

$i = 1$ denoting the group that becomes economically active in the recession period.[26]

In the one-asset economy, the welfare consequences of a recession are monotone in age, with older generations suffering more. For $\sigma = 3$, the loss for the oldest households is equivalent to a 12.7% decline in consumption. In the model this age group finances roughly half of consumption from income (evenly split between dividends and non-asset income) and finances the other half by selling assets. In a recession, income (output) declines by 8.3% and asset prices decline by 17%, translating into a 12.7% decline in consumption for this age group.

The welfare loss is much smaller for younger age groups for two reasons. First, welfare losses are expressed in units of lifetime consumption, so relatively small losses for younger households partly reflect the fact that one period of recession accounts for a smaller fraction of remaining lifetime for the young, relative to the elderly. Second, the flip side of large capital losses for older households is that young households get to buy stocks at fire-sale prices. As the economy recovers (with high probability) in subsequent periods, stock prices bounce back, and younger generations enjoy substantial capital gains. The lower is the intertemporal elasticity of substitution (and thus the larger is the recession-induced asset price decline), the more unevenly are welfare costs distributed across generations, with larger losses for the old and smaller losses for younger households.

To isolate the role of asset price movements in allocating welfare costs across age groups from the effect of differential remaining lifetime, we also consider a thought experiment in which

[26] Welfare gains are measured as the percentage increase in consumption (in all periods and all stages of life) under a no-recession scenario needed to make households indifferent between the current aggregate state being z_l rather than z_h. Under both the recession and no-recession scenarios, output realizations after the initial period are drawn from the same *iid* distribution described in Section 5.

Table 6: **Expected Welfare Gains from One-Period Recession**

Age i	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
	Single-Asset Economy		
1	−1.62%	−0.78%	−0.38%
2	−2.04%	−1.19%	−0.82%
3	−2.43%	−1.29%	−0.67%
4	−3.57%	−2.75%	−2.17%
5	−5.46%	−6.26%	−6.57%
6	−8.30%	−12.66%	−15.16%
	Fixed Portfolio Economy		
1	−1.76%	−0.66%	−0.03%
2	−2.63%	−2.14%	−1.93%
3	−2.66%	−1.63%	−1.03%
4	−3.56%	−2.72%	−2.12%
5	−5.10%	−5.92%	−6.25%
6	−7.81%	−12.20%	−14.83%
	Endogenous Portfolio Economy		
1	−1.62%	0.33%	2.98%
2	−2.04%	−2.69%	−3.08%
3	−2.43%	−1.97%	−0.91%
4	−3.57%	−3.75%	−3.66%
5	−5.46%	−6.15%	−7.34%
6	−8.30%	−9.20%	−11.42%

a recession lasts for six periods. Because all generations now spend the rest of their lives in a recession, in this experiment all age variation in welfare losses is driven by asset price movements. Table 7 reports the change in *realized* welfare (in the usual consumption equivalent metric) from experiencing six consecutive periods of depressed output instead of six consecutive periods of normal output. A comparison of Tables 7 and 6 illustrates that the remaining lifetime effect is quantitatively important. For example, in the logarithmic utility case ($\sigma = 1$), welfare losses from a one-period recession (Table 6) are much smaller for the young than for the old, whereas in the corresponding six-period recession simulation, the welfare cost is the same for all age groups. This finding simply reflects the content of Proposition 1 that with logarithmic utility, consumption for *each* age group is proportional to the aggregate shock.

For $\sigma > 1$, Table 7 documents that the welfare costs of recessions are disproportionately borne by the old, even when the recession outlives all age groups. Moreover, the age variation in welfare losses is large: from -5.7% of consumption for the youngest age group up to -12.7% for the oldest cohort, indicating that asset price movements are quantitatively very important for the age distribution of welfare losses.

Exogenous portfolios economy We now turn to the benchmark version of the model with two assets, in which we exogenously impose the substantial heterogeneity in portfolio composition across age groups observed in the Survey of Consumer Finances (see Section 2). Recall that in this version of the model, older households hold a significant share of their wealth in safe assets, while younger households are leveraged and thus more exposed to asset price declines. Again, Figure 7 shows the time path for aggregate net worth for this economy, while Table 5 breaks down the price declines for stocks and bonds in the period of the recession. Note that each bond pays off one unit of consumption in the period of the recession, as in every other period: this is the definition of a safe asset. At the same time, the equilibrium price of new bonds (and stocks) must both adjust so that markets clear, given agents' (optimal) saving choices and the (suboptimal) portfolio shares they are forced to adopt.

The two-asset model generates a slightly larger decline in asset values in a recession than the stock-only model described above. To understand why, recall that because younger households are leveraged in this economy, they suffer a larger decline in wealth in the recession period relative to the one-asset economy. Thus when the recession hits, younger households require an even larger decline in asset prices (relative to the one-asset model) in order to be willing to absorb the extra assets older households are selling.

Table 7: **Realized Welfare Gains from Six-Period Recession**

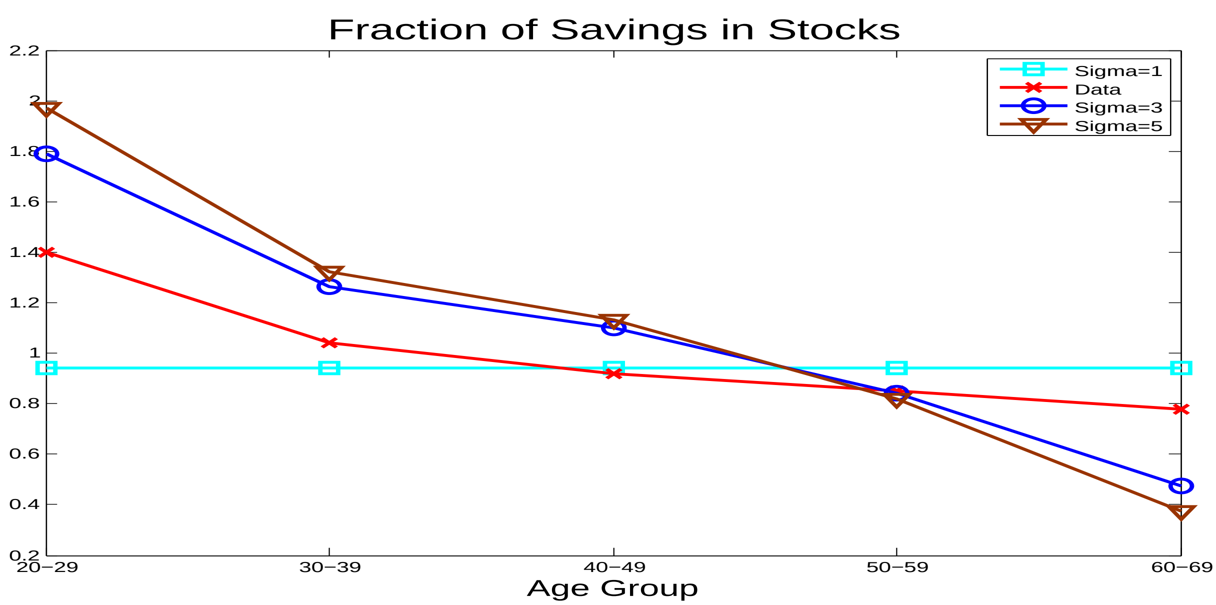
Age i	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
	Single-Asset Economy		
1	−8.30%	−5.48%	−4.00%
2	−8.30%	−5.99%	−4.29%
3	−8.30%	−6.90%	−4.90%
4	−8.30%	−9.27%	−9.22%
5	−8.30%	−11.81%	−14.42%
6	−8.30%	−12.66%	−15.16%
	Fixed Portfolio Economy		
1	−8.55%	−5.82%	−4.00%
2	−8.68%	−7.11%	−5.68%
3	−8.31%	−7.24%	−5.55%
4	−8.08%	−8.97%	−9.17%
5	−7.92%	−11.21%	−13.88%
6	−7.81%	−12.20%	−14.83%
	Endogenous Portfolio Economy		
1	−8.30%	−4.51%	3.93%
2	−8.30%	−8.60%	−7.69%
3	−8.30%	−7.70%	−4.94%
4	−8.30%	−9.00%	−9.51%
5	−8.30%	−9.43%	−11.35%
6	−8.30%	−9.20%	−11.42%

Table 6 indicates larger welfare losses for households aged 30–49 in the two-asset economy, relative to the one-asset model, while households aged 60 and older fare better in the two asset model. These differences are readily interpreted: because 30–49 year-old households are now leveraged, they take a bigger hit in the period of the crisis, while households aged 60 and older have more safe assets and suffer less. Finally, because of the larger asset price declines, newborn households fare slightly better in the benchmark model than in the one-asset economy: they face only modest welfare losses.

Endogenous portfolios economy We now move to our final economy, the economy in which households choose portfolios freely at each age. Recall that there are only two values for the aggregate shock, and thus the presence of two assets ensures complete risk sharing within the set of agents that is active in any two successive periods. However, aggregate shocks will still have asymmetric effects on the welfare of different generations, since the overlapping-generations structure of the model prevents the cohorts who are alive at a particular point in time from sharing aggregate risk with past or future generations.

In Figure 8 we plot how equilibrium portfolios in this model vary with age.[27] We do this for three values for risk aversion, $\sigma = 1$, 3, and 5.

Figure 8: **Portfolio Choices of Households in Data and Models**



[27]These are the portfolio shares after a long sequence of good shocks $z = z_h$.

In the log case ($\sigma = 1$) portfolios are age-invariant (see again Proposition 1). For $\sigma > 1$, aggregate asset values become more volatile than output, for the usual reason: when agents are less willing to substitute intertemporally, prices must adjust more to induce agents to tolerate fluctuations in consumption. Because asset prices fluctuate by more than output, younger households who have little wealth relative to earnings require a more leveraged portfolio (a higher equity-to-debt ratio) to face the same exposure to aggregate risk as older (and wealthier) households. Thus, the equilibrium share of equity in household portfolios is decreasing with age, consistent with the downward-sloping profile for the share of risky assets in net worth observed in the SCF. While this qualitative pattern is an important success for the model, for $\sigma = 3$ or $\sigma = 5$ age variation in portfolio composition is larger in the model than in the data. Thus, for these values for σ empirically observed portfolios do not vary quite enough with age to share risk efficiently across generations: older Americans are over-exposed to aggregate risk in the data, relative to what is optimal from the perspective of the model.

Figure 7 indicates (for $\sigma = 3$) that in the endogenous portfolio model, asset prices decline by 24% and thus by more than in the other two model economies. This reflects the fact that with endogenous portfolios, younger households are more heavily leveraged. Thus, compared to the other economies, younger households take a bigger hit in the period of the shock. This translates into a larger decline in equilibrium consumption for younger households and necessitates a larger fall in asset prices to preserve asset market clearing. Table 5 indicates that price changes in the complete markets economy are very similar to those that would emerge in a representative-agent economy, with a price to output decline elasticity roughly equal to σ (see Section 4.1).

Turning to welfare (Table 6), when portfolios are endogenous the oldest households suffer less (relative to the other economies) in a recession, because they hold more safe assets. Conversely, younger households (e.g., 30-49 year olds) generally fare worse, due to greater leverage. The differences in welfare losses by age between the models with exogenous and endogenous portfolios offer one metric for how far observed U.S. portfolios are from those that deliver efficient intergenerational risk sharing through the trade of financial assets.

Note that there is no way for agents alive in the period before the recession to share risk with the new cohort that enters the economy in the recession period itself. Since asset prices fall so significantly in the endogenous portfolios economy, the youngest households are actually better off entering the economy during a recession compared to normal times. For $\sigma = 3$ their welfare gain is equivalent to a one third of 1% increase in lifetime consumption, in an economy that features an empirically realistic fall in asset prices and an endogenously chosen portfolio allocation reasonably close to the data.

Thus the endogenous portfolios economy delivers an interesting twist: the fact that existing generations diversify aggregate risk efficiently magnifies asset price declines. These larger asset price declines in turn work to decrease effective risk sharing between generations alive in the previous period and new labor market entrants.

7 Results with Cyclical Age-Earnings Profile

So far we have assumed that in a recession, labor incomes of all age groups decline proportionally to the fall in output and the aggregate wage. However, in reality young households are especially vulnerable to unemployment, while non-cyclical social security benefits are an important part of the non-asset income of older age groups. To the extent that recessions have a disproportionate direct negative impact on younger households for these reasons, this will work to offset the benefits the young enjoy from sharp falls in asset prices. To quantify the importance of this offsetting and empirically relevant effect, we conduct an experiment in which recessions have asymmetric effects on the non-asset income of different age groups.

Because we have no recession data on income from the SCF we turn to the March CPS to estimate how the Great Recession affected labor incomes across age groups. Our CPS measure of non-asset income is conceptually close to the SCF measure of labor income we previously used to calibrate the life-cycle profile for model earnings . This measure includes all CPS income components except for dividends, interest, rents, and one-third of self-employment income. For the year 2007, the SCF and CPS life-cycle profiles for labor income align quite closely.[28] In calibrating the alternative experiment in this section, we assume that the life-cycle profile for labor productivity when aggregate productivity is high $\{\varepsilon_i(z_h)\}_{i=1}^{I}$ corresponds exactly to the 2007 SCF-based profile used in the baseline version of the model. We then measure the percentage decline in labor income by age group in the CPS between 2007 and 2009 (survey years 2008 and 2010).[29] Finally, we choose a life-cycle profile for the recession state $\{\varepsilon_i(z_l)\}_{i=1}^{I}$ such that when the model economy transits into a recession, it replicates the differentials across age groups in percentage labor income declines in the CPS. The declines in earnings by age group fed into the model are then as follows:

From the CPS data we observe that indeed the age-earnings profile shifts in favor of older

[28] The main difference is that the SCF shows higher non-asset income for the 50-59 and 60-69 year old age groups.

