WHEN DEMAND CREATES ITS OWN SUPPLY: SAVING TRAPS

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Abstract

The mechanism by which aggregate supply creates the income that generates its matching demand (called Say's Law), may not work in a general equilibrium with decentralized markets and savings in bonds or money. Full employment is an equilibrium, but convergence to that state is slow. A self-fulfilling precautionary motive to accumulate bonds (with a zero aggregate supply) can set the economy on an equilibrium path with a fast convergence towards a steady state with unemployment that may be an absorbing state from which no equilibrium path emerges to restore full employment.

Keywords: Say's law, saving trap, private debt, credit constraint, financial crisis.

JEL codes: E00, E10, E21, E41

1 Introduction

In the recent global crisis, the demand for precautionary savings and the reduction of consumption have played an aggravating role. The lower demand for goods may generate a self-fulfilling equilibrium with unemployment and individuals' uncertainty. This mechanism is proven here in a model of general equilibrium with decentralized markets for goods and an economy-wide market for bonds.

The mechanism invalidates the common view of "Say's Law" that the aggregate supply (*i.e.*, the capacity to supply goods) creates its matching demand and that full employment is the natural state of economies[1]. In general equilibrium, there is no coordination problem between demand and supply, hence an analytical model that addresses the problem of Say's Law should have an equilibrium with full employment. The model presented here has indeed such an equilibrium in which all individuals realize their transactions according to perfect foresight and there is no unused capacity of production. However, in addition to that full employment equilibrium, there is at least one other equilibrium where output falls short of capacity.

In an equilibrium with unemployment, demand is low because some individuals prefer to accumulate savings. The motive for precautionary saving is self-fulfilling[2] and arises because individuals are subject to a credit constraint and are uncertain about their opportunities to sell goods in future periods. There is no aggregate shock in the economy[3] and there is perfect foresight on the future path of unemployment. All uncertainty arises because of individuals' trades and individuals' heterogeneity.

In the full-employment equilibrium, the individuals' heterogeneity has no impact because as individuals are sure they can always sell in the future, they buy goods in each period, aggregate demand equals the production capacity and full employment is self-fulfilling. In the equilibrium with unemployment however, demand is less than capacity because some agents are credit constrained or prefer to accumulate precautionary saving if they have a lower utility for consumption during a period.

For clarity's sake, there is here no physical capital and goods are not storable. The

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[1]What Say (1803) really meant is probably different from the current interpretation. He was not familiar with general equilibrium.

[2]The present model is in the line of Ayagari (1994), but Ayagari assumed individual exogenous productivity shocks. There are no such shocks here.

[3]For uncertainty with aggregate shocks, see Bloom (2009).

impact of saving on the demand for goods through investment is obviously important, but that issue should be addressed separately. Here, the asset for saving or dissaving is a bond and in most of the paper, the net supply of bonds is zero. (Bonds could also be viewed as inside money). The debts of some agents are the assets of others. Although in an equilibrium with unemployment, agents want to accumulate precautionary savings, that cannot be satisfied in the aggregate. Higher savings by some agents entail the absence of sales by other agents who increase their debt.

The present model is related to the well known study of the Great Capitol Hill Baby Sitting Co-op crisis by Sweeney and Sweeney (1977) where an "insufficient" quantity of money generates the unemployment of resources in an economy without credit. It seems that unemployment arises because people are chasing a fixed quantity of money in a kind of musical chairs game. However, the critical feature is the credit constraint. A partial relaxation of the credit constraint is equivalent to an increase of money. That equivalence is discussed at the end of the paper.

At full employment, a shift of expectations[4] towards a future with unemployment generates a self-fulfilling behavior with a higher saving and a lower demand. The downturn is sudden and the convergence to the steady state with unemployment is fast: for a particular parametric case, the dynamics are one dimensional in the aggregate debt with a convergence rate that is strictly positive. The downturn is fast because agents are not constrained when they abstain from consumption. The asymmetry with an upturn is an important result of the analysis and the mechanism for this asymmetry may be a generic property in models of recoveries from financial crisis.

When the economy is in a state with unemployment, some agents are credit constrained. An upturn requires higher expectations about the future, but if expectations shift towards optimism, the constrained agents cannot increase their consumption. Agents become free from the constraint only if they sell. In an economy with decentralized markets, the process takes time as the aggregate demand trickles down gradually to the entire economy. If a transition toward full employment is possible, in equilibrium, the convergence is slow. In the one dimensional case of the model, the rate of convergence, while strictly positive, is asymptotically nil.

Under some parameters, the asymmetry between downturns and upturns takes a strong

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[4]Such a shift has to occur through coordination between agents. In actual economies, a shift may be triggered by real shocks that create their own impact on output. Here, we concentrate on the sole endogenous dynamics in the interaction between debt and demand. The analysis of the coordination of a shift in expectations is obviously beyond the scope of the present paper.

form: the steady state with unemployment is an absorbing state out of which no equilibrium path can emerge toward full employment.

Asymmetries between downturns and upturns of aggregate activity have been the subject of a large literature that cannot be acknowledged here[5]. The present model is not a model of business cycles, but it may provide an analytical framework that is related to the protracted impact of financial crises as discussed by Reinhart and Rogoff (2009), and Schularick and Taylor (2012).

The issues that are addressed in this paper are central in macroeconomics and hence not new. In policy circles, the positive impact on private consumption on growth during a recession is a triviality that has not been matched by a large number of formal analyses. The coordination of aggregate demand and supply has been discussed, among many others, by John Law (1705), Robertson (1892), and, of course, Keynes (1936). In the late 1960s, Leijhonhufvud (1968) emphasized again, with no analytical model, that in an economy where goods are traded with money, Say's Law may not hold.

The coordination of demand and supply was analyzed by Diamond (1982, 1984) in a model of search where individuals exchange their production, first directly for a good produced by another agent, which they consume, second, through the intermediation of money. The externality in exchanges is generated by the structural feature of search. Any equilibrium is inefficient compared to the first best. The demand is represented by the opportunities for meeting an agent who has produced in the recent past. With low production, the incentive to produce would be as low as going to an empty singles' bar. Supply and demand effects are therefore hard to separate and the model does not emphasize the critical role of saving which, in some cases, is socially unproductive in the aggregate, as argued by Keynes.

In this paper, there is no search and agents can always produced up to capacity but they are randomly matched in decentralized markets. The origin of that structure is found in Townsend (1980) and the picture of the two lanes of a turnpike of infinite length, with agents moving in opposite directions, trading goods and money with their vis-à-vis in a sequence of sales and purchases and never meeting the same agent again, as in Samuelson (1958). The closing of the Townsend turnpike with discrete placement on a circle of infinite diameter is equivalent to a continuum of agents on a finite circle

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[5]For a recent theoretical model where the asymmetry is generated by be real effects such as adjustment costs, see Jovanovic, (2009)

with random matching. That is the basis for the construction of Green and Zhou (1998, 2002), and Zhou (1999).

In Green and Zhou (2002), hereafter G-Z, infinitely lived agents are randomly matched in pairs and, to simplify, in each period an agent may be a either a buyer or a seller. Buyers are homogeneous in their utility of consumption. The random matching generates a demand for money. Except for the seller's maximum capacity of production of one, there is no indivisibility. In a symmetric equilibrium, sellers and buyers set the price of goods in money to a stationary p, for which the distribution of money holdings converges to a geometric distribution on the multiples of p. Under the price posting by the seller and the buyer in a match, there is a continuum of equilibrium prices p. A lower price generates higher real balances and is therefore Pareto improving.

In the present paper, the matching structure is such that in each period, an agent is both a seller and a buyer: within the period, each agent is a household with two heads, one buying and one producing and selling, with no communication during the day[6]. In addition, and that is a special feature here with respect to other studies, agents have heterogeneous utilities for consumption. In each period, an agent is set randomly (with independent draws through time) to have either a high or a low need for consumption. As in G-Z and other papers discussed below, trade takes place in decentralized markets between goods and bonds with a credit constraint. In an equilibrium, the price of goods in bonds has a stationary value p that is determined through a price posting that is similar (but not identical) to the process in G-Z.

An innovation in this paper is a particular case where the dynamic path is characterized through the evolution of one variable, the aggregate debt. This characterization is used to prove algebraically how the steady state with unemployment can be an absorbing state.

In full employment, the buying head knows that the producing head at home will sell during the day. He can therefore buy the good from the seller he meets in the match and be sure that his asset balance is the same at the end of the day as at the beginning. Since everyone is buying, everyone is producing and selling. There is no individual uncertainty and no motive for precautionary savings from one period to the next. As a difference with G-Z, the arbitrary trading price p can be normalized as in any model of Walrasian general equilibrium. There is no continuum of stationary price p.

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[6]This assumption is removed in a section where in each period, half the population buys in the morning and sells in the afternoon, with the reverse for the other half.

Suppose now that there is uncertainty on whether a matched customer will come during the period to purchase the good produced by the household. If the household has a low need for consumption during that period, it abstains from consumption in order to increase its savings, up to some endogenous maximum. By the matching process, the fraction of non consuming agents is the fraction of agents who do not sell, the unemployed. The probability of not making a sale during the period defines the unemployment rate. It generates the individual uncertainty and sustains the precautionary motive in a self-fulfilling process that leads to an equilibrium with unemployment.

The present study is also related to Guerrieri and Lorenzoni (2009) who show that in a liquidity (or credit) constrained economy with decentralized markets, aggregate shocks on the production may have a stronger impact on aggregate activity than in an economy with centralized markets. Their model is based on the structure of Lagos and Wright (2005) where each period is divided in three sub-periods with trading in decentralized markets[7] in the first two. In the last one, a centralized market enables agents to reconstitute their money balances. Because of the linear utility in that last sub-period, all agents have the same quantity of money for the next "round" and each of these rounds is effectively a sequence of three periods equilibria. This framework is not suitable for the analysis of the evolution of the distribution of money or bonds through time which is the central issue in G-Z and in the present paper. Furthermore, there is no aggregate shock here.

The model is presented in Section 2. For simplicity, goods are assumed to be indivisible, (an assumption that is removed in Section 7). Following the analysis in that section, prices between goods and bonds are normalized to one. In each period, an agent is determined by randomness to be either with a high need or a low need for consumption. High need agents always consume unless they are against their credit limit. Low need agents may accumulate savings. We can therefore analyze the equilibria according to the consumption function of the low need agents. By definition, a high regime takes place when all unconstrained agents consume. In the steady state, there is full employment. A low regime takes place when agents with a low need save up to some level that is endogenously determined. After setting the conditions for the optimal consumption function, the argument is developed in three steps, each with two propositions.

In Section 3, the equilibrium in the high regime is analyzed in two steps. First, the

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[7]Agents are not matched between pairs, but between islands with perfectly competitive markets. In each island there is a productivity shock.

regime is shown to converge to full employment. Second, after the path of unemployment is determined, the consumption function is shown to be optimal if the level of the aggregate debt is not too high. Convergence is slow, with a rate asymptotically equal to zero.

In Section 4, the same two-step analysis is carried for the low regime where agents save up to one unit of bond. The convergence rate to a steady state with unemployment is asymptotically strictly positive. Under some parameter conditions, the low regime path that converges to unemployment is an equilibrium.

Section 5 shows that under some conditions, there is a path from full employment to a steady state with unemployment and that the reverse is not true: the steady state with unemployment is an absorbing state.

Section 6 provides extensions and shows the robustness of the properties. The argument for the uniform price and stable equilibrium price p is reinforced in Section 6.1. In Section 6.2, there can be multiple steady states with different levels of precautionary savings ($N > 1$). Section 6.3 shows that the model with a higher credit limit is equivalent to a model with a positive level of aggregate assets and a credit limit of 1. (Outside money is a special case). In Section 6.4, the indivisibility assumption is removed. As in Green and Zhou (2002), the discrete support of the distribution of savings is stable when there is no indivisibility constraint. In Section 6.5, finally, households are not separated in two different heads during each period but the sequence of any agent's buying and selling, within a period, is determined randomly at the beginning of the period. In the concluding section, some issues of robustness and research are discussed. Given the abstract structure of the model, the policy remarks are suggestive and brief. Technical proofs are placed in the Appendix.

