Robust Dynamic Optimal Taxation and Environmental Externalities[*]

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Abstract

We study a dynamic stochastic general equilibrium model where agents are concerned about model uncertainty regarding climate change. An externality from greenhouse gas emissions adversely affects the economy's capital stock. We assume that the mapping from climate change to damages is subject to *uncertainty*, and we adapt and use techniques from robust control theory in order to study efficiency and optimal policy. We obtain a sharp analytical solution for the implied environmental externality, and we characterize dynamic optimal taxation. A small increase in the concern about model uncertainty can cause a significant drop in optimal energy extraction. The optimal tax which restores the social optimal allocation is Pigouvian. Under more general assumptions, we develop a recursive method and solve the model computationally. We find that the introduction of uncertainty matters qualitatively and quantitatively. We study optimal output growth in the presence and in the absence of concerns about uncertainty and find that these can lead to substantially different conclusions.

1   Introduction

We study optimal taxation in a dynamic stochastic general equilibrium model where agents are concerned about model uncertainty. We assume that an externality through global temperature changes resulting from greenhouse gas emissions *(GHG)* adversely affects the economy's capital stock and, thus, output. Its precise effects, however, are subject to uncertainty. In order to model the effect of the emissions created by economic activity on the environment, we employ the framework in Golosov, Hassler, Krusell, and Tsyvinski (GHKT, 2013).[1] While they assume that the mapping from climate change to damages is subject to *risk*, in our model this mapping is subject to *Knightian uncertainty.* We study the implications of this assumption using a robust control approach. We believe that this is an appropriate application of uncertainty in economic modeling. After all, man-made climate change is unprecedented, and there is an ongoing heated debate about its potential effects. Our approach can perhaps be thought of as a first step towards addressing the critique that economic models consistently under-assess risk (Stern, 2013). While our model does not include the risks of large-scale human migration or conflict resulting from climate change, it proposes a robust control approach as an alternative to standard probability distribution-based modeling. More specifically, concerned about model uncertainty, a social planner in our model maximizes social welfare under a "worst-case scenario."

In addition to taking into consideration model-uncertainty, there are two other differences between our assumptions and those in GHKT. First, we find it convenient to assume that the environmental externality affects output indirectly, through the capital stock. As a result, the theoretical analysis in our model brings different results, although the two assumptions lead to identical results if we assume 100% capital depreciation (as we do in the computational part). A second difference is that we use estimates about total fossil fuel supplies that are significantly larger than theirs. This is partly due to adding the supply of unconventional oil and gas, but mainly due to considering estimated methane hydrate resources.[2]

Under plausible assumptions, we obtain a sharp analytical solution for the implied pollution externality, and we characterize dynamic optimal taxation. A small increase in the concern about model uncertainty can cause a significant drop in optimal energy extraction. The optimal tax, which restores the social optimal allocation, is Pigouvian. Under more general assumptions, we develop a simple recursive method that allows us to solve the model computationally. We find that the introduction of uncertainty matters in the sense that our model produces results that are qualitatively different, for example, in terms of oil consumption, from GHKT. At the same time, we find that concerns about uncertainty do not affect renewable energy adoption. The reason is that the margin that determines short-term decisions regarding energy sources is driven by two factors: the trade-off between higher versus lower total energy consumption, and the choice of coal versus gas/oil, rather than

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[1] Acemoglu, Aghion, Bursztyn, and Hemous (2012) study related issues. See Nordhaus and Boyer (2000) and Stern (2007) for earlier work that also points to the importance of uncertainty.

[2] See Boswell and Collett (2011), Hartley, Medlock, Temzelides, and Zhang (2012), and references therein for a more detailed discussion on total estimated fossil fuel resources.

by renewable energy use. We find that oil-use in our model can be flat for some parametrizations. We study optimal output growth in the presence and in the absence of concerns about uncertainty and find that the results can be very different. In the worst case scenario, optimality implies that a small sacrifice in yearly output can prevent a large future welfare loss.

Since the green energy sector does not create emissions in our model, we find that the optimal path for the use of green energy does not directly depend on the level of concern about model uncertainty. However, since green energy, coal, and oil are substitutes, model uncertainty does affect the use of green energy indirectly, through its impact on coal and oil. We also find that an increase in the concern about model uncertainty causes a significant decline in the use of coal, while the use of oil is slightly delayed. Holding other parameters fixed, the optimal path of oil consumption is determined jointly by the resource scarcity effect and by the model uncertainty effect. Naturally, we do not find a significant difference in oil consumption when the scarcity effect dominates. However, when we consider a higher level of initial resources of fossil fuel, the concern about model uncertainty substantially discourages the use of oil.

As we mentioned, our work builds on the model in GHKT.[3] In addition, we rely on existing work in robust control theory from both economics and engineering. In the traditional stochastic control literature, uncertainties in the system are modeled using probability distributions. The goal there is to derive a policy that works best "on average." In contrast, given a bound on uncertainty, robust control is concerned with optimizing performance under a so-called worst-case scenario.[4] Hansen and Sargent (2001) introduced techniques from robust control theory to dynamic economic decision making problems.[5] They pointed out the connection between the max-min expected utility theory of Gilboa and Schmeidler (1989) and the applications of robust control theory proposed by Anderson et al. (2000) and Dupuis et al. (1998). Hansen, Sargent, Turmuhambetova and Williams (2005) give a thorough introduction to the robust control approach, and develop a variety of tools required to make it useful in an economics context. They discuss applications to a wide range of problems within the Linear-Quadratic-Gaussian world.[6]

As is standard in the robust control literature, our paper postulates the problem of optimal fossil fuel extraction as a two-person zero-sum dynamic game: in each stage, a social planner (a representative household in the decentralized version) maximizes social

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[3]See also Barrage (2013). Previous related work includes Hotelling (1931), Dasgupta and Heal (1974), Nordhaus (2000, 2008), Hoel (1978), Stern (2007), Sinn (2008), Gars, Golosov, and Tsyvinski (2009), Krusell and Smith (2009), and Ploeg and Withagen (2012, 2012). GHKT (2013) provide an excellent review of this literature.

[4]See, for example, Lewis (1986) and Chandrasekharan (1996).

[5]See Knight (1921), Savage (1954), Ellsberg (1961), Gilboa and Schmeidler (1989), Hansen and Sargent (2001 and 2010) for related research.

[6]Related work includes Hansen, Sargent and Tallarini (1999), Hansen and Sargent (2003), Colgey, Colacito, Hansen and Sargent (2008). See Williams (2008) for a review. In a recent paper, Bidder and Smith (2012) use robust control theory to study the implications of model uncertainty for business cycles generated through "animal spirits."

welfare (lifetime utility) by choosing the level of energy extraction, consumption, labor and capital investment. Then, a malevolent player chooses alternative distributions in order to minimize the respective payoff. Our work contributes to the existing literature of applications of robust control in economics in two ways. First, it explores a class of models under a non-quadratic objective and non-linear constraints. In that regard, we demonstrate that models of the type in GHKT (2013) can be restated in a robust control framework. We then derive some sharp analytical results, and compute the resulting model numerically. Second, we employ the exponential distribution as the approximating distribution. While existing studies usually employ the Linear-Quadratic model combined with Gaussian distributions in order to produce analytical solutions, our work shows that the approximating distribution for models with log-utility and full depreciation of capital can be drawn from either the normal or the exponential family.

The paper proceeds as follows. Section 2 presents the basic model. Section 3 studies the model analytically, while Section 4 presents our numerical and quantitative findings. A brief conclusion follows. The Appendix contains some technical material.

2   The Model

In order to characterize the optimal policy for the case where there is a concern about climate change and model uncertainty, we first formulate a general framework for the "robust planner's problem," a benchmark that we will subsequently compare to decentralized market solutions.

Time, t, is discrete and the horizon is infinite. The world economy is populated by a $[0, 1]$-continuum of infinite-lived representative agents with utility

$$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t). \tag{1}$$

The function u is a standard concave period utility function, C_t represents final-good consumption in period t, and $\beta \in (0, 1)$ is the discount factor. The final goods sector uses energy, E, capital, K, and labor, N, to produce output. Labor supply is inelastic. The economy's capital stock depreciates at rate $\delta \in (0, 1)$. Henceforth, $\tilde{K}$ represents the end-of-period capital (before interacting with the climate factor through the process described below). The feasibility constraint in the final goods sector is given by

$$C_t + \tilde{K}_{t+1} = Y_t + (1 - \delta)K_t. \tag{2}$$

There are four production sectors. The final-goods sector, indexed by $i = 0$, produces the consumption good. The corresponding production function is given by $Y = F(K, N_0, E)$. Thus, in addition to capital and labor, production of the final good requires the use of energy, E. The three energy-producing sectors for oil, coal, and green energy (labelled by $i = 1, 2, 3$, respectively) produce energy amounts E_1, E_2 and E_3 (measured in carbon equivalents). The

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oil sector is assumed to produce oil at zero cost. We denote by R the total oil energy stock, and we impose the resource constraint, $R_t \geq 0$, for all t. Both the coal and the green energy sectors use linear technologies

$$E_i = A_i N_i, \ i = 2, 3 \tag{3}$$

We follow GHKT in modeling a simplified carbon cycle as follows. The variable S (measured in units of carbon content) represents the GHG concentration in the atmosphere in excess of the pre-industrial level. We denote by P and T the permanent and temporary components of S, respectively. These evolve according to the following.

$$
\begin{align}
P' &= P + \phi_L(E_1 + E_2) \tag{4} \\
T' &= (1 - \phi)T + (1 - \phi_L)\phi_0(E_1 + E_2) \tag{5} \\
S' &= P' + T' \tag{6}
\end{align}
$$

We introduce model uncertainty regarding climate change through a stochastic variable, γ, which reduces the end-of-period capital stock $\tilde{K}'$ by a factor of $h(S', \gamma)$ to K'. That is, $K' = h(S', \gamma)\tilde{K}'$.[7] We use $\pi(\gamma)$ to denote the approximating distribution of γ, while $\hat{\pi}(\gamma)$ denotes the welfare-minimizing distribution, and $m(\gamma) = \frac{\hat{\pi}(\gamma)}{\pi(\gamma)}$ is the likelihood ratio. The distance, ρ, between $\hat{\pi}(\gamma)$ and $\pi(\gamma)$ is measured by relative entropy:

$$\rho(\hat{\pi}(\gamma), \pi(\gamma)) \equiv E[m(\gamma) \log m(\gamma)] \equiv \hat{E}[\log m(\gamma)] \equiv \int [m(\gamma) \log m(\gamma)]\pi(\gamma)\mathrm{d}\gamma \tag{7}$$

As is standard in robust control, the concern about model uncertainty is represented by a two-person zero-sum dynamic game in which, after observing the choice of a social planner, a malevolent player chooses the worst specification of the model in each period. This game proceeds as follows.[8] At the beginning of a period, the state; i.e., the value of (K, N, P, T, R) is revealed. Then, the planner chooses $(C, E_i, N_i, \tilde{K}', P', T', S', R')$ in order to maximize social welfare. After observing the planner's choice, nature (the "malevolent player") chooses an alternative distribution $\hat{\pi}(\gamma)$ or, equivalently, $m(\gamma)$, to minimize welfare. Note that any deviation from the approximating distribution will be penalized by adding $\alpha\rho(\hat{\pi}(\gamma), \pi(\gamma))$ to the objective function. Here, α represents the magnitude of the "punishment." A greater α means a greater penalty associated with the deviation of γ from its approximating distribution, thus, a lower concern about robustness.

