Stable Matching in Large Economies*

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Abstract

Complementarities of preferences have been known to jeopardize the stability of two-sided matching, yet they are a pervasive feature of many markets. We revisit the stability issue with such preferences in a large market. Workers have preferences over firms while firms have preferences over distributions of workers and may exhibit complementarity. We demonstrate that if each firm's choice changes continuously as the set of available workers changes, then there exists a stable matching even with complementarity. Building on this result, we show that there exists an approximately stable matching in any large finite economy. We apply our analysis to show the existence of stable matchings in probabilistic and time-share matching models with a finite number of firms and workers.

1 Introduction

Since the celebrated work by Gale and Shapley (1962), matching theory has taken a center stage in market design and more broadly, economic theory. In particular, its successful

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application in medical matching and school choice has fundamentally changed how these markets are organized. A key desideratum in the design of such matching markets is "stability"—that the mechanism admits no incentives for its participants to "block" (i.e., side-contracting around) the suggested matching. Stability is crucial for long-term sustainability of a market; unstable matching would be undermined by the parties side-contracting around it either during or after a market.[1] When one side of the market is under centralized control, as with school choice, blocking by a pair of agents on both sides is less of a concern; but even in this case, stability is desirable from a fairness standpoint, as it would eliminate justified envy—envy that cannot be explained away by the preferences of the agents on the other side. In the school choice application, if schools' preferences rest on the test score or other priority that a student feels entitled to, eliminating justified envy appears to be a necessary requirement.

Unfortunately, a stable matching exists only under limited market conditions. It is well known that existence of a stable matching is not generally guaranteed unless the preferences of participants, say firms, are substitutable.[2] In other words, complementarity can lead to nonexistence of a stable matching.

This is a serious limitation on the applicability of centralized matching mechanisms, since complementarities of preferences are a pervasive feature of many matching markets. Firms often seek to hire workers with complementary skills. For instance, in professional athletic leagues, teams demand athletes that complement one another in skills as well as in the positions they play. Some public schools in New York City seek diversity of their student bodies in their skill levels. US colleges tend to exhibit a desire to assemble a class that is complementary and diverse in terms of their aptitudes, life backgrounds, and demographics.

Unless we can get a handle on complementarities, we would not know how to organize such markets, and the applicability of centralized matching will remain severely limited. The limitation is particularly pertinent for many decentralized markets that may potentially benefit from centralization. College admissions and graduate admissions are obvious examples. Decentralized matching leaves much to be desired in terms of efficiencies and

[1] Table 1 in Roth (2002) shows that unstable matching algorithms tend to die out while stable ones survive the test of time.

[2] Substitutability here means that a firm's demand for a worker never grows with more workers being available. More precisely, if a firm does not wish to hire a worker from a set of workers, then it never prefers to hire that worker from a bigger (in the sense of set inclusion) set of workers. Existence of a stable matching under substitutable preferences is established by Sönmez and Ünver (2010), Hatfield and Milgrom (2005), Hatfield and Kojima (2008), and Hatfield and Kominers (2010).

fairness, and of the yield management burden put on the institutions (see Che and Koh (2013)). Despite the potential benefit from centralizing these markets, the exact benefit as well as the method of centralized matching remains unclear, given the instability that may arise from complementary preferences of the participants.

This paper takes a step toward accommodating complementarities and other forms of general preferences. The general impossibility means, however, that the notion of stability needs to be weakened in some way. Our approach is to consider a large market. Specifically, we consider a market which consists of a large number of workers/students on one side and a finite number of firms/colleges with large capacities on the other, and ask whether stability can be achieved in an "asymptotic" sense—i.e., whether participants' incentives for blocking disappears as the number of workers and firms' capacities grow large. Large markets we envision approximate college admissions and labor markets. Our stability notion also preserves the motivation behind the original notion of stability: as long as the incentive for blocking is sufficiently weak, the instability and fairness concerns will not be so serious as to jeopardize the mechanism.

We first consider a continuum model in which there are a finite number of firms and a continuum of workers. Each worker desires to match with at most one firm. Firms have preferences over groups of workers, and importantly, their preferences may exhibit complementarities. A matching is a distribution of workers across firms. The model generalizes Azevedo and Leshno (2011) who assume responsive preferences (a special case of substitutable preferences) for the firms.

Our main result is that there exists a stable matching if firms' preferences exhibit continuity—that is, the set of workers chosen by each firm varies continuously as the set of workers available to that firm changes. This result is quite general since continuity is satisfied by a rich class of preferences including those exhibiting complementarities.[3] The existence of a stable matching follows from two results: (i) a stable matching can be characterized as a fixed point of a suitably defined mapping over a functional space, and (ii) such a fixed point exists given the continuity assumption. The construction of our fixed point mapping differs from the existing matching literature such as Adachi (2000), Hatfield and Milgrom (2005), and Echenique and Oviedo (2006), among others. The existence of a fixed point is established by using the Kakutani-Fan-Glicksberg fixed point theorem—a generalization of Kakutani's fixed point theorem to functional spaces—which appears to be new to the matching literature.

[3]For instance, it allows for Leontief-type preferences with respect to alternative types of workers, desiring to hire all types in equal size (or density).

When firm preferences satisfy substitutability (but not necessarily continuity), we show that the set of stable matchings is nonempty and forms a complete lattice. In particular, there exist worker-optimal (firm-pessimal) and firm-optimal (worker-pessimal) stable matchings. A version of the rural hospital theorem also holds given an appropriate version of the law of aggregate demand. While these results are well known for finite markets and can thus be expected from the existing matching theory, we also provide a novel condition that generalizes the full support assumption of Azevedo and Leshno (2011) and guarantees the uniqueness of stable matching under substitutable preferences.

Building on our analysis on the continuum model, we show that there is a sense in which it serves as a legitimate approximation of large finite economies. More specifically we demonstrate that, for any large finite economy that is sufficiently close to our continuum economy (in terms of the distribution of worker types and firms' preferences), there exists an approximately stable matching in the sense that the incentives for blocking is arbitrarily small.

Although the basic model assumes that firm preferences are strict, our framework can be extended to allow for indifferences in the firms' preferences. Accommodating indifferences is particularly important in the school choice context, in which preferences are given by coarse priorities that put many students in the same priority class. To accommodate this extension, we represent a firm's preference as a choice correspondence (as opposed to a function). We then extend both the fixed point characterization (via a correspondence defined on a functional space) and the proof of the existence the existence result.

Equipped with this generalization, we can extend the "fractional" matching models to allow for general preferences. These models study how schools/firms and students/workers can share time or match probabilistically in a stable manner (see Sotomayor (1999), Alkan and Gale (2003), and Kesten and Ünver (2014), among others). Our continuum model lends itself to studying such a probabilistic/time share environment; we can simply interpret types in different subsets within the type space as probabilistic/time units belonging to alternative (finite) workers. Our novel contribution is to allow for more general preferences including complementarities as well as indifferences. As mentioned, accommodating indifferences is important in school choice design, and complementarities are also relevant since some schools (as those in NYC) seek diversity in their student bodies. We also establish existence of a strongly stable matching suggested defined by Kesten and Ünver (2014), in this more general environment.

Relationship with the Literature

The present paper is connected with several strands of literature. Most importantly, it is related to the growing literature on matching and market design. Since the seminal contributions by Gale and Shapley (1962) and Roth (1984), stability has been recognized as the most compelling solution concept in matching markets.[4] As argued and demonstrated by Sönmez and Ünver (2010), Hatfield and Milgrom (2005), Hatfield and Kojima (2008), and Hatfield and Kominers (2010) in various situations, the substitutability condition is necessary and sufficient for guaranteeing the existence of a stable matching when the number of agents is finite. Our paper contributes to this line of studies by showing that substitutability is not needed for the existence of a stable matching once there is a continuum of agents on one side of the market and, moreover, there exists an approximately stable matching in large finite markets.

Our study was inspired by a recent research on matching with a continuum of agents by Azevedo and Leshno (2011).[5] As in our paper, they assume that there are a finite number of firms and a continuum of workers and, among other things, show the existence and uniqueness of a stable matching in that setting. The crucial difference relative to the current work is that they assume firms have responsive preferences (which is a special case of substitutability). One of our contributions is that, while almost universally assumed in the literature, restrictions on preferences such as responsiveness or even substitutability are unnecessary for guaranteeing the existence of a stable matching in the continuum markets. Also, one of the uniqueness results by Azevedo and Leshno (2011) is obtained as a special case of our uniqueness result under substitutable, not necessarily responsive, preferences.

An independent study by Azevedo and Hatfield (2012) also analyzes matching with a continuum of agents.[6] Like the current paper, their study finds that a stable matching exists even when not all agents have substitutable preferences. There are a number of notable differences between their study and ours, however. First, they consider a large

[4]See Roth (1991) and Kagel and Roth (2000) for empirical and experimental evidence on the importance of stability in labor markets, and Abdulkadiroğlu and Sonmez (2003) for the interpretation of stability as a fairness concept in school choice.

[5]Also related, although formally different, are various recent studies on large matching markets, such as Roth and Peranson (1999), Immorlica and Mahdian (2005), Kojima and Pathak (2008), Kojima and Manea (2008), Manea (2009), Che and Kojima (2010), Lee (2012), Liu and Pycia (2013), Che and Tercieux (2013), Ashlagi, Kanoria and Leshno (2014), Kojima, Pathak and Roth (2013), and Hatfield, Kojima and Narita (2014*b*).

[6]Although less related, our study also has some analogy with Azevedo, Weyl and White (2012). They show the existence of competitive equilibrium in an exchange economy with continuum agents and indivisible objects.

number (more precisely, continuum) of firms each employing a finite number of workers, so they consider a continuum of agents on both sides of the market. By contrast, we consider a finite number of firms each employing a large number (continuum) of workers. These two models provide complementary approaches for studying large markets. In the school choice context, for example, in many school districts, there are usually a small number of schools each admitting hundreds of students, which fits well with our modeling approach. But in a large school district such as New York City, the number of schools is also large, so their model may offer a reasonably good approximation. Second, Azevedo and Hatfield (2012) assume that there are finite number of firm and worker types. This enables them to use Brouwer's fixed point theorem to characterize the stable matching. We put no restriction on the number of workers' types. The general preferences require a topological fixed point theorem from functional analysis. This type of mathematics has never been applied to discrete two-sided matching literature to our knowledge, and we view the introduction of these tools to the matching literature as one of our methodological contributions. Our model also has an advantage of subsuming the previous work by Azevedo and Leshno (2011) as well as many others mentioned above, which assume a continuum of worker types. Finally, they also consider many-to-many matchings, although our applications to time-share and probabilistic matching models allow for many-to-many matching. And they also consider matching with contracts, while our study focuses on the case with a fixed term of contract, as assumed in the standard matching literature.

Our methodological contribution is also related to another recent advance in matching theory based on the monotone method. In the one-to-one matching context, Adachi (2000) defines a certain operator whose fixed points are equivalent to stable matchings. His work has been generalized in many directions by such papers as Fleiner (2003), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), Ostrovsky (2008), and Hatfield and Kominers (2010). We also define an operator whose fixed points are equivalent to stable matchings. A crucial difference is, however, that these previous studies impose restrictions on preferences (e.g., responsiveness or substitutability) so that the operator is monotone, which enables one to apply Tarski's fixed point theorem to show existence of stable matchings. By contrast, we do not impose responsiveness or substitutability restrictions and instead rely on the continuum of workers, along with continuity of firms' preferences, to guarantee continuity of the operator (in an appropriately chosen topology). That approach allows us to use (a generalization of) Kakutani fixed point theorem, a more familiar tool in traditional economic theory such as the existence proofs of general equilibrium and the Nash equilibrium in mixed strategies.

The current paper is also related with the literature on matching with couples. Like a firm in our model, a couple can be seen as a single agent with complementary preferences over contracts, as pointed by Hatfield and Kojima [Any suggestion on the last phrase; should we cite??]. Roth (1984) and unpublished work by Sotomayor show that there does not necessarily exist a stable matching if there exists a couple. Klaus and Klijn (2005) provide a condition to guarantee the existence of stable matchings. A more recent work by **?** presents conditions under which the probability that a stable matching exists even in the presence of couples converges to one as the market becomes infinitely large, and similar conditions have been further analyzed by Ashlagi, Braverman and Hassidim (2014). Pycia (2012) and Echenique and Yenmez (2007) study many-to-one matching with complementarity as well as peer effect. Our paper is different from these studies in various respects, but it complements these papers by formalizing a sense in which finding a stable matching becomes easier in a large market even in the presence of complementarities.

The remainder of this paper is organized as follows. Section 2 provides an example that illustrates the main contribution of our paper. Section 3 describes a matching model in the continuum economy. Section 4 establishes the existence of a stable matching under general, continuous preferences and also under substitutable preferences. In Section 5, we use this existence result to show that an approximately stable matching can be found in any large finite economy. In Section 6, we extend our analysis to the case where firms may have multi-valued choice mappings (that is, choice correspondences), and apply it to time share/probabilistic matching models.

2 Illustrative Example

Before proceeding, we illustrate the main contribution of our paper using an example. We first illustrate how complementary preferences may lead to non-existence of a stable matching when there are a finite number of agents. To this end, suppose that there are two firms f_1 and f_2 and two workers θ and θ'. The agents have the following preferences:

$$\begin{aligned}
&\theta : f_1 \succ f_2; \\
&\theta' : f_2 \succ f_1; \\
&f_1 : \{\theta, \theta'\} \succ \emptyset; \\
&f_2 : \{\theta\} \succ \{\theta'\} \succ \emptyset.
\end{aligned}$$

That is, worker θ prefers f_1 to f_2, and worker θ' prefers f_2 to f_1; firm f_1 prefers employing both workers to employing no one, which the firm in turn prefers to employing only one of them; and firm f_2 prefers worker θ to θ', which it in turn prefers to employing neither.

Firm f_1 has a "complementary" preference, and this creates instability. To see this, recall stability requires that there be no blocking coalition. Due to f_1's complementary preference, it must employ either both workers or neither in any stable matching. The former case is unstable since worker θ' prefers firm f_2 to firm f_1, and f_2 prefers θ' to being unmatched, thus they can block the matching. The latter is also unstable since, in such a case, f_2 will only hire θ, leaving θ' unemployed; and this outcome will be blocked by f_1 forming a coalition with θ and θ', benefiting all members of the coalition.

Can stability be restored if the market becomes large? As long as the market remains finite, the answer is no. To see this, consider a scaled-up version of the above model: there are q workers of type θ and q workers of type θ', and they have the same preferences as above. Firm f_2 prefers type-θ workers to type-θ' workers, and wishes to hire in that order but at most up to q workers. Firm f_1 has a complementary preference for hiring exactly identical numbers of type-θ and type-θ' workers (with no capacity limit). Formally, if x and x' are the numbers of available workers of types θ and θ', respectively, then firm f_1 would choose $\min\{x, x'\}$ workers of each type.

As long as q is odd (including the original economy with $q = 1$), there exists no stable matching.[7] To see this, first note that if firm f_1 hires more than $q/2$ workers of each type, then firm f_2 has a vacant position, so f_2 can block with a type-θ' worker who prefers f_2 to f_1. If f_1 hires fewer than $q/2$ workers of each type, then some workers will remain unmatched (since f_2 hires at most q workers). If a type-θ worker is unmatched, then f_2 will form a blocking coalition with that worker. If a type-θ' worker is unmatched, then firm f_1 will form a blocking coalition by adding that worker along with a θ worker (possibly matched with f_2).

Consequently, "exact" stability is not guaranteed even in a large market. Nevertheless, one may hope to achieve approximate stability. This is indeed the case with the above example; the "magnitude" of instability diminishes as the economy grows large. To see this, let q be odd and consider a matching in which f_1 hires $\frac{q+1}{2}$ workers of each type while f_2 hires $\frac{q-1}{2}$ workers of each type. This matching is unstable because f_2 has one vacant position it wants to fill and a type-θ' worker who is matched to f_1 prefers f_2. However, note that this is the only possible block of this matching, and it involves only one worker. As the economy grows large, if the additional single worker becomes insignificant for firm f_2 relative to its size—and this is what the continuity of a firm's preference captures—, then the payoff consequence of forming such a block must also become insignificant, suggesting

[7] Here we sketch the argument, which is in Appendix A in fuller form. When q is even, a matching in which each firm hires $\frac{q}{2}$ of each type of workers is stable.

that the instability problem becomes insignificant as well.

This can be seen most clearly in the limit of the above economy. Suppose there is a unit mass of workers, half of whom (in Lebesgue measure, say) are of type θ and the other half are of type θ'. Their preferences are the same as before. And suppose firm f_1 wishes to maximize $\min\{x, x'\}$, where x and x' are the measures of type-θ and type-θ' workers, respectively. Firm f_2 can hire at most $\frac{1}{2}$, and prefers to fill as much of this quota as possible with type-θ workers and fill the remaining quota with type-θ' workers. In this economy, there is a (unique) stable matching in which each firm hires exactly one half of workers of each type. To see this, note that any blocking coalition involving firm f_1 requires taking away a positive, and identical, measure of type-θ' and type-θ workers from firm f_2, which is impossible since type-θ' workers will object to it. Also, any blocking coalition involving firm f_2 requires taking away a positive measure type-θ workers away from firm f_1 and replacing the same measure of type-θ' workers in its workforce, which is impossible since type-θ workers will object to it. Our analysis below will show that the continuity of firms' preferences, to be defined more clearly, is responsible for guaranteeing existence of a stable matching in the continuum economy and approximate stability in the large finite economies in this example.