[29] The differentials across age groups turn out to be very similar when considering a much narrower measure of labor income that only includes wage and salary income.

Table 8: Recession Decline in Model Earnings

	(%)
Per capita earnings	-8.3
Age of Head	
20-29	-11.0
30-39	-11.9
40-49	-8.8
50-59	-8.9
60-69	-6.2
70+	+1.6

generations: the decline in earnings in the recession disproportionately impacts younger generations. It is therefore to be expected that the welfare consequences are less favorable for the young and less detrimental for the old, relative to our benchmark in the previous section.

We now repeat the experiments from Section 6 with the stochastically fluctuating age-earnings profile. Figure 9 plots the endogenously chosen asset portfolios for $\sigma = 3$ in this version of the model (labelled "age-specific shocks") and compares it to the portfolio allocations in the data and in the previous version of the model in which aggregate shocks were age neutral. We observe that with age-specific shocks the portfolio share in stocks declines less steeply with age, and therefore matches the data better, relative to the case with age-neutral shocks. The reason for the lower leverage of the young generation is that these households now face more direct exposure to aggregate income risk (their labor incomes fall especially sharply in economic downturns), and thus they are less willing to take on additional indirect exposure by holding risky equities. The reverse is true for older households.[30]

We now briefly report on the welfare consequences of a one-period recession in this economy. Appendix G reports asset price elasticities and the additional welfare results.

From Table 9 (the counterpart of Table 6) we observe that the welfare losses of the elderly from a recession tend to be smaller now and that the young lose more. This tilting of welfare losses across the age distribution is due to the fact that now the young face significantly larger income falls in recessions.[31] Now young households, the net buyers of assets, have especially low income in

[30]In fact, for $\sigma = 1$ when previously portfolios were constant over the life cycle, the higher risk of labor incomes for the young now induces them to hold (counterfactually) a *safer* portfolio than older households.

[31]The asset price decline is steeper in this version of the model, relative to the one with age-neutral productivity shocks. Compare Table 5 with Table A-2 in the Appendix.

Figure 9: **Portfolio Choices of Households in Data and Models**

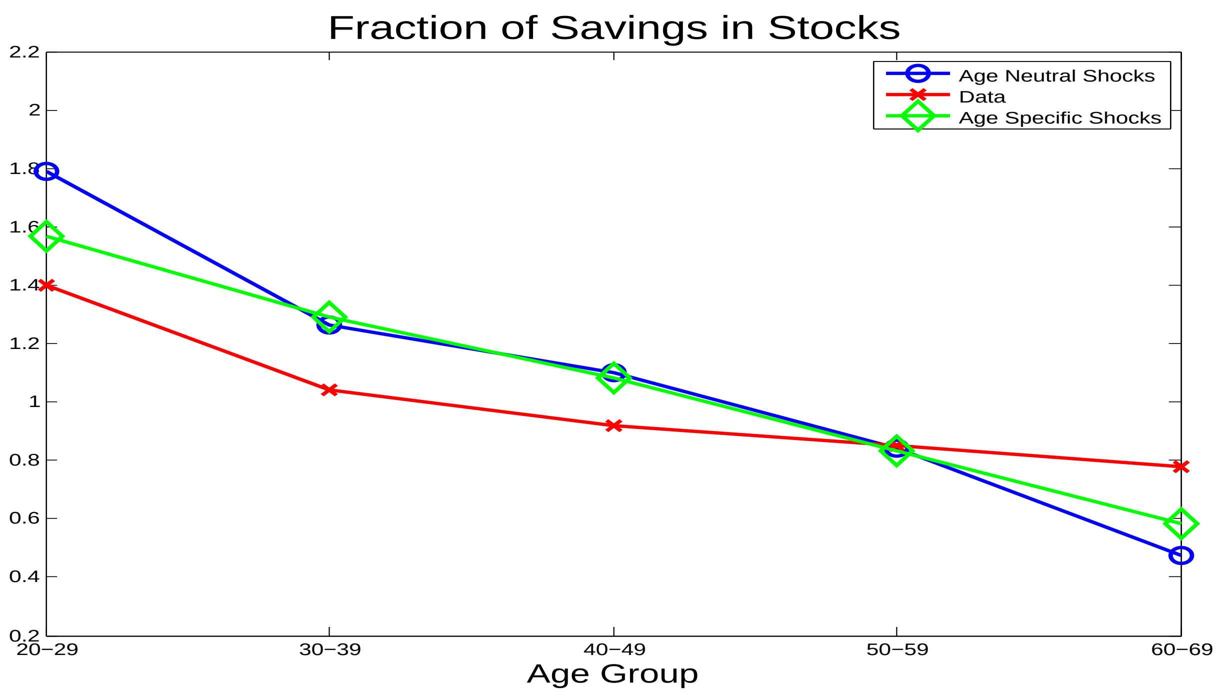


Table 9: **Expected Welfare Gains from One-Period Recession (cyclical age-earnings profile)**

Age i	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
	Single-Asset Economy		
1	−3.69%	−1.30%	−0.80%
2	−4.36%	−2.05%	−1.60%
3	−3.51%	−1.13%	−0.44%
4	−4.36%	−2.96%	−2.37%
5	−4.72%	−6.05%	−6.47%
6	−5.95%	−11.34%	−14.08%
	Fixed Portfolio Economy		
1	−3.71%	−1.20%	−0.45%
2	−4.75%	−3.08%	−2.83%
3	−3.69%	−1.49%	−0.81%
4	−4.30%	−2.93%	−2.32%
5	−4.45%	−5.68%	−6.12%
6	−5.38%	−10.69%	−13.57%
	Endogenous Portfolio Economy		
1	−2.43%	−0.44%	2.26%
2	−2.22%	−2.79%	−3.48%
3	−3.09%	−2.80%	−1.83%
4	−3.45%	−3.94%	−4.04%
5	−4.57%	−5.36%	−6.45%
6	−7.65%	−8.05%	−9.73%

recessions and therefore require larger price declines in order to be willing to buy the assets. This larger price decline partially compensates the young for the larger income declines they experience in recessions. Overall, for $\sigma = 3$ or $\sigma = 5$, welfare losses from a one-period recession still increase sharply with age.[32] When $\sigma = 5$ and portfolios are chosen endogenously, agents are still better off when becoming economically active during a recession rather than during normal times, despite the fact that their labor income is 11% lower during the first 10 years of their life.[33]

Overall, we conclude that our results from the previous section are qualitatively robust: welfare losses increase with age, and the oldest households lose the most from a severe recession. In addition, if the asset price decline is large relative to the fall in output and earnings (as was the case in the Great Recession), then the youngest households continue to benefit from becoming economically active in a recession, despite the sharp decline in labor income they experience.

8 Conclusion

In this paper we have analyzed the distributional consequences of a large recession across different age cohorts. In a quantitative version of our stochastic overlapping-generations economy restricted to match life-cycle income and asset profiles from the SCF, we find that older households suffer large welfare losses from a severe, long-lasting recession. Young households, in contrast, lose less and might even benefit from an economic downturn. The key statistic determining these welfare consequences is the size of the equilibrium asset price decline, relative to the fall in wages and output. If households have a low intertemporal elasticity of substitution, then middle-aged households are keen to sell their assets in a downturn in order to smooth consumption, putting additional downward pressure on asset prices. This generates larger welfare losses for older households and potential welfare gains for households that become economically active during the recession and buy assets at fire sale prices.

Knowledge about how the welfare costs of recessions are distributed across the age distribution can help inform the discussion of the appropriate policy response. Many of the policies that have been implemented in response to the Great Recession have obvious distributional consequences.

[32] Table A-3 in the appendix reports welfare results from a six-period recession in this version of the model. For $\sigma = 1$, the welfare costs of a very prolonged slump are now declining with age, while they were age-invariant in the version with age-neutral shocks (recall Table7). However, for $\sigma = 3$ or 5, welfare costs are still generally increasing with age.

[33] The large fall in labor income for the young in recessions partly reflects lower hours worked. To the extent that lower market hours translate into additional and valued leisure, our welfare estimates might actually understate welfare gains for the young.

First, financing an increasing share of the government budget through debt rather than taxation shifts the tax burden toward the young and future generations, benefitting older age groups. Second, the Troubled Asset Relief Program (TARP) and large-scale asset purchases by the Federal Reserve (LSAP) were policies that were more or less explicitly designed to support asset prices. To the extent that these policies were successful, these were policies that also benefitted older and therefore wealthier households.[34] From the perspective of the very asymmetric welfare results documented in this paper, a *distributional* argument can be made in favor of such policies. We view a full quantitative exploration of this hypothesis as an obvious next step in the analysis, but beyond the scope of the current paper.

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[34] Another important policy response was a dramatic extension of unemployment benefits. On the one hand, this favored younger workers, who bore the brunt of rising unemployment. On the other, older workers are more likely to be eligible to receive benefits. In 2010, the numbers of insured unemployed in the 25-34, 35-44, and 45-54 age groups were very similar (Department of Labor, Employment and Training Administration).

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Appendix

A Economy with Endogenous Portfolios and Complete Markets

Since in our applications the number of values the aggregate state can take is two, markets are sequentially complete when households can freely trade a bond and a stock. We exploit this for the purposes of characterizing equilibrium prices and quantities numerically. In particular, we solve the model assuming agents trade state-contingent claims to capital. We then reconstruct the equilibrium prices of conventional stocks and bonds as additional (effectively redundant) assets.[35]

Let $a_i(z')$ be shares of stock purchased by a household of age i. These shares represent a claim to fraction $a_i(z')$ of the capital stock if and only if aggregate state z' is realized in the next period. The state of the economy is the distribution of shares of stock A, given the current period shock z. We denote the state-contingent stock prices $P(z, A, z')$.

With this asset market structure the maximization problem of the households now reads as

$$v_i(z, A, a) = \max_{c \geq 0, a'(z')} \left\{ u(c) + \beta_{i+1} \sum_{z' \in Z} \Gamma_{z,z'} \; v_{i+1}\left(z', A'(z'), a'(z')\right) \right\} \tag{A-1}$$

$$\text{s.t.} \qquad c + \sum_{z'} \left[a'(z') - a\right] \; P(z, A, z') = \varepsilon_i(z) w(z) + d(z) a \tag{A-2}$$

$$A'(z') = G(z, A, z') \tag{A-3}$$

with solution $c_i(z, A, a)$, $a'_i(z, A, z', a)$.

Definition 2. *A recursive competitive equilibrium with complete markets, are value and policy functions* $\{v_i, c_i, a'_i\}$*, pricing functions* w, d, P *and an aggregate law of motion* G *such that:*

1. *Given the pricing functions and the aggregate law of motion, the value functions* $\{v_i\}$ *solve the recursive problem of the households and* $\{c_i, a'_i\}$ *are the associated policy functions.*

2. *Wages and dividends satisfy*

$$w(z) = (1 - \theta) z \qquad \text{and} \qquad d(z) = \theta z \tag{A-4}$$

[35] When securities pay off only in specific states of the world, the household portfolio choice problem is relatively simple. Solving the portfolio choice problem between bonds and stocks is much harder because the returns on the two assets are very collinear.

3. Markets clear

$$\sum_{i=1}^{I} c_i(z, A, A_i) = z \tag{A-5}$$

$$\sum_{i=1}^{I} a_i'(z, A, z', A_i) = 1 \qquad \forall z' \in Z. \tag{A-6}$$

4. The aggregate law of motion is consistent with individual optimization

$$\begin{aligned} G_1(z, A, z') &= 0 \\ G_{i+1}(z, A, z') &= a_i'(z, A, z', A_i) \qquad \forall z',\ i = 1, ..., I-1. \end{aligned} \tag{A-7}$$

We now describe how we reconstruct returns and prices for conventional stocks and bonds, given the prices of state-contingent shares, exploiting the equivalence between the two market structures when the aggregate shock takes only two values. Let $W(z, A)$ denote the value of the (unlevered) firm after it has paid out dividends. This is equal to the price of all state-contingent shares:

$$W(z, A) = \sum_{z' \in Z} P(z, A, z'). \tag{A-8}$$

In the presence of state-contingent shares, risk-free bonds and levered stocks are redundant assets, but they can still be priced. We now compute these prices $q(z, A)$ and $p(z, A)$ as functions of the state-contingent prices $P(z, A, z')$ and $W(z, A)$. There are two ways of securing one unit of the good unconditionally in the next period. One could either buy one unit of the risk free bond at price $q(z, A)$ or instead buy a bundle of state-contingent shares for each possible z', setting the state-specific quantity to $1/\left[W(z', G(z, A, z')) + \theta z'\right]$ so as to ensure a gross payout of one in each state. A no arbitrage argument implies that the cost of the two alternative portfolios must be identical:

$$q(z, A) = \sum_{z'} \frac{P(z, A, z')}{W(z', G(z, A, z')) + \theta z'}. \tag{A-9}$$

With the bond price in hand, the stock price can immediately be recovered from the condition that the value of the unlevered firm (in the economy with state-contingent shares) must equal the value of levered stocks and risk free bonds:

$$p(z, A) = W(z, A) - q(z, A)B. \tag{A-10}$$

B Computational Appendix

Even for a moderate number of generations the state space is large: $N - 2$ continuous state variables (plus z). Since we want to deal with big shocks local methods should be used with caution. Consequently in the economies with either only one asset or exogenous portfolios, we have used global methods based on sparse grids, as in Krueger and Kubler (2004, 2006) (see section B.1). For the economies with portfolio choice, we have solved the equilibrium using linear methods; (see Section B.2). Section B.3 discusses the accuracy of our solution algorithms, especially the perhaps more suspect one based on linearization. As we document there, the findings when using linear methods for the exogenous portfolio economy do not differ noticeably from those using global methods. Moreover, when we apply linear methods to the log case for which there is a closed form solution, the approximated allocations almost completely coincide with the exact ones.

