Stress Tests and Information Disclosure*

Itay Goldstein
University of Pennsylvania

Yaron Leitner
Federal Reserve Bank of Philadelphia

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Abstract

We study an optimal disclosure policy of a regulator who has information about banks' ability to overcome future liquidity shocks. We focus on the following trade-off: Disclosing some information may be necessary to prevent a market breakdown, but disclosing too much information destroys risk-sharing opportunities (Hirshleifer effect). We find that during normal times, no disclosure is optimal, but during bad times, partial disclosure is optimal. We characterize the optimal form of this partial disclosure. We also relate our results to the debate on the disclosure of stress test results.

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1 Introduction

In the new era of financial regulation following the crisis of 2008, central banks around the world will conduct periodic stress tests for financial institutions to assess their ability to withstand future shocks. A key question that occupies policymakers and bankers is whether the results of the stress tests should be disclosed and, if so, at what level of detail. The debate over this question is summarized in an article in the *Wall Street Journal* from March 2012. In this article, Fed Governor Daniel Tarullo expresses support for wide disclosure, saying that "the disclosure of stress-test results allows investors and other counterparties to better understand the profiles of each institution." On the other hand, the Clearing House Association expresses the concern that making the additional information public "could have unanticipated and potentially unwarranted and negative consequences to covered companies and U.S. financial markets."[1]

A classic concern about disclosure in the economics literature is based on the Hirshleifer effect (Hirshleifer, 1971). According to the Hirshleifer effect, greater disclosure might decrease welfare because it reduces risk-sharing opportunities for economic agents. This is indeed a relevant concern in the context of banks and stress tests. A large literature (e.g., Allen and Gale, 2000) studies risk-sharing arrangements among banks. If banks are exposed to random liquidity shocks, they will create arrangements among themselves or with outside markets to insure against such shocks. More recently, banks are known to hedge their risks with various derivative contracts. If more information about the state of each individual bank and its ability to withstand future shocks is publicly disclosed, then such risk-sharing and hedging opportunities will be limited, generating a welfare loss.

While this concern may provide credible content to the "unwarranted and negative consequences" referred to in the above quote from the Clearing House Association, it is hard to deny that greater disclosure that "allows investors and other counterparties to better understand the profiles of each institution" appears to be crucial at times. In

[1] See "Lenders Stress over Test Results," *Wall Street Journal*, March 5, 2012.

particular, as was clear during the recent financial crisis, when aggregate conditions seem bleak, the lack of disclosure might lead to a breakdown in financial activity. In the context of risk sharing and insurance, if the aggregate state of the financial sector is perceived to be weak, banks would not be able to insure themselves against undesirable outcomes (see, e.g., Leitner, 2005). In this case, some disclosure on certain banks might be necessary to enable some risk sharing and its welfare-improving effects.

In this paper, we study a model to analyze these forces and provide guidance for optimal disclosure policy in light of these forces. The model can address the debate on disclosure of stress test results but applies more generally to the issue of disclosing regulatory information even outside the stress-test arena. In the model, financial institutions suffer a loss if their future capital falls below a certain level. Part of the future capital of the financial institution can be forecasted based on current analysis and will become clear to policymakers conducting stress tests. However, there are also future shocks that cannot be forecasted with such an analysis. Financial institutions can engage in risk-sharing arrangements to guarantee that their capital does not fall below the critical level.

These risk-sharing arrangements work well if the overall state of the financial industry is perceived to be strong. In this case, no disclosure by the regulator is needed. Consistent with the Hirshleifer effect, disclosure can be even harmful because it prevents optimal risk-sharing arrangements from taking place. However, if, on average, banks are perceived to have capital below the critical level, then risk-sharing arrangements that insure them against falling below that level cannot arise without some disclosure. In this case, partial disclosure emerges as the optimal solution.

To study optimal disclosure rules in bad times, we distinguish between two different cases. First, we consider an environment where the information discovered by the regulator in the stress test is not already known to the bank. This is a reasonable assumption if the information involves assessment of bank exposure to aggregate conditions or to the state of other banks, and those are known to the regulator, who analyzes many banks, and not to the individual banks themselves. In this case, we

show that it is optimal to create two scores – a high score and a low score – and to give the high score to a group of banks whose average forecastable capital is equal to the critical level, and a low scores to other banks. This is similar to the Bayesian persuasion solution proposed by Kamenica and Gentzkow (2011).

By providing disclosure that separates banks into two groups, the regulator enables risk sharing among the banks that receive the high score. All banks whose forecasted capital is above the critical level receive the high score, but some banks with forecasted capital below the critical level also receive the high score. Importantly, for this to work, the regulator must not provide additional information about banks receiving the high score because with too much information, banks that are below the critical level cannot participate in risk sharing.

Interestingly, the optimal disclosure rule is not necessarily monotone; i.e., it is not always the case that banks below a certain threshold receive a low score and banks above the threshold receive a high score. There is a gain and a cost from giving a bank a high score. The gain is enabling the bank to participate in risk sharing, preventing a welfare-decreasing drop in capital. The cost is that giving a high score to one bank takes resources, thereby preventing other banks from receiving a high score. The allocation of banks into the "high-score" group depends on the gain-to-cost ratio, and this does not always generate a monotone rule; it depends on the distribution of shocks that banks are exposed to. We provide conditions under which the disclosure rule is monotone.

The second environment we consider is one where the information discovered by the regulator in the stress test is known to the bank itself but not to the outside market. In this case, pooling banks into two groups will not generally work. Banks whose forecastable level of capital is significantly above the critical level will refuse to participate in a risk-sharing arrangement with a group whose average forecastable capital is just at the critical level. Hence, in this case, the optimal disclosure rule has multiple scores. As before, one score is reserved for banks that are revealed to be below the critical capital level, and these banks are shunned from risk-sharing arrangements.

Other scores pool together banks below the critical level with a bank above the critical level to enable risk sharing. Different scores are required to accommodate the different reservation utilities of different banks above the critical level of capital.

Interestingly, in this environment, non-monotonicity becomes a general feature of optimal disclosure rules. When considering banks below the critical level of capital, it turns out that the stronger ones will be pooled with a bank whose level of capital is only slightly above the critical level (hence receiving a moderate score and sell at a moderate price), while the weaker ones will be pooled with a bank whose level of capital is significantly above the critical level (hence receiving a high score and sell at a high price). As we show in this paper, the increase in cost from pooling with a moderately strong bank to pooling with a very strong bank is not significant for the weakest banks but is significant for the moderately weak banks, and this leads to the non-monotonicity result.

The non-monotonicity result may lead to equilibrium outcomes in which lower types end up with higher expected payoffs. This may be plausible if the bank and the regulator learn the bank's type at the same time during the stress test, but may not be plausible if the bank learns its type before the regulator and can freely dispose assets (e.g., Innes, 1990). To explore the latter case, we solve for optimal disclosure rules when we add a constraint that higher types end up with higher equilibrium payoffs. Interestingly, the non-monotonicity result may continue to hold in this case, but to satisfy the constraint on equilibrium payoffs, the probability that low types participate in risk sharing is reduced. Another new insight is that banks that are above the critical level may be pooled into the same score and hence sell at prices that are above their reservation utilities. This increases the expected payoff for high types and relaxes the constraint on equilibrium payoffs for low types.

In summary, our paper generates the following results about optimal disclosure rules. First, no disclosure is optimal during good times, but partial disclosure is optimal during bad times. Second, partial disclosure takes the form of different scores pooling together banks of different levels of strength. The number of scores increases

as we move from a case in which banks do not already have the information revealed in the stress test to the case in which they do possess this information. Third, non-monotonicity appears to be a pervasive feature of optimal disclosure rules, such that a given score pools together strong banks with weak banks. This type of non-monotonicity may continue to hold even when we impose monotonicity on equilibrium expected payoffs.

1.1 Related literature

Our paper is related to the literature on Bayesian persuasion, going back to Kamenica and Gentzkow (2011). The solution for the first case in which the bank does not know its type is similar to the solution in Kamenica and Gentzkow (2011), but since we put more structure on the planner's objective function in the context of the banking industry, we obtain more results. In particular, we show that disclosure should be based on the gain-to-cost ratio and provide conditions under which a simple cutoff rule is optimal. The second case in which the bank knows its type is completely new to this literature and provides new results.[2]

The literature on disclosure of regulatory information is reviewed in a recent paper by Goldstein and Sapra (2013), which highlights the disadvantages of disclosure. Morris and Shin (2002) show that disclosure might be bad if economic agents share strategic complementarities and wish to act like each other even though it is not socially optimal. Providing a public signal then makes them place a too large weight on it because it provides information not only about fundamentals but also about what others know about the fundamentals. However, Angeletos and Pavan (2007) show that this conclusion may not hold when agents share strategic substitutes or when coordination is socially desirable. Leitner (2012) shows that disclosing information may reduce the regulator's ability to obtain information about contracts that banks

[2]In a different model of persuasion in the banking sector, Gick and Pausch (2014) study a game in which investors with heterogenous priors can take one of two actions and the regulator's objective is to get as close as possible to an outcome in which some predetermined fraction of investors take the first action. They show that in general, it is optimal for the regulator to choose a signal that is not too informative because full information induces investors to herd on the same action.

enter with one another. In his setting, it is optimal to reveal partial information: the regulator should set a position limit for each bank and reveal only whether a bank has reached its limit. The idea that disclosing information may reduce the regulator's ability to collect information from banks also appears in Prescott (2008). Bond and Goldstein (forthcoming) show that disclosure of information by the government to the market might harm the government's ability to learn from the market. Hence, the government may want to disclose information only on variables on which it cannot learn from the market. Increased disclosure might also be harmful due to the adverse effect it might have on the ex-ante incentives of bank managers, as in the traditional corporate-finance literature emphasizing the tension between ex-post and ex-ante optimal actions (e.g., Burkart, Gromb, and Panunzi, 1997). Morrison and White (2013) and Shapiro and Skeie (2013) study how the regulator's disclosure policy is affected by reputational concerns. Our paper analyzes a different tradeoff involving risk-sharing opportunities, which are at the heart of financial activity.

In a recent paper, Bouvard et al (2014) study how disclosure affects the possibility of bank runs when there are two types of banks and the regulator has private information about banks' types as well as the proportion of banks of each type. They show that during normal times disclosing information is undesirable because it can lead to bank runs, but during crises, disclosing information is desirable because it can prevent some runs. This result relates to one of our results but is based on completely different considerations. In addition, most of our results on the design of optimal disclosure rules are absent in their setting because they assume that there are only two types of banks.

In a related paper, Lizzeri (1999) studies the optimal disclosure policy of an intermediary who is hired by a firm to certify the quality of its products. Lizzeri (1999) shows that a monopolist intermediary may choose to restrict the flow of information and reveal only the minimum information that is required for an efficient exchange. Disclosing less information allows the intermediary to extract more rents from firms that are being rated. Instead, in our setting, providing less information allows for

better risk sharing.[3]

There is also an extensive literature that studies information disclosure by firms, particularly whether the regulator should mandate firms to disclose information.[4] Our paper contributes to this literature by illustrating a case in which the regulator would like to restrict information flow from firms. A strong firm ignores the fact that revealing information destroys risk-sharing opportunities for weak firms, but the regulator takes this negative externality into account.

In a different context, Marin and Rahi (2000) provide a theory of market incompleteness, which is based on the tradeoff between adverse selection and the Hirshleifer effect. Adverse selection favors an increase in the number of securities because it reduces information asymmetries among agents. The Hirshleifer effect favors a reduction in the number of securities. Our paper does not talk about security design but instead discusses how the regulator should pool banks into groups to enable risk sharing. Because the payoff function in our setting exhibits some convexity (a bank suffers a loss if its capital falls below a certain level), two groups may be necessary even when banks do not have private information. When banks have private information, more groups are necessary to accommodate the different reservation utilities of banks above the critical level.

Finally, the idea that risk-sharing arrangements may break down when aggregate conditions are bleak relates to Leitner (2005). He shows that in this case, it is optimal for banks to remain unlinked rather than form a financial network. In one interpretation of our model, we show how the disclosure policy affects the financial networks that banks form.

[3] See also Kartasheva and Yilmaz (2012), who extend Lizzeri's framework by adding different outside options for firms as well as information asymmetries among potential buyers. In their setting the first-best outcome is full disclosure, while in our setting, the first-best outcome typically involves pooling and, hence, only partial disclosure.

[4] A partial list of this literature includes Grossman (1981), Diamond (1985), Fishman and Hagerty (1990, 2003), Admati and Pfleiderer (2000), and a recent paper by Alvarez and Barlevy (2013).

2 A model

2.1 Economic environment

There is a bank, a regulator (i.e., a planner), and a perfectly competitive market. The bank has an asset, which yields a random cash flow $\tilde{\theta} + \tilde{\varepsilon}$, where $\tilde{\theta}$ is referred to as the bank's type and $\tilde{\varepsilon}$ is the bank's idiosyncratic risk, which is independent of type. The bank can sell its asset in the market for an amount x, which will be derived endogenously. Everyone is risk neutral, and the risk-free rate is normalized to be zero percent. Therefore, the price x is the expected value of the asset $\tilde{\theta} + \tilde{\varepsilon}$, conditional on the information available to the market (the information will depend on the disclosure regime). We use z to denote the bank's final cash holdings, and so $z = x$ if the bank sells the asset and $z = \tilde{\theta} + \tilde{\varepsilon}$ if the bank keeps the asset.

We assume that the bank derives the following final payoff as a function of z:

$$R(z) = \begin{cases} z & \text{if } z < 1 \\ z + r & \text{if } z \geq 1, \end{cases} \tag{1}$$

for a parameter $r > 0$. This payoff function captures the general idea that a bank derives some gains when its cash holdings are (weakly) above some threshold. One can think of several motivations: (1) The bank has a project that yields a positive net present value r but requires a minimum level of investment. For various reasons (e.g., projects cash flows are nonverifiable), the bank cannot finance the project if it does not have sufficient cash in hand. (2) The bank has a debt liability of 1. Not paying it leads to loss of future income r. (3) The bank faces a run if its cash holdings fall below some threshold.

Note that our results do not depend on the particular specification for $R(z)$ above. For example, our results extend to the case in which r depends on the bank's type (we discuss this more later). The results also extend to other payoff functions that exhibit discontinuity, such as assuming that the bank obtains az for some $a \in [0, 1)$ if $z < 1$, and $z + r$ if $z \geq 1$ (where r can be set to zero). The case $a = r = 0$ may best capture the idea that when the asset value falls below some threshold, there is a bank run and the bank is left with nothing. Key to all these specifications is the

discontinuity in payoffs.[5]

The bank chooses whether to keep its asset or sell it in the market. The bank does so in a way that maximizes its expected final payoff $R(z)$, conditional on the information available to it. As will be clear later, the bank will have a motive to sell its asset at a price of at least 1. This essentially provides insurance that the bank's cash holdings do not fall below the threshold. More generally, selling the asset can be thought of as engaging in risk sharing. In our model risk sharing takes a simple form: the bank replaces a random cash flow with a deterministic cash flow by selling its asset to the market.[6] The nature of our model continues to hold for other forms of risk sharing, including the case in which multiple banks share risk among themselves (see discussion in Section 5).

The bank's type $\tilde{\theta}$ is drawn from a finite set $\Theta \subset \mathbb{R}$ according to a probability distribution function $p(\theta) = \Pr(\tilde{\theta} = \theta)$. The idiosyncratic risk $\tilde{\varepsilon}$ is drawn from a continuous cumulative distribution function F that satisfies $E(\tilde{\varepsilon}) = 0$. The probability structure (i.e., the functions p and F) is common knowledge.

The focus of this paper is on the optimal disclosure policy of a regulator who has information about the bank. For example, the regulator could obtain information by maintaining examination staff at the bank or by conducting stress tests. The regulator can disclose information to the market before the bank can sell its asset. Hence, disclosure affects the terms of trade and the bank's ability and incentive to engage in risk sharing.

Specifically, we assume that the regulator observes the realization of $\tilde{\theta}$, which we denote by θ. The market does not observe θ. As for the bank, we focus on two cases: (1) The bank does not observe θ. (2) The bank observes θ. In both cases, we assume that no one observes the realization of $\tilde{\varepsilon}$ (denoted by ε), which is residual noise.

The first case captures the idea that the regulator may have some information advantage relative to banks. This is a plausible assumption when asset values depend

[5] A similar discontinuity in payoffs appears in Leitner (2005) and in Elliott et al. (forthcoming).

[6] The market is not affected by the discontinuity in payoffs as the bank and just gets $\theta + \varepsilon$ if it buys the asset. Hence, this transfer of risk can increase surplus.

on future regulatory actions or when asset values depend on interactions among banks, and the regulator's ability to collect information from multiple banks allows it to come up with better estimates. This case can also be relevant when the regulator has a better understanding of the bank's position based on past experience of dealing with other institutions. The second case captures the idea that the regulator and the bank share the same information, which is unobservable to other market participants. For example, the bank may know its ability to withstand future liquidity shocks, and the regulator can find out this information by conducting stress tests. Throughout most of the analysis, we assume that the bank cannot affect what the planner observes (i.e., θ is given), but in the second case we also analyze a situation in which the bank observes θ before the planner and can freely (and secretly) dispose assets, i.e., reduce θ (see Section 4.5).