2 The model and an overview of equilibria

2.1 The model

There is a continuum of goods and agents indexed on a circle by $i \in [0, 1)$. Time is discrete. Goods are perishable between periods and indivisible. (Goods are divisible in Section 6.4.). In each period, an agent is either of a high or low *type*. A type is revealed at the beginning of the period and is defined by a random variable $\theta_{it} \in \{0, 1\}$ that is independently distributed across agents and periods, with a probability of the high type α, $(0 < \alpha < 1)$, which is known by all agents.

The high type has a higher utility for consumption than the low type, or, equivalently, a higher penalty for no consumption. The welfare W_i of any agent i is defined by the discounted sum of expected utilities of consumption from all future periods:

$$W_i = E\Big[\sum_{t\geq 0} \beta^t u(x_{i,t}; \theta_{i,t})\Big], \qquad \text{with} \qquad u(x, \theta) = (1 + \theta c)x - \theta c, \qquad (1)$$

where $x_{i,t} \in \{0, 1\}$ is the consumption in period t, $\beta = 1/(1 + \rho)$ is the discount factor between periods[8]. The parameter c that is the penalty for no consumption, while being a high type, will play an important role in the model.

In any period, agent i is endowed with the capacity to produce one unit of good i, at no cost. As in any macroeconomic model, agents consume goods produced by others. In a standard Walrasian model with complete markets, production and consumption take place according to plan after the auctioneer has found the equilibrium prices between supply and demand for consumption. The fundamental feature of the present model is the absence of a central institution to coordinate production and consumption. The complexity of the coordination of consumption and production of different goods in a modern economy with decentralized markets is modeled by the following structure.

During each time period t, each agent $i \in [0, 1)$ is a two-headed household with a buyer and a seller. The household first decides whether to consume or not for that time period. If the household consumes, then the buyer goes out to purchase the good j that is determined randomly by a matching function $j = \phi_t(i)$ from $[0, 1)$ to $[0, 1)$. Without loss of generality, that matching function is defined by[9]

$$\phi_t(i) = \begin{cases} i + \xi_t, & \text{if } i + \xi_t < 1, \\[2mm] i + \xi_t - 1, & \text{if } i + \xi_t \geq 1, \end{cases} \qquad (2)$$

where the random variables $\xi_t \in (0, 1)$ that are i.i.d., with a uniform density[10] on $[0, 1)$.

The seller stays "at home" and waits for a customer during the day, that is the buyer from the random household $j' = \phi_t^{-1}(i)$. The seller has the capacity to supply one

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[8]The assumption of a finite capacity and a linear utility function could also be replaced by an linear production cost γ with no capacity limit and a utility function with a slope of 1 or $1 + c$ for $x \leq 1$ and γ' for $x \geq 1$ with $\gamma' < \gamma$.

[9]The assignment j is independent of the decision to consume.

[10]One could use other matching functions ϕ_t such that for any subset $\mathcal{H}$ of $[0, 1)$, $\mu(\mathcal{H}) = \mu\Big(\phi_t(\mathcal{H})\Big)$, where μ is the Lebesgue-measure on $[0, 1)$ for a uniform random matching.

unit of good i, at a zero cost of production. The terms of the transactions will be described later.

A key assumption is that the household makes its consumption decision with an income that is subject to idiosyncratic shocks. At the time of the consumption decision, the household does not know whether it will make a sale during that day. Here, buyer and seller do not communicate during the day (as in Guerrieri and Lorenzoni (2009)). The separation of consumption and production in decentralized markets could also be done through a staggered matching, which is more complicated. An extension in that direction is provided by Section 6.5 in which the main properties of the model hold[11].

Agents can transfer wealth between periods through a bond. In most of the paper, the net supply of bonds is zero. One could also view the bonds as inside money. The savings of some agents are the debts of others. (The case of a positive aggregate supply of bonds is considered in Section 6.3.)

Agents are subject to a credit constraint, which can be justified by standard arguments on the recoverability of debts. The credit limit should be set in terms of real goods. Here, we set this limit such that with probability one at the end of any period, an agent cannot have a debt greater than one in units of good. (That exogenous limit is increased in Section 6.3). An endogenous determination of the credit limit, for example with respect to the expected future income of an agent, is much beyond the scope of the present paper and should be the subject of further research. The credit limit could be enforced here by a financial institution. Within a period, that institution allows the individual to have a debt position higher than the constraint provided that the limit is not exceeded at the end of the period, with certainty. (Banks do actually behave that way, as clearing houses, when various transfers are scheduled within a period).

There are two sets of prices, the intra-temporal prices between goods and bonds in any period t, and the rate of return of bonds between periods. Since agents make decisions in a decentralized environment and are symmetric in their technology and matching, it is natural to consider equilibria in which for any period t, the price of goods in bonds is the same for any $i \in [0, 1)$, and is equal to p_t. We now address three issues in sequence: first, we focus on equilibria where $p_t = p$ is constant over time; second, p is an equilibrium price of goods in bonds; third, p can be normalized to 1.

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[11]In this extension, there is no division between buyer and seller inside households, but the population of agents is divided in two groups of equal size in each day (period): "morning agents" consume in the morning and produce in the afternoon, and "afternoon agents" do the reverse.

First, we assume that the "price level", p_t is constant over time, and hence that the real rate of bonds (in terms of the price level) is equal to 0. There may be other equilibria in which p_t varies over time, exactly as in the Samuelson model of overlapping generations, there are equilibria with a non stationary price level, in addition to the equilibrium with a stationary price. The existence of that type of multiple equilibria (or a continuum of equilibria), has been studied extensively over the last half century and is obviously not the subject here. Hence our assumption of a constant price $p_t \equiv p$. Given the price p and the real constraint of one unit of goods, the credit constraint in units of bonds is equal to p.

Following Green and Zhou (2002), in the bilateral relation between buyer and seller, we assume price posting by the sellers and by the buyers, with no bargaining. Throughout the paper, sellers cannot observe the bond holdings of the buyers. There are two steps.

Assume that all sellers post the same price p. Consider the deviation by an individual seller. If he posts a lower price, he cannot attract more sales because the matching process is exogenous. When the household decides at the beginning of the period whether to consume, it expects that the price of the good is p, with probability 1. A lower posted price does not generate new customer and it reduces the value of the sale that takes place if a customer shows up[12]. Given the strategy of other sellers (posting p), a price strictly below p is strictly dominated. This first step generates the well known and discussed property in macroeconomics that prices are rigid downwards.

In the second step, for the "upward rigidity" of prices, one can use one of two independent arguments. In the first one, all buyers are posting a maximum purchase price of p, with no bargaining[13]. Since all sellers post the price p, that value is rational. Given the posting by the buyers, an upwards price deviation by the seller would end in no sale, whereas the production cost is nil. In the second argument, one assumes that bonds and debts are registered with a financial institution that issues cash for the day, and that cash carrying entails a very small proportional cost. When all prices are equal to p, it is not rational to carry more than p to the market, and no seller would post a price higher than p, since he would lose his demand.

The upward rigidity of prices is reexamined in the Section 6.1 that shows that under some condition, when all sellers post the price p, the equilibrium distribution of bond

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[12]The setting with exogenous matching is a stylized representation of the property that the process to increase the volume of sales through a price reduction takes time.

[13]Green and Zhou (2002) assume that seller and buyer post prices and quantities for the transaction.

is such that a seller posting a price higher than p would get a strictly smaller payoff, even if the buyer pays whatever price is asked by the seller, subject to his own credit constraint. As a summary at this stage, a seller who lowers his offer from p faces no higher demand and therefore a smaller amount of revenue, and he loses his demand if his price is greater than p. That property, which is not an assumption, is an extreme form of a kinked demand curve. The impact of the price behavior on aggregate activity is discussed again in the concluding section.

We have argued previously that the credit constraint must be in real terms and it is equal to p. As in the standard model of general equilibrium, we can normalize one price. Given the previous discussion, we can set $p = 1$.

2.2 The first best

The first best allocation maximizes a social welfare function. Without loss of generality, that function can be taken as the sum (integral) of the agents' utilities, which is identical to the *ex ante* expected utility of any agent. Since goods are perishable, the optimization applies in each period independently of the others and we can omit the time subscript. Let $x_i(\theta)$ be the consumption i for type θ. From (1), the first best allocation maximizes the function

$$J = \int_0^1 \Big(\alpha u(x_i(1); 1)\Big) di + (1 - \alpha) u(x_i(0); 0)\Big) di, \qquad \text{with} \qquad x_i \in \{0, 1\}. \quad (3)$$

The optimal allocation is obviously $x_i(\theta_i) = 1$. The maximum level of utility from consumption is achieved when consumption takes place in every period. In this allocation, all agents produce up to capacity. There is no unemployment.

Lemma 1. *In the first best allocation, each agent consumes one unit in each period and there is no unemployment.*

The first best will turn out to be one of the equilibria, according to the equilibrium definition that is given in the next section.

2.3 The structure of equilibria

As all transactions are zero or one, the distribution of the holdings of bonds is discrete with the support contained in $\mathcal{K} = \{-1, 0, 1, 2, \ldots\}$. An agent with holdings k is said to be in state k. When $k = -1$, he is in debt. Let $\Gamma(t)$ be the vector of the distribution of agents at the beginning of period t across states:

$$\Gamma(t) = (\gamma_{-1}(t), \gamma_0(t), \ldots)',$$

where $\gamma_k(t)$ is the mass of agents in state k. It will be shown later that in any equilibrium, the support of the distribution is bounded and $\Gamma(t)$ is a vector of finite dimension. The population has a total mass equal to 1 and the net aggregate amount of bonds is zero:

$$\sum_{k \geq -1} \gamma_k(t) = 1, \qquad \sum_{k \geq -1} k\gamma_k(t) = 0. \tag{4}$$

Let $\pi(t)$ be the fraction of agents who do not demand goods in period t. The total demand is $1 - \pi(t)$ and

$$1 - \pi(t) = \int x_{i,t} di, \tag{5}$$

where $x_{i,t} \in \{0, 1\}$ is the demand of agent i in period t. Because of random matching, the probability that an agent makes no sale is period t is $\pi(t)$, which will be called the rate of unemployment.

Let $k_{i,t}$ be the bond balance of agent i at the end of period t. That holding evolves such that

$$\begin{cases} k_{i,t+1} = 1 + k_{i,t} - x_{i,t}, \text{ with probability } 1 - \pi(t), \\ \\ k_{i,t+1} = k_{i,t} - x_{i,t}, \qquad \text{with probability } \pi(t). \end{cases} \tag{6}$$

Let $\tilde{\pi}^t = \{\pi(\tau)\}_{\tau \geq t}$ be a path of unemployment rates for periods $\tau \geq t$. Suppose perfect foresight on that path. The consumption function of an agent in period t depends only on his state (his savings in bonds), his type (low or high) and the path of future unemployment rates $\tilde{\pi}^t$. Given the consumption decisions and the unemployment rate in period t, the distribution of bonds in period $t+1$ is deterministic. That distribution determines $\pi(t+1)$ and the consumption functions in that period. From period to period, the path of the unemployment rate is deterministic, and rational agents who know the structure of the model can compute that path. The assumption of perfect foresight is therefore justified.

We can now define an equilibrium of the economy. The definition does not have to include the determination of the prices that was discussed previously and is independent of the unemployment rate and the distribution of bonds[14].

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[14]However, the argument of the price stability in Section 6.1 does depend on the distribution of bonds.

Definition of an equilibrium

An equilibrium is defined by an initial distribution of bonds $\Gamma(0)$ that satisfies (4), a path of unemployment rates $\tilde{\pi}^0 = \{\pi(\tau)\}_{\tau \geq 0}$, a consumption function $x_t = x(k_t, \theta_t, \tilde{\pi}^t)$, and the evolution of the distribution of assets that is determined by (6). The path of unemployment rates satisfies (5) and the consumption function of any agent maximizes, in any period t, his utility function

$$E[\sum_{\tau \geq t} \beta^{\tau - t} u(x_t; \theta_t)],$$

for a given balance k_t at the beginning of period t and type θ_t, subject to the accumulation constraint(6), the credit constraint $k_{i,t} \geq -1$, and under perfect foresight about the path of unemployment after period t, $\tilde{\pi}(t) = \{\pi(\tau)\}_{\tau \geq t}$.

2.4 Optimal consumption functions

Agents of the high type have a simple behavior: unless they are under a credit constraint, they consume.