3 Model of a Continuum Economy

Agents and their measures. There exist a finite set $F = \{f_1, \ldots, f_n\}$ of firms and a unit mass of workers. Let $\emptyset$ be the null firm, representing the workers' option of not being matched with any firm, and define $\tilde{F} := F \cup \{\emptyset\}$. The workers are identified with types $\theta \in \Theta$, where Θ is a compact metric space. Let Σ denote a Borel σ-algebra of space Θ. Let $\overline{\mathcal{X}}$ be the set of all nonnegative measures such that for any $X \in \overline{\mathcal{X}}$, $X(\Theta) \leq 1$. Assume that the entire population of workers is distributed according to a finite, nonnegative (Borel) measure $G \in \overline{\mathcal{X}}$ on (Θ, Σ). That is, for any $E \in \Sigma$, $G(E)$ is the measure of workers belonging to E. Assume $G(\Theta) = 1$ for normalization. For illustration, the limit economy of the example in the previous section is a continuum economy with $F = \{f_1, f_2\}$, $\Theta = \{\theta, \theta'\}$, and $G(\theta) = G(\theta') = 1/2$. In the sequel, we shall use this as our leading example for the purpose of illustrating various concepts we develop.

Any subset of the population, or **subpopulation**, is represented by a nonnegative measure X on (Θ, Σ) such that $X(E) \leq G(E)$ for all $E \in \Sigma$. Let $\mathcal{X} \subset \overline{\mathcal{X}}$ denote the set of all subpopulations. We further say that a nonnegative measure $\tilde{X} \in \mathcal{X}$ is a **subpopulation of** $X \in \mathcal{X}$, denoted $\tilde{X} \sqsubset X$, if $\tilde{X}(E) \leq X(E)$ for all $E \in \Sigma$. We use $\mathcal{X}_X$ to denote the set of all subpopulations of X.

Given the order $\sqsubset$, for any $X, Y \in \mathcal{X}$, we define $X \vee Y$ (join) and $X \wedge Y$ (meet) to be the supremum and infimum of X and X', respectively.[8] That $X \vee Y$ and $X \wedge Y$ are well-defined, i.e. they are also measures belonging to $\mathcal{X}$, follows from the following lemma, whose proof is in Appendix B.1.

Lemma 1. *The partially ordered set* $(\mathcal{X}, \sqsubset)$ *is a complete lattice.*

The join and meet of X and Y in $\mathcal{X}$ can be illustrated via a couple of examples. In our leading example with two types of workers, given $X := (x, x'), Y := (y, y')$, we have $X \vee Y = (\max\{x, y\}, \max\{x', y'\})$, and $X \wedge Y = (\min\{x, y\}, \min\{x', y'\})$. Consider next a continuum economy with types $\Theta = [0, 1]$ and suppose the measure G admits a bounded density g for all $\theta \in [0, 1]$. In this case, it easily follows that for $X, Y \sqsubset G$, their densities x and x' are well defined,[9] and $Z := (X \vee Y)$ and $Z' := (X \wedge Y)$ admit densities z and z' defined by $z(\theta) = \max\{x(\theta), y(\theta)\}$ and $z'(\theta) = \min\{x(\theta), y(\theta)\}$ for all θ, respectively.

Further, for $X, Y \sqsubset G$, we let $(X + Y)(\cdot) := (X(\cdot) + Y(\cdot)) \wedge G(\cdot)$ and $(X - Y)(\cdot) := (X(\cdot) - Y(\cdot)) \vee 0$, which ensures that $X + Y, X - Y \in \mathcal{X}$.

Consider the space of all (signed) measures (of bounded variation) on (Θ, Σ). We endow this space with a weak* topology and its subspace $\mathcal{X}$ with the relative topology. Given a sequence of measures (X_k) and a measure X on (Θ, Σ), we write $X_k \xrightarrow{w^*} X$ to indicate that (X_k) converges to X as $k \to \infty$ under weak* topology, and simply say that (X_k) **weakly converges** to X.[10]

Agents' preferences. We now describe agents' preferences. Each worker is assumed to have a strict preference over $\tilde{F}$. Let $\mathcal{P}$ denote the (finite) set of all possible worker preferences, and let $P \in \mathcal{P}$ denote its generic element (i.e., a particular worker preference). We write $f \succ_P f'$ to indicate that f is strictly preferred to f' according to P. For each

[8] That is, $X \vee Y$ for instance is the smallest measure of which both X and Y are subpopulations. One can show that for all $E \in \Sigma$,

$$(X \vee Y)(E) = \sup_{D \in \Sigma} X(E \cap D) + Y(E \cap D^c),$$

which is a special case of Lemma 1 in Appendix D.

[9] Since $|X([0, \theta']) - X([0, \theta])| \leq |G([0, \theta']) - G([0, \theta])| \leq N|\theta' - \theta|$, where $N := \sup_s g(s)$, so $X([0, \theta))$ is Lipschitz continuous, and its density is well defined.

[10] We use the term "weak convergence" since it is common in statistics and mathematics, though weak* convergence is a more appropriate term from the perspective of functional analysis. As is well known, $X_k \xrightarrow{w^*} X$ if $\int_\Theta h dX_k \to \int_\Theta h dX$ *for all bounded continuous function* h. See Theorem 13 in Appendix C to see some implications of this convergence.

$P \in \mathcal{P}$, let $\Theta_P \subset \Theta$ denote the set of all worker types whose preference is given by P, and assume that Θ_P is measurable and $G(\partial\Theta_P) = 0$, where $\partial\Theta_P$ denotes the boundary of Θ_P.[11] Since all worker types have strict preferences, Θ can be partitioned into the sets in $\mathcal{P}_\Theta \equiv \{\Theta_P : P \in \mathcal{P}\}$.

We next describe firms' preferences. We do so indirectly by defining a firm f's **choice function**, $C_f : \mathcal{X} \to \mathcal{X}$, where $C_f(X)$ is a subpopulation of X for any $X \in \mathcal{X}$ and satisfies the following **revealed preference** property: for any $X, X' \in \mathcal{X}$ with $X' \sqsubset X$, if $C_f(X) \sqsubset X'$, then $C_f(X') = C_f(X)$.[12] Note we are assuming that the firm's demand is unique given any set of available workers. In Section 4.2, we consider a generalization of the model in which the firm's choice is not unique. Let $R_f : \mathcal{X} \to \mathcal{X}$ be a **rejection function** defined by $R_f(X) := X - C_f(X)$. By convention, we let $C_\varnothing(X) = X, \forall X \in \mathcal{X}$, meaning that $R_\varnothing(X)(E) = 0$ for all $X \in \mathcal{X}$ and $E \in \Sigma$. In our leading example, the choice functions of firms f_1 and f_2 are given respectively by $C_{f_1}(x_1, x_1') = (\min\{x_1, x_1'\}, \min\{x_1, x_1'\})$ and $C_{f_2}(x_2, x_2') = (\min\{x_2, \frac{1}{2}\}, \min\{\frac{1}{2} - x_2, x_2'\})$, when x_i of type θ-workers and x_i' of type-θ' workers are available to firm f_i, $i = 1, 2$.

In sum, continuum matching model is summarized as a tuple $(G, F, \mathcal{P}_\Theta, C_F)$.

Remark 1. Our model takes firms' choice functions as a primitive, which gives us some flexibility in describing their preferences, in particular preferences over alternatives that are not chosen. This approach is also adopted by other studies in matching theory, which include Alkan and Gale (2003) and Aygün and Sönmez (2013) among others. An alternative, albeit more restrictive, approach would be to assume that each firm is endowed with a complete, continuous preference relation over $\mathcal{X}$. Maximization with such a preference will result in an upper hemicontinuous choice correspondence defined over $\mathcal{X}$.[13] Assuming a unique optimal choice will then give us a choice function (which is also continuous), although, as will be shown in Section 4.2, our results generalize to the case in which each firm's choice is not unique.

[11] Formally, $\partial E := \overline{E} \cap \overline{E^c}$, where $\overline{E}$ and $\overline{E^c}$ are the closure of E and its complement, respectively. This means that if Θ is discrete as in our leading example, then we have $\overline{E} = E$ and $\overline{E^c} = E^c$, so $\overline{E} \cap \overline{E^c} = E \cap E^c = \emptyset$. Hence, the assumption is satisfied.

[12] This property must hold if the choice is made by a firm optimizing with a well-defined preference relation. See for instance Hatfield and Milgrom (2005), Fleiner (2003), and Alkan and Gale (2003) for some implicit or explicit use of the revealed preference property in matching theory literature. Recently, Aygün and Sönmez (2013) have clarified the role of this property in the context of matching with contracts.

[13] This also relies on the fact that the set of alternatives $\mathcal{X}$ is compact, a fact we establish in the proof of Theorem 3.

Matchings, and their efficiency and stability requirements. A **matching** is $M = (M_f)_{f\in\tilde{F}}$ such that $M_f \in \mathcal{X}$ for all $f \in \tilde{F}$ and $\sum_{f\in\tilde{F}} M_f = G$. Firms' choice functions can be used to define a partial order on firms' preferences over matchings. For any two matchings, M and M', we say that $M'_f \succeq_f M_f$ (or firm f prefers M'_f to M_f) if $M'_f = C_f(M'_f \vee M_f)$.[14] We also say $M'_f \succ_f M_f$ if $M'_f \succeq_f M_f$ and $M'_f \neq M_f$. The resulting preference (partial) order amounts to taking a minimal stance on the firms' preferences, limiting attention to those revealed via their choices. Given this preference order, we say $M' \succeq_F M$ if $M'_f \succeq_f M_f$ for each $f \in F$.

To discuss workers' welfare, fix any matching M and any firm f. Let

$$D^{\succeq f}(M) := \sum_{P\in\mathcal{P}} \sum_{f'\in\tilde{F}:f'\succeq_P f} M_{f'}(\Theta_P \cap \cdot) \text{ and } D^{\preceq f}(M) := \sum_{P\in\mathcal{P}} \sum_{f'\in\tilde{F}:f'\preceq_P f} M_{f'}(\Theta_P \cap \cdot) \quad (1)$$

denote the measure of workers assigned to firm f or better (according to their preferences) and the measure of workers assigned to firm f or worse (again according to their preferences), respectively, where $M_{f'}(\Theta_P \cap \cdot)$ denotes a measure that takes the value $M_{f'}(\Theta_P \cap E)$ for each $E \in \Sigma$. Starting from M as a default matching, the latter measures the number of workers who are available to firm f for possible rematching. Meanwhile, the former measure is useful for characterizing the workers' overall welfare. For any two matchings M and M', we say that $M' \succeq_\Theta M$ if for each $f \in \tilde{F}$, $D^{\succeq f}(M') \sqsubset D^{\succeq f}(M)$. That is, if, for each firm f, the measure of workers assigned to f or better is larger in one matching than the other, then we can say that the workers' overall welfare is higher in the former matching.

Equipped with these notions, we can define Pareto efficiency and stability.

Definition 1. *A matching M is **Pareto efficient** if there exists no matching $M' \neq M$ such that $M' \succeq_F M$ and $M' \succeq_\Theta M$.*

Definition 2. *A matching $M = (M_f)_{f\in\tilde{F}}$ is **stable** if*

1. *For all $P \in \mathcal{P}$, we have $M_f(\Theta_P) = 0$ for any f satisfying $\emptyset \succ_P f$; and for each $f \in F$, $M_f = C_f(M_f)$, and*
2. *There exist no $f \in F$ and $M'_f \in \mathcal{X}$ such that $M'_f \succ_f M_f$ and $M'_f \sqsubset D^{\preceq f}(M)$.*

Condition 1 of this definition, called **individual rationality**, means that each matched worker prefers the matching over being unmatched, and that each firm never wishes to unilaterally drop some of its matched workers. Condition 2, called **no blocking**, requires that there be no firm and a set of workers who are not matched together but want to do so. When Condition 2 is violated by f and M'_f, we say that f and M'_f **block** M.

[14] This is known as the Blair order in the literature. See Blair (1984).

Remark 2 (Equivalence to group stability)**.** We say that a matching M is **group stable** if Condition 1 of Definition 2 holds and, in addition,

2'. *There exist no* $F' \subseteq F$ *and* $M'_{F'} \in \mathcal{X}^{|F'|}$ *such that* $M'_f \succ_f M_f$ and $M'_f \sqsubset D^{\preceq f}(M_f)$ for all $f \in F'$.

This definition is a strengthening of our stability concept, as it requires that the matching be immune to blocks by coalitions potentially involving multiple firms. Such stability concepts with coalitional blocks are analyzed by Sotomayor (1999), Echenique and Oviedo (2006), and Hatfield and Kominers (2010), among others. Clearly any group stable matching is stable, because if Condition 2 of stability is violated by a firm f and M'_f, then Condition 2' of group stability is violated by a singleton set $F' = \{f\}$ and $M'_{\{f\}}$. The converse also holds. To see why, note that if Condition 2' of group stability is violated by $F' \subseteq F$ and $M'_{F'}$, then Condition 2 of stability is violated by any f and M'_f such that $f \in F'$ and $M'_f \neq M_f$, because $M'_f \succeq_f M_f$ and $D^{\preceq f}(M)$ by assumption.[15]

As in the standard finite market, stability implies Pareto efficiency:

Proposition 1. *If a matching is stable, then it is Pareto efficient.*

Proof. See Appendix B.2. ▮

4 Existence of s Stable Matching in the Continuum Economy

A key to finding a stable matching is to identify the "available" workers for each firm—namely, the workers willing to match with each firm. The optimal choice by a firm from the available workers then identifies those the firm hires in a stable matching, since any better set of workers the firm may approach for hiring—i.e., to form a blocking coalition

[15] By requiring $M'_f \sqsubset D^{\preceq f}(M_f)$ for all $f \in F'$ in Condition 2', our group stability concept is implicitly assuming that workers who are considering participating in a blocking coalition with $f \in F'$ use the current matching M_{-f} as the reference point. This means that workers are available to firm f as long as they prefer f to their current matching. However, given that a more preferred firm $f' \in F'$ may be making offers to workers in $D^{\preceq f}(M_f)$ as well, the set of workers available to f may be smaller. Such a consideration would result in a weaker notion of group stability. Any such concept, however, will be equivalent to our notion of stability, because in this remark we establish that even the most restrictive notion of group stability, i.e., the concept using $D^{\preceq f}(M_f)$ in Condition 2', is equivalent to stability, while stability is weaker than any group stability concept described above.

with—would include the workers that are unavailable to that firm (in the sense that they have a better choice than the firm), thus meeting a key requirement of stability.

How does one then identify available workers for a firm? The process of identifying available workers involves a fixed-point flavor: For a set (or more precisely a measure) of workers to be available to f, they must have no better choice than f, so to identify the former, one must identify the set of firms available to the workers. But this in turn requires one to identify the workers that are available to these latter firms.

It is thus natural to search for a stable matching as a fixed point of a mapping—or more intuitively, the stationary point of a process that repeatedly revises the set of available workers to the firms, based on the preferences of the workers and the firms. Formally, we define a map: $T : \mathcal{X}^{n+1} \to \mathcal{X}^{n+1}$, where $T(X) = (T_f(X))_{f\in\tilde{F}}$ for each $X \in \mathcal{X}^{n+1}$. For each $f \in \tilde{F}$, the map $T_f : \mathcal{X}^{n+1} \to \mathcal{X}$ is defined by

$$T_f(X)(E) := \sum_{P: f_-^P=\emptyset} G(\Theta_P \cap E) + \sum_{P: f_-^P\neq\emptyset} R_{f_-^P}(X_{f_-^P})(\Theta_P \cap E), \tag{2}$$

where $f_-^P \in \tilde{F}$, called the **immediate predecessor** of f, is a firm that is ranked immediately above firm f according to P.[16] The map can be interpreted as a tâtonnement process whereby an auctioneer quotes "budget" of workers that firms can choose from. The auctioneer, just like the classical Walrasian one, revises the budget quotes based on the preferences of the market participants, shrinking the budget for a firm f (i.e., making smaller work force available to it) when more workers are demanded by the firms that they rank ahead of f, and expanding otherwise. Once the process converges, reaching a fixed point, workers who are "truly" available to firms—in the sense of being compatible with the preferences of other market participants—will have been found.

Alternatively, the mapping can be seen as a process by which firms rationally adjust their beliefs about available workers based on the preferences of the other market participants. The fixed point of the mapping then captures the workers firms can iteratively rationalize as being available to them. To illustrate, fix a firm f. Consider first the worker types $\theta \in \Theta_P$ for which f is at the top of their preference T (i.e., $f_-^P = \emptyset$). Firm f can rationally believe all such workers are available to that firm, which explains the first term of (11). Consider next the worker types $\theta \in \Theta_P$ for which f is the second-best according to P. Firm f can rationally believe that among them only those who would be rejected by their top choice firm are available to it, which explains the second term of (11). Now, consider the worker types for which f is their third-best. Firm f analogously rationalizes as being available to

[16]An immediate predecessor of$f$ is formally defined such that $f_-^P \succ_P f$ and if $f' \succ_P f$ for $f' \in \tilde{F}$, then $f' \succeq_P f_-^P$.

it only those among them who would be rejected by their first-best and second-best firms. But the workers who would be rejected by the first-best are available to the second-best, according to the earlier rationalization. This in turn rationalizes f's belief that the workers available to f are precisely those who are available but unacceptable to the second-best firm. In general, for any worker types, the same iterative process of belief rationalization establishes the validity of (11).

Remark 3. *Our map can be rewritten to mimic Gale and Shapley's deferred acceptance algorithm, where the firms and workers take turns to reject dominated proposals in each round. Specifically, we can write $T = \Psi \circ \Phi$, where, for each profile $X = (X_f)_{f \in \tilde{F}} \in \mathcal{X}^{n+1}$ of workers, the map*

$$\Phi_f(X) := G - R_f(X_f)$$

returns the workers that are not rejected by each firm, and for $Y = (Y_f) \in \mathcal{X}^{n+1}$, the subsequent map

$$\Psi_f(Y) := G - \sum_{P \in \mathcal{P}} Y_{f_-^P}(\Theta_P \cap \cdot),$$

returns workers available for each firm f—more specifically, those that remain after removing the workers who would be accepted by firms they consider better than f. The map written in this way resembles those developed in the context of the finite matching markets (e.g., see Adachi (2000), Hatfield and Milgrom (2005), and Echenique and Oviedo (2006)), but the construction here differs due to dealing with a richer space of worker types. As will be also clear, this new construction is needed for a new method of proof for characterizing the fixed point.