B.1 Sparse Grid Approximations

Relative to the methods described in Krueger and Kubler (2004, 2006), there are two additional complications in the present model. The first is that, while the sparse grids used there are subsets of $(I-1)-$dimensional cubes, wealth shares used in this paper are defined on the $(I-2)$ dimensional simplex. We deal with this issue by defining the state space in levels of wealth rather than in shares, and then we map a generation's level of wealth to a share when evaluating the Euler equations. The second complication is that the prices of the assets cannot be read off of first-order conditions in this model (one can do this in production economies where factor prices equal marginal productivities), but must instead adjust so that excess demand for stocks and bonds is jointly zero. We therefore use an algorithm that nests an outer and an inner loop. In the outer loop we iterate over prices, and in the inner loop we iterate over policies. Specifically, we use the following algorithm to solve for the equilibrium in the single-asset economy and the fixed portfolio, two-asset economy:

1 Solve for the steady state prices and wealth *levels*: $\overline{p}, \overline{q}, \overline{W} = (\overline{W}_2, \overline{W}_3, ..., \overline{W}_{I-1}, \overline{W}_I)$. As

described above we work with an endogenous state space of dimension $I-1$ rather than $I-2$, and then map wealth levels into wealth shares.

2 Create a sparse grid around the steady state wealth distribution. Call this grid $\mathbb{W}$. We verify that the hyper-cube defined by the bounds of the grid contains the ergodic set of wealth levels.

3 We start with the outer loop over prices (this loop was unnecessary in Krueger and Kubler (2004)). At an outer loop iteration n we have guesses from the previous iteration for Chebyshev coefficients for prices, $(\alpha_z^{p,n}, \alpha_z^{q,n})$, which are used to compute the values of prices (p, q) for each realization of z and each point $W \in \mathbb{W}$ in the endogenous state space. We denote the vector of price values by $(\psi_{z,W}^{p,n}, \psi_{z,W}^{q,n})_{W\in\mathbb{W}}$. The Chebyshev coefficients $(\alpha_z^{p,n}, \alpha_z^{q,n})$ also determine the pricing function on the entire state space, denoted by $(\widehat{\psi}_z^{p,n}, \widehat{\psi}_z^{q,n})$, somewhat abusing notation.[36]

4 Given approximate price functions in the inner loop we iterate over household policies. In this loop we generate both the savings policy and the law of motion for the wealth distribution consistent with approximate price functions $(\widehat{\psi}_z^{p,n}, \widehat{\psi}_z^{q,n})$. The savings policy is indexed by generation and *current* state z, and so the current guess of the savings policy function at policy iteration m when the price iteration is n is determined by Chebyshev coefficients of the form $(\alpha_{z,i}^{y,n,m})$. These can be used to compute the optimal savings level at grid point W and is denoted by $(\psi_{z,i,W}^{y,n,m})$. As in the previous step the Chebyshev coefficients also determine the entire approximating savings functions $(\widehat{\psi}_{z,i}^{y,n,m})$. The law of motion for wealth is a function of savings, current prices, and future prices; it must therefore be indexed by current state z, generation i, and future state z'. Similarly, the Chebyshev coefficients $(\alpha_{z,i,z'}^{G,n,m})$ are used to compute the law of motion $(\psi_{z,i,z',W}^{G,n,m})$ for all points $W \in \mathbb{W}$ and to generate the approximating functions $\widehat{\psi}_{z,i,z'}^{G,n,m}$.

5 At this point we loop over each value of z and each point in $W \in \mathbb{W}$ and solve the $I-1$ Euler equations for the $I-1$ savings, $y_{i,z,W}$. The Euler equations that we solve to generate the updated savings are:

$$u'(c_i(y_{i,z,W}; W, z)) = \beta_i \mathbb{E}_z \widehat{R}_i^{n,m}(z') u'(\widehat{c}_{i+1}(W_+(z'), z'^{n,m}),$$

[36]Note that this notation implies $\widehat{\psi}_z^{p,n}(W) = \psi_{z,W}^{p,n}$.

where

$$\widehat{R}_i^{n,m}(z') = \left(\lambda_i \frac{\widehat{\psi}_{z'}^{p,n}(\psi_{z,z',W}^{G,n,m}) + \theta z' + \overline{B}\widehat{\psi}_{z'}^{q,n}(\psi_{z,z',W}^{G,n,m}) - \overline{B}}{\psi_{z,W}^{p,n}} + \frac{(1-\lambda_i)}{\psi_{z,W}^{q,n}}\right)$$

$$W_+(z') = \begin{bmatrix} W_{+,2}(z') \\ ... \\ W_{+,I} \end{bmatrix} = \begin{bmatrix} \left(\lambda_1 \frac{\widehat{\psi}_{z'}^{p,n}(\psi_{z,z',W}^{G,n,m}) + \theta z' + \overline{B}\widehat{\psi}_{z'}^{q,n}(\psi_{z,z',W}^{G,n,m}) - \overline{B}}{\psi_{z,W}^{p,n}} + \frac{(1-\lambda_1)}{\psi_{z,W}^{q,n}}\right) y_{1,z,W} \\ ... \\ \left(\lambda_{I-1} \frac{\widehat{\psi}_{z'}^{p,n}(\psi_{z,z',W}^{G,n,m}) + \theta z' + \overline{B}\widehat{\psi}_{z'}^{q,n}(\psi_{z,z',W}^{G,n,m}) - \overline{B}}{\psi_{z,W}^{p,n}} + \frac{(1-\lambda_{I-1})}{\psi_{z,W}^{q,n}}\right) y_{I-1,z,W} \end{bmatrix}$$

$$\begin{aligned}
c_1(y_{1,z}; W, z) &= (1-\theta)z\epsilon_1(z) - y_{1,z,W} \\
& \quad \text{for } i = 1, ..., I-1: \\
c_i(y_{i,z}; W, z) &= (1-\theta)z\epsilon_i(z) + \left(\widehat{\psi}_{z,W}^{p,n} + \theta z + \overline{B}\ \widehat{\psi}_{z,W}^{q,n}\right) \frac{W_i}{\sum_{l=2}^{I} W_l} - y_{i,z,W} \\
& \quad \text{for } i = 1, ..., I-2: \\
\widehat{c}_{i+1}(W_+(z'), z'^{n,m}) &= (1-\theta)z\epsilon_{i+1}(z') + W_{+,i+1}(z') - \widehat{\psi}_{z',i+1}^{y,n,m}(W_+(z')) \\
\widehat{c}_I(W_+(z'), z'^{n,m}) &= (1-\theta)z\epsilon_I(z') + W_{+,I}(z').
\end{aligned}$$

Note that in the calculation of the c_i's we switch from using wealth levels to using wealth shares to satisfy the requirement that only the latter are truly minimal state variables. This is another difference with the previous use of Smolyak polynomials in Krueger and Kubler (2004).

With the new savings in hand, we update the savings policies as $\psi_{z,i,W}^{y,n,m+1} = y_{i,z,W}$ and the law of motion for wealth levels is updated via

$$\psi_{z,i,z',W}^{G,n,m+1} = \left(\lambda_i \frac{\widehat{\psi}_{z'}^{p,n}(\psi_{z,z',W}^{G,n,m}) + \theta z' + \overline{B}\widehat{\psi}_{z'}^{q,n}(\psi_{z,z',W}^{G,n,m}) - \overline{B}}{\psi_{z,W}^{p,n}} + \frac{(1-\lambda_i)}{\psi_{z,W}^{q,n}}\right) \psi_{i,z,W}^{y,n,m+1}.$$

6 If $\max_{W\in\mathbb{W}} \max_z \max_{z'} \max_i |\psi_{z,i,W}^{y,n,m+1} - \psi_{z,i,W}^{y,n,m}|$ is below an acceptable tolerance level then we proceed to step [7]. Otherwise we return to [4] with the updated savings functions and aggregate law of motion for wealth denoted by $m+1$ above. We now generate new Chebyshev coefficients $\alpha_{z,i}^{y,n,m+1}$ by solving the system $\widehat{\psi}_{z,i}^{y,n,m+1}(W) = \psi_{z,i,W}^{y,n,m+1}$ for each $W \in \mathbb{W}$.

7 For each point in the grid $\mathbb{W}$ and each value of z, we check the market clearing conditions.

If:

$$\max_{W\in\mathbb{W}}\max_{z}|\sum_{i=1}^{I-1}\frac{\psi^{y,n,m+1}_{z,i,W}\lambda_i}{\psi^{p,n}_{z,W}}-1|+|\sum_{i=1}^{I-1}\frac{\psi^{y,n,m+1}_{z,i,W}(1-\lambda_i)}{\psi^{q,n}_{z,W}}-\overline{B}|$$

is below an acceptable tolerance level we stop. Otherwise, we update our guess of prices $\psi^{p,n+1}_{z,W}=\sum_{i=1}^{I-1}\lambda_i\psi^{y,n,m+1}_{z,i,W}$ and $\psi^{q,n+1}_{z,W}=\sum_{i=1}^{I-1}(1-\lambda_i)\psi^{y,n,m+1}_{z,i,W}/\overline{B}$ and return to step [3]. We now generate new Chebyshev coefficients $(\alpha^{p,n+1}_{z,i},\alpha^{q,n+1}_{z,i}$ by solving $\widehat{\psi}^{p,n+1}_{z}(W)=\psi^{p,n+1}_{z,W}$ and $\widehat{\psi}^{q,n+1}_{z}(W)=\psi^{q,n+1}_{z,W}$ for each value of z and each $W\in\mathbb{W}$.

B.2 Linearization

In the endogenous portfolios economy we look for affine functions[37] for state contingent laws of motion of wealth shares, H^G, and prices, H^p, that approximate the equilibrium functions p and G. Letting i stand for a generation, j stand for the index of the current state, and k stand for the index of the contingency state for the following period, the desired output of the algorithm are pricing and policy functions of the form

$$\begin{aligned}H^G_{ijk}(A) &= \overline{A}_{i+1,k}+\eta^G_{ijk1}+\sum_{m=2}^{I}\eta^G_{ijkm}A_m\\ H^p_{jk}(A) &= \overline{p}_{jk}+\eta^p_{jk1}+\sum_{m=2}^{I}\eta^p_{jkm}A_m,\end{aligned}$$

where $\overline{A}_{i+1,k}$ and $\overline{p}_{jk}$ are the steady state wealth shares and asset prices respectively (asset prices in steady state are just the conditional probabilities).

We write these functions succinctly in matrix notation. We stack policies first by generation, then by contingency state, then by current state. Letting $\vec{A}$ be the vector of wealth shares and J

[37] Note that, because we have a Markov chain, the functions are affine and not log-linear as is typical in the literature that uses AR(1) processes.

the number of exogenous aggregate states ($J = 2$ in the application), this yields:

$$H^G(A) = \begin{bmatrix} H^G_{1,1,1}(A) \\ ... \\ H^G_{I-1,1,1}(A) \\ H^G_{1,1,2}(A) \\ ... \\ H^G_{I-1,1,2}(A) \\ ... \\ H^G_{I-1,J,2}(A) \\ ... \\ H^G_{I-1,J,J}(A) \end{bmatrix} = \begin{bmatrix} \overline{A}_{21} \\ ... \\ \overline{A}_{I,1} \\ \overline{A}_{22} \\ ... \\ \overline{A}_{I,2} \\ ... \\ \overline{A}_{2,J} \\ ... \\ \overline{A}_{I,2} \end{bmatrix} + \eta^G \cdot \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix}$$

Then prices can be written by stacking first by contingency and then by current state:

$$H^p(A) = \begin{bmatrix} H^p_{11}(A) \\ ... \\ H^p_{1J}(A) \\ ... \\ H^p_{J1}(A) \\ ... \\ H^p_{JJ}(A) \end{bmatrix} = \begin{bmatrix} \overline{p}_{11} \\ .... \\ \overline{p}_{1J} \\ ... \\ \overline{p}_{J1} \\ ... \\ \overline{p}_{JJ} \end{bmatrix} + \eta^p \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix}$$

Stacking these equations gives a system of $I \times J \times J$ functions of the form:

$$\begin{bmatrix} H^G(A) \\ H^p(A) \end{bmatrix} = \begin{bmatrix} \overline{A} \\ \overline{p} \end{bmatrix} + \begin{bmatrix} \eta^G \\ \eta^p \end{bmatrix} \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix}$$

The method of solving for these coefficients amounts to proposing an iterative procedure from the future to the present as is standard going from $\{\eta^{Gn}, \eta^{pn}\}$ assumed to be the relevant laws of motion tomorrow and then linearizing the Euler equations and market clearing conditions in order to get linear laws of motion and prices today via an equation of the form:

$$\begin{bmatrix} \overline{A} \\ \overline{p} \end{bmatrix} + \begin{bmatrix} \eta^{Gn+1} \\ \eta^{pn+1} \end{bmatrix} \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix} = -D^{-1}(\eta^n) \begin{bmatrix} C(\eta^n), B(\eta^n) \end{bmatrix} \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix}$$

where D is a $(J \times J \times I) \times (J \times J \times I)$ matrix, C is a vector, and B is a $(J \times J \times I) \times I - 1$ matrix, all of which depend on the current guesses for coefficients and the gradient of the Euler equation evaluated at the steady states described below. Since this equation must hold for arbitrary $\vec{A}$, we get:

$$\begin{bmatrix} \overline{A} \\ \overline{p} \end{bmatrix} + \begin{bmatrix} \eta^{Gn+1} \\ \eta^{pn+1} \end{bmatrix} = -D^{-1}(\eta^n) \begin{bmatrix} C(\eta^n), B(\eta^n) \end{bmatrix} \tag{A-11}$$

