Investment and The Cross–Section of Equity Returns*

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Abstract

In the neoclassical model of investment – with decreasing returns to scale and mean–reverting idiosyncratic productivity – small firms earn a higher expected return than large firms as long as the term structure of equity is increasing. This is the case, since low–productivity firms owe a larger fraction of their valuation to future cash flows. With large enough operating leverage, the model also delivers a value premium, as firms with high book-to-market ratios (value) are riskier than their counterparts with low book-to-market (growth). Consistent with the evidence, growth firms have seen their idiosyncratic productivity grow in recent times and invest to take full advantage of their enhanced efficiency. On the other hand, value firms divest in order to catch up with declining idiosyncratic productivity. When calibrated to match key moments of the cross–sectional distribution of investment and the average book-to-market ratio, however, the model delivers a value premium that is much smaller than found in the data. This result holds true for different specifications of the stochastic discount factors and does not depend upon the magnitude of capital adjustment costs.

Key words: Asset Pricing, Value Premium, Operating Leverage.

JEL Codes: D24, D92, G12.

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1 Introduction

A main objective of the rapidly developing field of macro–finance is to develop models that can rationalize cross–firm variation in both quantities and prices. The contribution of this paper is to characterize the implications for the cross–section of equity returns of a particular model – known as the neoclassical model of investment – which has become standard in the macroeconomics and firm dynamics literature because of its ability to replicate key features of the investment process.

Firms produce by means of a decreasing returns to scale production function and are subject to capital adjustment costs. Their productivity depends upon a common and an idiosyncratic component – mean–reverting and orthogonal to each other. Future cash-flows are discounted by means of an exogenously given time–varying stochastic discount factor.

A robust finding is that as long as the term structure of equity returns is upward sloping, small firms earn higher expected returns than large firms. The intuition is straightforward. Since idiosyncratic productivity is mean–reverting, small firms owe a larger fraction of their valuation to cash–flows that will realize in the distant future.

A somewhat surprising result is that the plain–vanilla version of the model – one without operating leverage – delivers a value discount. This is the case because in such model, size and book–to–market ratio are counterfactually positively correlated, and there is substantial overlap between small firms and growth firms.

When firms incur a large enough fixed operating cost, the model does generate a value premium. Consistent with the evidence, the book-to-market criterion select as growth firms entities whose productivity has been rising. Value firms, on the contrary, are shrinking to adjust their capacity to a rapid decline of the idiosyncratic component of productivity.

When calibrated to match key moments of the cross–sectional distribution of investment, as well as Sharpe ratio and the first two moments of the risk–free rate, the model delivers a value premium which is substantially smaller than in the data. This the case for all sensible assumptions on the magnitude of capital adjustment costs and on the nature of the stochastic discount factor.

The remainder of the paper is organized as follows. In Section 2 we consider a simple 3-period version of our model, with the purpose of developing intuition that may help us comprehend the implications of the fully fledged infinite-horizon model introduced in Section 3. The resulting covariation of equity returns with size, book-to-market, and investment rate is analyzed in Section 4. The robustness of our results to alternative assumptions on capital adjustment costs and the nature of the stochastic discount factor is assessed in sections 5 and 6, respectively.

2 A Three–Period Model

In this section, we lay out a simple three–period model of investment and we explore analytically its implications for the cross–section of equity returns. The time periods are indexed by $t = -1, 0, 1, 2$. Firms produce output by means of $y_t = e^{s_t + z_t} k_t^{\alpha}$, where $\alpha \in (0, 1)$ and $k_t \geq 0$ denotes the capital stock. We assume one–period time–to–build and geometric depreciation. Let $\delta \in (0, 1)$ be the depreciation rate. Dividends equal cash flows minus investment.

The variables s_t and z_t denote the idiosyncratic and aggregate components of productivity, respectively. Both evolve according to first–order autoregressive processes and independent, normally distributed innovations. That is,

$$s_{t+1} = \rho_s s_t + \varepsilon_s, \quad \varepsilon_s \sim N(\mu_s, \sigma_s^2),$$
$$z_{t+1} = \rho_z z_t + \varepsilon_z, \quad \varepsilon_z \sim N(\mu_z, \sigma_z^2),$$

where $\rho_s, \rho_z \in (0, 1)$ and $\sigma_s, \sigma_z > 0$.

At any time t, firms evaluate cash flows at $t + 1$ according to the stochastic discount factor $M_{t+1} \equiv M(z_t, z_{t+1})$. It follows that, conditional on capital k_1 and productivity levels $\{s_1, z_1\}$, the value of equity at $t = 1$ is

$$V_1(k_1, s_1, z_1) \equiv \max_{k_2} \; e^{s_1+z_1} k_1^{\alpha} + k_1(1 - \delta) - k_2 + E_1[M_2[e^{s_2+z_2} k_2^{\alpha} + k_2(1 - \delta)]],$$

where the linear operator E_s denotes the expectation taken conditional on the information known at $t = s$. As of $t = 0$, the firm's optimization problem is

$$\max_{k_1} \; -k_1 + E_0[M_1 V_1(k_1, s_1, z_1)].$$

2.1 Characterization

Define the expected equity return at time t as the expected value of cash–flows to equity–holders divided by the ex–dividend market value. Then, we can write the expected return on equity at time $t = 0$ as

$$E_0[R_1] = \frac{E_0[V_1(k_1, s_1, z_1)]}{E_0[M_1 V_1(k_1, s_1, z_1)]}. \tag{1}$$

With some abuse of notation, rewrite (1) as

$$E_0[R_1] = \frac{E_0[y_1 + k_1(1-\delta)] + E_0[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]}{E_0[M_1[y_1 + k_1(1-\delta)]] + E_0[M_1[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]]}.$$

We can express equity as a portfolio of two risky assets whose payoffs at time $t = 1$ and $t = 2$, respectively, are listed in the table below. We will refer to them as current and continuation asset, respectively.

	Payoff at $t = 1$	Payoff at $t = 2$
Current Asset	$e^{s_1+z_1}k_1^\alpha + k_1(1-\delta)$	0
Continuation Asset	$-k_2$	$e^{s_2+z_2}k_2^\alpha + k_2(1-\delta)$

The loading on the current asset, which we denote as $x(s_0, z_0)$, is the fraction of equity value accounted for by the current asset, or

$$x(s_0, z_0) = \frac{E_0[M_1[y_1 + k_1(1-\delta)]]}{E_0[M_1[y_1 + k_1(1-\delta)]] + E_0[M_1[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]]}.$$

Since the expected returns on current and continuation assets are independent of idiosyncratic productivity, the latter influences expected returns on equity only via its impact on the loading x.

Idiosyncratic productivity being mean–reverting, its expected growth rate is decreasing in s_0. It follows that x is strictly increasing in s_0. The continuation asset accounts for a larger fraction of the value of small firms. These claims are formally stated in Lemma 1. All proofs are in the appendix.

Lemma 1 *1. Equity is a portfolio of current and continuation assets, which pay off exclusively at $t = 1$ and $t = 2$, respectively;*

2. The excess return of neither asset depends on idiosyncratic productivity;

3. The current asset is itself a portfolio of the riskless asset and a risky asset with expected returns $\frac{E_0(e^{\varepsilon}_{z,1})}{E_0(M_1 e^{\varepsilon}_{z,1})}$. The loadings are both positive and are function of the risk–free rate.

4. The loading on the current asset is an increasing function of idiosyncratic productivity.

In order to determine how expected equity returns vary with s_0, we need to assess the slope of the equity term structure. That is, we need to establish whether the current asset commands a greater or lower expected return than the continuation asset. In order to answer this question, we make functional assumptions on the stochastic discount factor.

2.1.1 The Stochastic Discount Factor

For the remainder of this section, we will assume that the stochastic discount factor is given by

$$\log M_{t+1} \equiv \log \beta - \gamma \varepsilon^z_{t+1},$$

where $\gamma > 0$ disciplines aversion to risk and $\beta > 0$ is the time discount factor. This choice allows us to make the most progress in the analytical characterization, as the risk-free rate is constant. In fact, for all $t \geq 0$,

$$R_{ft} = R_f = \frac{1}{E_t[M_{t+1}]} = \frac{1}{\beta} e^{-\frac{1}{2}\gamma^2 \sigma^2_z}.$$

The maximum Sharpe ratio is also constant:

$$\frac{std(M_{t+1})}{E(M_{t+1})} = \sqrt{e^{\gamma^2 \sigma^2_z} - 1}.$$

Under these assumptions, we have that

$$\frac{E_0(e^{\varepsilon}_{z,1})}{E_0(M_1 e^{\varepsilon}_{z,1})} = \frac{E_0(e^{\varepsilon}_{z,1})}{E_0[e^{(1-\gamma)\varepsilon_{z,1}}]} = e^{\gamma \sigma^2_z} R_f.$$

Furthermore, the expected return on the continuation asset equals

$$\begin{aligned} \frac{E_0[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]}{E_0[M_1[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]]} &= e^{\frac{1}{2}\gamma^2\sigma^2_z} \frac{E_0\left[e^{\frac{\rho_z}{1-\alpha}\varepsilon_{z,1}}\right]}{E_0\left[e^{\left(\frac{\rho_z}{1-\alpha}-\gamma\right)\varepsilon_{z,1}}\right]/R_f} \\ &= e^{\frac{\gamma\rho_z}{1-\alpha}\sigma^2_z} R_f \end{aligned}$$

Since the risk–free rate is constant by assumption, the only source of risk is the volatility of cash-flows. As long as $\rho_z > 1 - \alpha$, the continuation asset will command a higher expected return than the current asset. This parametric condition is rather intuitive.