Theorem 1. *A matching M is stable if and only if there is a fixed point $X = T(X)$ such that $M_f = C_f(X_f), \forall f \in \tilde{F}$. Also, any such X and M satisfy $X_f = C^{\preceq f}(M), \forall f \in \tilde{F}$.*

Proof. This result follows as a corollary of Theorem 12. For details, see Appendix C. ▮

Example 1. To illustrate how a stable matching can be found from T mapping, consider our leading example. We can denote candidate measures of available workers by 4-tuple: $(X_{f_1}, X_{f_2}) = (x_1, x_1'; x_2, x_2') \in [0, \frac{1}{2}]^4$, where $X_{f_1} = (x_1, x_1')$ are the measures of workers of types θ and θ' available to f_1 and $X_{f_2} = (x_2, x_2')$ are the measures of workers of types θ and θ' available to f_2. Since f_1 is most preferred by θ and f_2 is most preferred by θ', according to our T, all of these workers are available to the respective firms. So, we can without loss set $x_1 = G(\theta) = \frac{1}{2}$ and $x_2' = G(\theta') = \frac{1}{2}$ and consider $(\frac{1}{2}, x_1'; x_2, \frac{1}{2})$ as our candidate measures. To compute $T(\frac{1}{2}, x_1'; x_2, \frac{1}{2})$, we first consider the choice by each firm given the respective available worker sets. Note $C_{f_1}(\frac{1}{2}, x_1') = (x_1', x_1')$, so $R_{f_1}(\frac{1}{2}, x_1') = (\frac{1}{2} - x_1', 0)$.

Similarly, $C_{f_2}(x_2, \frac{1}{2}) = (x_2, \frac{1}{2} - x_2)$, so $R_{f_2}(x_2, \frac{1}{2}) = (0, x_2)$. Now applying our formula in (11), we get $T(\frac{1}{2}, x_1'; x_2, \frac{1}{2}) = (\frac{1}{2}, x_2; \frac{1}{2} - x_1', \frac{1}{2})$. So, $(\frac{1}{2}, x_1'; x_2, \frac{1}{2})$ is a fixed point of T only if $(\frac{1}{2}, x_1'; x_2, \frac{1}{2}) = (\frac{1}{2}, x_2; \frac{1}{2} - x_1', \frac{1}{2})$, or $x_1' = x_2 = \frac{1}{4}$. The optimal choice from the fixed point then gives a stable matching $M = (\frac{1}{4}, \frac{1}{4}; \frac{1}{4}, \frac{1}{4})$. One can easily see that this is the limit of the approximately stable matching $(\frac{q+1}{4q}, \frac{q+1}{4q}; \frac{q-1}{4q}, \frac{q-1}{4q})$ in the finite market example studied in Section 2 as $q \to \infty$.

4.1 General Preferences

We now introduce a condition on the firms' preferences that ensures existence of a stable matching.

Definition 3. *Firm f's preference is* ***continuous*** *if, for any sequence $(X_k)_{k\in\mathbb{N}}$ and X in $\mathcal{X}$ such that $X_k \xrightarrow{w^*} X$, it holds that $C_f(X_k) \xrightarrow{w^*} C_f(X)$.*

As suggested by the name, continuity of a firm's preferences means that the firm's choice changes continuously with the distribution of workers available to it. Under this assumption, we obtain a general existence result as follows:

Theorem 2. *If each firm's preference is continuous, then there exists a stable matching.*

Proof. This result follows as a corollary of Theorem 3. For details, see Appendix C. ∎

To gain some intuition, note that in a continuum economy, the measures of worker matched with firms can change continuously. Therefore, a matching can occur in a way that balances out the number of workers demanded by different firms even with complementary preferences. This helps eliminate blocks that lead to non-existence of stable matchings.

To prove that T admits a fixed point, we first demonstrate that continuity of firms' preferences implies that mapping T is also continuous. We also verify that $\mathcal{X}$ is a compact and convex set. Continuity of T and compactness and convexity of $\mathcal{X}$ allow us to apply the Kakutani-Fan-Glicksberg fixed-point theorem to guarantee that T has a fixed point. Then, the existence of a stable matching follows from Theorem 1, which shows the equivalence between the set of stable matchings and the set of the fixed points of T.

Many complementary preferences are compatible with continuous preferences and thus with existence of a stable matching. Recall Example 1. In that example, firm f_1 has a Leontief type preference, for it wishes to hire an equal measure of workers of types θ and θ' (so, in particular, the firm wants to hire type-θ workers only if type-θ' workers are also available, and vice versa). As seen in Example 1, a stable matching exists despite the extreme complementarity. And the reason has to do with the continuityAs the firm's preferences are clearly continuous in that example, the existence of a stable matching in that example is implied by Theorem 2.

A stable matching may fail to exist even in the continuum economy unless all firms have continuous preferences, as the following example illustrates.

Example 2 (Role of continuity)**.** Consider the following economy, which is modified from Example 1. There are worker types θ and θ' (each with measure $1/2$) and firms f_1 and f_2. Firm f_1 wants to hire only measure $1/2$ of each worker types together, and would like to be unmatched otherwise; meanwhile, firm f_2's choice function is continuous: it exhibits "responsive" preferences preferring type-θ workers to type-θ' workers and in turn prefers the latter to leaving a position vacant, and faces a capacity of measure $1/2$.

Then, C_{f_1} violates continuity, while C_{f_2} does not. As before, we assume

$$\theta : f_1 \succ f_2;$$
$$\theta' : f_2 \succ f_1.$$

No stable matching exists in this environment. To see this, consider the following two cases:

1. Suppose f_1 hires measure $1/2$ of each type of workers. For such a matching, none of the capacity of f_2 is filled. Thus such a matching is blocked by f_2 and type-θ' workers (note that every type-θ' worker is currently matched with f_1, so they are willing participate in the block).

2. Suppose f_1 hires no worker. Then, the only candidate for a stable matching is one in which f_2 hires measure $1/2$ of type-θ workers (or else, f_2 and unmatched workers of type θ would block the matching). Then, since f_1 is most preferred by all θ workers, and type-θ' workers prefer f_1 to $\emptyset$, the matching is blocked by a coalition of measure $1/2$ of type-θ workers, measure $1/2$ of type-θ' workers, and f_1.

The continuity assumption is important for existence of a stable matching, as this example shows that nonexistence can happen even if only one firm f has a discontinuous choice function. This example also suggests that non-existence can reemerge once some "lumpiness" is reintroduced into the continuum economy (i.e., one firm can only hire a certain minimum mass of workers). However, this kind of lumpiness is not very natural in a continuum economy, although it is a natural consequence of the fact that each worker is indivisible in a finite economy.

4.2 Stable Matching with Choice Correspondence

In this section, we extend our analysis to the case in which a firm's choice from a subpopulation of available workers is not necessarily unique. In other words, we allow a firm's choice to be a correspondence (multi-valued function). There are at least three motivations for this generalization. First, there is a continuum of workers in our environment, and in such a situation it is natural to assume that a firm may be indifferent between some distributions and choose more than one distribution as most preferred. Second, indifferences appear to be inherent in some applications. In school choice, for instance, schools are often required by law to regard many students to have the same priority,[17] in which case the choice is multi-valued. Lastly, as will be seen in Section 7, our model turns out to have a connection with probabilistic and time share matching models. In that model, a distribution of workers corresponds to time shares/probabilities with which workers are matched to firms, and indifferences naturally arise between distributions that represent the same matching in terms of time shares/probabilities.

Let $C_f : \mathcal{X} \rightrightarrows \mathcal{X}$ be a **choice correspondence**: i.e., for any $X \in \mathcal{X}$, $C_f(X) \subset \mathcal{X}$ is the set of subpopulations of X that are the most preferred by f among all subpopulations of X. By convention, we let $C_\varnothing(X) = \{X\}, \forall X \in \mathcal{X}$. Then, let $R_f(X) := X - C_f(X)$, or equivalently $R_f(X) = \{Y \in \mathcal{X} \mid Y = X - X' \text{ for some } X' \in C_f(X)\}$. We assume that for any $X \in \mathcal{X}$, $C_f(X)$ is nonempty. Assume further that $C_f(\cdot)$ satisfies the revealed preference property: For any $X, X' \in \mathcal{X}$ with $X' \sqsubset X$, if $C_f(X) \cap \mathcal{X}_{X'} \neq \emptyset$, then $C_f(X') = C_f(X) \cap \mathcal{X}_{X'}$ (recall $\mathcal{X}_{X'}$ is the set of subpopulations of X'). Define a function $D^{\preceq f} : \mathcal{X}^{n+1} \to \mathcal{X}$ for each $f \in F$ to be the same as the one in (1).

Definition 4. *A matching M is **stable** if*

1. *For all $P \in \mathcal{P}$ and $E \in \Sigma$, we have $M_f(\Theta_P \cap E) = 0$ for any f satisfying $\varnothing \succ_P f$; and for each $f \in F$, $M_f \in C_f(M_f)$, and*
2. *There exist no $f \in F$ and $M'_f \in \mathcal{X}$ such that $M'_f \sqsubset D^{\preceq f}(M)$, $M'_f \in C_f(M'_f \vee M_f)$, and $M_f \notin C_f(M'_f \vee M_f)$.*

Clearly, this definition is a generalization of Definition 2 to the case of choice correspondences. The existence of a stable matching then follows from imposing appropriate

[17] In the public school choice program in Boston, for instance, a student's priority at a school is based only on coarse criteria such as the student's residence and whether her sibling is currently enrolled at that school. Consequently, at each school, many students are equipped with the same priority (Abdulkadiroğlu et al., 2005).

conditions on the firms' choice correspondences that allow for the existence of a fixed-point for a correspondence operator defined similarly to the mapping T in Section 4.[18]

The notion of continuous preferences is naturally extended to the following:

Definition 5. *The firm f's choice correspondence C_f is* ***upper hemicontinuous*** *if, for any sequences $(X^k)_{k\in\mathbb{N}}$ and $(\tilde{X}^k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ such that $X^k \xrightarrow{w^*} X$, $\tilde{X}^k \xrightarrow{w^*} \tilde{X}$, and $\tilde{X}^k \in C_f(X^k), \forall k$, we have $\tilde{X} \in C_f(X)$.*

Theorem 3. *Suppose that for each $f \in F$, C_f is convex-, closed-valued, and upper hemicontinuous. Then, there exists a stable matching.*

Proof. See Appendix C. ∎

This result shows that our main result — that there exists a stable matching when there are a continuum of workers — does not hinge on the restrictive assumption that each firm's choice is unique. On the contrary, this result holds for a wide range of specifications that allow for indifferences and choice correspondences. As will be seen in the next section, this generalization turns out to be useful when analyzing a model of probabilistic and time-share matchings, which may appear to be unrelated to our model at a first glance.

4.3 Substitutable Preferences

A class of preferences studied extensively in the matching theory literature are substitutable preferences. A well-known set of results, including existence of a stable matching, obtain under these preferences. We show that the same set of results follow in our continuum economy model with a suitable formulation of substitutable preferences. Since the arguments establishing these results are by now fairly standard, we shall be brief in our treatment of this case. One novel issue, though, is the question of uniqueness of a stable matching. Azevedo and Leshno (2011) show that multiplicity of stable matchings disappear in the large economy if firms have rich preferences over workers or if their quotas are generic. This striking result is obtained with the restricted preference domain of "responsive" preferences. We provide a condition for uniqueness of a stable matching under general substitutable preferences. We begin by defining the class of preferences:

Definition 6. *Firm f's preference is* ***substitutable*** *if $R_f(X) \sqsubset R_f(X')$ whenever $X \sqsubset X'$.*

[18] For details, refer to Appendix C.

In words, substitutability means that a firm rejects more of any given worker types when facing a bigger set of workers. Importantly, the assumption excludes the kind of complementary preferences studied in the previous section. At the same time, the substitutable preferences are not a special case of the preferences considered in Section 4.1 either, since continuity of preferences need not be satisfied here.

Again, by Theorem 1, the fixed points of the map T characterize the stable matchings. Since we do not assume continuity of the choice mappings, however, Theorem 2 does not apply. Instead, as shown in the proof of the next theorem, substitutability of the firms' preferences implies that the map T is monotone increasing with respect to the partial order $\sqsubset_{\tilde{F}}$. Next, recall from Lemma 1 a partially ordered set $(\mathcal{X}, \sqsubset)$, which makes the partially ordered set $(\mathcal{X}^{n+1}, \sqsubset_{\tilde{F}})$ a complete lattice, where $X_{\tilde{F}} \sqsubset_{\tilde{F}} X'_{\tilde{F}}$ if $X_f \sqsubset X'_f$ for all $f \in \tilde{F}$. Hence, Tarski's fixed point theorem yields existence as well as the lattice structure of stable matchings.

To describe the lattice structure, it is also worth describing the extreme points based on the preference orders defined earlier. We say that a stable matching $\overline{M}$ is **firm-optimal** (resp., **firm-pessimal**) if $\overline{M} \succeq_F M$ (resp., $\overline{M} \preceq_F M$) for every stable matching M. A matching $\underline{M}$ is **worker-optimal** (resp., **worker-pessimal**) if $\underline{M} \succeq_\Theta M$ (resp., $\underline{M} \preceq_\Theta M$) for every stable matching M. The result is then stated as follows:

Theorem 4. *When the firms' preferences are substitutable, (i) the set $\mathcal{X}^*$ of fixed points of T is nonempty, and $(\mathcal{X}^*, \sqsubset_{\tilde{F}})$ is a complete lattice; and (ii) there exists a firm-optimal (and worker-pessimal) stable matching $\overline{M} = (C_f(\overline{X}_f))_{f \in F}$, where $\overline{X} = \sup_{\sqsubset_F} \mathcal{X}^*$, and a firm-pessimal (and worker-optimal) stable matching $\underline{M} = (C_f(\underline{X}_f))_{f \in F}$, where $\underline{X} = \inf_{\sqsubset_F} \mathcal{X}^*$.*

Proof. See Appendix D.1. ▮

As has been noted by Hatfield and Milgrom (2005), the algorithm finding the fixed point corresponds to the Gale and Shapley's deferred acceptance algorithm, although the algorithm may not terminate in finite rounds in our continuum model.

Consider an additional restriction on the preferences.

Definition 7. *Firm f's preference exhibits the **law of aggregate demand** if for any $X, X' \in \mathcal{X}$, with $X \sqsubset X'$, $C_f(X)(\Theta) \leq C_f(X')(\Theta)$.*[19]

This property simply ensures that a firm demands more workers (in terms of cardinality) when more workers (in terms of set inclusion) becomes available. This property is needed to obtain the next two results.

[19] This property is an adaptation of the same property to our continuum economy that appears in the literature such as Hatfield and Milgrom (2005), Alkan (2002), and Fleiner (2003).

Theorem 5 (Rural hospital theorem)**.** *If firms' preferences exhibit substitutability and the law of aggregate demand, then for any stable matching M, we have $M_f(\Theta) = \overline{M}_f(\Theta)$ for each $f \in F$ and $M_\emptyset = \overline{M}_\emptyset$.*

Proof. See Appendix D.2. ▮

The result implies that the measure of workers matched with each firm $f \in F$ as well as the measure of unmatched workers is identical across all stable matchings.

We next introduce a condition that would ensure uniqueness of a stable matching. The condition refers to some new notation. For any matching M and subset F' of firms, let $M^f_{F'}$ be a subpopulation of workers, defined by

$$M^f_{F'}(E) := \sum_{P \in \mathcal{P}} \sum_{f': f \succ_P f', f' \notin F'} M_{f'}(\Theta_P \cap E) \text{ for each } E \in \Sigma,$$

who are matched outside firms F' and available to firm f under M.

Definition 8 (Rich preferences)**.** *The firms' preferences are* ***rich*** *if for any individually rational matching $\hat{M} \neq \underline{M}$ such that $\hat{M} \succeq_F \underline{M}$, there exists $f^* \in F$ such that $\underline{M}_{f^*} \neq C_{f^*}(\underline{M}_{f^*} + \hat{M}^{f^*}_{\bar{F}})$, where $\bar{F} := \{f \in F | \hat{M}_f \succ_f \underline{M}_f\}$.*[20]

In words, the condition is explained as follows. Consider any (individually rational) matching $\hat{M}$ that is preferred to the worker-optimal matching $\underline{M}$ by all firms, strictly by firms in $\bar{F} \subset F$. Then, the richness condition requires that, at matching $\underline{M}$, there must exist a firm that would be happy to match with some workers who are not hired by the firms in $\bar{F}$ but are willing to match with the firm under $\hat{M}$. Since firms are more selective at $\hat{M}$ than at $\overline{M}$, it is intuitive that a firm would demand at the latter matching some workers that the more selective firms would not demand at the former matching. The presence of such worker types requires richness of the preference palette of firms as well as workers—hence the name. This point will be seen more clearly when one considers (a general class of) responsive preferences, and we shall illustrate this in an example later.