Lemma 2. *In any period of an equilibrium, a high-type agent consumes if he is not credit constrained (in state -1).*

The property is intuitive. Suppose a high type agent saves today. He incurs a penalty c. The best use of that savings is to consume it in the future when he is also a high type and he is credit constrained. Because of discounting, the expected value of the future penalty is smaller than the penalty today. The agent is better off by not saving. In the Appendix, the proof of the Lemma follows the intuitive argument.

It follows from Lemma 2 that the dynamics of the economy are driven by the behavior of the low-type agents. Since, in an equilibrium, the path $\tilde{\pi}^t$ is known with perfect foresight, we omit it from the notation. The optimization problem of an agent is standard. For a given path of future unemployment rates, let $V_k(t)$ denote the utility of an agent in state k (with a balance k), at the end of period t, after transactions have taken place in period t. We have the Bellman equations

$$\begin{cases} V_{-1}(t-1) = \beta E\Big[u(0; \theta) + \pi(t)V_{-1}(t) + (1 - \pi(t))V_0(t)\Big], \\ V_k(t-1) = \beta E\Big[max_{x \in \{0,1\}}\Big(u(x; \theta) + \pi(t)V_{k-x}(t) + (1 - \pi(t))V_{k-x+1}(t)\Big)\Big], \ k \geq 0, \end{cases}$$
(7)

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where expectations are taken with respect to the preference shock θ that is realized in period t. Note that on the first line, the consumption of a credit constrained agent (in state -1) is equal to 0.

A low-type agent who is not credit constrained (in state $k \geq 0$) consumes in period t if his utility of consumption is greater than the value of one more unit of bonds. If he consumes, his balance increases with probability $1 - \pi(t)$ from k to $k + 1$ and his expected utility of savings at the end of the period is $\pi(t)V_{k-1}(t) + (1 - \pi(t))V_k(t)$. If he saves instead, his expected utility is $\pi(t)V_k(t) + (1 - \pi(t))V_{k+1}(t)$. The difference between the two expected utilities, $\zeta_k(t)$, is called here, by an abuse of notation, *the marginal utility of savings*:

$$\zeta_k(t) \equiv \pi(t)\Big(V_k(t) - V_{k-1}(t)\Big) + (1 - \pi(t))\Big(V_{k+1}(t) - V_k(t)\Big). \tag{8}$$

Consumption is optimal for an agent of the low type and in state k if and only if[15]

$$\zeta_k(t) \leq 1. \tag{9}$$

For any fixed t, the function $V_k(t)$ is increasing in k. It is bounded by the discounted value of consumption in all future periods. Therefore, the increments $V_k(t) - V_{k-1}(t)$ are arbitrarily small when k is sufficiently large and the previous condition (9) is satisfied. We have the following Lemma (proven in the Appendix).

Lemma 3. *There exists $\bar{N}$ such that in any equilibrium and any period, any agent with a balance that is at least equal to $\bar{N}$ consumes. One can choose $\bar{N}$ such that $\beta^{\bar{N}+1}(1 + c)/(1 - \beta) < 1$.*

Since a sale is at most for one unit, any agent, or any type, with savings greater than $\bar{N}$ does not increase his savings. The following property follows immediately.

Lemma 4. *Assume that the support of the initial bond distribution is bounded by N_0. Then, in any period of an equilibrium, the support of that distribution is bounded by $Max(N_0, \bar{N})$.*

Without loss of generality, in the rest of the paper, $N_0 \leq \bar{N}$ and in any equilibrium for any period, the support of the distribution of bonds is bounded by $\bar{N}$.

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[15]In the standard optimization problem with continuous consumption, the optimal rule compares the marginal utility of consumption (here equal to one) and the marginal value of accumulated savings. Here, consumption is discrete and the marginal utility of savings is replaced by a weighted average of the increments of utilities when the balance increases by one unit from $k - 1$, (with probability $1 - \pi(t)$ when no sale is made in the period), and to k (with probability $\pi(t)$).

It is intuitive that in any period, the marginal utility of saving of an agent decreases with the level of saving. (This property is not taken as an assumption). If this property holds, high type agents consume if and only if their savings at the beginning of a period are at least equal to N, where N depends on the path of the unemployment rate in the future. We define such a function as an N-*consumption function*. This property is not an assumption but it will be shown to hold in an equilibrium. From the previous definitions, we have the following characterization.

Lemma 5. *The N_t-consumption function is optimal in period t if and only if*

$$\begin{cases} \zeta_k(t) \geq 1 \ for \ k \leq N_t - 1, \\[2mm] \zeta_k(t) \leq 1 \ for \ k > N_t - 1, \end{cases} \tag{10}$$

$$with \ \ \zeta_k(t) = \pi(t)\Big(V_k(t) - V_{k-1}(t)\Big) + (1 - \pi(t))\Big(V_{k+1}(t) - V_k(t)\Big).$$

From the previous result, we are led to define two regimes of consumption.

Definition of two regimes

In a high regime, all not credit constrained agents consume. In a low regime, a low-type agent consumes in a period t if and only his savings at the beginning of that period are at least equal to $N_t \geq 1$.

Note that the consumption function is unique in the high regime. In a low regime however, the equilibrium value of N_t may not be unique.

2.5 High and low regimes with three states

The precautionary motive is driven by the parameter c, and it increases with its level. We will see that, when the value of c is in some bounded interval, it is not optimal to save more than one unit. In that case, the dynamics are characterized by a variable of one dimension and they can be analyzed algebraically. When c increases beyond that interval, the "target" level of saving rises above 1, the vector of dynamic variables has a dimension higher than one, and one has to rely on numerical results. In order to keep an algebraic analysis, we focus in the next three sections on the cases of high and low regimes with individual savings bounded by 1. It should be clear that this upper bound on savings is not exogenous, and it will be shown that in equilibrium, no agent saves more than one unit of bonds.

When the support of the distribution is bounded by 1, that distribution is defined by

the vector

$$\Gamma(t) = (\gamma_{-1}(t), \gamma_0(t), \gamma_1(t))',$$

where $\gamma_k(t)$ is the mass of agents in state k at the beginning of period t. Because the total mass of agents is 1, and the net supply of bonds is 0,

$$\gamma_{-1}(t) + \gamma_0(t) + \gamma_1(t) = 1, \qquad \gamma_{-1}(t) = \gamma_1(t). \tag{11}$$

The number of degrees of freedom is therefore reduced to one, and the analysis of the dynamics can be characterized by one variable, which will be chosen as the amount of debt, $B(t) = \gamma_{-1}(t) = \gamma_1(t)$.

Let $x \in \{0,1\}$ be the consumption of a low-type agent in state 0: in the 0-consumption function, that agent consumes, $x = 1$; in the 1-consumption function, that agent saves, $x = 0$. That is the only difference between the high and low regimes when no agent holds more than one unit of savings. Therefore, the Bellman equations (7) can be written

$$\begin{cases} V_{-1} = \beta\Big(\alpha\Big(-c + \pi V_{-1} + (1-\pi)V_0\Big) + (1-\alpha)\Big(\pi V_{-1} + (1-\pi)V_0\Big)\Big), \\ V_0 = \beta\Big(\alpha\Big(1 + \pi V_{-1} + (1-\pi)V_0\Big) + (1-\alpha)\Big(x + \pi V_{-x} + (1-\pi)V_{1-x}\Big)\Big), \\ V_k = \beta\Big(1 + \pi V_0 + (1-\pi)V_1\Big), \quad \text{for} \quad k \geq 1, \end{cases} \tag{12}$$

with $x = 0$ in the low regime, and $x = 1$ in the high regime. In the next two sections, we assume that the economy is permanently on a path with high and with a low regime, respectively, and we will show that under some condition, these paths are equilibria.

3 The high regime

In the high regime, the agents who do not consume are at the credit constraint, in state -1, with a debt of 1. By the matching process, that fraction is equal to the fraction of agents who do not produce, the unemployment rate:

$$\pi(t) = B(t). \tag{13}$$

3.1 Dynamics

The evolution of the bond distribution $\Gamma(t) = (\gamma_{-1}(t), \gamma_0(t), \gamma_1(t))'$, and of the unemployment rate are determined by

$$\Gamma(t+1) = H(\pi_t).\Gamma(t), \qquad \text{with} \qquad \pi_t = \gamma_{-1}(t), \tag{14}$$
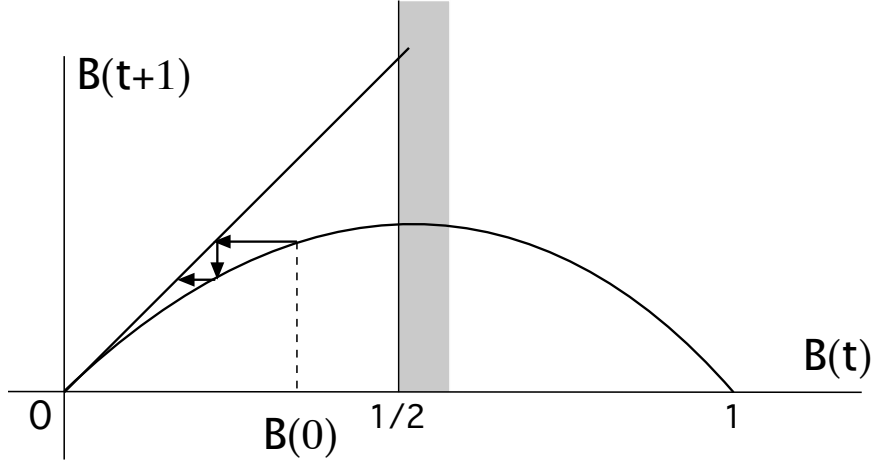
Figure 1: Dynamics of the aggregate debt (equal to the mass of credit constrained agents) in the high regime with three states.

---

and the transition matrix

$$H(\pi) = \begin{pmatrix} \pi(t) & \pi(t) & 0 \\ 1 - \pi(t) & 1 - \pi(t) & \pi(t) \\ 0 & 0 & 1 - \pi(t) \end{pmatrix}. \tag{15}$$

For example on the first line, agents are in state -1 at the end of period t: either because they were in state -1 at the beginning of the period, could not consume, and made no sale, hence the term $H_{11} = \pi(t)$; or because they were in state 0 at the beginning of period t, consumed, and made no sale, hence the term $H_{12} = \pi(t)$.

Since $\pi(t) = B(t)$, using (11), (14) and (15),

$$B(t + 1) = B(t)\Big(1 - B(t)\Big). \tag{16}$$

The evolution of $B(t)$ is represented in Figure 1. Because each bond is the asset of an agent and the liability of another, the total amount of bonds is bounded by $1/2$. For any $B(0)$, the value of $B(t)$ converges to 0 which defines the full employment steady state. That steady state is globally stable under the consumption function of the high regime. Whether that consumption is optimal on that path, and the path is an equilibrium, will be analyzed below.

Proposition 1. *If the support of the initial distribution of bonds is bounded by 1, in the high regime, the economy converges monotonically to the full-employment steady state with no debt. The rate of convergence of the unemployment rate is asymptotically equal to zero.*

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3.2 Equilibrium

Using the Bellman equations (12) with $x = 1$, the marginal utilities of savings $\zeta_k(t)$, $(t \geq 0)$, in state k, defined in (8), satisfy the equations of backward induction

$$
\begin{cases}
\zeta_0(t) = \beta\Big(\pi(t)(1 + \alpha c) + (1 - \pi(t))\zeta_0(t + 1)\Big), \\
\zeta_k(t) = \beta\Big(\pi(t)\zeta_{k-1}(t + 1) + (1 - \pi(t))\zeta_k(t + 1)\Big), \text{ for } k \geq 1.
\end{cases}
\tag{17}
$$

By repeated iterations of (17), and using $\pi(t) = B(t)$, we have

$$
\begin{cases}
\zeta_0(0) = (1 + \alpha c)\beta\Big(B(0) + \sum_{t \geq 0} \beta^{t+1}(1 - B(0))\ldots(1 - B(t))B(t + 1)\Big), \\
\qquad\qquad \text{with} \qquad B(0) = B, \qquad B(t + 1) = B(t)(1 - B(t)), \\
\zeta_k(t) = \dfrac{\beta\pi(t)}{1 - \beta(1 - \pi(t))}\zeta_{k-1}(t), \qquad \text{for } k \geq 1.
\end{cases}
\tag{18}
$$

The previous equation shows that $\zeta_0(0)$ is a function of $B(0)$ and c. We can take the period 0 as an arbitrary period and let $\zeta(B; c)$ be the marginal utility of saving for an agent with no savings in a period when the aggregate debt is equal to B. We can expect that a higher level of the aggregate debt B in some period induces higher unemployment rates on the future path of the high regime, and therefore a higher value of $\zeta(B; c)$. That intuition is confirmed by the following result.