The risk of the continuation asset is driven by the covariance between time-1 innovations to aggregate productivity (ε^z_1) and the time–1 conditional expectation of time–2 cash flows. Such moment is greater, the greater the autocorrelation of the process ρ_z and the lower the returns to scale in production.

Returns to scale are relevant, because they shape the elasticity of the capital choice k_2 to time-1 productivity innovations. For the remainder of the section, we decide to focus on the scenario for $\rho_z > 1 - \alpha$, as it is the empirically relevant one.

The impact of interest rate risk would depend on the sign of the covariance between interest rate and the innovation in the aggregate productivity shock. A countercyclical risk-free rate will magnify the risk of the continuation asset. Conversely, pro-cyclical risk-free rate will lower it.

2.1.2 Size, Book-to-Market, and Investment Rate

In the simple model under consideration, the only driver of cross–sectional heterogeneity in expected equity returns is the variation in idiosyncratic productivity. It follows that the firm–level variation in ex–dividend value of equity, or market size, is sufficient to completely characterize the distribution of expected returns.

The reason, very simply, is that market size at time t=0 is pinned down by the levels of aggregate and idiosyncratic productivity, z_0 and s_0. Given the absence of capital adjustment costs, the level of installed capital k_0 is irrelevant.

As a corollary, information on indicators such as book–to–market ratio and investment rate cannot improve upon our characterization of the cross–section of returns. Because of their popularity, however, and the role they will play in the rest of the paper, we characterize the model-implied correlations between the two and expected returns.

Both book-to-market ratio and the investment rate vary with the installed capital k_0. In order to compute the cross-sectional distribution of both quantities at time $t = 0$, we need to make assumptions about the distribution of k_0.

We posit that k_0 was chosen optimally by each firm at time $t = -1$, under the assumptions that $z_{-1} = z_0 = 0$ and $s_{-1} \sim N\left(0, \frac{\sigma_s^2}{1-\rho_s^2}\right)$. In other words, we consider the scenario in which the aggregate productivity realization was equal to its unconditional mean in both $t = -1$ and $t = 0$, and that the cross-sectional distribution of idiosyncratic productivity is equal to the unconditional distribution.

Under these assumptions, the average growth rate of capital installed by firms experiencing a realization of idiosyncratic productivity s_0 is

$$E\left[\log\left(\frac{k_1}{k_0}\right)|s_0\right] = \frac{\rho_s(1-\rho_s)}{1-\alpha}s_0.$$

See Lemma 4 in Appendix. It follows that in the cross-section the investment rate is increasing in s_0.

Finally, we want to understand how the book-to-market ratio, i.e.

$$\frac{E(k_0|s_0)}{E_0[M_1[y_1 + k_1(1-\delta)]] + E_0[M_1[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]]},$$

varies with s_0 in the cross section. Previous analysis reveals that both numerator and denominator are increasing in s_0. In Lemma 2 we prove that the denominator grows faster.

Lemma 2 *Assume that the discount factor is* $M_{t+1} = \beta e^{\gamma \varepsilon_{t+1}}$ *and that* $s_{-1} \sim N(0, \frac{\sigma_z^2}{1-\rho_s^2})$. *Along the path for the aggregate shock* $z_{-1} = z_0 = 0$,

1. *Size and investment rate are increasing in* s_0

2. *Book-to-market is decreasing in* s_0

A corollary of our results is that in the cross-section expected returns covary positively with book-to-market and negatively with size and investment rate. However – it is worth restating it – conditional on size, both book to market and investment rate are uncorrelated with expected returns. High book-to-market and low-investment rate firms earn higher returns because on average they have low market size.

2.2 Operating leverage

Now assume that at $t = 1$ firms incur a fixed operating cost $c_f > 0$. For simplicity, assume also that c_f is such that equity value is always non–negative.

How does the novel assumption affect the properties stated above? Our interest in answering this question stems from the role that operating leverage will play in the quantitative analysis to follow.

The expected return on equity at $t = 0$ becomes

$$E_0[R_1] = \frac{E_0[V_1(k_1, s_1, z_1)] - c_f}{E_0[M_1 V_1(k_1, s_1, z_1)] - c_f / R_f}.$$

Equity is now the combination of a short position on the risk–free asset and a long position on the current and continuation assets introduced above. The short position on the risk–free asset is

$$-\frac{c_f / R^f}{E_0[M_1[y_1 + k_1(1-\delta)]] + E_0[M_1[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]]},$$

which is strictly decreasing in c_f.

Everything else equal, raising the fixed cost is equivalent to expanding the short position – i.e. increasing leverage. It follows that the expected return on equity is increasing in c_f. The impact on returns will be larger, the lower is s_0.

A greater level of s_0 is equivalent to a decline in leverage, i.e. a smaller short position on the risk–free asset and a smaller long position on the portfolio of current and continuation assets. It follows that the expected equity return still falls with s_0. This property is stated formally in Lemma 3.

Lemma 3 *With operating leverage, as long as* $\rho_z > 1 - \alpha$*, the expected return on equity is still monotonically decreasing in the level of idiosyncratic productivity* s_0*.*

The fixed cost also affects the book–to–market ratio, which becomes

$$\frac{E(k_0|s_0)}{E_0[M_1[y_1 + k_1(1-\delta)]] + E_0[M_1[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]] - c_f/R_f}.$$

The slope of the mapping between size and book-to-market increases in absolute value.

Figure 1 illustrates the correlation patterns that arise between expected returns and size, book-to-market and investment rate by means of a parametric example, with and without operating leverage.

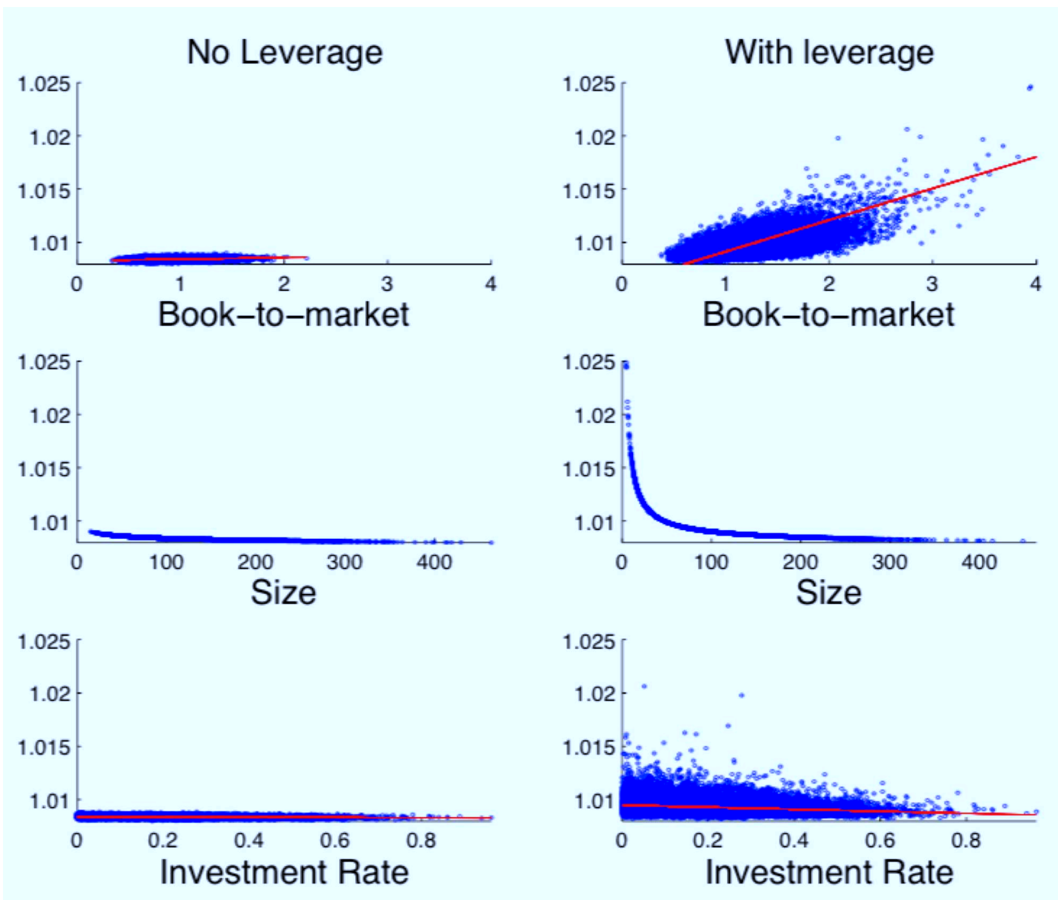

Figure 1: Cross–sectional Variation of Expected Returns.

