# A Duality Approach to Continuous-Time Contracting Problems with Limited Commitment*

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**Abstract**

We propose a duality approach to solving contracting models with either one-sided or two-sided limited commitment in continuous time. We establish weak and strong duality theorems and provide a dynamic programming characterization of the dual problem. The dual problem gives a linear Hamilton-Jacobi-Bellman equation with a known state space subject to free-boundary conditions, making analysis much more tractable than the primal problem. We provide two explicitly solved examples of a consumption insurance problem. We characterize the optimal consumption allocation in terms of the marginal utility ratio. We find that neither autarky nor full risk sharing can be an optimal contract with two-sided limited commitment, unlike in discrete-time models. We also derive an explicit solution for the unique long-run stationary distribution of consumption relative to income.

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# 1. Introduction

Many empirical studies find that idiosyncratic variation in consumption is systematically related to idiosyncratic variation in income, rejecting the hypothesis of full risk sharing (e.g., Cochrane (1991), Mace (1991), and Townsend (1994)). Instead of assuming exogenous market incompleteness, one important approach to reconciling this empirical evidence is to assume that individuals have limited commitment (e.g., Kocherlakota (1996), Alvarez and Jermann (2000), and Ligon, Thomas, and Worrall (2002)). This assumption is motivated by the fact that debt repayments are costly to enforce. Debt collection, litigation, and income garnishment are costly, and the debtor may default on debt. In this case, individual income risks are only incompletely shared.

Although discrete-time dynamic models with limited commitment have been widely applied in economics and finance,[1] these models are typically difficult to solve analytically and numerical solutions are needed. The main contribution of this paper is to propose a duality approach in a continuous-time setup, which permits analytical solutions. We consider a consumption insurance problem between a principal and an agent analogous to the problems analyzed by Thomas and Worrall (1988), Kocherlakota (1996), Alvarez and Jermann (2000), Ligon, Thomas, and Worrall (2002), and Ljungqvist and Sargent (2004).[2] The continuous-time setup is analytically convenient and allows us to derive sharp and transparent results. We find that the usual dynamic programming approach using the agent's continuation value as a state variable in the primal problem delivers a nonlinear Hamilton-Jacobi-Bellman (HJB) equation with state constraints. The state space of the continuation value is endogenous in models with two-sided limited commitment. Such a nonlinear HJB equation typically does not admit any analytical solution and is difficult to analyze even numerically. By contrast, the dual problem transforms the primal problem with participation constraints into an unconstrained problem, which delivers a linear HJB equation subject to free-boundary conditions. Technically, it is an instantaneous (or singular) control problem, similar to the problems analyzed in Harrison and Taksar (1983), Harrison (1985), and Stokey (2008).

We study the link between the dual and primal problems and establish the weak and strong

[1] Other examples include applications to wage contracts by Thomas and Worrall (1988), sovereign debt by Bulow and Rogoff (1989), Kletzer and Wright (2000), and Hellwig and Lorenzoni (2009), asset markets by Kehoe and Levine (1993) and Alvarez and Jermann (2000, 2001), optimal taxation by Chari and Kehoe (1993), business cycles by Cooley, Marimon, and Quadrini (2004), international business cycles by Kehoe and Perri (2002), consumption inequality by Krueger and Uhlig (2006) and Krueger and Perri (2006), the welfare effects of a progressive tax by Krueger and Perri (2011), political economy by Acemoglu, Golosov, and Tsyvinski (2011), and asset bubbles by Kocherlakota (2008) and Miao and Wang (2011, 2012).

[2] The principal and the agent can be interpreted in different ways in different contexts. They can be two households, a planner and a household, or a firm and a worker.

duality theorems. We provide a dynamic programming characterization of the dual problem using the usual state variables (individual incomes) together with additional costate variables. The costate variables are the cumulative amounts of the Lagrange multipliers associated with the intertemporal participation constraints, starting from pre-specified initial conditions. These costate variables are nonnegative and increasing processes. They are also the control variables in the dual problem and rise whenever the participation constraints bind.

In the case of one-sided limited commitment, there is only one costate variable, which is associated with the agent's participation constraints. To facilitate discussion, we first consider the case in which the principal and the agent have an identical discount rate. In this case, the costate variable is also equal to the ratio of the marginal utilities of the principal and the agent. The agent's current income and the marginal utility ratio constitute the state variables of the HJB equation for the dual problem. From the HJB equation, we derive a free boundary using the agent's binding participation constraints. The free boundary partitions the state space into two regions: the jump region and the no-jump region. When the initial promised value to the agent is higher than the outside value, the initial state of the marginal utility ratio and the agent's income must lie in the no-jump region. Subsequently, the marginal utility ratio and the agent's consumption remain constant in the interior of the no-jump region. They rise instantaneously whenever the agent's income increases and hits the free boundary. The marginal utility ratio keeps the agent's continuation value above the outside option value.

If the principal and the agent have different discount rates, then the solution is similar to that in the case of equal discount rates except that we must adjust the costate variable by the difference in the discount rates so that the adjusted costate variable is equal to the marginal utility ratio. This ratio and the agent's consumption are no longer constant in the no-jump region. They rise (fall) over time when the agent is more (less) patient than the principal.

In the case of two-sided limited commitment, we suppose for simplicity that the principal and the agent have an identical discount rate. There are two costate variables associated with the principal's and the agent's participation constraints, respectively. These two costate variables and the agent's income constitute the state variables of the HJB equation for the dual problem. We show that the HJB equation is linearly homogeneous and can be reduced to a two-dimensional problem using the agent's income and the marginal utility ratio as state variables. The marginal utility ratio is also equal to a suitably defined ratio of the two costate variables. From the HJB equation, we solve for the two free boundaries using the binding participation constraints of the principal and the agent. The two free boundaries partition the state space into three areas. The area between the two free boundaries is the no-jump region. The other two areas are the jump region. When the initial promised value to the agent is higher than his outside value and also not too large to push the principal's value below the principal's outside value, the initial state of the marginal utility ratio and the agent's income must lie in the no-jump region. Subsequently, the marginal utility ratio and the agent's consumption remain constant. Whenever the agent's income rises (falls) and hits the boundary determined by the agent's (the principal's) binding participation constraints, the marginal utility ratio and the agent's consumption rise (fall) instantaneously. The state processes of the marginal utility ratio and the agent's income will never move out of the no-jump region.

Another main contribution of this paper is to provide two explicitly solved examples with either one-sided or two-sided limited commitment. This contribution is important because, to the best of our knowledge, our paper is the first one that derives explicit closed-form solutions for dynamic models with two-sided limited commitment. Furthermore, our explicitly solved examples exhibit different risk-sharing dynamics than those in the discrete-time models. In particular, neither autarky nor full risk sharing can be an optimal contract in our examples.

The first example is a continuous-time version of the discrete-time models analyzed in Thomas and Worrall (1988), Krueger and Uhlig (2006), and in Chapter 19 of Ljungqvist and Sargent (2004). In this example, the principal is a risk-neutral planner and the agent is a household with a constant relative risk aversion utility function. Only the agent has limited commitment and may renege on the contract and enter autarky. We assume that the agent's income follows a geometric Brownian motion process and that the agent and the principal may have different subjective discount factors. In their discrete-time model, Ljungqvist and Sargent (2004) assume that the agent and the principal have an identical discount factor and show that the agent will be fully insured in the long run when his income is a bounded independently and identically distributed (IID) process.[3] By contrast, in our continuous-time model, the agent can never be fully insured. We show that the log consumption-income ratio is a one-sided regulated Brownian motion with a lower barrier (Harrison (1985) and Stokey (2008)). It has a unique long-run stationary distribution with an unbounded support, if the agent's income growth is sufficiently large.

In the second example, we incorporate the principal's limited commitment into the first example.[4] We suppose that the principal may also renege on the contract and take the autarky

[3]In a discrete-time model, Zhang (2013) allows the agent and the principal to have different subjective discount factors and provides a stopping time characterization of the optimal contract. But his approach does not admit an explicit solution.

[4]In Appendix C, we modify the second example by considering a symmetric setup in which both the principal and the agent have an identical constant absolute risk aversion utility function. The agent's income is modeled as an arithmetic Brownian motion and the principal's income is the negative of the agent's income so that these two incomes are perfectly negatively correlated. In this case, we show that the consumption-income difference is a two-sided regulated Brownian motion with two barriers and has a unique long-run stationary distribution. We also obtain a comparative statics result similar to that in the second example.

value of zero. This problem is a continuous-time version of the problems analyzed in Kocherlakota (1996), Alvarez and Jermann (2000), Ligon, Thomas, and Worrall (2002), and Chapter 20 of Ljungqvist and Sargent (2004). In a symmetric setup, Kocherlakota (1996) shows that the agent's consumption has a unique long-run stationary distribution when incomes follow a bounded IID process. By contrast, in our continuous-time model with a geometric Brownian motion income process, consumption itself has no long-run stationary distribution. But the log consumption-income ratio has a unique stationary distribution with a bounded support. The log consumption-income ratio is a two-sided regulated Brownian motion with two finite barriers (Harrison (1985) and Stokey (2008)). We call the interval between these two barriers the *risk-sharing band*. Under full risk sharing, consumption is constant and hence the band becomes the real line. The wider is the band, the more is the risk sharing.

In discrete-time models, Kocherlakota (1996), Alvarez and Jermann (2000), and Ligon, Thomas, and Worrall (2002) show that, depending on parameter values, there are three cases for an optimal contract: full risk sharing, autarky (no risk sharing), and limited risk sharing. In particular, Ligon, Thomas, and Worrall (2002) show that when the discount factor is sufficiently small, autarky is the only sustainable allocation, and when the discount factor is sufficiently large, full risk sharing can be achieved. By contrast, in our continuous-time model, only limited risk sharing can happen. This result reflects the difference in the nature of shocks and the difference in the continuous-time and discrete-time frameworks. In discrete-time models, the state space of shocks is typically finite. In our model, the shock is driven by a Brownian motion. Full risk sharing cannot be an optimal contract in our model because the unbounded Brownian motion shock can cause the autarky value to exceed any constant utility level from full risk sharing. This result also holds in a discrete-time model if the income process is unbounded.

Autarky cannot be an optimal contract in our model because the cost of staying in autarky is so high that participating in risk sharing is always mutually beneficial no matter how heavily the principal and the agent discount the future utility level.[5] This result is not due to the nonstationarity or unboundedness of the income process used in our example because we show that autarky is the only optimal contract in a discrete-time approximation of our model if the nonstationary income process is not too volatile or the principal and the agent are sufficiently impatient. In particular, we show that the net benefit from risk sharing depends on the length of the time interval. Thus, time frequency matters for the optimal contract.

We also conduct a comparative statics analysis with respect to the agent's risk aversion parameter, the volatility of the income process, and the subjective discount rate. We find

[5]The intuition is subtle. See Section 6.5 and the proof of Proposition 3 for an analysis of a discrete-time version of our model.

that the risk-sharing band expands when one of the following cases happens: (i) the subjective discount rate falls, (ii) the volatility of the income process rises, and (iii) the agent's coefficient of relative risk aversion rises. This result is intuitive. When contracting parties are more patient, cooperation and risk sharing are more likely to sustain. When either the income volatility or the degree of risk aversion is high, the autarky value is low, thereby reducing the agent's incentive to renege.

**Related Literature** The usual approach to solving dynamic contracting models is to use dynamic programming and adopt the agent's promised utility (or the continuation value) as a state variable. This approach is pioneered by Green (1987), Thomas and Worrall (1988), Spear and Srivastava (1987), and Abreu, Pearce, and Staccheti (1990).[6] DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007), Sannikov (2008), and Williams (2009, 2011) extend this approach to study continuous-time principal-agent problems with hidden action or hidden information. Miao and Rivera (2013) and Strulovici (2011) introduce robustness and renegotiation-proofness into this framework, respectively. Grochulski and Zhang (2011) apply this approach to study a consumption insurance problem with one-sided limited commitment in continuous time. They provide an explicit solution to the problem when the principal and the agent are equally patient. However, their analysis cannot be generalized to more general discount rates or to the case with two-sided limited commitment.

Our duality approach is closely related to that in the discrete-time setup proposed by Marcet and Marimon (1992, 1998), who build on the early work of Hansen, Epple, and Roberds (1985) and Kydland and Prescott (1980). The Marcet-Marimon approach has been extended by Messner, Pavoni, and Sleet (2011, 2012) in more general discrete-time contracting problems. Unlike in the discrete-time setup, our continuous-time approach allows us to derive transparent results and closed-form solutions. It requires different mathematical machinery and our results of the weak and strong duality theorems are nontrivial.

Our duality approach is also related to the mathematical finance literature on portfolio choice in continuous time (see, e.g., Xu and Shreve (1992), He and Pages (1993)). To the best of our knowledge, our paper is the first one to apply this approach to dynamic contracting problems with limited commitment in continuous time. Sannikov (2012) applies the duality approach to analyze a moral hazard model in which the agent's actions have long-run effects. His dual problem reduces to a standard optimal control problem rather than a singular control problem. As is well known, the duality approach is related to the maximum principle (e.g., Bismut (1973)). Williams (2009, 2011) applies the maximum principle to analyze the agent's

[6] Chapters 19 and 20 of Ljungqvist and Sargent (2004) provide an excellent introduction to this approach.

incentive problems in models with hidden action or hidden information. Neither Sannikov (2012) nor Williams (2009, 2011) studies models with limited commitment.

Our characterization of the optimal consumption policy in terms of the marginal utility ratio is similar to that of Thomas and Worrall (1988) and Ligon, Thomas, and Worrall (2002). In a discrete-time model with two-sided limited commitment, they derive a simple updating rule in terms of the marginal utility ratio. Each state of nature is associated with a particular interval of possible ratios of marginal utilities. Given the current state and the previous period's marginal utility ratio, the new ratio lies within the interval associated with the current state, such that the change in the ratio is minimized. The updating rule requires that the ratio of marginal utilities be kept constant whenever possible. However, if full risk sharing is not attainable, then the ratio must change to an endpoint of the current interval, and one of the households will be constrained, i.e., its participation constraints bind. Although the updating rule is intuitive, the discrete-time setup does not permit an explicit solution to the intervals of marginal utility ratios. Thus, numerical solutions are needed and they get messy when there are many states of shocks.

In our continuous-time model, the marginal utility ratio is also kept constant whenever possible. There is a band for marginal utility ratios associated with each income level. When the agent's income is sufficiently high to hit a boundary such that the agent's participation constraints bind, the marginal utility ratio rises continuously. But when the agent's income is sufficiently low to hit another boundary such that the principal's participation constraints bind, the marginal utility ratio falls continuously. The marginal utility ratio always lies within the band and moves continuously. The analytical power of our duality approach is that we are able to explicitly characterize the two boundaries of the band and the stationary distribution of consumption relative to income.

The remainder of the paper proceeds as follows. Section 2 presents a model with one-sided limited commitment. Section 3 presents a duality approach to solving this problem. Section 4 provides an example with one-sided limited commitment. Section 5 generalizes the duality approach to two-sided limited commitment. Section 6 provides an example with two-sided limited commitment. Section 7 concludes. Technical proofs are relegated to appendices.

## 2. One-Sided Limited Commitment

Consider a canonical contracting model with limited commitment in a continuous-time infinite-horizon environment. We fix a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ on which is defined a one-dimensional standard Brownian motion $\{B_t\}_{t\geq 0}$. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by this Brownian motion and $\mathcal{F}_0$ is trivial. For ease of exposition, we shall refer to the two contracting parties as the principal and the agent. The principal is risk neutral and discounts future cash flows at the rate $r > 0$. The agent is risk averse and has an income process $Y = \{Y_t\}_{t\geq 0}$ satisfying the stochastic differential equation:

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t, \quad Y_0 = y,$$

where $\mu : \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$.

**Assumption 1** *(i) For each $y$, there is a unique Ito process $\{Y_t\}_{t\geq 0}$ satisfying the above stochastic differential equation.*[7] *(ii) The expectation $E\left[\int_0^\infty e^{-rt} Y_t dt\right]$ is finite for $r > 0$.*

A consumption plan $C = \{C_t\}_{t\geq 0}$ is a nonnegative process such that the present value is finite,

$$E\left[\int_0^\infty e^{-rt} C_t dt\right] < \infty. \tag{1}$$

The agent derives utility from a consumption plan $C$ according to

$$U_0^a(C) \equiv E\left[\int_0^\infty e^{-\rho t} u(C_t)\, dt\right], \tag{2}$$

where $\rho > 0$ is the subjective discount rate and $u : \mathbb{R}_+ \to \mathbb{R}$. His continuation utility at date $t$ is given by

$$U_t^a(C) \equiv E_t\left[\int_t^\infty e^{-\rho(s-t)} u(C_s)\, ds\right]. \tag{3}$$

Assume that $u$ satisfies:

**Assumption 2** $u' > 0$, $u'' < 0$, $\lim_{c\downarrow 0} u'(c) = \infty$ *and* $\lim_{c\uparrow\infty} u'(c) = 0$.

By this assumption, there exists a strictly decreasing and continuously differentiable inverse function, $I : \mathbb{R}_{++} \to \mathbb{R}_{++}$, defined as $I(x) = (u')^{-1}(x)$, for all $x > 0$. Define $I(0) = \lim_{x\downarrow 0} I(x) = \infty$ and $I(\infty) = \lim_{x\uparrow\infty} I(x) = 0$.

The agent does not have access to financial markets. To insure himself again income risk, he writes a contract with the risk-neutral principal. The agent hands in his endowment $Y$ to

[7] Sufficient conditions are Lipschitz and growth conditions on $\mu$ and $\sigma$: there is a constant $k$ such that for any $x$ and $y$ in $\mathbb{R}$,

$$|\mu(x) - \mu(y)| \leq k|x - y|, \quad |\mu(y)| \leq k(1 + y^2),$$
$$|\sigma(x) - \sigma(y)| \leq k|x - y|, \quad |\sigma(y)| \leq k(1 + y^2).$$

See Duffie (1996).

the principal, who then returns consumption $C$ to the agent. The principal can freely access financial markets and derives utility according to

$$U^p(y, C) \equiv E\left[\int_0^\infty e^{-rt}(Y_t - C_t)\,dt\right].$$

Note that we allow $\rho \neq r$ in the model because when we interpret the principal as a financial intermediary, his discount rate $r$ is the interest rate. In a general equilibrium model, the endogenously determined interest rate is typically lower than the agent's subjective discount rate $\rho$ (see, e.g., Alvarez and Jermann (2000) and Krueger and Perri (2011)).

The key assumption of the model is that the agent has limited commitment. He can walk away from the contract and take an outside value at any time after signing the contract. Suppose that the outside value is given by $U_d(Y_t)$ at time $t$, where $U_d : \mathbb{R}_+ \to \mathbb{R}$ is a measurable function. One example is that the outside value is equal to the autarky value so that

$$U_d(Y_t) = E_t\left[\int_t^\infty e^{-\rho(s-t)}u(Y_s)\,ds\right]. \tag{4}$$

To ensure that the agent does not walk away, we impose the following participation constraint:

$$U_t^a(C) \geq U_d(Y_t), \quad \forall t \geq 0. \tag{5}$$

In addition, we also impose the following initial individual rationality constraint or the promise-keeping constraint:

$$U_0^a(C) = w, \tag{6}$$

where $w$ is an initial promised value to the agent. We call a consumption plan *enforceable* if it satisfies (5) and (6). Let $\Gamma(y, w)$ denote the set of all enforceable consumption plans. By (5), we must assume that $w \geq U_d(Y_0)$ throughout the analysis.

We can now state the contracting problem as follows:

**Primal problem (one-sided limited commitment):**

$$V(y, w) = \sup_{C \in \Gamma(y,w)} U^p(y, C). \tag{7}$$

We call this problem *the primal problem* and call $V$ the *primal value function*. The standard approach to solving this problem is to apply dynamic programming and use the agent's continuation value as a state variable (e.g., DeMarzo and Sannikov (2006), Sannikov (2008), Williams (2009, 2011) and Grochulski and Zhang (2011)). Let $W_t \equiv U_t^a(C)$ denote this state variable. By the Martingale Representation Theorem, there is a process $\{\sigma_t^W\}_{t\geq 0}$ such that $\{W_t\}_{t\geq 0}$ satisfies the following stochastic different equation:

$$dW_t = (\rho W_t - u(C_t))\, dt + \sigma_t^W dB_t. \tag{8}$$

The primal value function $V$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} rV(Y_t, W_t) &= \sup_{C_t, \sigma_t^W} \quad Y_t - C_t + V_y(Y_t, W_t)\,\mu(Y_t) + \frac{1}{2}V_{yy}(Y_t, W_t)\,\sigma^2(Y_t) \\ &\quad + V_w(Y_t, W_t)(\rho W_t - u(C_t)) + \frac{1}{2}V_{ww}(Y_t, W_t)\left(\sigma_t^W\right)^2 + V_{yw}(Y_t, W_t)\,\sigma_t^W \sigma(Y_t), \end{aligned}$$

subject to (8) and the participation constraint, $W_t \geq U_d(Y_t)$.[8]

After optimizing with respect to $C_t$ and $\sigma_t^W$, the HJB equation reduces to a nonlinear partial differential equation (PDE). Together with the participation constraint, the HJB equation is difficult to solve both analytically and numerically.

In the next section, we will use the duality method to solve this problem. Before doing so, we first present the solution to the first-best benchmark in which the participation constraint (5) is removed. We make the following assumption:

**Assumption 3** *The integral $\int_0^\infty e^{-\rho t} u\left(I\left(e^{(\rho-r)t}\right)\right) dt$ is finite.*[9] *The initial promised value $w$ satisfies*

$$\lim_{\phi \downarrow 0} \int_0^\infty e^{-\rho t} u\left(I\left(e^{(\rho-r)t}/\phi\right)\right) dt = \frac{u(0)}{\rho} < w < \frac{u(\infty)}{\rho} = \lim_{\phi \uparrow \infty} \int_0^\infty e^{-\rho t} u\left(I\left(e^{(\rho-r)t}/\phi\right)\right) dt.$$

By this assumption and Assumption 2, there exists a unique Lagrange multiplier $\phi^* > 0$ associated with the promise-keeping constraint (6) such that the first-best consumption is deterministic and is given by

$$C_t^{FB} = I\left(e^{(\rho-r)t}/\phi^*\right), \quad \text{for all } t \geq 0.$$

For instance, if $u(c) = c^\alpha/\alpha$, $0 \neq \alpha < 1$, then

$$C_t^{FB} = \phi^{*\frac{1}{1-\alpha}} e^{\frac{r-\rho}{1-\alpha}t}, \text{ where } \phi^* = \left[\frac{\alpha w(\rho - \alpha r)}{1-\alpha}\right]^{\frac{1-\alpha}{\alpha}}.$$

Assumption 3 is satisfied if and only if $\rho > \alpha r$ and $\alpha w > 0$. In the first best, the risk-neutral principal bears all uncertainty and fully insures the risk-averse agent. In particular, if $\rho = r$, then the first-best consumption plan is constant over time. If $r > (<)\, \rho$, the agent is more (less) patient than the principal so that the first-best consumption increases (decreases) over time.

[8] In models with two-sided limited commitment, there is also an endogenous upper bound on $W_t$. See the end of Section 5.2. for a discussion and the examples in Section 6 and Appendix C.

[9] This assumption implies that $\int_0^\infty e^{-\rho t} u\left(I\left(e^{(\rho-r)t}/\phi\right)\right) dt$ is finite for each $\phi > 0$.

# 3. Duality

We first set up the dual problem heuristically. We then study the relation between the dual problem and the primal problem by proving the weak and strong duality theorems, respectively. Finally, we provide a dynamic programming characterization of the dual problem.

## 3.1. Heuristic Derivation

In this subsection, we use informal heuristic arguments to derive the dual problem by ignoring some technical issues. We will provide formal results in the next two subsections, with rigorous proofs given in the appendix. First, similar to the Lagrange method in discrete time (e.g., Marcet and Marimon (1998) and Ljungqvist and Sargent (2004)), we write down the Lagrangian in continuous time:

$$\begin{aligned}\mathfrak{L} &= E\left[\int_0^\infty e^{-rt}\left(Y_t - C_t\right)dt\right] + \phi\left(E\left[\int_0^\infty e^{-\rho s}u\left(C_s\right)ds\right] - w\right)\\ &\quad + E\left[\int_0^\infty e^{-rt}\lambda_t\left(\int_t^\infty e^{-\rho(s-t)}u\left(C_s\right)ds - U_d\left(Y_t\right)\right)dt\right],\end{aligned}$$

where $e^{-rt}\lambda_t \geq 0$ is the Lagrange multiplier associated with the participation constraint (5) at each time $t \geq 0$ and $\phi > 0$ is the Lagrange multiplier associated with the promise-keeping constraint (6). It must be the case that $\phi > 0$ because raising the agent's promised value would increase the agent's consumption and reduce the principal's value.

