VCG in Theory and Practice

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May 2013
Revised: December 26, 2013

It is now common to sell online ads using an auction. Auctions are used for search ads by Google and Microsoft, for display ads by DoubleClick and other ad exchanges, and for social network ads by Facebook. However, different auction designs are used in each of these cases. Search ads use a Generalized Second Price (GSP) auction, display ad exchanges generally use a Vickrey (second price) auction, and Facebook uses a Vickrey-Clarke-Groves (VCG) auction.

It turns out that these auctions are all closely related. The VCG auction encompasses the traditional Vickrey auction as a special case. It has the attractive property that bidding the true value is a dominant strategy for all players and the equilibrium revenue should, in theory, be about the same as the GSP auction. However, it also has some drawbacks; see Ausubel and Milgrom [2006] and Rothkopf et al. [1990] for a list of issues.

In this note we describe two simple theoretical properties of the VCG auction in a search ad framework and some of the practical lessons learned in implementing a VCG auction for contextual ads.

1 Search ad auctions

In a search ad auction advertisers submit keywords and bids. When the advertiser's keyword matches a user's query, the advertiser enters an auction. The advertiser with the highest bid gets the most prominent slot, the advertiser with the second highest bid gets the second most prominent slot and so on.In the actual auction, the bids are adjusted by a "quality score," but we ignore this additional complexity in this exposition.

2 How the GSP auction works

Let v_s be the value of a click to an advertiser in slot $s = 1, \ldots, S$, and let x_s be the clicks (or clickthrough rate) associated with that slot. We assume that the slots have been ordered with the most prominent slots first, so that $x_1 > x_2 > \cdots > x_S$.

The GSP auction produces a price for each slot. These prices must satisfy the revealed preference conditions that an advertiser who purchases slot s prefers that slot to other slots it could have purchased:

$$v_s x_s - p_s x_s \geq v_s x_t - p_t x_t \tag{1}$$

It turns out that if these inequalities are satisfied for $t = s + 1$ they are satisfied for all slots. After some manipulation we find an the following system of inequalities that characterizes equilibrium prices.

$$v_s(x_s - x_{s+1}) + p_{s+1}x_{s+1} \geq p_s x_s \geq v_{s+1}(x_s - x_{s+1}) + p_{s+1}x_{s+1} \tag{2}$$

We note that these inequalities imply that

$$(v_s - v_t)(x_s - x_t) \geq 0, \tag{3}$$

so that advertisers with higher values get more prominent slots, which shows that the GSP equilibria are efficient.

The same manipulations work in reverse. That is, we can start with an efficient assignment of advertisers to slots, which must satisfy inequality (3) and show that that there must exists prices that satisfy the equilibrium inequalities (2). Thus this simple position auction has mini version of the First and Second Welfare Theorems.

There are many prices that satisfy these inequalities, but a particularly interesting equilibrium is the one with minimal revenue, where the right inequalities hold as an equalities. Writing these conditions out for the 3-slot case gives us this system:

$$p_1 x_1 = v_2(x_1 - x_2) + p_2 x_2 \tag{4}$$

$$p_2 x_2 = v_3(x_2 - x_3) + p_3 x_3 \tag{5}$$

$$p_3 x_3 = v_4 x_3 \tag{6}$$

Adding up the payments gives us a lower bound on revenue to the seller of

$$R_L = v_2(x_1 - x_2) + 2v_3(x_2 - x_3) + 3v_4 x_3. \tag{7}$$

We can perform the same sort of manipulations to get an upper bound on revenue.

3 How the VCG auction works

In the VCG auction, each bidder is required to pay the cost their presence imposes on the other bidders, using their stated bids as the value they place on the slots. If advertiser 1 participates in the auction the total payments by the other advertisers is $b_2 x_2 + b_3 x_3$. If advertiser 1 does not participate in the auction, the other advertisers all move up one position and so pay $b_2 x_1 + b_3 x_2 + b_4 x_3$. Thus the "harm" that advertiser 1 imposes on the other advertisers is the difference, $b_2(x_1 - x_2) + b_3(x_2 - x_3) b_3 x_4$, so this is the amount advertiser 1 is required to pay. It turns out that it in the VCG auction it is optimal for each advertiser to bid its true value per click. Writing out the VCG payments in the three-slot case, we have:

$$
\begin{aligned}
p_1 x_1 &= v_2(x_1 - x_2) + v_3(x_2 - x_3) + v_4 x_3 && (8)\\
p_2 x_2 &= \qquad\qquad\quad + v_3(x_2 - x_3) + v_4 x_3 && (9)\\
p_3 x_3 &= \qquad\qquad\qquad\qquad\qquad\quad + \; v_4 x_3 && (10)
\end{aligned}
$$

It is easy to check that this produces the same outcome as that in system (4)–(6). Hence the minimum revenue GSP equilibrium has the same revenue as the VCG equilibrium, a result noted by Edelman et al. [2007] and Varian [2007], and is a special case of a result derived by Demange and Gale [1985], Demange et al. [1986] in a different context. See Roth and Sotomayor [1990] for a unified treatment.