2.3 Capital Adjustment Costs

We now assume that firms adjusting their capital stock incur a cost equal to

$$g(k_t, k_{t+1}) \equiv \frac{\phi}{2}\left[\frac{k_{t+1}}{k_t} - (1-\delta)\right]^2 k_t,$$

with $\phi > 0$. Conditional on capital k_1 and productivity levels $\{s_1, z_1\}$, the value of equity at $t = 1$ is now

$$V_1(k_1, s_1, z_1) \equiv \max_{k_2} \; e^{s_1+z_1} k_1^{\alpha} + k_1(1-\delta) - k_2 - g(k_1, k_2) + E_1[M_2[e^{s_2+z_2} k_2^{\alpha} + k_2(1-\delta)]].$$

As of $t = 0$, the firm's optimization problem writes as

$$\max_{k_1} \; -k_1 - g(k_0, k_1) + E_0[M_1 V_1(k_1, s_1, z_1)].$$

Capital adjustment costs introduce a novel dimension of heterogeneity, as expected returns at $t = 0$ will no longer be pinned down by the level of idiosyncratic productivity s_0 and will depend non-trivially on the installed capital k_0.

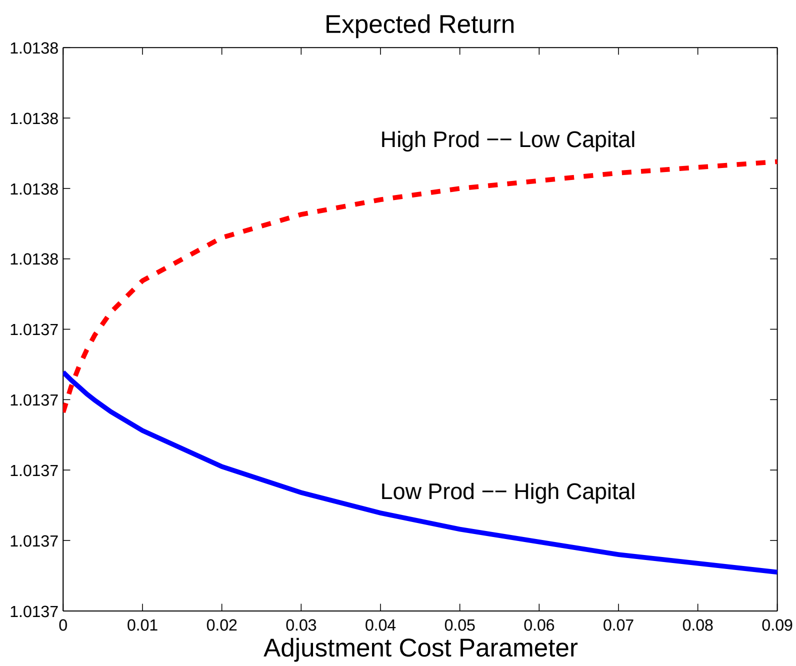

Figure 2: Comparative Statics of Expected Returns with Respect to the Parameter ϕ.

The impact of the adjustment cost on returns will be larger for those firms whose installed capital is farther from the static first-best level. Figure 2 shows the result of raising the value assigned to the parameter ϕ on the expected returns of two particular firms, in the case of a simple parametric example. One firm, which we will refer to as *growing*, is endowed with relatively high productivity and low capital. The other, which is *shrinking*, has low productivity and high capital. Interestingly for our purposes, the shrinking firm has a higher book-to-market ratio and a lower investment rate.

For $\phi = 0$, the model boils down to the scenario characterized above, where expected returns are pinned down by the levels of idiosyncratic productivity. Since it has lower productivity, the shrinking firm is riskier and therefore earns a higher expected return.

As the adjustment cost increases, however, the risk of the growing firm increases monotonically, while the risk of the shrinking firm declines. There is a threshold of the parameter ϕ, such that for higher values the growing firm earns a higher return. This result is interesting in that it hints that increasing the cost of adjusting the capital stock may lead to a decline in the risk spread between high- and low-book-to-market firms, eventually making it negative.

The conditional tense is warranted as we have neither assessed the generality of the result yet, nor evaluated its quantitative significance. The latter task requires a fully fledged model, such as that introduced in Section 3.

A complete analytical characterization of the comparative statics exercise is not readily available. Yet, we can make some progress by studying the impact of varying ϕ on the three elements that shape the expected return: The return on the current asset, the return on the continuation asset, and the share of total value arising from the former, respectively. The conditional payoffs of current and continuation assets at $t = 1$ are redefined as

$$e^{s_1+z_1} k_1^{\alpha} + k_1(1-\delta)$$

and

$$-k_2 - \frac{\phi}{2}\left[\frac{k_2}{k_1} - (1-\delta)\right]^2 k_1 + E_1\left[M_2\left[e^{s_2+z_2} k_2^{\alpha} + k_2(1-\delta)\right]\right],$$

respectively.

It is easy to show that, consistent with the top–left panel of Figure 3, the investment rate of the shrinking firm increases with ϕ – i.e. the optimal choice of k_1 is monotonically increasing in the parameter value. It follows that the expected return on the current asset also declines with ϕ. This is the case, because the loading of the current asset on the risk-free asset is a strictly increasing function of k_1. Conversely, since its investment rate is strictly decreasing in the parameter value, the return on the current asset of the growing firm increases with ϕ.

For the shrinking firm, the loading on the current asset is increasing in ϕ, as cash–flows are front-loaded. A higher fraction of the firm value is accounted by the current asset. The opposite occurs for the growing firm. Cash–flows are backloaded, so that the current asset accounts for a smaller fraction of total value.

Finally, as revealed in the bottom–left panel of Figure 3, increasing ϕ means more risk for the continuation asset of shrinking firms. The intuition is that for shrinking firms,

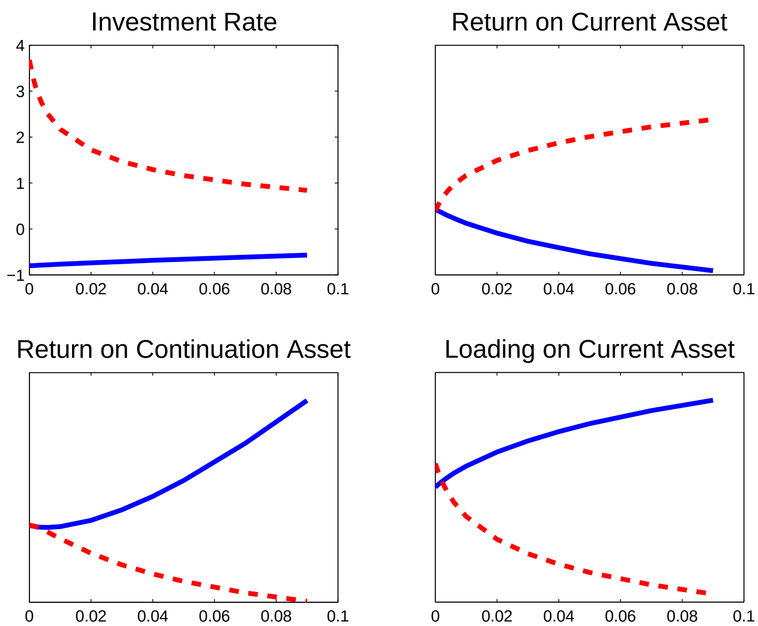

Figure 3: Comparative Statics with Respect to the Parameter ϕ.

a greater k_1 means higher value at $t = 1$ contingent on a good realization of aggregate productivity and lower value contingent on a bad realization. The covariance between payoffs at $t = 1$ and the stochastic discount factor increases in absolute value.

Conversely, for the growing firm, a smaller k_1 means that the payoff of the continuation asset at $t = 1$ is larger contingent on a bad realization of aggregate productivity and smaller contingent on a good realization. The covariance with the stochastic discount factor declines in absolute value. The expected return on the continuation asset is lower.

We conclude this section by considering the impact of operating leverage on the comparative statics exercise just described. Refer to Figure 4. Qualitatively, nothing changes. When $c_f > 0$, it is still the case that increasing the adjustment cost parameter ϕ leads to a decline in the excess return earned by the shrinking firm with respect to the growing firm.

However, consistent with the analysis conducted above, the spread in returns for $\phi = 0$ is greater. It follows that the range of values for the parameter ϕ that produces a positive spread between shrinking and growing firm is now larger. These considerations will be relevant when we evaluate the implications for asset returns of the fully fledged model we now introduce.