Using integration by parts, we can compute that[10]

$$E\left[\int_0^\infty e^{-rt}\lambda_t\left(\int_t^\infty e^{-\rho(s-t)}u\left(C_s\right)ds\right)dt\right] = E\left[\int_0^\infty\left(\int_0^t e^{(\rho-r)s}\lambda_s ds\right)e^{-\rho t}u\left(C_t\right)dt\right].$$

Plugging this equation into the Lagrangian, we obtain

$$\begin{aligned}\mathfrak{L} &= E\left[\int_0^\infty e^{-rt}\left(Y_t - C_t\right)dt\right] - E\left[\int_0^\infty e^{-rt}\lambda_t U_d\left(Y_t\right)dt\right]\\ &\quad + E\left[\int_0^\infty\left(\int_0^t e^{(\rho-r)s}\lambda_s ds + \phi\right)e^{-\rho t}u\left(C_t\right)dt\right] - \phi w.\end{aligned}$$

[10]Specifically,

$$\begin{aligned}E\left[\int_0^\infty e^{-rt}\lambda_t\left(\int_t^\infty e^{-\rho(s-t)}u\left(C_s\right)ds\right)dt\right] &= E\left[\int_0^\infty\left(\int_t^\infty e^{-\rho s}u\left(C_s\right)ds\right)d\left(\int_0^t e^{(\rho-r)s}\lambda_s ds\right)\right]\\ &= E\left[\left(\int_t^\infty e^{-\rho s}u\left(C_s\right)ds\right)\left(\int_0^t e^{(\rho-r)s}\lambda_s ds\right)\right]\Bigg|_0^\infty\\ &\quad - E\left[\int_0^\infty\left(\int_0^t e^{(\rho-r)s}\lambda_s ds\right)d\left(\int_t^\infty e^{-\rho s}u\left(C_s\right)ds\right)\right]\\ &= E\left[\int_0^\infty\left(\int_0^t e^{(\rho-r)s}\lambda_s ds\right)e^{-\rho t}u\left(C_t\right)dt\right].\end{aligned}$$

As in Marcet and Marimon (1998), we define a costate process $X$ as the cumulative amounts of the Lagrangian multipliers,

$$X_t \equiv \int_0^t e^{(\rho-r)s}\lambda_s ds + \phi, \quad t \geq 0. \tag{9}$$

This process is increasing, continuous, and satisfies

$$dX_t = e^{(\rho-r)t}\lambda_t dt. \tag{10}$$

Using this process, the Lagrangian becomes

$$\begin{aligned} \mathfrak{L} \quad = \quad & E\left[\int_0^\infty e^{-rt} Y_t dt\right] - E\left[\int_0^\infty e^{-\rho t} U_d\left(Y_t\right) dX_t\right] \\ & + E\left[\int_0^\infty e^{-rt}\left(X_t e^{-(\rho-r)t} u\left(C_t\right) - C_t\right) dt\right] - X_0 w. \end{aligned} \tag{11}$$

To derive the dual problem, we first choose consumption to maximize $\mathfrak{L}$. Define the *dual function* of $u$ as[11]

$$\tilde{u}(z) \equiv \max_{c>0}\{zu\left(c\right) - c\}, \text{ for } z > 0. \tag{12}$$

Since $u$ is strictly concave, the solution is $c^* = I\left(1/z\right)$. We can show that $I\left(1/z\right)$ is strictly increasing in $z$ and $\tilde{u}\left(z\right)$ is strictly convex in $z$.[12] Optimizing over $C_t$ in (11) yields

$$\mathcal{L}\left(X\right) \equiv E\left[\int_0^\infty e^{-rt}\left(Y_t + \tilde{u}\left(X_t e^{-(\rho-r)t}\right)\right) dt\right] - E\left[\int_0^\infty e^{-\rho t} U_d\left(Y_t\right) dX_t\right] - X_0 w. \tag{13}$$

We then choose the process $X$ to minimize $\mathcal{L}\left(X\right)$.

**Dual problem (one-sided limited commitment):**

$$\inf_{X\in\mathcal{I}} \quad \mathcal{L}\left(X\right), \tag{14}$$

where $\mathcal{I}$ denotes the set of all increasing, right continuous processes $X$ with left limits and starting at positive initial values such that

$$E\left[\int_0^\infty e^{-\rho t}\left|U_d\left(Y_t\right)\right| dX_t\right] \quad < \quad \infty, \tag{15}$$

$$E\left[\int_0^\infty e^{-rt}\left|\tilde{u}\left(X_t e^{-(\rho-r)t}\right)\right| dt\right] \quad < \quad \infty. \tag{16}$$

[11]Note that this dual function is not the same as the following convex conjugate function often defined in the literature:

$$\tilde{u}\left(y\right) = \sup_{x>0} u\left(x\right) - xy, \quad y > 0.$$

[12]This result follows from

$$\frac{dI\left(1/z\right)}{dz} = \frac{-1}{z^2 u''\left(I\left(1/z\right)\right)} > 0,$$

and the fact that $\tilde{u}'(z) = u\left(I\left(1/z\right)\right)$ increases in $z$.

Note that in this formulation of the dual problem, the set $\mathcal{I}$ of feasible processes contains all increasing and right continuous processes with left limits. A process $X \in \mathcal{I}$ is generally not absolutely continuous with respect to time $t$. Thus, equations (9) and (10) will not hold in a rigorous mathematical sense. Our previous derivation is purely heuristic and will not be used in our formal proofs.[13] But without using a heuristic derivation, it is far from routine to formulate the dual problem. We also emphasize that the infimum in (14) is taken with respect to the whole sample path $\{X_t\}_{t\geq 0}$, including the initial value $X_0 > 0$. Finally, the integrability conditions in (15) and (16) ensure that $\mathcal{L}(X)$ is finite.

### 3.2. Weak and Strong Duality

We break up the dual problem (14) into two sub-problems. First, define

$$L(y,x,X) \equiv E\left[\int_0^\infty e^{-rt}\left(Y_t + \tilde{u}\left(X_t e^{-(\rho-r)t}\right)\right)dt\right] - E\left[\int_0^\infty e^{-\rho t} U_d(Y_t)\, dX_t\right], \tag{17}$$

where the expectations are conditional on $X_0 = x$ and $Y_0 = y$. Define the dual value function as

$$\tilde{V}(y,x) \equiv \inf_{X \in \mathcal{I}(x)} \quad L(y,x,X) \quad \text{for any } x > 0, \tag{18}$$

where $\mathcal{I}(x)$ denotes the set of all processes in $\mathcal{I}$ starting at $x > 0$. Second, we study the problem:

$$\inf_{x>0} \tilde{V}(y,x) - xw. \tag{19}$$

The following property is useful.

**Proposition 1** *$\tilde{V}(y,x)$ is convex in $x$.*

Now, we study the relationship between the primal problem (7) and the dual problem (14).

**Theorem 1** *(weak duality) For every enforceable plan $C \in \Gamma(y,w)$, every $x > 0$, and every $X \in \mathcal{I}(x)$, the following inequality holds:*

$$U^p(y,C) \leq L(y,x,X) - xw. \tag{20}$$

*Equality holds if and only if for all $t \geq 0$,*

$$X_t e^{-(\rho-r)t} u'(C_t) - 1 = 0, \tag{21}$$

$$\int_0^t e^{-\rho s}(U_s^a(C) - U_d(Y_s))dX_s = 0. \tag{22}$$

[13] In fact, we will show later that the optimal $X$ is a regulated Brownian motion which is not absolutely continuous (Harrison (1985)).

This theorem shows that the objective function $L(y,x,X)-xw$ in the dual problem provides an upper bound on the objective function $U^p(y,C)$ in the primal problem. An immediate corollary is that the primal value function is weakly below the dual value function:

$$V(y,w) \leq \inf_{x>0} \tilde{V}(y,x) - xw. \tag{23}$$

This result is called weak duality.

Equations (21)-(22) give conditions under which equality in (20) holds. These conditions are analogous to the Kuhn-Tucker conditions in the discrete-time model analyzed in Marcet and Marimon (1998), Ljungqvist and Sargent (2004), and Zhang (2013). In particular, equation (21) is the first-order condition for consumption, and equation (22) is a continuous-time version of the complementary slackness condition for optimality.

The following theorem shows that a solution to the dual problem implies a solution to the primal problem and hence the equalities in (20) and (23) hold.

**Theorem 2** *(strong duality) Suppose that $X^* \in \mathcal{I}$ is a solution to the dual problem (14). Let*

$$C_t^* \equiv I\left(e^{(\rho-r)t}/X_t^*\right), \quad t \geq 0.$$

*If $C^*$ satisfies condition (1) and if condition (16) holds for the processes $X^\delta = X^* + \delta$ and $\bar{X}^{\pm\delta} = X^*(1 \pm \delta)$ for some small $\delta > 0$, then $C^*$ is a solution to the primal problem (7). In addition,*

$$V(y,w) = \inf_{x>0} \tilde{V}(y,x) - xw.$$

The idea of the proof of this theorem is to first show that $C^*$ is enforceable and then show that $C^*$ and $X^*$ satisfy (21)-(22). As a result, we can apply Theorem 1. To make this argument work, we use perturbation around $X^*$. The integrability conditions in the theorem ensure that certain functions are integrable after small perturbations. These are simple sufficient conditions used when we take limits in the proof. They can be easily verified in our examples presented later.

Theorem 2 shows that after solving the dual problem, optimal consumption in the primal problem can be completely characterized by the function $I\left(e^{(\rho-r)t}/X_t^*\right)$. By the previous analysis, this function is strictly increasing with $e^{(r-\rho)t}X_t^*$. By (21), this term is the ratio of the marginal utilities of the principal and the agent. This result can be generalized to the case of a risk-averse principal and to the case of two-sided limited commitment, as will be shown in Section 5. Alternatively, we can interpret $e^{(r-\rho)t}X_t^*$ as the "temporary relative Pareto weight" on the principal and the agent, as in Chapter 20 of Ljungqvist and Sargent (2004). In the next subsection, we provide a dynamic programming characterization of the dual problem.

### 3.3. Dynamic Programming

Since the exogenous state process $Y$ in our model is assumed to be Markovian, we can provide a dynamic programming characterization for the dual problem. We adopt the ratio of the marginal utilities of the principal and the agent as a state variable. This ratio is equal to the discount-rate-adjusted costate variable $Z_t \equiv e^{-(\rho-r)t} X_t$ and satisfies the dynamics:

$$dZ_t = Z_t/X_t dX_t - (\rho - r) Z_t dt, \quad X_0 = Z_0 = z > 0. \tag{24}$$

We then rewrite the problem (18) as

$$J(y,z) \equiv \inf_{X \in \mathcal{I}(z)} E\left[\int_0^\infty e^{-rt}(Y_t + \tilde{u}(Z_t))\, dt\right] - E\int_0^\infty e^{-\rho t} U_d(Y_t)\, dX_t, \tag{25}$$

subject to (24). This is a singular control or instantaneous control problem in control theory (e.g., Harrison and Taksar (1983), Fleming and Soner (2006), or Stokey (2008)), where $X$ is the control process and $Y$ and $Z$ are state processes. Note that $J$ and $\tilde{V}$ are related by

$$J(Y_0, Z_0) = \tilde{V}(Y_0, X_0).$$

We shall proceed heuristically to derive the HJB equation for the control problem (25). Suppose that $X$ satisfies (10). Substituting (10) into (24) and (25) and using the Principle of Optimality, we derive a discrete-time approximation of the Bellman equation:

$$rJ(Y_t, Z_t)\, dt = \inf_{\lambda_t \geq 0} [Y_t + \tilde{u}(Z_t) - \lambda_t U_d(Y_t)]\, dt + E_t[dJ(Y_t, Z_t)],$$

subject to

$$dZ_t = \lambda_t dt - (\rho - r) Z_t dt, \quad X_0 = Z_0 = z.$$

It follows from Ito's Lemma that

$$\begin{aligned} rJ(Y_t, Z_t)\, dt &= \inf_{\lambda_t \geq 0} \; [Y_t + \tilde{u}(Z_t) - \lambda_t U_d(Y_t)]\, dt + J_z(Y_t, Z_t)[\lambda_t + (r-\rho) Z_t]\, dt \\ &\quad + J_y(Y_t, Z_t)\mu(Y_t)\, dt + \frac{1}{2} J_{yy}(Y_t, Z_t)\sigma^2(Y_t)\, dt. \end{aligned}$$

Cancelling out $dt$, we obtain the following partial differential equation (PDE):

$$\begin{aligned} rJ(y,z) &= \inf_{\lambda \geq 0} \; y + \tilde{u}(z) + (r-\rho) z J_z(y,z) + J_y(y,z)\mu(y) + \frac{1}{2} J_{yy}(y,z)\sigma^2(y) \\ &\quad + \lambda [J_z(y,z) - U_d(y)]. \end{aligned}$$

It must be the case that $J_z(y,z) \geq U_d(y)$, otherwise the above minimization problem is not well-defined since $\lambda \geq 0$ can be made arbitrarily large. The solution is given by

$$\begin{aligned} J_z(y,z) = U_d(y) &\implies \lambda \geq 0, \\ J_z(y,z) > U_d(y) &\implies \lambda = 0. \end{aligned}$$

Formally, the HJB equation is formulated in terms of a variational inequality:

$$\min\left\{y+\tilde{u}\left(z\right)+\mathcal{A}J\left(y,z\right),\ J_z\left(y,z\right)-U_d\left(y\right)\right\}=0,\quad\left(y,z\right)\in\mathbb{R}_+\times\mathbb{R}_{++},\tag{26}$$

where

$$\mathcal{A}J\left(y,z\right)=\left(r-\rho\right)zJ_z\left(y,z\right)+J_y\left(y,z\right)\mu\left(y\right)+\frac{1}{2}J_{yy}\left(y,z\right)\sigma^2\left(y\right)-rJ\left(y,z\right).\tag{27}$$

The variational inequality (26) partitions the state space into two regions:

$$\begin{aligned}\Omega_1&=\left\{\left(y,z\right)\in\mathbb{R}_+\times\mathbb{R}_{++}:J_z\left(y,z\right)=U_d\left(y\right)\right\},\\ \Omega_2&=\left\{\left(y,z\right)\in\mathbb{R}_+\times\mathbb{R}_{++}:J_z\left(y,z\right)>U_d\left(y\right)\right\}.\end{aligned}$$

A free boundary $z=\varphi\left(y\right)$ defined by

$$\varphi\left(y\right)=\inf\left\{z'>0:J_z\left(y,z'\right)>U_d\left(y\right)\right\}$$

separates $\Omega_1$ and $\Omega_2$. See Figure 1 in Section 4 for an illustration.

If initially $\left(Y_0,Z_0\right)\in\Omega_1$, then $X$ should jump up immediately, such that $Z$ reaches the boundary. On the other hand, if $\left(y,z\right)\in\Omega_2$, then

$$y+\tilde{u}\left(z\right)+\mathcal{A}J\left(y,z\right)=0,$$

and $X$ must stay constant. Thus, we call $\Omega_1$ and $\Omega_2$ the jump and the no-jump regions, respectively. If $\left(Y_0,Z_0\right)$ starts inside the no-jump region, then $X$ will be a process that regulates $Z$ so that $\left(Y_t,Z_t\right)$ stays inside the no-jump region. The sample path of $X$ at the optimum must have the property that it increases only when $\left(Y_t,Z_t\right)$ hits the free boundary, at which time the participation constraints bind.

Following the standard dynamic programming theory, we shall state a verification theorem.

**Theorem 3** *(verification) Let $J\left(y,z\right)$ be a twice continuously differentiable solution to* (26) *such that for any $Z$ in (24) and $X\in\mathcal{I}\left(z\right)$, (i) the process defined by*

$$\int_0^t e^{-rs}J_y\left(Y_s,Z_s\right)\sigma\left(Y_s\right)dB_s,\quad t\geq 0,\tag{28}$$

*is a martingale, and (ii)*

$$\lim_{t\to\infty}E\left[e^{-rt}J\left(Y_t,Z_t\right)\right]=0.\tag{29}$$

*Suppose further that $Z_t^*=e^{-(\rho-r)t}X_t^*$, where $X^*\in\mathcal{I}\left(z\right)$ and $\left(y,z\right)\in\Omega_2$, is such that (i)*

$$Y_t+\tilde{u}\left(Z_t^*\right)+\mathcal{A}J\left(Y_t,Z_t^*\right)=0,\tag{30}$$

*for all* $t \geq 0$, *(ii) for all* $t \geq 0$,

$$\int_0^t e^{-\rho s} \left( J_z \left( Y_s, Z_s^* \right) - U_d \left( Y_s \right) \right) dX_s^* = 0. \tag{31}$$

*Then* $X^*$ *is the optimal solution to problem (25) and* $J$ *is the associated dual value function. Suppose further that* $J(y,z)$ *is strictly convex in* $z$ *on* $\Omega_2$, *there exists* $z^* > 0$ *such that* $J_z(y, z^*) = w$, *and the conditions in Theorem 2 hold. Then* $X_0^* = z^*$ *is the optimal solution to problem (19) and the primal value function is given by* $V(y,w) = J(y,z^*) - z^* w$. *The optimal consumption plan, continuation values, and the marginal utility ratio are, respectively, given by*

$$C_t^* = I\left(1/Z_t^*\right), \quad W_t^* = J_z\left(Y_t, Z_t^*\right), \quad Z_t^* = -V_w\left(Y_t, W_t^*\right). \tag{32}$$

Equation (30) is a linear PDE, which is easier to solve explicitly, as illustrated in the next section. Condition (28) is a technical condition used to verify the optimality of $X^*$ by the martingale method. Condition (29) is the transversality condition that usually appears in infinite-horizon control problems. Condition (31) indicates that $X^*$ increases if and only if $J_z(Y_t, Z_t^*) = U_d(Y_t)$. It is also related to (22) and may be interpreted as a complementary slackness condition associated with the participation constraints. The solution $X^*$ is related to the classical Skorokhod problem. As is well known (e.g., Harrison and Taksar (1983)), we can express $X^*$ as

$$X_t^* = \max\left\{ z^*, \max_{s\in[0,t]} \varphi\left(Y_s\right) e^{(\rho - r)s} \right\}.$$

In addition, $X^*$ also admits a local time characterization, which we will not pursue here.

Equation (32) shows that optimal consumption can be completely characterized by $Z_t^*$, the ratio of the marginal utilities of the principal and the agent, which, by the Envelope Theorem, is equal to the negative slope of the Pareto frontier given an income level. Equation (32) also shows that the agent's continuation value is equal to the partial derivative of the dual value function with respect to the marginal utility ratio, $Z_t^*$, given an income level. Thus, optimal consumption can be expressed as a function of the income level and the agent's continuation value, as in the literature (e.g., Ljungqvist and Sargent (2004)).

Since $J_z(y, \varphi(y)) = U_d(y)$ on the free boundary, it follows from $w \geq U_d(Y_0)$ that

$$J_z\left(Y_0, Z_0^*\right) = w \geq J_z\left(Y_0, \varphi\left(Y_0\right)\right).$$

When $J$ is strictly convex in $z$ on $\Omega_2$, we deduce that $X_0^* = Z_0^* \geq \varphi(Y_0)$, implying that the optimal starting value of $X^*$ or $Z^*$ is inside the no-jump region. Thus, there is no jump in $X^*$ or $Z^*$ and both processes are continuous.

# 4. Example I

We now introduce the participation constraint (5) to the example studied in Section 2. In this case, the first-best allocation cannot be achieved. Thus, the agent must also bear income uncertainty. To derive a closed-form solution, we assume that the agent's income $Y$ follows a geometric Brownian motion:

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t, \quad Y_0 = y > 0, \tag{33}$$

where $\sigma > 0$. We assume $r > \mu + \sigma^2/2$ so that the present value of income discounted at $r$ is finite. This assumption also allows us to check condition (28) in Theorem 3.[14]

Let $u(c) = c^\alpha/\alpha$ for $0 \neq \alpha < 1$. The log utility case corresponds to $\alpha = 0$. We assume $\rho > \alpha r$ so that the first-best allocation exists. Suppose that the agent's outside option is autarky so that

$$U_d(y) = E\left[\int_0^\infty e^{-\rho t} u(Y_t) dt | Y_0 = y\right] = \kappa y^\alpha, \tag{34}$$

where we define

$$\kappa \equiv \frac{1}{\alpha(\rho - \alpha\mu - \alpha(\alpha - 1)\sigma^2/2)}, \tag{35}$$

and assume that $\rho > \alpha\mu + \alpha(\alpha - 1)\sigma^2/2$ to ensure a finite autarky value.

## 4.1. Solution

We can derive that

$$\tilde{u}(z) = \frac{1-\alpha}{\alpha} z^{\frac{1}{1-\alpha}}, \text{ for } z > 0,$$

and the optimal consumption rule is $c^* = z^{\frac{1}{1-\alpha}}$. In the no-jump region, equation (30) becomes:

$$rJ(y,z) = y + \frac{1-\alpha}{\alpha} z^{\frac{1}{1-\alpha}} + (r-\rho) z J_z(y,z) + J_y(y,z)\mu y + \frac{\sigma^2}{2} y^2 J_{yy}(y,z). \tag{36}$$

This is a linear PDE. Given the two free-boundary conditions, $J_z(y,z) = U_d(y)$ and $J_{zz}(y,z) = 0$ (often called the value-matching and super-contact conditions in the literature, e.g., Dumas (1991)), we can derive the following general solution:[15]

$$J(y,z) = \frac{y}{r-\mu} + \frac{(1-\alpha)^2}{(\rho - \alpha r)\alpha} z^{\frac{1}{1-\alpha}} + A z^{\frac{1-\beta}{1-\alpha}} y^\beta, \tag{37}$$

[14] We will provide additional proofs for this example in Appendix B.

[15] For log utility, the dual value function is given by

$$J(y,z) = \frac{y}{r-\mu} + \frac{z\ln(z)}{\rho} + \frac{r-2\rho}{\rho^2} z + A z^{1-\beta} y^\beta,$$

where $A = \frac{b^\beta}{\rho(1-\beta)\beta}$ and $b = e^{(\mu - \sigma^2/2 - r)\rho^{-1} - \beta^{-1} + 1}$. The free boundary and consumption rule are the limits when $\alpha \to 0$.

where $A$ is a constant to be determined and $\beta$ is the positive root of the characteristic equation:

$$r = (r-\rho)\frac{1-\beta}{1-\alpha} + \mu\beta + \frac{\sigma^2}{2}\beta\left(\beta - 1\right). \tag{38}$$

We can also show that $\beta > 1$. We rule out the particular solution corresponding to the negative root because this solution makes $J\left(y,z\right)$ converge to infinity as $y \downarrow 0$. But $J\left(y,z\right)$ should converge to the finite first-best value since the autarky value is so small that the participation constraints will not bind when $y \downarrow 0$.

From (37) and the above two free-boundary conditions, we can derive that

$$\begin{aligned} \frac{1-\alpha}{\left(\rho - r\alpha\right)\alpha} z^{\frac{\alpha}{1-\alpha}} + A\frac{1-\beta}{1-\alpha} z^{\frac{\alpha-\beta}{1-\alpha}} y^{\beta} &= \kappa y^{\alpha}, \\ \frac{1}{\left(\rho - r\alpha\right)} z^{\frac{\alpha}{1-\alpha}-1} + A\frac{1-\beta}{1-\alpha}\frac{\alpha-\beta}{1-\alpha} z^{\frac{\alpha-\beta}{1-\alpha}-1} y^{\beta} &= 0. \end{aligned} \tag{39}$$

Substituting the second equation above into the first one, we can derive the free-boundary, $z = by^{1-\alpha}$, where

$$b = \left[\frac{\alpha(\rho - \alpha r)\left(\beta - \alpha\right)\kappa}{\beta(1-\alpha)}\right]^{\frac{1-\alpha}{\alpha}} > 0. \tag{40}$$

Here, the sign can be verified using the definition of $\kappa$ and the assumptions on parameter values.

Substituting the free boundary $z = by^{1-\alpha}$ into the super-contact condition $J_{zz}\left(y,z\right) = 0$ or (39), we can derive

$$A = -\frac{\left(1-\alpha\right)^2 b^{\frac{\beta}{1-\alpha}}}{\left(1-\beta\right)\left(\alpha-\beta\right)(\rho - \alpha r)} < 0.$$

We then obtain the no-jump region $\left\{(y,z) \in \mathbb{R}^2_{++} : z \geq by^{1-\alpha}\right\}$ and the dual value function in this region given in (37). In the jump region $\left\{(y,z) \in \mathbb{R}^2_{++} : z < by^{1-\alpha}\right\}$, we use $J_z\left(y,z\right) = U_d\left(y\right)$ and $\lim_{z \downarrow by^{1-\alpha}} J\left(y,z\right) = \lim_{z \uparrow by^{1-\alpha}} J\left(y,z\right)$ to derive that

$$J\left(y,z\right) = \left(z - by^{1-\alpha}\right) U_d\left(y\right) + J\left(by^{1-\alpha}, y\right), \quad \text{for } z < by^{1-\alpha}. \tag{41}$$

In Appendix B, we show that $J\left(y,z\right)$ is strictly convex in $z$ in the no-jump region. Using equation, $J_z\left(y,z\right) = w$, or

$$\frac{1-\alpha}{\left(\rho - r\alpha\right)\alpha} z^{\frac{\alpha}{1-\alpha}} + A\frac{1-\beta}{1-\alpha} z^{\frac{\alpha-\beta}{1-\alpha}} y^{\beta} = w, \tag{42}$$

we can derive a unique solution for $z$ when $w \in [U_d(y), \infty)$ for $\alpha > 0$ and $w \in [U_d(y), 0)$ for $\alpha < 0$. This solution is used as the initial value for the processes $Z_t^*$ and $X_t^*$.