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[7]In GHKT, γ directly affects output. For technical reasons, we find it convenient to assume that γ adversely affects the economy's capital stock. The two assumptions lead to identical results when there is 100% capital depreciation (as we assume for our numerical results).

[8]Our attention will be restricted to a particular type of equilibrium, the so-called Markov perfect (or *feedback*) equilibrium. This equilibrium is *strongly time-consistent.*

This leads to the following social planner's problem:

$$V(K, N, P, T, R) = \max_{\{C, E_i, N_i, \tilde{K}', P', T', S', R'\}} \min_{m(\gamma)}$$

$$\{u(C) + \beta \int [m(\gamma)V(K', N', P', T', R') + \alpha m(\gamma) \log m(\gamma)] \pi(\gamma)\mathrm{d}\gamma\} \tag{8}$$

s.t.

$$E_i = A_i N_i; \ i = 2, 3 \tag{9}$$
$$E = (\kappa_1 E_1^\rho + \kappa_2 E_2^\rho + \kappa_3 E_3^\rho)^{1/\rho} \tag{10}$$
$$N = N_0 + N_2 + N_3 \tag{11}$$
$$\tilde{K}' = F(K, N_0, E) + (1 - \delta)K - C \tag{12}$$
$$K' = h(S', \gamma)\tilde{K}' \tag{13}$$
$$R' = R - E_1 \geq 0 \tag{14}$$
$$N' = A_N N \tag{15}$$
$$P' = P + \phi_L(E_1 + E_2) \tag{16}$$
$$T' = (1 - \phi)T + (1 - \phi_L)\phi_0(E_1 + E_2) \tag{17}$$
$$S' = P' + T' \tag{18}$$
$$1 = \int m(\gamma)\pi(\gamma)\mathrm{d}\gamma \tag{19}$$

Under a set of additional assumptions, the social planner's problem can be solved analytically, and we will focus on the analytical solution first. We will discuss the decentralized problem and show that the socially optimal allocation can be restored by imposing appropriate fossil fuel taxes on the energy-producing sector.

3 The Analytical Solution

For the remainder of this section, we will make the following additional assumptions. While these assumptions are admittedly strong, they allow us to fully solve the model analytically. As we shall see, certain aspects of the solution remain instructive in the next Section, when the restrictive assumptions are dropped and the model is solved numerically.

(A1) The period utility function is given by $u(C) = \log(C)$.
(A2) Capital depreciates fully; i.e., $\delta = 1$.
(A3) The production function is given by $F(K, N_0, E) = A_0 K^\theta N_0^{1-\theta-\nu} E^\nu$.
(A4) The damage function is given by $h(S', \gamma) = e^{-S'\gamma}$.[9]
(A5) The approximating distribution for γ is exponential with mean λ^{-1} and variance λ^{-2}; i.e., $\pi(\gamma) = \lambda e^{-\lambda\gamma}$.[10]

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[9]There exists a constant, Δ, such that if the GHG concentration, S, is greater than $\frac{1}{\Delta}$, the system cannot be "robustified," in the sense that the value of the game goes to negative infinity. However, if the economy starts with an initial $S_0 < \frac{1}{\Delta}$, then S_t will converge to $\frac{1}{\Delta}$ as $t \to +\infty$.

[10]The exponential distribution with mean λ^{-1} is the maximum-entropy distribution among all continuous

(A6.1) $\phi_L = 0$.[11]

(A6.2) $\phi = 0$.

(A7) There is a single fossil energy sector producing oil at zero cost. Production is subject to a resource feasibility constraint: $R' \geq 0$. As a result, $N_1 = 0$ and $N_0 = N$.

(A8) There is no population growth, and the aggregate labor supply is normalized to 1. That is, $A_N = 1$ and $N = 1$ in all periods.

(A9) There is no technology improvement. That is, A_0 is constant over time. We normalize $A_0 = 1$.

(A10) The resource feasibility constraint is not binding.[12]

We will first solve the social planner's problem. We will then discuss the decentralized problem and show that the socially optimal allocation can be restored by implementing fossil fuel taxes on the energy-producing sector.

Under **A1**-**A10**, the social planner's problem can be rewritten as:

$$V(K, S)$$
$$= \max_{\{C, E, \tilde{K}', S'\}} \min_{m(\gamma)} \{ u(C) + \beta \int \left[m(\gamma) V(K', S') + \alpha m(\gamma) \log m(\gamma) \right] \pi(\gamma) \mathrm{d}\gamma \} \quad (20)$$

$$s.t.$$
$$\tilde{K}' = F(K, E) - C \quad (21)$$
$$K' = h(S', \gamma)\tilde{K}' \quad (22)$$
$$S' = S + \phi_0 E \quad (23)$$
$$1 = \int m(\gamma)\pi(\gamma)\mathrm{d}\gamma \quad (24)$$

where $h(S', \gamma) = e^{-S'\gamma}$ and $F(K, E) = K^\theta E^\nu$. To solve this problem, we first guess that $V(\cdot)$ takes the form

$$V(K', S') = f(S') + \bar{A}\log(K') + \bar{D} = f(S') + \bar{A}\log(h(S', \gamma)\tilde{K}') + \bar{D} \quad (25)$$

where $\bar{A}$ and $\bar{D}$ are undetermined coefficients. The functional form for $f(\cdot)$ will be derived when we solve the minimizing player's problem.

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distributions supported in $[0, \infty]$ that have mean λ^{-1}. The worst case distribution for γ is also exponential with mean $(\lambda^*)^{-1}$ and variance $(\lambda^*)^{-2}$, where $\lambda^* = \lambda(1 - \Delta S'^*) = \lambda(1 - \Delta\phi_0 c_E)(1 - \Delta S)$. That is, $\pi^*(\gamma) = \lambda^* e^{-\lambda^*\gamma}$. Since $\lambda^* = \lambda(1 - \Delta S'^*) < \lambda$, the worst case mean of γ, $(\lambda^*)^{-1}$, is strictly greater than the approximating mean, λ^{-1}.

[11]If $\phi_L > 0$, we need to depict the dynamics of P and T separately before we sum them in order to obtain the dynamics of S. Assuming that $\phi_L = 0$ allows us to express the dynamics of S without the need to consider P and T separately. That is, $S' = (1 - \phi)S + \phi_0 E$. Moreover, **(A6.1)** and **(A6.2)** imply that $S' = S + \phi_0 E$, which is necessary for an analytical solution.

[12]Later we provide a sufficient condition for **(A10)**.

First, we define the robustness problem (the *inner minimization problem*) by

$$\mathcal{R}(V)(\tilde{K}', S') \quad = \quad \min_{m(\gamma)} \int \left[m(\gamma)V(K', S') + \alpha m(\gamma)\log m(\gamma) \right] \pi(\gamma)\mathrm{d}\gamma \tag{26}$$

$$\text{s.t.}$$

$$K' = e^{-S'\gamma}\tilde{K}' \tag{27}$$

$$1 \quad = \quad \int m(\gamma)\pi(\gamma)\mathrm{d}\gamma \tag{28}$$

The F.O.N.C. for $m(\gamma)$ implies that

$$m^*(\gamma) = \frac{\exp(-\frac{V(K',S')}{\alpha})}{\int \exp(-\frac{V(K',S')}{\alpha})\pi(\gamma)\mathrm{d}\gamma} = (1 - \Delta S')e^{\Delta S'\lambda\gamma} \tag{29}$$

or, equivalently,

$$\hat{\pi}^*(\gamma) = m^*(\gamma)\pi(\gamma) = \lambda^* e^{-\lambda^*\gamma} \tag{30}$$

where we define $\Delta = \frac{\bar{A}}{\alpha\lambda}$ and $\lambda^* = \lambda(1 - \Delta S')$.[13] Thereby,

$$\mathcal{R}(V)(\tilde{K}', S') \quad = \quad \int \left[m^*(\gamma)V(K', S') + \alpha m^*(\gamma)\log m^*(\gamma) \right] \pi(\gamma)\mathrm{d}\gamma$$

$$= \quad -\alpha \log\left[\int \exp\left(-\frac{V(K', S')}{\alpha}\right)\pi(\gamma)\mathrm{d}\gamma \right] \tag{31}$$

Substituting equation(25) into equation(31), we obtain

$$\mathcal{R}(V)(\tilde{K}', S') = f(S') + \bar{A}\log(\tilde{K}') + \bar{D} + H(S'; \alpha, \bar{A}) \tag{32}$$

where $H(S'; \alpha, \bar{A})$, the robust version of the externality from carbon emissions, is given by

$$H(S'; \alpha, \bar{A}) = -\alpha \log\left[\int h^{-\frac{\bar{A}}{\alpha}}(S', \gamma)\pi(\gamma)\mathrm{d}\gamma \right] \tag{33}$$

It follows from **(A4)-(A5)** that

$$H(S'; \alpha, \bar{A}) = \alpha \log(1 - \Delta S') \tag{34}$$

Next, we define the optimal choice problem (the *outer maximization problem*). Using the analysis above, this problem can be written as

$$V(K, S) = \max_{\{C, E, \tilde{K}', S'\}} \left\{ \log(C) + \beta\mathcal{R}(V)(\tilde{K}', S') \right\} \tag{35}$$

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[13] The worst case distribution of γ remains exponential with a distorted mean $(\lambda^*)^{-1}$ and variance $(\lambda^*)^{-2}$.

or equivalently,

$$
\begin{aligned}
f(S) + \bar{A}\log(K) + \bar{D} \\
= \max_{C,E}\{\log(C) + \beta[f(S') + \bar{A}\log(\tilde{K}') + \bar{D} + H(S';\alpha,\bar{A})]\} \quad (36)
\end{aligned}
$$

s.t.

$$
\tilde{K}' = F(K,E) - C \quad (37)
$$

$$
S' = S + \phi_0 E \quad (38)
$$

$$
H(S';\alpha,\bar{A}) = \alpha\log(1 - \Delta S') \quad (39)
$$

The F.O.N.C. imply

$$
C = \frac{F(K,E)}{1 + \beta\bar{A}} \quad (40)
$$

$$
-\phi_0\left[\frac{\partial f(S')}{\partial S'} + \frac{\partial H(S';\alpha,\bar{A})}{\partial S'}\right] = \frac{1 + \beta\bar{A}}{\beta}\frac{\frac{\partial F(K,E)}{\partial E}}{F(K,E)} \quad (41)
$$

Noting that $H(S;\alpha,\bar{A})$ is a logarithmic function of S, we guess that $f(S) = \bar{B}\log(1 - \Delta S)$, where $\bar{B}$ is an undetermined coefficient. As a result, the above F.O.N.C. can be simplified to

$$
C = \frac{K^\theta E^\nu}{1 + \beta\bar{A}} \quad (42)
$$

$$
\frac{\beta\phi_0\Delta(\alpha + \bar{B})}{1 - \Delta S'} = \frac{\nu(\beta\bar{A} + 1)}{E} \quad (43)
$$

After some tedious derivations, we obtain

$$
\bar{A} = \frac{\theta}{1 - \beta\theta} \quad (44)
$$

$$
\bar{B} = \frac{1}{1 - \beta}[\alpha\beta + \frac{\nu}{1 - \beta\theta}] \quad (45)
$$

The expression for $\bar{D}$ is more complicated and less intuitive. Substituting $\bar{A} = \frac{\theta}{1 - \beta\theta}$ into the F.O.N.C., we obtain the optimal allocation. We summarize the above discussion in the following.