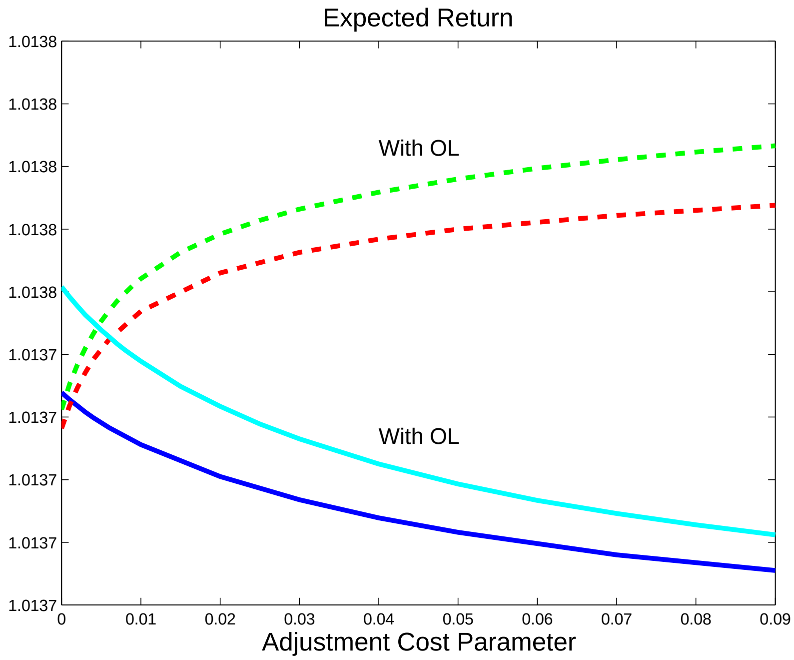

Figure 4: Effect of Operating Leverage on the Comparative Statics of Expected Returns.

3 A Fully Fledged Model

Time is discrete and is indexed by $t = 1, 2, \ldots$. The horizon is infinite. At every time t, a positive mass of firms produce an homogenous good by means of the production function $y_t = e^{z_t + s_t} k_t^{\alpha}$, with $\alpha \in (0,1)$. Here $k_t \geq 0$ denotes physical capital, which depreciates at the rate $\delta \in (0,1)$. The variables z_t and s_t are aggregate and idiosyncratic random disturbances, respectively. They are orthogonal to each other.

The common component of productivity z_t is driven by the stochastic process

$$z_{t+1} = \rho_z z_t + \sigma_z \varepsilon_{z,t+1},$$

where $\rho_z \in (0,1)$, $\sigma_z > 0$, and $\varepsilon_{z,t} \sim N(0,1)$ for all $t \geq 0$. The conditional distribution of z_{t+1} will be denoted as $J(z_{t+1}|z_t)$.

The dynamics of the idiosyncratic component s_t is described by

$$s_{t+1} = \rho_s s_t + \sigma_s \varepsilon_{s,t+1},$$

where $\rho_s \in (0,1)$, $\sigma_s > 0$, and $\varepsilon_{s,t} \sim N(0,1)$ for all $t \geq 0$. The conditional distribution of s_{t+1} will be denoted as $H(s_{t+1}|s_t)$.

Gross investment x requires firms to incur a cost $g(x, k_t)$, where

$$g(x, k_t) \equiv \chi(x)\phi_0 k_t + \phi_1 \left(\frac{x}{k_t}\right)^2 k_t, \quad \phi_0, \ \phi_1 \geq 0,$$

and where $\chi(x) = 0$ for $x = 0$ and $\chi(x) = 1$ otherwise. The first component of $g(x, k_t)$ reflects a fixed cost, scaled by capital in place, which the firm incurs if and only if gross investment is different from zero. We also assume that each period firms incur a fixed operating cost $c_f \geq 0$. Think of that as overhead.

Firms discount future cash flows by means of the discount factor $M(z_t, z_{t+1})$, with

$$\log M(z_t, z_{t+1}) \equiv \log \beta + \gamma_0 z_t + \gamma_1 z_{t+1},$$

where $\beta > 0$, $\gamma_0 > 0$, and $\gamma_1 < 0$. This specification implies that the conditional risk–free rate equals

$$R_{f,t} = \frac{1}{\beta} e^{-z_t[\gamma_0 + \rho_z \gamma_1]} e^{-\frac{1}{2}\gamma_1^2 \sigma_z^2}.$$

Notice that $R_{f,t}$ is counter–cyclical if and only if $\gamma_0 > -\rho_z \gamma_1$. The price of risk is constant, as

$$\frac{std(M_{t+1})}{E_t(M_{t+1})} = \sqrt{e^{\gamma_1^2 \sigma_z^2} - 1}.$$

Abandoning the time notation for expositional convenience, we denote the firm's value function as $V(z, k, s)$, where k, z, and s, are capital in place, aggregate productivity, and idiosyncratic productivity, respectively. $V(z, k, s)$ is the fixed point of the following functional equation:

$$V(z, k, s) = \max_x e^{s+z} k^\alpha - x - g(x, k) - c_f + \int_{\Re} \int_{\Re} M(z, z') V(z', k', s') dH(s'|s) dJ(z'|z),$$
$$\text{s.t. } k' = k(1 - \delta) + x.$$

Our main object of interest will be the expected return on equity, defined as the ratio of expected cum-dividend value at the next date to the current ex-dividend value. Conditional on a triplet of state variables (z, k, s), it is

$$R_e(z, k, s) = \frac{\int_{\Re} \int_{\Re} V(z', k^*, s') dH(s'|s) dJ(z'|z)}{\int_{\Re} \int_{\Re} M(z, z') V(z', k^*, s') dH(s'|s) dJ(z'|z)},$$

where k^* is the optimal choice of capital.

3.1 Calibration

One period is assumed to be one quarter. Consistent with most macroeconomics studies, we set $\delta = 0.030$. Following **?**, we let $\rho_z = 0.95$ and $\sigma_z = 0.007$.

The elasticity parameter in the production function, α, is set equal to 0.6. This is the elasticity with respect to capital that one would obtain with a more general specification of the production function where output also depended on labor, if returns to scale were 0.8 and the share of value added that accrued to capital was 0.3.

Table 1: Parameter Values

Description	Symbol	I $\alpha = 0.6$ NO OL	II $\alpha = 0.6$ OL	III $\alpha = 0.3$ OL
From other studies				
Capital share	α	0.600	0.600	0.300
Depreciation rate	δ	0.030	.	.
Persist. aggregate shock	ρ_z	0.950	.	.
Variance aggregate shock	σ_z	0.007	.	.
Calibrated				
Persist. idiosync. shock	ρ_s	0.900	.	.
Variance idiosync. shock	σ_s	0.060	0.060	0.105
Fixed operating cost	c_f	0.000	0.00135	0.0070
Fixed cost of investment	ϕ_0	0.000015	.	.
Variable cost of investment	ϕ_1	0.0054	0.0054	0.009
Parameter pricing kernel	β	0.970	.	.
Parameter pricing kernel	γ_0	31.850	.	.
Parameter pricing kernel	γ_1	-33.000	.	.

Table 2: Calibrated Targets

	Data	I	II	III
		$\alpha = 0.6$	$\alpha = 0.6$	$\alpha = 0.3$
Investment Rate		NO OL	OL	OL
Mean	0.041	0.053	0.053	0.040
Standard Deviation	0.096	0.090	0.090	0.091
Autocorrelation	0.266	0.278	0.278	0.271
Inaction Rate	0.144	0.150	0.150	0.169
Book–to–Market	0.721	0.491	0.717	0.564
Risk-Free Rate and Sharpe Ratio				
Mean	0.017	0.018	0.018	0.018
Standard Deviation	0.020	0.023	0.023	0.023
Sharpe Ratio	0.426	0.415	0.405	0.424
Mean Excess Return	0.074	0.047	0.048	0.054
St. Dev. Excess Return	0.170	0.113	0.120	0.127

The parameters of the process driving idiosyncratic productivity (ρ_s and σ_s), along with those governing the adjustment costs (ϕ_0 and ϕ_1), were chosen to match the mean and standard deviation of the investment rate, the autocorrelation of investment, and the rate of inaction. The target values are moments estimated from a large panel of public companies. The estimation procedure is detailed in Appendix **??**.

? show that a simpler version of the neoclassical investment model with lognormal disturbances – one without investment adjustment costs – has the interesting properties that (i) the mean investment rate is a simple non linear function of of the parameters ρ_s and σ_s and that (ii) the standard deviation of the investment rate is a simple non–linear function of the mean. It follows that in that framework, mean and standard deviation do not identify the pair $\{\rho_s, \sigma_s\}$. While these properties do not hold exact in our model, inspection reveals that a similar restriction between the two moments exists, leaving us with a degree of freedom.

We proceed to set $\rho_s = 0.9$, a value consistent with the value estimated by **?** for public firms, and set the remaining three parameters to minimize a weighted average of the distances between the moments and their targets.

The parameters governing the stochastic discount factor were chosen to match the first two unconditional moments of the risk–free, as well as the mean Sharpe ratio. Because of non–linearities in the map between parameters and moments, there are indeed to sets of parameters that match the targets. One produces a counter–cyclical risk-free rate, while the other generates a pro–cyclical rate. We decide to go with the former.

Finally, we begin our exploration by setting $c_f = 0$. Parameter values and moments are reported in Column I of Tables 1 and 2, respectively. The acronym "NO OL" stands for "no operating leverage."