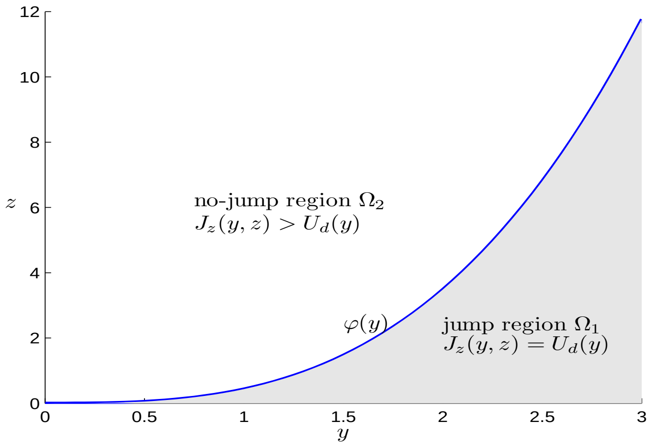


Figure 1: The state space for Example I. The curve $z = \varphi(y) = by^{1-\alpha}$ partitions the state space into a jump region $\Omega_1$ and a no-jump region $\Omega_2$. Parameter values are given by $\mu = 0.02$, $\sigma = 0.1$, $\rho = 0.04$, $r = 0.04$, and $\alpha = -2$.

### 4.2. Numerical Illustrations

Figure 1 plots the state space. The curve $z = by^{1-\alpha}$ partitions the state space into the jump and no-jump regions. Whenever the initial promised value $w \geq U_d(y)$, the initial state $(y, z)$ must lie in the no-jump region. The optimal $X^*$ ensures that $(Y_t, Z_t^*)$ will never leave the no-jump region. Whenever $Y_t$ is high enough and hits the free boundary, $X_t^*$ will rise instantaneously to make $(Y_t, Z_t^*)$ stay in the no-jump region.

Figure 2 plots the dual and primal value functions given three different values of $y$. The three dots on the left panel of this figure indicate the points $\left(y, by^{1-\alpha}\right)$ on the free boundaries. The lowest promised value to the agent for each $y$ is determined by $w_{\min}(y) = J_z\left(y, by^{1-\alpha}\right) = U_d(y)$, which is indicated by the dots on the right panel. This panel shows the value function $V(y, w)$, which gives the Pareto frontier conditional on $y$. This frontier is concave and decreasing. Note that when $w$ is small, the principal makes positive profits, but when $w$ is sufficiently large, the principal incurs losses. This gives the principal an incentive to renege on the contract if he lacks commitment. In the next section, we will analyze this case.

Figure 3 plots the simulated paths of incomes $Y_t$, optimal consumption $C_t^*$, and continuation values $W_t^* = J_z(Y_t, Z_t^*)$. The optimal consumption plan is given by $C_t^* = (Z_t^*)^{\frac{1}{1-\alpha}}$, where

$$X_t^* = \max\left\{X_0^*, \max_{s\in[0,t]} bY_s^{1-\alpha}e^{(\rho-r)s}\right\} \quad \text{and} \quad Z_t^* = e^{-(\rho-r)t}X_t^*. \tag{43}$$

Since $X^*$ is an increasing process and rises whenever $Y_t$ is high enough to hit the free boundary, $C^*$ also follows this pattern if $\rho = r$.

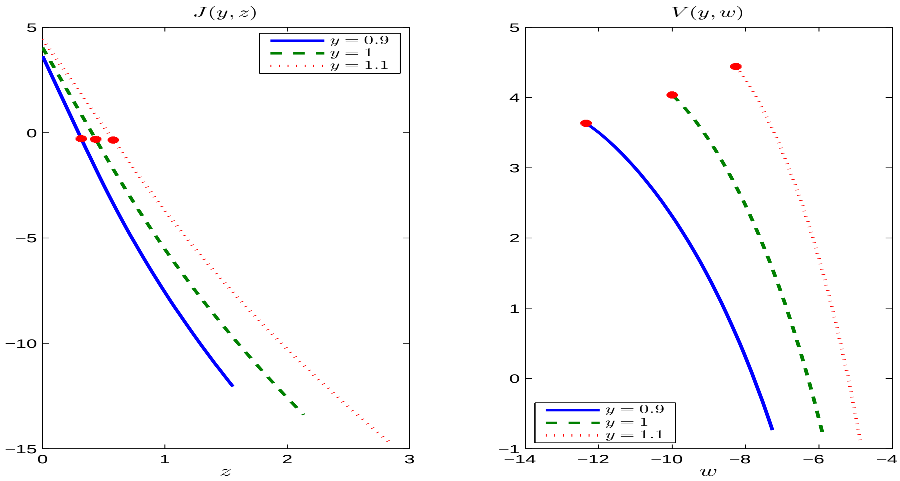


Figure 2: The dual and primal value functions for Example I. Parameter values are given by $\mu = 0.02$, $\sigma = 0.1$, $\rho = 0.04$, $r = 0.04$, and $\alpha = -2$.

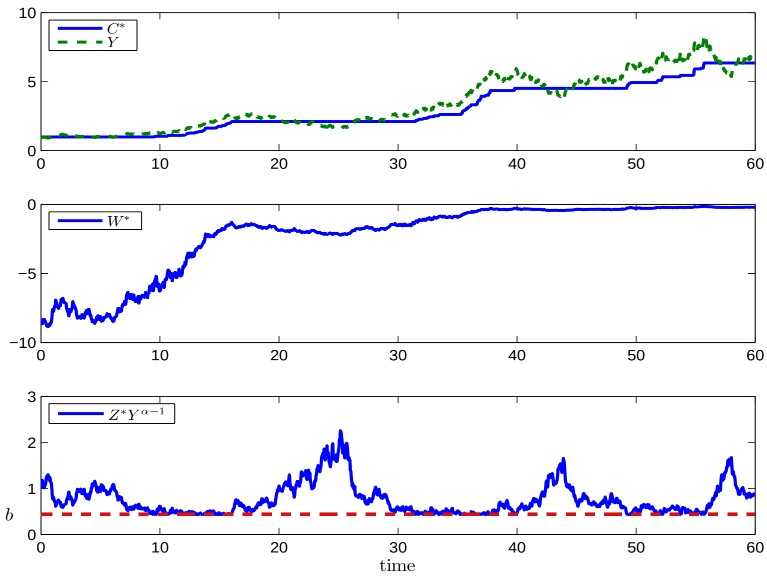


Figure 3: Simulated paths of consumption $C^*$, incomes $Y$, the agent's continuation values $W^*$, and the process $Z^*Y^{\alpha-1}$ for Example I. Parameter values are given by $\mu = 0.02$, $\sigma = 0.1$, $\rho = 0.04$, $r = 0.04$, and $\alpha = -2$.

### 4.3. Stationary Distribution

By Harrison (1985), the process $\ln\left(C_t^*/Y_t\right)$ is a regulated Brownian motion with drift $\sigma^2/2 - \mu + (1-\alpha)^{-1}(r-\rho)$ and volatility $-\sigma$ on $[(1-\alpha)^{-1}\ln b, \infty)$. By Proposition 5.5 in Harrison (1985) or Proposition 10.8 in Stokey (2008), when $(1-\alpha)^{-1}(r-\rho) + \sigma^2/2 < \mu$, $\ln\left(C_t^*/Y_t\right)$ has a unique stationary distribution with the density function:

$$p(x) = \frac{-\delta e^{\delta x}}{b^{\frac{\delta}{1-\alpha}}}, \quad x \in \left[\frac{\ln b}{1-\alpha}, \infty\right),$$

where $\delta \equiv \left(2(r-\rho)/(1-\alpha) + \sigma^2 - 2\mu\right)/\sigma^2$. This is in sharp contrast to the discrete-time case analyzed by Ljungqvist and Sargent (2004), who show that the optimal consumption plan converges to the first best in the long run if the agent's endowment follows a bounded IID process.

## 5. Two-Sided Limited Commitment

This section extends our methodology to the case of two-sided limited commitment. We extend the model in Section 2 in two respects. First, we allow the principal to be risk averse. That is, the principal derives utility according to

$$U^p(y, C) = E\left[\int_0^\infty e^{-rt} u^p\left(A(Y_t) - C_t\right) dt\right],$$

where $u^p : \mathbb{R}_+ \to \mathbb{R}$ satisfies $(u^p)' > 0$ and $(u^p)'' \leq 0$. When the principal is risk neutral ($(u^p)'' = 0$), we allow him to access financial markets so that his consumption can be negative. Here, $A(Y_t)$ represents the aggregate endowment and $C_t$ is the agent's consumption, where $A : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and increasing function. Define the principal's continuation value as

$$U_t^p\left(A(Y) - C\right) = E_t\left[\int_t^\infty e^{-r(s-t)} u^p\left(A(Y_s) - C_s\right) ds\right].$$

For simplicity, we have assumed that the agent's endowment, $Y$, is the only exogenous state process, and that the principal's endowment is $A(Y) - Y$.

Second, we allow the principal to have limited commitment. He can also walk away from the contract and take an outside value, which is given by $U_d^p\left(A(Y_t) - Y_t\right)$. The agent's utility and continuation value are still given by (2) and (3), respectively, and his outside option is denoted as $U_d(Y_t)$. For simplicity, assume that $\rho = r$. We henceforth use $\rho$ to denote the common discount rate.

A consumption plan $C$ is *sustainable* if both the participation constraints for the agent (5) and the following participation constraints for the principal hold:

$$E_t \left[ \int_t^\infty e^{-\rho(s-t)} u^p \left( A\left(Y_s\right) - C_s \right) ds \right] \geq U_d^p \left( A\left(Y_t\right) - Y_t \right), \quad \forall t \geq 0. \tag{44}$$

The initial promise-keeping constraint is given by (6).

Let $\Phi(y, w)$ denote the set of all sustainable consumption plans. We can now state the primal problem as follows:

**Primal problem (two-sided limited commitment):**

$$V(y, w) = \sup_{C \in \Phi(y,w)} U^p(y, C). \tag{45}$$

This problem may be viewed as a continuous-time version of the contracting models studied by Thomas and Worrall (1988) and Kocherlakota (1996) and discussed in Ljungqvist and Sargent (2004, Chapter 20). As in Thomas and Worrall (1988), we may interpret the principal as a risk-neutral firm and the agent as a worker. There is a competitive spot market for labor where a worker is paid $Y_t$ at time $t$. If the worker works for the firm, he is paid the wage $C_t$ at time $t$. The worker is free to walk away from the firm at any time and works in the spot market. The firm can also renege on a wage contract and buy labor at the spot market wage. Let the firm's net profit be $Y_t - C_t$ and the outside value be zero. In this case, we set $A(Y_t) = Y_t$. In Section 6, we will study an example of this type of models.

Following Ljungqvist and Sargent (2004, Chapter 20), we may alternatively interpret the principal and the agent as two households. Both are risk averse and the aggregate endowments $A(Y_t) = \bar{Y}$ are constant. In this case, the endowments of the two households are perfectly negatively correlated. The contract design problem is to find an insurance/transfer arrangement that reduces consumption risk while respecting the participation constraints for both households. In Appendix C, we will study an example of this type of models.

As in the case of one-sided limited commitment, we will solve the dual problem, and then study its relation to the primal problem.

### 5.1. Duality

As in Section 3.1, we first proceed heuristically to derive the dual problem. This heuristic derivation will not be used in our formal proofs. We write down the Lagrangian as follows,

$$\begin{aligned}\mathfrak{L} &= E\left[\int_0^\infty e^{-\rho t}u^p\left(A\left(Y_t\right)-C_t\right)dt\right]+E\left[\int_0^\infty e^{-\rho t}\lambda_t\left(\int_t^\infty e^{-\rho(s-t)}u\left(C_s\right)ds-U_d\left(Y_t\right)\right)dt\right]\\ &\quad +E\left[\int_0^\infty e^{-\rho t}\eta_t\left(\int_t^\infty e^{-\rho(s-t)}u^p\left(A\left(Y_s\right)-C_s\right)ds-U_d^p\left(A\left(Y_t\right)-Y_t\right)\right)dt\right]\\ &\quad +\phi\left(E\left[\int_0^\infty e^{-\rho t}u\left(C_t\right)dt\right]-w\right),\end{aligned}$$

where $e^{-\rho t}\lambda_t$, $e^{-\rho t}\eta_t$, and $\phi$ are the Lagrange multipliers associated with (5), (44), and (6), respectively. Integration by parts allows us to rewrite the Lagrangian as

$$\begin{aligned}\mathfrak{L} &= E\left[\int_0^\infty e^{-\rho t}u^p\left(A\left(Y_t\right)-C_t\right)dt\right]-E\left[\int_0^\infty e^{-\rho t}\lambda_t U_d\left(Y_t\right)dt\right]\\ &\quad +E\left[\int_0^\infty X_t e^{-\rho t}u\left(C_t\right)dt\right]-\phi w+E\left[\int_0^\infty H_t e^{-\rho t}u^p\left(A\left(Y_t\right)-C_t\right)dt\right]\\ &\quad -E\left[\int_0^\infty e^{-\rho t}\eta_t U_d^p\left(A\left(Y_t\right)-Y_t\right)dt\right],\end{aligned} \tag{46}$$

where the costate processes $\{X_t\}_{t\geq 0}$ and $\{H_t\}_{t\geq 0}$ are defined as

$$X_t \equiv \int_0^t \lambda_s ds+\phi, \quad H_t \equiv \int_0^t \eta_s ds. \tag{47}$$

Define a dual function as

$$\tilde{u}\left(y,x,h\right)=\max_{c>0}\ \left(1+h\right)u^p\left(A\left(y\right)-c\right)+xu\left(c\right). \tag{48}$$

The concavity assumption on $u$ and $u^p$ implies a unique solution to this problem, $c^*$, which satisfies the first-order condition:

$$\frac{x}{1+h}=\frac{\left(u^p\right)'\left(A\left(y\right)-c^*\right)}{u'\left(c^*\right)}.$$

We can express $c^*$ as a function of $y$ and $x/\left(1+h\right)$ :

$$c^*=c\left(y,\frac{x}{1+h}\right).$$

As in the one-sided case, we can interpret $x/\left(1+h\right)$ as the ratio of the marginal utilities of the principal and the agent (also see Ligon, Thomas, and Worrall (2002)). We can easily show that $c\left(y,x/\left(1+h\right)\right)$ increases with $x/\left(1+h\right)$. It also increases with $y$ if $A$ is an increasing function of $y$. In addition, $\tilde{u}(y,x,h)$ is strictly convex in $x$ and in $h$, respectively. It is also linearly homogeneous in $x$ and $1+h$.

Optimizing with respect to $C_t$ in (46) yields

$$\begin{aligned}\mathcal{L}(X,H) &= E\left[\int_0^\infty e^{-\rho t}\tilde{u}(Y_t,X_t,H_t)\,dt\right]-E\left[\int_0^\infty e^{-\rho t}U_d(Y_t)\,dX_t\right] \\ &\quad -E\left[\int_0^\infty e^{-\rho t}U_d^p(A(Y_t)-Y_t)\,dH_t\right]-X_0 w.\end{aligned} \tag{49}$$

For $\mathcal{L}(X,H)$ to be finite, we impose the following integrability conditions:

$$E\left[\int_0^\infty e^{-\rho t}\left|U_d(Y_t)\right|dX_t\right] < \infty, \tag{50}$$

$$E\left[\int_0^\infty e^{-\rho t}\left|U_d^p(A(Y_t)-Y_t)\right|dH_t\right] < \infty, \tag{51}$$

$$E\left[\int_0^\infty e^{-\rho t}\left|\tilde{u}(X_t,H_t,Y_t)\right|dt\right] < \infty, \tag{52}$$

$$E\left[\int_0^\infty e^{-\rho t}(1+H_t)\left|u^p(A(Y_t))\right|dt\right] < \infty. \tag{53}$$

We can now formulate the dual problem by suitably choosing sets of feasible choices.

**Dual problem (two-sided limited commitment):**

$$\inf_{X\in\mathcal{I},H\in\mathcal{I}(0)} \mathcal{L}(X,H), \tag{54}$$

where $\mathcal{I}$ ($\mathcal{I}(0)$) denotes the set of all increasing processes that satisfy conditions (50)-(53), are right continuous with left limits, and start at positive values (zero).

We emphasize that in the dual problem, a feasible choice for $X$ and $H$ may not be absolutely continuous with respect to time $t$ and hence equation (47) is purely heuristic and will not be used in our formal analysis.

We solve the dual problem in two steps. First, we consider the following dual problem:

$$\tilde{V}(y,x,h)=\inf_{X\in\mathcal{I}(x),H\in\mathcal{I}(h)} L(y,x,h,X,H), \tag{55}$$

where $L(y,x,h,X,H)$ is defined as

$$\begin{aligned}L(y,x,h,X,H) &\equiv E\left[\int_0^\infty e^{-\rho t}\tilde{u}(Y_t,X_t,H_t)\,dt\right]-E\left[\int_0^\infty e^{-\rho t}U_d(Y_t)\,dX_t\right] \\ &\quad -E\left[\int_0^\infty e^{-\rho t}U_d^p(A(Y_t)-Y_t)\,dH_t\right].\end{aligned}$$

In this problem, we fix the initial value for the controls $X$ and $H$ at $x > 0$ and $h \geq 0$, respectively. We then set $h = 0$ and select the initial value $x > 0$ by solving the following problem:

$$\inf_{x>0} \tilde{V}(y, x, 0) - xw. \tag{56}$$

As in the one-sided case, we can show that $\tilde{V}(y, x, h)$ is convex in $x$ by the convexity of $\tilde{u}$.

We now examine the relation between the dual and primal problems.

**Theorem 4** *(weak duality) For all $C \in \Phi(y, w)$, $X \in \mathcal{I}(x)$, $x > 0$, $H \in \mathcal{I}(h)$, and $h \geq 0$, the following inequality holds:*

$$U^p(y, C) \leq \frac{L(y, x, h, X, H)}{1+h} - \frac{x}{1+h} w. \tag{57}$$

*Equality holds if and only if for all $t \geq 0$,*

$$\begin{aligned} (1 + H_t)(u^p)'(A(Y_t) - C_t) - X_t u'(C_t) &= 0, & (58)\\ \int_0^t e^{-\rho s}(U_s^a(C) - U_d(Y_s))dX_s &= 0, & (59)\\ \int_0^t e^{-\rho s}(U_s^p(A(Y) - C) - U_d^p(A(Y_s) - Y_s))dH_s &= 0. & (60)\end{aligned}$$

Equation (58) is the first-order condition for optimal consumption. Equations (59) and (60) are complementary slackness conditions associated with the agent's and the principal's participation constraints, respectively. From (57), we can derive that

$$V(y, w) \leq \inf_{h>0, x>0} \frac{\tilde{V}(y, x, h)}{1+h} - \frac{x}{1+h} w.$$

Conditions (58)-(60) are crucial to establish equality in the above equation, which is the so-called strong duality studied below.

Since $\tilde{u}(y, x, h)$ is linearly homogeneous in $(x, 1+h)$ in that $\tilde{u}(y, x, h) = (1 + h)\tilde{u}\left(y, \frac{x}{1+h}, 0\right)$, so is $\tilde{V}(y, x, h)$, i.e.,

$$\tilde{V}(y, x, h) = (1 + h)\tilde{V}(y, x/(1 + h), 0). \tag{61}$$

We can then define the marginal utility ratio, $X_t/(1 + H_t)$, as a state variable. This property is useful to characterize the optimal contract.

The following theorem shows that the solution to the primal problem can be inferred from the solution to the dual problem.

**Theorem 5** *(strong duality) Let $X^* \in \mathcal{I}$ and $H^* \in \mathcal{I}(0)$ be a solution to the dual problem (54). Let $Z_t^* \equiv X_t^*/(1 + H_t^*)$ and $C_t^* \equiv c(Y_t, Z_t^*)$. Suppose the following conditions hold:*

*(i) $U^p(y,C^*)^- < \infty$. (ii) Given $H^*$, (52) holds for the processes $X^\delta \equiv X^* + \delta$ and $\bar{X}^{\pm\delta} \equiv X^*(1 \pm \delta)$ for some small $\delta > 0$. (ii) Given $X^*$, (52) holds for the processes $H^\delta \equiv H^* + \delta$ and $\bar{H}^{\pm\delta} \equiv H^*(1 \pm \delta)$ for some small $\delta > 0$. Then $C^*$ is a solution to the primal problem (45) and*

$$V(y,w) = \inf_{z>0} \tilde{V}(y,z,0) - zw = \inf_{x>0,h\geq 0} \frac{\tilde{V}(y,x,h)}{1+h} - \frac{x}{1+h}w. \tag{62}$$

This theorem shows that, after we solve for the optimal ratio of the marginal utilities $Z_t^*$ from the dual problem, optimal consumption in the primal problem can be completely characterized by the function $c(Y_t, Z_t^*)$. In addition, (62) provides the link between the primal and dual value functions. In the next subsection, we use dynamic programming to characterize the dual problem and derive the solution for the marginal utility ratio.

## 5.2. Dynamic Programming

We now proceed heuristically to derive the HJB equation for the dual problem (55). Suppose that $X$ and $H$ are absolutely continuous with respect to time $t$ so that $dX_t = \lambda_t dt$ and $H_t = \eta_t dt$, where $\lambda_t, \mu_t \geq 0$. If $\tilde{V}$ is sufficiently smooth, then we can derive the HJB equation as

$$\begin{aligned} \rho\tilde{V}(y,x,h) &= \min_{\lambda\geq 0,\eta\geq 0} \tilde{u}(y,x,h) + \tilde{V}_y(y,x,h)\mu(y) + \frac{1}{2}\tilde{V}_{yy}(y,x,h)\sigma(y)^2 \\ &\quad +\lambda\left[\tilde{V}_x(y,x,h) - U_d(y)\right] + \eta\left[\tilde{V}_h(y,x,h) - U_d^p(A(y)-y)\right]. \end{aligned} \tag{63}$$

The solution satisfies

$$\begin{aligned} \tilde{V}_x(y,x,h) > U_d(y) &\implies \lambda = 0, \\ \tilde{V}_x(y,x,h) = U_d(y) &\implies \lambda \geq 0, \end{aligned}$$

and

$$\begin{aligned} \tilde{V}_h(y,x,h) > U_d^p(A(y)-y) &\implies \eta = 0, \\ \tilde{V}_h(y,x,h) = U_d^p(A(y)-y) &\implies \eta \geq 0. \end{aligned}$$

The variational inequality is

$$0 = \min\left\{\tilde{V}_x(y,x,h) - U_d(y),\ \tilde{V}_h(y,x,h) - U_d^p(A(y)-y),\ \tilde{u}(y,x,h) + \mathcal{A}\tilde{V}(y,x,h)\right\}, \tag{64}$$

where

$$\mathcal{A}\tilde{V}(y,x,h) = \tilde{V}_y(y,x,h)\mu(y) + \frac{1}{2}J_{yy}(y,x,h)\sigma(y)^2 - \rho\tilde{V}(y,x,h).$$

Following Harrison and Taksar (1983), we construct an optimal policy as follows. There are two free boundaries satisfying, respectively,

$$\tilde{V}_x(y,x,h) = U_d(y), \tag{65}$$

$$\tilde{V}_h(y,x,h) = U_d^p(A(y)-y). \tag{66}$$

These two boundaries partition the state space into two types of regions. The no-jump region contains all states $(y,x,h)$ satisfying

$$\tilde{u}(y,x,h) + \mathcal{A}\tilde{V}(y,x,h) = 0,$$
$$\tilde{V}_x(y,x,h) > U_d(y), \text{ and } \tilde{V}_h(y,x,h) > U_d^p(A(y)-y),$$

and the jump region contains all states $(y,x,h)$ satisfying

$$\tilde{u}(y,x,h) + \mathcal{A}\tilde{V}(y,x,h) > 0,$$
$$\tilde{V}_x(y,x,h) = U_d(y), \text{ or } \tilde{V}_h(y,x,h) = U_d^p(A(y)-y).$$

See Figure 4 in Section 6 for an illustration.

Because $\tilde{V}$ is linearly homogeneous in $x$ and $1+h$, we can reduce (64) to a two-dimensional problem where the state variables are $y$ and $x/(1+h)$. We can also express the two free boundaries as

$$\frac{x}{1+h} = \varphi_1(y), \quad \frac{x}{1+h} = \varphi_2(y),$$

where $\varphi_1(y)$ and $\varphi_2(y) > \varphi_1(y)$ are determined by the value-matching conditions (65) and (66), respectively, together with super-contact conditions as illustrated in Section 6.

If the initial state lies in the jump region, it jumps immediately to the no-jump region. Once the state lies in the no-jump region, the processes $X$ and $H$ are regulators such that $(Y,X,H)$ will never leave the no-jump region. The process $X$ stays constant within the no-jump region and increases if and only if $(Y,X,H)$ hits the boundary $\varphi_1(Y_t)$ at some time $t$. The process $H$ also stays constant within the no-jump region and increases if and only if $(Y,X,H)$ hits the boundary $\varphi_2(Y_t)$ at some time $t$.

Formally, we establish the following Verification Theorem analogous to Theorem 3. The proof is also similar and hence is omitted.