Proposition 1. *Assume that (A1)-(A10) hold. The two-person zero-sum dynamic game described by eq(20)-eq(24) admits a feedback (Markov perfect) equilibrium. The equilibrium strategies are given by:*

$$
C^* = (1 - \beta\theta)K^\theta E^{*\nu} = (1 - \beta\theta)K^\theta[c_E(1 - \Delta S)]^\nu \quad (46)
$$

$$
E^* = c_E(1 - \Delta S) \quad (47)
$$

$$
S'^* = S + \phi_0 c_E(1 - \Delta S) \quad (48)
$$

$$
\hat{\pi}^*(\gamma) = \lambda^* e^{-\lambda^*\gamma} \quad (49)
$$

where $c_E = \frac{\nu(1-\beta)}{[\beta\alpha(1-\beta\theta)+\nu]\phi_0\Delta}$ and $\lambda^* = \lambda(1 - \Delta S'^*)$.

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A few technical remarks are in order. First, the function $V(K, S)$ is increasing in K, decreasing in S, and jointly concave in K and S. The value of $\bar{A}$ is the same as in the model without concern about model uncertainty. Both E^* and S'^* are affine functions of S. In addition, it can be shown that, given S, both E^* and S'^* are increasing functions of α. This is intuitive since a greater α implies a larger resulting penalty from a deviation of γ from its approximating distribution, thus, a lower concern about model-uncertainty. Note that C^* is affected by S only through E^*. This is due to logarithmic utility. As a result, a greater concern about model-uncertainty will lower both E^* and C^*. The value of the externality from one unit of emissions evaluated at E^* is given by

$$\lambda^s = -\beta \frac{\partial V(K', S')}{\partial E}\Big|_{K'^*, S'^*} = \frac{\beta \phi_0 \Delta(\bar{B} + \alpha)}{1 - \Delta S'^*} = \frac{\nu}{c_E(1 - \beta\theta)(1 - \Delta S)} = \frac{\nu}{(1 - \beta\theta)E^*} \quad (50)$$

Our model so far is similar to the oil regime in GHKT, except that we assume that the resource constraint is not binding. Since $S_{t+1} = S_t + \phi_0 E_t$, we arrive at the following expression for the aggregate oil extraction

$$\sum_{t=0}^{+\infty} E_t = \lim_{t \to +\infty} \phi_0^{-1}(S_t - S_0) = \phi_0^{-1}\left(\frac{1}{\Delta} - S_0\right) \quad (51)$$

Thus, the resource constraint is not binding if and only if the aggregate oil reserves are greater than $\phi_0^{-1}(\frac{1}{\Delta} - S_0)$. Figures 1, 2, and 3 below illustrate how E^* responds to a concern about model-uncertainty. Figures 1 and 2 show how E^* reacts to a change in the penalty parameter, α, in the multiplier version of the game.

We can also study the effect of change in concern about uncertainty on energy use. Figure 3 refers to the equivalent constraint game, in which $\hat{\pi}(\gamma)$ is constrained in a closed ball of radius δ centered at $\pi(\gamma)$, denoted by $B_\delta(\pi(\gamma))$. Direct calculation shows that the distance between $\hat{\pi}^*(\gamma)$ and $\pi(\gamma)$, as measured by entropy is given by

$$\rho(\hat{\pi}^*(\gamma), \pi(\gamma)) = \log(1 - \Delta S'^*) + \frac{\Delta S'^*}{1 - \Delta S'^*} \quad (52)$$

Since $\hat{\pi}^*(\gamma)$, which is chosen by the minimizing player, must be on the boundary of $B_\delta(\pi(\gamma))$, we have that $\rho(\hat{\pi}^*(\gamma), \pi(\gamma)) = \delta$. Recall that ρ measures the relative entropy of π and $\hat{\pi}$. Figure 3 shows how E^* changes as we relax δ, allowing for more uncertainty about the approximating model. In the Appendix we show that $\frac{\partial E^*}{\partial \delta}\big|_{\delta=0} = -\infty$. That is, even an infinitesimal concern about model uncertainty can cause a significant drop in the optimal energy extraction.

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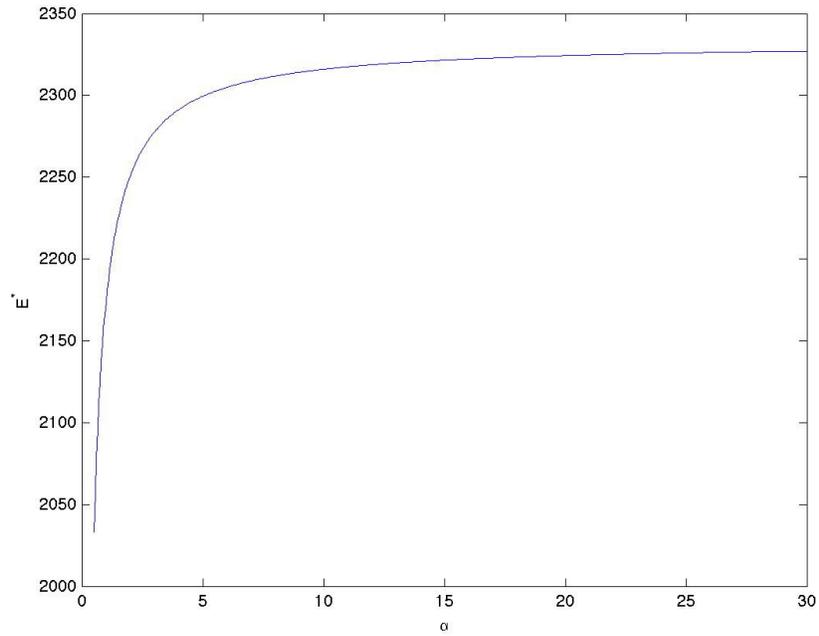
Figure 1: The Effect of Penalty Parameter α on Optimal Carbon Emissions, E
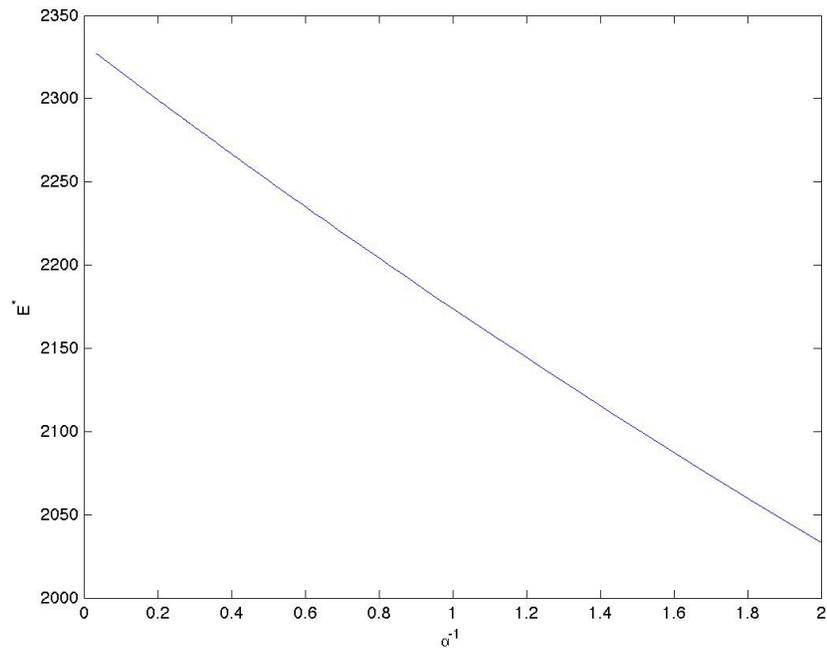
Figure 2: The Effect of α^{-1} on E

11
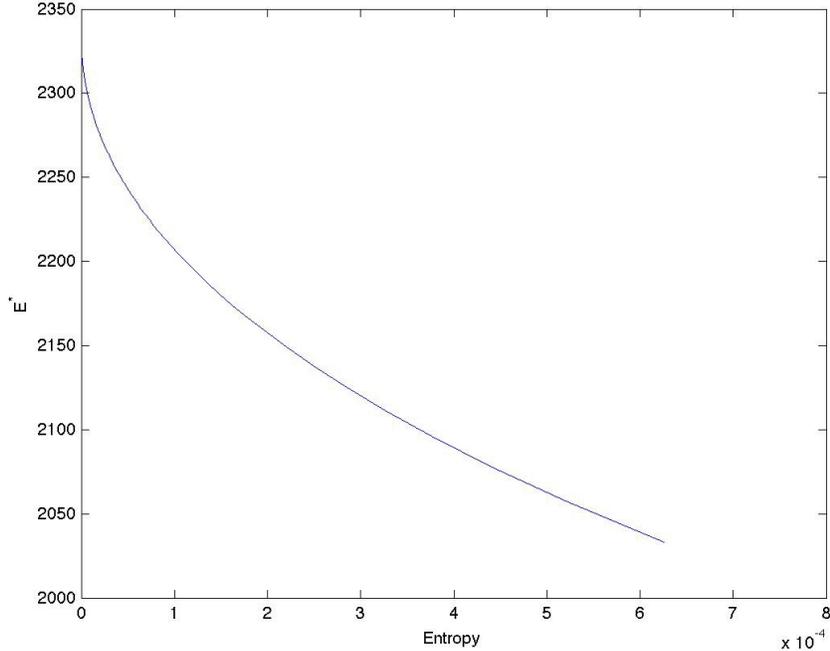
Figure 3: The Effect of Model Deviation as Measured by Entropy, δ, on E

Robust control modeling can be introduced in different ways. So far we used a closed-loop zero-sum dynamic game in which the social planner moves first in each period. Alternatively, we can construct a game with the same information structure by interchanging the order of max and min in eq(20). The two games differ only in terms of the timing protocol. However, both lead to the same (unique) feedback saddle-point equilibrium if certain conditions are satisfied. More precisely, if (A1)-(A10) hold, then the objective in (20) is strictly concave in C and E, and strictly convex in $m(\gamma)$. Consequently, the two closed-loop zero-sum dynamic games admit the same unique pure strategy saddle-point Nash equilibrium, which is the one described in Proposition 1.