Lemma 6. *In the high regime, the utility of a unit of additional savings is a function of the debt B, $\zeta(B; c)$, as defined in (18). It is continuous, strictly increasing in $B \in [0, 1/2]$, and $\zeta(0) = 0$.*

From (18), the value of $\zeta(B; c)$ increases linearly with the cost parameter c. Define c^* such that $\zeta(1/2; c^*) = 1$. Assume that $c \leq c^*$. From Lemma 6 and because $B(t)$ is decreasing with time (Proposition 1 and Figure 1), $\zeta_0(t) < 1$ for any $t \geq 0$. Using the second equation in (18), $\zeta_k(t) < 1$ for any $t \geq 0$ and $k \geq 1$. Using Lemma (5), the 0-consumption is optimal for the low-type agents: the high regime is an equilibrium.

Assume now that $c > c^*$. By continuity, there exists $\bar{B}$ such that $\zeta(\bar{B}; c) = 1$. The arguments in the previous paragraph apply for any $B < \bar{B}$. We have proven the following result.

Proposition 2.

If $c \leq c^*$, with $\zeta(1/2; c^*) = 1$ and $\zeta(\cdot; c)$ defined in (18), the full employment equilibrium is globally stable. From any level of aggregate debt, the high regime is an equilibrium.

If $c > c^*$, there exists $\bar{B}(c) < 1/2$ such that from an initial level of debt B, the high regime is an equilibrium if and only if $B \leq \bar{B}(c)$.

The full employment steady state can be the limit of an equilibrium path only if after some date, that path is in the high regime. In the context of the present paper, the main property in the previous result is probably the local stability that validates the relevance of the full employment state. The case $B > \bar{B}(c)$ may generate a saving trap and will be examined in Section 5.

4 The low regime

The only difference with the high regime is that low-type agents with savings of zero do not consume, but save instead. As in the previous section, the support of the distribution of bonds is bounded by 1 in the period 0 and therefore in every period, and the dynamics are analyzed through the distribution vector $\Gamma(t)$ of dimension 3.

4.1 Dynamics

The evolution of $\Gamma(t)$ is now determined by

$$\Gamma(t+1) = L(\pi(t)).\Gamma(t), \qquad \text{with} \qquad L(\pi) = \begin{pmatrix} \pi & \alpha\pi & 0 \\ 1 - \pi & a & \pi \\ 0 & b & 1 - \pi \end{pmatrix}, \qquad (19)$$

and

$$\begin{cases} a = (1 - \alpha)\pi + \alpha(1 - \pi), \\ b = (1 - \pi)(1 - \alpha). \end{cases} \qquad (20)$$

In any period t, the agents who do not consume are either credit constrained (of a mass equal to $\gamma_{-1}(t)$), or of low type and with no savings (of mass $(1 - \alpha)\gamma_0(t)$). Hence,

$$\pi(t) = \gamma_{-1}(t) + (1 - \alpha)\gamma_0(t).$$

Since the debt is equal to $B(t) = \gamma_1(t) = \gamma_{-1}(t)$ and $\gamma_0(t) = 1 - \gamma_{-1}(t) - \gamma_1(t)$, the unemployment rate is a linear function of $B(t)$:

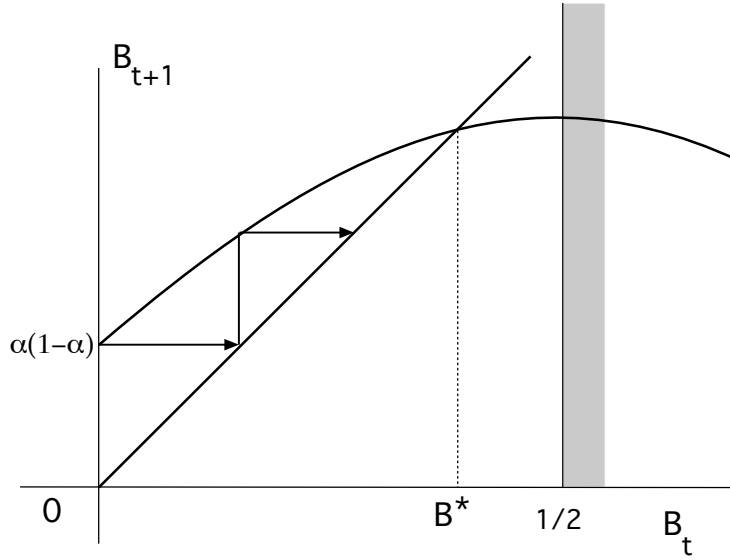$$\pi(t) = 1 - \alpha - (1 - 2\alpha)B(t). \qquad (21)$$

Figure 2: Dynamics of the fraction of liquidity-constrained agents in the low regime

Using (19), $B(t+1) = \gamma_1(t+1) = b\gamma_0(t) + (1-\pi)\gamma_1(t)$, and one finds the evolution of $B(t)$:

$$B(t+1) = P(B(t)), \text{ with } P(B) = -(1-2\alpha)^2 B^2 + (1-2\alpha)^2 B + \alpha(1-\alpha). \quad (22)$$

The polynomial $P(B)$ has its maximum at $B = 1/2$, with $P(1/2) \leq 1/4$. Hence, there is a unique value $B^* \in (0, 1/2)$ such that $P(B^*) = B^*$. For any initial value $B(0)$, the sequence $B(t+1) = P(B(t))$ converges monotonically to the fixed point B^*, as represented in Figure 2 that represents the important case of the path that begins at the full-employment steady state, with $B(0) = 0$.

By (21), the evolution of the unemployment rate is also monotone but we have two cases. When the fraction of high-type agents, α, is relatively large, *i.e.*, $\alpha > 1/2$, the unemployment rate increases over time with the debt $B(t)$.

When $\alpha < 1/2$, the fraction of high-type, who always consume, is relatively small. In the first period of a switch to the low regime, demand falls by a large amount since all the low-type agents (who have a greater mass) do not consume. The unemployment rate jumps up. As more agents accumulate the desired level of savings to consume, unemployment decreases over time.

In Figure 2, from the initial position of full employment with zero debt, the amount of debt increases over time to its steady state level. There is a remarkable difference

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with Figure 1: for any parameter of the model, the graph of the curve $P(B)$ is not tangent to the 45° line at the fixed point $B^* = P(B^*)$. As for any dynamics with one variable, the convergence to a steady state is asymptotically exponential and here with a strictly positive convergence rate. Recall that in the high regime, the asymptotic convergence rate is zero. In comparison, the convergence in the low regime is much faster than in the high regime. For the particular case with $\alpha = 1/2$, the convergence is instantaneous as half the individuals do not consume in period 0. One can easily verify that the stationary distribution of debt $(1/4, 1/2, 1/4)$ is attained at the end of period 0.

In general, when a full-employment economy switches to the low regime in period 0, the fraction of agents who do not consume is that of the low-type agents. It is equal to $1 - \alpha$ and the reduction of consumption is equal to the unemployment rate $\pi(0) = 1 - \alpha$. The evolution of the unemployment rate after period 0 depends on α.

Proposition 3. *If the support of the initial distribution of bonds is bounded by 1, in the low regime, the economy converges monotonically to a steady state with unemployment and debt.*

(i) The rate of convergence is asymptotically strictly positive.

(ii) From an initial position of no debt, in the low regime after the jump in period 0, the rate of unemployment increases with time if $\alpha > 1/2$, and decreases with time if $\alpha < 1/2$.

4.2 Equilibrium

As for the high regime, the marginal utilities of saving $\zeta_k(t)$ are obtained through the Bellman equations (12) where x is now replaced by 0. In parallel to equations (17), we have

$$
\begin{cases}
\zeta_0(t) = \beta\Big(\pi(t)\alpha(1+c) + b(t) + a(t)\zeta_0(t+1)\Big), \\
\zeta_1(t) = \beta\Big(\pi(t)(1-\alpha) + \pi(t)\alpha\zeta_0(t+1) + (1-\pi(t))\zeta_1(t+1)\Big), \\
\zeta_k(t) = \beta\Big(\pi(t)\zeta_{k-1}(t+1) + (1-\pi(t))\zeta_k(t+1)\Big), \text{ for } k \geq 2.
\end{cases} \tag{23}
$$

with $a(t) = \alpha(1 - \pi(t)) + \pi(t)(1-\alpha)$, $b(t) = (1-\pi(t))(1-\alpha)$.

From Lemma 5, the low regime is an equilibrium if and only if $\zeta_0(t) \geq 1$, $\zeta_k(t) \leq 1$, for $k \geq 1$.

The steady state

In the steady state, omitting the time argument,

$$
\begin{cases}
\zeta_0 = \beta \dfrac{\pi\alpha(1+c)+b}{1-\beta a}, \\[2ex]
\zeta_1 = \beta \dfrac{(1-\alpha)\pi + \alpha\pi\zeta_0}{1-\beta(1-\pi)}, \\[2ex]
\zeta_k = \lambda^{k-1}\zeta_1, \text{ with } \lambda = \dfrac{\beta\pi}{1-\beta(1-\pi)} < 1 \text{ for } k \geq 2.
\end{cases}
\tag{24}
$$

Let π^* be the unemployment rate of the low regime. $\zeta_0 > 1$ is equivalent to

$$
c > \underline{c}_1 = \frac{\rho}{\pi^*\alpha}.
\tag{25}
$$

This condition is equivalent to $\pi\alpha c/\rho \geq 1$, and it has an intuitive interpretation. A low regime is an equilibrium in the steady state only if low-type agents have an incentive to save when their balance is zero. The value of savings is the discounted value of the stream of payments, in each future period, where the payment is equal to the product of the penalty c, the probability of being a high type and the probability of not making a sale in that period. When that discounted value is greater than 1, which is the value of consumption, saving is optimal.

The values of ζ_0 and ζ_1 in (24) are linear increasing functions of c, with $\zeta_1 < 1$ if $\zeta_0 = 1$. Hence, there is a threshold $\bar{c}_1$ such that if $c \in (\underline{c}_1, \bar{c}_1)$, $\zeta_0 > 1 > \zeta_1$, and $\zeta_0 = 1 > \zeta_1$ for $c = \underline{c}_1$, $\zeta_0 > \zeta_1 = 1$ for $c = \bar{c}_1$. From (24), if $c \in [\underline{c}_1, \bar{c}_1]$, $\zeta_k < 1$ and the low regime steady state is an equilibrium.

If $c > \bar{c}_1$, the penalty of no consumption for the high type is sufficiently strong to accumulate more than one unit of savings. In this case, which will be analyzed in Section 6.2, the support of the distribution of bonds extends beyond 1. From the previous discussion and since the variables on the dynamic path are continuous functions of the initial condition, we have the following result.

Proposition 4. *Let π^* and B^* be the rate of unemployment and the level of the debt in the steady state of the low regime with the 1-consumption function (Proposition 3).*

(i) The steady state is an equilibrium if and only if $\rho/(\alpha\pi^*) = \underline{c}_1 \leq c \leq \bar{c}_1$, for some value $\bar{c}_1$.

(ii) If $c \in (\underline{c}_1, \bar{c}_1)$, there exists an open interval containing B^* such that if B is in that interval, the low regime with the 1-consumption function is an equilibrium.

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The second part of the proposition holds by a continuity argument: if $c \in (\underline{c}_1, \bar{c}_1)$, in a neighborhood of the steady state, $\zeta_0(t) > 1 > \zeta_1(t)$ and the dynamic path defines an equilibrium in a neighborhood of B^*. The steady state of that low regime is locally stable. The interesting issue, to which we now turn, is the global stability, in particular the transitions between steady states in the high and the low regime, respectively.

5 Transitions and traps

Under some parametric conditions, the economy at full employment can, at any time, move to a situation of unemployment but that the reverse may not be true.

5.1 Transition from full employment to unemployment

Assume that the economy is at full employment. To simplify the discussion, $\alpha = 1/2$. Under the parameter conditions in Proposition 4, $(c \in (\underline{c}_1, \bar{c}_1))$, the steady state under the 1-consumption function is an equilibrium. When $\alpha = 1/2$, $\underline{c}_1 = 4\rho$ and $\bar{c}_1 = 4\rho(3 + 4\rho)$. Since the transition path from full employment to that steady state takes one period, that transition is an equilibrium. Because paths are continuous functions of the parameters, we can extend the equilibrium property when α belongs to an open interval containing $1/2$.