What is left to describe is how to generate the matrices $D(\eta^n)$ and $B(\eta^n)$, as well as the vector $C(\eta^n)$. Recall that for each generation $i = 1, ..., I-1$ and each current shock z_j, we have J (non-linearized) Euler equations of the form:

$$0 = u'(c_i(A, z_j))p(z_j, A, z_k) - ...$$
$$\beta_i \Gamma_{jk} \left(\sum_{\ell=1}^{J} p(z_k, G(z_j, A, z_k), z_\ell) + \theta z_k \right) u'(c_{i+1}(G(z_j, A, z_k), z_k))$$

This system gives a total of $(I-1) \times J \times J$ conditions for any wealth distribution A. In addition there are $J \times J$ market clearing conditions of the form:

$$\sum_{i=1}^{I-1} G_i(z_j, A, z_k) = 1$$

For each shock z_j we compute the steady state price in an economy where total factor productivity is constant at that level z_j, call it $\overline{p}_j$ and let $\overline{p}_{j,k} = \overline{p}_j \Gamma_{jk}$. The steady state wealth distribution is then $\overline{A}_j = (\overline{A}_{2,j} ... \overline{A}_{I,j})$.[38] We linearize around these steady states by writing each Euler equation as a function of the agent's own wealth share and the share prices a, $\vec{p} = (p_1, ..., p_J)$, $\vec{a}' = (a'_1, ..., a'_J)$, $\vec{p}' = (p'_1, ..., p'_J)$, and $\vec{a}'' = (a''_1, ..., a''_J)$. This gives:

$$0 = \phi_{i,j,k}(a, \vec{p}, \vec{a}', \vec{p}', \vec{a}'') = ... u' \left((1-\theta) z_j \epsilon_i(z_j) + \left(\sum_\ell p_\ell + \theta z_j \right) a - \sum_\ell p_\ell a'_\ell \right) p_k - ...$$
$$\beta_i \Gamma_{jk} \left[\sum_\ell p'_\ell + \theta z_k \right] u' \left((1-\theta) z_k \epsilon_{i+1}(z_k) + \left(\sum_\ell p'_\ell + \theta z_k \right) a'_k - \sum_\ell p'_\ell a''_\ell \right)$$

[38] Note that in the economy where the labor efficiency units depend on the aggregate shocks, the implied steady state wealth shares are different.

Combine steady state arguments for the ϕ function via:

$$\overline{x}_{i,j,k} = [\overline{A}_{ij}, (\overline{p}_{j1}, ..., \overline{p}_{jJ}), (\overline{A}_{i+1,1}, ..., \overline{A}_{i+1,J}), (\overline{p}_{k1}, ..., \overline{p}_{kJ}), (\overline{A}_{i+2,1}, ..., \overline{A}_{i+1,J})]$$

Then the linearized Euler equation for generation i, when the current shock is z_j and next period's shock is z_k, is:

$$\begin{aligned}0 = \phi_{ijk}(\overline{x}_{ijk}) + \frac{\partial \phi_{ijk}}{\partial a}(\overline{x}_{ijk})(a - \overline{A}_{i,j}) + \sum_{\ell} \frac{\partial \phi_{ijk}}{\partial p_\ell}(\overline{x}_{ijk})(p_\ell - \overline{p}_{j,\ell}) + ...\\ \sum_{\ell} \frac{\partial \phi_{ijk}}{\partial a'_\ell}(\overline{x}_{ijk})(a'_\ell - \overline{A}_{i+1,\ell}) + \sum_{\ell} \frac{\partial \phi_{ijk}}{\partial p'_\ell}(\overline{x}_{ijk})(p'_\ell - \overline{p}_{k,\ell}) + \frac{\partial \phi_{ijk}}{\partial a''_\ell}(\overline{x}_{ijk})(a''_\ell - \overline{A}_{i+2,\ell})\end{aligned}$$

We now impose the representative agent condition, $a_i = A_i$, and we evaluate each of these Euler equations at an aggregate wealth distribution $A = (A_2, ..., A_I)$ using linearized equilibrium policy functions and prices $\{H^G, H^p\}$. We obtain:

$$\begin{aligned}0 = \phi_{ijk}(\overline{x}_{ijk}) + \frac{\partial \phi_{ijk}}{\partial a}(\overline{x}_{ijk})(A_i - \overline{A}_{i,j}) + \sum_{\ell} \frac{\partial \phi_{ijk}}{\partial p_\ell}(\overline{x}_{ijk})[H^p_{j\ell}(A) - \overline{p}_{j,\ell}] + ...\\ \sum_{\ell} \frac{\partial \phi_{ijk}}{\partial a'_\ell}(\overline{x}_{ijk})[H^G_{ij\ell}(A) - \overline{A}_{i+1,\ell}] + \sum_{\ell} \frac{\partial \phi_{ijk}}{\partial p'_\ell}(\overline{x}_{ijk})(H^p_{k\ell}[H^G_{j_k}(A)] - \overline{p}_{k,\ell}) + ...\\ \frac{\partial \phi_{ijk}}{\partial a''_\ell}(\overline{x}_{ijk})(H^G_{i+1k\ell}[H^G_{jk}(A)] - \overline{A}_{i+2,\ell}).\end{aligned} \quad \text{(A-12)}$$

In addition, the linearized market clearing conditions are:

$$\sum_{i=1}^{I-1} [H^G_{ijk}(A) - \overline{A}_{i+1,k}] = 0 \quad \text{(A-13)}$$

Now we use these equations to define an iterative procedure using the linearized prices and policies, noting that the linearization of the Euler equations guarantees that, given a guess of coefficients defining prices and policies, the resulting updated prices and policies are also linear. The idea is to iterate from the future to the present, in order to update prices and policies. Given current guesses

H^{Gn}, H^{pn} we find new guesses H^{Gn+1}, H^{pn+1} by solving:

$$0 = \phi_{ijk}(\overline{x}_{ijk}) + \frac{\partial \phi_{ijk}}{\partial a}(\overline{x}_{ijk})(A_i - \overline{A}_{i,j}) + \sum_{\ell} \frac{\partial \phi_{ijk}}{\partial H^p_{\ell}}(\overline{x}_{ijk}) \ \left[H^{pn+1}_{j,\ell}(A) - \overline{p}_{j,\ell}\right] + ...$$
$$\sum_{\ell} \frac{\partial \phi_{ijk}}{\partial a'_{\ell}}(\overline{x}_{ijk}) \left[H^{Gn+1}_{i,j,\ell}(A) - \overline{A}_{i+1,\ell}\right] + \sum_{\ell} \frac{\partial \phi_{ijk}}{\partial p'_{\ell}}(\overline{x}_{ijk}) \left[H^{pn}_{k,\ell}(H^{Gn+1}_{j,k}(A)) - \overline{p}_{k,\ell}\right] + ...$$
$$\frac{\partial \phi_{ijk}}{\partial a''_{\ell}}(\overline{x}_{ijk}) \ \left[H^{Gn}_{i+1,k,\ell}\left(H^{Gn+1}_{j,k}(A)\right) - \overline{A}_{i+2,\ell}\right]. \quad \text{(A-14)}$$

in conjunction with:

$$\sum_{i=1}^{I-1} \left(H^{Gn+1}_{ijk}(A) - \overline{A}_{i+1,k}\right) = 0. \quad \text{(A-15)}$$

Implementing this algorithm involves stacking these equations into a system of the form in equation (A-11). First, given a guess of the coefficients, η^n we can rewrite equation (A-14) as

$$0 = c^n_{ijk} + b^n_{ijk} A_i + [d^n_{ijk1}, .., d^n_{ijkJ}] \begin{bmatrix} \eta^{Gn+1}_{ij1} \\ \eta^{Gn+1}_{ijJ} \end{bmatrix} \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix} +$$
$$[e^n_{ijk2}, ..., e^n_{ijkI}] \begin{bmatrix} \eta^{Gn+1}_{1jk} \\ ... \\ \eta^{Gn+1}_{I-1,jk} \end{bmatrix} \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix} + [f^n_{ijk1}, .., f^n_{ijkJ}] \begin{bmatrix} \eta^{pn+1}_{j1} \\ ... \\ \eta^{pn+1}_{jJ} \end{bmatrix} \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix}$$

Suppressing the steady state arguments in partial derivatives, the new terms reflect the current guess of η^n:

$$\begin{aligned}
c^n_{ijk} &= \phi_{ijk}(\overline{x}_{ijk}) - \frac{\partial \phi_{ijk}}{\partial a}\overline{A}_{ij} + \sum_{\ell=1}^{J} \frac{\partial \phi_{ijk}}{\partial p'_\ell}\eta^{p,n}_{k\ell}\begin{bmatrix} 1 \\ \overline{A}_k \end{bmatrix} + \sum_{\ell=1}^{J} \frac{\partial \phi_{ijk}}{\partial a''_\ell}\eta^{Gn}_{i+1,k\ell}\begin{bmatrix} 1 \\ \overline{A}_k \end{bmatrix} \\
b^n_{ijk} &= \frac{\partial \phi_{ijk}}{\partial a} \\
d^n_{ijk\ell} &= \frac{\partial \phi_{ijk}}{\partial a'_\ell} \\
\left[e^n_{ijk2}, ..., e^n_{ijkI}\right] &= \left[\frac{\partial \phi_{ijk}}{\partial p'_1}, ..., \frac{\partial \phi_{ijk}}{\partial p'_J}\right]\begin{bmatrix} \eta^{pn}_{k1} \\ ... \\ \eta^{pn}_{kJ} \end{bmatrix}\begin{bmatrix} \mathbf{0}_{[1\times(I-1)]} \\ \mathbb{I}_{(I-1)\times(I-1)} \end{bmatrix} + \\
&\qquad \left[\frac{\partial \phi_{ijk}}{\partial a''_1}, ..., \frac{\partial \phi_{ijk}}{\partial a''_J}\right]\begin{bmatrix} \eta^{Gn}_{i+1,k1} \\ ... \\ \eta^{Gn}_{i+1,kJ} \end{bmatrix}\begin{bmatrix} \mathbf{0}_{[1\times(I-1)]} \\ \mathbb{I}_{(I-1)\times(I-1)} \end{bmatrix} \\
f^n_{ijk\ell} &= \frac{\partial \phi_{ijk}}{\partial p_\ell}.
\end{aligned}$$

Now we stack these equations. The first round is by generation and yields:

$$0 = C^n_{jk} + B^n_{jk}\ \vec{A} + \begin{bmatrix} D^n_{jk1} + E^n_{jk1} \\ \mathbf{1_{k1}} \end{bmatrix}\eta^{G,n+1}_{j1}\begin{bmatrix} 1 \\ \vec{A} \end{bmatrix} + ... + \begin{bmatrix} D^n_{jk1} + E^n_{jk1} \\ \mathbf{1_{k1}} \end{bmatrix}\eta^{G,n+1}_{jJ}\begin{bmatrix} 1 \\ \vec{A} \end{bmatrix} + F^n_{jk}\ \eta^{p,n+1}_{j}\begin{bmatrix} 1 \\ \vec{A} \end{bmatrix}$$

where the stacked policies and price matrices are just:

$$\eta^{G,n+1}_{jk} = \begin{bmatrix} \eta^{G,n+1}_{1jk} \\ ... \\ \eta^{G,n+1}_{I-1,jk} \end{bmatrix}, \qquad \eta^{p,n+1}_{j} = \begin{bmatrix} \eta^{p,n+1}_{j1} \\ ... \\ \eta^{p,n+1}_{jJ} \end{bmatrix}.$$

The stacked coefficient matrices are:

$$
\begin{aligned}
C_{jk}^{n} &= \begin{bmatrix} c_{1jk}^{n} \\ \ldots \\ c_{I-1,jk}^{n} \\ -1 \end{bmatrix} \\
B_{jk}^{n} &= \begin{bmatrix} 0, 0, \ldots, 0, 0 \\ b_{2jk}^{n}, 0, \ldots, 0, 0 \\ 0, b_{3jk}^{n}, \ldots, 0, 0 \\ .., .., .., .., .. \\ 0, 0, \ldots, b_{I-1,jk}^{n}, 0 \\ 0, 0, \ldots, 0, 0 \end{bmatrix} \\
D_{jk\ell}^{n} &= \begin{bmatrix} d_{1jk\ell}^{n}, 0, \ldots, 0 \\ 0, d_{2jk\ell}^{n}, \ldots, 0 \\ .., .., .., .. \\ 0, 0, \ldots, d_{I-1,jk\ell}^{n} \end{bmatrix} \\
& \text{if } l = k \text{ then} \\
E_{jkl}^{n} &= \begin{bmatrix} e_{1jk2}^{n}, \ldots, e_{1jkl}^{n} \\ .., .., .. \\ e_{I-1,jk2}^{n}, \ldots, e_{I-1,jkl}^{n} \end{bmatrix} \\
& \text{and} \\
\mathbf{1}_{kl} &= \begin{bmatrix} 1, \ldots, 1 \end{bmatrix} \\
& \text{and if } l \neq k \text{ then } E_{jkl}^{n} = 0_{[I-1 \times I-1]} \text{ and } \mathbf{1}_{kl} = 0_{[1 \times I-1]} \\
F_{jk}^{n} &= \begin{bmatrix} f_{1jk1}^{n}, \ldots, f_{1jkJ}^{n} \\ .., .. \\ f_{I-1,jk1}^{n}, \ldots, f_{I-1,jkJ}^{n} \\ 0, 0 \end{bmatrix}
\end{aligned}
$$