Let us now turn to the decentralized problem. Suppose a percentage tax, τ_t, is imposed on emissions, E_t. Since the extraction cost of energy (the cost of creating emissions) is zero, it must be true that

$$\tau_t = p_t = \frac{\partial F(K_t, E_t)}{\partial E_t} = \nu K_t^\theta E_t^{\nu-1} \tag{53}$$

The above equation captures the one-to-one relationship between E_t and τ_t. Therefore, to achieve the optimal emissions level, $E_t = c_E(1 - \Delta S)$ in eq(47), we must impose $\tau_t = \nu c_E^{\nu-1}(1 - \Delta S_t)^{\nu-1} K_t^\theta$. It is straightforward to show that $\tau_t = \frac{\lambda^s}{u'(C_t^*)}$, where C_t^* is the optimal consumption, given by eq(46). That is, the optimal tax on emissions is equal to the corresponding GHG externality measured in units of the consumption good. It remains to show that C_t^* can be recovered under the optimal tax. This can be shown using the representative

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household's problem as follows. Since we have established a one-to-one relationship between E_t and τ_t, we may assume without loss of generality that the planner chooses E_t. Further, assume that E_t is chosen as a function of S_t only.[14] Given $E = E(S)$, k, K, and S, a representative household solves:

$$V(k, K, S) = \max_{c, \tilde{k}'} \min_{\hat{\pi}(\gamma)} \left\{ u(c) + \beta \hat{E}_\gamma \left[V(k', K', S') + \alpha \log \left(\frac{\hat{\pi}(\gamma)}{\pi(\gamma)} \right) \right] \right\} \quad (54)$$

$$s.t.$$

$$c + \tilde{k}' = r(K, S)k + \tau(K, S)E(S) + \pi^{profit} \quad (55)$$

$$\tilde{K}' = G(K, S) \quad (56)$$

$$k' = e^{-\gamma S'} \tilde{k}' \quad (57)$$

$$K' = e^{-\gamma S'} \tilde{K}' \quad (58)$$

$$S' = S + \phi_0 E(S) \quad (59)$$

where $u(c) = \log(c)$, $r(K, S) = \theta K^{\theta-1}[E(S)]^\nu$, $\tau(K, S) = \nu K^\theta [E(S)]^{\nu-1}$, π^{profit} is the firm's profit, and $\tilde{K}' = G(K, S)$ is the equilibrium transition law for the aggregate capital stock. Here, (k, K, S) stands for the beginning-of-period and $(\tilde{k}', \tilde{K}', S')$ for the end-of-period state, respectively. Notice that $(\tilde{k}', \tilde{K}')$ is not equal to the beginning-of-next-period state, (k', K'), due to capital deterioration by a factor $e^{-\gamma S'}$. In addition, $\hat{E}_\gamma$ is calculated with respect to the worst case distribution for γ, $\hat{\pi}(\gamma)$, as chosen by the minimizing player. Since the minimizing player moves after the maximizing player, the worst distribution is, in general, conditional on the end-of-period state, $(\tilde{k}', \tilde{K}', S')$. It can be shown that the optimal consumption sequence satisfies the following Euler equation:

$$u'(c^*) = \beta \frac{\int e^{-\gamma S'} r(K', S') u'(c'^*) e^{-\frac{V(k', K', S')}{\alpha}} \pi(\gamma) d\gamma}{\int e^{-\frac{V(k', K', S')}{\alpha}} \pi(\gamma) d\gamma} \quad (60)$$

This yields the following Proposition.

Proposition 2. *Assume that (A1) - (A10) hold. The optimal energy consumption is $E = c_E(1 - \Delta S)$. The optimal tax is $\tau_t = \frac{\lambda^s}{u'(C^*)}$, with tax proceeds rebated lump-sum to the representative consumer. The resulting competitive equilibrium allocation coincides with the solution to the planner's problem. That is, $c^* = C^* = (1 - \beta\theta)K^\theta[c_E(1 - \Delta S)]^\nu$.*

4   The Computational Solution and Calibration

In this Section we first extend the analytical model by relaxing assumptions **(A6.1)** and **(A6.2)**. For our baseline model, we will assume that $\pi(\gamma)$, the approximating distribution of γ, is exponential. As we now allow for $\phi_L > 0$, we need to introduce two additional state

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[14] This is without loss of generality, since our goal is to recover the optimal emissions in eq(47), which only depends on S_t.

variables (P and T), since keeping track of the sum $S = P+T$ will no longer suffice. We will also relax **(A7)** by incorporating a "coal" and a "green" sector into the model. Furthermore, we will relax **(A8)** and **(A9)** by allowing $A_2 N_2$ and $A_3 N_3$ to grow at a rate of two percent per year. Last, we will drop **(A10)**.

The social planner's problem becomes:

$$V(K, N, P, T, R) = \max_{\{C, E_1, E_2, E_3, E, \tilde{K}', P', T', S', R'\}} \min_{m(\gamma)}$$

$$\{u(C) + \beta \int [m(\gamma) V(K', N', P', T', R') + \alpha m(\gamma) \log m(\gamma)] \pi(\gamma) d\gamma\} \quad (61)$$

s.t.

$$E = (\kappa_1 E_1^\rho + \kappa_2 E_2^\rho + \kappa_3 E_3^\rho)^{1/\rho} \quad (62)$$

$$\tilde{K}' = F\left(K, N(1 - \frac{E_2}{A_2 N} - \frac{E_3}{A_3 N}), E\right) - C \quad (63)$$

$$K' = h(S', \gamma)\tilde{K}' \quad (64)$$

$$A_2' N' = (1 + g) A_2 N \quad (65)$$

$$A_3' N' = (1 + g) A_3 N \quad (66)$$

$$R' = R - E_1 \geq 0 \quad (67)$$

$$P' = P + \phi_L(E_1 + E_2) \quad (68)$$

$$T' = (1 - \phi)T + (1 - \phi_L)\phi_0(E_1 + E_2) \quad (69)$$

$$S' = P' + T' \quad (70)$$

$$1 = \int m(\gamma) \pi(\gamma) d\gamma \quad (71)$$

To solve this problem we first argue that most of the analysis conducted in Section 3 carries over. The only difference is that the function $f(\cdot)$ no longer has a closed form expression. We will again apply the outer-inner loop method used in Section 3. The inner loop minimization problem is unchanged, while the outer loop maximization problem will be solved in parts. In that regard, it is important to note that solving the optimization problem for E_i, P', T', and R' can be carried out separately from solving for C and $\tilde{K}'$. Furthermore, the solution to the second optimization problem remains the same as in Section 3; i.e., $C^* = (1 - \beta\theta)Y^*$ and $\tilde{K}'^* = \beta\theta Y^*$, where Y^* denotes the optimal output level. After substituting for C^*, the optimization problem for E_i, P', T', and R' can be simplified, leading to the dynamic

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programming problem below:

$$f(N, P, T, R) = \max_{E_1, E_2, E_3, E, P', T', S', R'}$$

$$\left\{ \frac{1}{1 - \beta\theta} \log[(1 - \frac{E_2}{A_2 N} - \frac{E_3}{A_3 N})^{1-\theta-\nu} E^\nu] + \beta[f(N', P', T', R') + \alpha \log(1 - \Delta S')] \right\} \quad (72)$$

s.t.

$$E = (\kappa_1 E_1^\rho + \kappa_2 E_2^\rho + \kappa_3 E_3^\rho)^{1/\rho} \tag{73}$$
$$N' = (1 + g)N \tag{74}$$
$$R' = R - E_1 \geq 0 \tag{75}$$
$$P' = P + \phi_L(E_1 + E_2) \tag{76}$$
$$T' = (1 - \phi)T + (1 - \phi_L)\phi_0(E_1 + E_2) \tag{77}$$
$$S' = P' + T' \tag{78}$$

Next, we characterize the optimality conditions for E_3, E_2, and E_1, respectively. The first-order condition for E_3 implies

$$\frac{\nu\kappa_3}{E_3^{1-\rho}E^\rho} = \frac{1 - \theta - \nu}{A_3 N_0} \tag{79}$$

The first-order condition for E_2 gives

$$\frac{1 - \theta - \nu}{A_2 N_0}$$

$$= \frac{\nu\kappa_2}{E_2^{1-\rho}E^\rho} + (1 - \beta\theta)\beta \left[\phi_L \left(\frac{\partial f}{\partial P'} - \frac{\alpha\Delta}{1 - \Delta S'} \right) + (1 - \phi_L)\phi_0 \left(\frac{\partial f}{\partial T'} - \frac{\alpha\Delta}{1 - \Delta S'} \right) \right] \tag{80}$$

Applying the envelope theorem to P and T gives

$$\frac{\partial f}{\partial P} = \beta \left(\frac{\partial f}{\partial P'} - \frac{\alpha\Delta}{1 - \Delta S'} \right) \tag{81}$$

$$\frac{\partial f}{\partial T} = \beta(1 - \phi) \left(\frac{\partial f}{\partial T'} - \frac{\alpha\Delta}{1 - \Delta S'} \right) \tag{82}$$

Defining $\hat{\Lambda}^P = -(1 - \beta\theta)\frac{\partial f}{\partial P}$ and $\hat{\Lambda}^T = -(1 - \beta\theta)\frac{\partial f}{\partial T}$ to be the marginal values of the externality caused by P and T, respectively, the first-order condition for E_2 becomes

$$\frac{1 - \theta - \nu}{A_2 N_0} = \frac{\nu\kappa_2}{E_2^{1-\rho}E^\rho} - \left[\phi_L \hat{\Lambda}^P + \frac{(1 - \phi_L)\phi_0}{1 - \phi} \hat{\Lambda}^T \right] \tag{83}$$

It is easy to see that the marginal externality caused by E_2 (or E_1) is given by

$$\hat{\Lambda}^S = \phi_L \hat{\Lambda}^P + \frac{(1 - \phi_L)\phi_0}{1 - \phi} \hat{\Lambda}^T \tag{84}$$

Thus, we obtain

$$\frac{\nu \kappa_2}{E_2^{1-\rho} E^\rho} - \hat{\Lambda}^S = \frac{1 - \theta - \nu}{A_2 N_0} \tag{85}$$