**Theorem 6** *(verification) Suppose that there exists a twice continuously differentiable solution* $\tilde{V}: \mathbb{R}_+ \times \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ *to (64) such that for all* $X \in \mathcal{I}(x)$ *and* $H \in \mathcal{I}(h)$, $x > 0$ *and* $h \geq 0$, *(i) the process defined by*

$$\int_0^t e^{-\rho s}\tilde{V}_y(Y_s,X_s,H_s)\,\sigma(Y_s)\,dB_s, \quad t \geq 0,$$

*is a martingale, and (ii)* $\lim_{t\to\infty} E\left[e^{-\rho t}\tilde{V}(Y_t, X_t, H_t)\right] = 0$*. Suppose further that* $X^* \in \mathcal{I}(x)$ *and* $H^* \in \mathcal{I}(h)$ *are such that (i) for all* $t > 0$*,*

$$0 = \tilde{u}(Y_t, X_t^*, H_t^*) + \mathcal{A}\tilde{V}(Y_t, X_t^*, H_t^*); \tag{67}$$

*(ii) for all* $t \geq 0$*,*

$$\int_0^t e^{-\rho s}\left(\tilde{V}_x(Y_s, X_s^*, H_s^*) - U_d(Y_s)\right) dX_s^* = 0, \tag{68}$$

$$\int_0^t e^{-\rho s}\left(\tilde{V}_h(Y_s, X_s^*, H_s^*) - U_d^p(A(Y_s) - Y_s)\right) dH_s^* = 0. \tag{69}$$

*Then* $\tilde{V}$ *is the dual value function for problem (55) and* $X^* \in \mathcal{I}(x)$ *and* $H^* \in \mathcal{I}(h)$ *are the solutions to this problem. If* $\tilde{V}(y, x, h)$ *is strictly convex in* $x$ *and there exists* $x^* > 0$ *such that* $\tilde{V}_x(y, x^*, 0) = w$ *and if the conditions in Theorem 5 are satisfied, then the primal value function is given by* $V(y, w) = \tilde{V}(y, x^*, 0) - x^* w$*. The optimal consumption plan and the agent's continuation value for the primal problem (7) are, respectively, given by*

$$C_t^* = c(Y_t, Z_t^*), \qquad Z_t^* = -V_w(Y_t, W_t^*), \tag{70}$$
$$W_t^* = \tilde{V}_x(Y_t, Z_t^*, 0), \qquad V(Y_t, W_t^*) = \tilde{V}_h(Y_t, X_t^*, H_t^*),$$

*where* $Z_t^* = X_t^*/(1 + H_t^*)$*,* $X_0^* = x^*$*, and* $H_0^* = 0$*.*

Equation (70) shows that optimal consumption can be characterized by two state variables, the income level $Y_t$ and the marginal utility ratio $Z_t^* = X_t^*/(1 + H_t^*)$. Applying the envelope condition to (62), we deduce that the marginal utility ratio $Z_t^*$ is equal to the negative slope of the Pareto frontier, $-V_w(Y_t, W_t^*)$. Applying the first-order condition to (62), the agent's continuation value can be characterized by the partial derivative of the dual value function. This result is similar to that in the case of one-sided limited commitment.

Differentiating (61) with respect to $h$ yields:

$$\tilde{V}_h(y, x, h) = \tilde{V}(y, x/(1 + h), 0) - \tilde{V}_x(y, x/(1 + h), 0)\frac{x}{1 + h}.$$

Thus, we can also describe the principal's value using the partial derivative of the dual value function:

$$\tilde{V}_h(Y_t, X_t^*, H_t^*) = \tilde{V}(Y_t, Z_t^*, 0) - \tilde{V}_x(Y_t, Z_t^*, 0) Z_t^* = \tilde{V}(Y_t, Z_t^*, 0) - W_t^* Z_t^* = V(Y_t, W_t^*).$$

Similar to Theorem 3 in the one-sided limited commitment case, condition (68) indicates that the process $X^*$ increases if and only if $\tilde{V}_x(Y_t, X_t^*, H_t^*) = U_d(Y_t) = W_t^*$. In addition, condition

(69) indicates that $H_t^*$ increases if and only if $\tilde{V}_h(Y_t, X_t^*, H_t^*) = U_d^p(A(Y_t) - Y_t) = V(Y_t, W_t^*)$. These two conditions are the complementary slackness conditions associated with the agent's and the principal's participation constraints. We can equivalently express the solution $X^*$ and $H^*$ as

$$\begin{aligned} X_t^* &= \max\left\{x^*, \sup_{s\in[0,t]} \varphi_1(Y_s^*)(1+H_s^*)\right\}, \\ H_t^* &= \max\left\{0, \sup_{s\in[0,t]} \frac{X_s^*}{\varphi_2(Y_s)} - 1\right\}. \end{aligned}$$

Since $\tilde{V}_x(y, \varphi_1(y), 0) = U_d(y)$ on the lower boundary, it follows from $w \geq U_d(Y_0)$ that

$$\tilde{V}_x(Y_0, X_0^*, 0) = w \geq \tilde{V}_x(Y_0, \varphi_1(Y_0), 0).$$

If $\tilde{V}(y, x, h)$ is convex in $x$, then $X_0^* \geq \varphi_1(Y_0)$. If $w$ is sufficiently large, then $X_0^*$ is also sufficiently large by the convexity of $\tilde{V}(\cdot, x, \cdot)$ in $x$. Since we can show that $\tilde{V}_h(Y_0, X_0, 0)$ decreases in $X_0$,[16] $\tilde{V}_h(Y_0, X_0, 0)$ will fall below $U_d^p(A(Y_0) - Y_0)$ when $X_0$ is sufficiently large. In this case, the principal's participation constraint is violated. Thus, there must be an upper bound on $w$. The upper bound is given by $\bar{w} = \tilde{V}_x(Y_0, \varphi_2(Y_0), 0)$, because if $w$ exceeds $\bar{w}$, then $X_0 > \varphi_2(Y_0)$ and $\tilde{V}_h(Y_0, X_0, 0) < U_d^p(A(Y_0) - Y_0)$. Consequently, we must have $w \in [U_d(Y_0), \bar{w}]$. In this case, the state vector $(Y_t^*, X_t^*, H_t^*)$ always lies in the no-jump region and both $X^*$ and $H^*$ are continuous processes.

## 6. Example II

In this section, we introduce limited commitment from the principal's side in Example I of Section 4. Suppose the principal's outside option is autarky as well. The agent has income $Y_t$ given by (33) and the principal does not have any income. Thus, the risk-neutral principal's autarky value is zero and the agent's autarky value is given by (34). Recall that the agent's utility is given by $u(c) = c^\alpha/\alpha$, $0 \neq \alpha < 1$, and we still maintain the same assumptions on parameter values as in Section 4. Note that our solution for the policy functions in the dual

[16] Differentiating (61) with respect to $h$ yields:

$$\tilde{V}_h(y, x, h) = \tilde{V}(y, x/(1+h), 0) - \tilde{V}_x(y, x/(1+h), 0)\frac{x}{1+h}.$$

Differentiating with respect to $x$ yields:

$$\tilde{V}_{hx}(y, x, h) = \tilde{V}_x(y, x/(1+h), 0)\frac{1}{1+h} - \tilde{V}_{xx}\frac{x}{(1+h)^2} - \tilde{V}_x(y, x/(1+h), 0)\frac{1}{1+h} = -V_{xx}\frac{x}{(1+h)^2} < 0.$$

problem when $\alpha = 0$ applies to the case of logarithmic utility $u(c) = \ln c$.[17]

We can compute the dual function as

$$\tilde{u}(y, x, h) = \max_{c>0} \ (1+h)(y-c) + xu(c) = (1+h)y + \frac{1-\alpha}{\alpha} x^{\frac{1}{1-\alpha}} (1+h)^{\frac{-\alpha}{1-\alpha}},$$

and the optimal consumption rule as

$$c^* = \left(\frac{x}{1+h}\right)^{\frac{1}{1-\alpha}}.$$

### 6.1. Solution

Hinted by the solution in Section 4, we conjecture that the no-jump region is given by

$$\left\{(y, x, h) \in \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{+} : \frac{x}{1+h} \in \left[b_1 y^{1-\alpha}, b_2 y^{1-\alpha}\right]\right\},$$

where $0 < b_1 < b_2$ are to be determined. We can verify that the dual value function with the following form satisfies (67) in the no-jump region,

$$\begin{aligned} \tilde{V}(y, x, h) \quad = \quad & \frac{1-\alpha}{\alpha\rho}(1+h)\left(\frac{x}{1+h}\right)^{\frac{1}{1-\alpha}} + \frac{(1+h)\,y}{\rho-\mu} \\ & + A_1(1-\alpha)(1+h)\left(\frac{x}{1+h}\right)^{\frac{1-\beta_1}{1-\alpha}} y^{\beta_1} \\ & + A_2(1-\alpha)(1+h)\left(\frac{x}{1+h}\right)^{\frac{1-\beta_2}{1-\alpha}} y^{\beta_2}, \end{aligned} \tag{71}$$

where $\beta_1$ and $\beta_2 > \beta_1$ are the two roots of equation (38) for $r = \rho$ and $A_1$ and $A_2$ are constants to be determined. We can also verify that the dual value function in the jump region takes the following form: for $x/(1+h) > b_2 y^{1-\alpha}$,

$$\tilde{V}(y, x, h) = \tilde{V}\left(y, x, \frac{x}{b_2 y^{1-\alpha}} - 1\right),$$

and for $x/(1+h) < b_1 y^{1-\alpha}$,

$$\tilde{V}(y, x, h) = \tilde{V}\left(y, b_1 y^{1-\alpha}(1+h), h\right) + \left(x - b_1 y^{1-\alpha}(1+h)\right) U_d(y).$$

[17] For log utility, the dual value function is given by

$$\tilde{V}(y, x, h) = \frac{1}{\rho}\left[x \ln\left(\frac{x}{1+h}\right) - x\right] + \frac{(1+h)\,y}{\rho-\mu} + A_1 x^{1-\beta_1}(1+h)^{\beta_1} y^{\beta_1} + A_2 (1+h)^{\beta_2} x^{1-\beta_2} y^{\beta_2},$$

where $A_1$ and $A_2$ are constants determined by the boundary conditions. It is not the limit when $\alpha \to 0$.

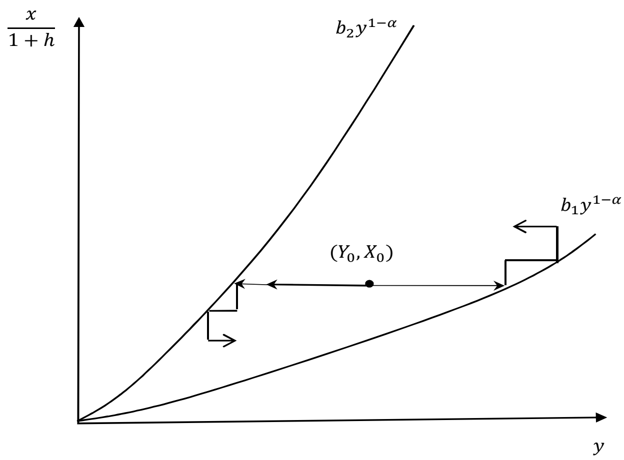


Figure 4: The state space for Example II. The two curves $x/(1+h) = b_1 y^{1-\alpha}$ and $x/(1+h) = b_2 y^{1-\alpha}$ partition the state space into three areas. The middle area is the no-jump region and the other two areas are the jump region.

We use the following four value-matching and super-contact conditions to determine the four constants $A_1$, $A_2$, $b_1$, and $b_2$ :

$$\lim_{\frac{x}{1+h}\downarrow b_1 y^{1-\alpha}} \tilde{V}_x(y,x,h) = U_d(y), \qquad \lim_{\frac{x}{1+h}\uparrow b_2 y^{1-\alpha}} \tilde{V}_h(y,x,h) = 0,$$
$$\lim_{\frac{x}{1+h}\downarrow b_1 y^{1-\alpha}} \tilde{V}_{xx}(y,x,h) = 0, \qquad \lim_{\frac{x}{1+h}\uparrow b_2 y^{1-\alpha}} \tilde{V}_{hh}(y,x,h) = 0.$$

We can simplify the above four equations to two equations for $b_1$ and $b_2$.

**Proposition 2** *Suppose that $\alpha < 1$, $\sigma > 0$, $\rho > \mu + \sigma^2/2$, and $\rho > \alpha\mu + \alpha(\alpha-1)\sigma^2/2$. Then there are two solutions for $b_1$ and $b_2$. One solution is such that $0 < b_1 < 1 < b_2 < \left(\frac{\rho}{\rho-\mu}\frac{1-\beta_1}{-\beta_1}\right)^{1-\alpha}$. The other solution is degenerate ($b_1 = b_2 = 1$).*

We rule out the degenerate solution since there are no two increasing processes $X^*$ and $H^*$ satisfying $X^* = (1+H^*)\,Y^{1-\alpha}$ for a geometric Brownian motion $Y$.

## 6.2. Numerical Illustrations

Figure 4 plots the state space. It shows that the two free boundaries $x/(1+h) = b_1 y^{1-\alpha}$ and $x/(1+h) = b_2 y^{1-\alpha}$ partition the state space into three areas. The area inside the two boundaries is the no-jump region and the other two areas are the jump region. The initial state $(Y_0, X_0^*)$ is inside the no-jump region. Consumption is constant in the interior of the no-jump

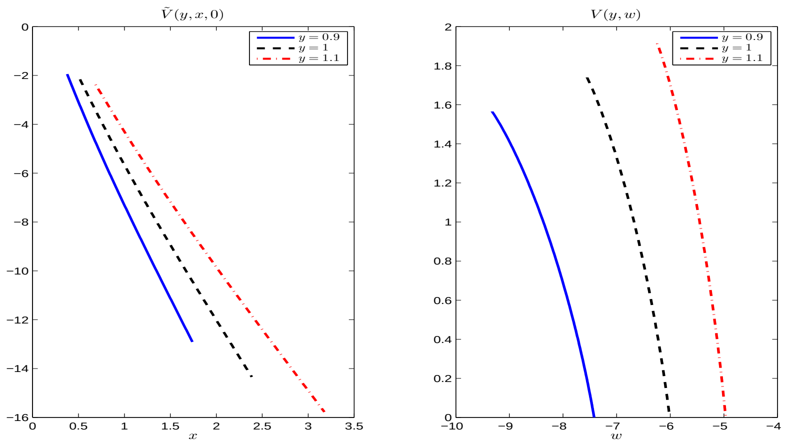


Figure 5: The dual and primal value functions in the no-jump region for Example II. Parameter values are given by $\mu = 0.02$, $\sigma = 0.1$, $\rho = 0.04$, and $\alpha = -2$.

region. Whenever $Y_t$ increases to the lower boundary, $X_t^*$ and $C_t^*$ rise, but $H_t^*$ does not change. Whenever $Y_t$ decreases to the upper boundary, $H_t^*$ rises and $C_t^*$ falls, but $X_t^*$ does not change.

Figure 5 plots the dual value function $\tilde{V}(y, x, 0)$ and the primal value function $V(y, w)$ for three values $y \in \{0.9, 1, 1.1\}$ in the no-jump region. This figure shows that $\tilde{V}(y, x, 0)$ is strictly convex in $x$ and $V(y, w)$ is strictly concave and decreasing in $w$. Note that the domains for both functions change with $y$. In particular, the domain of $V(y, w)$ for $w$ increases with $y$ because a larger promised value is needed to induce the agent's participation when his income is larger.

By Harrison and Taksar (1983), Harrison (1985), or Stokey (2008), we deduce that $\ln X^*$ and $\ln H^*$ regulate the reflected diffusion process,

$$(\alpha - 1) \ln Y_t + \ln X_t^* - \ln (1 + H_t^*), \quad t \geq 0,$$

within the band $[\ln b_1, \ln b_2]$. We can then express the solutions for $X^*$ and $H^*$ as

$$\begin{aligned} X_t^* &= \max \left\{ x^*, \sup_{s \in [0,t]} b_1 Y_s^{1-\alpha} (1 + H_s^*) \right\}, \\ H_t^* &= \max \left\{ 0, \sup_{s \in [0,t]} \frac{X_s^*}{b_2 Y_s^{1-\alpha}} - 1 \right\}, \end{aligned}$$

where $X_0^* = x^*$ and $H_0^* = 0$. Figure 6 plots the simulated paths of incomes $Y_t$, consumption $C_t^* = (X_t^* / (1 + H_t^*))^{\frac{1}{1-\alpha}}$, the continuation value $W_t^* = \tilde{V}_x (Y_t, X_t^* / (1 + H_t^*), 0)$, and $X_t^* / (1 + H_t^*) Y_t^{\alpha - 1}$. This figure shows intuitively how $C_t^*$ and $W_t^*$ move with incomes $Y_t$. Using (71), we can show that $W_t^*$ normalized by the autarky value $\kappa Y_t^\alpha$ is an invertible function of

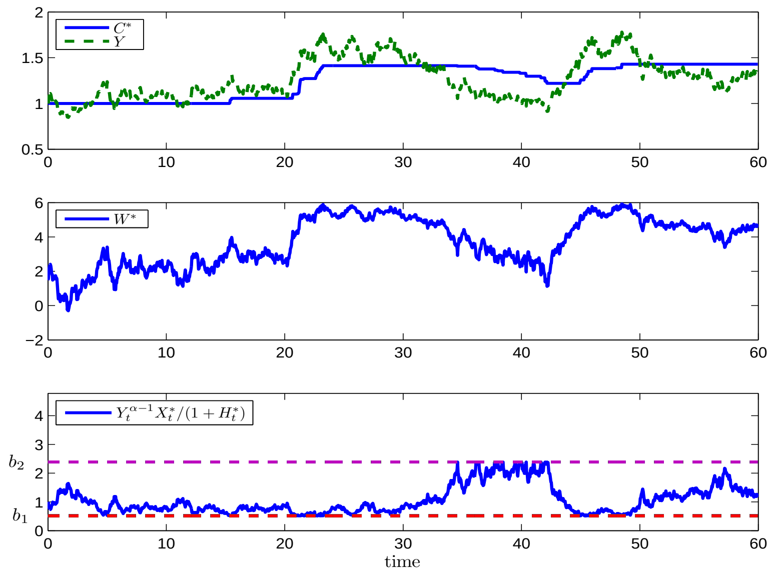


Figure 6: Simulated paths of the agent's optimal consumption $C_t^*$, incomes $Y_t$, continuation values $W_t^*$, and the process $Y_t^{\alpha-1}X_t^*/(1+H_t^*)$, $t \geq 0$, for Example II. Parameter values are given by $\mu = 0.02$, $\sigma = 0.1$, $\rho = 0.04$, and $\alpha = -2$.

$C_t^*/Y_t$. Thus, we can also write $C_t^*/Y_t$ as a function of $W_t^*/(\kappa Y_t^{\alpha})$, which may be derived as in the standard approach using the continuation value as a state variable.

### 6.3. Stationary Distribution

Since $C_t^* = (X_t^*/(1+H_t^*))^{\frac{1}{1-\alpha}}$, it follows that $\ln(C_t^*/Y_t)$ is a regulated Brownian motion with drift $\sigma^2/2 - \mu$ and volatility $-\sigma$ on $\left[(1-\alpha)^{-1}\ln b_1, (1-\alpha)^{-1}\ln b_2\right]$. By Proposition 5.5 in Harrison (1985) or Proposition 10.8 in Stokey (2008), $\ln(C_t^*/Y_t)$ has a stationary distribution with the density function:

$$p(x) = \frac{\delta e^{\delta x}}{b_2^{\frac{\delta}{1-\alpha}} - b_1^{\frac{\delta}{1-\alpha}}}, \quad x \in \left[(1-\alpha)^{-1}\ln b_1, (1-\alpha)^{-1}\ln b_2\right],$$

where $\delta \equiv 1 - 2\mu/\sigma^2$.

In a discrete-time setup with a symmetric IID endowment process, Kocherlakota (1996) shows that there is a unique long-run stationary distribution for the agent's continuation values. In our continuous-time model, the agent's endowment is a geometric Brownian motion process. We can show that the agent's continuation value $W_t^*$ normalized by his autarky value $\kappa Y_t^{\alpha}$ has a unique long-run stationary distribution because we have shown before that $W_t^*/(\kappa Y_t^{\alpha})$ can be written as a function of $C_t^*/Y_t$.

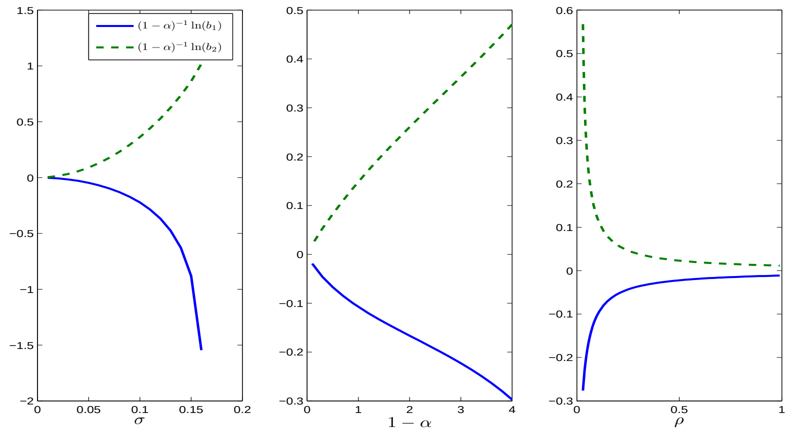


Figure 7: Comparative statics for Example II. Parameter values are given by $\mu = 0.02$, $\sigma = 0.1$, $\rho = 0.04$, and $\alpha = -2$, unless one of them is changed in the comparative statics.

By Proposition 13 in Chapter 5 of Harrison (1985) or Stokey (2008), the average increase and the average decrease in $\ln\left(C_t^*/Y_t\right)$ per unit of time are, respectively, given by

$$\frac{\left(\sigma^2/2-\mu\right) b_1^{\frac{\delta}{1-\alpha}}}{b_2^{\frac{\delta}{1-\alpha}} - b_1^{\frac{\delta}{1-\alpha}}} \quad \text{and} \quad \frac{\left(\sigma^2/2-\mu\right) b_2^{\frac{\delta}{1-\alpha}}}{b_2^{\frac{\delta}{1-\alpha}} - b_1^{\frac{\delta}{1-\alpha}}}.$$

It follows that the average increase per unit of time is higher than the average decrease per unit of time if and only if $\sigma^2/2 < \mu$.

## 6.4. Comparative Statics

First, we consider the effect of the volatility of the agent's income on risk sharing. Because the agent is risk averse, a larger volatility reduces his autarky value. A lower autarky value reduces the agent's incentive to default, and hence makes it easier to enforce risk-sharing contracts. In line with this intuition, the left panel of Figure 7 shows that the risk-sharing band expands with $\sigma$.

Next, we study how the agent's risk aversion affects risk sharing. When the agent is more risk averse (i.e., when $1-\alpha$ is higher), then he is less willing to go to autarky because in autarky he must face full income uncertainty. Similar to the above, risk sharing becomes easier when the agent's incentive to default is reduced. Consistent with this intuition, the middle panel of Figure 7 shows that the risk-sharing band expands as $1-\alpha$ increases. On the other hand, the band shrinks to a singleton (i.e., autarky) as $\alpha \to 1$. This is because the agent is risk-neutral if $\alpha = 1$, in which case autarky is the only enforceable allocation. When both the principal and the agent are risk neutral, any contract is a zero-sum game. That is, any contract that gives one party a positive net gain (over autarky) must impose a loss upon the other party. The latter would default.

Finally, we discuss the effect of the subjective discount rate $\rho$. Consider the party who is making a transfer to his partner. To satisfy his participation constraint, the benefit of future insurance must exceed the current loss. Hence, higher patience (i.e., lower $\rho$) increases the weight on future benefit and makes it easier to satisfy the participation constraint. Consistent with this intuition, the right panel of Figure 7 shows that the risk-sharing band expands as $\rho$ decreases. The result that patience enhances cooperation is well known in the literature on models with limited commitment, as well as in game theory.

### 6.5. Why Are Continuous-Time Models Different?

Our comparative statics results are generally consistent with those in discrete-time models. However, there is an important difference. In discrete-time models, Kocherlakota (1996), Alvarez and Jermann (2000, 2001), and Ligon, Thomas, and Worrall (2002) show that there may be three regimes for efficient allocation depending on parameter values: (i) full risk sharing forever is possible; (ii) only limited risk sharing is possible; or (iii) only autarky is possible. In particular, for high enough values of the discount factor, sufficient endowment risk, or enough degree of risk aversion, the first-best allocation is an optimal contract. In the opposite extreme, the autarky allocation is optimal. By contrast, Proposition 2 shows that the risk-sharing band is finite for any admissible parameter values satisfying the assumptions in this proposition. Thus, neither the first-best allocation nor autarky can be an optimal contract. This difference is due to our continuous-time Brownian motion environment. In the existing literature on discrete-time models, income processes are often assumed to be either IID or stationary Markovian with a bounded support. In our model, the income process is a nonstationary geometric Brownian motion.

To better understand the difference, we discretize Example II for the special case with $u(c) = \ln c$ and $\mu = 0$. A time interval is denoted by $dt$ and the income process is approximated by a binomial process,

$$\frac{y_{t+dt} - y_t}{y_t} = \begin{cases} \sigma\sqrt{dt} & \text{with probability } 0.5; \\ -\sigma\sqrt{dt} & \text{with probability } 0.5. \end{cases}$$

An important feature of this shock is that its standard deviation is $\sigma\sqrt{dt}$. The autarky value $U_d(y)$ satisfies the recursion:

$$U_d(y) = \ln(y)\,dt + \frac{1}{1+\rho dt}\left[0.5U_d((1+\sigma\sqrt{dt})y) + 0.5U_d((1-\sigma\sqrt{dt})y)\right].$$

Following Ljungqvist and Sargent (2004), we write down the discrete-time Bellman equation as:

$$
\begin{aligned}
U^p(y, v) = \max_{c, v_b, v_g} \quad & (y - c)dt + \frac{1}{1 + \rho dt}\left[0.5U^p((1 - \sigma\sqrt{dt})y, v_b) + 0.5U^p((1 + \sigma\sqrt{dt})y, v_g)\right] \\
\text{subject to} \quad & v = (\ln(c) - \ln(y))dt + \frac{1}{1 + \rho dt}(0.5v_b + 0.5v_g), \\
& v_b \in [0, \bar{U}^a((1 - \sigma\sqrt{dt})y)], \\
& v_g \in [0, \bar{U}^a((1 + \sigma\sqrt{dt})y)],
\end{aligned}
$$

where $U^p(y, v)$ is the principal's value function when the income is $y$ and the surplus of the agent's continuation value over his autarky value is $v$. Here, $v_b$ and $v_g$ denote the agent's surplus in the continuation contract after a bad and a good shock, respectively, and $\bar{U}^a(y)$ denotes the agent's highest surplus among all sustainable contracts when his income is $y$. Since $U^p(y, v)$ is decreasing in $v$, $U^p(y, \bar{U}^a(y)) = 0$ by the principal's participation constraint. When $dt = 1$, we obtain a discrete-time model.