Notice that the market clearing condition has been included at the bottom of each system of Euler equations, so that each of these matrices now has I rows rather than $I-1$ rows. We then stack by next period's aggregate TFP states. This gives:

$$0 = \begin{bmatrix} C_{11}^n \\ ... \\ C_{1J}^n \\ ... \\ C_{J1}^n \\ ... \\ C_{JJ}^n \end{bmatrix} + \begin{bmatrix} B_{11}^n \\ ... \\ B_{1J}^n \\ ... \\ B_{J1}^n \\ ... \\ B_{JJ}^n \end{bmatrix} \vec{A} + \begin{bmatrix} H_1 & ... & \mathbf{0} \\ & \ddots & \\ & H_j & \\ & & \ddots \\ \mathbf{0} & ... & H_J \end{bmatrix} \begin{bmatrix} \eta_{11}^{G,n+1} \\ ... \\ \eta_{1J}^{G,n+1} \\ ... \\ \eta_{J1}^{G,n+1} \\ ... \\ \eta_{JJ}^{G,n+1} \end{bmatrix} \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix} + \begin{bmatrix} F_1^n & ... & \mathbf{0} \\ & \ddots & \\ & F_j^n & \\ & & \ddots \\ \mathbf{0} & ... & F_J^n \end{bmatrix} \begin{bmatrix} \eta_1^{p,n+1} \\ ... \\ \eta_J^{p,n+1} \end{bmatrix} \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix}$$

where:

$$H_j^n = \begin{bmatrix} H_{j11}^n & ... & H_{j1J}^n \\ & ... & \\ H_{jJ1}^n & ... & H_{jJJ}^n \end{bmatrix}, \qquad H_{jk\ell}^n = \begin{bmatrix} D_{jk\ell}^n + E_{jk\ell}^n \\ \mathbf{1}_{k\ell} \end{bmatrix}, \qquad F_i^n = \begin{bmatrix} F_{i1}^n \\ ... \\ F_{iJ}^n \end{bmatrix}.$$

This yields, making dependence on current coefficients η^n explicit:

$$0 = \begin{bmatrix} C(\eta^n), B(\eta^n) \end{bmatrix} \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix} + \begin{bmatrix} H(\eta^n), F(\eta^n) \end{bmatrix} \begin{bmatrix} \eta^{G,n+1} \\ \eta^{p,n+1} \end{bmatrix} \begin{bmatrix} 1 \\ \vec{A} \end{bmatrix}$$

This system can be solved for new policies and prices. Noting that this expression has to hold for all $\vec{A}$ finally results in equation (A-11):

$$\begin{bmatrix} \eta^{G,n+1} \\ \eta^{p,n+1} \end{bmatrix} = - \begin{bmatrix} H(\eta^n), F(\eta^n) \end{bmatrix}^{-1} \begin{bmatrix} C(\eta^n), B(\eta^n) \end{bmatrix}$$

This expression is exactly the form of the equation needed to update the policy and price coefficients. The equilibrium coefficients are found by iterating on this procedure until convergence.

B.3 Numerical Accuracy

The analytical results available for the endogenous portfolio and single-asset economies, for the case when $\sigma = 1$ and the age profile of earnings does not vary with z, provide us with a useful test case to assess the numerical accuracy of our computational results. We now compare our numerical results to the theoretical prediction from Proposition 1, item by item. We make the following observations:

1 The distribution of wealth shares is constant along the simulation and is plotted in Figure A-1.

2 The bond and stock prices are proportional to z. This is verified in Figure A-2, which simply shows that in each state z the prices fall by exactly the same percentage as z for one period and then return to their previous values. In terms of numbers, $\bar{p} = p(z, \bar{A})/z = 0.5167$ and $\bar{q} = q(z, \bar{A})/z = 0.6468$ for all $z \in Z$.

3 In the main text we displayed that portfolios were constant across age groups with $\sigma = 1$. That they are equal to $\frac{\bar{p}}{\bar{p}+B\bar{q}}$ can be verified from point [2] since $\frac{\bar{p}}{\bar{p}+B\bar{q}} = 0.9428$, which is exactly the value for the portfolio share λ_i chosen by each household (see again the figure in the main text). The portfolio shares are also constant over the simulation path, as is shown in Figure A-3 and as predicted by Proposition 1.

4 The consumption profiles, normalized by total output z along the simulation path are plotted in Figure A-4. Note that they are independent of z, as the proposition indicates. They are also equal to the theoretical consumption levels given in the proposition.

5 Verification of part [5] of the proposition is guaranteed since [2] holds.

6 The equity premium in theory is 0.16%, given our calibration. Along the simulation path, the average equity premium is also exactly 0.16%.

When $\sigma = 1$ and the earnings distribution is independent of z, the single-asset economy generates allocations and dynamics that are identical to the endogenous portfolio economy. The price and consumption dynamics can be seen in Figures A-5 and A-6; they are in fact identical to the endogenous portfolio economy. The welfare losses are identical to the second decimal place, as can be seen in the text, but for completeness this table is replicated here, with four decimal places, as Table A-1.

Table A-1: **Expected Welfare Gains from One-Period Recession,** $\sigma = 1$

Generation	$20-29$	$30-39$	$40-49$	$50-59$	$60-69$	$70+$
Single Asset	-1.6218	-2.0437	-2.4263	-3.5737	-5.3652	-8.3011
Endog. Port.	-1.6218	-2.0446	-2.4272	-3.5738	-5.3643	-8.3000

C Proof of Proposition 1: Economy with Log-Utility and iid Shocks

We will verify that the conjectured expressions for prices and allocations satisfy households' budget constraints, households' intertemporal first-order conditions, and all the market clearing conditions. Recall that $\sum_{i=1}^{I} \varepsilon_i = 1$, $\overline{A}_1 = 0$ and $\sum_{i=1}^{I} \overline{A}_i = 1$. It is then straightforward to verify that the expressions in Proposition 1 for $\lambda_i(z, \overline{A}, \overline{A}_i)$, $c_i(z, \overline{A}, \overline{A}_i)$ and $y_i(z, \overline{A}, \overline{A}_i)$ satisfy the market clearing conditions for goods, stocks, and bonds.

The next step is to verify that the conjectured decision rules $c_i(z, \overline{A}, \overline{A}_i)$ and $y_i(z, \overline{A}, \overline{A}_i)$ satisfy the household budget constraints. Given identical portfolios, all households earn the return to saving. Substituting in the candidate expressions for prices and portfolio shares, the gross return to saving conditional on productivity being z_{-1} in the previous period and z in the current period is

$$\frac{\overline{\lambda}\left(p(z, \overline{A}) + d(z, \overline{A})\right)}{p(z_{-1}, \overline{A})} + \frac{(1-\overline{\lambda})}{q(z_{-1}, \overline{A})} = \frac{z\left(\overline{p} + \theta + \overline{q}B\right)}{z_{-1}\left(\overline{p} + \overline{q}B\right)}.$$

Given this expression for returns, consumption for a household of age i is

$$c_i(z, \overline{A}) = (1-\theta)\varepsilon_i z + y_{i-1}(z_{-1}, \overline{A})\left(\frac{z\left(\overline{p} + \theta + \overline{q}B\right)}{z_{-1}\left(\overline{p} + \overline{q}B\right)}\right) - y_i(z, \overline{A}).$$

Substituting in the candidate expression for $y_i(z, \overline{A})$ gives

$$c_i(z, \overline{A}) = z\left[(1-\theta)\varepsilon_i + \theta\overline{A}_i + \left(\overline{A}_i - \overline{A}_{i+1}\right)\left(\overline{p} + \overline{q}B\right)\right],$$

which is the conjectured expression for equilibrium consumption (Property 4).

The final condition to verify is that agents' intertemporal first-order conditions with respect to stocks and bonds are satisfied, i.e.

$$
\begin{aligned}
\frac{p(z,\overline{A})}{c_i(z,\overline{A},\overline{A}_i)} &= \beta_{i+1}\sum_{z'\in Z}\Gamma_{z,z'}\frac{\left(p(z',\overline{A})+d(z',\overline{A})\right)}{c(z',\overline{A},\overline{A}_i)} \qquad \forall i=1,...,I-1,\\
\frac{q(z,\overline{A})}{c_i(z,\overline{A},\overline{A}_i)} &= \beta_{i+1}\sum_{z'\in Z}\Gamma_{z,z'}\frac{1}{c(z',\overline{A},\overline{A}_i)} \qquad \forall i=1,...,I-1.
\end{aligned}
$$

Substituting in the conjectured expressions for $p(z,\overline{A})$, $q(z,\overline{A})$ and $c_i(z,\overline{A},\overline{A}_i)$ gives

$$
\overline{p}=\kappa_{i+1}(\overline{A})\sum_{z'\in Z}\Gamma_{z,z'}\frac{[z'\overline{p}+\theta z'-B+z'\overline{q}B]}{z'} \qquad \forall i=1,...,I-1, \tag{A-16}
$$

$$
\overline{q}=\kappa_{i+1}(\overline{A})\sum_{z'\in Z}\Gamma_{z,z'}\frac{1}{z'} \qquad \forall i=1,...,I-1, \tag{A-17}
$$

where

$$
\kappa_{i+1}(\overline{A})=\beta_{i+1}\frac{\left[(1-\theta)\varepsilon_i+\theta\overline{A}_i+\left(\overline{A}_i-\overline{A}_{i+1}\right)(\overline{p}+\overline{q}B)\right]}{\left[(1-\theta)\varepsilon_{i+1}+\theta\overline{A}_{i+1}+\left(\overline{A}_{i+1}-\overline{A}_{i+2}\right)(\overline{p}+\overline{q}B)\right]}.
$$

Note from these first order conditions that the conjectured expressions for $p(z,\overline{A})$ and $q(z,\overline{A})$ cannot be valid for shocks that are not *iid* over time, because absent *iid* shocks, the right-hand sides of (A-16) and (A-17) would be functions of z.

Adding the two first-order conditions for stocks and bonds gives

$$
\overline{p}+\overline{q}B=\kappa_{i+1}(\overline{A})\left(\overline{p}+\overline{q}B+\theta\right) \qquad \forall i=1,...,I-1. \tag{A-18}
$$

This is the first-order condition for pricing claims to capital for a non-stochastic life-cycle economy, in which the constant asset price is $\overline{p}+\overline{q}B$ and the constant asset income is θ. The $I-1$ equations (A-18) combined with $\overline{A}_1=0$ and the market clearing condition $\sum_{i=1}^{I}\overline{A}_i=1$ can be used to solve for $\left\{\overline{A}_i\right\}_{i=1}^{I}$ and $\overline{p}+\overline{q}B$. This system of equations has a solution, because it defines a standard non-stochastic general-equilibrium life-cycle economy. Property 5 in Proposition 1 follows from that fact that z does not appear anywhere in this system of equations, and the only place B appears is in the price term $\overline{p}+\overline{q}B$.

Given $\left\{\overline{A}_i\right\}_{i=1}^{I}$ and $\overline{p}+\overline{q}B$ the only remaining relative price still to be determined is the relative price of stocks and bonds. We need to verify that there exists a ratio $\overline{p}/\overline{q}$ which supports the conjectured portfolio split $\overline{\lambda}$ for each age group. From equations (A-16) and (A-17) the relative valuation of stocks and bonds is equal across age groups. Specifically (and now using the iid assumption to simplify the expressions):

$$\frac{\overline{p}}{\overline{q}} = \frac{\overline{p} + \theta + \overline{q}B - B \sum_{z' \in Z} \Gamma_{z'} \frac{1}{z'}}{\sum_{z' \in Z} \Gamma_{z'} \frac{1}{z'}}, \tag{A-19}$$

which is independent of age group i. This reflects the fact that consumption of all age groups is equally sensitive to aggregate productivity, and thus (given common constant relative risk aversion) all age groups require the same equity premium to support the conjectured (symmetric) portfolios. When the value of aggregate wealth is given by (A-18) and the ratio of stock to bond prices by (A-19), all agents' first-order conditions are satisfied.

The average equity premium is defined as

$$\sum_{z} \Gamma_z \left\{ \sum_{z'} \Gamma_{z'} \left[\frac{p(z', \overline{A}) + d(z', \overline{A})}{p(z, \overline{A})} \right] - \frac{1}{q(z, \overline{A})} \right\}$$

The expression for the equity premium in Proposition 1 (Property 6) follows from substituting the definition for dividends into the first term, substituting the expression for $q(z, \overline{A})$ in equation (A-19) into the denominator of the second term, and then replacing all remaining asset prices with the corresponding price expressions in Property 2.

D Asset Prices in the Representative Agent Economy

Suppose the representative agent invests an exogenous fraction λ of savings in stocks and fraction $1 - \lambda$ in bonds. Let $c(z, a)$ and $y(z, a)$ denote optimal consumption and savings as functions of the aggregate shock z and individual start-of-period wealth a, and let $p(z)$ and $q(z)$ be the equilibrium prices for stocks and bonds. Note that there is no need to keep track of aggregate wealth as a state: by assumption, the supply of capital is constant and equal to one. Thus prices can only depend on z.

The dynamic programming problem for a household is

$$v(z, a) = \max_{c \geq 0, y} \left\{ u(c) + \beta \sum_{z' \in Z} \Gamma_{z,z'} \; v\left(z', a'\right)\right) \right\}$$

subject to

$$c + y = (1 - \theta)z + (p(z) + d(z) + B)\, a$$

and the law of motion

$$a'\,[p(z') + d(z') + B] = \left(\frac{\lambda\ [p(z') + d(z')]}{p(z)} + \frac{(1-\lambda)}{q(z)} \right)\ y.$$

The solution to this problem yields decision rules $c(z,a)$, $y(z,a)$ and $a'(z', y(z,a))$ is the associated value for next period wealth.