This has the same form as the corresponding equation in GHKT, but under a different interpretation for $\hat{\Lambda}^S$. To see the difference, it is convenient to restore the time index, t. From eq(81) and eq(82) we have

$$\hat{\Lambda}_t^P = (1 - \beta\theta)\alpha\Delta \sum_{j=1}^{+\infty} \frac{\beta^j}{1 - \Delta S_{t+j}} = \theta\bar{\gamma} \sum_{j=1}^{+\infty} \frac{\beta^j}{1 - \Delta S_{t+j}} \tag{86}$$

$$\hat{\Lambda}_t^T = (1 - \beta\theta)\alpha\Delta \sum_{j=1}^{+\infty} \frac{[\beta(1 - \phi)]^j}{1 - \Delta S_{t+j}} = \theta\bar{\gamma} \sum_{j=1}^{+\infty} \frac{[\beta(1 - \phi)]^j}{1 - \Delta S_{t+j}} \tag{87}$$

The second equality in either equation is obtained by using $(1 - \beta\theta)\alpha\Delta = (1 - \beta\theta)\alpha\frac{\bar{A}}{\alpha\lambda} = \theta\lambda^{-1} = \theta\bar{\gamma}$, where $\lambda^{-1} = \bar{\gamma}$ is the mean of γ under the approximating model. It follows immediately that $\hat{\Lambda}_t^S$ can be expressed as

$$\hat{\Lambda}_t^S = \theta\bar{\gamma} \sum_{j=1}^{+\infty} \left[\phi_L \frac{\beta^j}{1 - \Delta S_{t+j}} + \frac{(1 - \phi_L)\phi_0}{1 - \phi} \frac{[\beta(1 - \phi)]^j}{1 - \Delta S_{t+j}} \right] \tag{88}$$

It is instructive to consider the case when $\alpha \to +\infty$; i.e., when there is no concern about model uncertainty. Observe that $\Delta \to 0$ as $\alpha \to +\infty$. Therefore,[15]

$$\begin{aligned} \lim_{\alpha \to +\infty} \hat{\Lambda}_t^S &= \theta\bar{\gamma} \sum_{j=1}^{+\infty} \left[\phi_L \beta^j + \frac{(1 - \phi_L)\phi_0}{1 - \phi} [\beta(1 - \phi)]^j \right] \\ &= \theta\bar{\gamma} \left[\frac{\phi_L \beta}{1 - \beta} + \frac{(1 - \phi_L)\phi_0 \beta}{1 - (1 - \phi)\beta} \right] \end{aligned} \tag{89}$$

Finally, the first-order condition for E_1 yields

$$\frac{\nu \kappa_1}{E_1^{1-\rho} E^\rho} - \hat{\Lambda}^S = \beta \left[\frac{\nu \kappa_1}{(E_1')^{1-\rho} (E')^\rho} - (\hat{\Lambda}^S)' \right] \tag{90}$$

Note that the operator $\mathbb{E}_t$ does not appear on the right-hand-side, as the planner optimizes under the worst case scenario, rather than averaging over all cases. As the planner's problem has a similar structure as in the analytical model, it can be shown that analogues of Propositions 1 and 2 hold in this environment. We numerically solve the above problem for the cases where $\alpha = 0.01$ and $\alpha = 100$. We use the same parameter values as in GHKT, except for R_0, which is set to 800 as in Rogner (1997). Figures 4 through Figure 6 plot the computed optimal paths.

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[15]Contrasting this with the corresponding equation in GHKT $\left(\hat{\Lambda}_t^S = \bar{\gamma} \left[\frac{\phi_L}{1-\beta} + \frac{(1-\phi_L)\phi_0}{1-(1-\phi)\beta} \right] \right)$, we identify two differences. First, eq(89) contains an additional term (θ). This is because GHG directly affect aggregate capital instead of output in our model. Second, the externality related to P and T is weighted by β in eq(89). This is because GHG in our model affect next period's capital rather than the current one.

Parameter	ϕ	ϕ_L	ϕ_0	θ	ν	β	ρ	$1+g$
Value	0.0228	0.2	0.393	0.3	0.04	0.985^{10}	-0.058	1.02^{10}
Parameter	P_0	T_0	R_0	κ_1	κ_2	$A_{2,0}$	$A_{3,0}$	λ^{-1}
Value	103	699	800	0.5008	0.08916	$7,693$	$1,311$	2.379×10^{-5}
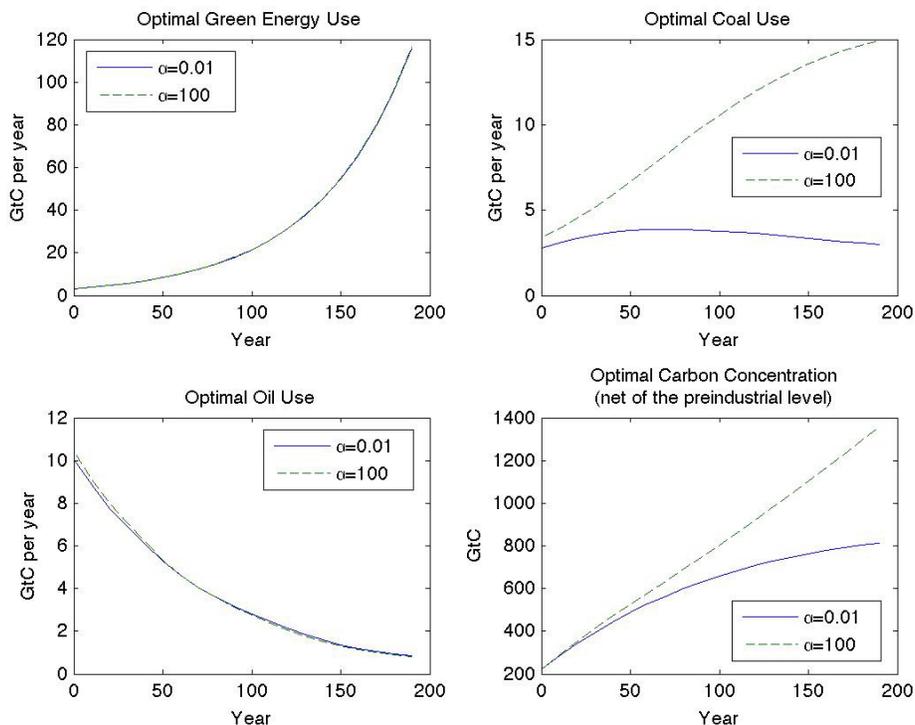
Figure 4: Optimal Use of Energy

Figure 4 describes the optimal paths for the use of green energy, coal, and oil, as well as the resulting carbon concentration in the atmosphere, conditional on different levels of concern about model uncertainty. For simplicity, we refer to the optimal path under $\alpha = 100$ as the "non-robust optimal path," and to the path under $\alpha = 0.01$ as the "robust optimal path." Since the green energy sector does not inject carbon into the atmosphere, the optimal path for the use of green energy does not directly depend on the level of concern about model uncertainty regarding the externality from carbon emissions. However, since green energy, coal, and oil are substitutes, model uncertainty considerations do affect the use of green energy indirectly, through its impact on the "dirty" energy sectors — coal and oil.

We find that an increase in the concern about model uncertainty causes a significant decline in the use of coal. In contrast, the use of oil is delayed, but only slightly. As the supply of oil is finite, the decline rate of oil-use depends not only on model uncertainty, but also on resource scarcity. As we will show in the next Section, an initial stock of oil equaling $R_0 = 800 GtC$ is low enough so that the resource scarcity effect overwhelms the

model uncertainty effect in determining the optimal use of oil in the economy. This explains why we do not observe a sharp decrease in the optimal use of oil when the concern about model uncertainty increases. Finally, straightforward calculation shows that the difference in energy use in the two optimal paths leads to a significant difference in the associated carbon accumulation. Our model predicts that if there is a "small" concern about model uncertainty ($\alpha = 100$), or if model uncertainty is not incorporated into the model ($\alpha = 0.01$), atmospheric carbon concentrations will reach a level as high as $1350GtC$ (net of preindustrial levels) after 180 years. However, this number is reduced by 40% to about $800GtC$ if concerns about model uncertainty are incorporated and addressed through the corresponding optimal tax, restoring the optimal energy path under $\alpha = 0.01$.
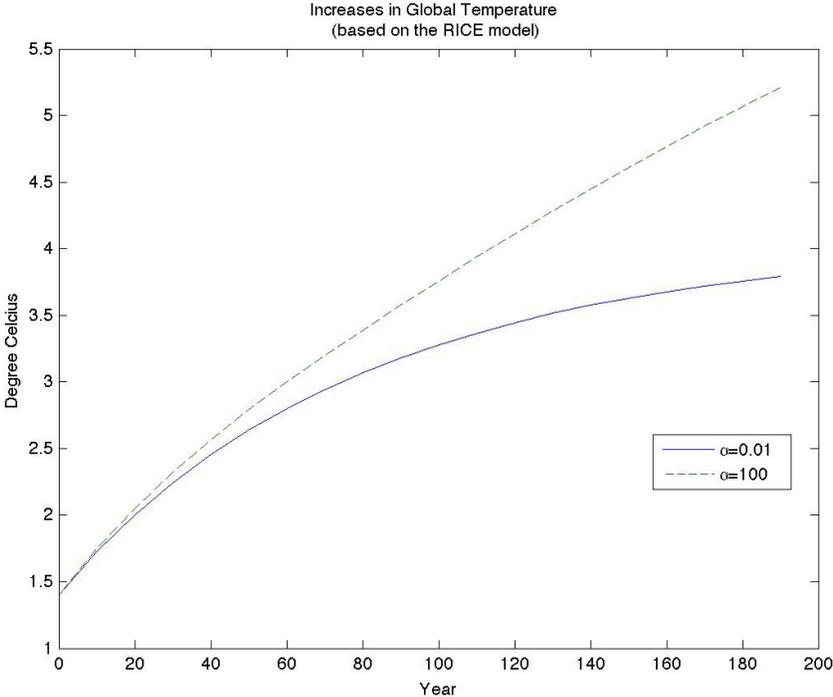
Figure 5: Increase in Global Temperatures

Figure 5 demonstrates a direct consequence of the above analysis: based on the mapping from carbon concentrations to global temperatures used in the RICE model, $T(S_t) = 3\ln(\frac{S_t}{\bar{S}})/\ln 2$, the global average temperature will rise by 3.8 degree Celsius 180 years from now if the concern about model uncertainty is addressed, and by 5.3 degrees Celsius otherwise.