Denote the types in Θ by $\theta_{\max} = \theta_1 > \theta_2 > ... > \theta_m = \theta_{\min}$. We assume that there are $k \geq 1$ types at or above 1. If information on θ was publicly available, these types could sell the asset at a price that guarantees their cash holdings to end up above the threshold of 1. We also assume that:

Assumption 1: $F(1 - \theta_{\min}) < 1$ and $F(1 - \theta_{\max}) > 0$.

Assumption 1 implies that even for the lowest type, there is a positive probability that the asset cash flow will be above 1, and even for the highest type, there is a positive probability that the asset cash flow will be below 1.

2.2 Disclosure rules

Before finding out the realization of $\tilde{\theta}$, the planner chooses and announces a disclosure rule to maximize expected total surplus. Since the market breaks even on average, maximizing expected total surplus is the same as maximizing the bank's expected payoff across the different possible types. The planner has the ability to commit to the chosen disclosure rule. As noted earlier, the focus of this paper is how disclosure affects the bank's ability to engage in risk sharing.

Formally, a *disclosure rule* is a set of "scores" S and a function that maps each

type to a distribution over scores. In our setting, the optimal disclosure rule can be implemented with a finite number of scores. Hence, there is no loss of generality in assuming that S is finite (or countable). We use $g(s|\theta)$ to denote the probability, according to the disclosure rule, that the planner assigns a score $s \in S$ when he observes type θ. That is, $g(s|\theta) = \Pr(\tilde{s} = s|\tilde{\theta} = \theta)$. Of course, for every $\theta \in \Theta$, the following has to hold: $\sum_{s \in S} g(s|\theta) = 1$.

To gain intuition on how disclosure rules work, note that full disclosure is obtained when for every type θ, the planner assigns some score $s_\theta \in S$ with probability 1, such that $s_\theta \neq s_{\theta'}$ if $\theta \neq \theta'$. No disclosure is obtained when the planner assigns the same distribution over scores to all types; e.g., each type obtains the same score.

For use below, denote $\mu(s) = E[\tilde{\theta} + \tilde{\varepsilon}|\tilde{s} = s)]$. $\mu(s)$ is the expected value of the bank's asset conditional on the bank obtaining score s. Since $\tilde{\varepsilon}$ is independent of $\tilde{\theta}$, and since $E(\tilde{\varepsilon}) = 0$, we obtain that:

$$\mu(s) = E[\tilde{\theta}|\tilde{s} = s] = \sum_{\theta \in \Theta} \theta \Pr(\tilde{\theta} = \theta|\tilde{s} = s) = \frac{\sum_{\theta \in \Theta} \theta p(\theta) g(s|\theta)}{\sum_{\theta \in \Theta} p(\theta) g(s|\theta)}, \tag{2}$$

where the last equality follows from Bayes' rule.

2.3 Sequence of events

The sequence of events is as follows:

1. The planner chooses a disclosure rule (S, g) and publicly announces it.

2. The bank's type θ is realized and observed by the planner. (In case 2, θ is also observed by the bank.)

3. The planner assigns the bank a score s according to the disclosure rule. (Recall that the planner can commit to assigning scores according to the disclosure rule chosen.) The planner publicly announces s.

4. The market offers to purchase the asset at a price $x(s)$.

5. The bank chooses whether to keep its asset or sell it for a price $x(s)$.

6. The residual noise ε is realized. As a result, the asset cash flow $\theta+\varepsilon$, the bank's cash holdings z, and the bank's final payoff $R(z)$ are determined. The market's payoff is $\theta+\varepsilon-x(s)$ if it purchases the asset, and 0 otherwise.

The planner's disclosure rule and assigned score specify a game between the bank and the market. We focus on perfect Bayesian equilibria of this game. Specifically, the bank chooses whether to sell or keep the asset to maximize its expected payoff conditional on its information, and the market chooses a price $x(s)$ that equals the expected value of the asset conditional on the publicly announced score, taking as given the bank's equilibrium strategy (i.e., whether the bank sells at this price or not). Formally, we assume Bertrand competition among at least two market participants. We assume that if the bank is indifferent between selling and not selling, it sells. The planner chooses a disclosure rule that maximizes the bank's expected payoff across different types, taking as given the equilibrium strategies of the market and the bank. We discuss the assumption that the planner can commit to a disclosure rule as well as other possible planner's objective functions in Section 5.

3 Bank does not observe its type

We start with the case in which the bank does not observe θ. So the bank observes only the score s assigned to it by the planner. We solve the game backward. One observation that simplifies the analysis is that the bank's decision of whether to sell the asset depends on s but not on θ or ε, which are unobservable to the bank. Hence, the decision of the bank to sell does not convey any additional information to the market. Consequently, the market sets a price $x(s)=\mu(s)$. It then follows from the payoff structure in (1) that:

Lemma 1 *In equilibrium, the bank sells the asset if and only if it obtains a score s such that $\mu(s)\geq 1$.*

The proof of Lemma 1 and all other proofs are in the appendix. The idea behind Lemma 1 is simple. If $\mu(s)>1$, selling guarantees that the bank's cash holding will

not fall below 1. Because of the penalty in the payoff structure when cash holdings fall below 1, the bank acts like a risk averse agent and is happy to replace the asset's random cash flow with its expected value. If instead, $\mu(s) < 1$, the bank prefers to keep the asset because if the bank sells the asset at a price below 1, the bank's cash holdings will surely be below 1, but if the bank keeps the asset, there is a positive probability that the asset's cash flow will turn out to be more than 1 (by Assumption 1). In this case the bank acts like a risk-loving agent.

The expected payoff for a bank of type θ, given disclosure rule (S, g), is then

$$u(\theta) \equiv \sum_{s:\mu(s)<1} [\theta + r \Pr(\tilde{\varepsilon} \geq 1 - \theta)] g(s|\theta) + \sum_{s:\mu(s)\geq 1} [\mu(s) + r] g(s|\theta). \tag{3}$$

The first term represents the cases in which the bank keeps the asset, and the second term represents the cases in which the bank sells the asset. The planner's problem is to choose a disclosure rule (S, g) to maximize the bank's ex-ante expected payoff $\sum_{\theta\in\Theta} p(\theta)u(\theta)$.

For use below, we refer to a score with $\mu(s) \geq 1$ as a "high" score and to a score with $\mu(s) < 1$ as a "low" score. Denote the probability that a bank of type θ obtains a high score by $h(\theta)$. That is, $h(\theta) = \sum_{s:\mu(s)\geq 1} g(s|\theta)$. This is the probability that the bank sells its asset (or, more broadly, engages in risk sharing).

Lemma 2 *The planner's problem reduces to finding a function $h : \Theta \to [0, 1]$ to maximize*

$$\sum_{\theta\in\Theta} p(\theta) \Pr(\tilde{\varepsilon} < 1 - \theta) h(\theta), \tag{4}$$

subject to the constraint

$$\sum_{\theta\in\Theta} p(\theta)(\theta - 1)h(\theta) \geq 0. \tag{5}$$

The intuition for Lemma 2 is as follows. The term $\Pr(\tilde{\varepsilon} < 1 - \theta)$ in the objective function (4) represents the gain from giving a high score to type θ. The gain is that type θ sells its asset for price $x \geq 1$ and hence can guarantee that its cash holdings are at least 1 even if the cash flow from its asset is less than 1 (when $\tilde{\varepsilon} < 1 - \theta$). The objective function is the weighted expected gain from giving high scores across types.

The planner can increase the value of the objective function by increasing $h(\theta)$ but faces the *resource constraint* (5). This constraint says that the average θ for types receiving a high score must be at least 1. It originates from the requirement that $\mu(s) \geq 1$ for every high score s. If this was not the case, the market would not be willing to pay a price $x \geq 1$ for an asset of a bank with a high score and there will be no benefit from giving a high score. Essentially, by giving a high score, the planner implements a cross subsidy from types with $\theta > 1$ to types with $\theta < 1$, so a high type sells its asset for less than what the asset is truly worth, and a low type sells its asset for more than what the asset is worth. This is beneficial because more types can ensure that their cash holdings are at least 1.

The solution to the planner's problem is as follows. If $E(\tilde{\theta}) \geq 1$, assigning $h(\theta) = 1$ for every $\theta \in \Theta$ satisfies the resource constraint and hence is optimal. Otherwise, if $E(\tilde{\theta}) < 1$, it is impossible to assign $h(\theta) = 1$ for every $\theta \in \Theta$ and so the resource constraint is binding, such that the average type getting a high score is exactly 1. The optimal disclosure rule then has to determine the probability with which each type gets a high score. This depends on comparing the "gain-to-cost ratio" from increasing $h(\theta)$ for different types. The gain from increasing $h(\theta)$ for a bank of type θ is the term $\Pr(\tilde{\varepsilon} < 1 - \theta)$ in the objective function (4). The cost is that type θ requires resources in the amount $1 - \theta$, as in equation (5). So the gain-to-cost ratio for type θ is:

$$G(\theta) \equiv \frac{\Pr(\tilde{\varepsilon} < 1 - \theta)}{1 - \theta}. \tag{6}$$

For types with $\theta \geq 1$, it is optimal to assign $h(\theta) = 1$ because there is no cost; these types provide resources. For types with $\theta < 1$, it follows from the linearity of the problem that it is optimal to set a cutoff G^* such that types with gain-to-cost ratio above the cutoff are assigned $h(\theta) = 1$, and types with gain-to-cost ratio below the cutoff are assigned $h(\theta) = 0$. The optimal G^* is the lowest cutoff possible that satisfies the resource constraint. For types with gain-to-cost ratio that equals G^*, the probability of obtaining a high score can be between 0 and 1 and is set such that the resource constraint is satisfied with equality.

The following proposition summarizes the optimal disclosure rule.

Proposition 1 *When a bank does not observe its type, the optimal disclosure rule is such that*

1. *If* $E(\tilde{\theta}) \geq 1$, *then* $h(\theta) = 1$ *for every* $\theta \in \Theta$.
2. *If* $E(\tilde{\theta}) < 1$, *then*

$$h(\theta) = \begin{cases} 1 \text{ if } \theta \geq 1 \text{ or if } \theta < 1 \text{ and } G(\theta) > G^* \\ 0 \text{ if } \theta < 1 \text{ and } G(\theta) < G^*, \end{cases} \tag{7}$$

where G^* *is the lowest* $G \in \{G(\theta)\}_{\theta<1}$ *that satisfies the resource constraint* $\sum_{\theta \geq 1} p(\theta)(\theta - 1) + \sum_{\theta<1:G(\theta)>G} p(\theta)(\theta - 1) \geq 0$. *If* $G(\theta) = G^*$, *then* $h(\theta) \in [0, 1)$.

An interesting question is whether and when full disclosure is optimal, and whether and when no disclosure is optimal. If $E(\tilde{\theta}) \geq 1$, we know that $h(\theta) = 1$ for every θ. The planner can implement this by giving all types the same score, i.e., with no disclosure. There are other ways to implement the optimal rule, assigning more than one score such that the average θ of types receiving each score is at least 1. In the special case $\theta_{\min} \geq 1$, the planner can even assign a different score to each type, i.e., provide full disclosure. In contrast, if $E(\tilde{\theta}) < 1$, the planner must assign at least two scores. Some disclosure is necessary because without disclosure the price would be less than 1 and no type would sell its asset. Yet, full disclosure is suboptimal because under full disclosure, only types above 1 sell their assets, while under the optimal disclosure rule, some types that are below 1 also sell their assets. The following result, characterizing circumstances under which full disclosure or no disclosure achieve the optimal rule, follows:

Corollary 1 *1. Full disclosure achieves the optimal outcome if and only if* $\theta_{\min} \geq 1$.

2. No disclosure achieves the optimal outcome if and only if $E(\tilde{\theta}) \geq 1$.

In general, the optimal disclosure rule can always be implemented with a maximum of two scores: a high score s_1 such that $g(s_1|\theta) = h(\theta)$ and a low score s_0 such that $g(s_0|\theta) = 1 - h(\theta)$. Then, if $E(\tilde{\theta}) \geq 1$, all types get the score s_1, and otherwise, if $E(\tilde{\theta}) < 1$, some types get s_1 and some types get s_0. Interestingly, in the case of

$E(\tilde{\theta}) < 1$ (summarized in the second part of Proposition 1), the types that obtain a low score are not necessarily the lowest. So, a simple cutoff rule that assigns a high score to high types and a low score to low types is not necessarily optimal. Intuitively, the gain from giving a high score is higher for lower types because low types are more likely to end up with low realizations of cash flow. That is, the numerator of (6) is decreasing in θ. But the cost of giving a high score to low types is also higher because low types require more resources. That is, the denominator of (6) is also decreasing in θ. Hence, it is unclear whether $G(\theta)$ is increasing or decreasing, or whether it is even monotone. The function $G(\theta)$, and hence the optimal disclosure rule, depends on the distribution of the idiosyncratic risk $\tilde{\varepsilon}$.

The optimal rule will involve a simple cutoff with respect to θ when $G(\theta)$ is increasing when $\theta < 1$. In this case types above the cutoff obtain a high score with probability 1, and types below the cutoff obtain a low score with probability 1. A sufficient condition for this to happen is that the cumulative distribution function of $\tilde{\varepsilon}$ satisfies Condition (1) below. This condition is satisfied by any cumulative distribution function that is concave on the positive region. Examples include a normal distribution and a uniform distribution (both with mean zero).

Condition 1 *$F(\varepsilon)/\varepsilon$ is decreasing when $\varepsilon > 0$.*

Corollary 2 *If $E(\tilde{\theta}) < 1$ and Condition (1) holds, the optimal disclosure rule involves a cutoff such that types below the cutoff obtain a low score and types above the cutoff obtain a high score.*

Another example in which the optimal rule involves a simple cutoff with respect to θ is when r in the payoff function (1) depends on θ according to some function $r(\theta)$, which is increasing in θ sufficiently strongly. This has a simple and intuitive economic interpretation: good banks have better investment opportunities in addition to having better assets in place. In this case, the gain from giving a high score is $r(\theta)\Pr(\tilde{\varepsilon} < 1 - \theta)$ and the gain-to-cost ratio is $r(\theta)G(\theta)$. So, no matter what shape $G(\theta)$ has, if $r(\theta)$ is increasing sufficiently strongly, the gain-to-cost ratio will

be monotonically increasing, and the disclosure rule will look like a cutoff rule. For example, if $r(\theta) = \frac{1}{\Pr(\tilde{\varepsilon}<1-\theta)}$, then $r(\theta)G(\theta) = \frac{1}{1-\theta}$, which is increasing in θ.

Finally, an example in which the optimal disclosure rule does not involve a simple cutoff as in Corollary 2 is when $G(\theta)$ is decreasing when $\theta \leq \theta_{k+1}$. In this case the optimal disclosure rule is nonmonotone in type. It includes a cutoff such that types below the cutoff and types above 1 obtain a high score, while types in the middle obtain a low score. A sufficient condition for this to happen is that $F(\varepsilon)/\varepsilon$ is increasing when $\varepsilon > 1 - \theta_{k+1}$.[7]

4 Bank observes its type

So far, we assumed that the bank does not observe its type. We showed that it is possible to implement the optimal disclosure rule with two scores, such that the planner pools everyone who sells under the same score. In this section, we show that this conclusion may no longer be true when the bank observes its type. The difference is that now each type has a "reservation price," i.e., a minimum price at which it is willing to sell. When different types have different reservation prices, the planner may need to assign more than two scores to distinguish among them. We also discuss how the planner should assign these multiple scores to low types who are pooled with high types.

4.1 Derivation of the planner's problem

We first derive banks' reservation prices. Define

$$\rho(\theta) = \begin{cases} \max\{1, \theta - r\Pr(\tilde{\varepsilon} < 1-\theta)\} \text{ if } \theta \geq 1 \\ \min\{1, \theta + r\Pr(\tilde{\varepsilon} \geq 1-\theta)\} \text{ if } \theta < 1. \end{cases} \tag{8}$$

[7] An example of a probability distribution function that satisfies the condition above is a truncated Cauchy distribution (Nadarajah and Kotz, 2006) on the interval $[-A, 0]$ minus its mean, where the lower bound A depends on the model parameters. Intuitively, for the sufficient condition above to hold, the probability distribution of $\tilde{\varepsilon}$ must put low weight on low values; that is, $F(1-\theta_{k+1}) < \frac{1-\theta_{k+1}}{1-\theta_{\min}} F(1-\theta_{\min}) < \frac{1-\theta_{k+1}}{1-\theta_{\min}}$. So when θ_{k+1} is close to 1, the distribution must have a fat tail to satisfy $E(\tilde{\varepsilon}) = 0$.

Lemma 3 *In equilibrium, a bank of type θ agrees to sell at price x if and only if $x \geq \rho(\theta)$.*

Lemma 3 is derived as follows. If type θ keeps its asset, its expected payoff is $E[R(\theta + \tilde{\varepsilon})] = \theta + r \Pr(\tilde{\varepsilon} \geq 1 - \theta)$. If type θ sells at price x, its payoff is $R(x)$, i.e., it is x when $x < 1$ and $x + r$ when $x \geq 1$. Hence, type θ agrees to sell if and only if $R(x) \geq \theta + r \Pr(\tilde{\varepsilon} \geq 1 - \theta)$. This reduces to $x \geq \rho(\theta)$.