Theorem 6. *If firms' preferences are rich and substitutable and exhibit the law of aggregate demand, then there exists a unique stable matching.*

Proof. See Appendix D.3. ▮

Checking the rich preference condition requires identifying the worker-optimal matching $\underline{M}$, which can be done by adapting the worker-proposing DA to the continuum economy

[20] Recall that $(\underline{M}_{f^*} + \hat{M}^{f^*}_{\bar{F}})(\cdot)$ is defined as $(\underline{M}_{f^*}(\cdot) + \hat{M}^{f^*}_{\bar{F}}(\cdot)) \wedge G(\cdot)$.

and running it in a given environment.[21] Once $\underline{M}$ is found, it is often straightforward to inspect the existence of $\hat{M}$ and f^* that satisfy the stated property, as Example 3 will illustrate later. The rich preference condition can also be useful for identifying (possibly stronger) sufficient conditions for uniqueness. Specifically, a full support condition that Azevedo and Leshno (2011) have shown to yield a unique stable matching when firms have responsive preferences will be shown to be sufficient for our rich preferences condition in a more general environment in which firms have responsive preferences but may face caps on the number of workers they can hire from different groups of workers. Such group specific quotas, typically based on socio-economic status or other characteristics, may arise from affirmative action or diversity considerations. As is pointed out by Abdulkadiroğlu and Sonmez (2003), the resulting preferences (or choice functions) may violate responsiveness but they nonetheless satisfy substitutability.

Responsive preferences with affirmative action. Assume that there is a finite set T of "ethnic types" that describe characteristics of a worker such as ethnicity, gender, and socio-economic status, such that type θ is mapped to T via some measurable function $\tau : \Theta \to T$. For each $t \in T$, a (measurable) set $\Theta^t := \{\theta \in \Theta | \tau(\theta) = t\}$ of agents has an ethnic type t. Each firm f faces (maximum) quota q_f for the workers and q_f^t for workers in ethnic type t. We assume $q_f \leq \sum_{t \in T} q_f^t$, allowing for the possibility that quota for some ethnic type may not bind. Aside from the quotas, a firm's preference is responsive and described by a continuous score function $s_f : \Theta \to [0, 1]$, with the interpretation that firm f prefers a type-θ' worker to a type-θ worker if and only if $s_f(\theta') > s_f(\theta)$. We assume that G is absolutely continuous and admits density g in the interior of Θ.[22] Such firms are said to have **responsive preferences with affirmative action**. The corresponding choice functions are defined formally in Appendix D.4. As shown there, the choice function exhibits substitutability and satisfies the law of aggregate demand. Consistent with Azevedo and Leshno (2011), the optimal choice by a firm is characterized by the cutoffs, possibly different across alternative ethnic types. The full support condition defined by Azevedo and Leshno (2011) in the context of pure responsive preferences is easily generalized to the current environment:

[21]Note that we may need to take $\underline{M}$ as the limit of the algorithm in case it does not finish in a finite time. See a leading example of Azevedo and Leshno (2011), for instance.

[22]This assumption is reasonable, and is implied by the firms' preferences to involve no ties over a positive measure of worker types. One can define a worker's type as $\theta = (P, t, s_1, ..., s_n)$, where P, t, and s_i are respectively the worker's preference, his ethnic type, and the firm i's score of the worker. For firm i's preference to involve no ties among a positive measure of worker types, the marginal distribution of its scores, s_i, must not involve a mass point. This requires the distribution of θ to be absolutely continuous.

Definition 9 (Full Support)**.** *Firms' preferences have **full support** if for each preference $P \in \mathcal{P}$, any ethnic type $t \in T$, and for any non-empty open cube $S \subset [0,1]^n$, the worker types*

$$\Theta_P^t(S) := \{\theta \in \Theta_P \cap \Theta^t \,|\, (s_f(\theta))_{f \in F} \in S\}$$

have a positive measure; i.e., $G(\Theta_P^t(S)) > 0$.

Our full support condition boils down to the full support condition of Azevedo and Leshno (2011) without affirmative action constraint, if T is a singleton set.

Proposition 2. *If firms have responsive preferences with affirmative action that satisfy the full support condition, then the preferences are rich.*

Proof. See Appendix D.5. ▮

Proposition 2 and Theorem 6 then imply the following:

Corollary 1. *Suppose the firms' preferences are responsive with affirmative action, and satisfy the law of aggregate demand. If the full support condition holds, then there exists a unique stable matching.*

Lastly, the next example demonstrates that the law of aggregate demand is also crucial for the uniqueness result: if the firm preferences violate the law of aggregate demand, then uniqueness of a stable matching does not necessarily hold even if the firm preferences are rich.

Example 3 (Necessity of LoAD for uniqueness)**.** Consider a continuum economy with worker types θ_1 and θ_2 (each with measure 1/2) and firms f_1 and f_2. Preferences are as follows:

1. Firm f_1 wants to hire as many workers of type θ_2 as possible if no worker of type θ_1 is available, but if any positive measure of type θ_1 workers is available, then f wants to hire only type θ_1 workers and no type θ_2 workers at all, and f wants to hire only up to measure 1/3 of type θ_1 workers.

2. The preference of firm f_2 is symmetric, changing the roles of worker types θ_1 and θ_2. More specifically, Firm f_2 wants to hire as many workers of type θ_1 as possible if no worker of type θ_2 is available, but if any positive measure of type θ_2 workers is available, then f wants to hire only type θ_2 workers and no type θ_1 workers at all, and f wants to hire only up to measure 1/3 of type θ_2 workers.

3. Worker preferences are as follows:

$$\theta_1 : f_2 \succ f_1 \succ \emptyset,$$
$$\theta_2 : f_1 \succ f_2 \succ \emptyset.$$

Clearly, the firm preferences are substitutable. To check the rich preference, note first that

$$\underline{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta_2 & \frac{1}{2}\theta_1 \end{pmatrix},$$

where the notation is such that measure 1/2 of type θ_1 workers are matched to f_2 and measure 1/2 of type θ_2 workers are matched to f_1.[23] Then, under any matching $\hat{M} \neq \underline{M}$ that satisfies $\hat{M}_f = C_f(\hat{M}_f \vee \underline{M}_f)$ for all f, some firm, say f_1, must be matched with a positive measure of θ_1 workers. Given that $\hat{M}$ is individually rational, this implies that f_1 is not matched with any θ_2 workers. Also, since f_2 is matched with no more than measure 1/3 workers of θ_2 under any individual rational matching, at least measure 1/6 of θ_2 workers are unemployed under $\hat{M}$, which means that these workers belong to $\hat{M}^{f_2}_{\bar{F}}$ since they prefer f_2 to $\emptyset$ and $\emptyset \notin \bar{F}$. If they are available to f_2 in addition to $\underline{M}_{f_2}$, then f_2 would choose not to be matched with any θ_1 workers, to whom it is matched under $\underline{M}_{f_2}$. Thus, the rich preference condition is satisfied. Finally, firm preferences violate the law of aggregate demand because, for instance, the choice of f_1 from measure 1/2 of θ_2 is to hire all of them, but even adding a measure $\epsilon < 1/2$ of type θ_1 workers rejects all θ_2 workers. As it turns out, there is a firm-optimal stable matching that is different from $\underline{M}$ and given as follow:

$$\overline{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{3}\theta_1 & \frac{1}{3}\theta_2 \end{pmatrix}.$$

5 Approximate Stability in Finite Economies

As we have seen in the illustrative example of Section 2, no matter how large the economy is, as long as it is finite, there does not necessarily exist a stable matching. This motivates us to look for an approximately stable matching in a large finite economy. In this section, we build on the existence of a stable matching in the continuum economy to demonstrate that an approximate stability can be achieved if the economy is finite but sufficiently large.

In order to analyze economies of finite sizes, we consider a sequence of economies $(\Gamma^q)_q$ indexed by a positive integer q. In each economy Γ^q, there is a set of n firms $f_1, \ldots, f_n$, which

[23]That this is a worker-optimal matching follows from the fact that the worker-proposing DA ends in the first round where each worker applies to his preferred firm while the firm accepts him.

is fixed across all q. There are also q workers, each with a type in Θ. The worker distribution is normalized with the economy's size. Formally, let the (normalized) population G^q of workers in Γ^q be defined so that $G^q(E)$ represents the number of workers with type in E divided by q. Any subpopulation $X^q \sqsubset G^q$ is then a discrete distribution over types that is similarly normalized, i.e. a step-function whose steps are multiples of $1/q$. Let $\mathcal{X}^q$ denote the set of all such subpopulations. Note that G^q, and thus every $X^q \in \mathcal{X}^q$, belongs to $\overline{\mathcal{X}}$, though it does not have to be an element of $\mathcal{X}$, i.e. subpopulation of G.

In order to formalize the approximate stability concept, we assume that in economy Γ^q, each firm f evaluates the set of workers it matches with using its preferences as in the continuum economy, but with the distribution of workers normalized by the economy's size. In particular, we first endow firms with cardinal utility functions over distributions of workers. Let $u_f : \overline{\mathcal{X}} \to \mathbb{R}$ denote the continuous utility function of firm f, with $u_f(X)$ being the firm's utility from matching with a subpopulation of workers $X \in \overline{\mathcal{X}}$.[24] That there is a single utility function for each firm defined on the space $\overline{\mathcal{X}}$, which includes worker distributions in any finite economy as well as continuum economy, means that the preferences of the firms remain constant (or consistent) over a sequence of economy $(\Gamma^q)_q$ and its limit Γ. Each firm's choice function in Γ^q is modified to choose the most preferred subpopulation among the discrete distributions — the chosen subpopulation should be a step function with each step being a multiple of $\frac{1}{q}$ that maximizes utility u_f of f.

Given any $\epsilon > 0$, we say that a matching is ϵ-stable if, for any block, the utility gain for any blocking firm is less than ϵ. Formally,

Definition 10. *A matching M in an economy Γ_q is ϵ-**stable** if, for any f and $M'_f \in \mathcal{X}^q$ that block M, $u_f(M'_f) < u_f(M_f) + \epsilon$.*

Remark 4. *For $\epsilon > 0$, we say that matching M is ϵ-**Pareto efficient** if there exists no matching $M' \neq M$ and firm $f \in F$ such that $M' \succeq_F$, $M' \succeq_\Theta M$, and $u_f(M'_f) \geq u_f(M_f) + \epsilon$. By an argument analogous to Pareto efficiency of a stable matching presented in Section 3, it is easy to see that any ϵ-stable matching is ϵ-Pareto efficient.*

Let us say that a sequence of economies $(\Gamma^q)_q$ converges to a continuum economy Γ if the measure G^q of worker types converges weakly to the measure G of the continuum economy, that is, $G^q \xrightarrow{w^*} G$.

[24] To guarantee the existence of such a utility function, we may assume as in Remark 1 that each firm is endowed with a complete, continuous preference relation. Then, since the set of alternatives, $\overline{\mathcal{X}}$, is a compact metric space, such a preference can be represented by a continuous utility function according to the Debreu representation theorem (Debreu, 1954).

Theorem 7. *Fix any $\epsilon > 0$ and a sequence of economies $(\Gamma^q)_q$ that converges to a continuum economy. For any sufficiently large q, there exists an ϵ-stable matching in Γ^q.*

Proof. See Appendix E. ∎

Below we revisit the example in Section 2 to give a concrete example of an approximately stable matching.

Example 4. Recall the finite economy described in Section 2.[25] If the index q is odd, then a stable matching does not exist. Let us consider the following matching: firm f_1 matches with $\frac{q+1}{2}$ workers of each type and firm f_2 matches with all remaining workers, i.e. $\frac{q-1}{2}$ workers of each type. Given this matching, it is straightforward to see that there is no blocking coalition involving firm f_1. Also, the only blocking coalition involving firm f_2 entails taking only a single worker of type θ' away from firm f_1. If q is large, then this deviation will only result in a small utility gain for firm f_2, so the above matching is ϵ-stable. In fact, any finite matching converging to the stable matching of the continuum economy, found in the example of Section 2, will suffice for our purpose.

6 Incentive Compatibility

In this section, we study incentives for truthful reporting under stable mechanisms. Following the literature, we adopt strategy-proofness as the criterion.

Let the type of each worker be a pair $\theta = (a, P)$, where a is interpreted as a "productivity type" while P is a "preference type"; a describes the worker's productivity or skills, while P is the preference of the worker over the firms and the outside option as before. We assume that a worker's preference does not affect firms' preferences and is private information, whereas the productivity type affects firms' preferences and is observable to the firms. Let A and $\mathcal{P}$ be the sets of productivity types and preference types, respectively, and $\Theta = A \times \mathcal{P}$. We assume that A is a finite set.[26] This implies that Θ is a finite set, so the population G

[25] With a slight abuse of notation, this example assumes that there are a total of $2q$ workers (q workers of θ and θ' each) rather than q. Of course, this is done for purely expositional purposes.

[26] Finiteness of A is necessitated by our use of weak* topology and the construction of strategy-proof mechanisms below. To illustrate the difficulty, suppose A were a unit interval, say, and G has a well defined density. Our construction below would require that the density associated with firms' choice mappings satisfy a certain population-proportionality property. Convergence in our weak* topology does not preserve this restriction on density. As a consequence, the operator T may violate upper hemicontinuity, which may result in the failure of the nonempty-valuedness of our mechanism. It may be possible to address this issue by strengthening the topology, but whether the resulting space satisfies conditions that would guarantee the existence of a stable matching is an open question.

of worker types is a discrete distribution. We say that a worker of type $\theta \in \Theta$ is **present** at G if $G(\theta) \neq 0$ and **absent** if $G(\theta) = 0$.

Adopting the model in Section 4.2, we assume that the choice behavior of each firm f is described by a choice correspondence C_f that satisfies upper hemi-continuity and the revealed-preference property. In addition, to formalize the notion that firms' preferences depend solely on workers' productivity types, we assume that each firm has a unique choice about the worker distribution in terms of productivity types, but regards all worker populations with the same distributions of productivity types as equally desirable. Formally, we assume that for any $Y \in C_f(X)$ and $Z \sqsubseteq X$, we have $Z \in C_f(X)$ if and only if $\sum_{P \in \mathcal{P}} Z(a, P) = \sum_{P \in \mathcal{P}} Y(a, P) := \Gamma^a_f(X)$ for all $a \in A$. It follows that C_f satisfies convex-valuedness and closed-valuedness.[27]

As before, a matching is described by a profile $M = (M_f)_f$ of subpopulations of workers matched with alternative firms or the outside option (note that given discreteness of G, each matching M can be expressed as a profile of discrete distributions). We assume that all workers of the same (reported) type are treated ex ante identically. Hence, given matching M, a worker of type (a, P) who is present at G is matched to $f \in \tilde{F}$ with probability $\frac{M_f(a,P)}{G(a,P)}$. Note that $\sum_{f \in \tilde{F}} \frac{M_f(a,P)}{G(a,P)} = 1$ holds by construction, giving rise to a valid probability distribution over $\tilde{F}$.

A **mechanism** is a function φ from the set of worker populations to the set of matchings. We assume that the workers evaluate lotteries via the partial order given by first-order stochastic dominance, and say that mechanism φ is **strategy-proof for workers** if, for each population G, productivity type $a \in A$, preference types P and P' in $\mathcal{P}$ such that both (a, P) and (a, P') are present at G, and $f \in \tilde{F}$, we have

$$\sum_{f' : f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} \geq \sum_{f' : f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P')}{G(a, P')}. \tag{3}$$

Some comments on our modeling assumptions are in order. First, a worker can misreport only her preference type, and not her productivity type: this assumption is the same as in the standard setting in the literature. Second, unlike in finite population models, the worker cannot alter the population G by unilaterally misreporting her preferences, because there

[27] Closed-valuedness of C_f is straightforward. To see that C_f is convex-valued, let $Y, Z \in C_f(X)$ and $\lambda \in (0, 1)$ and consider a convex combination $W := \lambda Y + (1 - \lambda) Z$. Then

1. $\sum_{P \in \mathcal{P}} W(a, P) = \lambda \sum_{P \in \mathcal{P}} Y(a, P) + (1 - \lambda) \sum_{P \in \mathcal{P}} Z(a, P) = \lambda \Gamma^a_f(X) + (1 - \lambda) \Gamma^a_f(X) = \Gamma^a_f(X)$,
2. For each $\theta \in \Theta, Y(\theta) \leq X(\theta)$ and $Z(\theta) \leq X(\theta)$, so $W(\theta) = \lambda Y(\theta) + (1 - \lambda) Z(\theta) \leq X(\theta)$.

Therefore the convex combination W is also in $C_f(X)$, implying convex-valuedness of C_f.

are a continuum of workers. Third, we only impose the restriction (3) for types (a, P) and (a, P') who are present at G. For the true worker type (a, P), this is the same assumption as in the standard strategy-proofness concept for finite markets. For the misreported type (a, P'), this assumption is made for expositional simplicity, because the concept of the mechanism in our model does not specify allocations for absent types. Note, however, that if φ is a mechanism that is individually rational (which is the case for stable mechanisms) and strategy-proof for workers in the above sense, then it is easy to specify the outcomes for absent types under φ so as to eliminate incentives for misreporting to be an absent type. Specifically, one can simply specify that the mechanism assigns any worker of an absent type to the outside option with probability one.

Now we are ready to analyze incentive properties of stable mechanisms in large markets. First, the following example shows that an arbitrary stable mechanism need not be strategy-proof for workers even in a market with continuum of workers.

Example 5. Let there be two firms f and f', as well as continuum (with measure one) of workers with one productivity type, $A = \{a\}$. Let

$$P : f, f'; \quad P' : f,$$

and G be a distribution such that $G(a, P) = G(a, P') = 1/2$. The choice correspondences of f and f' are such that f chooses up to capacity $1/2$ of workers while f' rejects every worker (the firm f is indifferent about which type of workers to choose because there is only one productivity type a).[28] Consider a stable matching mechanism which maps the aforementioned worker population to matching M such that $M_f(a, P) = M_{\o}(a, P') = 1/2$ and $M_f(a, P') = M_{f'}(a, P) = M_{f'}(a, P') = M_{\o}(a, P) = 0$. Then, a worker whose type is (a, P') is matched with $\o$ with probability one if she reports her true preferences P', while she is matched to f with probability one if she reports preferences P. Therefore, each type (a, P') worker has incentives to misreport her preferences to be P. Thus, this mechanism is not strategy-proof for workers.