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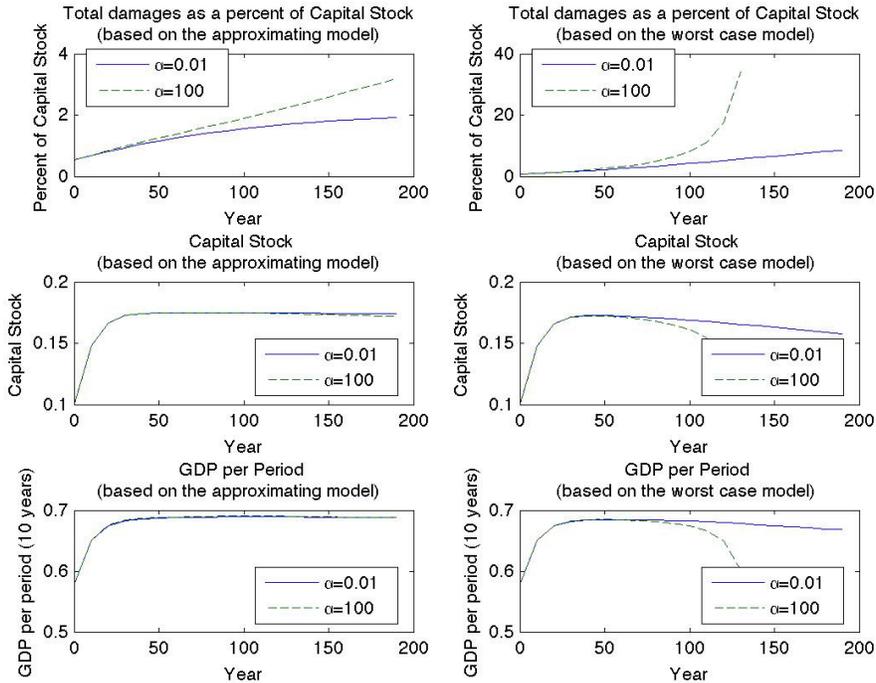
Figure 6: Capital Stock and Output

The graphs in the first (second) column in Figure 6 describe the paths of total damages as a percentage of the capital stock, and as a function of the capital stock, and of output, respectively, assuming that the approximating model (worst case model) for γ is the true model.[16] In each graph, the green-dashed line (blue-solid line) represents the outcome when energy is extracted based on the non-robust (robust) optimal path. The main findings can be summarized as follows. If the approximating model for γ is the true model, pursuing the robust optimal path for energy consumption would further reduce total damages by an additional 1 percent 180 years from now. However, due to a more conservative use of oil and coal in the final good sector, such a policy will also reduce both capital stock and output in the long run. Since utility depends only on consumption (which is proportional to output), this implies that the welfare loss from over-estimating the concern about uncertainty would be rather small. In contrast, if the true distribution of γ evolves according to the worst case model in each period (second column of Fig. 6), the cost of implementing the non-robust optimal policy is rather large. In fact, the non-robust policy, which overlooks concerns about model uncertainty, will dramatically reduce the entire capital stock in 120 years, resulting in a large reduction in output and welfare.[17]

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[16]To obtain smooth paths, γ is set to be the expected mean of the approximating (worst case) distribution(s) in each period.

[17]The dramatic effects on capital, output, and social welfare are partly due to the assumption that the approximating distribution of γ is exponential. As we discuss next, the losses are somewhat reduced, though

4.1 Varying the Approximating Distribution

Here we further explore the implications of assumption **(A5)**. To this end, we now assume that the approximating distribution of γ is normal with mean $\bar{\gamma}$ and variance σ^2; i.e., $\pi(\gamma) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(\gamma-\bar{\gamma})^2}{2\sigma^2}}$. This creates two key differences. First, the normal distribution provides us with two degrees of freedom: the mean, $\bar{\gamma}$, reflecting the planner's prior expectation regarding damages, and the variance, σ^2, indicating the prior regarding model uncertainty. In comparison, recall that the exponential distribution only used one parameter, λ, which determined both the mean and the variance of γ.[18]

We have:

$$H(S';\alpha,\bar{A}) = -(\bar{\gamma}+\frac{\bar{A}\sigma^2}{2\alpha}S')\bar{A}S' \tag{91}$$

$$\hat{\pi}^*(\gamma) \sim \mathcal{N}(\bar{\gamma}+\frac{\bar{A}\sigma^2}{\alpha}S'^2,\sigma^2) \tag{92}$$

It is straightforward to show that $H(\cdot)$ is strictly negative, strictly increasing in α, and strictly decreasing in both $\bar{\gamma}$ and σ^2. In addition, the worst case distribution for γ also follows a normal distribution, and $\hat{\pi}^*(\gamma)$ and $\pi(\gamma)$ differ only in their means. That is, when choosing the worst case model, nature only alters the mean of γ, rather than its variance. As a by-product, the relative entropy of $\hat{\pi}^*(\gamma)$ with respect to $\pi^*(\gamma)$ is given by

$$\rho(\hat{\pi}^*(\gamma),\pi^*(\gamma)) = \frac{\bar{A}^2\sigma^2 S'^2}{2\alpha^2} \tag{93}$$

To complete the model, we need to replace the term $\alpha\log(1-\Delta S')$ in eq(72) with $-(\bar{\gamma}+\frac{\bar{A}\sigma^2}{2\alpha}S')\bar{A}S'$. Accordingly, the optimality conditions for E_1, E_2, and E_3 remain intact, expect that the values of the externality associated with P, T, and E_2 (or E_1), respectively, are now as follows:

$$\hat{\Lambda}_t^P = \frac{\beta\theta\bar{\gamma}}{1-\beta} + \frac{\theta\bar{A}\sigma^2}{\alpha}\sum_{j=1}^{+\infty}\beta^j S_{t+j} \tag{94}$$

$$\hat{\Lambda}_t^T = \frac{\beta(1-\phi)\theta\bar{\gamma}}{1-\beta(1-\phi)} + \frac{\theta\bar{A}\sigma^2}{\alpha}\sum_{j=1}^{+\infty}[\beta(1-\phi)]^j S_{t+j} \tag{95}$$

$$\hat{\Lambda}_t^S = \phi_L\hat{\Lambda}_t^P + \frac{(1-\phi_L)\phi_0}{1-\phi}\hat{\Lambda}_t^T \tag{96}$$

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still large, if the approximating distribution of γ is assumed to be normal. The exponential distribution is one way to capture the extreme effects in Stern (2013) in the context of our model.

[18] As we shall see below, assuming that γ is normally distributed can also eliminate the "breaking point" for S, which is always present when γ follows an exponential. This is because the exponential distribution has a "fat" tail, thus, allowing more room for nature to create a worst-case-scenario given a level of penalty, α.

Note that $\hat{\Lambda}_t^S$ reduces to the previous expression as $\alpha \to +\infty$, or as $\sigma^2 \to 0$. That is,

$$\hat{\Lambda}_t^S = \theta\bar{\gamma} \left[\frac{\phi_L \beta}{1 - \beta} + \frac{(1 - \phi_L)\phi_0 \beta}{1 - (1 - \phi)\beta} \right], \text{ as } \alpha \to +\infty, \text{ or } \sigma^2 \to 0 \tag{97}$$

We will consider three cases regarding the initial stock of fossil fuel: $R_0 = 253.8 GtC$, $R_0 = 8000 GtC$, and $R_0 = \infty$. While the $R_0 = \infty$ case is for expository purposes only, the other two cases are of interest. Indeed, the total stock of oil and gas is estimated to exceed $8000 GtC$ if methane hydrates are included.[19] For each case, we numerically solve the above problem for $\alpha = 0.01$ and for $\alpha = +\infty$.[20]

Below we plot the same quantities as those shown in Fig. 4 through Fig. 6, but under the assumption that the approximating distribution of γ is normal. Our focus here is to compare the effects of model uncertainty on optimal oil-use under different values of R_0. As we have discussed earlier, holding other parameters fixed, the optimal path of oil consumption is determined jointly by the resource scarcity effect and the model uncertainty effect. First, note that we can hardly identify a difference between the robust and the non-robust optimal path for oil-consumption when the scarcity effect dominates, that is, when R_0 is sufficiently small. Figure 7 shows that when $R_0 = 253.8 GtC$, the non-robust optimal paths replicate their counterparts in GHKT. In this case, model uncertainty delays the optimal use of oil only slightly. However, Fig. 10 displays an altogether different pattern. When R_0 is set to $8000 GtC$, although both paths are still decreasing over time, model uncertainty discourages the use of oil substantively. Finally, as R_0 goes to infinity, as shown in Fig. 12, we observe a qualitative difference between the two paths. On the one hand, the non-robust optimal path allows the use of oil to grow unboundedly, partially due to the technological progress in the coal and green sectors. On the other hand, the increasing trend in oil consumption is curbed due to the externality caused by carbon emissions.

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[19] Estimated resources of methane hydrates vary, but they alone can amount to as much as $2.1 \times 10^4 GtC$. Of course, only a small fraction of these resources is recoverable using today's technologies. See Boswell and Collett (2011). See also Hartley, Medlock, Temzelides, and Zhang (2012) and references therein.

[20] To draw an even closer comparison with GHKT, we have re-scaled γ by a factor of $1/\theta$, where θ is the share of capital. The reason is that, given a Cobb-Douglas specification in final goods production, and given 100% depreciation of capital, a proportional damage of $e^{-\gamma S'}$ on capital is equivalent to a proportional damage of $e^{-\theta\gamma S'}$ on output. Accordingly, the mean and variance of γ in the approximating model are set to $\bar{\gamma} = 7.93 \times 10^{-5}$ and $\sigma^2 = 2.65 \times 10^{-8}$, respectively.
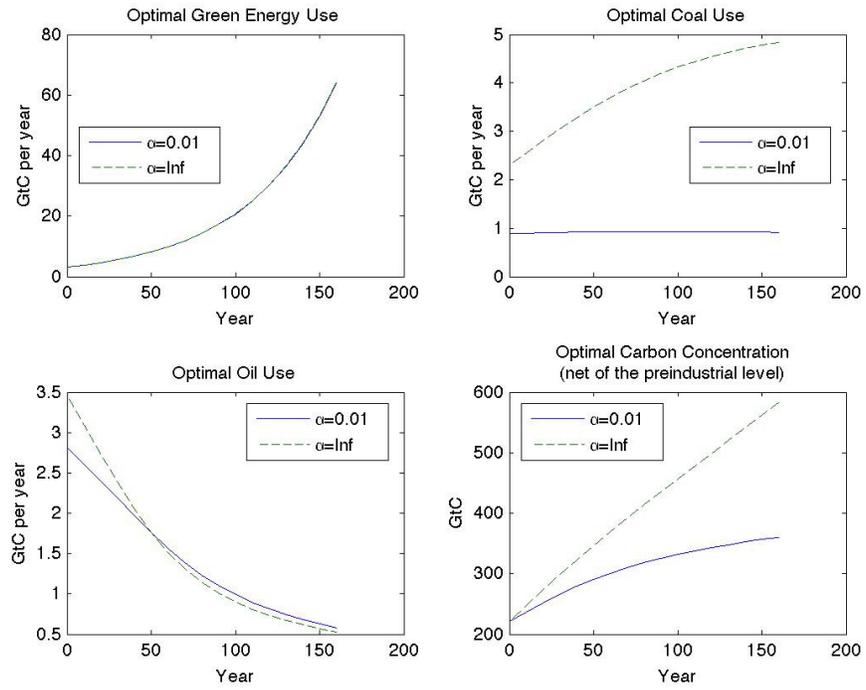
Figure 7: Optimal Use of Energy when $R_0 = 253.8$
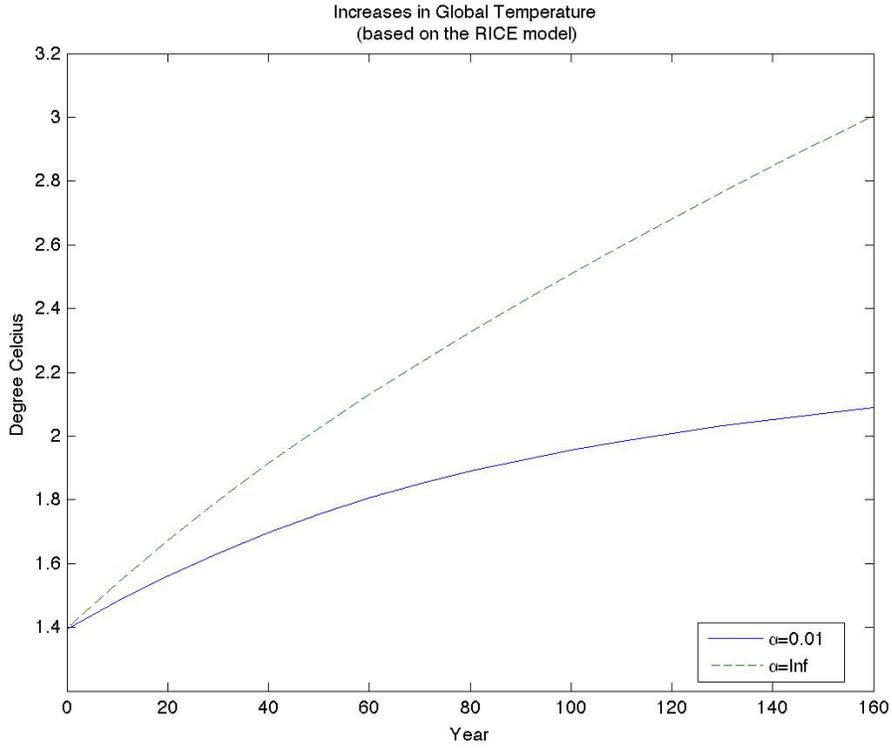
Figure 8: Increase in Global Temperatures when $R_0 = 253.8$