We refer to $\rho(\theta)$ as type θ's reservation price and denote $\rho_i = \rho(\theta_i)$. As illustrated in Figure 1, the reservation price satisfies two properties, which we use later. First, $\rho(\theta)$ is increasing in θ. Second, $\rho(\theta) < \theta$ when $\theta > 1$. The intuition for the second property is that types above 1 are willing to sell below their true valuation (i.e., at a discount) to guarantee that their cash holdings do not fall below 1. This is the insurance motive. The figure also shows that a very low type will agree to sell its asset for less than 1, but only if the price is above θ. Intuitively, the bank will demand compensation for losing the option value of ending up with cash holdings above 1. As emphasized before, in this range the bank is essentially risk loving. However, as we show below, in equilibrium such transactions will not happen as the market is not willing to pay a price above the expected type. Overall, as in the previous section, a bank never sells in equilibrium for a price below 1.

We now derive some properties that have to hold under the optimal disclosure rule.[8]

Lemma 4 *Under an optimal disclosure rule:*

1. Every type $\theta_i \geq 1$ sells its asset with probability 1.

2. Whenever type $\theta_i \geq 1$ receives score s, the price is $x(s) = \mu(s)$.

3. If the highest type that obtains score s is below 1, then every type keeps its asset upon obtaining score s.

The first part in Lemma 4 follows because if a type $\theta \geq 1$ did not sell its asset, the planner could strictly increase type θ's payoff, without affecting the payoffs of other

[8] We establish existence of an optimal disclosure rule below.

types, by fully revealing θ's type. Then the market would offer to buy type θ's asset at a price θ, and type θ would accept the offer.

The second part follows from the first part and the observation that the reservation price is increasing in θ. These imply that every type sells its asset upon obtaining score s, as long as score s is also obtained by type $\theta_i \geq 1$. Hence, selling at this score does not convey any additional information to the market, and the market prices the asset at the expected value given the score: $x(s) = \mu(s)$.

The third part reflects the fact that if no type above 1 obtains score s, the price $x(s)$ must be less than 1. Then, the bank will sell only if the price is strictly above the true value. But this cannot be an equilibrium outcome because the market would overpay in expectation. The third part holds under any disclosure rule, not only an optimal one.

It follows from Lemmas 3 and 4 that under an optimal rule a bank sells its asset upon receiving score $s \in S$ if and only if $\mu(s) \geq 1$. Hence, the equilibrium payoff for type θ is $u(\theta)$, as in equation (3) in the previous section. It also follows that if the highest type that obtains score s is type $\theta_i \geq 1$, then $\mu(s) \geq \rho_i$, so that θ_i agrees to sell. Formally, denote by S_i the set of scores that type $\theta_i \geq 1$ obtains with a positive probability but higher types do not obtain; that is, $S_i = \{s \in S : g(s|\theta_i) > 0$ and $g(s|\theta') = 0$ for every $\theta' > \theta\}$. Then

$$\mu(s) \geq \rho_i \text{ for all } s \in S_i \text{ and } i \in \{1, ..., k\}. \tag{9}$$

(Recall that types 1 through k are above 1.)

The planner's problem reduces then to finding a disclosure rule (S, g) to maximize the expected payoff across types $\sum_{\theta \in \Theta} p(\theta)u(\theta)$, just as in the previous section, such that equation (9) holds. This equation is a generalization of the condition for selling $\mu(s) \geq 1$ in Lemma 1, but now to satisfy the reservation prices of different types, there are different conditions for different scores.

Denote $h_i(\theta) = \sum_{s \in S_i} g(s|\theta)$. So $h_i(\theta)$ is the probability that θ obtains a score such that the highest type that obtains the score is $\theta_i \geq 1$. The probability that a

bank sells its asset is then $\sum_{i=1}^{k} h_i(\theta)$. We can now write down the planner's problem as follows:

Lemma 5 *When the bank observes its type, the planner's problem reduces to choosing a set of functions* $\{h_i : \Theta \longrightarrow [0,1]\}_{i=1,...,k}$ *to maximize*

$$\sum_{\theta \in \Theta} p(\theta) \Pr(\tilde{\varepsilon} < 1 - \theta) \sum_{i=1}^{k} h_i(\theta), \tag{10}$$

such that

$$\sum_{\theta \in \Theta} p(\theta)(\theta - \rho_i) h_i(\theta) \geq 0 \text{ for every } i \in \{1, ..., k\}, \tag{11}$$

$$\sum_{i=1}^{k} h_i(\theta) \leq 1 \text{ for every } \theta \in \Theta, \tag{12}$$

and

$$h_i(\theta) = 0 \text{ for every } i \in \{1, ..., k\} \text{ and } \theta > \theta_i. \tag{13}$$

The derivation of Lemma 5 follows directly from the discussion above and from the analysis in the previous section. The objective function (10) is as in Lemma 2, noting that the probability that type θ sells its asset is $h(\theta) = \sum_{i=1}^{k} h_i(\theta)$. Again, the planner wants to maximize the expected gain from giving banks a score that enables them to sell at a price above 1 when otherwise they would end up with cash holdings below 1. Equation (11) is a generalization of the resource constraint (5) and follows from (9).[9] Now there are k resource constraints required to satisfy the reservation price of all the types above 1. Equation (12) simply says that the probability that type θ sells its asset is at most 1. Finally, Equation (13) says that the highest type that obtains a score $s \in S_i$ is type θ_i, by definition.

The problem in Lemma 5 is a linear programming problem. Because the feasible region is bounded ($h_i(\theta) \in [0,1]$) and nonempty,[10] a solution exists. The solution can

[9] Specifically, from equation (2), $\mu(s) \geq \rho_i$ reduces to $\sum_{\theta \in \Theta} p(\theta)(\theta - \rho_i) g(s|\theta) \geq 0$. Summing over all $s \in S_i$, we obtain (11).

[10] $h_i(\theta) = \begin{cases} 1 & \text{if } \theta = \theta_i \\ 0 & \text{if } \theta \neq \theta_i \end{cases}$ satisfies all the constraints.

be implemented with $k+1$ scores, which we denote by $s_0, s_1, ..., s_k$. Specifically,

$$g(s_i|\theta) = \begin{cases} h_i(\theta) & \text{if } i \in \{1, ..., k\} \\ 1 - \sum_{j=1}^{k} h_j(\theta) & \text{if } i = 0 \end{cases} . \tag{14}$$

As in Corollary 1, full disclosure can be an optimal rule if and only if there are no types below 1. No disclosure can be optimal if and only if there are sufficient resources, so that every type sells in equilibrium if there is no disclosure. However, the condition for no disclosure to be optimal changes to $E(\tilde{\theta}) \geq \rho_1$, and so it is stricter than in the previous section. Essentially, there must be sufficient resources so that every type can sell its asset at the reservation price of the highest type.[11] Hence, disclosure may be necessary even if $E(\tilde{\theta}) \geq 1$.

4.2 Properties of optimal disclosure rules

In this section, we focus on two properties of optimal disclosure rules, which must hold when resources are scarce.[12] We say that resources are scarce if it is impossible to implement an outcome in which every type sells its asset with probability 1. A sufficient condition for this to happen is that $E(\tilde{\theta}) < 1$.[13] In this case, all k resource constraints in (11) are binding.[14] So if the highest type that obtains score s is $\theta_i \geq 1$, the price is ρ_i.

The first property is as follows:

Proposition 2 *Suppose resources are scarce. Under an optimal disclosure rule, types above* 1 *that have different reservation prices obtain different scores.*

Intuitively, if two types above 1 have different reservation prices but the same score, the sale price equals to the reservation price of the higher type. This means that the lower type sells for more than its reservation price and, therefore, ends up

[11] Formally, since $h_1(\theta_1) = 1$ (Lemma 4), under no disclosure, $h_1(\theta) = 1$ for every $\theta \in \Theta$. This satisfies the resource constraint (11) if and only if $E(\tilde{\theta}) \geq \rho_1$.

[12] When resources are not scarce, there is always an optimal rule that satisfies the two properties, but there are also optimal disclosure rules that do not satisfy these properties.

[13] See more details in the appendix. A necessary and sufficient condition is that a_k in Lemma A-2 in the appendix is well defined.

[14] See Lemma A-1 in the appendix.

with more resources than it requires. But this is a waste of resources without any gain. The planner can do better by assigning the lower type its own score, so that this type ends up with less resources. This frees up resources that can be used to cross subsidize types below 1. This, in turn, increases the probability that low types will end up with cash holdings that are at least 1.

It follows that when resource are scarce and $\rho_1 > \rho_2 > ... > \rho_k$, the planner must assign at least $k+1$ scores. So without loss of generality, the planner assigns scores $s_0, s_1, ..., s_k$, such that score s_i ($i \in \{1, ..., k\}$) pools together type θ_i, which is above 1, with a type (or types) that are below 1, and score s_0 pools together only types that are below 1. When a bank obtains score s_0, the bank does not sell its asset. When a bank obtains score $s_i \neq s_0$, the bank sells its asset for price ρ_i.[15]

Corollary 3 *If resources are scarce and $\rho_1 > \rho_2 > ... > \rho_k$, the planner must assign at least $k+1$ scores.*

The second property is that pooling between types below 1 and types above 1 is non-monotone. Among the types below 1 that are pooled with types above 1, the lowest types below 1 are pooled with the highest types above 1. Formally,

Proposition 3 *Suppose resources are scarce, $\theta' < \theta'' < 1 < \theta_i < \theta_j$, and $\rho_i < \rho_j$. Under an optimal disclosure rule, if a positive probability exists that type θ' is pooled together with type θ_i (i.e., $h_i(\theta') > 0$), then type θ'' cannot be pooled together with type θ_j (i.e., $h_j(\theta'') = 0$).*

This result seems surprising. To understand the intuition, we need to go back to the gain-to-cost ratio that was guiding the allocation of scores in the case the bank did not know its type in the previous section (see Equation (6)). When giving scores to types below 1 in the case studied in this section, the planner will have a similar gain-to-cost ratio in mind, but now there will be different gain-to-cost ratios for the

[15] It is also possible to implement the optimal outcome with more than $k+1$ scores. For example, instead of assigning score s_i where $i \in \{1, .., k\}$, the planner can assign multiple scores with $\mu(s) = \rho_i$ such that $h_i(\theta) = \sum_{s:\mu(s)=\rho_i} g(s|\theta)$.

k different scores that guarantee selling in equilibrium. In particular, the gain-to-cost ratio from pooling type $\theta < 1$ with type $\theta_i \geq 1$ is:

$$\frac{\Pr(\tilde{\varepsilon} < 1 - \theta)}{\rho_i - \theta}. \tag{15}$$

The gain from pooling type $\theta < 1$ with any of the types above 1 is the same. Type θ's cash holdings will surely be at least 1, so the gain is $\Pr(\tilde{\varepsilon} < 1 - \theta)$ as before. But the cost is different and depends on which score is allocated. It is more costly to pool with a higher type because the sale price is higher and θ ends up with more resources. In particular, when θ is pooled with type $\theta_i > 1$, θ sells at price ρ_i, and so the cost is $\rho_i - \theta$.

To understand the non-monotonicity result in Proposition 3, it is useful to break up the cost $\rho_i - \theta$ into two components $\rho_i - 1$ and $1 - \theta$. The latter is the cost of bringing the bank up to the threshold of 1 and the former is the cost of bringing it up farther from 1 to the reservation price ρ_i of the bank with which it is pooled. For types $\theta < 1$ that are close to 1, the second component is negligible, while the first component is first order. In contrast, for very low types below 1, the second component is first order. Hence, to save on resources, it is more beneficial to reduce the first component for the types that are close to 1. This can be done by pooling these types with types above 1 that have a low reservation price ρ_i. Then, the result in the proposition follows: Higher types below 1 are pooled together with lower types above 1.

An immediate corollary of Proposition 3 is that among the types below 1 that sell their assets, lower types sell for higher prices. Formally,

Corollary 4 *Suppose resources are scarce and $\theta' < \theta'' < 1$. Under an optimal disclosure rule, if a positive probability exists that type θ' sells at price x' and type θ'' sells at price x'', then $x'' \leq x'$.*

Corollary 4 implies that the sale price is nonmonotone in type. Among the types below 1 that sell their assets, lower types sell for higher prices. Note, however, that

among types above 1, the opposite is true, as these types end up selling exactly for their reservation price, which is increasing in type.

4.3 Closed-form solutions and examples

The results in the previous subsection provide general properties of the optimal disclosure rule and also a general algorithm that can be used to determine the optimal disclosure rule for every set of parameters and distribution functions covered by our model. To get a better idea of how the disclosure rule works, in this subsection we illustrate the solution to the planner's problem in some special cases and provide a few examples. For use below, define $G_i(\theta) = \frac{\Pr(\tilde{\varepsilon}<1-\theta)}{\rho_i-\theta}$, which is the gain-to-cost ratio from pooling together type $\theta < 1$ with type $\theta_i > 1$, as in (15).

Case 1. $\rho_1 = 1$: Here, the highest reservation price is 1. As we know from (8), this can be consistent with having multiple types above 1, but either r is sufficiently high or $\theta_{\max}$ is sufficiently low, so they are willing to sell the asset at a price of 1. It follows immediately from Proposition 1 and from the analysis in the previous subsection that in this case, the optimal disclosure rule is essentially identical to the one when the bank does not observe its type, as characterized in Proposition 1.

Case 2. $k = 1$: Here, there is only one type above 1. It can be easily shown that in this case the optimal disclosure rule is similar to that in Proposition 1 (when the bank does not know its type), except that the gain-to-cost ratio is $G_1(\theta)$ instead of $G(\theta)$. Then, Corollary 2 describing when the disclosure rule will feature monotonicity will hold only if ρ_1 is sufficiently small. Otherwise, $G_1(\theta)$ is decreasing when $\theta < 1$ even if Condition 1 is satisfied.[16] Intuitively, when ρ_1 is very high, the cost $(\rho_1 - \theta)$ of pulling type $\theta < 1$ with the high type is relatively high no matter how high θ is, and so the dominant factor in deciding which types should be included in risk sharing is that the gain $\Pr(\tilde{\varepsilon} < 1 - \theta)$ is decreasing in θ. So when ρ_1 is sufficiently high, low types and the type above 1 obtain the high score, while types in the middle obtain

[16] Formally, $G_1(\theta)$ is increasing when $\theta < 1$ if and only if $-F'(1-\theta)(\rho_1 - \theta) + F(1-\theta) \geq 0$. This reduces to $\rho_1 \leq \max_{\theta:\theta<1}\{\theta + \frac{F(1-\theta)}{F'(1-\theta)}\}$.

the low score.

Case 3. $k \geq 2$ and $G_i(\theta)$ is increasing in θ for every $\theta < 1$ and every $i \in \{1, ..., k\}$:[17] Using similar logic as in Section 3, one can show that the lowest types are excluded from risk sharing. The optimal disclosure rule can be found by applying Propositions 2 and 3. First, pool the lowest type above 1 (type θ_k) with the highest types below 1 until all the resources from type θ_k are exhausted. Next, pool the second lowest type above 1 (type θ_{k-1}) with the remaining highest types below 1 until the resources from type θ_{k-1} are exhausted. And so on, until we exhaust the resources from the highest type θ_1.[18,19] The following example illustrates the solution:

Example 1 There are five types $\theta_1 > \theta_2 > 1 > \theta_3 > \theta_4 > \theta_5$ and $\rho_1 > \rho_2$. So without loss of generality there are 3 scores s_0, s_1, s_2. Suppose $G_i(\theta)$ is increasing in θ for every $\theta < 1$ and $i \in \{1, 2\}$. Suppose in addition that:

$$p(\theta_2)(\theta_2 - \rho_2) = p(\theta_3)(\rho_2 - \theta_3) \tag{16}$$

$$p(\theta_1)(\theta_1 - \rho_1) = p(\theta_4)(\rho_1 - \theta_4). \tag{17}$$

Then the optimal disclosure rule is as follows (an element in the table is the probability that type θ_j obtains score s_i):

	θ_5	θ_4	θ_3	θ_2	θ_1
score s_1 (sell at price ρ_1)		1			1
score s_2 (sell at price ρ_2)			1	1	
score s_0 (keep asset)	1				

In particular, equation (16) implies that θ_3 gets score s_2 with probability 1, so that the resource constraint for score s_2 is satisfied with equality. Equation (17) implies that θ_4 gets score s_1 with probability 1, so that the resource constraint for score s_1 is satisfied with equality. As we can see, θ_1 and θ_4 are pooled together at the highest

[17] A sufficient condition for this to happen is that Condition 1 holds and $\rho_1 < \max_{\theta;\theta<1}\{\theta + \frac{F(1-\theta)}{F'(1-\theta)}\}$.

[18] Proposition A-1 in the appendix provides a closed-form solution.

[19] Note that, in general, when resources are scarce, the optimal disclosure rule uniquely determines the probability $h_j(\theta)$, and hence the price at which type θ sells its asset. However, there is more than one way to implement it (e.g., footnote 15).

price; θ_2 and θ_3 are pooled together at the lower price; and θ_5 does not sell and does not participate in risk sharing.

Case 4. $k \geq 2$ and $G_i(\theta)$ is decreasing in θ for every $\theta < 1$ and every $i \in \{1, ..., k\}$:[20] In this case, types in the "middle" are excluded from risk sharing, and the optimal disclosure rule can be found as follows: Pool the highest type θ_1 with the lowest types until all the resources from type θ_1 are exhausted. Next, pool the second highest type θ_2 with the remaining lowest types, and so on, until all the resources of types above 1 are exhausted.

Case 5. $k \geq 2$ and there exists $\hat{k} \in \{1, ..., k\}$ such that for every $\theta < 1$, $G_i(\theta)$ is decreasing in θ if $i \in \{1, ..., \hat{k}\}$ and increasing in θ if $i \in \{\hat{k}+1, ..., k\}$:[21] In this case, the optimal disclosure rule can by found be combining the procedures in cases 3 and 4. We illustrate in the example below.