The above example notwithstanding, we next show that there exists a stable mechanism that is strategy-proof for workers. To do so, we first define a correspondence B from $\mathcal{X}$ to

[28] In this example, firm f' plays no role except making the incentive problem nontrivial: note that without f', there would be only one preference type. We chose this, perhaps somewhat artificial, example only for simplicity. It is straightforward to make an example in which f' finds some worker subpopulations acceptable.

itself that is given by

$$B(X) := \{X' \sqsubset X \mid \text{ for each } a \in A, \text{ there is some } \alpha^a \in [0,1] \text{ such that} \\ X'(a,P) = \min\{X(a,P), \alpha^a G(a,P)\} \text{ for all } P \in \mathcal{P}\}. \quad (4)$$

The correspondence $B(X)$ requires each firm to hire workers with different preference types (and the same productivity type) in proportion to their population sizes at G, namely proportion α^a of $G(a,P)$ from each worker type (a,P), except for worker types (a,P) whose available measure $X(a,P)$ falls short of $\alpha^a G(a,P)$, of which the entire measure is chosen. We then modify the choice correspondence of each firm $f \in F$ to

$$\tilde{C}_f(X) = C_f(X) \cap B(X), \quad (5)$$

while we let $\tilde{C}_\emptyset = C_\emptyset$.

In Section 7, we study a setup that is more general than the current one in that the sets of indifferent workers (or indifference classes) are allowed to vary across firms. Under the more general setup, we establish that $\tilde{C}_f$ is a singleton correspondence (i.e., a function) and satisfies the revealed preference property (Lemma 6 of Appendix G). Given the revealed preference property, one can define stability in the model $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ with each $\tilde{C}_f$ defined in equation (5). Then, Theorem 9 of Section 7 shows that a matching M in the model $(G, F, \mathcal{P}_\Theta, C_F)$ is stable if M is stable in the model $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$.[29] Finally, Theorem 10 of Section 7 shows that there exists a stable matching in the model $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$.

These results guarantee that there exists a stable matching in the model $(G, F, \mathcal{P}_\Theta, C_F)$ that corresponds to choices by firms whose choice functions are given by $\tilde{C}_F$. Intuitively, a stable matching we just identified satisfies the additional property that each firm seeks to select workers with different preference types and common productivity type in proportion to the sizes of their populations. Given these results, let φ be a mechanism that finds a stable matching identified in Theorem 10, i.e., a stable matching corresponding to $\tilde{C}_F$, and refer to φ as a **population-proportional stable mechanism**.

Theorem 8. *Any population-proportional stable mechanism is strategy-proof for workers.*

Proof. Denote the population-proportional stable mechanism by φ. Suppose for contradiction that there exist a, P, P', and f for which inequality (3) fails. Then, let f be the most preferred firm (or the outside option) at P among those for which inequality (3) fails. Then,

$$\sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a,P)}{G(a,P)} < \sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a,P')}{G(a,P')}, \quad (6)$$

[29]In fact, Theorem 9 establishes a stronger result that M is *strongly* stable in $(G, F, \mathcal{P}_\Theta, C_F)$.

while

$$\sum_{f':f'\succeq_P f^P_-} \frac{\varphi_{f'}(G)(a,P)}{G(a,P)} \geq \sum_{f':f'\succeq_P f^P_-} \frac{\varphi_{f'}(G)(a,P')}{G(a,P')},$$

so it follows that

$$\frac{\varphi_f(G)(a,P)}{G(a,P)} < \frac{\varphi_f(G)(a,P')}{G(a,P')}. \tag{7}$$

By the definition of a population-proportional stable mechanism, inequality (7) holds only if

$$\varphi_f(G)(a,P) = X(a,P), \tag{8}$$

where

$$X(a,P) = \sum_{f':f\succeq_P f'} \varphi_{f'}(G)(a,P) \tag{9}$$

by the characterization theorem of stable matchings, Theorem 1. Equalities (8) and (9) imply

$$\sum_{f':f\succ_P f'} \varphi_{f'}(G)(a,P) = 0,$$

and thus, because $\sum_{f'\in\tilde{F}} \frac{\varphi_{f'}(G)(a,P)}{G(a,P)} = 1$ as $\varphi(G)$ is a matching, we obtain

$$\sum_{f':f'\succeq_P f} \frac{\varphi_{f'}(G)(a,P)}{G(a,P)} = 1.$$

This inequality contradicts inequality (6) because the right hand side of inequality (6) cannot be strictly larger than 1 as $\varphi(G)$ is a matching, which completes the proof. ▮

Remark 5. Even with a continuum of workers, no stable mechanism is strategy-proof for the firms. To show this fact, consider the following example.[30] Let $F = \{f_1, f_2\}$, $\Theta = \{\theta, \theta'\}$, and $G(\theta) = G(\theta') = 1/2$. Worker preferences are given as follows:

$$\theta : f_2 \succ f_1 \succ \emptyset,$$
$$\theta' : f_1 \succ f_2 \succ \emptyset.$$

[30] This example is a continuum-population variant of an example in Section 3 of Hatfield, Kojima and Narita (2014*a*). See also Azevedo (2014) who shows that stable mechanisms are manipulable via capacities even in markets with continuum of workers.

Firm preferences are responsive; f_1 prefers θ to θ' to vacant positions, and wants to be matched with workers of up to measure 1; and f_2 prefers θ' to θ to vacant positions, and wants to be matched with workers of up to measure 1/2.

Let φ be any stable mechanism. Given the above input, the following matching is the unique stable matching:

$$M \equiv \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta' & \frac{1}{2}\theta \end{pmatrix}.$$

Matching M is clearly stable because it is individually rational and every worker is matched to her most preferred firm. To see the uniqueness, note first that in any stable matching, every worker has to be matched to a firm (if there is a positive measure of unmatched workers, then there is also a vacant position in firm f_1, and they block the matching). All workers of type θ' are matched with f_1, because otherwise f_1 and θ' workers who are not matched with f_1 block the matching (note that f_1 has vacant positions to fill with θ' workers). Given this, all workers of type θ are matched with f_2, because otherwise f_2 and θ workers who are not matched with f_2 block the matching (note that f_2 has vacant positions to fill with type θ workers).

Now, assume that f_1 misreports its preferences, declaring that θ is the only acceptable worker type, while it still has a responsive preferences and wants to be matched to the same measure as before, 1. And assume that preferences for other agents are unchanged. Then, it is easy to verify that the unique stable matching is

$$M' \equiv \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta & \frac{1}{2}\theta' \end{pmatrix}.$$

Therefore, firm f_1 prefers its outcome at M' to the one at M, proving that no stable mechanism is strategy-proof for the firms.

7 Time Share/Probabilistic Matching Models

As mentioned above, our model turns out to have a connection with time share and probabilistic matching models. In both models, there is a *finite* set of workers with a set of firms remaining to be finite. In the time-share model, a matching between firms and workers corresponds to the time that they spend together while, in the probabilistic matching model, it corresponds to the probabilities that they are matched to one another. Probabilistic matching is often used in allocation problems without money such as school choice, while time-share models have been proposed as a solution to labor matching markets in which part-time jobs are available (see Biró, Fleiner and Irving (2013) for instance). We will follow the time share interpretation for the remainder of this Section.

To describe the time share matching model, we use the same framework as in the continuum matching model, while assuming that the type space Θ is finite and $\Sigma = 2^{\Theta}$ (that is, the power set of Θ). In the time share model, each $\theta \in \Theta$ represents an individual worker while the (discrete) measure $G(\theta)$ represents his endowment of time.[31] Likewise, firm f being matched with a subpopulation $X \sqsubset G$ means that it hires each worker $\theta \in \Theta$ for the amount of time equal to $X(\theta)$.

Each worker $\theta \in \Theta$ has a strict preference over firms, denoted as $\succ_\theta$. Then, $\Theta_P = \{\theta \in \Theta : \succ_\theta = P\}$ for each $P \in \mathcal{P}$ and $\mathcal{P}_\Theta = \{\Theta_P : P \in \mathcal{P}\}$, as defined earlier. For firms' preferences, we allow each firm to be indifferent among a set of workers. To be concrete, we let a partition $\{\Theta_f^1, ..., \Theta_f^{K_f}\}$ of Θ denote the set of indifference classes for each firm f. The partition means that firm f is indifferent to redistributing the total time contracted with workers within each group Θ_f^k. Let $I_f = \{1, ..., K_f\}$ be the associated index set. Then, the firm f's preference is represented by a choice correspondence $C_f : \mathcal{X} \rightrightarrows \mathcal{X}$ satisfying the revealed preference and the following property: For any $Y \in C_f(X)$ and $Z \sqsubseteq X$, $Z \in C_f(X)$ if and only if $\sum_{\theta \in \Theta_f^k} Z(\theta) = \sum_{\theta \in \Theta_f^k} Y(\theta)$ for all $k \in I_f$. In other words, each firm f has a unique optimal choice up to the time share allocation within each indifference class. This preference is analogous to the one described in Section 6 where each indifference class consists of the workers with the same productivity type so all firms have the same indifference classes. In contrast to that model, indifferent classes are allowed to vary across firms in the current setup. By extending the notation $\Gamma_f^a(X)$ in Section 6, we let $\Gamma_f^k(X)$ denote the firm f's unique choice of total time share with workers in the indifference class Θ_f^k: That is, $\Gamma_f^k(X) = \sum_{\theta \in \Theta_f^k} Y(\theta)$ for any $Y \in C_f(X)$. Let $\Gamma_f(X) = (\Gamma_f^k(X))_{k \in I_f}$.

A time share model described so far is denoted by $(G, F, \mathcal{P}_\Theta, C_F)$. Notice that our time share model has the same structure as the *fractional matching model* that is also known as *random or aggregate matching model* in the literature.[32]

Given the choice correspondence framework, the notion of stability follows the one in Definition 4. Notice that the stability in the time share model corresponds to ex ante

[31]In the probabilistic matching model, it is natural to assume that $G(\theta) = 1/|\Theta|$ for all $\theta \in \Theta$ (after normalization to ensure $G(\Theta) = 1$), since $G(\theta)$ represents the total probability with which the agent θ can be matched with firms.

[32]This literature includes Hylland and Zeckhauser (1979), Roth, Rothblum and Vande Vate (1993), Alkan and Gale (2003), Baïou and Balinski (2000), Echenique et al. (2013), and Kesten and Ünver (2014) among others.

stability, if one interprets the time share as probability share. That is, if any worker (or student) were to shift shares from a less preferred to a more preferred firm (school), the latter must necessarily be worse off. In this sense, stability, in particular no blocking coalition condition, involves the notion of fairness or no justified envy. What this condition reduces to when preferences are all responsive (as is the case in the existing literature) is that whenever a worker θ enjoys a positive share with a firm f, then all workers preferred by f more than θ have zero share with firms they prefer less than f.

The existence of a stable matching in the time share model follows from Theorems 3 in a straightforward manner.

Corollary 2. *Suppose that Γ_f is continuous for each $f \in F$. Then, there exists a stable matching in the time share model.*

Proof. By Theorem 3, it suffices to show that C_f is convex- and closed-valued, and upper hemicontinuous.

To show first that C_f is convex-valued, for any given X, consider any $X', X'' \in C_f(X)$. Note first that $X', X'' \sqsubseteq X$ implies $\lambda X' + (1-\lambda)X'' \sqsubseteq X$. Also, for any $\lambda \in [0,1]$ and $k \in I_f$,

$$\sum_{\theta \in \Theta_f^k} (\lambda X' + (1-\lambda)X'')(\theta) = \lambda \sum_{\theta \in \Theta_f^k} X'(\theta) + (1-\lambda) \sum_{\theta \in \Theta_f^k} X''(\theta) = \Gamma_f(X),$$

where the second equality holds since the assumption that $X', X'' \in C_f(X)$ implies $\Gamma_f^k(X) = \sum_{\theta \in \Theta_f^k} X'(\theta) = \sum_{\theta \in \Theta_f^k} X''(\theta)$. Thus, $\lambda X' + (1-\lambda)X'' \in C_f(X)$.

To show the upper hemicontinuity, consider two sequences $(X^\ell)_{\ell \in \mathbb{N}}$ and $(\tilde{X}^\ell)_{\ell \in \mathbb{N}}$ converging to some X and $\tilde{X}$, respectively, such that for each ℓ, $\tilde{X}^\ell \in C_f(X^\ell)$, i.e. $\tilde{X}^\ell \sqsubseteq X^\ell$ and $\Gamma_f^k(X^\ell) = \sum_{\theta \in \Theta_f^k} \tilde{X}^\ell(\theta), \forall k \in I_f$. Since Γ_f is continuous, we have $\Gamma_f^k(\tilde{X}) = \lim_{\ell \to \infty} \Gamma_f^k(\tilde{X}^\ell) = \lim_{\ell \to \infty} \sum_{\theta \in \Theta_f^k} \tilde{X}^\ell(\theta) = \sum_{\theta \in \Theta_f^k} \tilde{X}(\theta)$, which (together with the fact that $\tilde{X} \sqsubseteq X$) means that $\tilde{X} \in C_f(X)$, establishing the upper hemicontinuity of C_f. The proof of closed-valuedness is similar and hence omitted. ▌

As mentioned before, a matching can be interpreted as a profile of probability shares. In particular, for probabilistic matching, say in the school choice context, stability involves the notion of fairness or no justified envy in an ex ante sense. In such an environment with indifference of school preferences/priority, the following stronger notion of fairness, proposed by Kesten and Ünver (2014), is of interest.

Definition 11. *A matching M in the time share model is **strongly stable** if (i) it is stable and (ii) for any $f \in F$ and any $\theta, \theta' \in \Theta_f^k$, if $\frac{M_f(\theta)}{G(\theta)} < \frac{M_f(\theta')}{G(\theta')}$, then $\sum_{f' \in \tilde{F}: f' \prec_\theta f} M_{f'}(\theta) = 0$.*

In words, strong stability means the following. Suppose a worker θ is in the same indifference class as θ' with respect to firm f's preference. If θ worker enjoys a strictly lower share of her time endowment with f than θ' does, then it must be that θ has no share with any firm she prefers strictly less than f. In that sense, the workers in the same priority class should not be discriminated against one another. This is an additional requirement not implied by the stability alone (which requires fairness across workers that a firm is not indifferent between).

In the remainder of this section, we establish the existence of a strongly stable matching. To do so, we extend the correspondence B in (4), and denote it by B_f, by imposing the population proportionality on each firm f's indifference classes. That is, for any $X \sqsubset G$ and $X' \in B_f(X)$, there is some $\alpha^k \in [0,1]$ for each $k \in I_f$ such that $X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\}$ for all $\theta \in \Theta_f^k$. Note that this mapping differs from B only in that the indifference classes can vary across firms. Then, as in Section 6, we modify each firm f's choice correspondence C_f into $\tilde{C}_f(z) = C_f(z) \cap B_f(z)$. We also let $\tilde{C}_\varnothing = C_\varnothing$.

Using this choice correspondence, we establish the equivalence between the set of strongly stable matchings in the original time share model and the set of stable matchings in a modified time share model:

Theorem 9. *A matching M in the time share model $(G, F, \mathcal{P}_\Theta, C_F)$ is strongly stable if and only if M is stable in the time share model $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$.*

Proof. See Section G. ▮

Given this equivalence result, we are now ready to show the existence of a strongly stable matching.

Theorem 10. *Suppose that Γ_f is continuous. Then, there exists a strongly stable matching in the time share model $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$.*

Proof. See Section G. ▮

This result generalizes the existence result of a strongly stable matching in the school choice context as studied by Kesten and Ünver (2014). They consider a school choice problem in which schools may regard multiple students as having the same priority. In that environment, they define probabilistic matchings that satisfy our strong stability property (which they call strong ex ante stability), and show their existence, in the environment in which schools have responsive preferences with ties. Our contribution here lies in formalizing a strongly stable matching and establishing its existence with general preferences that may violate responsiveness or even substitutability. Our result could be useful for school choice environments in which the schools may need a balanced student body in terms of gender, ethnicity, income, or skill levels. For example, in New York City, the so-called Education Option (EdOpt) school programs are required to assign 16 percent, 68 percent, and another 16 percent of the seats to the top performers, middle performers, and the lower performers, respectively (Abdulkadiroğlu, Pathak and Roth, 2005). Strong stability will ensure ex ante fairness in the sense that it not only implements the desired mix of students but also will ensure fairness in the treatment of students relative to the schools' objective.

8 Matching with Contracts

The preceding sections assumed that the terms of employment contracts are exogenously given. In many applications, however, they are decided endogenously. To study such a situation, we generalize our basic model by introducing a continuum-population version of the "matching with contracts" model due to Hatfield and Milgrom (2005).

Let Ω denote a finite set of all available contracts with its typical element denoted as ω. Assume that Ω is partitioned into subsets, $\{\Omega_f\}_{f\in\tilde{F}}$, where Ω_f is a set of contacts for firm $f \in \tilde{F}$ and $\Omega_\varnothing = \{\omega_\varnothing\}$. Think of each contract as containing information about associate firm and terms of hire. Let $f(\omega) \in \tilde{F}$ denote the identity of firm associated with contract ω. Thus, $f(\omega) = f$ if and only if $\omega \in \Omega_f$. We use $P \in \mathcal{P}$ to denote workers' ranking or preference defined over Ω. Let $\omega_-^P \in \Omega$ denote a contract that is an immediate predecessor of ω according to preference P, that is, ω_-^P is the contract with the property $\omega_-^P \succ_P \omega$ and $\omega' \succeq_P \omega_-^P$ for all $\omega' \succ_P \omega$ (if such a contract exists). As before, Θ_P denotes a subset of types in Θ whose preference is given as P.