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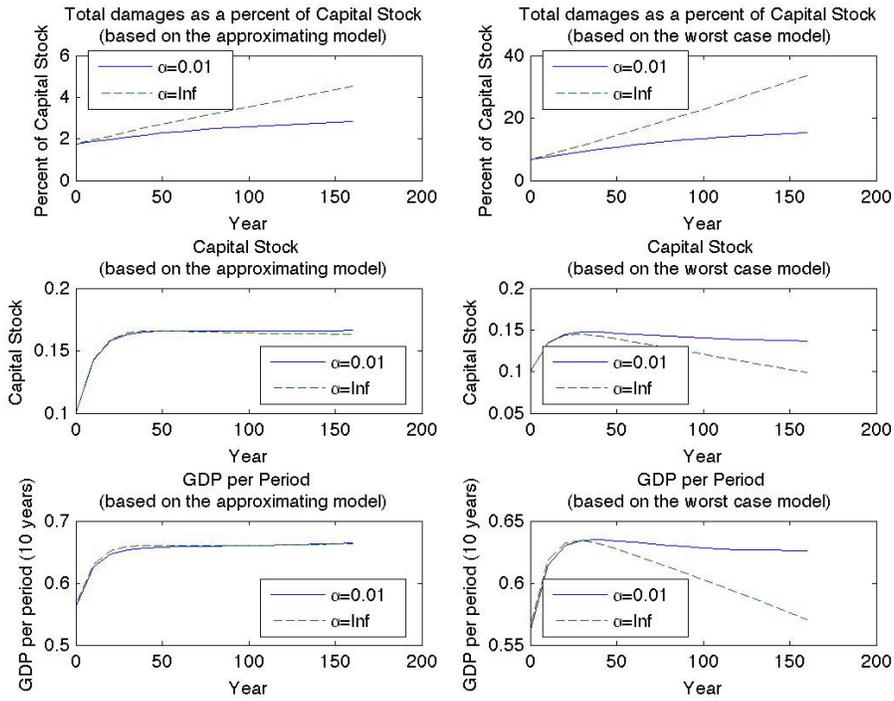
Figure 9: Capital Stock and Output when $R_0 = 253.8$
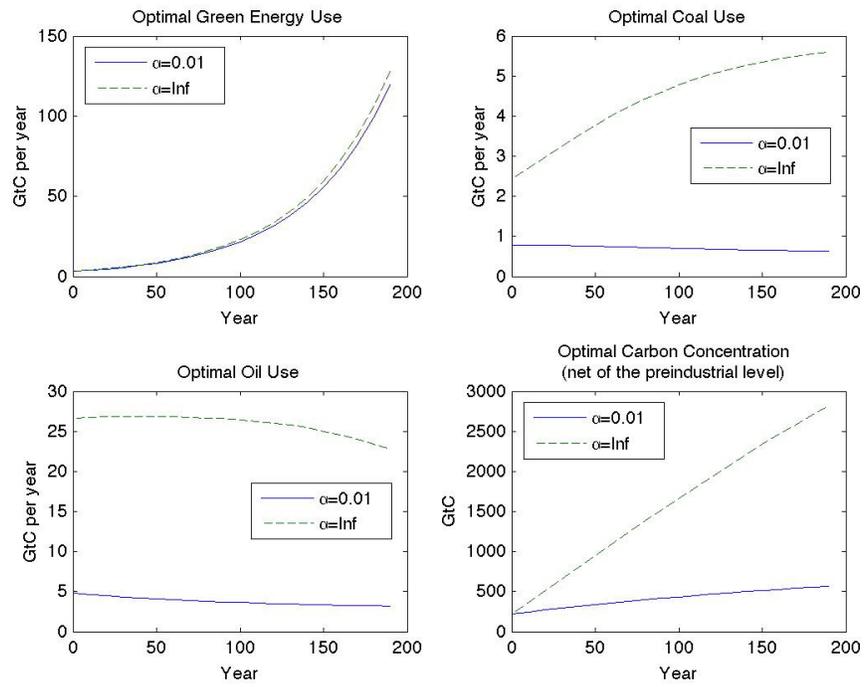
Figure 10: Optimal Use of Energy when $R_0 = 8000$
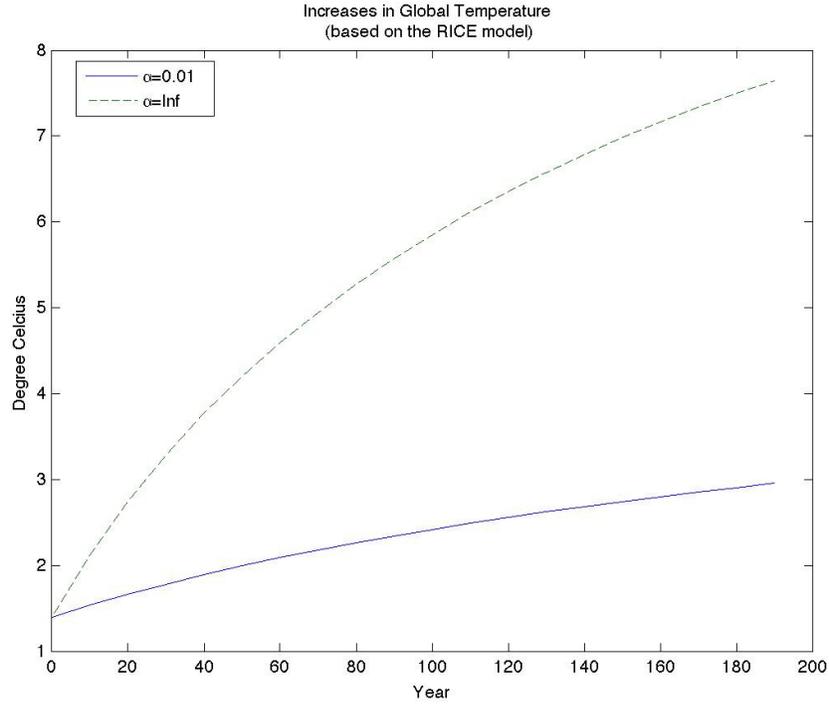
Figure 11: Increase in Global Temperatures when $R_0 = 8000$
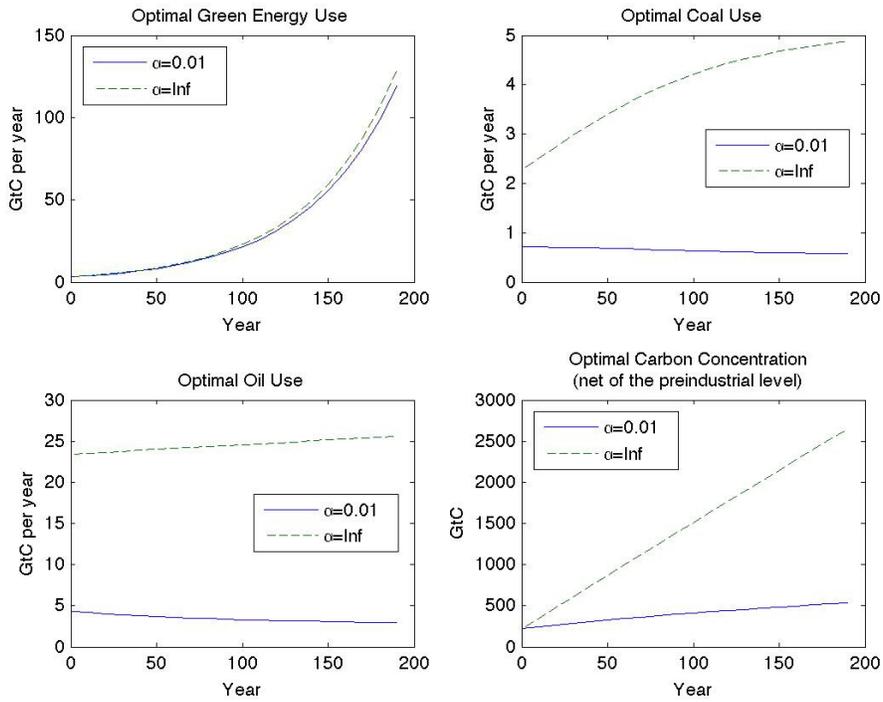
Figure 12: Optimal Use of Energy when $R_0 = \infty$

We now turn to a comparative analysis of the damages resulting from fossil fuel consumption. GHKT assume $R_o = 253.8GtC$ and estimate damages of *$56.9/ton of carbon* using an annual discount rate of *1.5%* and *$496/ton* under a rate of *0.1%*. When $\beta = 0.985^{10}$, and if there is no concern about model uncertainty ($\alpha = \infty$), the welfare loss implied by our model equals $0.985^{10} \times 56.4 = \$48.5/ton$. This number is independent of the approximating distribution for γ, the initial stock of oil, and of the future path of the GHG concentration. When $\alpha = 0.01$, however, these factors can matter substantially, as seen below. If the approximating distribution is normal, the losses are given in the following Table.