Example 2 In Example 1 suppose that $G_1(\theta)$ is decreasing when $\theta < 1$ (rather than increasing) and $p(\theta_1)(\theta_1 - \rho_1) = p(\theta_5)(\rho_1 - \theta_5)$. The optimal disclosure rule is as follows.

	θ_5	θ_4	θ_3	θ_2	θ_1
score s_1 (sell at price ρ_1)	1				1
score s_2 (sell at price ρ_2)			1	1	
score s_0 (keep asset)		1			

4.4 Discussion on non-monotonicity

Optimal disclosure rules may exhibit two forms of non-monotonicity. First, the probability of selling the asset may be nonmonotone in type (Example 2). Second, the sale price may be nonmonotone in type (Examples 1 and 2); related, pooling between types above 1 and types below 1 is nonmonotone.

The first form of non-monotonicity arises when the gain-to-cost ratio is decreasing in θ. A necessary condition for this is that the gain is decreasing in θ. That is, for

[20] A sufficient condition for this to happen is that $\rho_k > \max_{\theta:\theta<1}\{\theta + \frac{F(1-\theta)}{F'(1-\theta)}\}$.

[21] A sufficient condition for this to happen is that Condition 1 holds and $\rho_{\hat{k}} \geq \max_{\theta:\theta<1}\{\theta + \frac{F(1-\theta)}{F'(1-\theta)}\} > \rho_{\hat{k}+1}$.

a given cost, the planner has a preference for helping low types. In our model this happens because every type may end up with cash holdings above 1 and obtain r even without selling its asset, but lower types are less likely to be in that position. So, the gain from having a low type sell its asset is higher. This will not be true in a variation of our model in which the bank obtains $R(z)$ if it sells the asset, and 0 otherwise. For example, the bank may have an investment opportunity that expires before the asset cash flows are obtained. In this case, the gain from selling is the same for all types, and the planner's objective becomes $\sum_{\theta\in\Theta} p(\theta) \sum_{i=1}^{k} h_i(\theta)$. Then the gain-to-cost ratio becomes $\frac{1}{\rho_i - \theta}$, which is increasing in θ.[22]

The second form of non-monotonicity follows from the payoff function (1), which induces pooling between types above 1 and types below 1, and because the sale price reflects the reservation price of the highest type in the pool, which means that pooling with a higher type is more costly. As a result, types below 1 can sell at a price above the price obtained by some types above 1, and moreover, as explained above, there is nonmonotonicity in which lower types below 1 are pooled with higher types above 1. This nonmonotonicity continues to hold under different planner's objective functions. Examples are when the planner's objective is to maximize $\sum_{\theta\in\Theta} p(\theta) \sum_{i=1}^{k} h_i(\theta)$, as discussed above, or when the planner's objective reflects some (unmodeled) social gain from having type θ_i sell its asset.

4.5 Non-monotonicity and free disposal

The two forms of non-monotonicity can lead to equilibrium outcomes in which low types end up with higher expected payoff than high types. For example, in Example 1, type θ_4 ends up with a higher expected payoff than type θ_3 because both types sell with probability 1 but type θ_4 sells for a higher price. Such an outcome is plausible if the bank and the regulator learn θ at the same time and the bank cannot affect θ. However, if the bank learns its θ before the regulator and can freely (and secretly)

[22] In the alternative model, type θ's reservation price changes to $\rho(\theta) = \begin{cases} \max\{1, \theta - r\} \text{ if } \theta \geq 1 \\ \theta, \text{ if } \theta < 1 \end{cases}$. Propositions 2 and 3, continue to hold.

dispose assets, the equilibrium above breaks down because a high type has strong incentives to increase its equilibrium payoff by destroying assets. We now explore optimal disclosure rules that are not exposed to free disposal when such free disposal is a possibility.

Specifically, we solve the planner's problem with the additional constraint that the bank's equilibrium payoff is weakly increasing in type. We refer to this constraint as the monotonicity constraint and to the solution to the constrained problem as an optimal monotone rule. We show in the appendix that optimal monotone rules continue to satisfy Lemma 4.[23] Hence, type θ's expected payoff is $u(\theta)$, as in equation (3). The monotonicity constraint is that for every two types $\theta' < \theta$,

$$u(\theta') \leq u(\theta). \tag{18}$$

We also show that when $E(\tilde{\theta}) < 1$, optimal monotone rules continue to satisfy the resource constraints in (11) with equality, or equivalently, for every $i \in \{1, ..., k\}$, the price for score $s \in S_i$ is ρ_i.[24] Hence, $u(\theta)$ reduces to

$$u(\theta) = \theta + r \Pr(\tilde{\varepsilon} \geq 1 - \theta) + \sum_{i=1}^{k} [\rho_i - \theta + r \Pr(\tilde{\varepsilon} < 1 - \theta)] h_i(\theta), \tag{19}$$

which is a linear combination of $\{h_i(\theta)\}_{i \in \{1,...,k\}}$. The term $\theta + r \Pr(\tilde{\varepsilon} \geq 1 - \theta)$ in (19) is the payoff that type θ obtains without selling its asset, and the coefficient of $h_i(\theta)$ is the extra payoff from selling at price ρ_i.

The planner's problem reduces to the problem in Lemma 5, with the additional constraint (18), where $u(\theta)$ is given by (19). This is a linear programming problem. As before, a solution exists and can be implemented with $k + 1$ scores such that (14) holds.

In the examples below, we illustrate the solutions for two special cases. We show that for some parameter values, optimal monotone rules continue to exhibit the two forms of nonmonotonicity discussed in the previous subsection. We also show that for some parameter values, the result in Proposition 2 no longer holds.

[23] See Lemma A-3.

[24] See Lemma A-5 in the appendix.

Example 3 Suppose there are two types above 1, as in Example 1. We show in the appendix that if $p(\theta_5)$ is sufficiently large, there exists a scalar $\bar{\alpha} > 0$ and functions $\gamma(\alpha)$, $\hat{\beta}(\alpha)$, $\Gamma(\alpha)$ that depend on the model parameters[25] such that the optimal monotone rule is given by

	θ_5	θ_4	θ_3	θ_2	θ_1
score s_1 (sell at price ρ_1)	$\gamma(\alpha^*)$	$\hat{\beta}(\alpha^*)$	α^*	α^*	1
score s_2 (sell at price ρ_2)			$1-\alpha^*$	$1-\alpha^*$	
score s_0 (keep asset)	$1-\gamma(\alpha^*)$	$1-\hat{\beta}(\alpha^*)$			

where $\alpha^* = \begin{cases} 0, \text{ if } \Gamma(0) > \Gamma(\bar{\alpha}) \\ \bar{\alpha} \text{ if } \Gamma(0) < \Gamma(\bar{\alpha}) \end{cases}$. (If $\Gamma(0) = \Gamma(\bar{\alpha})$, both $\alpha^* = 0$ and $\alpha^* = \bar{\alpha}$ are optimal.) Moreover, $0 < \hat{\beta}(0) < \hat{\beta}(\bar{\alpha}) < 1$ and $0 < \gamma(\bar{\alpha}) < \gamma(0) < 1$.

Example 3 illustrates two properties of optimal monotone rules.[26] First, for some parameter values ($\Gamma(0) > \Gamma(\bar{\alpha})$), lower types continue to sell at higher prices (types θ_4 and θ_5 sell at a price above the one obtained by types θ_2 and θ_3). However, to satisfy the monotonicity constraint, so that they do not have an incentive to destroy assets, they sell with probability that is less than 1. Second, for other parameter values ($\Gamma(0) < \Gamma(\bar{\alpha})$), it is no longer optimal that types above 1 with different reservation prices obtain different scores. Instead it is optimal to pool type θ_2 with type θ_1 so that type θ_2 sells its asset at a price above its reservation price. This increases the payoff for type θ_2, which is beneficial because it relaxes the monotonicity constraint for lower types. In the extreme case $\bar{\alpha} = 1$, all types that sell obtain the same score.

In the next example, optimal monotone rules exhibit the first type of nonmonotonicity (in probability of sale).

Example 4 Suppose there is only one type above 1 and the gain-to-cost ratio is decreasing in type. We show in the appendix that under the optimal monotone rule, the probability of selling the asset continues to be nonmonotone in θ: Lower types sell with higher probability than middle type. Relative to the case in which we do not

[25] The proof in the appendix contains the exact expressions.

[26] The result extends to a more general case in which there are two types above 1 and $G_i(\theta)$ is increasing in θ for every $\theta < 1$ and $i \in \{1, 2\}$.

impose the monotonicity constraint, the probability that low types sell is lower, while the probability that types in the middle sell is higher. In other words, the increase in sale probability as type decreases is moderated in order to satisfy the monotonicity constraint, but overall nonmonotonicity in probability of sale remains part of the solution.

5 Discussion

In this section, we discuss some of the assumptions, interpretations, and possible extensions of the model.

1. In our model risk sharing takes a simple form: a bank sells its asset to a competitive market. We obtain similar results if we assume instead that the bank can enter into more complicated derivative contracts, under which the bank replaces a random cash flow with a deterministic cash flow. Such derivative contracts are quite common in today's banking industry. The nature of our results also remains the same if banks can create risk-sharing arrangements among themselves such as in the traditional interbank market (see more below).

2. An interesting extension of our model would allow the regulator to provide funds to banks. Such an extension would suggest that in some cases, it is optimal to inject money not only to weak banks but also to strong banks, so that the market cannot distinguish among them.[27] For example, suppose there are two banks: strong ($\theta_1 = 1.2$, $\rho_1 = 1$) and weak ($\theta_2 = 0.4$), and the planner has a bailout fund in the amount of 0.4. Giving all the money to the weak bank identifies the bank as weak. Because the value of the weak bank after the cash injections is 0.8, it will not be able to sell its asset for a price of at least 1. So the weak bank will continue to face the risk that its cash holdings are below 1. Splitting the money equally between the two banks leads to a better outcome. Now, after the cash injection, the value of the strong bank is 1.4 and the value of the weak bank is 0.6. Since the market cannot

[27] This might have been part of the logic behind having all nine large banks obtain Troubled Asset Relief Program (TARP) funds during the financial crisis in 2009.

distinguish between the two banks, each bank will be able to sell its assets for a price of 1, which is the average value of both banks. Then both banks could guarantee cash holdings of 1.

3. In our model, all the economic surplus is captured by the banking sector, and so the regulator sets a disclosure rule aiming to maximize the surplus in the banking sector. However, our model can also easily capture externalities imposed by the banks on the rest of society. Suppose, for example, that when a bank of type θ fails (i.e., when $\theta + \varepsilon < 1$), society suffers some exogenous loss $l(\theta)$. Then the social gain from having a bank sell its asset is higher by $l(\theta)$. We can include this gain in the planner's objective function and take it into account in the design of the disclosure rule. In particular, we can capture this gain by adding $l(\theta)$ to the coefficient of $h_i(\theta)$ in the planner's objective function (10). Our main results, including the nonmonotonicity results, continue to hold in this case. Clearly, now the planner will have a stronger motive to help banks with a high $l(\theta)$. This may capture the familiar "too big to fail" argument, whereby regulators try to save institutions, whose failure will cause a big damage to the economy.

4. As in any standard mechanism design problem, we assumed that the regulator can commit to assigning scores according to the announced disclosure rule. It is important to note that in many cases this commitment would arise endogenously. For example, if banks create risk-sharing arrangements among themselves rather than with a third party, the regulator cannot gain by deviating from the optimal ex-ante disclosure rule, e.g., saying things are better than they are, since then banks will have insufficient funds to honor the agreements and they will all fail. Similarly, even if risk sharing is done with the market, as is explicitly the case in our model, the regulator cannot gain by deviating ex post in case there is a continuum of banks and the probability $p(\theta)$ of being a type θ represents the proportion of banks of this type in the continuum. In this case, maximizing the bank's ex-ante expected payoff is the same as maximizing the sum of banks' ex-post payoffs. Since the regulator is interested in that he has no incentive to deviate ex post and say that some banks are

doing better because this will come at the expense of other banks. In this sense, the regulator is very different from a single bank, and we indeed cannot expect a single bank to be able to commit on a disclosure rule. The bank cares only about its own payoff ex post and so will always want to deviate and disclose a better type. Hence, studying disclosure by the regulator and not by individual banks is very natural in the context of our model.

5. One objection people may have for the disclosure rule derived in our paper is that pooling strong banks with weak banks might be perceived as "cheating" by the regulator, which is unlikely to pass public scrutiny. It is important to note, however, that the regulator does not cheat or lie in our model. A high score does not necessarily mean that the bank is strong. It only means that, on average, the bank's cash flow is above the critical level, which is indeed correct. When the regulator gives a score, the regulator can also announce the average cash flow for banks receiving the score. It is then known that for a high score, these banks are not expected to fail, which ends up being true in equilibrium, and so there should not be any public objection to this practice used by the regulator. In addition, one can think of scores more broadly than just grades. Scores separate banks into groups, and assigning scores is isomorphic to recommending banks which groups to form. For example, one can think of scores as suggesting mergers among banks or joint liability arrangements as in Leitner (2005). We solved for the optimal design of groups under the constraint that each bank prefers to join the recommended group rather than stay in autarky, and under the assumption that idiosyncratic risk is fully diversified within a group. This might be the case if there is a continuum of banks of each type, or more realistically, if the regulator provides insurance against idiosyncratic risk within a group.[28]

6. While our model focuses on the optimal disclosure policy by a regulator, we believe that it can be used as a benchmark to think of credit rating agencies. For example, our model suggests that low types receiving high rating may be a feature of a socially optimal outcome. An interesting question is how the optimal disclosure

[28] We do abstract, however, from other issues of group formation, such as whether a bank receiving one score will attempt to form a link with a bank receiving a different score.

rule looks like when the regulator faces competition from credit rating agencies, or whether it is possible to implement risk sharing when the regulator and credit rating agencies have a different objectives.

7. Finally, our results could also be applied to other settings of Bayesian persuasion. The novelty in our setting relative to the broad Bayesian persuasion literature (aside from the micro foundations for the banking context) is that agents, whose type is being disclosed by a planner, know their type and so have different reservation prices. This generates interesting implications for the optimal disclosure rule, which are explored in Section 4. This is applicable for many other settings studied with Bayesian persuasion tools. For example, consider schools that grade students with different abilities, and potential employers who care about the average ability of students they hire. Suppose that students know their own abilities, and students can open their own business instead of getting hired. In this case, a student's reservation price is the benefit from opening his own business. Our analysis here can shed light on the way the school will communicate the information about the students.[29]

6 Conclusion

We provide a model of an optimal disclosure policy of a regulator, who has information about banks, and whose disclosure policy affects whether banks can take corrective actions, particularly whether banks can engage in risk sharing to ensure that their capital does not fall below some critical level. We show that during normal times, no disclosure is necessary, but during bad times (insufficient liquidity), partial disclosure is needed. Partial disclosure takes the form of different scores pooling together banks of different levels of strength. Two scores are sufficient if banks do not have the information that the regulator has. In this case, the optimal disclosure rule may take a simple form, such that banks whose forecasted capital is below some threshold obtain the low score and banks whose forecasted capital is above the threshold obtain the high score; we provide conditions for this to happen. More than two scores may

[29] Ostrovsky and Schwarz (2010) study a similar problem but without such reservation prices.

be needed if a bank shares the same information that the regulator has about the bank. In this case, the optimal disclosure rule is non-monotone. Among the strong banks, the stronger banks obtain higher scores, which reflect a higher asset value, on average, but among the weak banks that are pooled with strong banks, the weaker banks obtain higher scores when they are pooled with strong banks. This type of non-monotonicity continues to hold even if we impose monotonicity on equilibrium payoffs.

References

[1] Admati, Anat, and Paul Pfleiderer (2000). Forcing Firms to Talk: Disclosure Regulation and Externalities. *Review of Financial Studies*, 13, 479-519.

[2] Allen, Franklin, and Douglas Gale (2000). Financial Contagion. *Journal of Political Economy*, 108, 1-33.

[3] Alvarez, Fernando, and Gadi Barlevy. Mandatory Disclosure and Financial Contagion. Working paper.

[4] Angeletos, George-Marios, and Alessandro Pavan (2007). Efficient Use of Information and Social Value of Information. *Econometrica*, 75, 1103-1142.

[5] Bond, Philip, and Itay Goldstein (2014). Government Intervention and Information Aggregation by Prices. *Journal of Finance*, forthcoming.

[6] Bouvard, Matthieu, Pierre Chaigneau, and Adolfo De Motta (2014). Transparency in the Financial System: Rollover Risk and Crises. *Journal of Finance*, forthcoming.

[7] Burkart, Mike, Denis Gromb, and Fausto Panunzi (1997). Large Shareholders, Monitoring, and the Value of Firms. *Quarterly Journal of Economics*, 112, 693-728.

[8] Diamond, Douglas (1985). Optimal Release of Information by Firms. *Journal of Finance*, 60, 1071-1094.

[9] Elliott, Matthew, Benjamin Golub, and Matthew Jackson (forthcoming). Financial Networks and Contagion. *American Economic Review*.

[10] Fishman, Michael, and Kathleen Hagerty (1990). The Optimal Amount of Discretion to Allow in Disclosure. *Quarterly Journal of Economics*, 427-444.