Any full risk sharing allocation gives a constant continuation value to the agent. When $y$ is sufficiently large, the autarky value $U_d(y)$ may exceed this value, violating the agent's participation constraint. Thus, full risk sharing is not sustainable even in discrete time. This result is generally true when the income process is unbounded. However, it is more subtle to understand why autarky is never optimal in our continuous-time model but it may be optimal in the discrete-time model depending on parameter values. In Appendix A, we prove the following result.

**Proposition 3** *In the discretized model, a non-autarkic sustainable risk sharing contract exists if and only if $\sigma\sqrt{dt} > 2\rho dt$.*

This proposition states that in the discretized model, a non-autarkic risk sharing contract exists if and only if the income volatility, $\sigma\sqrt{dt}$, is sufficiently large or the agent is sufficiently patient (i.e., $\rho dt$ is so small that the discount factor $1/(1 + \rho dt)$ is sufficiently close to 1). This means that autarky is the only optimal contract if and only if the income process is not too volatile or the agent is sufficiently impatient. This result is consistent with the finding in the literature of discrete-time models and does not depend on whether the income process is stationary or not. By contrast, when the time interval is sufficiently small, the condition $\sigma\sqrt{dt} > 2\rho dt$ is always satisfied, and hence autarky cannot be an optimal contract.[18]

[18]Note that we do not take the continuous time limit as $dt \to 0$. It is technically involved to prove the convergence from the discretized model to the continuous-time model. Such an analysis is beyond the scope of this paper.

The idea of the proof is as follows. We construct a particular risk sharing contract in which the principal receives a small amount of income from the agent if a good shock realizes, but transfers a small amount of income to the agent, otherwise. This contract is such that it always satisfies the agent's participation constraints, but whether it satisfies the principal's participation constraints depends on the time interval $dt$. Specifically, we show in Appendix A that the net benefit over autarky to the principal from the constructed contract is positively related to $\sigma\sqrt{dt} - 2\rho dt$. When $dt$ is sufficiently small, the net benefit is always positive.

## 7. Conclusion

In this paper, we have proposed a duality approach to solving continuous-time contracting problems with either one-sided or two-sided limited commitment. We have established the weak and strong duality theorems and provided a dynamic programming characterization of the dual problem. We have also provided explicit solutions for two examples of a consumption insurance problem. We have demonstrated how our approach is analytically convenient and how the optimal contracts in continuous time are different from those in discrete time. In particular, we have shown that neither autarky nor full risk sharing can be an optimal contract with two-sided limited commitment, unlike in discrete-time models. An important advantage of our approach over the standard approach using the continuation value as a state variable is that our state space is the positive orthant, but the state space for the continuation value is endogenous under two-sided limited commitment. The other advantage is that the HJB equation in the dual problem is a linear PDE, while that in the primal problem is a nonlinear PDE with state constraints. An interesting direction of future research is to apply the duality approach to other contracting environments, such as those with moral hazard or adverse selection.

# Appendices

## A Proofs

The following lemma will be repeatedly used in later proofs.

**Lemma 1** *Define a process $M$ by*

$$M_t = E_t\left[\int_t^\infty e^{-\rho(s-t)} N_s ds\right].$$

*If one of the two conditions is satisfied for $X \in \mathcal{I}$,*

1. $E\left[\int_0^\infty e^{-\rho t} X_t\,|N_t|\,dt\right] < \infty$,
2. $\{N_t\}_{t\geq 0}$ *is nonnegative and* $E\left[\int_0^\infty e^{-\rho t} M_t dX_t\right] < \infty$,

*then*

$$E\left[\int_0^\infty e^{-\rho t} X_t N_t dt\right] = X_0 M_0 + E\left[\int_0^\infty e^{-\rho t} M_t dX_t\right].$$

**Proof:** For any finite $T > 0$, integration by parts yields:

$$\begin{aligned}
& E\left[\int_0^T e^{-\rho t} X_t N_t dt\right] \\
= & -E\left[\int_0^T X_t d\left(\int_t^T e^{-\rho s} N_s ds\right)\right] \\
= & -E\left[\left(X_t \int_t^T e^{-\rho s} N_s ds\right)\Big|_0^T - \int_0^T \left(\int_t^T e^{-\rho s} N_s ds\right) dX_t\right] \\
= & X_0 E\left[\int_0^T e^{-\rho s} N_s ds\right] + E\left[\int_0^T e^{-\rho t} E_t\left[\int_t^T e^{-\rho(s-t)} N_s ds\right] dX_t\right],
\end{aligned}$$

where the last equality follows from the Law of Iterative Expectations. If $\{N_t\}_{t\geq 0}$ is nonnegative and $E\left[\int_0^\infty e^{-\rho t} M_t dX_t\right] < \infty$, then the Monotone Convergence Theorem implies that

$$\begin{aligned}
E\left[\int_0^\infty e^{-\rho t} X_t N_t dt\right] &= \lim_{T\to\infty} E\left[\int_0^T e^{-\rho t} X_t N_t dt\right] \\
&= X_0 E\left[\int_0^\infty e^{-\rho s} N_s ds\right] + \lim_{T\to\infty} E\left[\int_0^T e^{-\rho t} E_t\left[\int_t^T e^{-\rho(s-t)} N_s ds\right] dX_t\right] \\
&= X_0 E\left[\int_0^\infty e^{-\rho s} N_s ds\right] + E\left[\int_0^\infty e^{-\rho t} E_t\left[\int_t^\infty e^{-\rho(s-t)} N_s ds\right] dX_t\right].
\end{aligned}$$

If $E\left[\int_0^\infty e^{-\rho t} X_t\,|N_t|\,dt\right] < \infty$, then the same conclusion follows from the Dominated Convergence Theorem. Q.E.D.

**Proof of Proposition 1:** Let $\theta \in (0,1)$. For any $x_1, x_2 > 0$ and any two processes $X_1 \in \mathcal{I}(x_1)$ and $X_2 \in \mathcal{I}(x_2)$, define $x^\theta \equiv \theta x_1 + (1-\theta) x_2$ and define the process $X^\theta$ as $X_t^\theta \equiv \theta X_{1t} + (1-\theta) X_{2t}$. By the definition of $\tilde{u}$ and the convexity of $\tilde{u}$,

$$X_t^\theta e^{-(\rho-r)t} u(C_t) - C_t \leq \tilde{u}(e^{-(\rho-r)t} X_t^\theta) \leq \theta \tilde{u}(e^{-(\rho-r)t} X_{1t}) + (1-\theta) \tilde{u}(e^{-(\rho-r)t} X_{2t})$$

for any $C \in \Gamma(y, w)$. From the proof of Theorem 1 below, we have

$$E\left[\int_0^\infty e^{-\rho t} X_{1t} \left|u(C_t)\right| dt\right] < \infty \quad \text{and} \quad E\left[\int_0^\infty e^{-\rho t} X_{2t} \left|u(C_t)\right| dt\right] < \infty.$$

In addition, it follows from (16) that

$$E\left[\int e^{-rt} \left|\tilde{u}(e^{-(\rho-r)t} X_{1t})\right| dt\right] < \infty \quad \text{and} \quad E\left[\int e^{-rt} \left|\tilde{u}(e^{-(\rho-r)t} X_{2t})\right| dt\right] < \infty.$$

Thus, $X^\theta$ satisfies the integrability condition (16). It is also trivial to verify that $X^\theta$ satisfies the integrability condition (15). Consequently, $X^\theta \in \mathcal{I}\left(x^\theta\right)$.

It follows from the convexity of $\tilde{u}$ that

$$L\left(y, x^\theta, X^\theta\right) \leq \theta L(y, x_1, X_1) + (1-\theta) L(y, x_2, X_2).$$

Since $X^\theta \in \mathcal{I}\left(x^\theta\right)$,

$$\tilde{V}\left(y, x^\theta\right) \leq L\left(y, x^\theta, X^\theta\right) \leq \theta L(y, x_1, X_1) + (1-\theta) L(y, x_2, X_2).$$

Taking infimum yields

$$\begin{aligned}
\tilde{V}\left(y, x^\theta\right) &\leq \theta \inf_{X_1 \in \mathcal{I}(x_1)} L(y, x_1, X_1) + (1-\theta) \inf_{X_2 \in \mathcal{I}(x_2)} L(y, x_2, X_2) \\
&= \theta \tilde{V}(y, x_1) + (1-\theta) \tilde{V}(y, x_2),
\end{aligned}$$

as desired. Q.E.D.

**Proof of Theorem 1:** By the definition of $\tilde{u}$ in (12),

$$X_t e^{-(\rho-r)t} u(C_t) - C_t \leq \tilde{u}(e^{-(\rho-r)t} X_t), \tag{A.1}$$

for any $C \in \Gamma(y, w)$. By (A.1), (1), and (16), we deduce that $E\left[\int_0^\infty e^{-\rho t} X_t (u(C_t))^+ dt\right] < \infty$, where we use $u^+$ and $u^-$ to denote the positive and negative parts, respectively, of any $u \in \mathbb{R}$. By Lemma 1,

$$E\left[\int_0^\infty \left(\int_t^\infty e^{-\rho s} (u(C_s))^+ ds\right) dX_t\right] < \infty.$$

It follows from $U_t^a(C) \geq U_d(Y_t)$ and $E\left[\int_0^\infty e^{-\rho t}|U_d(Y_t)|dX_t\right] < \infty$ that

$$\begin{aligned}
& E\left[\int_0^\infty \left(\int_t^\infty e^{-\rho s}(u(C_s))^- ds\right) dX_t\right] \\
\leq \; & E\left[\int_0^\infty \left(\int_t^\infty e^{-\rho s}(u(C_s))^+ ds\right) dX_t\right] - E\left[\int_0^\infty e^{-\rho t} U_d(Y_t) dX_t\right] \\
< \; & \infty.
\end{aligned}$$

Applying Lemma 1 again yields:

$$E\left[\int_0^\infty e^{-\rho t} X_t (u(C_t))^- dt\right] < \infty.$$

Thus, we obtain

$$E\left[\int_0^\infty e^{-\rho t} X_t \left|u(C_t)\right| dt\right] < \infty.$$

It follows from Lemma 1 that

$$E\left[\int_0^\infty e^{-\rho t} X_t u(C_t) dt\right] = X_0 w + E\left[\int_0^\infty e^{-\rho t} U_t^a(C)\, dX_t\right], \tag{A.2}$$

where $w = U_0^a(C)$.

Multiplying $e^{-rt}$ and taking expectations on both sides of (A.1), we obtain

$$\begin{aligned}
U^p(y, C) &= E\left[\int_0^\infty e^{-rt}(Y_t - C_t) dt\right] \\
&\leq E\left[\int_0^\infty e^{-rt}(Y_t + \tilde{u}(e^{-(\rho-r)t} X_t)) dt\right] - E\left[\int_0^\infty e^{-\rho t} X_t u(C_t) dt\right].
\end{aligned} \tag{A.3}$$

Substituting (A.2) into (A.3) yields:

$$\begin{aligned}
U^p(y, C) &\leq E\left[\int_0^\infty e^{-rt}(Y_t + \tilde{u}(e^{-(\rho-r)t} X_t)) dt\right] - X_0 w - E\left[\int_0^\infty e^{-\rho t} U_t^a(C)\, dX_t\right] \\
&= L(y, x, X) - X_0 w - E\left[\int_0^\infty e^{-\rho t}\left(U_t^a(C) - U_d(Y_t)\right) dX_t\right] \\
&\leq L(y, x, X) - xw,
\end{aligned}$$

where the last inequality follows from the fact that $C \in \Gamma(y, w)$ and $X \in \mathcal{I}(x)$ and $X_0 = x$. Equalities hold if and only if (21)-(22) hold. Q.E.D.

**Proof of Theorem 2:** First, we show that $C^*$ defined in the theorem satisfies the participation and promise-keeping constraints. Define $X^\varepsilon \equiv X^* + \varepsilon$ for $\varepsilon \in (0, \delta)$. The convexity of $\tilde{u}$ implies that

$$e^{-(\rho-r)t} u(C_t^*) \leq \frac{\tilde{u}\left(X_t^\varepsilon e^{-(\rho-r)t}\right) - \tilde{u}\left(X_t^* e^{-(\rho-r)t}\right)}{\varepsilon} \leq \frac{\tilde{u}\left(X_t^\delta e^{-(\rho-r)t}\right) - \tilde{u}\left(X_t^* e^{-(\rho-r)t}\right)}{\delta}.$$

By assumption, $E\left[\int_0^\infty e^{-rt}\left|\tilde{u}\left(X_t^\delta e^{-(\rho-r)t}\right)\right|dt\right]<\infty$. Furthermore,

$$\begin{aligned}
E\left[\int_0^\infty e^{-\rho t}\left|u\left(C_t^*\right)\right|dt\right] &\leq E\left[\int_0^\infty e^{-rt}\frac{\left|X_t^*e^{-(\rho-r)t}u\left(C_t^*\right)\right|}{X_0^*}dt\right]\\
&\leq E\left[\int_0^\infty e^{-\rho t}\frac{\left|\tilde{u}\left(X_t^*e^{-(\rho-r)t}\right)\right|+C_t^*}{X_0^*}dt\right]<\infty.
\end{aligned}$$

It follows from $E\left[\int_0^\infty e^{-\rho t}\left|u\left(C_t^*\right)\right|dt\right]<\infty$ and $E\left[\int_0^\infty e^{-rt}\left|\tilde{u}\left(X_t^\delta e^{-(\rho-r)t}\right)\right|dt\right]<\infty$ that $E\left[\int_0^\infty e^{-\rho t}\left|\tilde{u}\left(X_t^\varepsilon e^{-(\rho-r)t}\right)\right|dt\right]<\infty$. Therefore, $X^\varepsilon\in\mathcal{I}\left(X_0^*+\varepsilon\right)$ and $\mathcal{L}\left(X^\varepsilon\right)\geq\mathcal{L}\left(X^*\right)$. This implies

$$\lim_{\varepsilon\downarrow 0}\frac{\mathcal{L}\left(X^\varepsilon\right)-\mathcal{L}\left(X^*\right)}{\varepsilon}\geq 0,$$

or, equivalently,

$$\lim_{\varepsilon\downarrow 0}E\left[\int_0^\infty e^{-rt}\frac{\tilde{u}\left(X_t^\varepsilon e^{-(\rho-r)t}\right)-\tilde{u}\left(X_t^*e^{-(\rho-r)t}\right)}{\varepsilon}dt\right]-w\geq 0.$$

By the Dominated Convergence Theorem,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}E\left[\int_0^\infty e^{-rt}\frac{\tilde{u}\left(X_t^\varepsilon e^{-(\rho-r)t}\right)-\tilde{u}\left(X_t^*e^{-(\rho-r)t}\right)}{\varepsilon}dt\right]\\
=\;& E\left[\int_0^\infty e^{-rt}\lim_{\varepsilon\downarrow 0}\frac{\tilde{u}\left(X_t^\varepsilon e^{-(\rho-r)t}\right)-\tilde{u}\left(X_t^*e^{-(\rho-r)t}\right)}{\varepsilon}dt\right]=E\left[\int_0^\infty e^{-\rho t}u\left(C_t^*\right)dt\right].
\end{aligned}$$

It follows that

$$U_0^a\left(C^*\right)=E\left[\int_0^\infty e^{-\rho t}u\left(C_t^*\right)dt\right]\geq w. \tag{A.4}$$

Define $X^\varepsilon\left(\omega,s\right)=X^*\left(\omega,s\right)+\varepsilon\mathbf{1}_{A\times[t,\infty)}\left(\omega,s\right)$ for $\varepsilon>0$, $t>0$ and $A\in\mathcal{F}_t$, where $\mathbf{1}$ denotes an indicator function. A similar argument shows that $X^\varepsilon\in\mathcal{I}\left(X_0^*\right)$. Since $\mathcal{L}\left(X^\varepsilon\right)\geq\mathcal{L}\left(X^*\right)$, we obtain

$$\lim_{\varepsilon\downarrow 0}\frac{\mathcal{L}\left(X^\varepsilon\right)-\mathcal{L}\left(X^*\right)}{\varepsilon}\geq 0.$$

By a similar argument, we can show that

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{\mathcal{L}\left(X^\varepsilon\right)-\mathcal{L}\left(X^*\right)}{\varepsilon}\\
=\;&\lim_{\varepsilon\downarrow 0}E\left[\int_0^\infty e^{-rt}\frac{\tilde{u}\left(X_t^\varepsilon e^{-(\rho-r)t}\right)-\tilde{u}\left(X_t^*e^{-(\rho-r)t}\right)}{\varepsilon}dt\right]-E\left[1_Ae^{-\rho t}U_d\left(Y_t\right)\right]\\
=\;&E\left[\mathbf{1}_A\int_t^\infty e^{-\rho s}u\left(C_s^*\right)ds\right]-E\left[\mathbf{1}_Ae^{-\rho t}U_d\left(Y_t\right)\right]\geq 0.
\end{aligned}$$

Because $A$ is an arbitrary subset in $\mathcal{F}_t$, it follows that

$$E_t\left[\int_t^\infty e^{-\rho s}u\left(C_s^*\right)ds\right]\geq e^{-\rho t}U_d\left(Y_t\right),$$

or, equivalently,

$$E_t\left[\int_t^{\infty} e^{-\rho(s-t)}u\left(C_s^*\right)ds\right] \geq U_d\left(Y_t\right).$$

Multiplying $e^{-\rho t}$, integrating with respect to $X$, and taking expectations on both sides of the above inequality, we can derive that

$$E\left[\int_0^{\infty} e^{-\rho t}U_t^a\left(C^*\right)dX_t\right] \geq E\left[\int_0^{\infty} e^{-\rho t}U_d\left(Y_t\right)dX_t\right]. \tag{A.5}$$

Second, we show that (A.4) and (A.5) must hold with equality. To prove this, consider $\bar{X}^{\varepsilon} = X^*(1+\varepsilon)$ for $\varepsilon \in (-\delta, \delta)$. The convexity of $\tilde{u}$ implies that

$$\begin{aligned}\frac{\tilde{u}\left(\bar{X}_t^{-\delta}e^{-(\rho-r)t}\right) - \tilde{u}\left(X_t^* e^{-(\rho-r)t}\right)}{-\delta} &\leq \frac{\tilde{u}\left(\bar{X}_t^{\varepsilon}e^{-(\rho-r)t}\right) - \tilde{u}\left(X_t^* e^{-(\rho-r)t}\right)}{\varepsilon} \\ &\leq \frac{\tilde{u}\left(\bar{X}_t^{\delta}e^{-(\rho-r)t}\right) - \tilde{u}\left(X_t^* e^{-(\rho-r)t}\right)}{\delta},\end{aligned}$$

which implies $\bar{X}^{\varepsilon} \in \mathcal{I}\left(X_0^*(1+\varepsilon)\right)$. Since $\mathcal{L}\left(\bar{X}^{\varepsilon}\right) \geq \mathcal{L}\left(X^*\right)$, we obtain

$$\lim_{\varepsilon \downarrow 0}\frac{\mathcal{L}\left(\bar{X}^{\varepsilon}\right) - \mathcal{L}\left(X^*\right)}{\varepsilon} \geq 0 \text{ and } \lim_{\varepsilon \uparrow 0}\frac{\mathcal{L}\left(\bar{X}^{\varepsilon}\right) - \mathcal{L}\left(X^*\right)}{\varepsilon} \leq 0. \tag{A.6}$$

By the Dominated Convergence Theorem,

$$\lim_{\varepsilon \to 0}\frac{\mathcal{L}\left(\bar{X}^{\varepsilon}\right) - \mathcal{L}\left(X^*\right)}{\varepsilon} = E\left[\int_0^{\infty} e^{-\rho t}X_t^* u(C_t^*)dt\right] - E\left[\int_0^{\infty} e^{-\rho t}U_d\left(Y_t\right)dX_t^*\right] - X_0^* w,$$

which, according to (A.6), should be both nonnegative and nonpositive. It follows that

$$E\left[\int_0^{\infty} e^{-\rho t}X_t^* u(C_t^*)dt\right] - E\left[\int_0^{\infty} e^{-\rho t}U_d\left(Y_t\right)dX_t^*\right] - X_0^* w = 0. \tag{A.7}$$

By Lemma 1,

$$E\left[\int_0^{\infty} e^{-\rho t}X_t^* u(C_t^*)dt\right] = X_0^* U_0^a\left(C^*\right) + E\left[\int_0^{\infty} e^{-\rho t}U_t^a\left(C^*\right)dX_t^*\right].$$

Plugging this equation into (A.7) yields:

$$X_0^*\left(U_0^a\left(C^*\right) - w\right) + E\left[\int_0^{\infty} e^{-\rho t}\left(U_t^a\left(C^*\right) - U_d\left(Y_t\right)\right)dX_t^*\right] = 0.$$

Thus, (A.4) and (A.5) must hold with equality since $X_0^* > 0$.

Finally, we show that $C^*$ is optimal in the primal problem. By Theorem 1,

$$U^p\left(y, C^*\right) \leq \sup_{C \in \Gamma(y,w)} U^p\left(y, C\right) \leq \inf_{X \in \mathcal{I}(x), x>0} L\left(y, x, X\right) - xw \leq L\left(y, x^*, X^*\right) - x^* w. \tag{A.8}$$

Since $C^*$ and $X^*$ satisfy (21)-(22), it follows from Theorem 1 that all the inequalities in (A.8) must hold with equalities, and hence $C^*$ is in fact the optimal solution to the primal problem. In addition,

$$V\left(y, w\right) = \inf_{x>0}\ \tilde{V}\left(y, x\right) - xw,$$

as desired. Q.E.D.

**Proof of Theorem 3:** **Step 1.** Define a process

$$G_t^X = \int_0^t e^{-rs}\left(Y_s + \tilde{u}\left(Z_s\right)\right) ds - \int_0^t e^{-\rho s} U_d\left(Y_s\right) dX_s + e^{-rt} J\left(Y_t, Z_t\right),$$

for any $X \in \mathcal{I}(z)$ and $Z_t = e^{-(\rho - r)t} X_t$ with $X_0 = Z_0 = z$. We show that $G_t^X$ is a submartingale.