4 Broad match

We said that the ad is eligible for the auction if the user's query matches the advertiser's keyword. But what counts as a match? It turns out that search engines use several types of match including "exact match" and "broad match." A keyword [dog food] would be an exact match for the query "dog food" but a broad match for the query"pet food."

The value of a click from a broad match could be somewhat different from the value of an exact-match click, but not radically so. To capture this, we will assume that the value of a broad match click is δv_s to the advertiser in slot s, where δ may be somewhat larger or smaller than 1.

Advertisers who choose broad match have to pick a single bid that applies for a whole range of auctions which, in principle, could contain different of competitors, different positions, and so on.

Note that the VCG auction works just fine in this case: the advertisers can each state their value for a visitor and the payments are calculated as described above, with the auctioneer applying the appropriate broad match adjustment. Everything works out neatly.

The GSP equilibrium, by contrast, can be quite messy since advertisers can appear in different positions in each auction. However, if the change in value from broad match is small, in the sense that it does not change the *ordering* of the advertisers in the different auctions, then the GSP auction works out neatly as well.

To see this let us two auctions for cat food and dog food (c and d), with the same advertisers, in the same order, but with potentially different values due to broad match. For simplicity we assume there are equal number of queries on "cat food" and "dog food." The equilibrium conditions in each auction for exact match are:

$$\begin{aligned} v_s^c x_s - p_s^c x_s &\geq v_s^c x_t - p_t^c x_t \\ v_s^d x_s - p_s^d x_s &\geq v_s^d x_t - p_t^d x_t \end{aligned}$$

In the case of broad match, such as bidding on the keyword [pet food], the equilibrium prices (p_s^{cd}) satisfy

$$\delta(v_s^c + v_s^d)x_s - p_s^{cd} x_s \geq \delta(v_s^c + v_s^d)x_t - p_s^{cd} x_t \tag{11}$$

In this case, the advertisers are getting half of their clicks for [dog food], and half for [cat food], so the value of the visitors to their site just adds up in a linear way, giveing us a simple expression for the equilibrium prices. The formula will be slightly more complex if δ can vary across positions, but as long as the same advertisers are ranked the same way in each of the broad match auctions, all works out neatly.

To summarize: the VCG auction handles broad match in general, while the GSP auction does so only under rather special circumstances. This makes the VCG auction attractive by comparison.

5 Unknown click through rates

It would seem that in order to compute payments for in the VCG auction we would need to know the clicks (or clickthrough rates) associated with each position. However, that is not the case. I provided an overly brief sketch of how this can be accomplished in Varian [2009] but spell out the argument in greater detail here.

Consider the following algorithm to compute advertiser 1's net payment.

1. Each time there is a click on position 1, *charge* advertiser 1 te amount b_2

2. Each time there is a click on position $s > 1$, *pay* advertiser 1 the amount $b_s - b_{s+1}$

At the end of the day there will be x_1 clicks on position 1, which results in a payment *from* advertiser 1 of $b_2 x_1$. There will be x_2 clicks on position 2, resulting in a payment *to* advertiser 1 of $(b_2 - b_3)x_2$. And finally, there will be x_3 clicks on position 3, yielded a payment *to* advertiser 1 of $(b_3 - b_4)x_3$.

The total payment by advertiser 1 is then

$$b_2 x_1 - (b_2 - b_3)x_2 - (b_3 - b_4)x_3,$$

which is simply a rearrangement of the payment in equation (8).

In turns out that each advertiser is still paying the cost it imposes on the other advertisers, just as in the original VCG argument, but now on a click-by-click basis. Suppose a click arrives on position 1. If advertiser 1 is present, the advertiser in position 2 gets no benefit *from that click*. If advertiser 1 is not present, then the advertiser who was in position 2 would now be in position 1 and would get b_2 from that click. Advertiser 3 would get zero on *that click* whether or not advertiser 1 was present.