For any $\omega \in \Omega_f$, let $X_\omega \in \mathcal{X}$ denote the subpopulation of workers who are available to firm f under the contract ω. Given any profile $X_f = (X_\omega)_{\omega\in\Omega_f} \in \mathcal{X}^{|\Omega_f|}$, each firm f's choice is described by a map $X_f \mapsto C_f(X_f) = (C_\omega(X_f))_{\omega\in\Omega_f} \in \mathcal{Y}_f(X_f)$, where

$$\mathcal{Y}_f(X_f) := \{(Y_\omega)_{\omega\in\Omega_f} \in \mathcal{X}^{|\Omega_f|} \mid \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega_f:\omega'\preceq_P\omega} Y_{\omega'}(\Theta_P \cap \cdot) \sqsubset X_\omega(\cdot), \forall\omega \in \Omega_f\}.$$

For any profile of subpoulations in $\mathcal{Y}_f(X_f)$, the measure of workers who are hired by f under any contract $\omega \in \Omega_f$ or worse cannot exceed the measure of workers, X_ω, who are available under ω. The requirement that the output of C_f should belong to $\mathcal{Y}_f$ is based on the premise that each firm f is aware of workers' preferences and also believes (correctly) that only those workers who are available under $\omega \in \Omega_f$ can be hired under the contracts that are weakly inferior to ω, and thus put an upper bound on the measure of workers hired under the latter contracts. As before, we let $C_{\emptyset}(X_{\emptyset}) = X_{\emptyset}$, and let $R_\omega(X_f) = X_\omega - C_\omega(X_f)$ for each $\omega \in \Omega_f$ and $R_f(X_f) = (R_\omega(X_f))_{\omega\in\Omega_f}$.

A **matching** is $M = (M_\omega)_{\omega\in\Omega}$ such that $M_\omega \in \mathcal{X}$ for all $\omega \in \Omega$ and $\sum_{\omega\in\Omega} M_\omega = G$. Let $M_f = (M_\omega)_{\omega\in\Omega_f} \in \mathcal{X}^{|\Omega_f|}$ denote a profile of subpopulations who are matched with f. From M_f, one can derive a subpopulation of workers *within* firm f that are available for f to match with under each contract $\omega \in \Omega_f$:

$$M_f^{\preceq\omega}(\cdot) := \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega_f:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap \cdot). \tag{10}$$

That is, all workers matched with f under the contracts that are no better than ω are available for matching under ω. Let $M_f^{\preceq} = (M_f^{\preceq\omega})_{\omega\in\Omega_f}$. Note that $M_f^{\preceq}$ does *not* take into account those workers who are matched with firms other than f and available for f. A subpopulation of *all* workers—not only those within firm f—who are available to $f \in \tilde{F}$ under contract $\omega \in \Omega_f$ is denoted as before by

$$D^{\preceq\omega}(M)(\cdot) = \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap \cdot).$$

Let $D^{\preceq f}(M) = (D^{\preceq\omega}(M))_{\omega\in\Omega_f}$.

Definition 12. *A matching $M = (M_\omega)_{\omega\in\Omega}$ is **stable** if*

1. *$M_\omega(\Theta_P) = 0$ for all $P \in \mathcal{P}$ and $\omega \in \Omega$ satisfying $\omega_\emptyset \succ_P \omega$; and for each $f \in F$, $M_f = C_f(M_f^{\preceq})$, and*

2. *There exist no $f \in F$ and $\widetilde{M}_f \in \mathcal{X}^{|\Omega_f|}, \widetilde{M}_f \neq M_f$ such that*

$$\widetilde{M}_f = C_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq}) \text{ and } \widetilde{M}_f^{\preceq} \sqsubset D^{\preceq f}(M).$$

Note that this definition reduces to the notion of stability in Definition 2.

Let us now define a fixed-point map $T = (T_\omega)_{\omega\in\Omega} : \mathcal{X}^{|\Omega|} \to \mathcal{X}^{|\Omega|}$: For each $\omega \in \Omega$ and $E \in \Sigma$,

$$T_\omega(X)(E) := \sum_{P\in\mathcal{P}} R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap E), \tag{11}$$

where $R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap E) = G(\Theta_P \cap E)$ if $\omega_-^P = \emptyset$.[33]

[33] Note that $f(\omega_-^P)$ is a firm associated with the immediate predecessor of ω, which may or may not be the same as $f(\omega)$.

Theorem 11. *If $M = (M_\omega)_{\omega\in\Omega}$ is stable, then $X = (D^{\preceq_\omega}(M))_{\omega\in\Omega}$ is a fixed point of T. Conversely, if $X = (X_\omega)_{\omega\in\Omega}$ is a fixed point of T, then there is a stable matching $M = (M_\omega)_{\omega\in\Omega}$ such that $M_f = C_f(X_f), \forall f \in \tilde{F}$.*

Proof. See Appendix F. ∎

Given this characterization result, the existence of stable matching with contracts follows from assuming that for each $f \in F$, $C_f : \mathcal{X}^{|\Omega_f|} \to \mathcal{X}^{|\Omega_f|}$ is continuous, since it guarantees that $T : \mathcal{X}^{|\Omega|} \to \mathcal{X}^{|\Omega|}$ is also continuous.

9 Discussions

Our analysis has so far focused on the existence of a desirable matching mechanism. Existence is clearly necessary to find a desired mechanism, but for practical implementation one needs an algorithm that is computable and fast. Our map T not only characterizes stable matchings but also is interpretable as a tâtonnement adjustment process for revising the "budget quotes" that has a potential to be used as an actual mechanism. In fact, in case of substitutable preferences T entails a monotonic process that converges regardless of the initial state, and in particular corresponds to the celebrated and largely successful Gale and Shapley's deferred acceptance (DA) algorithms when the initial measures are set at either the largest (firm-proposing DA) or at the smallest measures (worker-proposing DA). Even for non-substitutable preferences, T may at least provide some meaningful clue for finding a practical algorithm.

A reasonable starting point of this inquiry is to ask how often—or more precisely how large is the set of initial states from which—the adjustment process of T converges. Our leading example suggests that the answer to this question can be "almost never," or more precisely "except when the initial state itself is stable." Recall from Example 1 that, starting from any state $(\frac{1}{2}, x_1'; x_2, \frac{1}{2})$, T moves us to a state $(\frac{1}{2}, \frac{1}{2} - x_2; x_1', \frac{1}{2})$. One can continue further on the adjustment process, and notice that the process cycles back to the initial state (chosen arbitrarily) in a few additional steps:

$$(\tfrac{1}{2}, x_1'; x_2, \tfrac{1}{2}) \to (\tfrac{1}{2}, x_2; \tfrac{1}{2} - x_1', \tfrac{1}{2}) \to (\tfrac{1}{2}, \tfrac{1}{2} - x_1'; \tfrac{1}{2} - x_2, \tfrac{1}{2}) \to (\tfrac{1}{2}, \tfrac{1}{2} - x_2; x_1', \tfrac{1}{2}) \to (\tfrac{1}{2}, x_1'; x_2, \tfrac{1}{2}).$$

This example suggests at least there are situations in which T, without modification, may not be relied upon to produce a desirable mechanism. The example also suggests that some procedure for breaking a cycle is needed for the adjustment process like T to produce a desirable outcome.

A Analysis of the Example in Section 2

Let r be the number of workers with each of the two types who are matched to f. We consider the following cases:

1. Suppose $r > q/2$. For any such matching, at least one position is vacant at firm f' because f' has q positions, but strictly more than q workers are matched to f out of the total of $2q$ workers. Thus such a matching is blocked by f' and a type θ' worker who is currently matched to f.

2. Suppose $r < q/2$. Consider the following cases.

   (a) Suppose that there exists a type θ worker who is unmatched. Then such a matching is unstable because that worker and firm f' block it (note that f' prefers θ most).

   (b) Suppose that there exists no type θ worker who is unmatched. This implies that there exists a type θ' worker who is unmatched (because there are $2q$ workers in total, but firm f is matched to strictly fewer than q workers by assumption, and f' can be matched to at most q workers in any individually rational matching). Then, since f is the most preferred by all θ workers, a θ' worker prefers f to $\emptyset$, and there is some vacancy at f because $r < q/2$, the matching is blocked by a coalition of a type θ worker, a type θ' worker, and f.

B Preliminaries for the Continuum Economy Model

B.1 Proof of Lemma 1

For any subset $\mathcal{Y} \subset \mathcal{X}$, define

$$\overline{Y}(E) := \sup\{\sum_i Y_i(E_i) \,|\, \{E_i\} \text{ is a finite partition of } E \text{ in } \Sigma \text{ and } \{Y_i\} \text{ is a finite collection of measures in } \mathcal{Y}, \forall i\}, \forall E.$$

and $\underline{Y}$ analogously (by replacing "sup" with "inf"). We prove the lemma by showing that $\overline{Y} = \sup \mathcal{Y} \in \mathcal{Y}$ and $\underline{Y} = \inf \mathcal{Y} \in \mathcal{X}$.

First of all, note that $\overline{Y}$ and $\underline{Y}$ are monotonic, i.e. for any $E \subset D$, we have $\overline{Y}(D) \geq \overline{Y}(E)$ and $\underline{Y}(D) \geq \underline{Y}(E)$, whose proof is straightforward and thus omitted.

We next show that $\overline{Y}$ and $\underline{Y}$ are measures. We only prove the countable additivity of $\overline{Y}$, since the other properties are straightforward to prove and also since a similar argument applies to $\underline{Y}$. To this end, consider any countable collection $\{E_i\}$ of disjoint sets in Σ and let $D = \cup E_i$. We need to show that $\overline{Y}(D) = \sum_i \overline{Y}(E_i)$. For doing so, consider any finite partition $\{D_i\}$ of D and any finite collection of measures $\{Y_i\}$. Letting $E_{ij} = E_i \cap D_j$, for any i, the collection $\{E_{ij}\}_j$ is a finite partition of E_i in Σ. Thus, we have

$$\sum_i Y_i(D_i) = \sum_i \sum_j Y_i(E_{ij}) \leq \sum_i \overline{Y}(E_i).$$

Since this inequality holds for any finite partition $\{D_i\}$ of D and collection $\{Y_i\}$, we must have $\overline{Y}(D) \leq \sum_i \overline{Y}(E_i)$. To show that the reverse inequality also holds, for each E_i, we consider any finite partition $\{E_{ij}\}_j$ of E_i in Σ and collection of measures $\{Y_{ij}\}_j$ in $\mathcal{Y}$. We prove that $\overline{Y}(D) \geq \sum_i \sum_j Y_{ij}(E_{ij})$, which would imply $\overline{Y}(D) \geq \sum_i \overline{Y}_{ij}(E_i)$ as desired since the partition $\{E_{ij}\}_j$ and collection $\{Y_{ij}\}_j$ are arbitrarily chosen for each i. Suppose not for contradiction. Then, we must have $\overline{Y}(D) < \sum_{i=1}^k \sum_j Y_{ij}(E_{ij})$ for some k. Letting $E := \cup_{i=1}^k (\cup_j E_{ij})$, this implies $\overline{Y}(D) < \sum_{i=1}^k \sum_j Y_{ij}(E_{ij}) \leq \overline{Y}(E)$, where the second inequality holds by the definitioin of $\overline{Y}$. This contradicts with the monotonicity of $\overline{Y}$ since $E \subset D$.

We now show that $\overline{Y}$ and $\underline{Y}$ are the supremum and infimum of $\mathcal{Y}$, respectively. It is straightforward to check that for any $Y \in \mathcal{Y}$, $Y \sqsubset \overline{Y}$ and $\underline{Y} \sqsubset Y$. Consider any $X, X' \in \mathcal{X}$ such that for all $Y \in \mathcal{Y}$, $Y \sqsubset X$ and $X' \sqsubset Y$. We show that $\overline{Y} \sqsubset X$ and $X' \sqsubset \underline{Y}$. First, if $\overline{Y} \not\sqsubset X$ to the contrary, then there must be some $E \in \Sigma$ such that $\overline{Y}(E) > X(E)$. This means there are a finite partition $\{E_i\}$ of E and a collection of measures $\{Y_i\}$ in $\mathcal{Y}$ such that $\overline{Y}(E) \geq \sum Y_i(E_i) > X(E) = \sum X(E_i)$. Thus, for at least one i, we have $Y_i(E_i) > X(E_i)$, which contradicts the assumption that for all $Y \in \mathcal{Y}$, $Y \sqsubset X$. An analogous argument can be used to show $X' \sqsubset \underline{Y}$.

B.2 Proof of Proposition 1

Suppose that matching M is not Pareto efficient. Then by definition of Pareto efficiency, there exists $M' \neq M$ such that $M' \succeq_F M$ and $M' \succeq_\Theta M$. Let $f \in F$ be a firm such that $M'_f \neq M_f$. By assumption, $M' \succeq_f M$.

Next, since $M' \succeq_\Theta M$, for each f, $D^{\succeq f}(M') \sqsupset D^{\succeq f}(M)$, or

$$\sum_{f':f' \succeq_P f} M'_{f'}(\Theta_P \cap E) \geq \sum_{f':f' \succeq_P f} M'_{f'}(\Theta_P \cap E), \forall E \in \Sigma.$$

This implies that

$$\sum_{f':f' \succeq_P f_-^P} M'_{f'}(\Theta_P \cap E) \geq \sum_{f':f' \succeq_P f} M_{f_-^P}(\Theta_P \cap E), \forall E \in \Sigma,$$

where f_{-}^{P} refers to the firm that is ranked immediately above f according to P (whenever it is well defined),[34] or equivalently

$$\sum_{f':f' \succ_P f} M'_{f'}(\Theta_P \cap E) \geq \sum_{f':f' \succ_P f} M_{f_{-}^{P}}(\Theta_P \cap E), \forall E \in \Sigma.$$

This in turn implies that, for each f and P,

$$\sum_{f':f' \preceq_P f} M'_{f'}(\Theta_P \cap E) \leq \sum_{f':f' \preceq_P f} M_{f_{-}^{P}}(\Theta_P \cap E), \forall E \in \Sigma,$$

or equivalently, for each f,

$$D^{\preceq f}(M') \sqsubset D^{\preceq f}(M).$$

By definition, $M'_f \sqsubset D^{\preceq f}(M')$, so we have $M'_f \sqsubset D^{\preceq f}(M)$.

Collecting the observations made so far, we conclude that f and M'_f block M, implying that M is not stable. Therefore, we have established that stability implies Pareto efficiency.

C Existence of Stable Matchings: Proof of Theorems 1, 2 and 3

Since Theorem 1 and 2 follow as corollaries from their counterparts (Theorem 12 and 3) in the correspondence case, we only consider the case where C_f and thus R_f are correspondences.

Let us extend the mapping T to the case of correspondence as follows: For any $X \in \mathcal{X}^{n+1}$,

$$T(X) = \{\tilde{X} \in \mathcal{X}^{n+1} \,|\, \tilde{X}_f(\cdot) = \sum_{P \in \mathcal{P}} Y_{f_{-}^{P}}(\Theta_P \cap \cdot), \forall f \in \tilde{F},$$
$$\text{for some } (Y_f)_{f \in \tilde{F}} \text{ such that } Y_f \in R_f(X_f), \forall f \in \tilde{F}, \},$$

where $Y_{f_{-}^{P}}(\Theta_P \cap \cdot) = G(\Theta_P \cap \cdot)$ if f is a top-ranked firm at P.

First, we obtain the characterization of stable matchings as fixed points of T, from which Theorem 1 follows as a corollary since the above T mapping coincides with that in (11), as can be easily verified.

Theorem 12. *A matching $M \in \mathcal{X}^{n+1}$ is stable if and only if there is a fixed point $X \in T(X)$ such that $M_f \in C_f(X_f), \forall f \in \tilde{F}$. Also, any such X and M satisfy $X_f = D^{\preceq f}(M), \forall f \in \tilde{F}$.*

[34] This is defined later as an immediate predecessor. Formally, $f_{-}^{P} \succ_P f$ and if $f' \succ_P f$, then $f' \succeq_P f_{-}^{P}$.

Proof. **("Only if" part):** Suppose M is a stable matching in $\mathcal{X}^{n+1}$. Define $X = (X_f)_{f \in \tilde{F}}$ as

$$X_f(E) = D^{\preceq f}(M)(E) = \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma.$$

We prove that X is a fixed point of T. Let us first show that for each $f \in \tilde{F}$, $X_f \in \mathcal{X}$. It is clear that as each $M_{f'}(\Theta_P \cap \cdot)$ is nonnegative and countably additive, so is $X_f(\cdot)$. It is also clear that since $(M_f)_{f \in \tilde{F}}$ is a matching, $X_f \sqsubset G$. Thus, we have $X_f \in \mathcal{X}$.

We next claim that $M_f \in C_f(X_f)$ for all $f \in \tilde{F}$. This is immediate for $f = \emptyset$ since $M_\emptyset \sqsubset X_\emptyset = C_\emptyset(X_\emptyset)$. To prove the claim for $f \neq \emptyset$, suppose for a contradiction that $M_f \notin C_f(X_f)$, which means that there is some $M'_f \in C_f(X_f)$ such that $M'_f \neq M_f$. Note that $M_f \sqsubset X_f$ and thus $(M'_f \vee M_f) \sqsubset X_f$. Then, by the revealed preference, we have $M_f \notin C_f(M'_f \vee M_f)$ and $M'_f \in C_f(M'_f \vee M_f)$ or $M'_f \succ_f M_f$, which means that M is unstable since $M'_f \sqsubset X_f = D^{\preceq f}(M)$, yielding the desired contradiction.