[11] Fishman, Michael, and Kathleen Hagerty (2003). Mandatory Versus Voluntary Disclosure in Markets with Informed and Uninformed Customers. *Journal of Law, Economics, and Organization*, 19, 45-63.

[12] Gick, Wolfgang, and Thilo Pausch (2014). Bayesian Persuasion by Stress Test Disclosure. Working paper. Deutsche Bundesbank.

[13] Goldstein, Itay, and Haresh Sapra (2013). Should Banks' Stress Test Results be Disclosed? An Analysis of the Costs and Benefits. *Foundations and Trends in Finance*, 8, 1-54.

[14] Grossman, Sanford (1981). The Informational Role of Warranties and Private Disclosure About Product Quality. *Journal of Law and Economics*, 24, 461-483.

[15] Hirshleifer, Jack (1971). The Private and Social Value of Information and the Reward to Inventive Activity. *American Economic Review*, 61, 561-574.

[16] Innes, Robert D. (1990). Limited liability and incentive contracting with ex-ante action choices. *Journal of Economic Theory*, 52, 45–67.

[17] Kamenica, Emir, and Matthew Gentzkow (2011). Bayesian Persuasion. *American Economic Review*, 101, 2590-2615.

[18] Kartasheva, Anastasia, and Bilge Yilmaz (2012). Precision of Ratings, manuscript.

[19] Leitner, Yaron (2005). Financial Networks: Contagion, Commitment, and Private Sector Bailouts. *Journal of Finance*, 60, 2925-2953.

[20] Leitner, Yaron (2012). Inducing Agents to Report Hidden Trades: A Theory of an Intermediary. *Review of Finance*, 16, 1013-1042.

[21] Lizzeri, Alessandro (1999). Information Revelation and Certification Intermediaries. *Rand Journal of Economics*, 30, 214-231.

[22] Marin, Jose M., and Rohit Rahi (2000). Limited Information Revelation and Market Incompleteness. *Review of Economic Studies*, 67, 455-481.

[23] Morris, Stephen, and Hyun Song Shin (2002). Social Value of Public Information. *American Economic Review*, 92, 1521-1534.

[24] Morrison, Alan, and Lucy White (2013). Reputational Contagion and Optimal Regulatory Forbearance. *Journal of Financial Economics*, 110, 642–658.

[25] Nadarajah, Saralees, and Kotz, Samuel (2006). A truncated Cauchy distribution. *International Journal of Mathematical Education in Science and Technology*, 37, 605-608.

[26] Ostrovsky, Michael and Michael Schwarz (2010). Information Disclosure and Unraveling in Matching Markets. *American Economic Journal: Microeconomics* 2, 34–63.

[27] Prescott, Edward Simpson (2008). Should Bank Supervisors Disclose Information About Their Banks? Federal Reserve Bank of Richmond *Economic Review,* 94, 1-16.

[28] Shapiro, Joel, and David Skeie (2013). Information Management in Banking Crises. Working paper

Appendix

Proof of Lemma 1. From the text, the equilibrium price is $x(s) = \mu(s)$. Conditional on the bank's information, the bank's expected payoff is $E[R(\tilde{\theta} + \tilde{\varepsilon}|\tilde{s} = s)] = \mu(s) + r \Pr(\tilde{\theta} + \tilde{\varepsilon} \geq 1|\tilde{s} = s)$ if it keeps the asset and $R(\mu(s))$ if it sells. Hence, if $\mu(s) \geq 1$, it is optimal for the bank to sell because $R(\mu(s)) = \mu(s) + r > E[R(\tilde{\theta} + \tilde{\varepsilon}|\tilde{s} = s)]$. If $\mu(s) < 1$, it is optimal to keep the asset because $R(\mu(s)) = \mu(s) < E[R(\tilde{\theta} + \tilde{\varepsilon}|\tilde{s} = s)]$. The strict inequality follows from Assumption 1.

Proof of Lemma 2. The planner's problem is to choose a disclosure rule (S, g) to maximize the bank's ex-ante expected payoff $\sum_{\theta\in\Theta} p(\theta)u(\theta)$. From the law of iterated expectations,

$$\sum_{\theta\in\Theta} p(\theta) \sum_{s:\mu(s)\geq 1} \mu(s)g(s|\theta) = \sum_{\theta\in\Theta} p(\theta) \sum_{s:\mu(s)\geq 1} \theta g(s|\theta).$$

Hence,

$$\begin{aligned}\sum_{\theta\in\Theta} p(\theta)u(\theta) &= \sum_{\theta\in\Theta} p(\theta) \sum_{s:\mu(s)<1} [\theta + r \Pr(\tilde{\varepsilon} \geq 1 - \theta)]g(s|\theta) + \sum_{\theta\in\Theta} p(\theta) \sum_{s:\mu(s)\geq 1} [\theta + r]g(s|\theta) \\ &= \sum_{\theta\in\Theta} p(\theta)[\theta + r \Pr(\tilde{\varepsilon} \geq 1 - \theta)] \sum_{s:\mu(s)<1} g(s|\theta) + \sum_{\theta\in\Theta} p(\theta)[\theta + r] \sum_{s:\mu(s)\geq 1} g(s|\theta)\end{aligned}$$

Since $\sum_{s:\mu(s)<1} g(s|\theta) = 1 - \sum_{s:\mu(s)\geq 1} g(s|\theta)$, we obtain:

$$\sum_{\theta\in\Theta} p(\theta)u(\theta) = \sum_{\theta\in\Theta} p(\theta)[\theta + r \Pr(\tilde{\varepsilon} \geq 1 - \theta)] + r \sum_{\theta\in\Theta} p(\theta) \Pr(\tilde{\varepsilon} < 1 - \theta) \sum_{s:\mu(s)\geq 1} g(s|\theta). \quad (1)$$

Only the second term in the right-hand side of (1) is affected by the disclosure rule. Hence, maximizing $\sum_{\theta\in\Theta} p(\theta)u(\theta)$ is the same as maximizing $\sum_{\theta\in\Theta} p(\theta) \Pr(\tilde{\varepsilon} < 1 - \theta) \sum_{s:\mu(s)\geq 1} g(s|\theta)$.

Next observe that if (S, g) is a disclosure rule, $h(\theta) = \sum_{s\in S:\mu(s)\geq 1} g(s|\theta)$ satisfies (5), as follows. For every high score $s \in S$, $\mu(s) \geq 1$, which from equation (2) reduces to $\sum_{\theta\in\Theta} p(\theta)(\theta - 1)g(s|\theta) \geq 0$. Summing over all high scores, we obtain constraint (5).

Conversely, if $\{h(\theta)\}_{\theta\in\Theta}$ solves the problem in Lemma 2, a disclosure rule (S, g) such that $S = \{s_0, s_1\}$ and $h(\theta) = g(s_1|\theta))$ is optimal. The result follows.

Proof of Proposition 1. 1. Setting $h(\theta) = 1$ for every $\theta \in \Theta$ achieves the maximal attainable value for (4) and satisfies the resource constraint (5). Any other $h : \Theta \to [0, 1]$ reduces the value of (4), by Assumption 1.

2. Suppose h solves the problem in Lemma 2. By Assumption 1, $h(\theta) = 1$ for every $\theta \geq 1$. Now consider a type $\hat{\theta} < 1$ s.t. $h(\hat{\theta}) > 0$. We must have $h(\theta) = 1$ for every $\theta \in \Theta$ s.t. $G(\theta) > G(\hat{\theta})$ because if a type $\theta_i < 1$ exists s.t. $h(\theta_i) < 1$ and $G(\theta_i) > G(\hat{\theta})$, we obtain a contradiction to the optimality of h by defining an alternate solution $\tilde{h}(\theta) = \begin{cases} h(\theta) \text{ if } \theta \notin \{\theta_i, \hat{\theta}\} \\ h(\theta) + \Delta \text{ if } \theta = \theta_i \\ h(\theta) - \frac{p(\theta_i)(1-\theta_i)}{p(\hat{\theta})(1-\hat{\theta})}\Delta \text{ if } \theta = \hat{\theta} \end{cases}$. In particular, if $\Delta > 0$ is sufficiently small, $\tilde{h}$ is a function from Θ to $[0, 1]$, which satisfies (5) and increases the value of (4) by $p(\theta_i)\Pr(\tilde{\varepsilon} < 1 - \theta_i)\Delta - p(\hat{\theta})\Pr(\tilde{\varepsilon} < 1 - \hat{\theta})\frac{p(\theta_i)(1-\theta_i)}{p(\hat{\theta})(1-\hat{\theta})}\Delta$, which equals $\Delta p(\theta_i)(1 - \theta_i)[G(\theta_i) - G(\hat{\theta})] > 0$. Consequently, if $h(\hat{\theta}) > 0$, we must have $\sum_{\theta \geq 1} p(\theta)(\theta - 1) + \sum_{\theta < 1 : G(\theta) > G(\hat{\theta})} p(\theta)(\theta - 1) \geq 0$ to satisfy constraint (5). Since the coefficient of $h(\theta)$ in (4) is positive, h is given by (7).

Proof of Corollary 1. Under full disclosure, $\mu(s) = \theta$ for every $s \in S$ such that $g(s|\theta) > 0$. Hence, $h(\theta) = \begin{cases} 1 \text{ if } \theta \geq 1 \\ 0 \text{ if } \theta < 1 \end{cases}$. By Proposition 1, this outcome is optimal if $\theta_{\min} \geq 1$ and suboptimal if $\theta_{\min} < 1$. Under no disclosure, $\mu(s) = E(\tilde{\theta})$ for every $s \in S$. If $E(\tilde{\theta}) \geq 1$, the outcome is $h(\theta) = 1$ for every $\theta \in \Theta$, which is optimal. If $E(\tilde{\theta}) < 1$, the outcome is $h(\theta) = 0$ for every $\theta \in \Theta$, which is suboptimal.

Proof of Corollary 2. Let $\theta_i < \theta_j < 1$. Then $1 - \theta_i > 1 - \theta_j > 0$. From condition 1, $G(\theta_i) < G(\theta_j)$. The result then follows from Proposition 1.

Proof of Lemma 3. The proof is in the text.

Proof of Lemma 4. 1. Consider an optimal disclosure rule (S, g), a type $\theta' \geq 1$, and a score $s' \in S$ s.t. $g(s'|\theta') > 0$. Suppose to the contrary that θ' does not sell its asset upon obtaining s'. Consider an alternate rule defined by $\tilde{S} = S \cup \{\tilde{s}\}$, $\tilde{g}(s|\theta') = \begin{cases} g(s'|\theta') \text{ if } s = \tilde{s} \\ 0 \text{ if } s = s' \\ g(s|\theta) \text{ if } s \notin \{s', \tilde{s}\} \end{cases}$, and for $\theta \neq \theta'$, $\tilde{g}(s|\theta) = \begin{cases} g(s|\theta) \text{ if } s \neq \tilde{s} \\ 0 \text{ if } s = \tilde{s} \end{cases}$. Under

$(\tilde{S}, \tilde{g})$, the only type that obtains score $\tilde{s}$ is θ'. So the equilibrium price for score $\tilde{s}$ is θ'. By Lemma 3, θ' sells upon obtaining score $\tilde{s}$. Equilibrium prices for all other scores remain unchanged. Hence, the alternate rule increases the expected payoff for type θ' by $rg(s'|\theta)\Pr(\tilde{\varepsilon} < 1 - \theta')$, while keeping the payoffs for all other types unchanged.

2. Consider an optimal disclosure rule (S, g), a type $\theta \geq 1$, and a score $s \in S$ s.t. $g(s|\theta) \geq 0$. From part 1, type θ sells upon obtaining score s. So by Lemma 3, $\rho(\theta) \leq x(s)$. To show $x(s) = \mu(s)$, we show that every type θ' s.t. $g(s|\theta') > 0$ sells upon obtaining score s, so selling does not convey additional information to the market. If $\theta' > \theta$, type θ' sells from part 1. If $\theta' < \theta$, then $\rho(\theta') \leq \rho(\theta) \leq x(s)$, and type θ' sells by Lemma 3.

3. Consider a disclosure rule (S, g), not necessarily an optimal one, and suppose $g(s|\theta) = 0$ for every $\theta \geq 1$. Suppose to the contrary that in the equilibrium that is induced by (S, g), some types sell upon obtaining score s. Suppose the highest type that sells is $\tilde{\theta} < 1$. So $\tilde{\theta} < \rho(\tilde{\theta})$. Since the market does not expect to lose money, the sale price must satisfy $x(s) \leq \tilde{\theta}$. But then $x(s) < \rho(\tilde{\theta})$, which contradicts Lemma 3.

Lemma A-1 *If resources are scarce, any solution to the problem in Lemma 5 satisfies* $\sum_{\theta\in\Theta} p(\theta)(\theta - \rho_i)h_i(\theta) = 0$ *for every* $i \in \{1, ..., k\}$.

Proof of Lemma A-1. Suppose to the contrary that $\{h_i\}_{i=1,...,k}$ solves the problem in Lemma 5 and there exists $i \in \{1, ..., k\}$ s.t. $\sum_{\theta\in\Theta} p(\theta)(\theta - \rho_i)h_i(\theta) > 0$. Since resources are scarce, a type $\tilde{\theta} < 1$ exists such that $h(\tilde{\theta}) < 1$. Let $\Delta \in (0, 1 - h(\tilde{\theta})]$ and $\tilde{h}_j(\theta) = \begin{cases} h_j(\theta) + \Delta \text{ if } j = i \text{ and } \theta = \tilde{\theta} \\ h_j(\theta) \text{ otherwise} \end{cases}$. Then $\{\tilde{h}_i\}_{i=1,...k}$ satisfies the constraints in Lemma 5 and strictly increases the value of the objective function, leading to a contradiction to the optimality of $\{h_i\}_{i=1,...,k}$.

Remarks for footnote 13. We show that $E(\tilde{\theta}) < 1$ implies that resources are scarce, as follows. Summing up all resource constraints and changing the order of summation, we obtain $\sum_{\theta\in\Theta} p(\theta)(\theta - \rho_i)\sum_{i=1}^{k} h_i(\theta) \geq 0$. Since $\rho_i \geq 1$ for every

$i \in \{1, ..., k\}$, it follows that $\sum_{\theta\in\Theta} p(\theta)(\theta - 1)\sum_{i=1}^{k} h_i(\theta) \geq 0$. Since $E(\tilde{\theta}) < 1$, $\sum_{\theta\in\Theta} p(\theta)(\theta - 1) < 0$. Hence, a type $\theta \in \Theta$ exists for which $\sum_{i=1}^{k} h_i(\theta) < 1$.

Proof of Proposition 2. Consider an optimal disclosure rule (S, g) and two types $\theta_j > \theta_i > 1$ s.t. $\rho_j > \rho_i$. Let $\{h_z(\theta)\}_{z\in\{1,...,k\}}$ be the corresponding solution to Lemma 5. That is, for every $z \in \{1, ..., k\}$, $h_z(\theta) = \sum_{s\in S_i} g(s|\theta)$.

Suppose to the contrary that a score $s' \in S$ exists such that $g(s'|\theta_i) > 0$ and $g(s'|\theta_j) > 0$. Without loss of generality, $g(s'|\theta) = 0$ for every $\theta > \theta_j$. Then $h_j(\theta_i) > 0$ and $h_j(\theta_j) > 0$. From Lemma A-1, $\sum_{\theta\in\Theta} p(\theta)(\theta - \rho_z)h_z(\theta) = 0$ for every $z \in \{1, ..., k\}$. We obtain a contradiction to Lemma A-1 by constructing an alternate solution $\{\tilde{h}_z\}_{z=1,...,k}$ to the problem in Lemma 5, such that at least one of the resource constraint is satisfied with strict inequality.

Specifically, if $\rho_j \geq \theta_i$, construct $\tilde{h}_z(\theta) = \begin{cases} h_z(\theta) + \Delta \text{ if } z = i \text{ and } \theta = \theta_i \\ h_z(\theta) - \Delta \text{ if } z = j \text{ and } \theta = \theta_i \\ h_z(\theta) \text{ otherwise} \end{cases}$. It is easy to verify that if Δ is sufficiently small, $\tilde{h}_s(\theta)$ solves the problem in Lemma 5. In particular, resource constraint i is satisfied because $\sum_{\theta\in\Theta} p(\theta)(\theta - \rho_i)h_i(\theta) = 0$ and $\theta_i \geq \rho_i$ imply that $\sum_{\theta\in\Theta} p(\theta)(\theta - \rho_i)\tilde{h}_i(\theta) = \Delta p(\theta_i)(\theta_i - \rho_i) \geq 0$. Resource constraint j is satisfied because $\sum_{\theta\in\Theta} p(\theta)(\theta - \rho_j)\tilde{h}_j(\theta) = -\Delta p(\theta_i)(\theta_i - \rho_j) \geq 0$. Moreover, either $\theta_i > \rho_i$ or $\theta_i = \rho_i < \rho_j$. So at least one resource constraint is satisfied with strict inequality.