Proposition 5. *If $c \in (4\rho, 4\rho(3 + 4\rho))$, then there is an open interval $\mathcal{J}$ containing $1/2$ such that if $\alpha \in \mathcal{J}$, there is an equilibrium path with the 1-consumption function, in the low regime, from the full-employment steady state to the steady state with unemployment. That steady state is an equilibrium.*

The conditions on the penalty c have a simple interpretation. $c > 4\rho$ is the condition (25) for the low regime to be an equilibrium: the penalty must be sufficiently high for agents with zero savings to save. The condition $c < 4\rho(3 + 4\rho)$ is only technical: it guarantees that any agent with one unit of savings chooses to consume. If $c > 4\rho(3 + 4\rho)$, low-type agents save more than one unit. They have an N_t-consumption function where $N_t \geq 2$, may not be constant over time. There is no algebraic characterization of the dynamic path. A numerical analysis of the equilibrium steady states with savings of more than one unit is presented in Section 6.2. The same upper-bound condition will be used in the next two results and it is likewise a technical assumption.

In Proposition 5, the interval $\mathcal{J}$ may not be small. When $\alpha \neq 1/2$, the convergence to the low regime steady state is relatively fast and the optimality conditions of the

1-consumption function hold when they hold in the steady state[16].

From the previous result, if the economy is at full employment with no debt at time 0, there are at least two equilibrium paths. In the first, unemployment is maintained for ever. In the second, self-fulfilling pessimism sets the economy on a path to unemployment[17].

5.2 Saving trap

When expectations switch from full employment to a path with unemployment, because individuals can reduce their consumption, the expectation of unemployment is self-fulfilling. The reverse does not hold. In an economy with unemployment, some agents are credit constrained. If expectations shift toward optimism, these agents cannot consume and are unaffected by expectations. These agents impose an "inertia" that may prevent the take-off on a path toward full employment. Proposition 2 showed that, for some parameters, there is no high regime equilibrium path that converges to full employment if the debt exceeds a critical level. Such a situation may occur in the steady state of the low regime with the 1-consumption function, as shown in the next result.

Proposition 6. *Assume that the level of the aggregate debt is as in the low regime steady state of Proposition 4. If c belongs to the interval $(6 + 8\rho, 4\rho(3 + 4\rho))$, then there is an open interval $\mathcal{J}'$ containing $1/2$ such that if $\alpha \in \mathcal{J}'$, there is no equilibrium path from the low regime steady state to the full employment steady state.*

The previous result requires that $6 + 8\rho < 4\rho(3 + 4\rho)$, which holds if ρ is greater than some value ρ^*. In this case, the unemployment in the near future has sufficient weight such that agents with zero savings have a marginal utility of bonds greater than one. Since a switch to the high regime requires their consumption, a take-off of the economy cannot occur. If c satisfies the condition in Proposition 6, then it satisfy the condition in Proposition 5. Redefining the intervals $\mathcal{J}$ and $\mathcal{J}'$, we can combine the two previous results

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[16]For example, the transition from full employment with the 1- consumption function is an equilibrium if $\alpha = \beta = 0.8, c = 3.1$.

[17]There is obviously a continuum of other equilibrium path when the switch between regimes is driven by a Poisson process.

Proposition 7. *If c belongs to the interval $(6 + 8\rho, 4\rho(3 + 4\rho))$, then there is an open interval $\mathcal{J}^*$ containing $1/2$ such that if $\alpha \in \mathcal{J}^*$, (i) there is an equilibrium path from full employment to the steady state with unemployment; (ii) there is no equilibrium path from the steady state with unemployment to full employment.*

As discussed after Proposition 5, the upper bound of the interval for c, $4\rho(3 + 4\rho)$ is technical, in order to ensure that agents do not save more than one units. The main sufficient condition is the lower bound $c > 6 + 8\rho$.

When for some distribution of bonds there are multiple equilibrium paths, a switch from one path to another can occur only if the expectations of agents shift, with perfect coordination, to the new path. In an actual economy, such a switch may be induced by a real shock that also introduces its own impact. In the absence of real shocks—they are not in the analytical framework of this study—the well-known technical artifice of a coordination through "sunspots" is convenient but it requires a coordination between individuals that is not credible. The device does not take into account the inertia that is created by the observation of history, among other issues. A proper framework for the analysis of self-fulfilling switches is the removal of the assumption of common knowledge but that task is beyond the scope of the present paper[18].

6 Extensions

6.1 Prices

We have seen in Section 2.1 that when all sellers post a price of 1, the posting of a price smaller than 1 is strictly inferior. We now show that under some conditions, an upward deviation from 1 is also inferior for a seller when other sellers post the price of 1 and the economy is in a steady state of the low regime as in Section 4, with the support of the distribution of bonds $\{-1, 0, 1\}$.

Proposition 8. *Assume that the price of goods in bonds is stationary at 1. There is an open interval $\mathcal{I}$ that contains $1/2$ such that if $\alpha \in \mathcal{I}$, (i) posting a sales price $p \neq 1$ yields a strictly smaller utility than posting $p = 1$ when the economy is in a low regime path from an initial position of full employment; (ii) if the debt in period 0 is smaller than some value $\hat{B}$, the same property holds on a high regime path that converges to full employment.*

---

[18]Regime switches without common knowledge, in an economy with small real perturbations, are analyzed in Chamley (1999).

6.2 High savings

In the previous sections, agents save no more than one unit of bond because the cost of non consumption for the high type, c, is within some range $[\underline{c}_1, \bar{c}_1]$, (Proposition 4). When $c > \bar{c}_1$, the precautionary motive induces agents to save more than 1. The main properties hold but the analysis cannot be done with algebra. Here, we consider only steady states. In an equilibrium steady state, the consumption function is an N-consumption where agents accumulate savings up to the level N. Near the steady state, by continuity, the optimal consumption function will also be an N-consumption function. We now show that for any N, such a consumption function generates a steady state that will be shown to be an equilibrium for some value of c.

Multiple steady states

Without loss of generality, we can assume that the support of the bond distribution is $\{-1, 0, 1, \ldots, N\}$. That distribution is defined by a column vector $\Gamma(t) = (\gamma_{-1}(t), \gamma_0(t), \gamma_1(t), \ldots, \gamma_N(t))'$ of dimension $N + 2$. In any period t of the low regime with an N-consumption function, the agents who consume are either of the high type with no constraint, $\alpha(1 - \gamma_{-1}(t))$, or of the low type in state N, $(1 - \alpha)\gamma_N(t)$. Since the aggregate consumption is $1 - \pi(t)$, the unemployment rate is

$$\pi(t) = 1 - \Big(\alpha(1 - \gamma_{-1}(t)) + (1 - \alpha)\gamma_N(t)\Big). \tag{26}$$

Extending (19) to the dimension $N + 2$, the evolution of the distribution $\Gamma(t)$ is given by

$$\Gamma(t + 1) = L(\pi(t)).\Gamma(t), \tag{27}$$

where the transition matrix $L(\pi)$ of dimension $(N + 2) \times (N + 2)$ is defined by

$$L(\pi) = \begin{pmatrix} \pi & \alpha\pi & 0 & 0 & \ldots & 0 \\ 1 - \pi & a & \alpha\pi & 0 & \ldots & 0 \\ 0 & b & a & \alpha\pi & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & b & a & \alpha\pi & 0 \\ 0 & \ldots & 0 & b & a & \pi \\ 0 & 0 & 0 & \ldots & b & 1 - \pi \end{pmatrix}, \tag{28}$$

with $a = (1 - \alpha)\pi + \alpha(1 - \pi), \quad b = (1 - \pi)(1 - \alpha)$.

The matrix $L(\pi)$ has 1 as an eigenvalue of order 1 (Lemma 8 in the Appendix). Let $w(\pi) = (w_{-1}(\pi), \ldots, w_N(\pi))'$ be the eigenvector associated to the unit eigenvalue, and define $S_N(\pi) = \sum_{k=-1}^{N} k w_k(\pi)$ as the aggregate level of wealth associated to
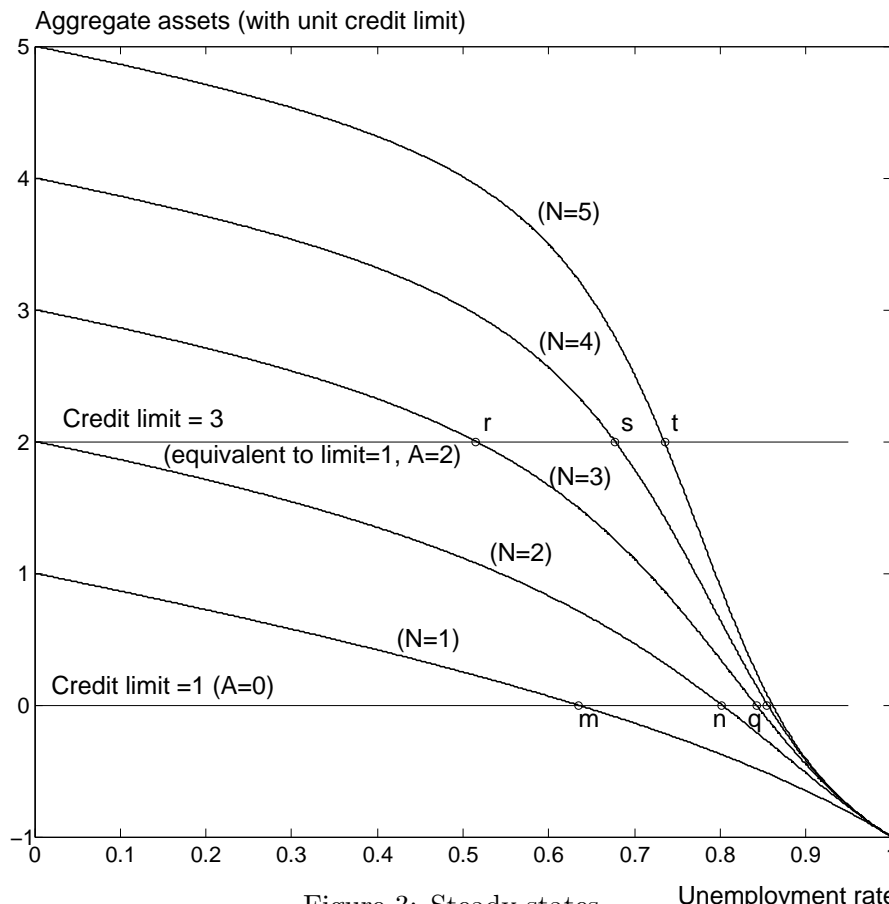
25

Figure 3: Steady states

Parameter values: $\beta = 0.9$, $\alpha = 0.25$.

In the base case with a credit limit of 1 and zero aggregate wealth, the points m, n and q represent steady states with savings up to $N = 1$, 2, and 3 units. The point m is an equilibrium with $\pi = 0.634$ for $c \in [0.701, 3.1075]$, the point n with $\pi = 0.802$ for $c \in [1.826, 4.334]$, and the point q with $\pi = 0.842$ for $c \in [3.642, 7.225]$.

On the upper line, the credit limit is equal to 3 with zero aggregate wealth or, equivalently, the limit is 1 with aggregate wealth equal to 2 (see Section 6.3). For $c = 60$, the points r, s and tg are equilibria. When the credit limit is $A \geq 1$, the maximum savings is $N - A + 1$. For example, at the point r, agents save up to one unit.

$w(\pi)$. By assumption, the level of aggregate wealth is zero, and if π^* is a steady state unemployment rate, $S_N(\pi^*) = 0$.

One verifies that $S_N(0) = N$ and $S_N(1) = -1$. The function $S_N(\pi)$ is declining in π and continuous in π. Hence, there exists a value π_N such that $S_N(\pi_N) = 0$. This value of π_N defines a steady state for the N-consumption function. The graphs of the function are represented in Figure 3 for different values of N. One verifies that the steady state unemployment rate rises with N: when agents accumulate a larger amount of precautionary savings, demand is lower and unemployment higher.