Given the preferences and technology described above, the market clearing conditions are simply

$$\begin{aligned} \lambda y(z,1) &= p(z), \\ (1-\lambda)y(z,1) &= q(z)B, \\ c(z,1) &= z. \end{aligned}$$

The individual and aggregate consistency condition is

$$a'(z', y(z,1)) = 1$$

Now suppose the process for z is a two-state Markov chain. There are just two equity prices to solve for: $p(z) \in \{p(z_l), p(z_h)\}$. The two market clearing conditions for stocks and bonds imply a parametric relationship between $q(z)$ and $p(z)$:

$$q(z) = \frac{p(z)(1-\lambda)}{\lambda B}.$$

Thus stock and bond prices must be equally sensitive to aggregate shocks. The realized gross real return to saving is given by

$$\lambda \frac{[p(z') + d(z')]}{p(z)} + (1-\lambda)\frac{1}{q(z)} = \frac{p(z') + \lambda\theta z'}{p(z)}$$

where the second equality follows from substituting in $d(z') = \theta z' + q(z')B - B$ and the expression for $q(z)$.as a function of $p(z)$. Thus the equilibrium equity prices are defined by the solutions to the two inter-temporal first order conditions:

$$\begin{aligned} p(z_h)u'(c(z_h,1)) &= \beta \sum_{z'\in Z} \Gamma_{z_h,z'} \left[u'(c(z',1)) \left[\lambda\theta z' + p(z')\right]\right], \\ p(z_l)u'(c(z_l,1)) &= \beta \sum_{z'\in Z} \Gamma_{z_l,z'} \left[u(c(z',1)) \left[\lambda\theta z' + p(z')\right]\right]. \end{aligned}$$

which, using the market clearing condition for consumption and the CRRA preference specification, can be written as

$$\begin{aligned} p(z_h)z_h^{-\sigma} &= \beta\Gamma_h\, z_h^{-\sigma}\left[\lambda\theta z_h + p(z_h)\right] + \beta(1-\Gamma_h)z_l^{-\sigma}\left[\lambda\theta z_l + p(z_l)\right], \\ p(z_l)z_l^{-\sigma} &= \beta\Gamma_l\, z_l^{-\sigma}\left[\lambda\theta z_l + p(z_l)\right] + \beta(1-\Gamma_l)z_h^{-\sigma}\left[\lambda\theta z_h + p(z_h)\right]. \end{aligned}$$

where $\Gamma_h = \Gamma_{z_h,z_h}$ and $\Gamma_l = \Gamma_{z_l,z_l}$. From the second pricing equation

$$p(z_h) = \frac{\beta\Gamma_h\lambda\theta z_h + \beta(1-\Gamma_h)\frac{z_l^{-\sigma}}{z_h^{-\sigma}}\left(\lambda\theta z_l + p(z_l)\right)}{(1-\beta\Gamma_h)}.$$

Substituting this into the first pricing equation,

$$p(z_l) = \frac{\beta\Gamma_l\, z_l^{-\sigma}\lambda\theta z_l + \beta(1-\Gamma_l)z_h^{-\sigma}\left(\lambda\theta z_h + \frac{\beta\Gamma_h\lambda\theta z_h + \beta(1-\Gamma_h)\frac{z_l^{-\sigma}}{z_h^{-\sigma}}\lambda\theta z_l}{(1-\beta\Gamma_h)}\right)}{z_l^{-\sigma}\left(\frac{(1-\beta)(1+\beta(1-\Gamma_h-\Gamma_l))}{(1-\beta\Gamma_h)}\right)}.$$

Since the expression for $p(z_h)$ is symmetric, we can take the ratio to express the ratio of prices across states as a function of fundamentals:

$$\frac{p(z_l)}{p(z_h)} = \frac{z_l}{z_h}\left(\frac{(1-\Gamma_l)\, z_h^{1-\sigma} z_l^{\sigma-1} + (\beta + \Gamma_l - \beta\Gamma_h - \beta\Gamma_l)}{(1-\Gamma_h)\, z_l^{1-\sigma} z_h^{\sigma-1} + (\beta + \Gamma_h - \beta\Gamma_h - \beta\Gamma_l)}\right).$$

Note that λ and θ have dropped out here: the *ratio* of stock prices across states does not depend on either λ or θ, though the *levels* of prices do. If aggregate shocks are *iid*, then $1-\Gamma_l = \Gamma_h$ and the expression above simplifies to

$$\frac{p(z_l)}{p(z_h)} = \left(\frac{z_l}{z_h}\right)^{\sigma}.$$

It is straightforward to verify that the same result is obtained even without the *iid* assumption in two special cases: $\sigma = 1$ or $\beta = 1$.

E Asset Prices in the Two-Period Overlapping-Generations Economy

Let $\widetilde{p} = \frac{p(z_h)}{p(z_l)}$, $\widetilde{z} = \frac{z_h}{z_l}$, and $\widetilde{x} = \frac{z_l}{p_l}$. In terms of these variables, the intertemporal first-order conditions, conditional on the current state being z_l and z_h are, respectively:

$$\begin{aligned}
((1-\theta)\widetilde{x}-1)^{-\sigma} &= \beta\Gamma_{z_l,z_l}\ (\theta\widetilde{x}+1)^{1-\sigma} + \beta\Gamma_{z_l,z_h}\ (\theta\widetilde{z}\widetilde{x}+\widetilde{p})^{1-\sigma} \\
\widetilde{p}\left((1-\theta)\widetilde{z}\widetilde{x}-\widetilde{p}\right)^{-\sigma} &= \beta\Gamma_{z_h,z_l}\ (\theta\widetilde{x}+1)^{1-\sigma} + \beta\Gamma_{z_h,z_h}\ (\theta\widetilde{z}\widetilde{x}+\widetilde{p})^{1-\sigma}
\end{aligned}$$

Our goal is to solve for $\widetilde{p}$ as a function of $\widetilde{z}$. However, except for the special case $\sigma = 1$, this system of equations cannot be solved in closed form. So instead we will linearize these equations and look for an approximate solution for relative prices as a linear function of relative productivity. We proceed as follows:

1. Take first-order Taylor-series approximations to these two first-order conditions around the non-stochastic steady state values for $\widetilde{p}$, $\widetilde{z}$, and $\widetilde{x}$, which we denote P, Z, and X (where $Z = P = 1$). This gives a system of two equations in three first-order terms $(\widetilde{x}-X)$, $(\widetilde{z}-Z)$, and $(\widetilde{p}-P)$:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \begin{pmatrix} (\widetilde{x}-X) \\ (\widetilde{z}-Z) \\ (\widetilde{p}-P) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A-20}$$

where

$$\begin{aligned}
A_{11} &= -\sigma\left((1-\theta)X-1\right)^{-\sigma-1}(1-\theta) - \\
&\qquad \left((1-\sigma)\beta\Gamma_{z_l,z_l}(\theta X+1)^{-\sigma}\theta + (1-\sigma)\beta\Gamma_{z_l,z_h}(\theta X+1)^{-\sigma}\theta\right) \\
A_{12} &= -(1-\sigma)\beta\Gamma_{z_l,z_h}(\theta X+1)^{-\sigma}\theta X \\
A_{13} &= -(1-\sigma)\beta\Gamma_{z_l,z_h}(\theta X+1)^{-\sigma} \\
A_{21} &= -\sigma\left((1-\theta)X-1\right)^{-\sigma-1}(1-\theta) - \\
&\qquad \left((1-\sigma)\beta\Gamma_{z_h,z_l}(\theta X+1)^{-\sigma}\theta + (1-\sigma)\beta\Gamma_{z_h,z_h}(\theta X+1)^{-\sigma}\theta\right) \\
A_{22} &= -\sigma\left((1-\theta)X-1\right)^{-\sigma-1}(1-\theta)X - (1-\sigma)\beta\Gamma_{z_h,z_h}(\theta X+1)^{-\sigma}\theta X \\
A_{23} &= \sigma\left((1-\theta)X-1\right)^{-\sigma-1} + \left((1-\theta)X-P\right)^{-\sigma} - (1-\sigma)\beta\Gamma_{z_H,z_H}(\theta X+1)^{-\sigma}
\end{aligned}$$

2. Use the first equation in A-20 to solve for $(\widetilde{x} - X)$ as a linear function of $(\widetilde{z} - Z)$ and $(\widetilde{p} - P)$:

$$(\widetilde{x} - X) = -\frac{A_{12}}{A_{11}} (\widetilde{z} - Z) - \frac{A_{13}}{A_{11}} (\widetilde{p} - P) .$$

Then substitute this solution into the second equation in (A-20), and solve for $(\widetilde{p} - P)$ as a function of $(\widetilde{z} - Z)$:

$$\begin{aligned} (\widetilde{p} - P) &= -\frac{A_{21}}{A_{23}} (\widetilde{x} - X) - \frac{A_{22}}{A_{23}} (\widetilde{z} - Z) \\ &= -\frac{A_{21}}{A_{23}} \left(-\frac{A_{12}}{A_{11}} (\widetilde{z} - Z) - \frac{A_{13}}{A_{11}} (\widetilde{p} - P) \right) - \frac{A_{22}}{A_{23}} (\widetilde{z} - Z) . \end{aligned}$$

Thus

$$\xi^{2p} \approx \frac{\widetilde{p} - P}{\widetilde{z} - Z} = \frac{A_{21}A_{12} - A_{22}A_{11}}{A_{23}A_{11} - A_{21}A_{13}}.$$

3. Now assume productivity shocks are iid., so that $\Gamma_{z_l,z_h} = \Gamma_{z_h,z_h} = \Gamma_{z_h}$ and $\Gamma_{z_l,z_l} = \Gamma_{z_h,z_l} = 1 - \Gamma_{z_h}$. Under this iid. assumption, $A_{11} = A_{21}$ and thus

$$\xi^{2p} \approx \frac{A_{21}A_{12} - A_{22}A_{21}}{A_{23}A_{11} - A_{21}A_{13}} = \frac{A_{12} - A_{22}}{A_{23} - A_{13}} = X \frac{\sigma(1-\theta)}{((X - X\theta - 1) + \sigma)}$$

Recall that X is the inverse of the steady state stock price, so we can equivalently write this elasticity in terms of the steady state gross interest rate R, where $R = \theta X + 1$:

$$\xi^{2p} \approx \frac{\sigma(1-\theta)}{1 - \theta\frac{(R-\sigma)}{(R-1)}}. \tag{A-21}$$

This is the expression given in the text. Note that for $\sigma = 1$, $\xi^{2p} = 1$.

4. We want to show that $1 < \xi^{2p} < \xi^{RA}$ for $\sigma > 1$. First, note that in any equilibrium, a positive stock price implies $R > 1$. Then

$$\frac{1}{\xi^{2p}} = \frac{1 - \theta\frac{(R-\sigma)}{(R-1)}}{\sigma(1-\theta)} = \frac{1}{\sigma}\left(1 + \frac{(\sigma-1)\frac{\theta}{R-1}}{(1-\theta)}\right) > \frac{1}{\sigma} = \frac{1}{\xi^{RA}}$$

Thus $\xi^{2p} < \xi^{RA}$.

Given that $\xi^{2p} = 1$ when $\sigma = 1$, showing that ξ^{2p} is strictly increasing in σ is sufficient to prove that $\xi^{2p} > 1$.

$$\frac{\partial}{\partial \sigma}\left(\frac{\sigma(1-\theta)}{1-\theta\frac{(R-\sigma)}{(R-1)}}\right) = (\theta-1)(R-1)\frac{R\theta - R + 1}{(R - R\theta + \theta\sigma - 1)^2}$$

It follows that ξ^{2p} is strictly increasing in σ if and only if $R > \frac{1}{1-\theta}$. But in any equilibrium, positive consumption for the young requires exactly this condition:

$$(1-\theta) - \frac{\theta}{R-1} > 0 \Leftrightarrow R > \frac{1}{1-\theta}$$

We conclude that $\xi^{2p} > 1$.

5. In the special case in which θ is such that $R = \frac{1}{\beta}$, the expression for ξ^{2p} simplifies further. The steady state value for R is an endogenous variable, and has to satisfy the steady state version of the inter-temporal first-order condition, where the steady state stock price is $\theta/(R-1)$:

$$\frac{\theta}{R-1}\left((1-\theta) - \frac{\theta}{R-1}\right)^{-\sigma} = \beta\left(\theta + \frac{\theta}{R-1}\right)^{-\sigma}\left(\theta + \frac{\theta}{R-1}\right). \tag{A-22}$$

When $\beta = \frac{1}{R}$, equation A-22 can be solved in closed form to give $R = \frac{1}{1-2\theta}$. Thus we have $R = \frac{1}{\beta} = \frac{1}{1-2\theta}$ which implies $\theta = \frac{1}{2}(1-\beta)$. Substituting $R = \frac{1}{\beta}$ and $\theta = \frac{1}{2}(1-\beta)$ into equation. A-21 gives

$$\xi^{2p} \approx \frac{\sigma(1-\theta)}{1-\theta\frac{(R-\sigma)}{(R-1)}} = \frac{\sigma(\beta+1)}{\sigma\beta+1},$$

which is the expression for this special case given in footnote 16.