4 Results

We illustrate the model's implications by means of a simple methodology commonly used in the empirical asset pricing literature. We assume that our economy is populated by a large number of firms and simulate their behavior for a very large number of periods. In every quarter, we form portfolios of firms based on the values assumed by certain firm–level characteristics, and we compute their realized returns. Finally, we report and compare the time–series means of the returns earned by the different portfolios.

In Table 3, we list unconditional mean returns for portfolios sorted on size, i.e. the ex–dividend firm value. Stocks are classified as small if they belong to the bottom two deciles of the size distribution in the period of portfolio formation. They are classified as large if they belong to the top two deciles. Alternatively, they are included in the medium–size category. For each portfolio, we also report mean values of size, book–to–market, investment rate, capital in place, and idiosyncratic productivity.

Consistent with the empirical evidence, on average small firms earn higher returns. This is the case because, as it was the case in the simple model analyzed in Section 2, small firms have a lower idiosyncratic productivity, which in turn is associated with higher risk.

In Table 4, we report the implications for portfolios sorted on the book–to–market ratio. Growth stocks belong to the bottom two deciles of the distribution of book-to-market. Value stocks belong to the top two deciles.

The model generates a counterfactual value discount, as value firms earn a lower return than growth firms. This is the case, because the book-to-market criterion identifies as *growth*, firms that have low capital and low idiosyncratic productivity. Conversely, it identifies as *value*, firms with high idiosyncratic productivity and high installed capital.

Contrary to the empirical evidence, in the stationary distribution size and book-to-market are positively associated. Most growth firms are also small firms, and value firms tend to be large firms.

Figures 5 and 6 provide more evidence in support of the claim that in this model, the

Table 3: Size Sorted Portfolios

	I	II	III
	$\alpha = 0.6$	$\alpha = 0.6$	$\alpha = 0.3$
Excess Returns	NO OL	OL	OL
Small Firms	1.732	1.871	2.109
Average Size	1.595	1.624	1.730
Large Firms	1.454	1.421	1.478
Large–Small	-0.278	-0.450	-0.631
Size			
Small Firms	0.301	0.208	0.171
Average Size	0.383	0.291	0.250
Large Firms	0.502	0.411	0.360
Large–Small	0.201	0.203	0.189
Book–to–Market			
Small Firms	0.424	0.689	0.579
Average Size	0.489	0.716	0.560
Large Firms	0.556	0.746	0.562
Large–Small	0.131	0.057	-0.018
Investment Rate			
Small Firms	0.030	0.030	0.021
Average Size	0.054	0.054	0.040
Large Firms	0.069	0.069	0.059
Large–Small	0.039	0.039	0.038
Capital			
Small Firms	0.154	0.154	0.098
Average Size	0.222	0.222	0.142
Large Firms	0.322	0.322	0.207
Large–Small	0.168	0.168	0.109
Idiosyncratic Shock			
Small Firms	-0.195	-0.195	-0.336
Average Size	-0.004	-0.004	-0.001
Large Firms	0.183	0.183	0.332
Large–Small	0.379	0.379	0.668

Table 4: Book–to–Market Sorted Portfolios

	I	II	III
	$\alpha = 0.6$	$\alpha = 0.6$	$\alpha = 0.3$
Excess Returns	NO OL	OL	OL
Growth Firms	1.708	1.709	1.718
Average BM	1.593	1.626	1.743
Value Firms	1.486	1.558	1.818
Value–Growth	-0.222	-0.151	0.100
Size			
Growth Firms	0.331	0.279	0.262
Average BM	0.388	0.299	0.257
Value Firms	0.453	0.320	0.250
Value–Growth	0.123	0.041	-0.012
Book–to–Market			
Growth Firms	0.399	0.619	0.477
Average BM	0.486	0.711	0.558
Value Firms	0.584	0.817	0.657
Value–Growth	0.185	0.198	0.180
Investment Rate			
Growth Firms	0.118	0.144	0.176
Average BM	0.057	0.054	0.039
Value Firms	-0.013	-0.053	-0.076
Value–Growth	-0.132	-0.197	-0.252
Capital			
Growth Firms	0.162	0.188	0.128
Average BM	0.224	0.227	0.146
Value Firms	0.303	0.273	0.164
Value–Growth	0.141	0.086	0.036
Idiosyncratic Shock			
Growth Firms	-0.130	-0.023	0.075
Average BM	0.000	0.000	0.005
Value Firms	0.110	0.019	-0.081
Value–Growth	0.240	0.042	-0.156

book-to-market criterion identifies as either growth or value, firms that in important respects are much unlike their empirical counterparts. The two figures display the dynamics of productivity around portfolio formation for both types of firm, as implied by our model and by our data on public firms, respectively.

According to the model, productivity of value firms rises ahead of the formation date and declines thereafter. Productivity of growth firms declines ahead of the formation date, to recover in the aftermath. The data suggests the opposite.

4.1 Operating Leverage

From the analysis conducted in Section 2, we know that introducing operating leverage in the model has two distinct effects: 1) it raises returns by increasing cash–flow risk, the more so the lower is idiosyncratic productivity, and 2) it affects the variation of book–to–market over the state space.

We start by setting the cost of operation c_f in order to generate an average book-to-market ratio equal to its empirical counterpart of about 0.7. Since no firm decision depends upon the value of c_f, no further change to the calibration is warranted.

Expected returns rise across all size categories, but grow faster for small firms. The size premium increases.

Comparing Columns I and II in Table 4 shows that operating leverage changes substantially the set of firms identified as growth and value, respectively. However, the model still produces a value discount.

In order to accommodate a greater value for c_f without generating a counterfactual average book-to-market value, we lower the parameter α – the elasticity of the production function – to 0.3. The fixed cost is set at the highest value among those consistent with non-negativity of the firm's value function in our numerical approximation.

Figure 7 illustrates the location on the state space of firms identified as small and large. Figure 8 does the same for growth and value firms. The color code identifies the magnitude of the returns, with warm colors signaling high returns, and cold colors being associated with low returns.

The characteristics of the firms selected as either growth or value are now radically different from the case without operating leverage. On average, growth firms have higher productivity and lower capital than value firms. The model generates a value premium, although its magnitude is limited. See Column III in Table 4. As a benchmark, consider that the average monthly equally weighted value premium over the period 1976:m1–2013:m12