Equilibria

Consider a steady state with the $N-$consumption function. Using the Bellman equations (7) in the steady state, the notation $v_k = V_k - V_{k-1}$ for any $k \geq 0$, and recalling the marginal utility of savings $\zeta_k = \pi v_k + (1 - \pi) v_{k+1}$, we have for $N \geq 2$,

$$\begin{cases} v_0 = \beta \Big(\alpha(1 + c) + (1 - \alpha) \zeta_0 \Big) \\ v_k = \beta \Big(\alpha \zeta_{k-1} + (1 - \alpha) \zeta_k \Big) \text{ for } 1 \leq k \leq N - 1, \\ v_N = \beta \Big(1 - \alpha + \alpha \zeta_{N-1} \Big), \\ v_{N+k} = \beta \zeta_{N+k-1}, \text{ for } k \geq 1. \end{cases}$$

The vector $\zeta' = (\zeta_0, \dots, \zeta_{N-1})'$ satisfies the stationary equation[19]

$$\zeta = \beta Q_N(\pi)\zeta + \beta R_N(\pi), \tag{29}$$

where the matrix $Q_N(\pi)$ and the vector $R_N(\pi)$ are of dimension N:

$$Q_N(\pi) = \begin{pmatrix} a & b & 0 & 0 & \dots & 0 \\ \pi\alpha & a & b & 0 & \dots & 0 \\ 0 & \pi\alpha & a & b & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \pi\alpha & a & b \\ 0 & \dots & 0 & 0 & \pi\alpha & a \end{pmatrix}, \qquad R_N(\pi) = \begin{pmatrix} \pi\alpha(1 + c) \\ 0 \\ \vdots \\ \vdots \\ 0 \\ b \end{pmatrix}.$$

Given the vector ζ, one finds that

$$\begin{cases} \zeta_N = \dfrac{\beta\pi}{1 - \beta(1 - \pi)}(1 - \alpha + \alpha\zeta_{N-1}), \\ \zeta_{N+k} = \Big(\dfrac{\beta\pi}{1 - \beta(1 - \pi)} \Big)^k \zeta_N, \text{ for } k \geq 1. \end{cases} \tag{30}$$

From the second equation, if $\zeta_N \leq 1$, then $\zeta_k < 1$ for any $k > N$. The necessary and sufficient condition for an equilibrium is $\zeta_k \geq 1$ for $k = 0, \dots, N - 1$ and $\zeta_N \leq 1$. In order to determine whether a steady state with an N-consumption function is an

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[19]On a transition with an N-consumption function, $\zeta(t) = \beta Q_N(\pi(t))\zeta(t + 1) + \beta R_N(\pi(t))$.

equilibrium, one first determines the unemployment rate. Then one computes the vector ζ, verifies that all its components are greater than 1, and that ζ_N in (30) is smaller than 1. These inequalities depend on the cost parameter c.

Numerical results are reported in Figure 3. Given fixed α and β, for each N, there is an interval $[\underline{c}_N, \bar{c}_N]$ such that the steady state under the N-consumption function is an equilibrium if and only $c \in [\underline{c}_N, \bar{c}_N]$. As can be expected, the higher N, the higher the interval. The intervals overlap. For example, if $c = 4$, both the points n and q are equilibria. The multiplicity of equilibria has a simple interpretation. A higher unemployment rate generates a higher precautionary motive and more savings. The higher savings depress demand and generates the self-fulfilling higher unemployment rate.

6.3 Higher credit limits and positive aggregate supply of bonds

The case of a higher credit limit A, with $A \geq 2$, turns out to be equivalent to a credit limit of 1 with an aggregate wealth equal to $A - 1$.

By the argument of Lemma 4, the support of the distribution is bounded and we can assume that in a steady state, this support is the set $\{-A, -A+1, \ldots, Z\}$. Extending the previous notation, the distribution has to satisfy the two conditions that the mass of agents is 1 and the aggregate wealth is 0:

$$\sum_{k=-A}^{k=Z} \gamma_k = 1, \qquad \sum_{k=-A}^{k=Z} k\gamma_k = 0. \tag{31}$$

Since $\sum \gamma_k = 1$, adding $A - 1$ on both sides of the second equation,

$$\sum_{k=-A}^{k=Z} (k + A - 1)\gamma_k = A - 1.$$

Define $\tilde{\gamma}_k = \gamma_{k+A-1}$. The previous equation can be written

$$\sum_{k=-1}^{k=N} k\tilde{\gamma}_k = A - 1,$$

with

$$N = Z + A - 1. \tag{32}$$

A steady state is the same as in the case of a credit limit of 1 with an aggregate wealth $A - 1$. Multiple steady states are represented in Figure 3 for $A = 3$ along the upper horizontal line. There is no steady state with an N-consumption function, $N \leq 3$. At

the steady state represented by the point r with $N = 3$ and $A = 3$, agents save up to one unit of bond ($Z = 1$ in (32)). In the steady states represented by the points s and t, unemployment is higher with more precautionary savings. Agents save up to two and three units, respectively. At the point r, unemployment is 0.735 and the debt is 1.04. That point represents an equilibrium when $c \in [21.44, 87.5]$. For $c = 60$, the points s and t are also equilibria.

The previous analysis can be used to analyze the impact of a positive supply of bonds, for example, fiat money. Assume that each agent owns a quantity of bonds $M \geq 1$, that is an integer, and that prices are equal to 1. The model is isomorphic to the previous model with $A = M$. The existence and the characterization of an unemployment equilibrium steady state, and the value of the maximum amount of savings N, depend on the credit limit and the cost parameter c.

The case of a zero credit limit is just a special case which is a formal representation of the issue discussed by Sweeney and Sweeney (1977) for the Capitol's baby co-op where members used vouchers to get babysitters and received vouchers by babysitting[20]. A crisis occurs when there is a self-fulfilling perception of a shortage of vouchers. The present model shows how multiple equilibria can arise in such a setting.

6.4 Non-indivisibility

The indivisibility of goods is now removed. As in the other parts of the paper, there is no indivisibility for bonds. An agent i can now produce his good x_i, at no cost, for any level in the interval $[0, 1]$. The utility function of any agent is the same as in the previous sections with the definition in (1). It is illustrated by Figure 4. The matching process is the same as in the previous sections. Each agent buys only from one agent and sells to one agent. The quantity of trade is now in the interval $[0.1]$. The price is determined by the postings by the seller and the buyer of a reservation price for the supply and the demand.

The analysis shows first that the main property of the discrete model, multiple equilibrium steady states with and without unemployment, and a discrete distribution of bond holdings, still holds. Second, the discrete distribution of bond holdings in the steady state is locally stable: after a perturbation that spreads the support of the distribution to a continuous interval, the distribution converges back to the initial discrete distribution. Green and Zhou (2002) prove a similar property in their model,

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[20]A similar issue probably motivates the work of Green and Zhou (2002).
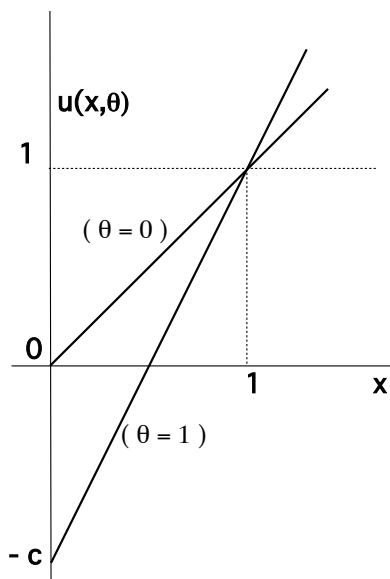
29

Figure 4: Utility functions for the high type ($\theta = 1$) and the low type ($\theta = 0$).

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but their result is more general since it has global stability and the proof requires much more technique.

The 1-consumption function of the discrete case is extended to the following function

$$c(k, \theta) = \begin{cases} Min(1 + k, 1) & \text{for the high type, } (\theta = 1), \\ \\ Min(Max(k, 0), 1) & \text{for the low type, } \quad (\theta = 0). \end{cases} \tag{33}$$

A high-type agent consumes as much as possible up to 1 and a low-type consumes any "surplus" of his savings over 0 up to a maximum of 1. (Higher threshold levels could be analyzed, as for the economy with indivisibilities).

Assume that the economy is in a steady state with unemployment rate π, that the bond distribution is discrete at $\{-1, 0, 1\}$, and that all sales are equal to 1. It is a simple exercise to show that the distribution is stationary under the continuous consumption function in (33), and that it generates the same unemployment rate as in the discrete model: agents make a sale of 1 with probability $1 - \pi$ and no sale with probability π. We now show that under the same parameter conditions as in the main model, the above consumption function is optimal.

Consider an agent with wealth $k = k + x$ with $x \in [0, 1)$ at the end of a period, and let $V_k(x)$ be his level of utility. Under the consumption function in (33),

$$
\begin{cases}
V_{-1}(x) = \beta\Big(\alpha\Big(c(x-1) + \pi V_{-1}(0) + (1-\pi)V_0(0)\Big) \\
\qquad\qquad\qquad\qquad + (1-\alpha)\Big(\pi V_{-1}(x) + (1-\pi)V_0(x)\Big)\Big), \\
V_0(x) = \beta\Big(\alpha\Big(1 + \pi V_{-1}(x) + (1-\pi)V_0(x)\Big) \\
\qquad\qquad\qquad\qquad + (1-\alpha)\Big(x + \pi V_0(0) + (1-\pi)V_1(0)\Big)\Big), \\
V_1(x) = \beta\Big(1 + \pi V_0(x) + (1-\pi)V_1(x)\Big), \\
V_k(x) = \beta\Big(1 + \pi V_{k-1}(x) + (1-\pi)V_k(x)\Big), \quad \text{for } k \geq 2.
\end{cases}
\tag{34}
$$

If $x = 0$, the first three equations determines $(V_{-1}(0), V_0(0), V_1(0))$ that are identical to (V_{-1}, V_0, V_1) as determined by (12). Likewise, for any $k \geq 2$, $V_k(0) = V_k$. Hence, the system (34) determines the continuous function $V(k)$ such that for any integer $k \geq -1$, $V(k) = V_k(0) = V_{k-1}(1)$.

High-type agents always consume as much as possible, following the argument in Lemma 2. Consider a low-type agent with a savings of $k \in (-1, 0)$ at the beginning of a period. His marginal utility of consumption is 1. The marginal impact of a consumption reduction on this expected utility at the end of the period is $\pi V'_{-1} + (1 - \pi)V'_0$, where V'_k is the derivative of V_k with respect to x. The zero consumption is optimal if

$$
1 \leq \pi V'_{-1} + (1 - \pi)V'_0. \tag{35}
$$

Because $V_k(x)$ is linear with $V_k(0) = V_k$ and $V_k(1) = V_{k+1}$, (35) is equivalent to

$$
1 \leq \pi(V_0 - V_{-1}) + (1 - \pi)(V_1 - V_0) = \zeta_0
$$

The condition is the same as in Lemma 5. A similar argument applies when $k \geq 0$.

The stability of the discrete distribution of wealth

Assume the following perturbation in the discrete wealth distribution. A small mass of agents is spread on the interval $[-1, 2)$. The spreading can be with atoms or atomless. By continuity, when the mass of the spreaded agents is sufficiently small, the consumption function (33) remains optimal. We now show through a heuristic argument that the distribution of bond holdings converges to a discrete distribution on the support $\{-1, 0, 1\}$.

Project the distribution of wealth on the interval $[0, 1)$ by the congruence operator with modulus 1. Assume an agent has a wealth 0 (mod. 1) at the beginning of the period. After his consumption decision, his wealth is still 0 (mod. 1) and his consumption does not change the wealth of the agent he buys from (mod. 1). An examination of the various possible types and states shows that at the end of the period, either the two agents have switched their wealth (mod. 1), or they have the same wealth (mod. 1). The match of these agents does not change the projection of the wealth distribution on $[0, 1)$. To summarize, there is no attrition of the mass of agent with wealth 0 at the beginning of a period.

We now show that the mass of agents with wealth 0 increases if the mass of wealths in $(0, 1)$ is strictly positive. Consider an agent with a wealth $k \in (0, 1)$ (mod. 1) who consumes k. Let k' be the wealth of his seller (before trading). If $k' = 0$, we are back to the previous case where agents exchange their wealth (mod. 1), and there is no impact on the wealth distribution (mod. 1). If $k' \in (0, 1)$, then the wealth of the seller increases to $k + k'$ (mod. 1). The interesting case is $k + k' = 0$: neither of the two agents had a wealth equal to 0 before the match and both of them have a wealth equal to 0 after the exchange. That match increases the mass of agents with a wealth 0. Such a match occurs with a strictly positive probability as long as the mass of agents with wealth $k \in (0, 1)$ is strictly positive[21].