F Asset Prices in the Economy with Exogenous Portfolios

In this section we briefly relate asset prices in the two-asset economy with exogenous portfolios to the price of the stock in the one-asset (stock only) economy. We will use hats ˆ to denote prices in the two-asset economy to differentiate these prices from those in the stock-only economy. First, note that in the two-asset economy, if all age groups were to have the *same* exogenous portfolio allocations, $\lambda_i = \lambda$, then the two-asset and one-asset economy differ with respect to prices, but are identical with respect to allocations. In particular, if $\lambda_i = \lambda\ \forall i$ then:

1. Real allocations $\widehat{c}_i(z, A, A_i)$ and $\widehat{y}_i(z, A, A_i)$ in the two-asset economy are identical to those in the one-asset economy, $c_i(z, A, A_i)$ and $y_i(z, A, A_i)$.

2. Equilibrium stock prices in the two-asset economy are proportional to those in the one-asset economy:

$$\widehat{p}(z, A) = \lambda p(z, A).$$

3. Equilibrium bond prices in the two-asset economy are proportional to stock prices in the one-asset economy:

$$\widehat{q}(z, A) = \frac{(1-\lambda)p(z, A)}{B}.$$

4. The conditional equity premium in the two-asset economy (conditional on the current aggregate state (z, A)) is

$$\sum_{z'} \Gamma_{z,z'} \left(\frac{\widehat{p}(z', A') + \widehat{d}(z', A')}{\widehat{p}(z, A)} \right) - \frac{1}{\widehat{q}(z, A)} = \sum_{z'} \Gamma_{z,z'} \left(\frac{p(z', A') + \theta z' - B}{\lambda p(z, A)} \right) - \frac{B}{(1-\lambda)p(z, A)}.$$

5. The conditional equity premium is decreasing in B and λ.

The intuition for these results is straightforward. Property 1 reflects the fact that when all households must choose the same portfolio split, the partitioning of asset income into risky and safe components is an artificial veil that has no effect on equilibrium quantities. Property 2 follows from the fact that aggregate savings are identical in the two-asset and one-asset economies, and the fraction of savings exogenously devoted to equity is λ. Property 3 follows from the fact that aggregate savings in the two-asset economy, $\widehat{q}(z, A)B + \widehat{p}(z, A)$ equals aggregate savings $p(z, A)$ in the one-asset economy. Property 4 follows from substituting the expressions for $\widehat{p}(z, A)$ and $\widehat{q}(z, A)$ and the definition for dividends $\widehat{d}(z, A) = \theta z - B + \widehat{q}(z, A)B$ into the definition of the conditional equity premium. Note that in this expression for the equity premium, all the endogenous price terms correspond to the one-asset economy and are thus independent of B and λ. Property 5 follows immediately.

Intuitively, more debt reduces bond prices and thus increases bond returns, and at the same time more debt reduces levered stock returns. Increasing λ has a similar effect: a smaller share of savings flowing into bonds reduces bond prices and increases bond returns. In addition, a larger λ has the effect of raising stock prices and reducing stock returns, further reducing the equity premium.

Now consider the general case, in which the exogenous portfolio shares λ_i vary with age. This will induce age variation in the returns to saving, and thus allocations will no longer coincide with the one-asset (stock only) economy. Denote aggregate savings and the fraction of aggregate savings in stocks

$$\widehat{Y}(z, A) = \sum_{i=1}^{I} \widehat{y}_i(z, A, A_i),$$

$$\widehat{s}(z, A) = \frac{\sum_{i=1}^{I} \lambda_i \, \widehat{y}_i(z, A, A_i)}{\widehat{Y}(z, A)} = \frac{cov\left(\lambda_i, \widehat{y}_i(z, A, A_i)\right)}{E\left[\widehat{y}_i(z, A, A_i)\right]} + E\left[\lambda_i\right].$$

The market clearing conditions for stocks and bonds are

$$\begin{aligned} \widehat{p}(z, A) &= \widehat{s}(z, A)\widehat{Y}(z, A), \\ \widehat{q}(z, A)B &= (1 - \widehat{s}(z, A))\widehat{Y}(z, A), \end{aligned}$$

which imply

$$\frac{\widehat{p}(z, A)}{\widehat{q}(z, A)} = \frac{\widehat{s}(z, A)B}{1 - \widehat{s}(z, A)}.$$

This expression suggests that the relative price of stocks to bonds will vary with $\{\lambda_i\}$ and B in a similar to fashion to the case with age-independent portfolio shares. However, in contrast to the economy with identical portfolio shares by age group, equilibrium prices cannot be readily related to the stock price in the one-asset economy, and in general will depend on both B and the entire distribution $\{\lambda_i\}$.

G Additional Tables with Results

Table A-2: **Relative Price Decline** $\xi = \left(\frac{\ln(p_0/p_{-1})}{\ln(z_0/z_{-1})} \right)$ **for Each Economy**

Economy	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
Single Asset	1.13	2.29	2.94
Fixed Portfoliios			
–Stock	1.18	2.45	3.19
–Bond	0.86	2.52	3.53
–Wealth	1.15	2.46	3.21
Endogenous Portfolios			
–Stock	1.07	2.98	5.00
–Bond	1.05	3.0	5.01
–Wealth	1.07	2.98	5.00

Table A-3: **Realized Welfare Gain from Recession (cyclical age-earnings profile)**

Age i	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
	Single-Asset Economy		
1	-10.06%	-7.28%	-6.30%
2	-9.32%	-7.48%	-6.87%
3	-8.06%	-7.30%	-7.06%
4	-7.71%	-8.67%	-9.17%
5	-6.89%	-10.03%	-11.61%
6	-6.50%	-11.34%	-14.08%
	Fixed Portfolio Economy		
1	-10.17%	-7.65%	-6.60%
2	-9.57%	-8.27%	-7.77%
3	-8.01%	-7.31%	-7.00%
4	-7.43%	-8.31%	-8.76%
5	-6.40%	-9.48%	-11.10%
6	-5.38%	-10.69%	-13.57%
	Endogenous Portfolio Economy		
1	-9.33%	-6.25%	1.73%
2	-8.57%	-9.7%	-9.82%
3	-8.43%	-8.66%	-6.62%
4	-7.58%	-8.48%	-9.15%
5	-7.27%	-8.19%	-9.86%
6	-7.65%	-8.05%	-9.73%