We next prove $X \in T(X)$. The fact that $M_f \in C_f(X_f), \forall f \in \tilde{F}$ means that $X_f - M_f \in R_f(X_f), \forall f \in \tilde{F}$. We set $Y_f = X_f - M_f$ for each $f \in \tilde{F}$ and obtain for any $E \in \Sigma$

$$\begin{aligned}
\sum_{P \in \mathcal{P}} Y_{f_-^P}(\Theta_P \cap E) &= \sum_{P \in \mathcal{P}} \Big(X_{f_-^P}(\Theta_P \cap E) - M_{f_-^P}(\Theta_P \cap E) \Big) \\
&= \sum_{P \in \mathcal{P}} \left(\sum_{f' \in \tilde{F}: f' \preceq_P f_-^P} M_{f'}(\Theta_P \cap E) - M_{f_-^P}(\Theta_P \cap E) \right) \\
&= \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E) = X_f(E),
\end{aligned}$$

where the second and fourth equality follows from the definition of $X_{f_-^P}$ and X_f, respectively, while the third from the fact that f_-^P is an immediate predecessor of f. The above equation holds for every firm $f \in \tilde{F}$, we conclude that $X \in T(X)$, i.e. X is a fixed point of T.

("If" part): Let us first introduce some notations. Let f_+^P denote an **immediate successor** of $f \in \tilde{F}$ at $P \in \mathcal{P}$: that is, $f_+^P \prec_P f$, and for any $f' \prec_P f$, $f' \preceq_P f_+^P$. Also, let $X_{f_+^P}(\Theta_P \cap \cdot) \equiv 0$ for the firm f that is ranked last at P. Note that for any $f, \tilde{f} \in \tilde{F}$, $f = \tilde{f}_-^P$ if and only if $\tilde{f} = f_+^P$.

Suppose now that $X \in \mathcal{X}^{n+1}$ is a fixed point of T. For each firm $f \in \tilde{F}$ and $E \in \Sigma$, define

$$M_f(E) = X_f(E) - \sum_{P \in \mathcal{P}} X_{f_+^P}(\Theta_P \cap E). \tag{12}$$

We first verify that for each $f \in \tilde{F}$, $M_f \in \mathcal{X}$. First, it is clear that for each $f \in \tilde{F}$, as both $X_f(\cdot)$ and $X_{f_+^P}(\Theta_P \cap \cdot)$ are countably additive, so is M_f. It is also clear that for each $f \in \tilde{F}$, $M_f \sqsubset X_f$.

Let us next show that for all $f \in \tilde{F}$, $P \in \mathcal{P}$, and $E \in \Sigma$,

$$X_f(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E), \tag{13}$$

which means that $X_f = D^{\preceq f}(M)$. To do so, consider first a firm f that is ranked last at P. By (12) and the fact that $X_{f_+^P}(\Theta_P \cap \cdot) \equiv 0$, we have $M_f(\Theta_P \cap E) = X_f(\Theta_P \cap E)$. Hence, (13) holds for such f. Consider now any $f \in \tilde{F}$ which is not ranked last, and assume for an inductive argument that (13) holds true for f_+^P, so $X_{f_+^P}(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \preceq_P f_+^P} M_{f'}(\Theta_P \cap E)$. Then, by (12), we have

$$\begin{aligned} X_f(\Theta_P \cap E) = M_f(\Theta_P \cap E) + X_{f_+^P}(\Theta_P \cap E) &= M_f(\Theta_P \cap E) + \sum_{f' \in \tilde{F}: f' \preceq_P f_+^P} M_{f'}(\Theta_P \cap E) \\ &= \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E), \end{aligned}$$

as desired.

To show that $M = (M_f)_{f \in \tilde{F}}$ is a matching, let f be a top-ranked firm at P and note that by definition of T, if $\tilde{X} \in T(X)$, then $\tilde{X}_f(\Theta_P \cap \cdot) = G(\Theta_P \cap \cdot)$. Since $X \in T(X)$, we have for any $E \in \Sigma$

$$G(\Theta_P \cap E) = X_f(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E) = \sum_{f' \in \tilde{F}} M_{f'}(\Theta_P \cap E),$$

where the second equality follows from (13). Since the above equation holds for every $P \in \mathcal{P}$, M is a matching.

Let us now fix any $f \in \tilde{F}$ and show that $M_f \in C_f(X_f)$, which is equivalent to showing $X_f - M_f \in R_f(X_f)$. Recall that $X_{f_+^P}(\Theta_P \cap E) = Y_f(\Theta_P \cap E)$ for all $P \in \mathcal{P}$ and $E \in \Sigma$. Then, (12) implies

$$X_f(\cdot) - M_f(\cdot) = \sum_{P \in \mathcal{P}} X_{f_+^P}(\Theta_P \cap \cdot) = \sum_{P \in \mathcal{P}} Y_f(\Theta_P \cap \cdot) = Y_f(\cdot) \in R_f(X_f),$$

as desired, where the last inclusion relationship follows from the definition of T.

We now prove that $(M_f)_{f \in \tilde{F}}$ is stable. To prove the first part of Condition 1 of Definition 4, note first that $M_\emptyset \in C_\emptyset(X_\emptyset) = \{X_\emptyset\}$. Then, for every $P \in \mathcal{P}$ and $E \in \Sigma$,

$$\sum_{f: f \prec_P \emptyset} M_f(\Theta_P) = \sum_{f: f \preceq_P \emptyset} M_f(\Theta_P) - M_\emptyset(\Theta_P) = X_\emptyset(\Theta_P) - M_\emptyset(\Theta_P) = 0,$$

where the middle equality follows from (13). The above equation means that for each $f \prec_P \emptyset$, we have $M_f(\Theta_P) = 0$, as desired. The second part of Condition 1 of Definition 4 (i.e. $M_f \in C_f(M_f)$) follows from the revealed preference property since $M_f \sqsubset X_f$ by (13) and $M_f \in C_f(X_f)$.

It only remains to check Condition 2 of Definition 4. Suppose for a contradiction that it fails. Then, there exist f and M'_f such that

$$M'_f \sqsubset D^{\preceq f}(M),\ M'_f \in C_f(M'_f \vee M_f),\ \text{and } M_f \notin C_f(M'_f \vee M_f). \tag{14}$$

So $M'_f \sqsubset D^{\preceq f}(M) = X_f$. Since then $M_f \sqsubset (M'_f \vee M_f) \sqsubset X_f$ and $M_f \in C_f(X_f)$, the revealed preference property implies $M_f \in C_f(M'_f \vee M_f)$, contradicting (14). We have thus proven that M is stable. ▌

We now prove Theorem 3, from which Theorem 2 follows as a corollary since, if T is a single-valued mapping, then the convex- and closed-valuedness hold trivially while the upper hemi-continuity is equivalent to continuity. To prove existence, by Theorem 12, it suffices to show that the mapping T has a fixed point. To this end, we establish a series of Lemmas.

We now establish the compactness of $\mathcal{X}$ and the upper-hemi continuity of T. Recall that $\mathcal{X}$ is endowed with weak* topology. The notion of convergence in this topology, i.e. weak convergence, can be stated as follows: A sequence of measures $(X_k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ weakly converges to a measure $X \in \mathcal{X}$, written as $X_k \xrightarrow{w^*} X$, if and only if $\int_\Theta h dX_k \to \int_\Theta h dX$ for all $h \in C(\Theta)$, where $C(\Theta)$ is the space of all continuous functions defined on Θ. The next result provides some conditions that are equivalent to weak convergence.

Theorem 13. *Let X and $(X_k)_{k\in\mathbb{N}}$ be finite measures on Σ. The following conditions are equivalent:*[35]

(a) $X_k \xrightarrow{w^*} X$;

(b) $\int_\Theta h dX_k \to \int_\Theta h dX$ *for all* $h \in C_u(\Theta)$, *where* $C_u(\Theta)$ *is the space of all uniformly continuous functions defined on* Θ.

(c) $\liminf_k X_k(A) \geq X(A)$ *for every open set* $A \subset \Theta$, *and* $X_k(\Theta) \to X(\Theta)$;

(d) $\limsup_k X_k(A) \leq X(A)$ *for every closed set* $A \subset \Theta$, *and* $X_k(\Theta) \to X(\Theta)$;

[35]This theorem is a modified version of "Portmanteau Theorem" that is modified to deal with any finite (i.e. not necessarily probability) measures. See Theorem 4.5.1 of Ash (1977) for this result, for instance.

(e) *$X_k(A) \to X(A)$ for every set $A \in \Sigma$ such that $X(\partial A) = 0$ (∂A denotes the boundary of A).*

Lemma 2. *The space $\mathcal{X}$ is convex and compact. Also, for any $X \in \mathcal{X}$, $\mathcal{X}_X$ is compact.*

Proof. Convexity of $\mathcal{X}$ follows trivially.

To prove the compactness of $\mathcal{X}$, let $C(\Theta)^*$ denote the dual (Banach) space of $C(\Theta)$ and note that $C(\Theta)^*$ is the space of all (signed) measures on (Θ, Σ), given Θ is a compact metric space.[36] Then, by Alaoglu's Theorem, the closed unit ball of $C(\Theta)^*$, denoted U^*, is weak* compact.[37] Clearly, $\mathcal{X}$ is a subspace of U^* since for any $X \in \mathcal{X}$, $\|X\| = X(\Theta) \leq G(\Theta) = 1$. The compactness of $\mathcal{X}$ will thus follow if $\mathcal{X}$ is shown to be a closed set. To prove this, we prove that for any sequence $(X_k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ and $X \in C(\Theta)^*$ such that $X_k \xrightarrow{w^*} X$, we must have $X \in \mathcal{X}$, which will be shown if we prove that $0 \leq X(E) \leq G(E)$ for any $E \in \Sigma$. Let us first make the following observation: every finite (Borel) measure X on the metric space Θ is normal,[38] which means that for any set $E \in \Sigma$,

$$X(E) = \inf\{X(O) : E \subset O \text{ and } O \in \Sigma \text{ is open}\} \tag{15}$$

$$= \sup\{X(F) : F \subset E \text{ and } F \in \Sigma \text{ is closed}\}. \tag{16}$$

To show first that $X(E) \leq G(E)$, consider any open set $O \in \Sigma$ such that $E \subset O$. Then, since $X_k \in \mathcal{X}$ for every k, we must have $X_k(O) \leq G(O)$ for every k, which, combined with (c) of Theorem 13 above, implies that $X(O) \leq \liminf_k X_k(O) \leq G(O)$. Given (15), this means that $X(E) \leq G(E)$.

To show next that $X(E) \geq 0$, consider any closed set $F \in \Sigma$ such that $F \in \Sigma$. Since $X_k \in \mathcal{X}$ for every k, we must have $X_k(F) \geq 0$, which, combined with (d) of Theorem 13 above, implies that $X(F) \geq \limsup_k X_k(F) \geq 0$. Given (16), this means $X(E) \geq 0$.

The proof for the compactness of $\mathcal{X}_X$ is analogous and hence omitted. ▮

[36] More precisely, $C(\Theta)^*$ is isometrically isomorphic to the space of all signed measures on (Θ, Σ) according to the Riesz Representation Theorem (see **?** for instance).

[37] The closed unit ball is defined as $U^* := \{X \in C^*(\Theta) : \|X\| \leq 1\}$, where $\|X\|$ is the dual norm, i.e.,

$$\|X\| = \sup\{\left|\int_\Theta h dX\right| : h \in C(\Theta) \text{ and } \max_{\theta\in\Theta} |h(\theta)| \leq 1\}.$$

If X is a nonnegative measure, then the supremum is achieved by taking $h \equiv 1$, and thus $\|X\| = X(\Theta)$. It is well known (see Riesz's Theorem in page 261 of **?** for instance) that if $C(\Theta)^*$ is infinite dimensional, then U^* is not compact under the norm topology (i.e., the topology induced by the dual norm). On the other hand, U^* is compact under the weak* topology, which follows from Alaoglu's Theorem (see **?** for instance).

[38] See Theorem 12.5 of Aliprantis and Border (2006).

Lemma 3. *The map T is a correspondence from $\mathcal{X}^{n+1}$ to itself. Also, it is convex-valued, upper hemi-continuous, and closed-valued.*

Proof. To show that T maps from $\mathcal{X}^{n+1}$ to itself, observe that for any $X \in \mathcal{X}^{n+1}$ and $\tilde{X} \in T_f(X)$, there is $Y_f \in R_f(X_f)$ for each $f \in \tilde{F}$ such that for all $E \in \Sigma$,

$$\tilde{X}(E) = \sum_{P \in \mathcal{P}} Y_{f_-^P}(\Theta_P \cap E) \leq \sum_{P \in \mathcal{P}} X_{f_-^P}(\Theta_P \cap E) \leq \sum_{P \in \mathcal{P}} G(\Theta_P \cap E) = G(E),$$

which means that $\tilde{X} \in \mathcal{X}$, as desired.

To prove that T is convex-valued, it suffices to show that for each $f \in \tilde{F}$, R_f is convex-valued. Consider any $X \in \mathcal{X}$ and $Y', Y'' \in R_f(X)$. There are some $X', X'' \in C_f(X)$ satisfying $Y' = X - X'$ and $Y'' = X - X''$. Then, the convexity of $C_f(X)$ implies that for any $\lambda \in [0, 1]$, $\lambda X' + (1-\lambda)X'' \in C_f(X)$ so $\lambda Y' + (1-\lambda)Y'' = X - (\lambda X' + (1-\lambda)X'') \in R_f(X)$.

To establish the upper hemi-continuity of T, we first establish the following claim:

Claim 1. *For any sequence $(X_k)_{k\in\mathbb{N}} \subset \mathcal{X}$ that weakly converges to $X \in \mathcal{X}$, a sequence $(X_k(\Theta_P \cap \cdot))_{k\in\mathbb{N}}$ also weakly converges to $X(\Theta_P \cap \cdot)$ for all $P \in \mathcal{P}$.*

Proof. Let X^P and X_k^P denote $X(\Theta_P \cap \cdot)$ and $X_k(\Theta_P \cap \cdot)$, respectively. Note first that for any $X \in \mathcal{X}$, we have $X^P \in \mathcal{X}$ for all $P \in \mathcal{P}$. Due to Theorem 13, it suffices to show that X^P and $(X_k^P)_{k\in\mathbb{N}}$ satisfy the condition (c) of Theorem 13. To do so, consider any open set $O \subset \Theta$. Then, letting Θ_P° denote the interior of Θ_P,

$$\begin{aligned}\liminf_k X_k^P(O) &= \liminf_k X_k(\Theta_P^\circ \cap O) + X_k(\partial\Theta_P \cap O) \\ &= \liminf_k X_k(\Theta_P^\circ \cap O) \geq X(\Theta_P^\circ \cap O) = X^P(O),\end{aligned}$$

where the second equality follows from the fact that $X_k(\partial\Theta_P \cap O) \leq X_k(\partial\Theta_P) \leq G(\partial\Theta_P) = 0$, the lone inequality from $X_k \xrightarrow{w^*} X$, (c) of Theorem 13, and the fact that $\partial\Theta_P^\circ \cap O$ is an open set, and the last equality from repeating the first two equalities with X instead X_k. Also, we have

$$X_k^P(\Theta) = X_k(\Theta_P) \to X(\Theta_P) = X^P(\Theta),$$

where the convergence is due to $X_k \xrightarrow{w^*} X$, (e) of Theorem 13, and the fact that $X(\partial\Theta_P) \leq G(\partial\Theta_P) = 0$. Thus, the two requirements in condition (c) of Theorem 13 are satisfied, so $X_k^P \xrightarrow{w^*} X^P$, as desired. ▌

It is also straightforward to observe that if C_f is upper hemi-continuous, then R_f is also upper hemi-continuous.

We now prove the upper hemi-continuity of T by considering any sequences $(X_k)_{k\in\mathbb{N}}$ and $(\tilde{X}_k)_{k\in\mathbb{N}}$ in $\mathcal{X}^{n+1}$ weakly converging to some X and $\tilde{X}$ in $\mathcal{X}^{n+1}$, respectively, such that $\tilde{X}_k \in T_f(X_k)$ for each k. To show that $\tilde{X} \in T(X)$, let $X_{k,f}$ and $\tilde{X}_{k,f}$ denote the components of X_k and $\tilde{X}_k$, respectively, that correspond to $f \in \tilde{F}$. Then, we can find $Y_{k,f} \in R_f(X_{k,f})$ for each k and f such that $\tilde{X}_{k,f}(\cdot) = \sum_{P\in\mathcal{P}} Y_{k,f_-^P}(\Theta_P \cap \cdot)$, which implies that for all $f \in \tilde{F}$ and $P \in \mathcal{P}$, $\tilde{X}_{k,f_+^P}(\Theta_P \cap \cdot) = Y_{k,f}(\Theta_P \cap \cdot)$ since f is the immediate predecessor of f_+^P at P. As $\tilde{X}_{k,f} \xrightarrow{w^*} \tilde{X}_f, \forall f \in \tilde{F}$, by assumption, we have $\tilde{X}_{k,f_+^P}(\Theta_P \cap \cdot) \xrightarrow{w^*} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$ by Claim 1, which means that $Y_{k,f}(\Theta_P \cap \cdot) \xrightarrow{w^*} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$ for all $f \in \tilde{F}$. This in turn implies that $Y_{k,f}(\cdot) = \sum_{P\in\mathcal{P}} Y_{k,f}(\Theta_P \cap \cdot) \xrightarrow{w^*} \sum_{P\in\mathcal{P}} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$. Now let $Y_f(\cdot) = \sum_{P\in\mathcal{P}} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$. We then have $\tilde{X}_f(\Theta_P \cap \cdot) = Y_{f_-^P}(\Theta_P \cap \cdot)$ and thus $\tilde{X}_f(\cdot) = \sum_{P\in\mathcal{P}} Y_{f_-^P}(\Theta_P \cap \cdot)$. Also, since $X_{k,f} \xrightarrow{w^*} X_f$ and $Y_{k,f} \xrightarrow{w^*} Y_f$, we must have $Y_f \in R_f(X_f)$ by the upper hemi-continuity of R_f, which means $\tilde{X} \in T(X)$, as desired.