By the generalized Ito's Lemma (e.g., Harrison (1985)),

$$\begin{aligned}
e^{rt} dG_t^X &= \left(Y_t + \tilde{u}\left(Z_t\right)\right) dt - e^{(r-\rho)t} U_d\left(Y_t\right) dX_t + dJ\left(Y_t, Z_t\right) - rJ\left(Y_t, Z_t\right) dt \\
&= \left(Y_t + \tilde{u}\left(Z_t\right)\right) dt - e^{(r-\rho)t} U_d\left(Y_t\right) dX_t^c + J_y\left(Y_t, Z_t\right) \mu\left(Y_t\right) dt \\
&\quad + J_y\left(Y_t, Z_t\right) \sigma\left(Y_t\right) dB_t + \frac{1}{2} J_{yy}\left(Y_t, Z_t\right) \sigma^2\left(Y_t\right) dt \\
&\quad + J_z\left(Y_t, Z_t\right)\left(e^{(r-\rho)t} dX_t^c - (\rho - r) Z_t dt\right) - rJ\left(Y_t, Z_t\right) dt \\
&\quad + J\left(Y_t, Z_t\right) - J\left(Y_t, Z_{t-}\right) - e^{(r-\rho)t} U_d\left(Y_t\right) \Delta X_t \\
&= \left(Y_t + \tilde{u}\left(Z_t\right)\right) dt - (\rho - r) Z_t J_z\left(Y_t, Z_t\right) dt + J_y\left(Y_t, Z_t\right) \mu\left(Y_t\right) dt \\
&\quad + \frac{1}{2} J_{yy}\left(Y_t, Z_t\right) \sigma^2\left(Y_t\right) dt - rJ\left(Y_t, Z_t\right) dt \\
&\quad + e^{(r-\rho)t}\left(J_z\left(Y_t, Z_t\right) - U_d\left(Y_t\right)\right) dX_t^c + J_y\left(Y_t, Z_t\right) \sigma\left(Y_t\right) dB_t \\
&\quad + J\left(Y_t, Z_t\right) - J\left(Y_t, Z_{t-}\right) - e^{(r-\rho)t} U_d\left(Y_t\right) \Delta X_t,
\end{aligned}$$

where $\Delta X_t \equiv X_t - X_{t-}$ and $X^c$ is the continuous part of $X$. Thus, for $T \geq t$,

$$\begin{aligned}
G_T^X &= G_t^X + \underbrace{\int_t^T e^{-rs}\left(Y_s + \tilde{u}\left(Z_s\right) + \mathcal{A}J\left(Y_s, Z_s\right)\right) ds}_{I_1(T)} \\
&\quad + \underbrace{\int_t^T e^{-\rho s}\left(J_z\left(Y_s, Z_s\right) - U_d\left(Y_s\right)\right) dX_s^c}_{I_2(T)} + \underbrace{\int_t^T e^{-rs} J_y\left(Y_s, Z_s\right) \sigma\left(Y_s\right) dB_s}_{I_3(T)} \\
&\quad + \underbrace{\sum_{t \leq s \leq T} e^{-rs}\left(J\left(Y_s, Z_s\right) - J\left(Y_s, Z_{s-}\right) - e^{(r-\rho)s} U_d\left(Y_s\right) \Delta X_s\right)}_{\Sigma(T)}.
\end{aligned}$$

Taking expectations conditional on the information at time $t$, we obtain

$$E_t[G_T^X] = G_t^X + E_t\left[I_1(T)\right] + E_t\left[I_2(T)\right] + E_t\left[I_3(T)\right] + E_t\left[\Sigma(T)\right].$$

By the variational inequality and $dX_t^c \geq 0$, we can show that $E_t\left[I_1(T)\right] \geq 0$ and $E_t\left[I_2(T)\right] \geq 0$. By condition (28), $E_t\left[I_3(T)\right] = 0$. Using the variational inequality, we can also show that

$$J\left(Y_s, Z_s\right) - J\left(Y_s, Z_{s-}\right) - e^{(r-\rho)s} U_d\left(Y_s\right) \Delta X_s = \int_{Z_s - \Delta Z_s}^{Z_s} \left(J_z\left(Y_s, z\right) - U_d\left(Y_s\right)\right) dz \geq 0,$$

where $\Delta Z_s = e^{(r-\rho)s}\Delta X_s$. Thus, $E_t[\Sigma(T)] \geq 0$. It follows that $\{G_t^X\}_{t\geq 0}$ is a submartingale. This implies that

$$J(Y_0, Z_0) = G_0^X \leq E\left[G_t^X\right],$$

for all $t$. Taking limits and using $\lim_{t\to\infty} E\left[e^{-rt}J(Y_t, Z_t)\right] = 0$, we have

$$J(Y_0, Z_0) \leq E\left[\int_0^\infty e^{-rt}\left(Y_t + \tilde{u}(Z_t)\right)dt - \int_0^\infty e^{-\rho t}U_d(Y_t)\,dX_t\right].$$

**Step 2.** Show that $\{G_t^{X^*}\}_{t\geq 0}$ is a martingale. Note that $X^*$ is continuous (e.g., Harrison and Taksar (1983) and Harrison (1985)). We can use assumptions in the theorem to verify $E_t[I_1(T)] = E_t[I_2(T)] = E_t[\Sigma(T)] = 0$ for $X^*$.

The above two steps imply that

$$J(y, z) = \inf_{X\in\mathcal{I}(z)} E\left[\int_0^\infty e^{-rt}\left(Y_t + \tilde{u}(Z_t)\right)dt\right] - E\left[\int_0^\infty e^{-\rho t}U_d(Y_t)\,dX_t\right],$$

and $X^*$ attains the minimum. By assumption, $J(y,z)$ is strictly convex in $z$ and $J_z(y, z^*) = w$. Thus, $z^*$ achieves the minimum of $J(y,z)$. This implies that $X^*$ with $X_0^* = z^*$ solves the dual problem (14).

**Step 3.** Show that $\{C_t^*\}_{t\geq 0}$ is optimal. This follows from Theorem 2. The proof of the remaining results is trivial. Q.E.D.

**Proof of Theorem 4:** By the integrability condition (53),

$$E\left[\int_0^\infty e^{-\rho t}(1+H_t)\left(u^p(A(Y_t) - C_t)\right)^+ dt\right] < E\left[\int_0^\infty e^{-\rho t}(1+H_t)\left(u^p(A(Y_t))\right)^+ dt\right] < \infty.$$

By the principal's participation constraint (44) and an argument similar to that in the proof of Theorem 1, we can show that

$$E\left[\int_0^\infty e^{-\rho t}(1+H_t)\left(u^p(A(Y_t) - C_t)\right)^- dt\right] < \infty.$$

Thus,

$$E\left[\int_0^\infty e^{-\rho t}\left|u^p(A(Y_t) - C_t)\right| dt\right] \leq E\left[\int_0^\infty e^{-\rho t}(1+H_t)\left|u^p(A(Y_t) - C_t)\right| dt\right] < \infty.$$

It follows from Lemma 1 that

$$E\left[\int_0^\infty e^{-\rho t}H_t u^p\left(A(Y_t) - C_t\right)dt\right] = H_0 U^p(y, C) + E\left[\int_0^\infty e^{-\rho t}U_t^p\left(A(Y_t) - C\right)dH_t\right], \quad \text{(A.9)}$$

where $H_0 = h$.

By the definition of $\tilde{u}$ in (48),

$$u^p\left(A\left(Y_t\right)-C_t\right) \leq \tilde{u}\left(Y_t, X_t, H_t\right)-H_t u^p\left(A\left(Y_t\right)-C_t\right)-X_t u\left(C_t\right), \tag{A.10}$$

for any $C \in \Phi(y, w)$. It follows that

$$X_t u\left(C_t\right) \leq \tilde{u}\left(Y_t, X_t, H_t\right)-(1+H_t) u^p\left(A\left(Y_t\right)-C_t\right).$$

Since each term on the right-hand side of the above inequality is integrable, we deduce that

$$E\left[\int_0^{\infty} e^{-\rho t} X_t\left(u\left(C_t\right)\right)^{+} dt\right]<\infty.$$

By an argument similar to that in the proof of Theorem 1, we can derive (A.2).

Multiplying $e^{-\rho t}$ and taking expectations on both sides of (A.10), we obtain

$$\begin{aligned}
U^p(y, C) & \leq E\left[\int_0^{\infty} e^{-\rho t}\left(\tilde{u}\left(Y_t, X_t, H_t\right)-H_t u^p\left(A\left(Y_t\right)-C_t\right)-X_t u\left(C_t\right)\right) dt\right] \\
& = L(y, x, h, X, H)-E\left[\int_0^{\infty} e^{-\rho t} H_t u^p\left(A\left(Y_t\right)-C_t\right) dt\right]-E\left[\int_0^{\infty} e^{-\rho t} X_t u\left(C_t\right) dt\right] \\
& \quad +E\left[\int_0^{\infty} e^{-\rho t} U_d\left(Y_t\right) dX_t\right]+E\left[\int_0^{\infty} e^{-\rho t} U_d^p\left(A\left(Y_t\right)-Y_t\right) dH_t\right].
\end{aligned} \tag{A.11}$$

Plugging (A.2) and (A.9) into (A.11), we obtain

$$\begin{aligned}
U^p(y, C) & \leq L(y, x, h, X, H)-x w-E\left[\int_0^{\infty} e^{-\rho t} U_t^a(C) dX_t\right] \\
& \quad -h U^p(y, C)-E\left[\int_0^{\infty} e^{-\rho t} U_t^p(A(Y)-C) dH_t\right] \\
& \quad +E\left[\int_0^{\infty} e^{-\rho t} U_d\left(Y_t\right) dX_t\right]+E\left[\int_0^{\infty} e^{-\rho t} U_d^p\left(A\left(Y_t\right)-Y_t\right) dH_t\right] \\
& = L(y, x, h, X, H)-x w-h U^p(y, C)-E\left[\int_0^{\infty} e^{-\rho t}\left(U_t^a(C)-U_d\left(Y_t\right)\right) dX_t\right] \\
& \quad -E\left[\int_0^{\infty} e^{-\rho t}\left(U_t^p(A(Y)-C)-U_d^p\left(A\left(Y_t\right)-Y_t\right)\right) dH_t\right] \\
& \leq L(y, x, h, X, H)-x w-h U^p(y, C),
\end{aligned}$$

where the last inequality follows from the fact that $C \in \Phi(y, w)$. Thus, we obtain (57). Equalities hold if and only if (21)-(22) hold. Q.E.D.

**Proof of Theorem 5:** Because the idea for this proof is similar to that for Theorem 2, we shall sketch the main steps only. First, we show that $C^*$ defined in the theorem is sustainable and satisfies the participation constraint. Define $X^{\varepsilon} \equiv X^*+\varepsilon$ for $\varepsilon \in(0, \delta)$. The convexity of $\tilde{u}$

implies $E\left[\int_0^\infty e^{-\rho t}\left|\tilde{u}(X_t^\varepsilon, H_t^*, Y_t)\right| dt\right] < \infty$. Define $\mathcal{L}(X^\varepsilon, H^*)$ as in (49). Since $\mathcal{L}(X^\varepsilon, H^*) \geq \mathcal{L}(X^*, H^*)$, we obtain

$$\lim_{\varepsilon \downarrow 0} \frac{\mathcal{L}(X^\varepsilon, H^*) - \mathcal{L}(X^*, H^*)}{\varepsilon} \geq 0.$$

By the Dominated Convergence Theorem and a similar argument in the proof of Theorem 2, we can show that

$$U_0^a(C^*) = E\left[\int_0^\infty e^{-\rho t} u(C_t^*)\right] \geq w. \tag{A.12}$$

Define $X^\varepsilon = X^* + \varepsilon \mathbf{1}_{A \times [t,\infty)}$ for $\varepsilon > 0$, $t > 0$ and $A \in \mathcal{F}_t$. By a similar argument in the proof of Theorem 2,

$$\lim_{\varepsilon \downarrow 0} \frac{\mathcal{L}(X^\varepsilon, H^*) - \mathcal{L}(X^*, H^*)}{\varepsilon} = E\left[\mathbf{1}_A \int_t^\infty e^{-\rho s} u(C_s^*)\, ds\right] - E\left[\mathbf{1}_A e^{-\rho t} U_d(Y_t)\right] \geq 0.$$

Because $A$ is an arbitrary subset in $\mathcal{F}_t$, it follows that

$$U_t^a(C^*) = E_t\left[\int_t^\infty e^{-\rho(s-t)} u(C_s^*)\, ds\right] \geq U_d(Y_t). \tag{A.13}$$

Multiplying $e^{-\rho t}$, integrating with respect to $X$, and taking expectations on both sides of the inequality, we can derive that

$$E\left[\int_0^\infty e^{-\rho t} U_t^a(C^*)\, dX_t\right] \geq E\left[\int_0^\infty e^{-\rho t} U_d(Y_t)\, dX_t\right]. \tag{A.14}$$

Similarly, define $H^\varepsilon = H^* + \varepsilon \mathbf{1}_{A \times [t,\infty)}$ for $\varepsilon > 0$, $t > 0$ and $A \in \mathcal{F}_t$. By a similar argument, we can show that

$$U_t^p(A(Y) - C^*) \geq U_d^p(A(Y_t) - Y_t), \tag{A.15}$$

and

$$E\left[\int_0^\infty e^{-\rho t} U_t^p(A(Y) - C^*)\, dH_t\right] \geq E\left[\int_0^\infty e^{-\rho t} U_d^p(A(Y_t) - Y_t)\, dH_t\right]. \tag{A.16}$$

Second, we show below that (A.12), (A.14), and (A.16) must hold with equality. To prove this, consider $\bar{X}^\varepsilon = X^*(1+\varepsilon)$ for small $\varepsilon \in (-\delta, \delta)$. Since $\mathcal{L}(\bar{X}^\varepsilon, H^*) \geq \mathcal{L}(X^*, H^*)$, we obtain

$$\lim_{\varepsilon \downarrow 0} \frac{\mathcal{L}(\bar{X}^\varepsilon, H^*) - \mathcal{L}(X^*, H^*)}{\varepsilon} \geq 0 \text{ and } \lim_{\varepsilon \uparrow 0} \frac{\mathcal{L}(\bar{X}^\varepsilon, H^*) - \mathcal{L}(X^*, H^*)}{\varepsilon} \leq 0.$$

By the Dominated Convergence Theorem,

$$E\left[\int_0^\infty e^{-\rho t} X_t^* u(C_t^*) dt\right] - E\left[\int_0^\infty e^{-\rho t} U_d(Y_t)\, dX_t^*\right] - X_0^* w = 0.$$

Since

$$E\left[\int_0^\infty e^{-\rho t} X_t^* u(C_t^*) dt\right] = X_0^* U_0^a(C^*) + E\left[\int_0^\infty e^{-\rho t} U_t^a(C^*)\, dX_t^*\right],$$

we obtain

$$X_0^*\left(U_0^a\left(C^*\right)-w\right)+E\left[\int_0^\infty e^{-\rho t}\left(U_t^a\left(C^*\right)-U_d\left(Y_t\right)\right)dX_t^*\right]=0.$$

Thus, (A.12) and (A.14) must hold with equality. Similarly, we can define $\bar{H}^\varepsilon = H^*(1+\varepsilon)$ and use a similar argument to show that (A.16) holds with equality.

Finally, we show that $C^*$ is optimal in the primal problem. Since $C^* \in \Phi(y,w)$, it follows from Theorem 4 that

$$\begin{aligned} U^p\left(y,C^*\right) &\leq \sup_{C\in\Phi(y,w)} U^p\left(y,C\right) \leq \inf_{X\in\mathcal{I}(x),H\in\mathcal{I}(h),x>0,h\geq 0} \frac{L\left(y,x,h,X,H\right)}{1+h}-\frac{x}{1+h}w \\ &= \inf_{x>0,h\geq 0}\frac{\tilde{V}\left(y,x,h\right)}{1+h}-\frac{x}{1+h}w \\ &= \inf_{z>0}\tilde{V}\left(y,z,0\right)-zw = \inf_{X\in\mathcal{I},H\in\mathcal{I}(0)}\mathcal{L}\left(X,H\right) \\ &= L\left(y,x^*,0,X^*,H^*\right)-x^*w, \end{aligned}$$

where the first equality in the third line follows from the linear homogeneity of $\tilde{V}$ in $(x,1+h)$ and the change of variables $z=x/\left(1+h\right)$. By Theorem 4, all inequalities hold with equalities. Thus, $C^*$ is the optimal solution to the primal problem. In addition,

$$V\left(y,w\right)=\inf_{z>0}\tilde{V}\left(y,z,0\right)-zw=\inf_{x>0,h\geq 0}\frac{\tilde{V}\left(y,x,h\right)}{1+h}-\frac{x}{1+h}w,$$

as desired. Q.E.D.

**Proof of Proposition 2:** By the value-matching and super-contact conditions, we can derive a system of four nonlinear equations for four unknowns $(b_1,b_2,A_1,A_2)$ :

$$\frac{1}{\rho\alpha}b_1^{\frac{\alpha}{1-\alpha}}+A_1(1-\beta_1)b_1^{\frac{\alpha-\beta_1}{1-\alpha}}+A_2(1-\beta_2)b_1^{\frac{\alpha-\beta_2}{1-\alpha}} = \kappa, \tag{A.17}$$

$$\frac{1}{\rho}+A_1(1-\beta_1)(\alpha-\beta_1)b_1^{\frac{-\beta_1}{1-\alpha}}+A_2(1-\beta_2)(\alpha-\beta_2)b_1^{\frac{-\beta_2}{1-\alpha}} = 0, \tag{A.18}$$

$$\frac{1}{\rho-\mu}-\frac{b_2^{\frac{1}{1-\alpha}}}{\rho}+A_1(\beta_1-\alpha)b_2^{\frac{1-\beta_1}{1-\alpha}}+A_2(\beta_2-\alpha)b_2^{\frac{1-\beta_2}{1-\alpha}} = 0, \tag{A.19}$$

$$\frac{1}{\rho}+A_1(1-\beta_1)(\alpha-\beta_1)b_2^{\frac{-\beta_1}{1-\alpha}}+A_2(1-\beta_2)(\alpha-\beta_2)b_2^{\frac{-\beta_2}{1-\alpha}} = 0. \tag{A.20}$$

The proof of the proposition contains five steps.

**Step 1.** We can solve for $A_1$ and $A_2$ for any $b_2$ using (A.19) and (A.20) as follows. (Figure 8 plots $A_1(b_2)$ and $A_2(b_2)$.) Plugging (A.20) into (A.19), we rewrite (A.19) and (A.20) as

$$\begin{aligned} \frac{1}{\rho-\mu}+A_1\beta_1(\beta_1-\alpha)b_2^{\frac{1-\beta_1}{1-\alpha}}+A_2\beta_2(\beta_2-\alpha)b_2^{\frac{1-\beta_2}{1-\alpha}} &= 0, \\ \frac{1}{\rho}+A_1(\beta_1-1)(\beta_1-\alpha)b_2^{-\frac{\beta_1}{1-\alpha}}+A_2(\beta_2-1)(\beta_2-\alpha)b_2^{-\frac{\beta_2}{1-\alpha}} &= 0. \end{aligned}$$

This linear system of equations in $(A_1, A_2)$ gives

$$\begin{aligned}
\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} &= \begin{pmatrix} \beta_1(\beta_1-\alpha)b_2^{\frac{1-\beta_1}{1-\alpha}} & \beta_2(\beta_2-\alpha)b_2^{\frac{1-\beta_2}{1-\alpha}} \\ (\beta_1-1)(\beta_1-\alpha)b_2^{-\frac{\beta_1}{1-\alpha}} & (\beta_2-1)(\beta_2-\alpha)b_2^{-\frac{\beta_2}{1-\alpha}} \end{pmatrix}^{-1} \begin{pmatrix} -\frac{1}{\rho-\mu} \\ -\frac{1}{\rho} \end{pmatrix} \\
&= \begin{pmatrix} \rho(\beta_2-1)(\beta_2-\alpha)b_2^{\frac{\beta_1-1}{1-\alpha}} - (\rho-\mu)\beta_2(\beta_2-\alpha)b_2^{\frac{\beta_1}{1-\alpha}} \\ -\rho(\beta_1-1)(\beta_1-\alpha)b_2^{\frac{\beta_2-1}{1-\alpha}} + (\rho-\mu)\beta_1(\beta_1-\alpha)b_2^{\frac{\beta_2}{1-\alpha}} \end{pmatrix} m^{-1},
\end{aligned}$$

where $m \equiv -(\beta_1-\alpha)(\beta_2-\alpha)(\beta_2-\beta_1)\rho(\rho-\mu) > 0$. We show some properties of $(A_1(b_2), A_2(b_2))$ to be used later. First, $A_1 < 0$ if $b_2 > 1$. Because $\beta_2 > 1 > \alpha$, it is sufficient to verify that

$$\rho(\beta_2-1) - (\rho-\mu)\beta_2 b_2^{\frac{1}{1-\alpha}} \leq \rho(\beta_2-1) - (\rho-\mu)\beta_2 = \mu\beta_2 - \rho = -\frac{\sigma^2}{2}\beta_2(\beta_2-1) < 0.$$

Second, $A_2 < 0$ for $b_2 \in \left(1, \bar{b}\right)$, where $\bar{b} \equiv \left(\frac{\rho}{\rho-\mu}\frac{1-\beta_1}{-\beta_1}\right)^{1-\alpha}$. Because $\beta_1 < \alpha$, it is sufficient to verify that

$$-\rho(1-\beta_1) + (\rho-\mu)(-\beta_1)b_2^{\frac{1}{1-\alpha}} < 0,$$

which follows from $b_2 < \bar{b}$. Third, both $A_1$ and $A_2$ increase in $b_2 > 1$. The sign of $A_1'(b_2)$ is the same as the sign of $\rho(\beta_2-1)(\beta_1-1) - (\rho-\mu)\beta_2\beta_1 b_2^{\frac{1}{1-\alpha}}$, which is negative because

$$\begin{aligned}
\rho(\beta_2-1)(\beta_1-1) - (\rho-\mu)\beta_2\beta_1 b_2^{\frac{1}{1-\alpha}} &< \rho(\beta_2-1)(\beta_1-1) - (\rho-\mu)\beta_2\beta_1 \\
&= \rho + \mu\beta_2\beta_1 - \rho(\beta_2+\beta_1) \\
&= \rho - \mu\frac{\rho}{1/2\sigma^2} + \rho\left(\frac{\mu}{1/2\sigma^2} - 1\right) = 0.
\end{aligned}$$

Using the same steps, we can verify that $A_2'(b_2) < 0$.

**Step 2.** We show that there is a unique solution $b_1 \in (0,1)$ to equation (A.18) for any $b_2 \in \left(1, \bar{b}\right)$ and $(A_1(b_2) < 0, A_2(b_2) < 0)$. First, when $A_1$ and $A_2$ are fixed, then the function

$$f(A_1, A_2, b) \equiv \frac{1}{\rho} + A_1(1-\beta_1)(\alpha-\beta_1)b^{\frac{-\beta_1}{1-\alpha}} + A_2(1-\beta_2)(\alpha-\beta_2)b^{\frac{-\beta_2}{1-\alpha}}$$

is single-peaked in $b$ because

$$\frac{\partial f}{\partial b} = \frac{b^{\frac{-\beta_2}{1-\alpha}-1}}{1-\alpha}\left(-A_1\beta_1(1-\beta_1)(\alpha-\beta_1)b^{\frac{\beta_2-\beta_1}{1-\alpha}} - A_2\beta_2(1-\beta_2)(\alpha-\beta_2)\right).$$

Because $A_1 < 0$ and $\beta_1 < 0$, the sign of $\frac{\partial f}{\partial b}$ turns from positive to negative only once. That $f$ is single-peaked in $b$ implies that $f(A_1, A_2, b) = 0$ pins down two solutions, one of which is $b_2$ (as

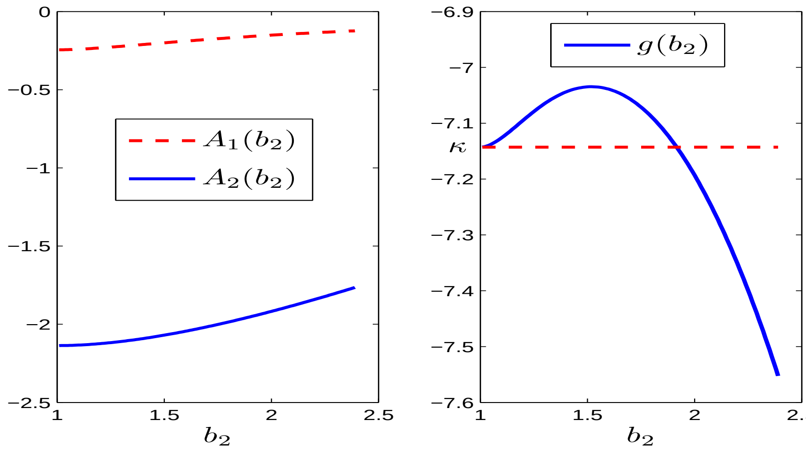


Figure 8: The functions $A_1(b_2)$, $A_2(b_2)$, and $g(b_2)$.

we already see in equation (A.20)). Second, we verify that $b_2$ is on the downside of $f$, which would imply that $b_1$ is on the upside of $f$. We have

$$\begin{aligned}
& \left.\frac{\partial f(A_1,A_2,b_2)}{\partial b}\right|_{A_1=A_1(b_2),A_2=A_2(b_2)} \\
= & \frac{1}{1-\alpha}\left(A_1\beta_1(\beta_1-1)(\alpha-\beta_1)b_2^{\frac{-\beta_1}{1-\alpha}-1}+A_2\beta_2(\beta_2-1)(\alpha-\beta_2)b_2^{\frac{-\beta_2}{1-\alpha}-1}\right) \\
= & \frac{1}{m(1-\alpha)}\Bigg(\left(\rho(\beta_2-1)(\beta_2-\alpha)b_2^{\frac{\beta_1-1}{1-\alpha}}-(\rho-\mu)\beta_2(\beta_2-\alpha)b_2^{\frac{\beta_1}{1-\alpha}}\right)\beta_1(\beta_1-1)(\alpha-\beta_1)b_2^{\frac{-\beta_1}{1-\alpha}-1} \\
& +\left(-\rho(\beta_1-1)(\beta_1-\alpha)b_2^{\frac{\beta_2-1}{1-\alpha}}+(\rho-\mu)\beta_1(\beta_1-\alpha)b_2^{\frac{\beta_2}{1-\alpha}}\right)\beta_2(\beta_2-1)(\alpha-\beta_2)b_2^{\frac{-\beta_2}{1-\alpha}-1}\Bigg) \\
= & \frac{b_2^{\frac{\alpha-2}{1-\alpha}}(\beta_2-\alpha)(\beta_1-\alpha)}{m(1-\alpha)}\Big(-\rho(\beta_2-1)(\beta_1-1)\beta_1+(\rho-\mu)\beta_2\beta_1(\beta_1-1)b_2^{\frac{1}{1-\alpha}} \\
& +\rho(\beta_1-1)\beta_2(\beta_2-1)-(\rho-\mu)\beta_1\beta_2(\beta_2-1)b_2^{\frac{1}{1-\alpha}}\Big) \\
= & \frac{b_2^{\frac{\alpha-2}{1-\alpha}}(\beta_2-\alpha)(\beta_1-\alpha)(\beta_2-\beta_1)}{m(1-\alpha)}\left(\rho(\beta_2-1)(\beta_1-1)-(\rho-\mu)\beta_2\beta_1b_2^{\frac{1}{1-\alpha}}\right) \\
< & \frac{b_2^{\frac{\alpha-2}{1-\alpha}}(\beta_2-\alpha)(\beta_1-\alpha)(\beta_2-\beta_1)}{m(1-\alpha)}\Big(\rho(\beta_2-1)(\beta_1-1)-(\rho-\mu)\beta_2\beta_1\Big)=0.
\end{aligned}$$

Therefore, $b_1$ is on the upside of the single-peaked function, i.e., $\frac{\partial f(A_1,A_2,b_1)}{\partial b_1}>0$.