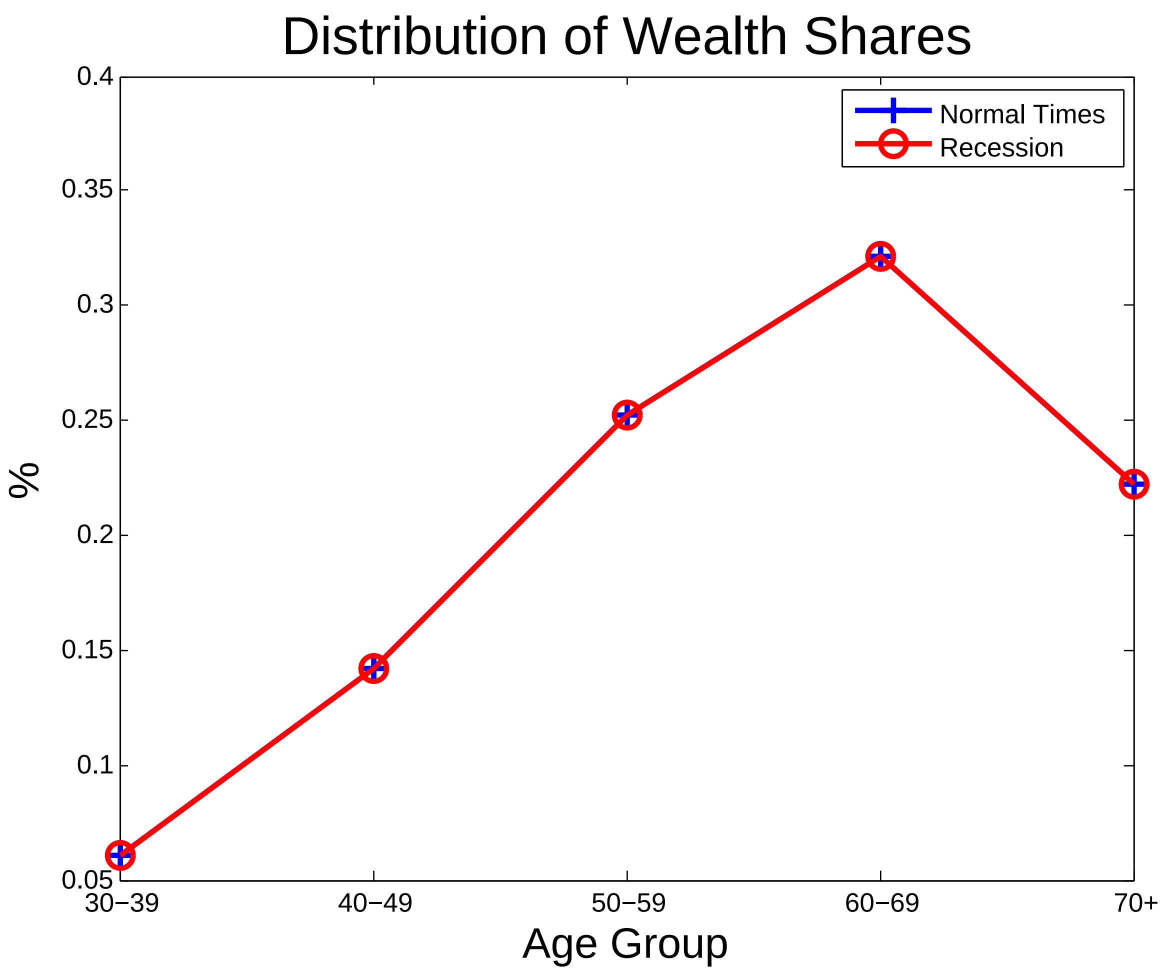

Figure A-1: Wealth Distribution Dynamics in Endogenous Portfolios Economy

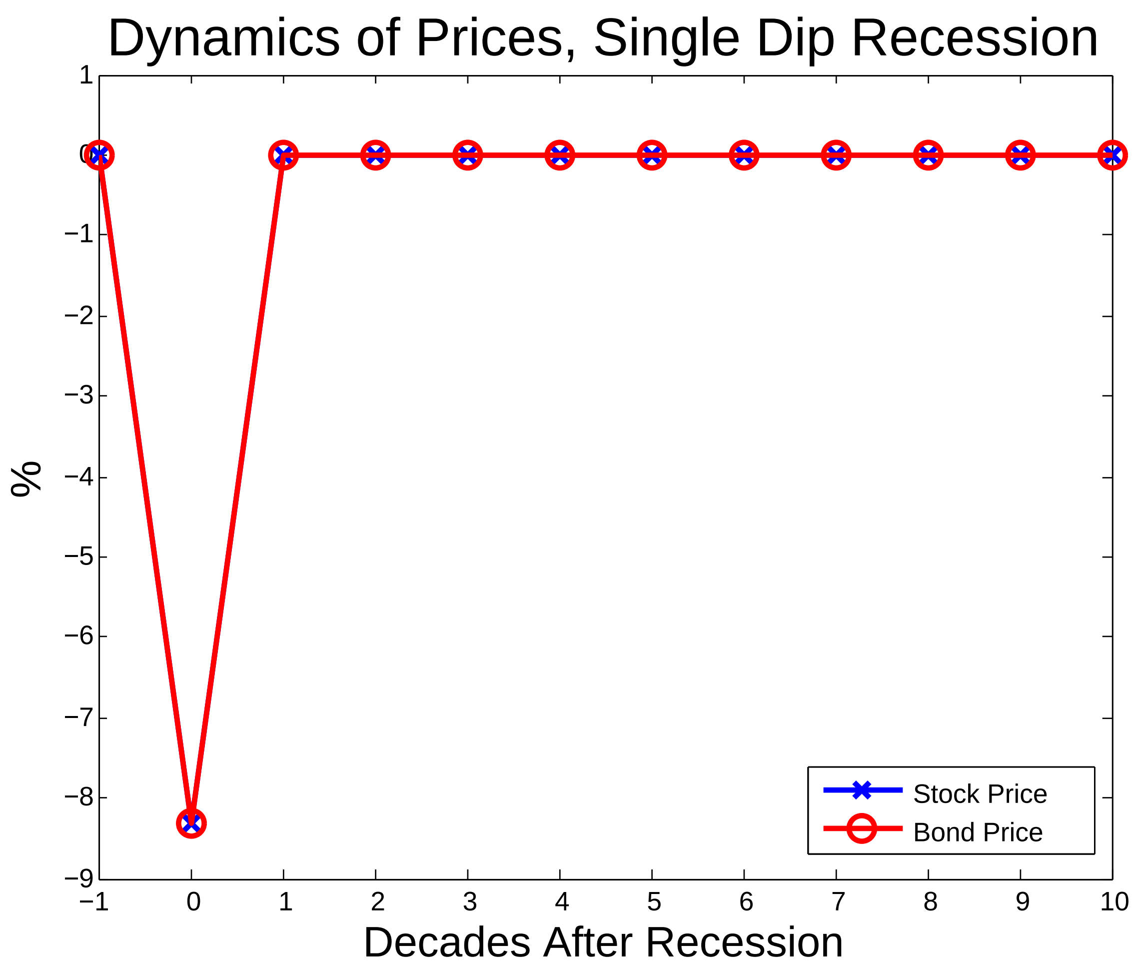

Figure A-2: Price Dynamics of Stock and Bond in Endogenous Portfolio Economy

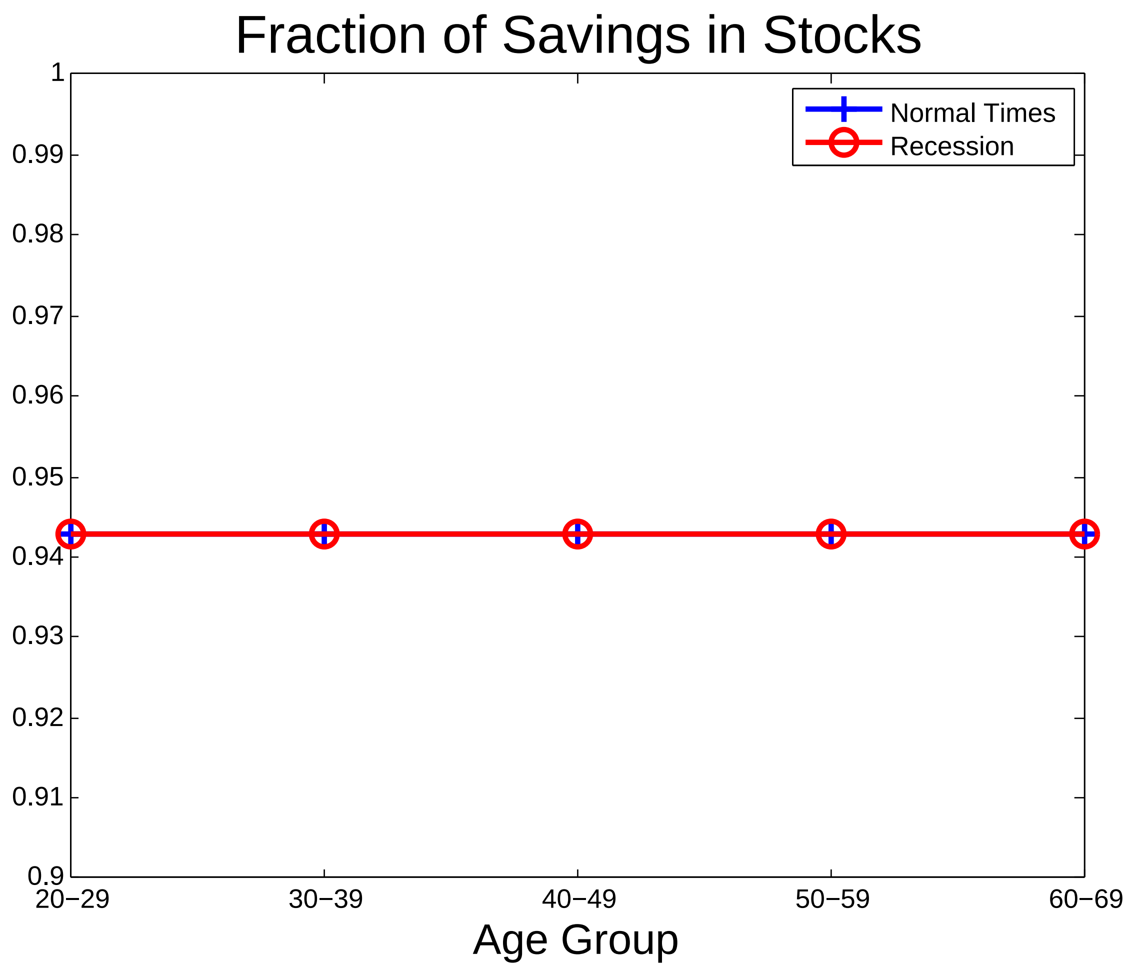

Figure A-3: Portfolio Dynamics in Endogenous Portfolio Economy

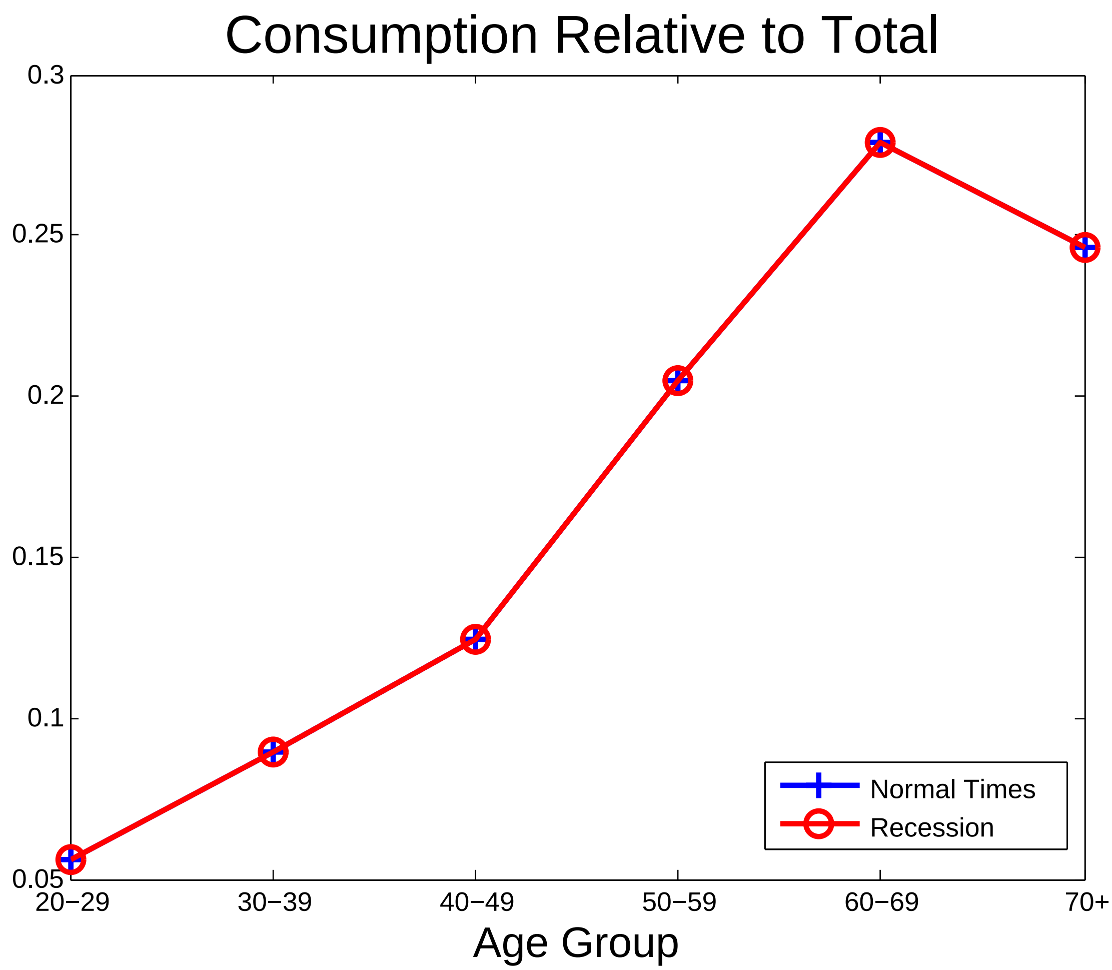

Figure A-4: Consumption Dynamics in Endogenous Portfolio Economy

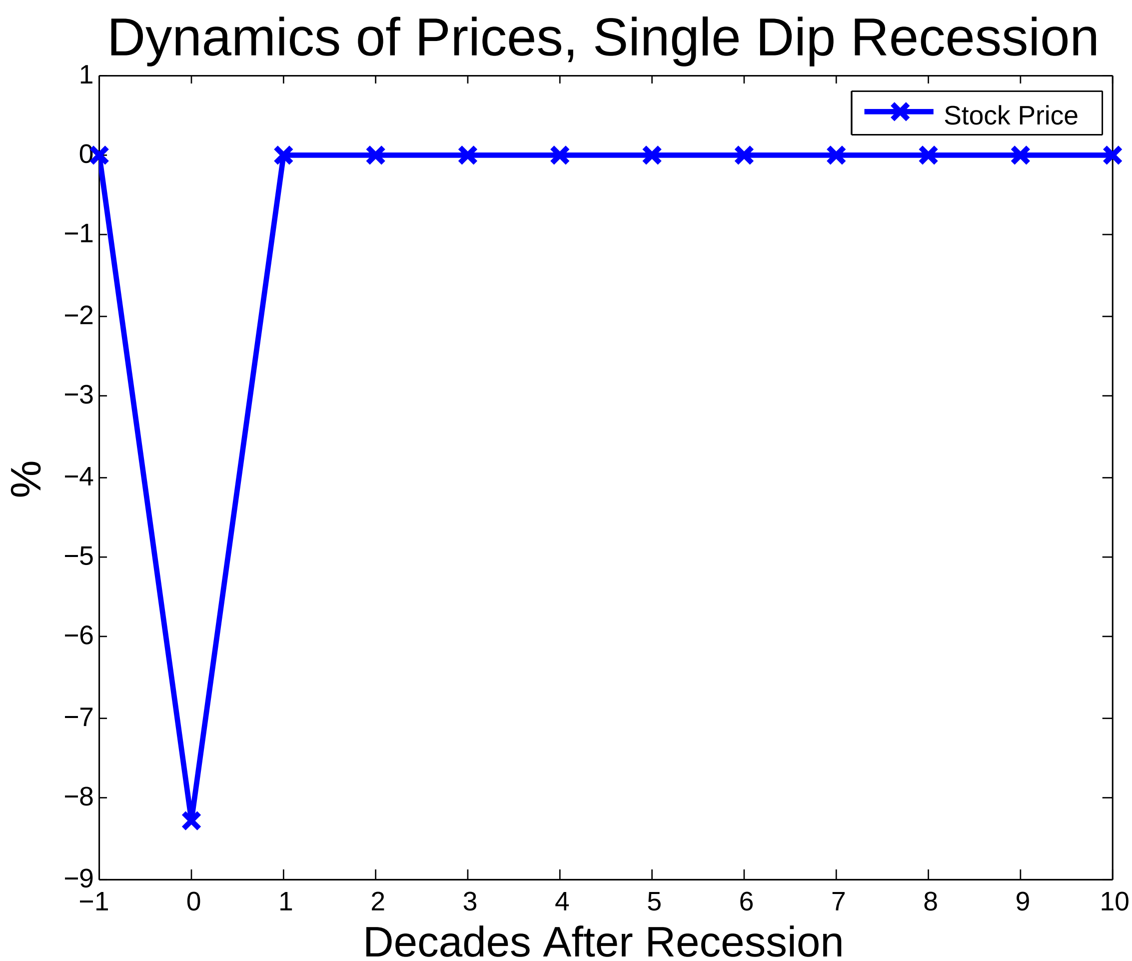

Figure A-5: Price Dynamics of Wealth in Single-Asset Economy

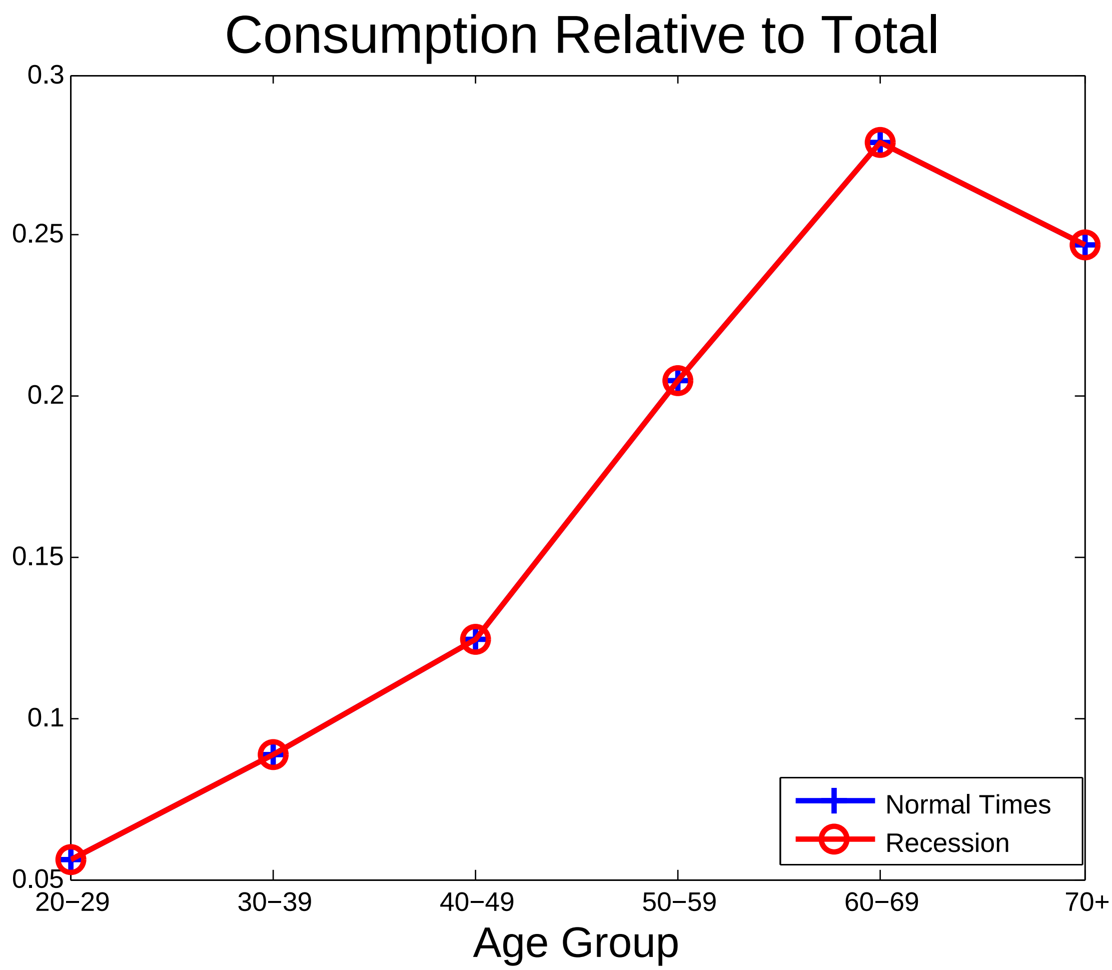

Figure A-6: Consumption Dynamics in Single-Asset Economy