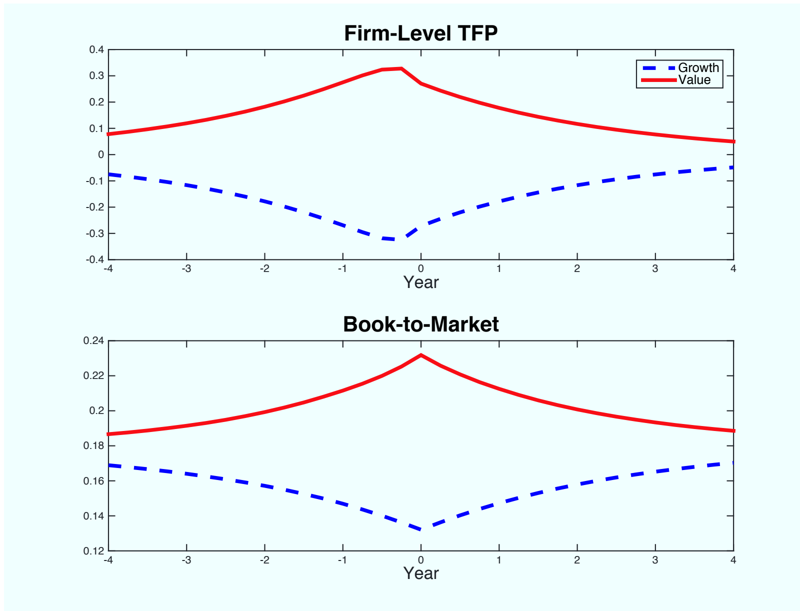

Figure 5: Dynamics Around Portfolio Formation – No operating leverage

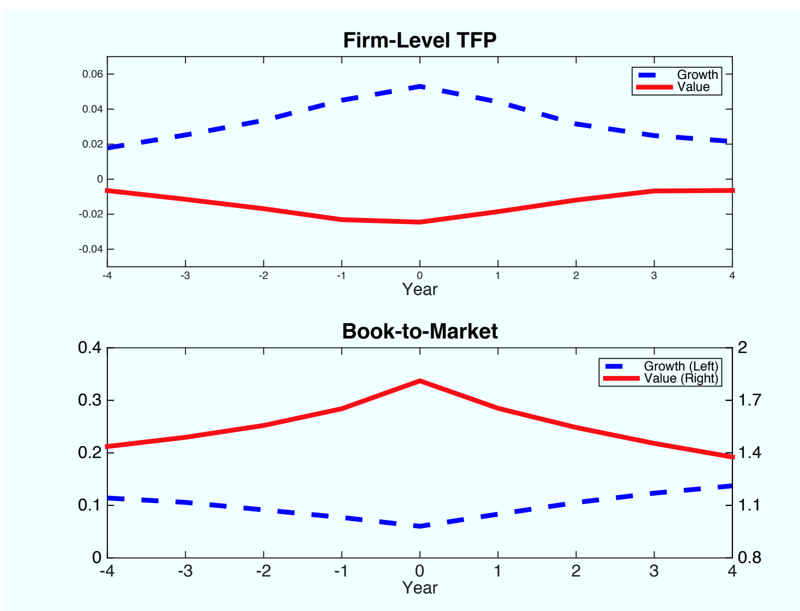

Figure 6: Dynamics Around Portfolio Formation – Data

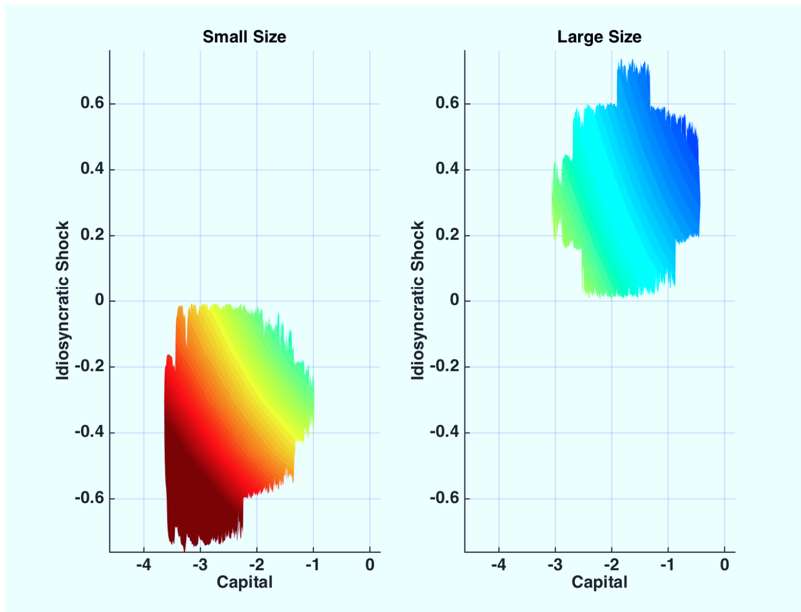

Figure 7: Location of small and large firms over the state space

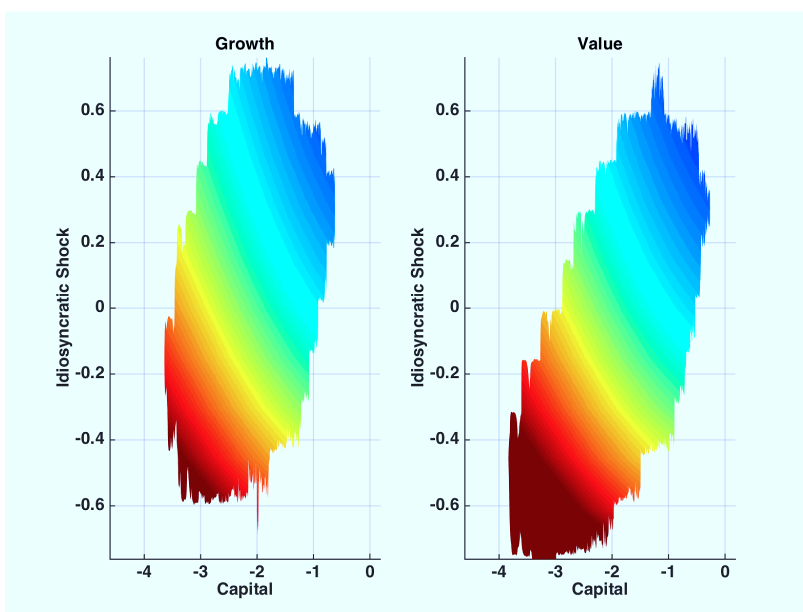

Figure 8: Location of growth and value firms over the state space

is 0.95%, which corresponds to a quarterly return of 2.88%.[1]

Small firms earn, on average, an equally weighted excess return of around 0.6% per quarter over large firms, a value close to empirical estimates. The average monthly equally weighted size premium over the period 1976:m1–2013:m12 is 0.36%, which corresponds to a quarterly return of 1.08%.

An investment strategy that is long on value firms and short on growth firms yields a lower expected return than a long-short position on small and large firms, for two reasons. First, idiosyncratic productivity covaries less with book-to-market than with size. Second, and more interesting, on average value firms shed capital after portfolio formation – paying it out as dividend – regardless of the aggregate state of nature. This makes them less risky. Growth firms, on the other hand, tend to invest – drawing resources from shareholders – in all aggregate states. This feature makes them riskier.

With the help of Figure 9, we see that growth firms have been investing ahead of portfolio formation, striving to achieve the efficient size dictated by their growing productivity. Value firms, on the other hand, have been divesting, prompted by declining productivity. These implications are consistent with the empirical evidence illustrated in Figure 6.

This also explains why, consistent with the empirical findings of Xing (2008) among others, an investment strategy calling for a long position on low investment–rate stocks and a short position on high investment–rate stocks yields a positive return on average. See Table 8.

So far we have restricted our analysis to the characterization of unconditional correlations between firm-level characteristics and expected returns. With the help of Table 5, we now illustrate the model's implications for the conditional relation between expected return and size and book-to-market, respectively.

The methodology, known as double sorting, is a simple extension of the single sorting employed above. In every periods stocks are sorted in nine different portfolio, depending on size and book-to-market.

As long as market value is monotone increasing in capital and idiosyncratic productivity, conditional on size, portfolios with higher book–to–market must be characterized by greater average capital and lower productivity. The risk – and expected return – increases. Conditional on book-to-market, larger firms must have greater capital and idiosyncratic productivity on average. The risk – and expected return – decreases.

[1] Data on equity returns are from Kenneth French's website: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

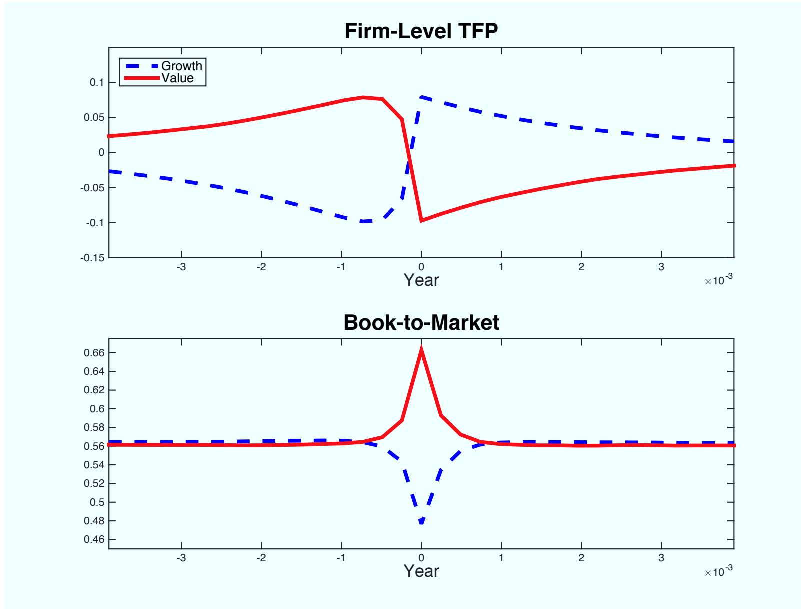

Figure 9: Dynamics Around Portfolio Formation – With Operating Leverage

5 Capital Adjustment Costs

In this section, we explore the role of quadratic adjustment costs in shaping the cross–section of equity returns. Starting from the benchmark model – model III above – we progressively lower ϕ_1 until it reaches zero, keeping all other parameters constant.

For the sake of completeness, we report all simulated moments in Table 9. With the help of Table 6, we survey how book–to–market sorted portfolios change as we decrease ϕ_1. As adjusting becomes cheaper, the mean and volatility of the investment rate increase, while the autocorrelation declines. Indeed, the autocorrelation is negative in the case without quadratic adjustment cost – a well–known result.

Now refer to Table 6. As we lower ϕ_1, we record an increase in the cross–sectional dispersion of all variables across book–to–market sorted portfolios, except for the idiosyncratic productivity.

In particular, the value premium increases. This finding can be rationalized with the intuition gained in Section 2. As ϕ_1 drops, the capital in place is closer to the efficient level at all times. In turn, this means that, everything else equal, value firms will pay out less dividends – this feature makes them riskier. Growth firms, on the other hand, will require less investment from shareholders – this makes them less risky.