If instead $\rho_j < \theta_i$, then since $\sum_{\theta\in\Theta} p(\theta)(\theta - \rho_j)h_j(\theta) = 0$, a type $\tilde{\theta} < \rho_j$ exists such that $h_j(\tilde{\theta}) > 0$. The alternate solution is similar to $\tilde{h}$ but for type $\tilde{\theta}$, $\tilde{h}_i(\tilde{\theta})$ changes to $\tilde{h}_i(\tilde{\theta}) + \tilde{\Delta}$ and $\tilde{h}_j(\tilde{\theta})$ changes to $\tilde{h}_j(\tilde{\theta}) - \tilde{\Delta}$, where $\tilde{\Delta} = \frac{p(\theta_i)(\theta_i - \rho_j)}{p(\tilde{\theta})(\rho_j - \tilde{\theta})}\Delta > 0$. Again, it is easy to verify that the alternate solution solves the problem in Lemma 5. Resource constraint j continues to be binding because $\sum_{\theta\in\Theta} p(\theta)(\theta - \rho_j)\tilde{h}_j(\theta) = -p(\theta_i)(\theta_i - \rho_j)\Delta - p(\tilde{\theta})(\tilde{\theta} - \rho_j)\tilde{\Delta} = 0$. Resource constraint i is satisfied with strict inequality because

$$\begin{aligned} \sum_{\theta\in\Theta} p(\theta)(\theta - \rho_i)\tilde{h}_i(\theta) &= p(\theta_i)(\theta_i - \rho_i)\Delta + p(\tilde{\theta})(\tilde{\theta} - \rho_i)\tilde{\Delta} \\ &> p(\theta_i)(\theta_i - \rho_i)\Delta + p(\tilde{\theta})(\tilde{\theta} - \rho_j)\tilde{\Delta} = p(\theta_i)(\rho_j - \rho_i)\Delta > 0. \end{aligned}$$

Proof of Corollary 3. From Proposition 2, we need at least k scores, so that each type above 1 obtains a different score. Since resources are scarce, we need another score for types below 1 that do not sell their assets.

Proof of Proposition 3. Consider an optimal disclosure rule (S, g) and four types $\theta' < \theta'' < 1 < \theta_i < \theta_j$, such that $\rho_i < \rho_j$. (Clearly, $\rho_i \geq 1$.) Let $\{h_z(\theta)\}_{z\in\{1,...,k\}}$ be the corresponding solution to Lemma 5. That is, for every $z \in \{1, ..., k\}$, $h_z(\theta) = \sum_{s\in S_i} g(s|\theta)$. From Proposition 2, if θ' is pooled together with type θ_i, then $h_i(\theta') > 0$. Suppose to the contrary that θ'' is pooled together with type θ_j. So from Proposition 2, $h_j(\theta'') > 0$. From Lemma A-1, $\sum_{\theta\in\Theta} p(\theta)(\theta - \rho_i)h_i(\theta) = 0$ for every $i \in \{1, ..., k\}$. We obtain a contradiction to Lemma A-1 by constructing an alternate solution $\{\tilde{h}\}_{i=1,...,k}$ to the problem in Lemma 5 that satisfies at least one of the resource constraints with strict inequality. Specifically, construct $\tilde{h}$ from h as follows. For type θ'', reduce $h_j(\theta'')$ and increase $h_i(\theta'')$, both by a small $\Delta > 0$. For type θ', reduce $h_i(\theta')$ and increase $h_j(\theta')$, both by $\Delta_1 = \frac{p(\theta'')(\theta''-\rho_j)}{p(\theta')(\theta'-\rho_j)}\Delta > 0$. Clearly, $\{\tilde{h}\}_{i=1,...,k}$ keeps the value of the objective function unchanged. The resource constraint j continues to be binding because $-\Delta p(\theta'')(\theta''-\rho_j)+\Delta_1 p(\theta')(\theta'-\rho_j) = 0$. The resource constraint i is loosened because by simple algebra, $\Delta p(\theta'')(\theta''-\rho_i)-\Delta_1 p(\theta')(\theta'-\rho_i) = \Delta p(\theta'')\frac{(\rho_j-\rho_i)(\theta'-\theta'')}{(\theta'-\rho_j)} > 0$. Clearly, $\tilde{h}$ also satisfies the other constraints in Lemma 5.

Proof of Corollary 4. From Lemma 4 and Proposition 2, we know that there are types $\theta_j \geq 1$ and $\theta_i \geq 1$ with reservation prices $\rho_i = x'$ and $\rho_j = x''$, such that $h_i(\theta') > 0$ and $h_j(\theta'') > 0$. We must have $x'' \leq x'$ because otherwise $\theta_i < \theta_j$ and Proposition 3 implies that $h_j(\theta'') = 0$, which is a contradiction.

Lemma A-2 *If resources are scarce, the following expressions are well defined:* $a_1 = \max\{\theta \in \Theta : \sum_{\theta}^{\theta_{k+1}} p(\theta)(\rho_k - \theta) > p(\theta_k)(\theta_k - \rho_k)\}$, $b_1 = \frac{p(\theta_k)(\theta_k-\rho_k)-\sum_{\theta:\theta\in(a_1,\theta_{k+1}]}(\rho_k-\theta)}{p(a_1)(\rho_k-a_1)}$, $c_1 = 1 - b_1$*; and for* $i > 1$*, define recursively* a_i *to be the largest type* $\theta' \leq a_{i-1}$*, such that* $\sum_{\theta:[\theta',a_{i-1})} p(\theta)(\rho_{k-i} - \theta) + c_{i-1}p(a_{i-1})(\rho_{k-i} - a_{i-1}) > p(\theta_{k-i})(\theta_{k-i} - \rho_{k-i})$,

$$b_i = \frac{p(\theta_{k-i})(\theta_{k-i} - \rho_{k-i}) - \sum_{\theta:\theta\in(a_i,a_{i-1})} p(\theta)(\rho_{k-i} - \theta) - c_{i-1}p(a_{i-1})(\rho_{k-i} - a_{i-1})}{p(a_i)(\rho_{k-i} - a_i)},$$

and $c_i = \begin{cases} c_{i-1} - b_i & \text{if } a_i = a_{i-1} \\ 1 - b_i & \text{otherwise} \end{cases}$

Proposition A-1 *If resources are scarce and* $G_i(\theta)$ *is increasing in* θ *for every* $\theta < 1$ *and* $i \in \{1, ..., k\}$*, the optimal disclosure rule is such that for every* $i \in \{1, ..., k\}$*,*

$$h_{k+1-i}(\theta) = \begin{cases} 1 & if \quad \theta = \theta_i \text{ or } \theta \in (a_i, a_{i-1}) \\ c_{i-1} & if \quad \theta = a_{i-1} \text{ and } a_i < a_{i-1} \\ b_i & if \quad \theta = a_i \\ 0 & if \quad \theta < a_i \end{cases}$$

where $a_i, \dot{b}_i$*, and* c_i *are defined in Lemma A-2.*

Lemma A-3 *Lemma 4 continues to hold when we restrict attention to monotone rules.*

Proof of Lemma A-3. Suppose (S, g) is an optimal monotone rule with equilibrium prices $x(s)$. By Lemma 3, type θ's expected payoff is

$$\bar{V}(\theta) \equiv \sum_{s:x(s)<\rho(\theta)} [\theta + r \Pr(\tilde{\varepsilon} \geq 1 - \theta)] g(s|\theta) + \sum_{s:x(s)\geq\rho(\theta)} [x(s) + r] g(s|\theta). \qquad (2)$$

The probability that type θ sells its asset is $\bar{h}(\theta) = \sum_{s:x(s)\geq\rho(\theta)} g(s|\theta)$. The price equals the expected cash flow of types purchasing the asset conditional on obtaining the score. Hence, $\sum_{\theta:\rho(\theta)\leq x(s)} p(\theta)[\theta - x(s)]g(s|\theta) = 0$ for every $s \in S$. The monotonicity constraint is that for every two types $\theta' < \theta$, $\bar{V}(\theta') \leq \bar{V}(\theta)$. We prove part 1 below. Parts 2 and 3 follow as in Lemma 4.

Consider a type $\theta_i > 1$. Suppose to the contrary that $\bar{h}(\theta_i) < 1$. So by Lemma 3, a score s' exists such that $g(s'|\theta_i) > 0$ and $x(s') < \rho_i$. Without loss, $\theta_i = \max\{\theta : \bar{h}(\theta) < 1\}$. If $\theta_i = \theta_1$ or $\bar{V}(\theta_i) < \bar{V}(\theta_{i-1})$, apply the proof of Lemma 4, but the probability that θ_i gets its own score must be sufficiently low so that monotonicity is preserved.

The remainder of the proof applies when $\theta_i < \theta_1$ and $\bar{V}(\theta_i) = \bar{V}(\theta_{i-1})$. Let $\theta_j = \max\{\theta : \bar{V}(\theta) = \bar{V}(\theta_i)\}$. Let $x_{\min} = \min\{x(s) : g(s|\theta_j) > 0\}$. Since $\bar{h}(\theta_j) = 1$,

$\bar{V}(\theta_j) \geq x_{\min} + r$. Observe that $x_{\min} + r \geq \rho(\theta_j) + r \geq \theta_j + r \Pr(\tilde{\varepsilon} \geq 1 - \theta_j) > \theta_i + r \Pr(\tilde{\varepsilon} \geq 1 - \theta_i)$. Hence, to satisfy $\bar{V}(\theta_i) = \bar{V}(\theta_j)$, there must be scores s_i and s_j, such that $g(s_i|\theta_i) > 0$, $g(s_j|\theta_j) > 0$, and $x(s_i) > x(s_j) \geq \rho_j > \rho_i$, because otherwise $\bar{V}(\theta_i) < \bar{V}(\theta_j)$.

Case 1: $\theta_i \geq x(s_i)$. Apply the logic from Lemma 4, but to satisfy the monotonicity constraint, reduce $g(s_i|\theta_i)$ and increase $g(s_j|\theta_i)$ so that θ_i's payoff is unchanged. To keep prices unchanged, increase $g(s_i|\theta_j)$ and reduce $g(s_j|\theta_j)$, so that the resources that type θ_j does not provide for score s_i are provided by type θ_i, and the resources that θ_j does not provide for score s_j are provided by θ_i. Formally, for a given $\Delta > 0$, let $\Delta_1 = \frac{p(\theta_j)}{p(\theta_i)} \frac{\theta_j - x(s_i)}{\theta_i - x(s_i)} \Delta$, $\Delta_2 = \frac{p(\theta_j)}{p(\theta_i)} \frac{\theta_j - x(s_j)}{\theta_i - x(s_j)} \Delta$, $\Delta_3 = \frac{\Delta_1[x(s_i) - \theta_i] - \Delta_2[x(s_j) - \theta_i] + (\Delta_1 - \Delta_2)[r \Pr(\tilde{\varepsilon} < 1 - \theta_1)]}{r \Pr(\tilde{\varepsilon} < 1 - \theta_1)}$. Observe that $\Delta_1 > \Delta_2 > 0$ and $\Delta_3 > 0$. Consider $(\tilde{S}, \tilde{g})$ defined by $\tilde{S} = S \cup \{\tilde{s}\}$,

$$\tilde{g}(s|\theta_i) = \begin{cases} g(s|\theta_i) - \Delta_1 \text{ if } s = s_i \\ g(s|\theta_i) + \Delta_2 \text{ if } s = s_j \\ \Delta_3 \text{ if } s = \tilde{s} \\ g(s|\theta_j) + \Delta_1 - \Delta_2 - \Delta_3 \text{ if } s = s' \\ g(s|\theta) \text{ if } s \notin \{s_i, s_j, s', \tilde{s}\} \end{cases}, \quad \tilde{g}(s|\theta_j) = \begin{cases} g(s|\theta_j) + \Delta \text{ if } s = s_i \\ g(s|\theta_j) - \Delta \text{ if } s = s_j \\ 0 \text{ if } s = \tilde{s} \\ g(s|\theta) \text{ if } s \notin \{s_i, s_j\} \end{cases},$$

and for $\theta \notin \{\theta_i, \theta_j\}$, $\tilde{g}(s|\theta) = \begin{cases} g(s|\theta) \text{ if } s \neq \tilde{s} \\ 0 \text{ if } s = \tilde{s} \end{cases}$. If Δ is sufficiently small, $(\tilde{S}, \tilde{g})$ is a disclosure rule. Clearly, prices for scores $s \notin \{s', \tilde{s}, s_i, s_j\}$ are the same under (S, g) and $(\tilde{S}, \tilde{g})$. Prices for scores s_i and s_j are also the same under both rules because the average cash flow conditional on obtaining each score and purchasing the asset remains unchanged. Formally, since $-p(\theta_i)[\theta_i - x(s_i)]\Delta_1 + p(\theta_j)[\theta_j - x(s_i)]\Delta = 0$, it follows that $\sum_{\theta:\rho(\theta) \leq x(s_i)} p(\theta)[\theta - x(s_i)]\tilde{g}(s|\theta) = \sum_{\theta:\rho(\theta) \leq x(s_i)} p(\theta)[\theta - x(s_i)]g(s|\theta) = 0$, and since $p(\theta_i)[\theta_i - x(s_j)]\Delta_2 - p(\theta_j)[\theta_j - x(s_j)]\Delta = 0$, it follows that $\sum_{\theta:\rho(\theta) \leq x(s_j)} p(\theta)[\theta - x(s_j)]\tilde{g}(s|\theta) = \sum_{\theta:\rho(\theta) \leq x(s_j)} p(\theta)[\theta - x(s_j)]g(s|\theta) = 0$. The price for score s' remains $x(s')$ because if Δ is sufficiently small, the average cash flow for score s' remains below ρ_i even if we include type θ_i, and so type θ_i continues not to sell upon obtaining s'. The price for score $\tilde{s}$ is Δ_3. Type θ_j's payoff increases by $\Delta[x(s_i) - x(s_j)]$, but if Δ is sufficiently small, monotonicity is preserved. Type θ_i's payoff remains unchanged from the definition of Δ_3. Clearly, payoffs for all other types remain unchanged. Hence, a contradiction to the optimality of (S, g).

Case 2: $\theta_i < x(s_i)$. Now type θ_i takes resources from s_i, so a higher type ex-

ists that provides resources. To satisfy the monotonicity constraint for type θ_i, we reduce $g(s_i|\theta_i)$. To keep the price for s_i unchanged, we reduce the probability that the higher type obtains score s_i. Formally, let $\theta_z = \max\{\theta : g(s_i|\theta) > 0\}$. So $\theta_z > x(s_i)$, $g(s'|\theta_z) = 0$, and $x(s_i) \geq \rho_z \geq \theta_z - r\Pr(\tilde{\varepsilon} < 1 - \theta_z) > \theta_i - r\Pr(\tilde{\varepsilon} < 1 - \theta_i)$. For a given $\Delta' > 0$, let $\Delta_4 = \frac{p(\theta_i)[x(s_i)-\theta_i]}{p(\theta_z)[\theta_z - x(s_i)]}\Delta'$, $\Delta_5 = \frac{r\Pr(\tilde{\varepsilon}<1-\theta_i)+x(s_i)-\theta_i}{r\Pr(\tilde{\varepsilon}<1-\theta_i)}\Delta'$, $\Delta_6 = \frac{r\Pr(\tilde{\varepsilon}<1-\theta_z)+x(s_i)-\theta_z}{r\Pr(\tilde{\varepsilon}<1-\theta_z)}\Delta_4$. Then $\Delta_4 > \Delta_6 > 0$, $\Delta' > \Delta_5 > 0$. Consider $(\hat{S}, \hat{g})$ defined by $\hat{S} = S \cup \{\tilde{s}_i, \tilde{s}_z\}$, $\hat{g}(s|\theta_i) = \begin{cases} g(s|\theta_i) - \Delta' \text{ if } s = s_i \\ \Delta_5 \text{ if } s = \tilde{s}_i \\ g(s|\theta_i) + \Delta' - \Delta_5 \text{ if } s = s' \\ 0 \text{ if } s = \tilde{s}_z \\ g(s|\theta_i) \text{ if } s \notin \{s_i, \tilde{s}_i, s'\} \end{cases}$, $\hat{g}(s|\theta_z) = \begin{cases} g(s|\theta_z) - \Delta_4 \text{ if } s = s_i \\ \Delta_6 \text{ if } s = \tilde{s}_z \\ \Delta_4 - \Delta_6 \text{ if } s = s' \\ 0 \text{ if } s = \tilde{s}_i \\ g(s|\theta_z) \text{ if } s \notin \{s_i, \tilde{s}_z, s'\} \end{cases}$ and for $\theta \notin \{\theta_i, \theta_z)$, $\hat{g}(s|\theta) = \begin{cases} g(s|\theta) \text{ if } s \notin \{\tilde{s}_i, \tilde{s}_z\} \\ 0 \text{ if } s \in \{\tilde{s}_i, \tilde{s}_z\} \end{cases}$. If Δ' is sufficiently small, $(\hat{S}, \hat{g})$ is a disclosure rule. The cash flow conditional on score s' remains below ρ_i even if we include type θ_i and θ_z, so these types continue not sell upon obtaining score s', and the price remains $x(s')$. The price for score $\tilde{s}_i$ is θ_i, and the price for $\tilde{s}_z$ is θ_z. The prices for all other scores are the same under (S, g) and $(\hat{S}, \hat{g})$. For score s_i, this follows because $-p(\theta_i)[\theta_i - x(s_i)]\Delta' - p(\theta_z)[\theta_z - x(s_i)]\Delta_4 = 0$. Type θ_i's payoff remains unchanged because $-\Delta'[x(s_i) - \theta_i + r\Pr(\tilde{\varepsilon} < 1 - \theta_i)] + \Delta_5 r\Pr(\tilde{\varepsilon} < 1 - \theta_i) = 0$. Type θ_z's payoff remains unchanged because $-\Delta_4[x(s_i) - \theta_z + r\Pr(\tilde{\varepsilon} < 1 - \theta_i)] + \Delta_6 r\Pr(\tilde{\varepsilon} < 1 - \theta_z) = 0$. Clearly, payoffs for all types also remain unchanged. The probability that θ_z sells its asset is less than 1 because $\hat{g}(s'|\theta_z) > 0$. Restart the proof of this Lemma for the problem in which θ_z is the highest type above 1 that sells with probability less than 1. Since there is a finite number of types, the process ends in a finite number of steps leading a contradiction to the optimality of (S, g).