We show below that $\frac{db_1}{db_2}<0$. Because

$$\frac{\partial f(A_1,A_2,b_1)}{\partial A_1}\frac{dA_1}{db_2}+\frac{\partial f(A_1,A_2,b_1)}{\partial A_2}\frac{dA_2}{db_2}+\frac{\partial f(A_1,A_2,b_1)}{\partial b_1}\frac{db_1}{db_2}=0,$$

we have

$$\frac{\partial f(A_1,A_2,b_1)}{\partial b_1}\frac{db_1}{db_2} = -\frac{\partial f(A_1,A_2,b_1)}{\partial A_1}\frac{dA_1}{db_2} - \frac{\partial f(A_1,A_2,b_1)}{\partial A_2}\frac{dA_2}{db_2} < 0,$$

where $\frac{\partial f(A_1,A_2,b_1)}{\partial A_1} > 0$, $\frac{dA_1}{db_2} > 0$, $\frac{\partial f(A_1,A_2,b_1)}{\partial A_2} > 0$, and $\frac{dA_2}{db_2} > 0$.

**Step 3.** We show that $b_1 = b_2 = 1$ is a solution. If $b_2 = 1$, then it follows from step 1 that

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} (\beta_2-\alpha)(\mu\beta_2-\rho) \\ -(\beta_1-\alpha)(\mu\beta_1-\rho) \end{pmatrix} m^{-1}.$$

Substituting the above and $b_1 = 1$ into (A.17) yields

$$\begin{aligned}
& \frac{1}{\rho\alpha} + A_1(1-\beta_1) + A_2(1-\beta_2) \\
= & \frac{1}{\rho\alpha} + A_1(1-\beta_1) + A_2(1-\beta_2) \\
= & \frac{1}{\rho\alpha} + \frac{(\beta_2-\alpha)(\mu\beta_2-\rho)(1-\beta_1) - (\beta_1-\alpha)(\mu\beta_1-\rho)(1-\beta_2)}{-(\beta_1-\alpha)(\beta_2-\alpha)(\beta_2-\beta_1)\rho(\rho-\mu)} \\
= & \frac{1}{\rho\alpha} + \frac{\mu(\beta_2+\beta_1) - (\alpha\mu+\rho) - \mu\beta_1\beta_2 + \alpha\rho}{-(\beta_1-\alpha)(\beta_2-\alpha)\rho(\rho-\mu)} \\
= & \frac{1}{\rho\alpha} + \frac{\mu + \frac{\sigma^2(\alpha-1)}{2}}{(\rho - \alpha\mu - \frac{\sigma^2\alpha(\alpha-1)}{2})\rho} = \kappa.
\end{aligned}$$

Therefore, $b_1 = b_2 = 1$ is a solution.

**Step 4.** We show that there is a unique solution $b_2 \in (1, \bar{b})$. First, the function

$$g(b_2) \equiv \frac{1}{\rho\alpha}(b_1(b_2))^{\frac{\alpha}{1-\alpha}} + (1-\beta_1)A_1(b_2)(b_1(b_2))^{\frac{\alpha-\beta_1}{1-\alpha}} + (1-\beta_2)A_2(b_2)(b_1(b_2))^{\frac{\alpha-\beta_2}{1-\alpha}}$$

is single-peaked in $b_2$, when variables $A_1$, $A_2$, $b_1$ are interpreted as functions of $b_2$. (Figure 8 plots the function $g(b_2)$.) We have

$$g'(b_2) = A_1'(b_2)(1-\beta_1)b_1^{\frac{\alpha-\beta_1}{1-\alpha}} + A_2'(b_2)(1-\beta_2)b_1^{\frac{\alpha-\beta_2}{1-\alpha}},$$

where

$$\begin{aligned}
A_1'(b_2) &= b_2^{\frac{\beta_1-1}{1-\alpha}-1}\frac{\beta_2-\alpha}{1-\alpha}\left(\rho(\beta_2-1)(\beta_1-1) - (\rho-\mu)\beta_2\beta_1 b_2^{\frac{1}{1-\alpha}}\right)m^{-1}, \\
A_2'(b_2) &= -b_2^{\frac{\beta_2-1}{1-\alpha}-1}\frac{\beta_1-\alpha}{1-\alpha}\left(\rho(\beta_2-1)(\beta_1-1) - (\rho-\mu)\beta_2\beta_1 b_2^{\frac{1}{1-\alpha}}\right)m^{-1}.
\end{aligned}$$

Hence the sign of $g'(b_2)$ equals that of

$$(\beta_2-\alpha)(1-\beta_1)b_1^{\frac{\beta_2-\beta_1}{1-\alpha}} + (\alpha-\beta_1)(1-\beta_2)b_2^{\frac{\beta_2-\beta_1}{1-\alpha}}, \tag{A.21}$$

which decrease in $b_2$ because $b_1^{\frac{\beta_2-\beta_1}{1-\alpha}}$ decreases in $b_2$, $b_2^{\beta_2-\beta_1}$ increases in $b_2$, and $(\alpha-\beta_1)(1-\beta_2) < 0$. Second, $g(b_2) > \kappa$ when $b_2 - 1 > 0$ is small. Because $g(1) = \kappa$, it is sufficient to show that $g'(b_2 = 1) > 0$. The sign of $g'(b_2)$ is positive because setting $b_1 = b_2 = 1$ in (A.21) yields

$$(\beta_2 - \alpha)(1 - \beta_1) + (\alpha - \beta_1)(1 - \beta_2) = (1 - \alpha)(\beta_2 - \beta_1) > 0.$$

Third, we show that

$$\lim_{b_2 \uparrow \bar{b}} g(b_2) < \kappa.$$

It follows from the formula for $A_2$ that $\lim_{b_2 \uparrow \bar{b}} A_2 = 0$. It follows from $b_1^{\frac{\beta_2}{1-\alpha}} \leq \rho(-A_2)(1-\beta_2)(\alpha - \beta_2)$ that $\lim_{b_2 \uparrow \bar{b}} b_1 = 0$. Thus,

$$\lim_{b_2 \uparrow \bar{b}} g(b_2) = \lim_{b_2 \uparrow \bar{b}} \frac{1}{\rho\alpha} b_1^{\frac{\alpha}{1-\alpha}} + A_2(1-\beta_2) b_1^{\frac{\alpha-\beta_2}{1-\alpha}} = \lim_{b_2 \uparrow \bar{b}} \left[ \frac{1}{\rho\alpha} + \frac{1}{\rho(\beta_2 - \alpha)} \right] b_1^{\frac{\alpha}{1-\alpha}}.$$

If $\alpha > 0$, then the above limit is zero and $\kappa > 0$. If $\alpha < 0$, then $[\frac{1}{\rho\alpha} + \frac{1}{\rho(\beta_2-\alpha)}] < 0$ and the above limit is $-\infty$. In both cases, the limit is less than $\kappa$. The Intermediate Value Theorem implies the existence of $b_2 \in \left(1, \bar{b}\right)$ such that $g(b_2) = \kappa$. This solution is unique because $g(b_2)$ is single-peaked and $g(1) = \kappa$.

**Step 5.** If there is a solution such that $b_2 > \bar{b}$, then according to step 1, $A_2 > 0$. The function $f$ in step 2 would be monotonically decreasing because

$$\frac{\partial f}{\partial b} = \frac{b^{\frac{-\beta_2}{1-\alpha}-1}}{1-\alpha} \left( -A_1\beta_1(1-\beta_1)(\alpha-\beta_1) b^{\frac{\beta_2-\beta_1}{1-\alpha}} - A_2\beta_2(1-\beta_2)(\alpha-\beta_2) \right) < 0.$$

This implies that $f(A_1, A_2, b) = 0$ has a unique solution and hence $b_1 = b_2 > 1$. If $b_1 = b_2$, then $g(b_2)$ in step 4 increases in $b_2$. This is because $g'(b_2)$ has the same sign as

$$(\beta_2 - \alpha)(1 - \beta_1) b_2^{\frac{\beta_2-\beta_1}{1-\alpha}} + (\alpha - \beta_1)(1 - \beta_2) b_2^{\frac{\beta_2-\beta_1}{1-\alpha}} = b_2^{\frac{\beta_2-\beta_1}{1-\alpha}} (\beta_2 - \beta_1)(1 - \alpha) > 0.$$

Therefore, $g(b_2) > g(1) = \kappa$, which means that $b_1 = b_2 > 1$ violates (A.17). Q.E.D.

**Proof of Proposition 3:** (necessity) Suppose that a non-autarkic risk sharing contract exists. First, we observe that $\bar{U}^a(y) = \bar{U}^a(1)$ and $U^p(y, v) = yU^p(1, v)$ for all $y > 0$. This is because the income process is homogeneous of degree one in its initial condition. More specifically, consider two income processes with initial conditions 1 and $y$, respectively. Denote the former as $(\{y_t\}_{t\geq 0}, y_0 = 1)$ and the latter as $\{yy_t\}_{t\geq 0}$. If $\{c_t\}_{t\geq 0}$ is a contract under $(\{y_t\}_{t\geq 0}, y_0 = 1)$, then $\{yc_t\}_{t\geq 0}$ is a contract under income $\{yy_t\}_{t\geq 0}$ such that the agent's surplus remains unchanged because $\ln(yc_t) - \ln(yy_t) = \ln(c_t) - \ln(y_t)$, and the principal's profit is $y$ times his profit under the contract $\{c_t\}_{t\geq 0}$ because $yy_t - yc_t = y(y_t - c_t)$.

Next, setting $y = 1$ in the Bellman equation presented in Section 6.4. and using the above homogeneity property, we obtain

$$
\begin{aligned}
U^p(1,v) = \max_{v_b,v_g} \quad & (1 - e^{(v - \frac{1}{1+\rho dt}(0.5v_b + 0.5v_g))/dt})dt \\
& + \frac{1}{1+\rho dt}\left[0.5(1-\sigma\sqrt{dt})U^p(1,v_b) + 0.5(1+\sigma\sqrt{dt})U^p(1,v_g)\right] \\
\text{subject to} \quad & v_b \in [0, \bar{U}^a(1)], \quad v_g \in [0, \bar{U}^a(1)].
\end{aligned}
$$

By the fact that $1 - x \leq e^{-x}$ for any $x$,

$$
(1 - e^{(v - \frac{1}{1+\rho dt}(0.5v_b + 0.5v_g))/dt})dt \leq \frac{1}{1+\rho dt}(0.5v_b + 0.5v_g) - v,
$$

with equality if and only if $\frac{1}{1+\rho dt}(0.5v_b + 0.5v_g) = v$. Thus, $U^p(1,v)$ is below the value function $M(v)$ defined in the following problem:

$$
\begin{aligned}
M(v) = \max_{v_b,v_g} \quad & \frac{1}{1+\rho dt}(0.5v_b + 0.5v_g) - v \\
& + \frac{1}{1+\rho dt}\left[0.5(1-\sigma\sqrt{dt})M(v_b) + 0.5(1+\sigma\sqrt{dt})M(v_g)\right] \\
\text{subject to} \quad & v_b \in [0, \bar{U}^a(1)], \quad v_g \in [0, \bar{U}^a(1)].
\end{aligned}
$$

Solving the above linear Bellman equation yields

$$
M(v) = M(0) - v = \frac{0.5\sigma\sqrt{dt}}{\rho dt}\bar{U}^a(1) - v.
$$

In addition, the solution satisfies $\frac{1}{1+\rho dt}(0.5v_b + 0.5v_g) \neq v$. Thus, $U^p(1,v) < M(v)$.

Because $U^p(1, \bar{U}^a(1)) = 0$ and $U^p(1, \bar{U}^a(1)) < M(\bar{U}^a(1))$, we have

$$
0 < M(\bar{U}^a(1)) = \frac{0.5\sigma\sqrt{dt} - \rho dt}{r\sqrt{dt}}\bar{U}^a(1).
$$

When a non-autarkic risk sharing contract exists, $\bar{U}^a(1) > 0$. The above inequality implies $\sigma\sqrt{dt} > 2\rho dt$.

(sufficiency) Suppose $\sigma\sqrt{dt} > 2\rho dt$. Consider a contract (not necessarily optimal) in which the agent's consumption satisfies

$$
\ln(c_t) - \ln(y_t) = \begin{cases} -\epsilon, & \text{if } \frac{y_t}{y_{t-dt}} = 1 + \sigma\sqrt{dt}; \\ (1+2\rho dt)\epsilon, & \text{if } \frac{y_t}{y_{t-dt}} = 1 - \sigma\sqrt{dt}. \end{cases}
$$

Under a good shock (i.e., $\frac{y_t}{y_{t-dt}} = 1+\sigma\sqrt{dt}$), denote the principal and the agent's continuation values as $U_g^p(y_t,\epsilon)$ and $U_g^a(y_t,\epsilon)$. Similarly, denote the continuation values under a bad shock as $U_b^p(y_t,\epsilon)$ and $U_b^a(y_t,\epsilon)$. They satisfy the Bellman equations:

$$\begin{aligned}
U_g^p(y,\epsilon) &= \left(1-e^{-\epsilon}\right)ydt+\frac{1}{1+\rho dt}\left[\frac{1}{2}U_g^p((1+\sigma\sqrt{dt})y,\epsilon)+\frac{1}{2}U_b^p((1-\sigma\sqrt{dt})y,\epsilon)\right],\\
U_b^p(y,\epsilon) &= \left(1-e^{(1+2\rho dt)\epsilon}\right)ydt+\frac{1}{1+\rho dt}\left[\frac{1}{2}U_g^p((1+\sigma\sqrt{dt})y,\epsilon)+\frac{1}{2}U_b^p((1-\sigma\sqrt{dt})y,\epsilon)\right],\\
U_g^a(y,\epsilon) &= \left(\ln(y)-\epsilon\right)dt+\frac{1}{1+\rho dt}\left[\frac{1}{2}U_g^a((1+\sigma\sqrt{dt})y)+\frac{1}{2}U_b^a((1-\sigma\sqrt{dt})y)\right],\\
U_b^a(y,\epsilon) &= \left(\ln(y)+(1+2\rho dt)\epsilon\right)dt+\frac{1}{1+\rho dt}\left[\frac{1}{2}U_g^a((1+\sigma\sqrt{dt})y)+\frac{1}{2}U_g^a((1-\sigma\sqrt{dt})y)\right].
\end{aligned}$$

We can verify that $U_g^a(y,\epsilon)=U_d(y)$ and $U_b^a(y,\epsilon)=U_d(y)+2(1+\rho dt)\epsilon dt$, hence the agent's participation constraints are satisfied. Next we examine the principal's participation constraints $U_g^p(y,\epsilon)\geq 0$ and $U_b^p(y,\epsilon)\geq 0$. Since $U_g^p(y,\epsilon)>U_b^p(y,\epsilon)$, it is sufficient to check whether $U_b^p(y,\epsilon)\geq 0$ only.

We can easily guess and verify that $U_g^p(y,\epsilon)=U_g^p(1,\epsilon)y$ and $U_b^p(y,\epsilon)=U_b^p(1,\epsilon)y$ for all $y>0$. Hence, the above Bellman equations can be rewritten as

$$\begin{aligned}
U_g^p(y,\epsilon) &= \left(1-e^{-\epsilon}\right)ydt+\frac{1}{1+\rho dt}\left(\frac{1}{2}(1+\sigma\sqrt{dt})U_g^p(y,\epsilon)+\frac{1}{2}(1-\sigma\sqrt{dt})U_b^p(y,\epsilon)\right),\\
U_b^p(y,\epsilon) &= \left(1-e^{(1+2\rho dt)\epsilon}\right)ydt+\frac{1}{1+\rho dt}\left(\frac{1}{2}(1+\sigma\sqrt{dt})U_g^p(y,\epsilon)+\frac{1}{2}(1-\sigma\sqrt{dt})U_b^p(y,\epsilon)\right).
\end{aligned}$$

Solving $U_b^p(y,\epsilon)$ yields

$$U_b^p(y,\epsilon)=\left((1+\rho dt)\left(1-e^{(1+2\rho dt)\epsilon}\right)+\frac{1+\sigma\sqrt{dt}}{2}\left(e^{(1+2\rho dt)\epsilon}-e^{-\epsilon}\right)\right)\frac{y}{\rho}.$$

When $\epsilon=0$, $U_b^p(y,\epsilon)=0$. We can then compute that

$$\begin{aligned}
\left.\frac{\partial U_b^p(y,\epsilon)}{\partial\epsilon}\right|_{\epsilon=0} &= \left(-(1+\rho dt)(1+2\rho dt)+\frac{1+\sigma\sqrt{dt}}{2}(2+2\rho dt)\right)\frac{y}{\rho}\\
&= \frac{y(1+\rho dt)}{r}\left(\sigma\sqrt{dt}-2\rho dt\right)>0.
\end{aligned}$$

This shows that the non-autarkic risk-sharing contract constructed before is enforceable and the principal is better off. Q.E.D.

# B Additional Proofs for Section 4

In this appendix, we verify that the solution in Section 4 satisfies the conditions in Theorem 3. This consists of six steps.

**Step 1.** We verify the transversality condition (29). For $y \leq (z/b)^{\frac{1}{1-\alpha}}$,

$$\begin{aligned} |J(y,z)| &= \left| \frac{y}{r-\mu} + \frac{(1-\alpha)^2}{(\rho - \alpha r)\alpha} z^{\frac{1}{1-\alpha}} + A z^{\frac{1-\beta}{1-\alpha}} y^{\beta} \right| \\ &\leq \left( \frac{1}{(r-\mu) b^{\frac{1}{1-\alpha}}} + \frac{(1-\alpha)^2}{(\rho - \alpha r)|\alpha|} + \frac{|A|}{b^{\frac{\beta}{1-\alpha}}} \right) z^{\frac{1}{1-\alpha}}. \end{aligned}$$

The integrability condition (16) implies that

$$E\left[ \int_0^{\infty} e^{-rt} \left| \frac{1-\alpha}{\alpha} Z_t^{\frac{1}{1-\alpha}} \right| dt \right] < \infty.$$

We then deduce that

$$\begin{aligned} 0 &= \lim_{t\to\infty} E\left[ \int_t^{\infty} e^{-rs} Z_s^{\frac{1}{1-\alpha}} ds \right] = \lim_{t\to\infty} E\left[ \int_t^{\infty} e^{-rs} \left( e^{(r-\rho)s} X_s \right)^{\frac{1}{1-\alpha}} ds \right] \\ &\geq \lim_{t\to\infty} E\left[ X_t^{\frac{1}{1-\alpha}} \int_t^{\infty} e^{-rs + \frac{r-\rho}{1-\alpha}s} ds \right] = \frac{\rho - \alpha r}{1-\alpha} \lim_{t\to\infty} E\left[ e^{-rt} Z_t^{\frac{1}{1-\alpha}} \right], \end{aligned}$$

where the inequality follows from the fact that $X$ is a nonnegative increasing process and where we have used the fact that $\rho > \alpha r$. Thus,

$$\lim_{t\to\infty} E\left[ e^{-rt} Z_t^{\frac{1}{1-\alpha}} \right] = 0.$$

implying that the transversality condition $\lim_{t\to\infty} E\left[e^{-rt} J(Y_t, Z_t)\right] = 0$ holds for $Y_t \leq (Z_t/b)^{\frac{1}{1-\alpha}}$. For $z < by^{1-\alpha}$, equation (41) implies that

$$|J(y,z)| < Ky,$$

for some constant $K > 0$. Thus, the transversality condition holds for $Y_t > (Z_t/b)^{\frac{1}{1-\alpha}}$ by Assumption 1.

**Step 2.** We check condition (28). It is sufficient to show that

$$E\left[ \int_0^{\infty} \left( e^{-rt} J_y(Y_t, Z_t) \sigma Y_t \right)^2 dt \right] < \infty .$$

We can show that

$$|J_y(y,z)| = \left| \frac{1}{r-\mu} + A z^{\frac{1-\beta}{1-\alpha}} y^{\beta - 1} \beta \right| \leq \left| \frac{1}{r-\mu} \right| + |A| \, b^{\frac{1-\beta}{1-\alpha}}, \quad \text{for } z \geq by^{1-\alpha},$$

and

$$|J_y(y,z)| = \left| z U_d'(y) + K_1 \right| \leq K_2, \qquad \text{for } z < by^{1-\alpha},$$

where $K_1$ and $K_2$ are some constant terms. Thus, we only need to verify

$$E\left[ \int_0^{\infty} \left( e^{-rt} Y_t \right)^2 dt \right] < \infty.$$

This is true if $r > \mu + \sigma^2/2$.

**Step 3.** We verify that the dual value function $J$ in (41) satisfies the variational inequality in the jump region. That is, if $z < by^{1-\alpha}$, then

$$rJ\left(y,z\right) < y + \frac{1-\alpha}{\alpha} z^{\frac{1}{1-\alpha}} + (r-\rho) z J_z\left(y,z\right) + J_y\left(y,z\right)\mu y + \frac{\sigma^2}{2} y^2 J_{yy}\left(y,z\right).$$

Because the left-hand side equals the right-hand side at $z = by^{1-\alpha}$, it is sufficient to show that the derivative of the left-hand side with respect to $z$ is above that of the right-hand side with respect to $z$. That is,

$$rJ_z\left(y,z\right) > \frac{z^{\frac{\alpha}{1-\alpha}}}{\alpha} + (r-\rho)J_z\left(y,z\right) + (r-\rho)zJ_{zz}\left(y,z\right) + J_{yz}\left(y,z\right)\mu y + \frac{\sigma^2}{2} y^2 J_{yyz}\left(y,z\right).$$

It follows from $J_z\left(y,z\right) = U_d(y)$ that $J_{zz}\left(y,z\right) = 0$, $J_{yz}\left(y,z\right) = U_d'(y)$, and $J_{yyz}\left(y,z\right) = U_d''(y)$. The above inequality becomes

$$rU_d(y) > \frac{z^{\frac{\alpha}{1-\alpha}}}{\alpha} + (r-\rho)U_d(y) + U_d'(y)\mu y + \frac{\sigma^2}{2} y^2 U_d''(y),$$

which, after simplification, is

$$\frac{y^\alpha}{\alpha} > \frac{z^{\frac{\alpha}{1-\alpha}}}{\alpha}.$$

To prove this inequality, we will show $b < 1$, which implies that $y^{1-\alpha} > by^{1-\alpha} > z$ as desired. By (35) and (40), to prove that

$$b = \left(\frac{\left(\rho - \alpha r\right)\left(\beta - \alpha\right)}{\beta(1-\alpha)(\rho - \alpha\mu - \alpha\left(\alpha - 1\right)\sigma^2/2)}\right)^{\frac{1-\alpha}{\alpha}} < 1,$$

it is sufficient to show that $\frac{(\rho-\alpha r)(\beta-\alpha)}{\beta(1-\alpha)(\rho-\alpha\mu-\alpha(\alpha-1)\sigma^2/2)}$ is less than 1 when $\alpha > 0$, and is greater than 1 when $\alpha < 0$. Because both the numerator and the denominator are positive, it is equivalent to proving that $\beta(1-\alpha)(\rho - \alpha\mu - \alpha\left(\alpha-1\right)\sigma^2/2) - \left(\rho - \alpha r\right)\left(\beta - \alpha\right)$ has the same sign as $\alpha$. We can show that

$$\begin{aligned}
& \beta(1-\alpha)(\rho - \alpha\mu - \alpha\left(\alpha - 1\right)\sigma^2/2) - \left(\rho - \alpha r\right)\left(\beta - \alpha\right) \\
= \; & \beta\alpha(1-\alpha)(-\mu + (1-\alpha)\sigma^2/2) + \beta(1-\alpha)\rho - \left(\rho - \alpha r\right)\left(\beta - \alpha\right) \\
= \; & \beta\alpha(1-\alpha)(-\mu + (1-\alpha)\sigma^2/2) + \alpha((1-\beta)(\rho - r) + (1-\alpha)r) \\
= \; & \beta\alpha(1-\alpha)\left(-\mu + (1-\alpha)\sigma^2/2 + \frac{(1-\beta)(\rho-r)}{(1-\alpha)\beta} + \frac{r}{\beta}\right) \\
= \; & \beta\alpha(1-\alpha)\left(-\mu + (1-\alpha)\sigma^2/2 + \mu + \sigma^2\left(\beta - 1\right)/2\right) \\
= \; & \beta\alpha(1-\alpha)\sigma^2\left(\beta - \alpha\right)/2,
\end{aligned}$$

where the fourth equality uses (38). The sign of the last line is determined by $\alpha$ because $\beta(1-\alpha)\sigma^2(\beta-\alpha)>0$.

**Step 4.** We verify that $J_z(y,z)\geq U_d(y)$ for all $(z,y)$ in the no-jump region for $J$ defined in (37). If $z\geq by^{1-\alpha}$, then $J_z(y,z)\geq J_z(y,by^{1-\alpha})=U_d(y)$, where the inequality follows from $J_{zz}\geq 0$ and the equality follows from the value-matching condition.