Table 5: Double Sorted Portfolios on Size and Book–to–Market ($\alpha = 0.3$ and OL)

	Low BM	Average BM	High BM	H-L	Low BM	Average BM	High BM	H-L
		Equity Returns				**Size**		
Small Size	2.003	2.112	2.136	0.134	0.185	0.167	0.169	-0.016
Average Size	1.713	1.731	1.746	0.033	0.256	0.250	0.242	-0.014
Large Size	1.494	1.462	1.504	0.010	0.355	0.368	0.343	-0.012
L–S	-0.509	-0.650	-0.632		0.170	0.202	0.174	
		Book-to-Market				**Investment Rate**		
Small Size	0.479	0.562	0.666	0.187	0.162	0.030	-0.087	-0.250
Average Size	0.477	0.557	0.653	0.176	0.174	0.038	-0.084	-0.258
Large Size	0.476	0.560	0.647	0.171	0.200	0.055	-0.064	-0.264
L–S	-0.003	-0.002	-0.019		0.038	0.025	0.023	
		Capital				**Idiosyncratic Shock**		
Small Size	0.090	0.094	0.112	0.022	-0.224	-0.349	-0.383	-0.160
Average Size	0.125	0.142	0.159	0.034	0.063	-0.002	-0.067	-0.130
Large Size	0.174	0.210	0.224	0.050	0.368	0.355	0.244	-0.124
L–S	0.083	0.116	0.111		0.592	0.704	0.627	
		Mass of Firms						
Small Size	0.042	0.094	0.061	0.020				
Average Size	0.099	0.402	0.099	-0.000				
Large Size	0.048	0.103	0.052	0.005				
L–S	0.006	0.009	-0.009					

6 The Stochastic Discount Factor

Recall that our benchmark model features a a countercyclical risk–free rate. In order to gauge the role of this assumption in generating our results, we now consider a scenario with constant risk–free rate. To that end, we set $\gamma_0 = -\rho_z \gamma_1$.

The maximum Sharpe ratio is unchanged, as it depends on γ_1 alone. The mean of the risk-free rate changes very slightly. See Table 10.

Table 10 also shows that switching from a countercyclical to a constant risk–free rate has little impact on investment moments and on the cross-sectional mean of book–to–market. On the other hand, the effects on excess equity returns are large, due to a sizeable decline in the volatility of returns.

The lower mean and volatility of realized equity returns are reflected in the book–to–market sorted portfolios reported in Table 7. The value premium takes an annualized

Table 6: Comparative Statics w.r.t. ϕ_1 – Book–to–Market Sorted Portfolios

	I	II	III	IV
	$\alpha = 0.3$; OL	$\alpha = 0.3$; OL	$\alpha = 0.3$; OL	$\alpha = 0.3$; OL
Excess Returns	$\phi_1 = 0.009$	$\phi_1 = 0.006$	$\phi_1 = 0.003$	$\phi_1 = 0.000$
Growth Firms	1.708	1.712	1.701	1.701
Average BM	1.743	1.744	1.748	1.783
Value Firms	1.818	1.817	1.814	1.851
Value–Growth	0.100	0.105	0.113	0.150
Size				
Growth Firms	0.262	0.266	0.273	0.280
Average BM	0.257	0.258	0.259	0.251
Value Firms	0.250	0.254	0.259	0.257
Value–Growth	-0.012	-0.012	-0.013	-0.023
Book–to–Market				
Growth Firms	0.477	0.474	0.470	0.454
Average BM	0.558	0.559	0.560	0.561
Value Firms	0.657	0.662	0.670	0.684
Value–Growth	0.180	0.188	0.200	0.230
Investment Rate				
Growth Firms	0.176	0.193	0.223	0.302
Average BM	0.039	0.040	0.041	0.042
Value Firms	-0.076	-0.085	-0.097	-0.119
Value–Growth	-0.252	-0.278	-0.320	-0.421
Capital				
Growth Firms	0.128	0.129	0.131	0.130
Average BM	0.146	0.147	0.148	0.144
Value Firms	0.164	0.168	0.174	0.176
Value–Growth	0.036	0.039	0.043	0.046
Idiosyncratic Shock				
Growth Firms	0.075	0.072	0.071	0.0.073
Average BM	0.005	0.001	-0.004	-0.010
Value Firms	-0.081	-0.070	-0.054	-0.038
Value–Growth	-0.156	-0.142	-0.125	-0.111

value which is only 1/4 of the benchmark value.

Table 7: The role of the Stochastic Discount Factor – Book–to–Market Sorted Portfolios

Excess Returns	i) Countercyclical R^f ii) Constant Price of Risk	i) Constant R^f ii) Constant Price of Risk	i) Constant R^f ii) Countercyclical Price of Risk
Growth Firms	1.718	0.616	0.985
Average BM	1.743	0.622	1.017
Value Firms	1.818	0.641	1.085
Value–Growth	0.100	0.024	0.100
Size			
Growth Firms	0.262	0.233	0.243
Average BM	0.257	0.219	0.234
Value Firms	0.250	0.194	0.229
Value–Growth	-0.012	-0.039	-0.013
Book–to–Market			
Growth Firms	0.477	0.472	0.440
Average BM	0.558	0.554	0.515
Value Firms	0.657	0.653	0.604
Value–Growth	0.180	0.181	0.164
Investment Rate			
Growth Firms	0.176	0.173	0.172
Average BM	0.039	0.032	0.033
Value Firms	-0.076	-0.088	-0.083
Value–Growth	-0.252	-0.260	-0.255
Capital			
Growth Firms	0.128	0.110	0.106
Average BM	0.1146	0.121	0.120
Value Firms	0.1164	0.126	0.136
Value–Growth	0.036	0.016	0.030
Idiosyncratic Shock			
Growth Firms	0.075	0.096	0.066
Average BM	0.005	0.011	-0.001
Value Firms	-0.081	-0.122	-0.055
Value–Growth	-0.156	-0.219	-0.121

Following the lead of Zhang (2006), we also explore a scenario featuring a countercyclical price of risk. We accomplish this task by adopting the pricing kernel specification of Jones and Tuzel (2013):

$$\log(M_{t+1}) = \log\beta - \frac{1}{2}\gamma_t^2\sigma_z^2 - \gamma_t\sigma_z\varepsilon_{z,t+1},$$
$$\log\gamma_t = \gamma_0 + \gamma_1 z_t.$$

We set $\beta = 0.996$, $\gamma_0 = 3.275$, and $\gamma_1 = -15.75$ to match the first two unconditional moments of the risk–free rate, as well as the mean Sharpe ratio. Table 10 shows that the unconditional moments of the investment rate and the average book–to–market are the same as in the case where the price of risk and the risk–free rate are both constant. On the other hand, assuming a countercyclical price of risk helps in generating a larger equity premium.

In this last case, the unconditional means for book–to–market sorted portfolios are virtually the same as in the case with constant price of risk, except for equity returns. Having a countercyclical price of risk helps in generating a larger value premium, which is now the same as in the benchmark case.

7 Conclusion

TBA

A Proofs

Proof of Lemma 1.

The current asset is itself a portfolio of two assets. One is conditionally riskless, since it pays $k_1(1-\delta)$ regardless of the state of nature. The other has a payoff $e^{s_1+z_1}k_1^\alpha$. The time–0 expected return of the latter is

$$\frac{E_0[e^{s_1+z_1}k_1^\alpha]}{E_0\left[M_1 e^{s_1+z_1}k_1^\alpha\right]} = \frac{E_0[e^{\varepsilon_{z,1}}]}{E_0[M_1 e^{\varepsilon_{z,1}}]}. \tag{2}$$

It follows that the expected return on the current asset is a weighted average of the conditional risk-free rate $R_{f,0}$ and (2), where the weight on the latter is

$$\frac{E_0\left[M_1[e^{s_1+z_1}k_1^\alpha]\right]}{E_0\left[M_1[e^{s_1+z_1}k_1^\alpha + k_1(1-\delta)]\right]} = \frac{R_{f,0}-(1-\delta)}{R_{f,0}-(1-\delta)(1-\alpha)}.$$

The weight on the short asset is

$$\frac{E_0\left[M_1[e^{s_1+z_1}k_1^\alpha + k_1(1-\delta)]\right]}{E_0\left[M_1[e^{s_1+z_1}k_1^\alpha + k_1(1-\delta)]\right] + E_0[M_1[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]]}.$$

The weight will be increasing in s_0 as long as the following quantity is decreasing:

$$\frac{E_0[M_1[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]]}{E_0\left[M_1[e^{s_1+z_1}k_1^\alpha + k_1(1-\delta)]\right]}.$$

Tedious algebra reveals that the latter can be rewritten as

$$e^{s_0\frac{\rho_s(\rho_s-1)}{1-\alpha}} \frac{\left[E[e^{\rho_s\varepsilon_s}]\right]^{1/(1-\alpha)}(1-\alpha)E_0\left[M_1\left(\frac{\alpha}{1-\frac{1-\delta}{R_{f,1}}}\right)^{\frac{\alpha}{1-\alpha}}\left[E_2[M_2e^{z_2}]\right]^{\frac{1}{1-\alpha}}\right]}{\left[E_0(M_1e^{z_1})\right]^{\frac{1}{1-\alpha}}\left[\left(\frac{\alpha}{1-\frac{1-\delta}{R_{f,0}}}\right)^{\frac{\alpha}{1-\alpha}} + \frac{1-\delta}{R_{f,0}}\left(\frac{\alpha}{1-\frac{1-\delta}{R_{f,0}}}\right)^{\frac{1}{1-\alpha}}\right]},$$

which is clearly decreasing in s_0, as $\rho_s in(0,1)$.