Now suppose a click arrives on position 2. If advertiser 1 is present, advertiser 2 gets b_2 from that click. If advertiser 1 is not present, advertiser

2 would be in the first slot and advertiser 3 would receive the click that went to the second slot. So advertiser 1's presence has imposed a net benefit of $(b_2 - b_3)$ on the other advertisers.

Finally, if a click arrives on position 3, then advertiser 1's presence yields a benefit of b_3 to advertiser 3. If advertiser 1 were absent, the advertiser 4 would receive that click, so the net benefit that advertiser 1's presence imposes on the other advertisers is $(b_3 - b_4)$.

6 Implementing the VCG auction

Google designed the GSP auction in the Fall of 2001 and implemented it in February of 2002. A few months later, Eric Veach, the main architect of the original GSP auction, came up with a way to create a truthful auction for clicks and showed it to Hal, who recognized it immediately as a VCG auction.

We thought very seriously about changing the GSP auction to a VCG auction during the summer of 2002. There were three problems: 1) the GSP auction was growing very rapidly and required a lot of engineering attention, making it difficult to develop a new auction; 2) the VCG auction was harder to explain to advertisers; 3) the VCG auction required advertisers to raise their bids above those they had become accustomed to in the GSP auction. The combination of these issues led to shelving the VCG auction in 2002.

In 2012, we reconsidered the VCG auction (or something close to it) for use with our contextual ads. These are ads that are displayed based on the textual content on the page; for example, pages about dogs might display dog food ads. Contextual ads can be displayed in a variety of formats, but a common format is an "ad block" of 4 ads, arranged either horizontally or vertically.

The primary reason for considering the VCG auction for contextual ads was that it is a) flexible and b) truthful.

6.1 Flexible

In addition to the well-known search ad system, Google offers "contextual ads." These ads are related to the contents on the page where the ad is being shown. Someone looking at a web page about dogs could see a contextually targeted ad for "dog food."

However, by 2012 there were other important treatments that could be applied to ads. One particularly useful ad treatment is known as "dynamic resizing." It turns out that if you have one highly relevant and three so-so ads, you get more total clicks by enlarging the relevant ad and showing it alone. Choosing when to do this and how much to charge was quite difficult with the GSP auction but could be handled easily by VCG.

6.2 Truthful

The fact that the dominant bidding strategy in the VCG was truthful was also important. This is because the contextual ads can participate in other auctions that have different rules. In particular, we mentioned above that display ads run through a (traditional) Vickrey auction. When a publisher doesn't have an ad to show, it can request ads in an ad exchange where contextual ads may compete with display ads.

Since ad exchanges are often run using a classic Vickrey auction, the dominant strategy is truthtelling. But equilibrium bids in the GSP auctions are generally not truthful. Changing the GSP auction to a VCG auction resolved this inconsistency and enabled the contextual ads to compete on an equal footing with other ads.

Truthful bidding also helps simplify the advertisers' decisions. We mentioned earlier that ads can be shown in a variety of formats, such as a horizontal list or a vertical list. The clickthrough rates for a horizontal list don't vary much from position to position, but can vary quite a bit in a vertical list. It turns out that for the GSP the equilibrium bid depends on the advertisers' estimates of these position effects—but they don't know what configurations will actually occur. The VCG solves this neatly, since the advertiser only has to reveal its value per click which is generally independent of position.

This is not to say that VCG (even in its pure form) does not have some problems. It is incentive compatible for the advertisers but not necessarily for the publishers. In fact, as the celebrated Myerson-Satterthwaite theorem shows, there is generally no mechanism that is incentive compatible for both sellers and buyers at the same time. Ausubel and Milgrom [2006] and Rothkopf et al. [1990] describe some other problematic issues, but most of these are not relevant for the particular situation we face. All auction forms have advantages and disadvantages so choosing the "best" mechanism will involve tradeoffs of one sort or another.

The attractive feature of the VCG auction is that the bids are true structural parameters that do not change as other features of the auction change. This is a consequence of our assumption that the value of a visitor to the advertiser's web page is constant. In a more general model where the probability of purchase could vary depending on auction design this may not be true. However, it appears to be a good approximation in practice.

6.3 Implementation

The design of the Vickrey auction is so elegant, one might hope that it would be relatively easy to implement. Alas, it is not so. There were many edge cases that needed to be dealt with, adding to design complexity. On the other hand, once the system was built, other aspects of the ad auction, such as dynamic resizing, became much simpler.

The final system, which rolled out in late 2012, cannot be considered a "pure" Vickrey auction, but it reasonably close to one given the design challenges involved. From what we can tell, it seems to be working pretty well.

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