**Step 5.** By the solution in Section 4, (30) and (31) hold. We need to check that $X^*\in\mathcal{I}(z)$ and the integrability conditions stated in the theorem hold. Since $\tilde{u}(z)=\frac{1-\alpha}{\alpha}z^{\frac{1}{1-\alpha}}$, we need to show

$$E\left[\int_0^\infty e^{-rt}\left(Z_t^*\right)^{\frac{1}{1-\alpha}}dt\right] < \infty, \tag{B.1}$$

$$E\left[\int_0^\infty e^{-\rho t}\left|U_d\left(Y_t\right)\right|dX_t^*\right] < \infty, \tag{B.2}$$

$$\begin{aligned}
E\left[\int_0^\infty e^{-rt}\left(e^{(r-\rho)t}\left(X_t^*+\delta\right)\right)^{\frac{1}{1-\alpha}}dt\right] &< \infty,\\
E\left[\int_0^\infty e^{-rt}\left(e^{(r-\rho)t}\left(X_t^*\left(1+\delta\right)\right)\right)^{\frac{1}{1-\alpha}}dt\right] &< \infty,\\
E\left[\int_0^\infty e^{-rt}\left(e^{(r-\rho)t}\left(X_t^*\left(1-\delta\right)\right)\right)^{\frac{1}{1-\alpha}}dt\right] &< \infty.
\end{aligned}$$

It is sufficient to check (B.1) and (B.2) since the last two integrals can be similarly checked using $Z_t^*=e^{(r-\rho)t}X_t^*$ and since we can derive

$$\begin{aligned}
E\left[\int_0^\infty e^{-rt}\left(e^{(r-\rho)t}\left(X_t^*+\delta\right)\right)^{\frac{1}{1-\alpha}}dt\right] &< E\left[\int_0^\infty e^{-rt}\left(e^{(r-\rho)t}\left(X_t^*\left(1+\delta/X_0^*\right)\right)\right)^{\frac{1}{1-\alpha}}dt\right]\\
&= \left(1+\delta/z\right)^{\frac{1}{1-\alpha}}E\left[\int_0^\infty e^{-rt}\left(Z_t^*\right)^{\frac{1}{1-\alpha}}dt\right].
\end{aligned}$$

To check (B.1), it suffices to use (43) to show

$$E\left[\int_0^\infty e^{-\frac{\rho-\alpha r}{1-\alpha}t}M_t dt\right]<\infty,\quad \text{where } M_t=\sup_{s\in[0,t]}Y_s e^{\frac{(\rho-r)s}{1-\alpha}}.$$

We will show that

$$E\left[\int_0^\infty e^{-\frac{\rho-\alpha r}{1-\alpha}t}M_t dt\right]=\frac{E\left[\int_0^\infty M_t d\left(-e^{-\frac{\rho-\alpha r}{1-\alpha}t}\right)\right]}{\frac{\rho-\alpha r}{1-\alpha}}<\infty.$$

Pick some $\epsilon>0$, and for $n=0,1,2,...$ define a sequence of stopping times $\tau_n\equiv\inf_{t\geq 0}\{t:M_t=Y_0(1+\epsilon)^n\}$. Since $M_t\leq Y_0(1+\epsilon)^{n+1}$ for $t\in[\tau_n,\tau_{n+1})$,

$$\begin{aligned}
E\left[\int_0^\infty M_t d\left(-e^{-\frac{\rho-\alpha r}{1-\alpha}t}\right)\right] &\leq E\left[\sum_{n=0}^\infty Y_0(1+\epsilon)^{n+1}\left(e^{-\frac{\rho-\alpha r}{1-\alpha}\tau_n}-e^{-\frac{\rho-\alpha r}{1-\alpha}\tau_{n+1}}\right)\right]\\
&= \sum_{n=0}^\infty Y_0(1+\epsilon)^{n+1}\left(E\left[e^{-\frac{\rho-\alpha r}{1-\alpha}\tau_n}\right]-E\left[e^{-\frac{\rho-\alpha r}{1-\alpha}\tau_{n+1}}\right]\right).
\end{aligned}$$

By Harrison (1985) or Stokey (2008), we can compute

$$E\left[e^{-\frac{\rho-\alpha r}{1-\alpha}\tau_n}\right] = (1+\epsilon)^{-\beta n},$$

where $\beta > 1$ satisfies (38). Therefore,

$$\begin{aligned} &\sum_{n=0}^{\infty} Y_0(1+\epsilon)^{n+1}\left(E\left[e^{-\frac{\rho-\alpha r}{1-\alpha}\tau_n}\right] - E\left[e^{-\frac{\rho-\alpha r}{1-\alpha}\tau_{n+1}}\right]\right) \\ = \quad & Y_0((1+\epsilon)^{\beta}-1)\sum_{n=0}^{\infty}\left((1+\epsilon)^{1-\beta}\right)^{n+1} < \infty, \end{aligned}$$

as desired.

To show (B.2), we use $U_d(y) = \kappa y^{\alpha}$ and define

$$G(y,z) \equiv E\left[\int_0^{\infty} e^{-\rho t} Y_t^{\alpha} dX_t^*\right], \quad \text{for } z \geq b y^{1-\alpha}.$$

Then, as in the proof of Theorem 3 for the HJB equation, $G$ satisfies

$$rG(y,z) = G_z(y,z)(r-\rho)z + G_y(y,z)\mu y + \frac{\sigma^2}{2} G_{yy}(y,z) y^2,$$

subject to

$$G_z(y,z)|_{z=by^{1-\alpha}} = -y^{\alpha}.$$

Solving yields:

$$G(y,z) = b^{\frac{\beta-\alpha}{1-\alpha}} \frac{1-\alpha}{\beta-1} z^{\frac{1-\beta}{1-\alpha}} y^{\beta}.$$

We also need to check that this solution satisfies the transversality condition:

$$\lim_{t\to\infty} E\left[e^{-rt} G(Y_t, Z_t^*)\right] = 0.$$

We only need to show that

$$\lim_{t\to\infty} E\left[e^{-rt}(Z_t^*)^{\frac{1-\beta}{1-\alpha}} Y_t^{\beta}\right] = 0.$$

This follows from

$$\lim_{t\to\infty} E\left[e^{-rt}(Z_t^*)^{\frac{1-\beta}{1-\alpha}} Y_t^{\beta}\right] \leq \lim_{t\to\infty} E\left[e^{-rt}\left(bY_t^{1-\alpha}\right)^{\frac{1-\beta}{1-\alpha}} Y_t^{\beta}\right] = \lim_{t\to\infty} E\left[b^{\frac{1-\beta}{1-\alpha}} e^{-rt} Y_t\right] = 0,$$

where the last equality follows from $r > \mu$ and where we have used the fact that $\beta > 1$ and $Z_t^* \geq bY_t^{1-\alpha}$.

**Step 6.** We show that $J(y,z)$ is strictly convex in $z$ in the no-jump region. We also derive a unique solution to (42).

The convexity follows from the fact that

$$
\begin{aligned}
J_{zz}(y,z) &= \frac{1}{(\rho - r\alpha)} z^{\frac{2\alpha-1}{1-\alpha}} + A\frac{1-\beta}{1-\alpha}\frac{\alpha-\beta}{1-\alpha} z^{\frac{2\alpha-\beta-1}{1-\alpha}} y^{\beta} \\
&> \frac{1}{(\rho - r\alpha)} z^{\frac{2\alpha-1}{1-\alpha}} + A\frac{1-\beta}{1-\alpha}\frac{\alpha-\beta}{1-\alpha} z^{\frac{2\alpha-\beta-1}{1-\alpha}} \left(\frac{z}{b}\right)^{\frac{\beta}{1-\alpha}} = 0,
\end{aligned}
$$

for $z > by^{1-\alpha}$, where the last equality uses the super-contact condition. Because $J_z(y,z)$ is strictly increasing in $z$ and

$$
\lim_{z\to\infty} J_z(y,z) = \begin{cases} \infty & \text{if } \alpha > 0; \\ 0 & \text{if } \alpha < 0, \end{cases}
$$

it follows from the Intermediate Value Theorem that the solution for $z$ exists in equation (42) as long as $w$ belongs to the range of the utility function (i.e., $w \in [U_d(y), \infty)$ when $\alpha > 0$ and $w \in [U_d(y), 0)$ when $\alpha < 0$).

# C Example III

In this appendix, we solve an example from Ljungqvist and Sargent (2004, Chapter 20) in which both the principal and the agent are risk averse. We consider a symmetric setup. Let $u^p(c) = u(c) = -e^{-\gamma c}$, where $\gamma > 0$ represents the coefficient of absolute risk aversion. The agent and the principal have incomes $Y_t = \sigma B_t$ and $-Y_t = -\sigma B_t$, respectively, where $B_t$ is a standard Brownian motion. In this case, their incomes are perfectly negatively correlated. This example does not satisfy some assumptions in our general theory developed before. In particular, consumption can be negative. Nevertheless, our key insights still apply and we shall proceed to derive the efficient contract since exponential utility is widely used in the contracting literature.

For this example, the dual function is given by

$$
\tilde{u}(x,h) = \max_{c} \; (1+h)\, u(-c) + x u(c) = -2\sqrt{x(1+h)},
$$

and the optimal consumption rule is given by

$$
c^* = \frac{1}{2\gamma} \ln\left(\frac{x}{1+h}\right).
$$

Let the outside value be the autarky value so that

$$
U_d(y) = \kappa e^{-\gamma y}, \quad U_d^p(-y) = \kappa e^{\gamma y}, \quad y \in \mathbb{R},
$$

where $\kappa \equiv -\left(\rho - \gamma^2\sigma^2/2\right)^{-1} < 0$. We assume $\rho > \gamma^2\sigma^2/2$ so that the autarky value is finite.

Conjecture that the no-jump region is given by

$$\left\{(y,x,h)\in\mathbb{R}\times\mathbb{R}_{++}\times\mathbb{R}_{+}:\frac{x}{1+h}\in\left[be^{2\gamma y},b^{-1}e^{2\gamma y}\right]\right\},$$

where $0<b<1$ is a constant to be determined. We can verify that the dual value function in the no-jump region takes the following form:

$$\tilde{V}\left(y,x,h\right)=\frac{-2\sqrt{x\left(1+h\right)}}{\rho}+Ax^{\frac{1-\beta}{2}}(1+h)^{\frac{1+\beta}{2}}e^{\gamma\beta y}+A(1+h)^{\frac{1-\beta}{2}}x^{\frac{1+\beta}{2}}e^{-\gamma\beta y},$$

where $\beta\equiv\sqrt{2\rho}\left(\gamma\sigma\right)^{-1}>1$ and $A$ is a constant to be determined.

In the jump region, we can verify that for $x<be^{2\gamma y}\left(1+h\right)$,

$$\tilde{V}\left(y,x,h\right)=\left(x-\left(1+h\right)be^{2\gamma y}\right)U_d\left(y\right)+\tilde{V}\left(y,\left(1+h\right)be^{2\gamma y},h\right),$$

and for $x>b^{-1}e^{2\gamma y}\left(1+h\right)$,

$$\tilde{V}\left(y,x,h\right)=\left(1+h-xbe^{-2\gamma y}\right)U_d^p\left(-y\right)+\tilde{V}\left(y,x,xbe^{-2\gamma y}-1\right).$$

The constants $A$ and $b$ are determined by the value-matching and super-contact conditions. Due to symmetry, we only need to use these conditions on one of the two boundaries. Without loss of generality, we use the lower boundary. By the value-matching condition, $\lim_{\frac{x}{1+h}\downarrow be^{2\gamma y}}\tilde{V}_x\left(y,x,h\right)=U_d\left(y\right)$, and the super-contact condition, $\lim_{\frac{x}{1+h}\downarrow be^{2\gamma y}}\tilde{V}_{xx}\left(y,x,h\right)=0$, we can derive

$$\begin{aligned}-\frac{2}{\rho}+A\left((1-\beta)\,b^{\frac{-\beta}{2}}+(1+\beta)\,b^{\frac{\beta}{2}}\right)&=\kappa,\\ \frac{2}{\rho}+A(\beta^2-1)(b^{\frac{\beta}{2}}+b^{-\frac{\beta}{2}})&=0.\end{aligned}$$

Simplifying yields one equation for $b$,

$$\frac{2\beta}{\beta^2-1}\frac{1}{1+b^{\beta}}-\rho\kappa b^{\frac{1}{2}}-\frac{\beta}{\beta-1}=0. \tag{C.1}$$

**Proposition 4** *Suppose that $\gamma>0$, $\sigma>0$, and $\rho>\gamma^2\sigma^2/2$. Then there are two solutions to the above equation. One satisfies $b\in(0,1)$ and the other is degenerate ($b=1$).*

**Proof:** Set $l=b^{\frac{1}{2}}$ and rewrite (C.1) as

$$\frac{2\beta}{\beta^2-1}\frac{1}{1+l^{2\beta}}-\rho\kappa l-\frac{\beta}{\beta-1}=0. \tag{C.2}$$

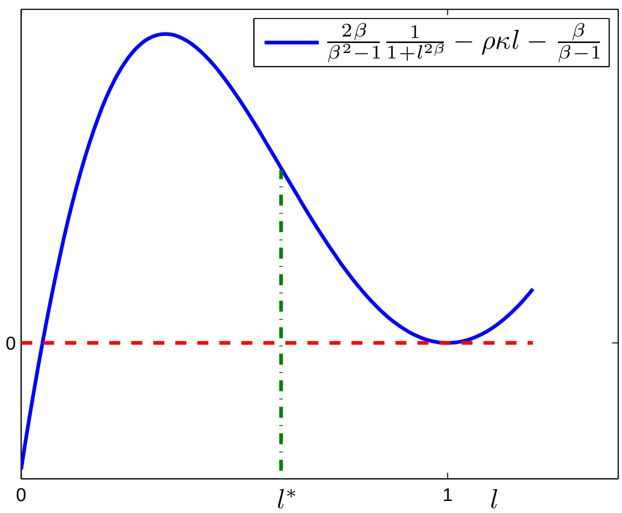


Figure 9: The left-hand side of (C.2)

**Step 1.** We verify that $l = 1$ is a solution:

$$\left(\frac{2\beta}{\beta^2-1}\frac{1}{1+l^{2\beta}} - \rho\kappa l - \frac{\beta}{\beta-1}\right)\Big|_{l=1} = \frac{\beta}{\beta^2-1} - \rho\kappa - \frac{\beta}{\beta-1}$$

$$= \frac{-\beta^2}{\beta^2-1} - \rho\kappa = \frac{-\beta^2}{\beta^2-1} + \frac{\frac{2\rho}{(\gamma\sigma)^2}}{\frac{2\rho}{(\gamma\sigma)^2}-1} = 0.$$

**Step 2.** The left-hand side of (C.2) is concave on $[0, l^*]$ and convex on $[l^*, \infty)$ where $l^* = \left(\frac{2\beta-1}{2\beta+1}\right)^{\frac{1}{2\beta}} \in (0, 1)$. (Figure 9 plots the left-hand side of (C.2).) To prove this, compute the first derivative of $\frac{1}{1+l^{2\beta}}$ as $-\frac{2\beta l^{2\beta-1}}{(1+l^{2\beta})^2}$ and the second derivative as

$$\frac{2\beta l^{2\beta-2}(1+l^{2\beta})}{(1+l^{2\beta})^4}\left((2\beta+1)l^{2\beta} - (2\beta-1)\right).$$

**Step 3.** The slope of the left-hand side of (C.2) is zero at $l = 1$ because

$$\left(\frac{2\beta}{\beta^2-1}\frac{1}{1+l^{2\beta}} - \rho\kappa l - \frac{\beta}{\beta-1}\right)'\Big|_{l=1} = -\frac{2\beta}{\beta^2-1}\frac{2\beta l^{2\beta-1}}{(1+l^{2\beta})^2}\Big|_{l=1} - \rho\kappa = \frac{-\beta^2}{\beta^2-1} - \rho\kappa = 0.$$

This implies that the equation (C.2) has no solution above one because it is convex above one.

**Step 4.** We show that (C.2) has a unique solution $l \in (0, 1)$. The convexity of the left-hand side of (C.2) on $[l^*, 1]$ and the zero-slope condition shown in step 3 imply that

$$\frac{2\beta}{\beta^2-1}\frac{1}{1+l^{2\beta}} - \rho\kappa l - \frac{\beta}{\beta-1} > 0, \text{ for all } l \in [l^*, 1).$$

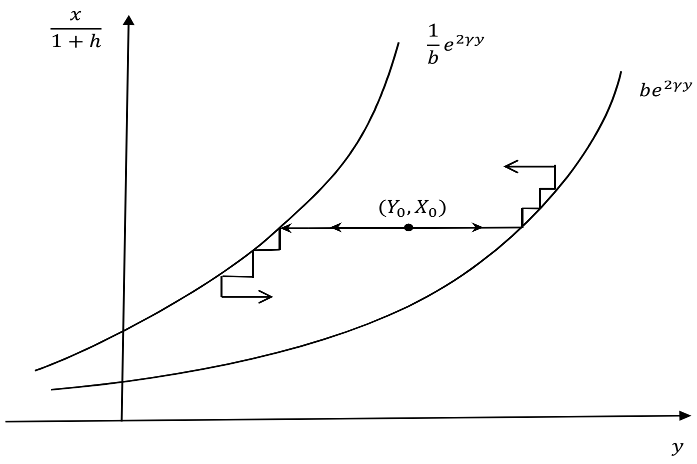


Figure 10: The state space for Example III. The two curves $\frac{x}{1+h} = be^{2\gamma y}$ and $\frac{x}{1+h} = \frac{1}{b}e^{2\gamma y}$ partition the state space into three areas. The middle area is the no-jump region and the other two areas are the jump region.

Further, the left-hand side of (C.2) is below zero at $l = 0$ because

$$\left(\frac{2\beta}{\beta^2-1}\frac{1}{1+l^{2\beta}} - \rho\kappa l - \frac{\beta}{\beta-1}\right)\Big|_{l=0} = \frac{2\beta}{\beta^2-1} - \frac{\beta}{\beta-1} = \frac{-\beta}{\beta+1} < 0.$$

The Intermediate Value Theorem implies the existence of a solution $l \in (0,1)$. Next, we show the uniqueness of $l$. By contradiction, suppose there are two solutions, $0 < l_1 < l_2 < l^*$. Because

$$\begin{aligned} \frac{2\beta}{\beta^2-1}\frac{1}{1+l_1^{2\beta}} - \rho\kappa l_1 - \frac{\beta}{\beta-1} &= 0, \\ \frac{2\beta}{\beta^2-1}\frac{1}{1+(l^*)^{2\beta}} - \rho\kappa l^* - \frac{\beta}{\beta-1} &> 0, \end{aligned}$$

the concavity of the left-hand side of (C.2) on $[0, l^*]$ implies that

$$\frac{2\beta}{\beta^2-1}\frac{1}{1+l_2^{2\beta}} - \rho\kappa l_2 - \frac{\beta}{\beta-1} > 0,$$

which contradicts the fact that $l_2$ is a solution to (C.2). Q.E.D.

As in Example II, we rule out the degenerate solution. Figure 10 plots the state space. It shows that the two boundaries $x(1+h)^{-1} = be^{2\gamma y}$ and $x(1+h)^{-1} = b^{-1}e^{2\gamma y}$ partition the state space into three areas. The area inside the two boundaries is the no-jump region and the other two areas are the jump region. The initial state $(Y_0, X_0^*)$ is inside the no-jump region. Figure 11 plots the dual value function $\tilde{V}(y,x,0)$ and the primal value function $V(y,w)$ for three values $y \in \{-0.1, 0, 0.1\}$ in the no-jump region. This figure shows that $\tilde{V}(y,x,0)$ is strictly

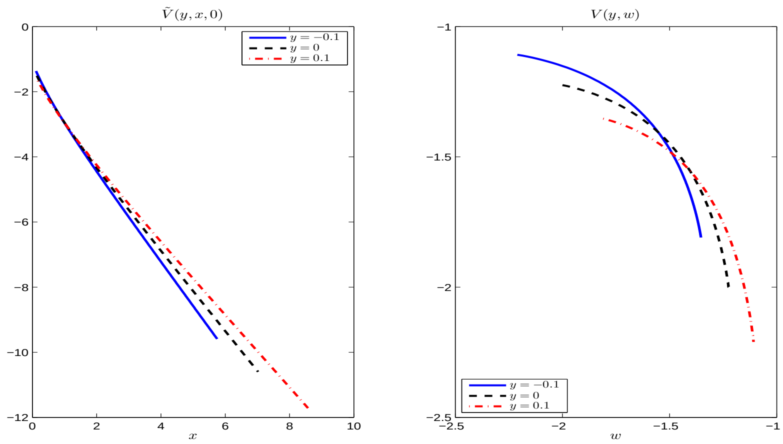


Figure 11: The dual and primal value functions in the no-jump region for Example III. Parameter values are given by $\rho = 1$, $\gamma = 1$, and $\sigma = 1$.

convex in $x$ and $V(y, w)$ is strictly concave and decreasing in $w$. Note that both functions are non-monotonic with $y$ and the domains change with $y$. In particular, the domain of $V(y, w)$ for $w$ increases with $y$ because a larger promised value is needed to induce the agent's participation when his income is larger.

Figure 12 plots the simulated paths of incomes $Y_t$, consumption $C_t^* = (2\gamma)^{-1} \ln (X_t^*/1 + H_t^*)$, the continuation value $W_t^* = V_x(Y_t, X_t^*/(1 + H_t^*), 0)$, and $X_t^*/(1 + H_t^*) e^{-2\gamma Y_t^*}$. This figure shows intuitively how $C_t^*$ and $W_t^*$ move with incomes $Y_t$. Since

$$C_t^* - Y_t = -Y_t + (2\gamma)^{-1} (\ln X_t^* - \ln (1 + H_t^*)),$$

$C^* - Y$ is a regulated Brownian motion with drift zero and diffusion $-\sigma$ on $\left[\frac{1}{2\gamma} \ln b, -\frac{1}{2\gamma} \ln b\right]$. It follows from Proposition 5.5 in Harrison (1985) or Proposition 10.8 in Stokey (2008) that $C^* - Y^*$ has a unique stationary distribution which is uniform on $\left[\frac{1}{2\gamma} \ln b, -\frac{1}{2\gamma} \ln b\right]$.

Figure 13 presents comparative static results. As in Example II, the risk-sharing band $\left[\frac{1}{2\gamma} \ln b, -\frac{1}{2\gamma} \ln b\right]$ expands when one of the following cases happens: (i) the common coefficient of relative risk aversion rises, (ii) the volatility of the income process rises, or (iii) the common subjective discount rate falls. In addition, neither autarky nor the first-best allocation is an optimal contract for any admissible parameter values satisfying the assumption in Proposition 3. The intuition is also similar to that for Example II.

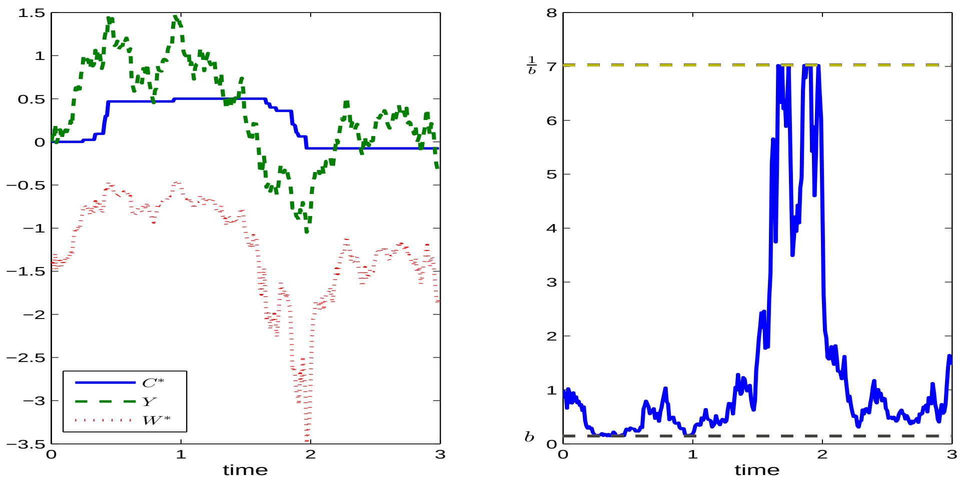


Figure 12: Simulated paths of the agent's optimal consumption $C_t^*$, incomes $Y_t$, continuation values $W_t^*$, and the process $e^{-2\gamma Y_t} X_t^* / (1 + H_t^*)$, $t \geq 0$, for Example III. Parameter values are given by $\rho = 1$, $\gamma = 1$, and $\sigma = 1$.

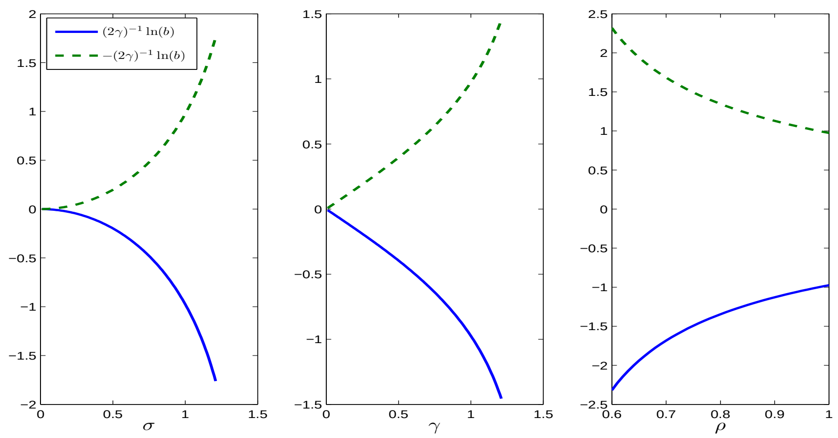


Figure 13: Comparative statics for Example III. Parameter values are given by $\sigma = 1$, $\rho = 1$, and $\gamma = 1$, unless one of them is changed in the comparative statics.

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