■

Proof of Lemma 2.

We limit ourselves to show that the book-to-market is decreasing in s_0. Rewrite it as

$$\begin{aligned} BM =& \frac{E(k_0|s_0)}{k_1}\frac{k_1}{E_0[M_1[e^{s_1+z_1}k_1^\alpha + k_1(1-\delta)]] + E_0[M_1[-k_2 + E_1[M_2[y_2 + k_2(1-\delta)]]]]} \\ & \frac{E(k_0|s_0)}{k_1}\widetilde{BM}. \end{aligned}$$

Table 8: Investment Sorted Portfolios

	I	II	III
	$\alpha = 0.6$	$\alpha = 0.6$	$\alpha = 0.3$
Excess Returns	NO OL	OL	OL
Low IK	1.600	1.644	1.804
Average IK	1.599	1.642	1.770
High IK	1.578	1.604	1.707
High–Low	-0.022	-0.044	-0.097
Size			
Low IK	0.374	0.282	0.236
Average IK	0.384	0.293	0.251
High IK	0.407	0.314	0.274
High–Low	0.033	0.033	0.038
Book–to–Market			
Low IK	0.557	0.828	0.667
Average IK	0.494	0.725	0.570
High IK	0.457	0.655	0.502
High–Low	-0.100	-0.173	-0.166
Investment Rate			
Low IK	-0.085	-0.085	-0.097
Average IK	0.033	0.033	0.023
High IK	0.149	0.149	0.138
High–Low	0.233	0.233	0.235
Capital			
Low IK	0.248	0.248	0.158
Average IK	0.228	0.228	0.146
High IK	0.222	0.222	0.141
High–Low	-0.026	-0.026	-0.018
Idiosyncratic Shock			
Low IK	-0.063	-0.063	-0.107
Average IK	-0.017	-0.017	-0.022
High IK	0.050	0.050	0.085
High–Low	0.113	0.113	0.192

Table 9: Comparative Statics w.r.t. ϕ_1 – Calibrated Targets

	I	II	III	IV
	$\alpha = 0.3$; OL	$\alpha = 0.3$; OL	$\alpha = 0.3$; OL	$\alpha = 0.3$; OL
Investment Rate	$\phi_1 = 0.009$	$\phi_1 = 0.006$	$\phi_1 = 0.003$	$\phi_1 = 0.000$
Mean	0.040	0.042	0.046	0.056
Standard Deviation	0.091	0.100	0.114	0.147
Autocorrelation	0.271	0.209	0.118	-0.063
Inaction Rate	0.169	0.184	0.219	0.290
Book–to–Market	0.564	0.565	0.566	0.567
Risk-Free Rate and Sharpe Ratio				
Mean	0.018	0.018	0.018	0.018
Standard Deviation	0.023	0.023	0.023	0.023
Sharpe Ratio	0.424	0.423	0.422	0.416
Mean Excess Return	0.054	0.054	0.054	0.053
St. Dev. Excess Return	0.127	0.127	0.127	0.128

Table 10: The role of Stochastic Discount Factor – Calibrated Targets

Investment Rate	i) Countercyclical R^f ii) Constant Price of Risk	i) Constant R^f ii) Constant Price of Risk	i) Constant R^f ii) Countercyclical Price of Risk
Mean	0.040	0.034	0.034
Standard Deviation	0.091	0.092	0.092
Autocorrelation	0.271	0.257	0.257
Inaction Rate	0.169	0.228	0.236
Book–to–Market	0.564	0.558	0.512
Risk-Free Rate and Sharpe Ratio			
Mean	0.018	0.017	0.017
Standard Deviation	0.023	0.000	0.000
Sharpe Ratio	0.424	0.449	0.411
Mean Excess Return	0.054	0.008	0.025
St. Dev. Excess Return	0.127	0.018	0.061

We have that

$$\log \widetilde{BM} = -\log\left(\frac{E_0[M_1[e^{s_1+z_1}k_1^{\alpha}+k_1(1-\delta)]]}{k_1}+\frac{E_0[M_1[-k_2+E_1[M_2[y_2+k_2(1-\delta)]]]]}{k_1}\right).$$

Since

$$\frac{E_0[M_1[e^{s_1+z_1}k_1^{\alpha}+k_1(1-\delta)]]}{k_1}=\left[\frac{\alpha}{1-\frac{1-\delta}{R_f}}\right]^{-1}+\frac{1-\delta}{R_f},$$

the first addendum in parenthesis does not depend on s_0. It follows that

$$\frac{\partial\log(\widetilde{BM})}{\partial s_0}=\frac{\rho_s(1-\rho_s)}{1-\alpha}\frac{E_0[M_1[-k_2+E_1[M_2[y_2+k_2(1-\delta)]]]]}{E_0[M_1[e^{s_1+z_1}k_1^{\alpha}+k_1(1-\delta)]]+E_0[M_1[-k_2+E_1[M_2[y_2+k_2(1-\delta)]]]]}.$$

By Lemma 4, $s_{-1}|s_0$ is normally distributed with mean $\rho_s s_0$ and variance σ_s^2. It follows that

$$\begin{aligned}\log\left[\frac{E(k_0|s_0)}{k_1}\right]&=\log\left[E\left(e^{\frac{\rho_s s_{-1}}{1-\alpha}}|s_0\right)\right]-\frac{\rho_s s_0}{1-\alpha}\\&=\frac{\rho_s(\rho_s-1)}{1-\alpha}s_0+\frac{1}{2}\left(\frac{\rho_s\sigma_s}{1-\alpha}\right)^2\end{aligned}$$

Finally,

$$\frac{\partial\log(BM)}{\partial s_0}=\frac{\rho_s(1-\rho_s)}{1-\alpha}\left[\frac{E_0[M_1[-k_2+E_1[M_2[y_2+k_2(1-\delta)]]]]}{E_0[M_1[e^{s_1+z_1}k_1^{\alpha}+k_1(1-\delta)]]+E_0[M_1[-k_2+E_1[M_2[y_2+k_2(1-\delta)]]]]}-1\right]<0.$$

■

Proof of Lemma 3.

Think of equity as being a portfolio consisting of the risk–free–asset as long as a composite of current and continuation business assets. The weight of the risk–free asset $-\frac{c_f/R^f}{E_0[M_1y_1]+E_0[M_1[-k_2+E_1[M_2y_2]]]-c_f/R^f}$ is negative and clearly increasing in s_0. That is, the short position on the risk–free asset declines with s_0. It follows that the long position on the composite of current and continuation assets also declines. Then the result follows from Lemma 1, which ensures that the return on the composite portfolio declines with s_0.

■

Lemma 4 *Let* $s_{t-1}\sim N\left(0,\frac{\sigma^2}{1-\rho^2}\right)$ *and* $s_t=\rho s_{t-1}+\varepsilon$, *with* $\varepsilon\sim N(0,\sigma^2)$, $\sigma>0$ *and* $\rho\in(0,1)$. *Then,* $E[s_{t-1}|s_t]=\rho s_t$.

Proof. For simplicity, let f denote the density of a Normal distribution with parameters $\left(0,\frac{\sigma^2}{1-\rho^2}\right)$. Let also g denote the density of a Normal distribution with parameters $\left(0,\sigma^2\right)$. It follows that

$$E[s_{t-1}|s_t]=\int\frac{s_{t-1}f(s_{t-1})g(s_t-\rho s_{t-1})ds_{t-1}}{\int f(s_{t-1})g(s_t-\rho s_{t-1})ds_{t-1}}.$$

To simplify notation further, let $\eta^2 \equiv \frac{\sigma^2}{1-\rho^2}$. Then,

$$f(s_{t-1})g(s_t - \rho s_{t-1}) = \frac{1}{2\pi\sigma\eta} \exp\left[-\frac{1}{2}\left(\frac{s_{t-1}^2}{\eta^2} + \frac{(\rho s_{t-1} - s_t)^2}{\sigma^2}\right)\right].$$

Algebraic manipulations yield

$$\begin{aligned} f(s_{t-1})g(s_t - \rho s_{t-1}) =& \frac{1}{2\pi\sigma\eta} \exp\left(-\frac{1}{2}\frac{(s_{t-1} - \rho s_t)^2}{\sigma^2}\right) \exp\left(-\frac{1}{2}\frac{s_t^2}{\sigma^2 + \eta^2\rho^2}\right) \\ =& \frac{1}{\sqrt{2\pi}\eta} \exp\left(-\frac{1}{2}\frac{s_t^2}{\sigma^2 + \eta^2\rho^2}\right) \times \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\frac{(s_{t-1} - \rho s_t)^2}{\sigma^2}\right). \end{aligned}$$

The latter expression is the product of a constant and the density of a Normal with mean ρs_t and variance σ^2. It follows that

$$E[s_{t-1}|s_t] = \int s_{t-1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\frac{(s_{t-1} - \rho s_t)^2}{\sigma^2}\right] = \rho s_t.$$

■