Lemma A-4 *Suppose $E(\tilde{\theta}) < 1$. Under an optimal monotone rule, a type $\theta_i < 1$ exists such that $h(\theta_i) < 1$ and $u(\theta_i) < u(\theta_{i-1})$.*

Proof of Lemma A-4 Suppose to the contrary that for every type $\theta_i < 1$, either

$h(\theta_i) = 1$ or $u(\theta_i) = u(\theta_{i-1})$. From Lemma A-3, $h(\theta_i) = 1$ for every $i \in \{1, ..., k\}$. By induction on i, $u(\theta_i) \geq 1 + r$ for every $i \in \{1, .., m\}$. Hence, $\sum_{\theta \in \Theta} p(\theta)u(\theta) \geq 1 + r$. But since the market breaks even, $\sum_{\theta \in \Theta} p(\theta)u(\theta) \leq E(\tilde{\theta}) + r < 1 + r$.

Lemma A-5 *Suppose $E(\tilde{\theta}) < 1$. Under an optimal monotone rule, if $s' \in S_j$ and $j \in \{1, ..., k\}$, then $x(s') = \rho_j$.*

Proof of Lemma A-5. Consider an optimal monotone rule (S, g) and a score $s' \in S_j$, where $j \in \{1, ..., k\}$. From Lemmas 3 and A-3, $x(s') = \mu(s') \geq \rho_j$. Suppose to the contrary that $\mu(s') > \rho_j$. Let $\theta_i = \min\{\theta \in \Theta : g(s'|\theta) > 0\}$. Without loss, $g(s'|\theta) = 0$ if $\theta \notin \{\theta_i, \theta_j\}$. Hence,

$$p(\theta_i)g(s'|\theta_i)[\mu(s') - \theta_i] = p(\theta_j)g(s'|\theta_j)[\theta_j - \mu(s')]. \tag{3}$$

Since $E(\tilde{\theta}) < 1$, a type $\theta_z < 1$ exists such that $h(\theta_z) < 1$ and $V(\theta_z) < V(\theta_{z-1})$ (Lemma A-4). Hence, a score $s_0 \in S$ exists such that $g(s_0|\theta_z) > 0$ and $x(s_0) < \rho(\theta_z)$, so type θ_z does not sell upon obtaining s_0.

Case 1. $h(\theta_i) < 1$. Then there exists a score $\tilde{s}_0 \in S$, such that $g(\tilde{s}_0|\theta_i) > 0$ and $x(\tilde{s}_0) < \rho_i$. From Lemma A-3, $\theta_i < 1 < \theta_j$. We construct an alternate monotone rule that increases type θ_z's payoff and keeps the payoffs of all other types unchanged. Under the alternate rule, the price for score s' drops to $x(s') - \varepsilon$, and $g(s'|\theta_z)$ increases. To keep θ_i's payoff unchanged, we increase $g(s'|\theta_i)$, and to keep θ_j's payoff unchanged, we assign it its own score. Formally, for a given $\varepsilon > 0$, let Δ solve

$$[g(s'|\theta_j) - \Delta][\theta_j - \mu(s') + \varepsilon] = g(s'|\theta_j)[\theta_j - \mu(s')], \tag{4}$$

Δ_1 solve

$$[g(s'|\theta_i) + \Delta_1][\mu(s') - \varepsilon - \theta_i] + \Delta_1 r F(1 - \theta_i) = g(s'|\theta_i)[\mu(s') - \theta_i], \tag{5}$$

and $\tilde{\Delta}$ solve

$$\begin{aligned} & p(\theta_i)[g(s'|\theta_i) + \Delta_1][\mu(s') - \varepsilon - \theta_i] + \tilde{\Delta} p(\theta_z)[\mu(s') - \varepsilon - \theta_z] \\ = \; & p(\theta_j)[g(s'|\theta_j) - \Delta][\theta_j - \mu(s') + \varepsilon]. \end{aligned} \tag{6}$$

Then $\Delta = \frac{g(s'|\theta_j)\varepsilon}{\theta_j - \mu(s') + \varepsilon} > 0$, $\Delta_1 = \frac{g(s'|\theta_i)\varepsilon}{\mu(s') - \varepsilon - \theta_i + rF(1-\theta_i)}$, and $\tilde{\Delta} = \frac{p(\theta_i)\Delta_1 rF(1-\theta_i)}{p(\theta_z)[\mu(s') - \varepsilon - \theta_z]}$.[30] Consider an alternate rule $(\tilde{S}, \tilde{g})$, defined by $\tilde{S} = \{S, \tilde{s}_j\}$, $\tilde{g}(s'|\theta) = \begin{cases} g(s'|\theta) + \Delta_1 \text{ if } \theta = \theta_i \\ \tilde{\Delta} \text{ if } \theta = \theta_z \\ g(s'|\theta) - \Delta \text{ if } \theta = \theta_j \\ 0 \text{ if } \theta \notin \{\theta_i, \theta_j, \theta_z\} \end{cases}$, $\tilde{g}(s_0|\theta) = \begin{cases} g(s_0|\theta) - \tilde{\Delta} \text{ if } \theta = \theta_z \\ g(s_0|\theta) \text{ if } \theta \neq \theta_z \end{cases}$, $\tilde{g}(\tilde{s}_0|\theta) = \begin{cases} g(\tilde{s}_0|\theta) - \Delta_1 \text{ if } \theta = \theta_i \\ g(\tilde{s}_0|\theta) \text{ if } \theta \neq \theta_i \end{cases}$, $\tilde{g}(\tilde{s}_j|\theta) = \begin{cases} \Delta \text{ if } \theta = \theta_j \\ 0 \text{ if } \theta \neq \theta_j \end{cases}$, and for $s \notin \{s, s_0, \tilde{s}_0, \tilde{s}_j\}$, $\tilde{g}(s|\theta) = g(s|\theta)$ for every $\theta \in \Theta$. If ε is sufficiently small, $\Delta_1 > 0$, $\tilde{\Delta} > 0$, and $(\tilde{S}, \tilde{g})$ is a disclosure rule. From (6), the expected cash flow conditional on score s' is $\mu(s') - \varepsilon > \rho_j$. Hence, types $\theta_i, \theta_j, \theta_z$ sell upon obtaining score s', and the price is $\mu(s') - \varepsilon$. The price for score $\tilde{s}_j$ is Δ. Clearly, prices for all other scores are the same as under (S, g). Type θ_z's payoff increases by $\tilde{\Delta}[\mu(s') - \varepsilon + rF(1-\theta_z)]$, but if ε is sufficiently small, monotonicity is preserved. The payoffs for types θ_i and θ_j remain unchanged by equations (5) and (4), respectively.

Case 2. $h(\theta_i) = 1$ and $u(\theta_i) > u(\theta_{i+1})$. Since maximizing $\sum_{\theta \in \Theta} p(\theta)u(\theta)$ is the same as maximizing $\sum_{\theta \in \Theta} p(\theta) \Pr(\tilde{\varepsilon} < 1 - \theta)h(\theta)$, to obtain a contradiction, it is sufficient to construct an alternate monotone rule that increases $h(\theta_z)$ and for every $\theta \neq \theta_z$, keeps $h(\theta)$ unchanged. If $\theta_j = \theta_i$, then $\theta_i > 1$, and the alternate rule assigns to type θ_z score s' instead of s_0, with a small probability ε. Type θ_i's payoff drops by $\varepsilon g(s'|\theta_i)$, but if ε is sufficiently small, monotonicity is preserved. If instead $\theta_j > \theta_i$, define $(\tilde{S}, \tilde{g})$ as in case 1 but set $\Delta_1 = 0$ and $\tilde{\Delta} = \frac{p(\theta_i)g(s'|\theta_i)\varepsilon}{p(\theta_z)[\mu(s') - \varepsilon - \theta_z]}$. Again, Type θ_i's payoff drops but monotonicity is preserved.

Case 3. $h(\theta_i) = 1$ and $u(\theta_i) = u(\theta_{i+1})$. Let $\theta' = \min\{\theta : u(\theta) = u(\theta_i)\}$. Suppose the lowest score (that with lowest price) that θ_i obtains is s'' and the the highest score that θ' obtains is s'''.[31] We must have $\mu(s''') \geq \mu(s'')$ because $\mu(s''') < \mu(s'')$ implies $u(\theta') < u(\theta_i)$.

Case 3.1. $\mu(s'') < \mu(s')$, $\theta_i > \mu(s'')$. Then a type $\theta'' < \mu(s'')$ exists such that

[30] To derive $\tilde{\Delta}$, observe that from (5), $p(\theta_i)[g(s'|\theta_i) + \Delta_1][\mu(s') - \varepsilon - \theta_i] = p(\theta_i)g(s'|\theta_i)[\mu(s') - \theta_i] - p(\theta_i)\Delta_1 rF(1-\theta_i)$, and from (3) and (4), $p(\theta_j)[g(s'|\theta_j) - \Delta][\theta_j - \mu(s') + \varepsilon] = p(\theta_j)g(s'|\theta_j)[\theta_j - \mu(s')] = p(\theta_i)g(s'|\theta_i)[\mu(s') - \theta_i]$.

[31] Formally, $g(s''|\theta_i) > 0$, $g(s'''|\theta') > 0$; $x(s) \geq x(s'')$ for every $s \in S$ s.t. $g(s|\theta_i) > 0$; and $x(s) \leq x(s''')$ for every $s \in S$ s.t. $g(s|\theta') > 0$.

$g(s''|\theta'') > 0$. Without loss of generality, $g(s''|\theta) = 0$ for $\theta \notin \{\theta_i, \theta''\}$. Hence,

$$p(\theta_i)g(s''|\theta_i)[\theta_i - \mu(s'')] = p(\theta'')g(s''|\theta'')[\mu(s'') - \theta''] \tag{7}$$

As before, construct an alternate monotone rule that reduces $x(s')$. To keep θ_i's payoff unchanged, increase $g(s'|\theta_i)$ and reduce $g(s''|\theta_i)$. To keep the price for s'' unchanged, reduce $g(s''|\theta'')$. To keep the payoff of θ'' unchanged, increase $g(s'|\theta'')$. We focus on the case in which $\theta_j > \theta_i$. If $\theta_j = \theta_i$, apply the same as if θ_j does not exist, that is, set $p(\theta_j) = 0$.

Formally, for a given $\varepsilon > 0$, let $\Delta_6 = \frac{\varepsilon g(s'|\theta_i)}{\mu(s') - \varepsilon - \mu(s'')}$, $\Delta_7 = \frac{p(\theta_i)[\theta_i - \mu(s'')]}{p(\theta'')[\mu(s'') - \theta'']}\Delta_6$, $\Delta_8 = \frac{\mu(s'') - \theta'' + rF(1-\theta'')}{\mu(s') - \varepsilon - \theta'' + rF(1-\theta'')}\Delta_7$, and Δ_9 solve

$$\begin{aligned} & p(\theta_i)[g(s'|\theta_i) + \Delta_6][\mu(s') - \varepsilon - \theta_i] \\ & +\Delta_8 p(\theta'')[\mu(s') - \varepsilon - \theta''] + \Delta_9 p(\theta_z)[\mu(s') - \varepsilon - \theta_z] \\ = \; & p(\theta_j)[g(s'|\theta_j) - \Delta][\theta_j - \mu(s') + \varepsilon], \end{aligned} \tag{8}$$

$$\text{Let } \tilde{S} = \{S, \tilde{s}_j\}, \tilde{g}(s'|\theta) = \begin{cases} g(s'|\theta) - \Delta \text{ if } \theta = \theta_j \\ g(s'|\theta) + \Delta_6 \text{ if } \theta = \theta_i \\ g(s'|\theta) + \Delta_8 \text{ if } \theta = \theta'' \\ \Delta_9 \text{ if } \theta = \theta_z \\ g(s'|\theta) \text{ if } \theta \notin \{\theta_j, \theta_i, \theta_z\} \end{cases}, \tilde{g}(s_0|\theta) = \begin{cases} g(s_0|\theta) - \Delta_9 \text{ if } \theta = \theta_z \\ g(s_0|\theta) + \Delta_7 - \Delta_8 \text{ if } \theta = \theta'' \\ g(s_0|\theta) \text{ if } \theta \notin \{\theta_z, \theta''\} \end{cases},$$

$$\tilde{g}(\tilde{s}_j|\theta) = \begin{cases} \Delta \text{ if } \theta = \theta_j \\ 0 \text{ if } \theta \neq \theta_j \end{cases}, \tilde{g}(s''|\theta) = \begin{cases} g(s''|\theta) - \Delta_6 \text{ if } \theta = \theta_i \\ g(s''|\theta) - \Delta_7 \text{ if } \theta = \theta'' \\ g(s''|\theta) \text{ if } \theta \notin \{\theta_i, \theta''\} \end{cases}, \text{ and for } s \notin \{s', s_0, \tilde{s}_j, s''\},$$

$\tilde{g}(s|\theta) = g(s|\theta)$. Consider $(\tilde{S}, \tilde{g})$. If ε is sufficiently small, $\Delta_6 > 0$, $\Delta_7 > \Delta_8 > 0$, and $\Delta_9 > 0$.[32] Hence, $(\tilde{S}, \tilde{g})$ is a disclosure rule. From equation (8), the expected cash

[32] $\Delta_9 > 0$, as follows. From (3) and (4), $p(\theta_j)[g(s'|\theta_j) - \Delta][\theta_j - \mu(s') + \varepsilon] = p(\theta_i)g(s'|\theta_i)[\mu(s') - \theta_i]$. Hence, (8) reduces to

$$\begin{aligned} & p(\theta_i)\Delta_6[\mu(s') - \varepsilon - \theta_i] + \Delta_9 p(\theta_z)[\mu(s') - \varepsilon - \theta'] \\ & +\Delta_8 p(\theta'')[\mu(s') - \varepsilon - \theta''] = p(\theta_i)g(s'|\theta_i)\varepsilon, \end{aligned} \tag{9}$$

From definition of Δ_6,

$$\varepsilon g(s'|\theta_i) = \Delta_6[\mu(s') - \varepsilon - \theta_i)] + \Delta_6[\theta_i - \mu(s'')]$$

So

$$\Delta_9 p(\theta_z)[\mu(s') - \varepsilon - \theta'] = \Delta_6 p(\theta_i)[\theta_i - \mu(s'')] - \Delta_8 p(\theta'')[\mu(s') - \varepsilon - \theta'']$$

To show that $\Delta_9 > 0$, we need to show that

flow conditional on score s' is $\mu(s')-\varepsilon > \rho_j$. Hence, types $\theta_i, \theta_j, \theta_z$ sell upon obtaining score s', and the price is $\mu(s')-\varepsilon$. The price for score $\tilde{s}_j$ is Δ. The price for score s'' remains $\mu(s'')$ because $\Delta_7 p(\theta'')[\mu(s'')-\theta''] = \Delta_6 p(\theta_i)[\theta_i-\mu(s'')]$. Clearly, prices for all other scores remain the same. Type θ_z's payoff increases by $\Delta_9[\mu(s')-\varepsilon+rF(1-\theta_z)]$, but if ε is sufficiently small, monotonicity is preserved. The payoffs for types θ_i and θ'' remain unchanged from the definition of Δ_6 and Δ_8, respectively. Type θ_j's payoff remains unchanged by equation (4).

Case 3.2. $\mu(s'') < \mu(s')$, $\theta_i \leq \mu(s'')$. The alternate rule is similar to that in case 3.1, but now type θ_i takes resources from score s'', so the keep the price for s'' unchanged, increase $g(s''|\theta_z)$. Formally, let $\Delta_{10} = \frac{p(\theta_i)[\mu(s'')-\theta_i]}{p(\theta'')[\mu(s'')-\theta_z]}\Delta_6$, and Δ_{11} solve

$$\begin{aligned} & p(\theta_i)[g(s'|\theta_i)+\Delta_6][\mu(s')-\varepsilon-\theta_i] + \Delta_{11}p(\theta_z)[\mu(s')-\varepsilon-\theta_z] \\ = \;& p(\theta_j)[g(s'|\theta_j)-\Delta][\theta_j-\mu(s')+\varepsilon], \end{aligned}$$

If $\theta_j > \theta_i$, the alternate rule is defined by $\tilde{S} = \{S, \tilde{s}_j\}$, $\tilde{g}(s'|\theta) = \begin{cases} g(s'|\theta)-\Delta \text{ if } \theta = \theta_j \\ g(s'|\theta)+\Delta_6 \text{ if } \theta = \theta_i \\ \Delta_{11} \text{ if } \theta = \theta_z \\ g(s'|\theta) \text{ if } \theta \notin \{\theta_j, \theta_i, \theta_z\} \end{cases}$,

$\tilde{g}(s_0|\theta) = \begin{cases} g(s_0|\theta)-\Delta_{10}-\Delta_{11} \text{ if } \theta = \theta_z \\ g(s_0|\theta) \text{ if } \theta \notin \{\theta_z, \theta''\} \end{cases}$, $\tilde{g}(\tilde{s}_j|\theta) = \begin{cases} \Delta \text{ if } \theta = \theta_j \\ 0 \text{ if } \theta \neq \theta_j \end{cases}$, $\tilde{g}(s''|\theta) = \begin{cases} g(s''|\theta)-\Delta_6 \text{ if } \theta = \theta_i \\ g(s''|\theta)+\Delta_{10} \text{ if } \theta = \theta_z \\ g(s''|\theta) \text{ if } \theta \notin \{\theta_i, \theta_z\} \end{cases}$, and for $s \notin \{s', s_0, \tilde{s}_j, s''\}$, $\tilde{g}(s|\theta) = g(s|\theta)$. If $\theta_j = \theta_i$, ignore type θ_j, that is, set $p(\theta_j) = 0$.