R_o/α	0.01	0.1	1	100	∞
253.8 GtC	239.60	70.65	50.85	48.52	48.49
8000 GtC	276.60	90.60	55.08	48.57	48.49
∞	318.70	103.06	63.42	56.49	48.49

$$(98)$$

4.2 Varying the Resource Feasibility Constraint

In order to further explore the model's implications, we now report the results for the case where oil is in infinite supply, while coal is constrained under an initial stock $R_{coal} = 666GtC$. This case demonstrates that the optimal use of oil mimics that in the case when both oil and coal are in infinite supply. In addition, the use of coal increases steadily at the beginning and then starts to drop.
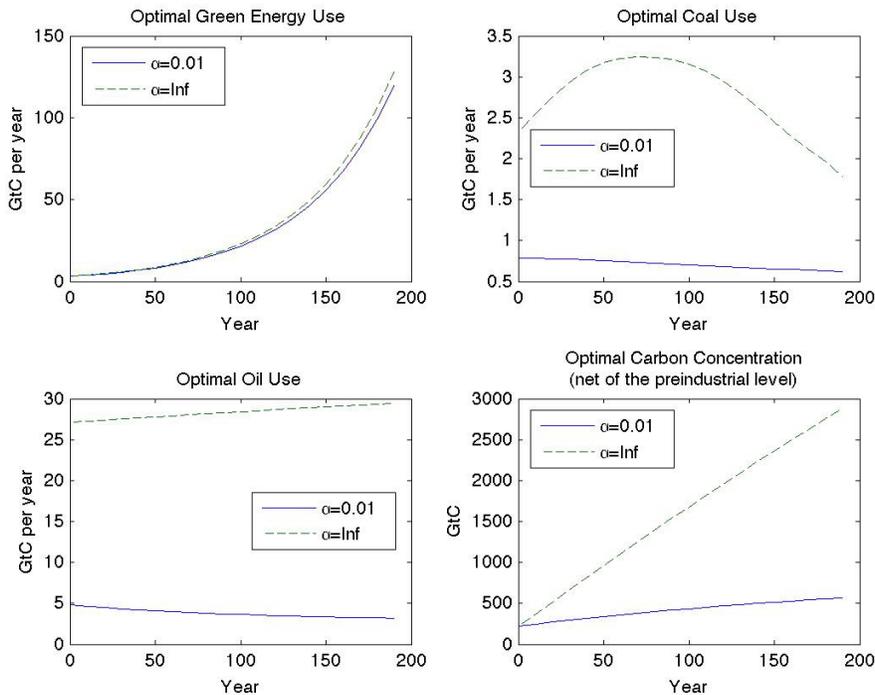
Figure 13: Optimal Use of Energy when $R_{coal} = 666$
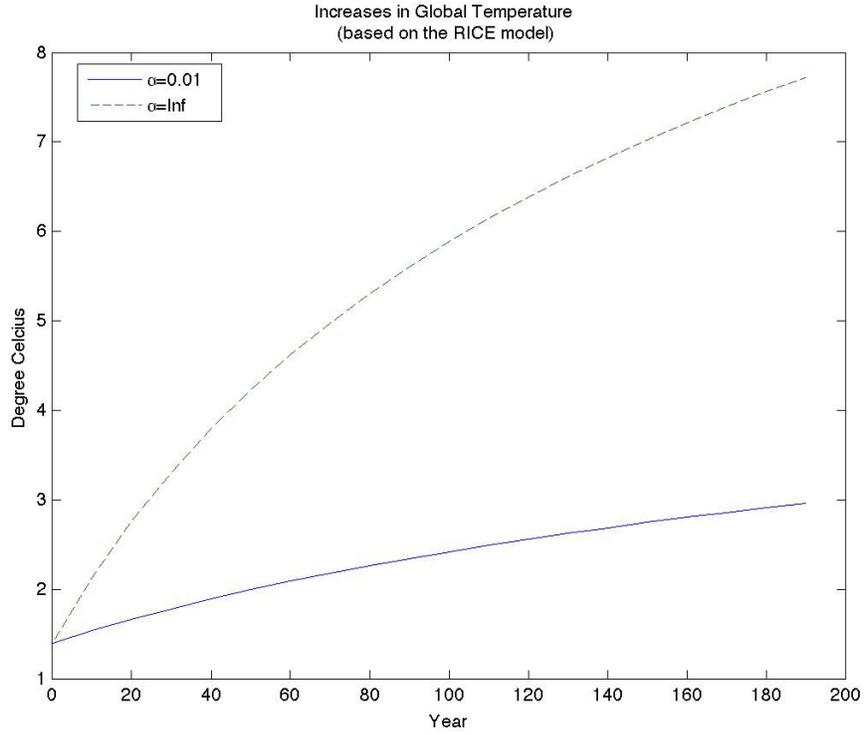
Figure 14: Increase in Global Temperature when $R_{coal} = 666$
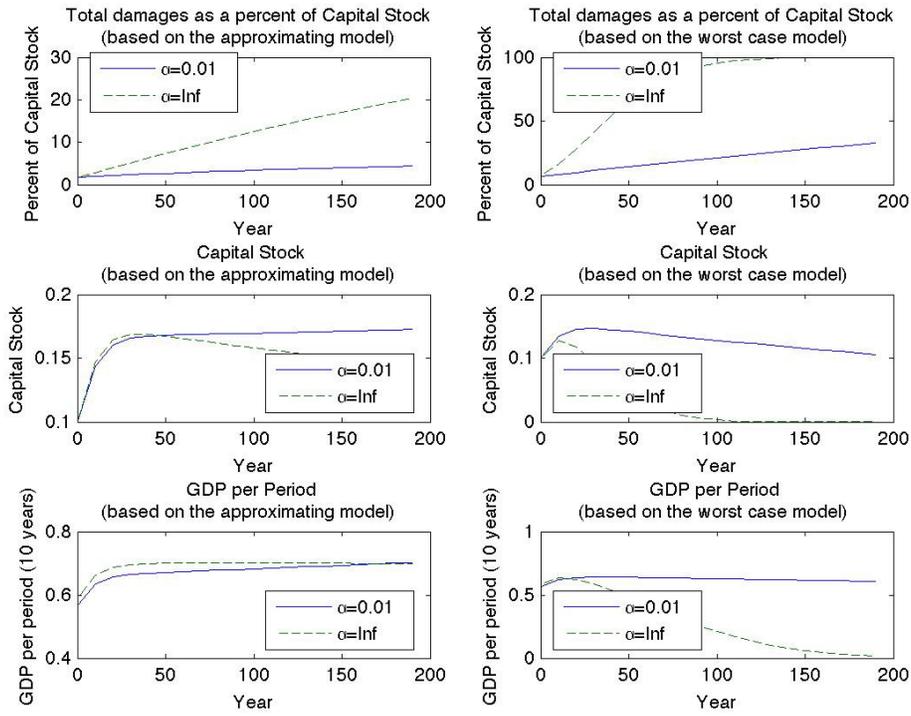
5   Conclusion

We studied optimal taxation in a dynamic stochastic general equilibrium model where agents are concerned about model uncertainty regarding climate change. Our model builds on (GHKT, 2013). We used robust control theory in order to model the uncertainty associated with climate change. In addition, we used an estimate of fossil fuel that includes methane hydrates as part of the supply of unconventional natural gas. While this huge resource is not readily available with today's technology, we believe that it is appropriate to include it given the long-term modeling that we follow throughout this exercise. Finally, we assumed a fat-tailed distribution of damages as a way to capture the extreme effects discussed in Stern (2013).

We obtained a sharp analytical solution for the implied externality, and we characterized the optimal tax. We found that a small increase in the concern about model uncertainty can cause a significant drop in optimal energy extraction. The optimal tax which restores the social optimal allocation was shown to be Pigouvian. Under more general assumptions, we developed a recursive method that allowed us to solve the model computationally. We showed that the introduction of uncertainty matters in a number of ways, both qualitatively and quantitatively. This dependence relies heavily on specific assumptions about the magnitude of fossil fuel reserves.

Our model can be extended in many ways. In the current version, the growth rate of renewables is assumed to be independent from the concern about model uncertainty. It would be interesting to endogenize growth in renewable energy productivity. A related extension could involve using a distortionary tax on labor to subsidize R&D in renewables in order to study the effects on energy composition and growth. Additionally, we could study a benchmark case where coal supply is constrained, while assuming infinite supply of gas and oil.

6   Appendix

Here we demonstrate that the optimal level of GHG, E^*, has the following properties: $\frac{\partial E^*}{\partial \delta} < 0$ and $\frac{\partial E^*}{\partial \delta}|_{\delta=0} = -\infty$, where δ is the upper bound for entropy allowed in the constraint game.

Proof. Recall that $E^* = c_E(1-\Delta S)$ and $\delta = \log(1-\Delta S'^*) + \frac{\Delta S'^*}{1-\Delta S'^*}$, where $S'^* = S + \phi_0 c_E(1-\Delta S)$. Define $a = \alpha^{-1}$ and $b = 1 - \Delta S'^* = (1 - \Delta\phi_0 c_E)(1 - \Delta S)$. It follows immediately that E^* is decreasing in a. In addition, since both Δ and c_E are functions of a, it follows that b is a function of a:

$$b(a) = [1 - \Delta(a)\phi_0 c_E(a)][1 - \Delta(a)S] \tag{99}$$

It is easy to see that b is decreasing in a. Thus, it defines a as an implicit function of b, with a negative slope. Moreover, we can rewrite δ as:

$$\delta = \log b + \frac{1-b}{b} \tag{100}$$

which defines b as an implicit function of δ. Direct calculation shows that $\frac{\partial b}{\partial \delta} = -\frac{b^2}{1-b} < 0$, as $b \in (0,1)$. Thus,

$$\frac{\partial E^*}{\partial \delta} = \frac{\partial E^*}{\partial a} \frac{\partial a}{\partial b} \frac{\partial b}{\partial \delta} < 0 \tag{101}$$

Evaluating this at $\delta = 0$, we obtain

$$\frac{\partial E^*}{\partial \delta}|_{\delta=0} = \left(\frac{\partial E^*}{\partial a}|_{a=0}\right)\left(\frac{\partial a}{\partial b}|_{b=1}\right)\left(\frac{\partial b}{\partial \delta}|_{\delta=0}\right) \tag{102}$$

It is straightforward to show that the first two terms on the right hand side in the above expression are strictly negative and finite, and the last term goes to $-\infty$. Therefore, $\frac{\partial E^*}{\partial \delta}|_{\delta=0} = -\infty$. $\qquad\square$

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