Although there is no indivisibility in consumption, the wealth distribution in the steady state is still discrete because of unit capacity of the suppliers.

6.5 Overlapping demands

The structure of trades is now modified such that the population is divided into two subgroups with overlapping demands and production. Goods are indivisible as before Section 6.4. Each period, a day, is divided in two halves, morning and afternoon, and in each spot $i \in [0, 1)$, there are two agents, $(i, 1)$ and $(i, 2)$. At the beginning of a period, one of the agents randomly becomes a *morning agent* who consumes in the morning and produces in the afternoon, and the other agent is an *afternoon agent*,

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[21]This is a point where one relies on heuristics. To be more formal, one can either discretize the interval $[0, 1)$ in an arbitrarily large number K of sub-intervals (the convergence could be observed in numerical simulations), or one can assume a density function $f_t(k)$ of the distribution of v and show that after an exchange, the density at the point 0 increases by $\int_0^1 f_t(k_t) f_t(1 - k_t) dt$, which is strictly positive as long as $f_t(k)$ is not identical to 0. Numerical simulations also show that under the consumption function (33), from an arbitrary initial wealth distribution, the distribution converges to a discrete distribution.

who does the reverse. The random draws are independent across spots and periods. At the same time, an agent learns his type, high or low, with the same probabilities as in the main model. There is no subjective discounting within a day and β is the discount factor between consecutive days. The agents have the same utilities as in the main model.

Morning and afternoon consumers and producers are matched by functions $\phi_{t,1}(i,1) = i + \xi_{t,1}$ (mod. 1), and $\phi_{t,2}(i,1) = i + \xi_{t,2}$, as in (2), with the random variables $\xi_{t,j}$ that are i.i.d., with a uniform density on $[0,1)$. The morning agents are very similar to the agents of the model in the text. They have to make a consumption decision in a period before knowing whether they will make a sale in the same period. The afternoon agents know whether they have sold in the morning. When there is unemployment, they still have a motive to sell because they may be a morning person in the next period.

Convergence to the low regime steady state

The low regime is defined, as in the model of the previous sections, such that the only agents who save are either credit constrained with a debt of 1, or with no debt and of the low type. As for the model of the text, the analysis is in two steps. First, the regime determines the wealth distribution that depends only on α. Second, given that distribution, there are parameters β and c such that the consumption function of the low regime is optimal. In the second step, we consider only the steady state.

Let $\gamma(t)$ be the distribution of agents at the beginning of period t. As in the text, we assume that no savings is of more than one unit. That property is stable in the dynamic process. At the beginning of a period, let the wealth distributions of the morning and the afternoon agents be $\hat{\gamma}$ and $\tilde{\gamma}$, respectively. Omitting the time subscript for these distributions,

$$\hat{\gamma} = \tilde{\gamma} = .5\gamma(t).$$

The rate of employment in the morning is $1 - \pi' = 2(\alpha\hat{\gamma}_0 + \hat{\gamma}_1)$. The factor 2 arises because the demand is from a mass of $1/2$ in aggregate, but the ratio between consumers and producers is 1. The distributions of the morning and afternoon agents at the end of the morning are $\hat{\gamma}'$ and $\tilde{\gamma}'$ with

$$\hat{\gamma}' = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1-\alpha & 1 \\ 0 & 0 & 0 \end{pmatrix} \hat{\gamma}, \qquad \tilde{\gamma}' = \begin{pmatrix} \pi' & 0 & 0 \\ 1-\pi' & \pi' & 0 \\ 0 & 1-\pi' & \pi' \\ 0 & 0 & 1-\pi' \end{pmatrix} \tilde{\gamma}.$$

Note that the afternoon agents who begin the period with one unit of savings have two units at the beginning of the afternoon if they make a sale in the morning. The vector $\tilde{\gamma}'$

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has 4 components. The unemployment rate in the afternoon is $\pi'' = 1 - 2(\alpha\tilde{\gamma}_0' + \tilde{\gamma}_1' + \tilde{\gamma}_2')$, and the distributions of the morning and afternoon agents at the end of the afternoon are $\hat{\gamma}''$ and $\tilde{\gamma}''$ with

$$\hat{\gamma}'' = \begin{pmatrix} \pi'' & 0 & 0 \\ 1 - \pi'' & \pi'' & 0 \\ 0 & 1 - \pi'' & 1 \end{pmatrix} \hat{\gamma}', \qquad \tilde{\gamma}'' = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 - \alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tilde{\gamma}'.$$

At the end of the day, the distribution of agents is equal to

$$\gamma(t+1) = \hat{\gamma}'' + \tilde{\gamma}''.$$

Using the property that $\gamma(t) = (B(t), 1 - 2B(t), B(t))$ where $B(t)$ is the total quantity of debt, one can show that $B(t+1)$ is a polynomial function of $B(t)$ (as in (22), but an order higher than 2). It is sufficient here to compute numerically the steady state by convergence of the dynamic process[22].

Optimal consumption of the low regime steady state

We consider the steady state in the low regime. Let V be the vector of state-dependent utilities of an agent at the end of a period, after all consumptions and sales have taken place. That agent becomes a morning or afternoon agent with equal probabilities at the beginning of next period. Hence,

$$V = \beta \frac{\hat{U} + \tilde{U}}{2}, \tag{39}$$

where $\hat{U}$ and $\tilde{U}$ are the vectors of state-dependent utilities, at the beginning of a period, of morning and afternoon agents, respectively. These utilities are defined before the agent learns his type (high or low).

1. A morning agent is like an agent in the main model. When he holds k units, he saves if $1 + \pi'' V_{k-1} + (1 - \pi'') V_k \leq \pi'' V_k + (1 - \pi'') V_{k+1}$. Using the notation $\zeta_k = \pi''(V_k - V_{k-1}) + (1 - \pi'')(V_{k+1} - V_k)$, saving is optimal if $1 \leq \zeta_0$. Consumption is optimal if $1 \geq \zeta_k$. The condition for his consumption function in the low regime is therefore

$$\zeta_0 \geq 1 \geq \zeta_k \quad \text{for any } k \geq 0. \tag{40}$$

We need to consider only the first four states for the utility of the morning agent before he learns his type, $\hat{U} = (\hat{U}_{-1}, \hat{U}_0, \hat{U}_1, \hat{U}_2)'$. This vector satisfies the

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[22]Starting from $\gamma(1) = (0, 1, 0)'$, the process converges (up to the 4th decimal) before the 20th iteration if $\alpha \in (.15, .85)$.

backward induction equation

$$\hat{U} = \hat{M}V + \begin{pmatrix} -\alpha c \\ \alpha \\ 1 \\ 1 \end{pmatrix}, \qquad \text{with} \qquad \hat{M} = \begin{pmatrix} \pi'' & 1-\pi'' & 0 & 0 \\ \alpha\pi'' & a & b & 0 \\ 0 & \pi'' & 1-\pi'' & 0 \\ 0 & 0 & \pi'' & 1-\pi'' \end{pmatrix}.$$

2. An agent who consumes in the afternoon is in one of four possible states at the beginning of the afternoon, $-1, 0, 1,$ or 2. (He is in state 2 if he enters the day in state 1 and makes a sale in the morning). The agent knows his state at the end of the period. The condition for the low regime consumption function (where he saves only if he is of the low type with no savings), is

$$V_1 - V_0 \geq 1 \geq V_{k+1} - V_k, \quad \text{for any } k \geq 1. \tag{41}$$

In the low regime, the afternoon agent has, at the beginning of the afternoon, (before knowing his type), the vector of utilities $\tilde{U}' = (\tilde{U}'_{-1}, \tilde{U}'_0, \tilde{U}'_1, \tilde{U}'_2)'$ that satisfies the backward induction equation

$$\tilde{U}' = \tilde{M}V + \begin{pmatrix} -\alpha c \\ \alpha \\ 1 \\ 1 \end{pmatrix}, \qquad \text{with} \qquad \tilde{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1-\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When this agent enters the period in the morning, he is in one of three states -1, 0 or 1, and the vector of utilities satisfies

$$\tilde{U} = \mathcal{M}\tilde{U}', \qquad \text{with} \qquad \mathcal{M} = \begin{pmatrix} \pi' & 1-\pi' & 0 & 0 & 0 \\ 0 & \pi' & 1-\pi' & 0 & 0 \\ 0 & 0 & \pi' & 1-\pi' & 0 \\ 0 & 0 & 0 & \pi' & 1-\pi' \end{pmatrix},$$

Hence,

$$\tilde{U} = \mathcal{M}\tilde{M}V + \mathcal{M}\begin{pmatrix} -\alpha c \\ \alpha \\ 1 \\ 1 \end{pmatrix}.$$

Using (39),

$$V = \frac{\beta}{2}(\hat{U} + \tilde{U}) = \beta\mathcal{A}V + \beta\mathcal{B}, \qquad \text{with}$$

$$\mathcal{A} = \frac{1}{2}\left(\hat{M} + \mathcal{M}\tilde{M}\right), \qquad \mathcal{B} = \frac{1}{2}\begin{pmatrix} -\alpha c \\ \alpha \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2}\mathcal{M}\begin{pmatrix} -\alpha c \\ \alpha \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

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The vector V is finally determined by

$$V = (I - \beta \mathcal{A})^{-1} \mathcal{B}.$$

The conditions (40) and (41) for the equilibrium in the low regime are verified numerically. Taking $\beta = 0.8$, the conditions are equivalent $c \in (3.71, 7.18)$ for $\alpha = 0.5$, or $c \in (3.21, 9.63)$ for $\alpha = 0.9$.

7 Conclusion

In the present analysis, there is no aggregate productivity shock and the critical assumption is the credit constraint that restrains the demand for goods. Multiple equilibria arise with self-fulfilling expectations of heterogeneous consumers' regarding their future opportunities to trade in decentralized markets. These expectations affect individuals' choices between consumption and saving, and aggregate consumption determines aggregate output, contrary to Say's Law.

In the model, prices are set endogenously by agents in an environment of one-to-one matching that is a strong form of imperfect competition and generates a stylized form of a kinked demand curve: when the price of goods in bond or money is the same in all markets, a seller who lowers his price does not attract more sales to increase revenues, and he looses too many sales if he increases the price. In equilibrium, endogenous prices are "rigid" both downwards and upwards. Such rigidities seem to be empirically relevant in the short- to medium-term. But a higher flexibility of prices may not be a sufficient condition to bring the economy to full employment. It may even make things worse.

Consider briefly what would happen if a planner, or self-fulfilling expectations (with remarkable coordination), would change all the prices. Assume for example, that a situation of unemployment would lead to decrease of all prices by the same amount (in terms of the bonds). Each agent would obviously not experience any change in the ratio between the prices of his output and the goods that he consumes. The level of assets in terms of goods would be higher, but the level of existing liabilities would also increase. The model would have to be altered technically to deal with this issue, but note that in the previous sections, the dynamics towards unemployment were driven by the agents against their credit constraint. On may conjecture that this problem would be amplified if because of excess capacity, prices would fall and liabilities increase. In any case, more research on this topic would be welcome.

As a first step here, the credit constraint is exogenous. The recent crisis has shown

how credit can actually be suddenly restricted in a downturn, thus amplifying the mechanism that is analyzed here. To make the endogenous credit limit depend on expected future trades would be a useful task.

The model presented here is an abstract analytical tool and it is not directly useable for practical policy prescriptions. In the context of the present model where an important role is played by the propagation of debt reductions in the path toward full employment, that an effective policy may be a subsidy of current consumption, or a short-term tax on the rate of return to saving. If agents save in money, inflation would do the job. Such a policy may induce all agents who are not credit constrained to consume, and thus generate the high regime which converges to full employment.

APPENDIX: Proofs

Lemma 2

Assume that an agent i is not credit-constrained in period 0, is of the high type and consumes in period 0. Consider an agent j identical to agent i, (with the same wealth at the beginning of period 0), who instead of consuming, saves in period 0. After that divergence of behavior in period 0, both agents face the same distribution of types and sales in all periods $t \geq 1$ and they optimize in each period.