Case 3.3. $\mu(s'') = \mu(s') = \mu(s''')$. First, combine scores s' and s''' into one score $\bar{s}$. That is, create a rule $(S, \bar{g})$, where $\bar{g}(s|\theta) = \begin{cases} g(s'|\theta)+g(s'''|\theta) \text{ if } s = \bar{s} \\ 0 \text{ if } s \in \{s', s'''\} \\ g(s|\theta) \text{ if } s \notin \{\bar{s}, s', s'''\} \end{cases}$. Clearly,

---

$$\Delta_6 p(\theta_i)[\theta_i - \mu(s'')] > \Delta_8 p(\theta'')[\mu(s') - \varepsilon - \theta'']$$

This reduces to

$$\Delta_7 p(\theta'')[\mu(s'') - \theta''] > p(\theta'')[\mu(s') - \varepsilon - \theta''] \frac{\mu(s'') - \theta'' + rF(1-\theta'')}{\mu(s') - \varepsilon - \theta'' + rF(1-\theta'')} \Delta_7$$

$$\frac{\mu(s'') - \theta''}{\mu(s') - \varepsilon - \theta''} > \frac{\mu(s'') - \theta'' + rF(1-\theta'')}{\mu(s') - \varepsilon - \theta'' + rF(1-\theta'')}$$

which follow since $\mu(s') - \varepsilon > \mu(s'')$.

$(S, \bar{g})$ is an optimal monotone rule, and the average cash flow for score $\bar{s}$ is $\mu(s')$. Since $\theta' \leq \mu(s') \leq \theta_j$, there is an optimal monotone rule $(\bar{S}, \bar{g}')$ and a score $s \in \bar{S}$ with price $\mu(s')$, such that the only types that obtains that score are θ' and θ_j. We can then apply case 1 or case 2 to obtain a contradiction.

Case 3.4. $\mu(s'') = \mu(s') < \mu(s''')$. We construct an alternate monotone rule under which the price for score s' drops to $x(s') - \varepsilon$, and $g(s'|\theta_z)$ increases, as in case 1. To keep θ_i's payoff unchanged, we increase $g(s'''|\theta_i)$ and reduce $g(s''|\theta_i)$. To keep prices unchanged, we reduce $g(s'''|\theta')$ and increase $g(s''|\theta')$. Formally, for a given $\varepsilon > 0$, let $\Delta_2 = \frac{\varepsilon g(s'|\theta_i)}{\mu(s''') - \mu(s'')}$, $\Delta_3 = \frac{p(\theta_i)[\mu(s''') - \theta_i]}{p(\theta')[\mu(s''') - \theta']}\Delta_2$, $\Delta_4 = \frac{p(\theta_i)[\mu(s'') - \theta_i]}{p(\theta')[\mu(s'') - \theta']}\Delta_2$, and Δ_5 solve

$$\begin{aligned} & (\Delta_3 - \Delta_4)p(\theta')[\mu(s') - \varepsilon - \theta'] + \Delta_5 p(\theta_z)[\mu(s') - \varepsilon - \theta'] \\ & +p(\theta_i)g(s'|\theta_i)[\mu(s') - \varepsilon - \theta_i] \\ = \;& p(\theta_j)[g(s'|\theta_j) - \Delta][\theta_j - \mu(s') + \varepsilon]. \end{aligned} \tag{10}$$

Then $\Delta_2 > 0$, $\Delta_3 > \Delta_4 > 0$, and $\Delta_5 > 0$.[33] Consider an alliterate rule $(\hat{S}, \hat{g})$, where

$$\hat{S} = \{S, \tilde{s}_j\}, \hat{g}(s'|\theta) = \begin{cases} g(s'|\theta) - \Delta \text{ if } \theta = \theta_j \\ \Delta_3 - \Delta_4 \text{ if } \theta = \theta' \\ \Delta_5 \text{ if } \theta = \theta_z \\ g(s'|\theta) \text{ if } \theta \notin \{\theta_j, \theta', \theta_z\} \end{cases}, \hat{g}(s_0|\theta) = \begin{cases} g(s_0|\theta) - \Delta_5 \text{ if } \theta = \theta_z \\ g(s_0|\theta) \text{ if } \theta \neq \theta_z \end{cases},$$

$$\hat{g}(\tilde{s}_j|\theta) = \begin{cases} \Delta \text{ if } \theta = \theta_j \\ 0 \text{ if } \theta \neq \theta_j \end{cases}, \hat{g}(s''|\theta) = \begin{cases} g(s''|\theta) - \Delta_2 \text{ if } \theta = \theta_i \\ g(s''|\theta) + \Delta_4 \text{ if } \theta = \theta' \\ g(s''|\theta) \text{ if } \theta \notin \{\theta_i, \theta'\} \end{cases}, \hat{g}(s'''|\theta) = \begin{cases} g(s'''|\theta) + \Delta_2 \text{ if } \theta = \theta_i \\ g(s'''|\theta) - \Delta_3 \text{ if } \theta = \theta' \\ g(s'''|\theta) \text{ if } \theta \notin \{\theta_i, \theta'\} \end{cases},$$

and for $s \notin \{s', s_0, \tilde{s}_j, s'', s'''\}$, $\hat{g}(s|\theta) = g(s|\theta)$. If ε is sufficiently small, $(\hat{S}, \hat{g})$ is a disclosure rule. From (8), the expected cash flow conditional on score s' is $\mu(s') - \varepsilon > \rho_j$. Hence, the price for score s' is $\mu(s') - \varepsilon$. The price for score $\tilde{s}_j$ is Δ. The prices for

[33] To see why $\Delta_5 > 0$ observe that from (3) and (4),

$$p(\theta_j)[g(s'|\theta_j) - \Delta][\theta_j - \mu(s') + \varepsilon] = p(\theta_j)g(s'|\theta_j)[\theta_j - \mu(s')] = p(\theta_i)g(s'|\theta_i)[\mu(s') - \theta_i],$$

Hence.

$$\Delta_5 = \frac{p(\theta_i)\varepsilon g(s'|\theta_i) - (\Delta_3 - \Delta_4)p(\theta')[\mu(s') - \varepsilon - \theta']}{p(\theta_z)[\mu(s') - \varepsilon - \theta']}.$$

$\Delta_5 > 0$ follows because

$$\begin{aligned} & (\Delta_3 - \Delta_4)p(\theta')[\mu(s') - \varepsilon - \theta'] < \Delta_3 p(\theta')[\mu(s''') - \theta'] - \Delta_4 p(\theta')[\mu(s'') - \theta'] \\ = \;& \Delta_2 p(\theta_i)[\mu(s''') - \theta_i] - \Delta_2 p(\theta_i)[\mu(s'') - \theta_i] = \Delta_2 p(\theta_i)[\mu(s''') - \mu(s'')] = p(\theta_i)\varepsilon g(s'|\theta_i) > 0 \end{aligned}$$

all other scores under $(\hat{S}, \hat{g})$ are the same as under (S, g). For score s'', this follows because $\Delta_2 p(\theta_\iota)[\mu(s'') - \theta_i] = \Delta_4 p(\theta')[\mu(s'') - \theta']$ and because θ' agrees to sell at price $\mu(s'')$. For s''', this follows because $\Delta_2 p(\theta_\iota)[\mu(s''') - \theta_i] = \Delta_3 p(\theta')[\mu(s''') - \theta']$ and because θ_i agrees to sell at price $\mu(s''')$. Relative to (S, g), under $(\hat{S}, \hat{g})$, the payoff for type θ' falls by $\Delta_3[\mu(s''') - \mu(s'')]$, and the payoff for type θ_z increases by $\Delta_5[\mu(s') - \varepsilon - \theta_5 + rF(1 - \theta_5)]$, but if ε is sufficiently small, monotonicity is preserved. The payoffs for all other types remain unchanged. For θ_i this follows because $\varepsilon g(s'|\theta_i) = \Delta_2[\mu(s''') - \mu(s'')]$. For θ_j, this follows from (4). The value of the planner's objective function increases because the probability of selling the asset weakly increases for every type and strictly increases for θ_z.

Lemma A-6 *Suppose $E(\tilde{\theta}) < 1$ and $G_i(\theta)$ is increasing in θ for $i \in \{1, ..., k\}$. Under an optimal monotone rule, if $\theta_i < 1$, $h(\theta_i) < 1$, and type θ_i ever sells its asset at price x, then lower types never sell at prices below x.*

Proof of Lemma A-6. The idea behind Lemma A-6 is that if type θ_i sells at price x and type $\theta' < \theta_i$ sells at price $x' < x$, the planner can increase the value of the objective function, as in Proposition 3. This could violate monotonicity because the payoff of type θ_i falls and that of type θ' increases. But because $G_i(\theta)$ is increasing, the planner can restore monotonicity (and increase the value of the objective function even further) by reducing $h(\theta')$ and increasing $h(\theta_i)$.

Proof of Proposition 3. Let

$$
\begin{aligned}
\hat{\beta}(\alpha) &\equiv \frac{\rho_2 + \alpha(\rho_1 - \rho_2) - [\theta_4 + r \Pr(\tilde{\varepsilon} \geq 1 - \theta_4)]}{\rho_1 - \theta_4 + r \Pr(\tilde{\varepsilon} < 1 - \theta_4)}, \\
\check{\beta}(\alpha) &\equiv \frac{p(\theta_1)(\theta_1 - \rho_1) - \alpha[p(\theta_2)(\rho_1 - \theta_2) + p(\theta_3)(\rho_1 - \theta_3)]}{p(\theta_4)(\rho_1 - \theta_4)}, \\
\gamma(\alpha) &\equiv \frac{p(\theta_1)(\theta_1 - \rho_1) - \alpha[p(\theta_2)(\rho_1 - \theta_2) + p(\theta_3)(\rho_1 - \theta_3)] - \min\{\hat{\beta}(\alpha), \check{\beta}(\alpha)\} p(\theta_4)(\rho_1 - \theta_4)}{p(\theta_5)(\rho_1 - \theta_5)}, \\
\hat{\gamma}(\alpha) &\equiv \frac{\rho_2 + \alpha(\rho_1 - \rho_2) - [\theta_5 + r \Pr(\tilde{\varepsilon} \geq 1 - \theta_5)]}{\rho_1 - \theta_5 + r \Pr(\tilde{\varepsilon} < 1 - \theta_5)}, \\
\Gamma(\alpha) &\equiv p(\theta_4) \Pr(\tilde{\varepsilon} < 1 - \theta_4) \hat{\beta}(\alpha) + p(\theta_5) \Pr(\tilde{\varepsilon} < 1 - \theta_5) \gamma(\alpha).
\end{aligned}
$$

Let $\breve{\alpha}$ be the unique solution to $\hat{\beta}(\alpha) = \breve{\beta}(\alpha)$ and $\bar{\alpha} = \min\{1, \breve{\alpha}\}$. Assume $\gamma(0) \leq \hat{\gamma}(0)$ (e.g., $p(\theta_5)$ is sufficiently large). Observe that $\bar{\alpha} > 0$, $\hat{\beta}(0) < \hat{\beta}(\bar{\alpha}) < 1$ and $\gamma(\bar{\alpha}) < \gamma(0) < 1$, where the last inequality follows from equation (17).

The derivation of the optimal monotone rule is as follows. Since Lemma 4 continues to hold, an $\alpha \in [0, 1]$ exists such that $h_1(\theta_1) = 1$, $h_1(\theta_2) = \alpha$, and $h_2(\theta_2) = 1 - \alpha$. From the resource constraint for score s_2, $h_2(\theta_2)p(\theta_2)(\theta_2 - \rho_2) \geq h_3(\theta_3)p(\theta_3)(\rho_2 - \theta_3)$. From equation (16), $h_2(\theta_3) \leq 1 - \alpha$. It is suboptimal to set $h_2(\theta_3) < 1 - \alpha$, as follows. Suppose by contradiction that $h_2(\theta_3) < 1 - \alpha$. To satisfy $u(\theta_3) \leq u(\theta_2)$, we must have $h_1(\theta_3) + h_2(\theta_3) < 1$, and from the resource constraint for score s_2, $h_2(\theta) > 0$ for some $\theta < \theta_3$. Hence, from Lemma A-6, $h_1(\theta_3) = 0$. That is, if type $\theta < \theta_3$ sells at price ρ_2, type θ_3 cannot sell at price $\rho_1 > \rho_2$. But then $u(\theta_3) < u(\theta_2)$, and since the gain-to-cost ratio is increasing, the planner can increase the value of the objective by transferring resources from the lowest type that sells with a positive probability to type θ_3. Hence, a contradiction. Consequently, $h_2(\theta_3) = 1 - \alpha$, and from equation (16), types θ_4 and θ_5 can obtain only scores s_0 and s_1. Since the gain-to-cost ratio is increasing, it is optimal to set $h_1(\theta_3) = \alpha$. As for $h_1(\theta_4)$, the planner would like to set it as high as possible, subject to the monotonicity constraint $u(\theta_4) \leq u(\theta_3)$ and the resource constraint for score s_1. The monotonicity constraint reduces to $h_1(\theta_4) \leq \hat{\beta}(\alpha)$. The resource constraint reduces to $h_1(\theta_4) \leq \breve{\beta}(\alpha)$. Hence, $h_1(\theta_4) = \min\{\hat{\beta}(\alpha), \breve{\beta}(\alpha)\}$. All remaining resources from type θ_1 are allocated to type θ_5 so that the resource constraint for type θ_1 is satisfied with equality. Hence, $h_1(\theta_5) = \gamma(\alpha)$. The monotonicity constraint $u(\theta_5) \leq u(\theta_4)$ reduces to $\gamma(\alpha) \leq \hat{\gamma}(\alpha)$ and is not binding, from the assumption $\gamma(0) \leq \hat{\gamma}(0)$ and the observation that $\gamma(\alpha)$ is decreasing in α and $\hat{\gamma}(\alpha)$ is increasing. Hence, the planner's problem reduces to choosing $\alpha \in [0, 1]$ to maximize $p(\theta_4)\Pr(\tilde{\varepsilon} < 1 - \theta_4)h_1(\theta_4) + p(\theta_5)\Pr(\tilde{\varepsilon} < 1 - \theta_5)\gamma(\alpha)$, such that $h_1(\theta_4) = \min\{\hat{\beta}(\alpha), \breve{\beta}(\alpha)\}$. Since $h_1(\theta_4)$ decreases in α when $\alpha > \breve{\alpha}$, it follows from the linearity of the problem that it is optimal to choose either $\alpha = 0$ or $\alpha = \bar{\alpha}$. The result follows.

Proposition A-2 *Suppose there is only one type above 1, $E(\tilde{\theta}) < \rho_1$, and $G(\theta)$ is decreasing in θ when $\theta < 1$. Let $\delta_\theta(\alpha) \equiv \frac{\theta_m + r\Pr(\tilde{\varepsilon} \geq 1-\theta_m) + \alpha[\rho_1 - \theta_m + r\Pr(\tilde{\varepsilon} < 1-\theta_m)] - [\theta + r\Pr(\tilde{\varepsilon} \geq 1-\theta)]}{\rho_1 - \theta + r\Pr(\tilde{\varepsilon} < 1-\theta)}$ and α^* be the (unique) α that solves $p(\theta_1)(\theta_1 - \rho_1) = \alpha p(\theta_m)(\rho_1 - \theta_m) + \sum_{i=2}^{m-1} p(\theta_i)(\rho_1 - \theta_i)\max\{0, \delta_\theta(\alpha)\}$.*

(i) Under the optimal monotone rule, type θ_1 sells with probability 1, type θ_m sells with probability α^*, type $\theta \in (\theta_m, \theta_2)$ sells with probability $\max\{0, \delta_\theta(\alpha^*)\}$, which is decreasing in θ.

(ii) The probability that low types sell is lower relative to the unconstrained benchmark (i.e., the problem without the monotonicity constraint), while the probability that high types (below 1) sell is higher.

Proof of Proposition A-2. From Lemmas A-3 and A-5, type θ_1 sells with probability 1, and the sale price is ρ_1. If the lowest type θ_m sells with probability α, the monotonicity constraint (18) implies that type $\theta \in (\theta_m, \theta_2)$ sells with probability of at $\delta_\theta(\alpha)$. Since $E(\tilde{\theta}) < \rho_1$ (i.e. resources are scarce) and $G(\theta)$ is decreasing in θ when $\theta < 1$, it is optimal that $\theta \in (\theta_m, \theta_2)$ sells with probability $\max\{0, \delta_\theta(\alpha)\}$. The optimal α satisfies the resource constraint with equality and is given by α^*. Part (ii) follows because in the problem without constraint (18), the optimal rule involves a cutoff, such that types below the cutoff and types above 1 sell with probability 1, and types in the middle sell with probability 0.

Figure 1: The figure illustrates the reservation price $\rho(\theta)$ as a function of θ.