Let $k_{i,t}$ and $k_{j,t}$ the savings of agent i and j in period t after consumption for the same path of the realization of types and sales. By definition of i and j, $k_{j,1} = 1 + k_{i,t}$.

For a given history of types and sales, define T as the maximum of τ such that for $0 \leq t \leq \tau - 1$, $k_{j,t} \geq 1 + k_{i,t}$ and $k_{j,\tau} = k_{i,\tau}$. The value of T may be infinite. In some sense, agent j does not consume his saving of period 0 before period T. Since agent i and j have the same history of sales in that time interval, the consumption path of agent j can be reproduced by agent i.

By definition of T, agent j consumes in period T and agent i does not. The difference of utilities in that period T is not greater than c. For the subsequent periods $t > T$, the utilities of the two agents are the same since they have the same bond holdings at the end of period T and the same paths of types and sales. Let U^i and U^j be the utilities of these agents from period 0 on. Since agent i can reproduce the consumption path of agent j for the periods before T, (if $T = 1$, these periods are omitted in the argument), we have

$$ U^i \geq U^j + c + E[-\beta^T c] \geq U^j + c(1 - \beta) > U^j, $$

where the expectation is taken over all paths of sales for $t \geq 0$ and of types for $t \geq 1$. Hence, consumption is strictly optimal in period 0. $\square$

Lemma 3

The utility of any agent is bounded above by that of perpetual consumption. The utility of an agent who holds k units of savings at the end of a period is bounded below by the utility of the path with consumption in the next k periods and no consumption thereafter while being of the high type:

$$ \frac{\beta}{1 - \beta}(1 - \beta^k) - \frac{\beta^{k+1}}{1 - \beta}c \leq V_k(t) \leq \frac{\beta}{1 - \beta}. $$

Hence,

$$V_k(t) - V_{k-1}(t) \le \frac{\beta^{k+1}}{1-\beta}(1+c).$$

A sufficient condition for (9) is $\beta^{k+1}\dfrac{1+c}{1-\beta} < 1$, which holds for $k \ge \bar{N}$ where $\bar{N}$ is defined in the Lemma. $\square$

Lemma 6

We prove here the monotonicity property. Let $B(t)$ and $B'(t)$ the paths of the debt in the high regime for two different levels debt in period 0, $B(0)$ and $B'(0)$, with $B(0) > B'(0)$. Recall that the sequences $B(t)$ and $B'(t)$ are monotone decreasing with $B'(t) < B(t)$ for any $t \ge 0$. Since $B(t)$ and $B'(t)$ are strictly smaller than 1,

$$1 > B(t) + \beta(1 - B(t))B(t+1) > B'(t) + \beta(1 - B'(t))B'(t+1). \tag{42}$$

Define for $0 \le T_1 \le T_2$, the sum

$$S_{T_1,T_2}(B(T_1)) = B(T_1) + \beta \sum_{t=T_1}^{T_2} \beta^{t-T_1}(1 - B(T_1))\dots(1 - B(t))B(t+1)$$

Fix T_2 arbitrary and set $T_1 = T_2 - 1$. Using (42),

$$\text{if } B(T_1) > B'(T_1), \quad 1 > S_{T_1,T_2}(B(T_1)) > S_{T_1,T_2}(B'(T_1)).$$

Proceed by backward induction on T_1 up to $T_1 = 0$ to show that $1 \ge S_{0,T_2}(B(0)) \ge S_{0,T_2}(B'(0))$ for any $T_2 \ge 1$. Then take the limit for $T_2 \to \infty$. It follows that

$$\zeta(B(0);c) = \beta(1 + \alpha c)S_{0,\infty}(B(0)) > \beta(1 + \alpha c)S_{0,\infty}(B'(0)) = \zeta(B'(0);c).$$

The continuity of $\zeta(B;c)$ is proven by the continuity of the limit $S_{0,\infty}(B)$. $\square$

Proposition 6

We use the following Lemma that is proven after the proposition.

Lemma 7. *Let $\alpha = 1/2$. If low-type agents with no savings (in state 0) save in period 0, then in period 1, $B(1) \ge 1/4$.*

Proposition 6 is now proven by contradiction. Assume first $\alpha = 1/2$. In the steady state with full employment, the marginal utility of savings for the low-type of agents is strictly smaller than 1. By continuity, if there is an equilibrium path to full employment, by continuity of the marginal utility of savings, there exists T such that for any $t \ge T$, low-type agents in state 0 (with no savings) consume.

If $T \geq 1$, redefine period $T - 1$ as period 0. By definition of T and Lemma 7, $B(0) \geq 1/4$. Using (18), the marginal utility of savings of the low-type agents in state 0, satisfies the inequality

$$\zeta_0(0) \geq (1 + \frac{c}{2})\frac{1}{4(1 + \rho)}. \tag{46}$$

If $c > 6 + 8\rho$, the right-hand side of the inequality is strictly greater than 1, which is a contradiction that the low-type agents consume (Lemma 5). By continuity, the argument holds if $\alpha \in \mathcal{J}'$ where $\mathcal{J}'$ is a neighborhood of $1/2$. $\square$

Proof of Lemma 7

Let π be the unemployment rate in period 0. Let K be the upper bound of the support of the distribution of bonds in period 0. Assume that agents in period 0 have an N-consumption function with $N \geq 1$ since low-type agents with no savings do not consume. The consumption, $1 - \pi$ is equal to $\alpha(\gamma_0 + \ldots \gamma_{N-1}) + \gamma_N + \ldots + \gamma_K$. Using $\alpha = 1/2$,

$$\pi = \frac{1}{2}(1 + B_0 - \gamma_N - \ldots - \gamma_K),$$

Since, $B(1) = (B(0) + \frac{\gamma_0}{2})\pi$,

$$B(1) = (B(0) + \frac{\gamma_0}{2})\Big(B(0) + \frac{1}{2}(1 - B_0 - \gamma_N - \ldots - \gamma_K)\Big).$$

Adding the following two equations

$$\begin{cases} B_0 = \gamma_1 + 2\gamma_2 + \ldots + K\gamma_K, \\ \gamma_0 = 1 - B(0) - \gamma_1 - \gamma_2 - \ldots - \gamma_K, \end{cases}$$

$\gamma_0 \geq 1 - 2B(0)$, and substituting in the equation of $B(1)$,

$$B(1) \geq \frac{1}{4}(1 + B(0) - \gamma_N - \ldots - \gamma_K).$$

The proof is concluded by noting that $B(0) \geq \gamma_N + \ldots + \gamma_K$.

$\square$

Proposition 8

Assume first that the economy is an steady state with the low regime. If a seller in state -1 who has the highest marginal utility of savings and the highest potential gain from an increase of revenues. If he increases his prices by $\eta < 1$, any buyer decreases effectively his state by two units because other sellers have a price of 1. Therefore, the

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effect on his demand is the same as if increases his price by 1. Assume therefore that he increases his price from 1 to 2. Consider a customer who is in state 0. If he buys, he incurs a debt strictly greater than 1 with probability 1 at the end of the period. At the price of 2, such an agent is credit constrained and cannot buy.

Consider now a low-type customer who is in state 1. Would he buy at the price of 2? No. He would be back to state -1. That would be the same outcome as buying at the price of 1 if he were in state 0. But by definition of the low regime, he does not buy in this case.

The customers that buy at the price of 2 are, at most, the high-type agents in state 1. The probability to face such an agent is $\alpha\gamma_1 = \alpha B$, where B is the aggregate debt. The expected utility gain from the strategy is bounded above by $\alpha B(V_1 - V_{-1})$, where V_k is determined by the Bellman equations (7) with $x = 0$. On the other hand, the posting of 1 generates a sale with probability $\alpha(1 - 2B) + B$, in which case the agent has a utility gain of $V_0 - V_{-1}$. A sufficient condition for no deviation from the price of 1 is therefore

$$\alpha B(V_1 - V_{-1}) < \Big(\alpha(1 - 2B) + B\Big)(V_0 - V_{-1}). \tag{47}$$

One verifies through straightforward algebra that this inequality holds for $\alpha = 1/2$. By continuity, it holds when α is in an open interval that contains $1/2$ and when the economy is in a neighborhood of the steady state. One proves in the same way that sellers in state 0 or 1 do not deviate from posting a price of 1.

We have seen in Section 4 that the low regime from full employment to unemployment has a transition of one period. The argument applies if in period 0, the economy is at full employment: in that period, no agent has any savings and a posting higher than 1 reduces the probability of a sale to 0. Hence the argument applies for the two periods of the path of the low regime. By continuity, it applies when α is in an open interval $\mathcal{I}$ that contains $1/2$, when the dynamic path does not converge in one period.

Consider now a high regime that converges full employment. The value of B converges to 0. The previous argument can be used with equation (47) replaced by

$$B(V_1 - V_{-1}) < (1 - B)(V_0 - V_{-1}). \tag{48}$$

Using the Bellman equations (12) with $x = 1$ in the high regime, when $B = \pi$ (in the high regime), tends to 0, at the limit, $V_0 = V_1 = \beta/(1 - \beta)$. One verifies that at the limit, $V_{-1} = \beta(-\alpha c - \beta V_0) < V_0$. The previous inequality becomes asymptotically

equivalent to

$$B(V_0 - V_{-1}) < (1 - B)(V_0 - V_{-1}), \qquad \text{with} \qquad V_0 - V_{-1} > 0.$$

which is satisfied when $B \to 0$.

$\square$

Lemma 8. *For any $\pi \in [0, 1]$, the matrix $L(\pi)$, defined in (28) has an eigenvalue equal to 1 that is of order 1. There is an eigenvector, w, associated to the unit eigenvalue such that $e'.w = 1$ where e is the unit column vector with $N + 2$ components.*

We replace $L(\pi)$ by L in the proof. Recall that the matrix L is square and of dimension $N + 2$. Call e the column-vector of ones of dimension $N + 2$. Since $e'.L = e'$ (the transition matrix preserves the mass of the distribution), 1 is an eigenvalue of L. To show that it is of order 1, consider the matrix

$$L - I = \begin{pmatrix} \pi - 1 & \alpha\pi & 0 & 0 & \dots & 0 \\ 1 - \pi & a - 1 & \alpha\pi & 0 & \dots & 0 \\ 0 & b & a - 1 & \alpha\pi & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & b & a - 1 & \alpha\pi & 0 \\ 0 & \dots & 0 & b & a - 1 & \pi \\ 0 & 0 & 0 & \dots & b & -\pi \end{pmatrix}, \tag{49}$$

where I is the identity matrix of dimension $N + 2$. Let Δ_j be the determinant for the first j rows and columns of this matrix. Replacing the last row of Δ_N by the sum of all rows of Δ_{N+1} and using $a + b - 1 = -\alpha\pi$, $\Delta_{N+1} = -b\Delta_N$ with $b = (1 - \alpha)(1 - \pi)$. By induction, $\Delta_{N+1} = (-b)^N(\pi - 1) \neq 0$. The unit eigenvalue is of order 1. $\square$

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ALGEBRA FOR SECTION 6.1

When $\alpha = 1/2$, $B = 1/4$, $\pi = 1/2$, and the values of V_k are determined by (7).

For $\alpha = 1/2$, the inequality (47) is equivalent to

$$V_1 - V_{-1} < 4(V_0 - V_{-1}). \tag{A}$$

The Bellman equations (7) become

$$\begin{cases} V_{-1} = \dfrac{\beta}{2}(-c + V_{-1} + V_0), \\[2mm] V_0 = \beta(\dfrac{1}{2} + \dfrac{V_{-1} + V_0}{4} + \dfrac{V_0 + V_1}{4}), \\[2mm] V_1 = \beta(1 + \dfrac{V_0 + V_1}{2}). \end{cases}$$

Taking the difference between the third and the first equation,

$$V_1 - V_{-1} = \frac{\beta(1 + \dfrac{c}{2})}{1 - \dfrac{\beta}{2}}. \tag{B}$$

Taking the difference between the second and the first equation,

$$V_0 - V_{-1} = \beta(\frac{1 + c}{2} + \frac{V_1 - V_{-1}}{4}).$$

The inequality (A) is therefore equivalent to

$$(1 - \beta)(V_1 - V_{-1}) < 2\beta(1 + c),$$

which, using (B), is equivalent to

$$(1 - \beta)(1 + \frac{c}{2}) < (2 - \beta)(1 + c),$$

which is satisfied.