The proof for the closed-valuedness of T is analogous to that for the upper hemi-continuity of T and hence omitted. ▌

Proof of Theorem 3. Thanks to Lemmas 2 and 3, we can invoke Kakutani-Fan-Glicksberg's fixed point theorem to conclude that the mapping T, which is defined on a convex set $\mathcal{X}^{n+1}$, has a nonempty set of fixed points.[39] ▌

D Substitutable Preferences Case

D.1 Proof of Theorem 4

The part (i) immediately follows from Tarski's fixed point theorem and the fact that each R_f is monotonic in $\sqsubset_{\tilde{F}}$ due to the substitutability of f's preference and thus T is monotonic as well.

We next prove part (ii). To see that the stable matching $\overline{M}$ is firm-optimal, observe first that for any stable matching M, there is some $X \in \mathcal{X}^*$ such that $M_f = C_f(X_f)$ for all $f \in \tilde{F}$. Thus, we have $M_f \sqsubset X_f \sqsubset \overline{X}_f$, which implies that $\overline{M}_f = C_f(M_f \vee \overline{M}_f)$ by the revealed preference since $\overline{M}_f = C_f(\overline{X}_f)$ and $(M_f \vee \overline{M}_f) \sqsubset \overline{X}_f$. Thus, $\overline{M}_f \succeq_f M_f$ for each $f \in F$, as desired. To show that $\overline{M}$ is worker-pessimal, fix any stable matching M. Then, by Theorem 1, there is some $X \in \mathcal{X}^*$ (i.e., a fixed point of T) such that $M_f = C_f(X_f)$ and

[39]For Kakutani-Fan-Glicksberg's fixed point theorem, refer to Theorem 16.12 and Corollary 16.51 in Aliprantis and Border (2006).

$X_f = D^{\preceq f}(M)$ for all $f \in \tilde{F}$. Thus, for each $P \in \mathcal{P}$ and $E \in \Sigma$,

$$T_{f_+^P}(X)(\Theta_P \cap E) = X_{f_+^P}(\Theta_P \cap E) = D^{\preceq f_+^P}(M)(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \prec_P f} M_{f'}(\Theta_P \cap E).$$

Similarly, for $\overline{X}$, we have $T_{f_+^P}(\overline{X})(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \prec_P f} \overline{M}_{f'}(\Theta_P \cap E)$. Since T_f is monotonic and $X \sqsubset \overline{X}$, we obtain

$$\begin{aligned} \sum_{f' \in \tilde{F}: f' \succeq_P f} \overline{M}_{f'}(\Theta_P \cap E) =& G(\Theta_P \cap E) - T_{f_+^P}(\overline{X})(\Theta_P \cap E) \\ \leq& G(\Theta_P \cap E) - T_{f_+^P}(X)(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \succeq_P f} M_{f'}(\Theta_P \cap E) \quad (17) \end{aligned}$$

for all $P \in \mathcal{P}$, $E \in \Sigma$, and $f \in \tilde{F}$, as desired.

D.2 Proof of Theorem 5

Let M be a stable matching. Then, there exists $X \in \mathcal{X}^*$ such that $M_f = C_f(X_f)$ for each $f \in F$. Since $X \sqsubset \overline{X}$, by the law of aggregate demand, we have

$$\overline{M}_f(\Theta) = C_f(\overline{X}_f)(\Theta) \geq C_f(X_f)(\Theta) = M_f(\Theta), \forall f \in F. \quad (18)$$

Next since $\overline{M}$ is worker pessimal, (17) holds for any $f \in \tilde{F}$. Let $w_P := \o_-^P$ be the immediate predecessor of $\o$ (i.e., the worst individually rational firm) for types in Θ_P. Then, setting $f = w_P$ in (17), we obtain

$$\begin{aligned} \sum_{f' \in F} \overline{M}_{f'}(\Theta_P \cap E) =& \sum_{f' \in \tilde{F}: f' \succeq_P w_P} \overline{M}_{f'}(\Theta_P \cap E) \\ \leq& \sum_{f' \in \tilde{F}: f' \succeq_P w_P} M_{f'}(\Theta_P \cap E) = \sum_{f' \in F} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma, \end{aligned}$$

or equivalently

$$\sum_{f' \in F} \overline{M}_{f'}(E) \leq \sum_{f' \in F} M_{f'}(E), \forall E \in \Sigma. \quad (19)$$

Since this inequality must hold with $E = \Theta$, combining it with (18) implies that $M_f(\Theta) = \overline{M}_f(\Theta)$ for all $f \in F$, as desired.

Further, we must have $\sum_{f \in F} \overline{M}_f = \sum_{f \in F} M_f$, which means that $\overline{M}_\o = M_\o$. To prove this, suppose otherwise. Then, by (19), we must have $\sum_{f' \in F} \overline{M}_{f'}(E) < \sum_{f' \in F} M_{f'}(E)$ for some $E \in \Sigma$. Also, by (19), $\sum_{f' \in F} \overline{M}_{f'}(E^c) \leq \sum_{f' \in F} M_{f'}(E^c)$. Combining these two inequalities, we obtain $\sum_{f' \in F} \overline{M}_{f'}(\Theta) < \sum_{f' \in F} M_{f'}(\Theta)$, which contradicts with (18).

D.3 Proof of Theorem 6

Suppose otherwise. Then there exists a stable matching M that differs from the worker-optimal stable matching $\underline{M}$. Let X and $\underline{X}$ be respectively fixed points of T such that $M_f = C_f(X_f)$, $\underline{M}_f = C_f(\underline{X}_f)$ and $\underline{X}_f \sqsubset X_f$, for each $f \in F$.

First of all, by Theorem 5, $\sum_{f\in F} M_f = \sum_{f\in F} \underline{M}_f$. Next, since $\underline{X}_f \sqsubset X_f$ for each $f \in F$, we have $(\underline{M}_f \vee M_f) \sqsubset X_f$. Revealed preference then implies that, for each $f \in F$,

$$M_f = C_f(\underline{M}_f \vee M_f)$$

or $M \succeq_F \underline{M}$. Moreover, since $M \neq \underline{M}$, the set $\bar{F} := \{f \in F | M_f \succ_f \underline{M}_f\}$ is nonempty. But then by the rich preferences, there exists $f^* \in F$ such that

$$\underline{M}_{f^*} \neq C_{f^*}(\underline{M}_{f^*} + M^{f^*}_{\bar{F}}).$$

For each $f \in F\backslash\bar{F}$, $M_f = \underline{M}_f$, by definition of $\bar{F}$, and Theorem 5 guarantees that $M_\varnothing = \underline{M}_\varnothing$. Consequently, we have for each $E \in \Sigma$, that

$$M^{f^*}_{\bar{F}}(E) = \sum_{P\in\mathcal{P}} \sum_{f': f^* \succ_P f', f' \notin \bar{F}} M_{f'}(\Theta_P \cap E) = \sum_{P\in\mathcal{P}} \sum_{f': f^* \succ_P f', f' \notin \bar{F}} \underline{M}_{f'}(\Theta_P \cap E) = \underline{M}^{f^*}_{\bar{F}}(E).$$

It then follows that

$$\underline{M}_{f^*} \neq C_{f^*}(\underline{M}_{f^*} + \underline{M}^{f^*}_{\bar{F}}). \tag{20}$$

Letting $\hat{M}_{f^*} := C_{f^*}(\underline{M}_{f^*} + \underline{M}^{f^*}_{\bar{F}})$, we have $\hat{M}_{f^*} \sqsubset (\underline{M}_{f^*} \vee \hat{M}_{f^*}) \sqsubset (\underline{M}_{f^*} + \underline{M}^{f^*}_{\bar{F}})$. The revealed preference condition then implies that

$$\hat{M}_{f^*} = C_{f^*}(\underline{M}_{f^*} \vee \hat{M}_{f^*}).$$

By (20), we then have $\hat{M}_{f^*} \succ \underline{M}_{f^*}$. Further, $\hat{M}_{f^*} \sqsubset (\underline{M}_{f^*} + \underline{M}^{f^*}_{\bar{F}}) \sqsubset D^{\preceq f^*}(\underline{M})$. We therefore have a contradiction to the stability of $\underline{M}$.

D.4 Choice Functions Representing Responsive Preferences with Affirmative Action

For any firm f and subpopulation $X \sqsubset G$ available to f, we can define firm f's optimal choice from X as a solution to the following problem:

$$[C] \qquad \max_{Y \sqsubset X} \int s_f(\theta) dY$$

subject to

$$Y(\Theta) \leq q_f, \text{ and } Y(\Theta^t) \leq q^t_f, \forall t \in T.$$

As in Theorem 2, one can show that the feasible set is compact. Since its objective function is continuous in X (by the definition of weak convergence, given continuity of $s(\cdot)$), the maximum is well defined. Further, the set of optimal choices is closed, so it is compact.

We next show that an optimal choice can be found in a class of feasible subpopulations with a cutoff structure. For each $t \in T$, let $\tilde{p}_f^t := \inf\{s \in [0,1] | X(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) \geq s\}) \leq q_f^t\}$. Then, we say Y is the **optimal cutoff rule for firm f at X** if $Y(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) \geq p_f^t\}) = X(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) \geq p_f^t\})$, and $Y(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) < p_f^t\}) = 0$, where $p_f^t := \max\{p_f, \tilde{p}_f^t\}$, and "the common cutoff" p_f is the supremum of the set of common cutoffs that maximize $Y(\Theta)$ subject to $Y(\Theta) \leq q_f$ and $Y(\Theta^t) \leq q_f^t, \forall t \in T$. Note that the optimal cutoff rule is uniquely determined.

Claim 2. *The optimal cutoff rule for firm f at X is its optimal choice from X.*

Proof. For any feasible solution Y to $[C]$, consider a cutoff rule $\hat{Y}$, given by $\hat{Y}(\Theta^t \cap \{\theta | s_f(\theta) \geq p_f^t\} \cap E) = X(\Theta^t \cap \{\theta | s_f(\theta) \geq p_f^t\} \cap E)$ for each $E \in \Sigma$, and $\hat{Y}(\Theta^t \cap \{\theta | s_f(\theta) < p_f^t\} \cap E) = 0$, for each $E \in \Sigma$, for some cutoff score p_f^t, for each $t \in T$. In words, a cutoff rule selects all workers above a certain cutoff score and rejects all workers below that score. As the cutoff score p_f^t rises, $\hat{Y}(\Theta^t)$ falls continuously (since $\hat{Y}$, being a subpopulation of G, is absolutely continuous), and it equals $X(\Theta^t)$ when $p_f^t = 0$ and zero when $p_f^t = 1$. Hence, there exists $p_f^t \in [0,1]$ such that $\hat{Y}(\Theta^t) = Y(\Theta^t)$.

Since both Y and $\hat{Y}$, being subpopulation of G which has density, have density functions say y and $\hat{y}$, respectively. In that case, $\hat{y}(\theta) = x(\theta) \geq y(\theta)$ if $s_f(\theta) \geq p_f^t$ and $\hat{y}(\theta) = 0 \leq y(\theta)$ if $s_f(\theta) < p_f^t$. Hence,

$$\int_{\Theta^t} s_f(\theta)\hat{y}(\theta)d\theta - \int_{\Theta^t} s_f(\theta)y(\theta)d\theta = \int_{\Theta^t} s_f(\theta)(\hat{y}(\theta) - y(\theta))d\theta$$
$$\geq \int_{\Theta^t} p_f^t(\hat{y}(\theta) - y(\theta))d\theta = p_f^t[\hat{Y}(\Theta^t) - Y(\Theta^t)] = 0.$$

In short, $\hat{Y}$ is feasible and yields a weakly higher value of objective than does Y. It follows that an optimal choice can be found in the class of cutoff rules. Moreover, if Y differs from $\hat{Y}$ for a positive measure, the inequality is strict. This implies that an optimal choice must coincide with a cutoff rule almost everywhere (i.e., for every positive measure set).

Fix any optimal choice Y that is a cutoff rule. If $Y(\Theta^t) < q_f^t$ for some t, then there exists an optimal cutoff rule in which $p_f^t \leq p_f^{t'}$ for all $t' \neq t$. To see this, suppose an optimal choice has $p_f^t > p_f^{t'}$, where t' is the ethnic type with the lowest cutoff at the optimal choice. We can assume without loss of generality that $p_f^{t'} = \inf\{s_f(\theta) | \theta \in \Theta^{t'}, y(\theta) > 0\}$, or else we can raise $p_f^{t'}$ slightly without any consequence. If $X(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) \in [p_f^{t'}, p_f^t)\}) = 0$, then we can lower p_f^t without consequence to $p_f^{t'}$, so the claim holds. If $X(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) \in [p_f^{t'}, p_f^t)\}) > 0$, then we can slightly lower p_f^t and slightly raise $p_f^{t'}$ so as to keep all constraints satisfied, which increases the value of the objective, producing a contradiction. This observation implies that there exists a common cutoff p_f that applies to all t whose quota is not binding, and the cutoffs for ethnic types with binding quotas are weakly higher than p_f. Hence, we can write $p_f^t := \max\{p_f, \tilde{p}_f^t\}$, where $\tilde{p}_f^t$ is defined above.

The common cutoff p_f should be chosen to maximize $Y(\Theta)$ subject to $Y(\Theta) \leq q_f$ and $Y(\Theta^t) \leq q_f^t, \forall t \in T$, or else it can be lowered to increase the employment (and thus increase the value of the objective). Let P_f^t be the set of maximizers,[40] and let $\bar{p}_f^t := \sup P_f^t$. Then, $\bar{p}_f^t \in P_f^t$ due to the compactness of the optimal choices: Any sequence of the optimal cutoff rules with common cutoff $p_f^t \in P_f^t$ converging to the cutoff rule with common cutoff $\bar{p}_f^t$ must be optimal as well, so its limit must be optimal given the compactness of the optimal choices. ∎

Based on Claim 2, we define a choice function C_f to be an optimal cutoff rule. The resulting choice function then satisfies the revealed preference axiom: If $X \sqsubset X'$ and $C_f(X') \sqsubset X$, then $C_f(X')$ is also an optimal cutoff rule at X.

Next, it is routine to see that C_f satisfies the law of aggregate demand. If $X \sqsubset \hat{X}$, then the optimal cutoff rule at the latter leads to the firm choosing a weakly higher mass of workers than the optimal cutoff rule at X.

It is also easy to see C_f exhibits substitutability. Again fix $X \sqsubset \hat{X}$. We show that $R_f(X) \sqsubset R_f(\hat{X})$, where R_f is defined before. For non-triviality, assume $R_f(X)(\Theta) > 0$. Let $(\hat{p}_f^t)_t$ be the cutoffs associated with $C_f(\hat{X})$ and let $(p_f^t)_t$ be the cutoffs associated with $C_f(X)$. Note first if the quota for t is binding at the optimal choice from X, we can only have $\hat{p}_f^t \geq p_f^t$, or else the quota for t will be violated at $\hat{X}$. There are two cases. First, suppose first $C_f(X)(\Theta) < q_f$. In this case, no mass of agents from X is rejected at $C_f(X)$ except for violating quotas for ethnic types, and those who are rejected for violating the ethnic type quotas must be rejected as well at $C_f(\hat{X})$ since their cutoffs are weakly higher. Hence, $R_f(X) \sqsubset R_f(\hat{X})$. Suppose next $C_f(X)(\Theta) = q_f$. In this case, the common cutoff $\hat{p}_f$ at $C_f(\hat{X})$ must be weakly higher than the common cutoff p_f at $C_f(X)$. If not, then feasibility of $C_f(\hat{X})$ implies that there exists t such that its quota is binding at $C_f(X)$

[40]The set P_f^t may not be a singleton. Suppose for instance that the measure of available workers is strictly smaller than the capacity of a firm, and say the firm has no affirmative action constraint and the infimum score of the available workers is say $s_m > 0$. Then any $p_f^t \in [0, s_m]$ will be an optimal cutoff, since selecting all available workers is optimal for the firm.

but not at $C_f(\hat{X})$. But then $\hat{p}_f^t > p_f^t \geq p_f > \hat{p}_f$, which implies that $C_f(\hat{X})$ violates the property of the optimal cutoff rule at $\hat{X}$. Since all cutoffs are uniformly higher at $C_f(\hat{X})$, we conclude that $R_f(X) \sqsubset R_f(\hat{X})$.

D.5 Proof of Proposition 2

To simplify notation, let $M = \underline{M}$, i.e., worker-optimal matching. Fix any individually rational matching $\hat{M}$ such that $\hat{M} \succeq_F M$ and assume that $\bar{F} := \{f' \in F | \hat{M}_{f'} \succ_f M_{f'}\}$ is nonempty. For any f, t, let $M_f^t := M_f(\Theta^t \cap \cdot)$ and $\hat{M}_f^t := \hat{M}_f(\Theta^t \cap \cdot)$. Since G is absolutely continuous, for any f, t, both M_f^t and $\hat{M}_f^t$, being its subpopulations, admit densities, denoted respectively by m_f^t and $\hat{m}_f^t$. Let p_f^t and $\hat{p}_f^t$ respectively denote the optimal cutoffs associated with $M_f = C_f(M_f)$ and $\hat{M}_f = C_f(M_f \vee \hat{M}_f)$.

Because C_f satisfies the law of aggregate demand (as established above in Appendix D.4), $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ and $M_f = C_f(M_f)$ imply $M_f(\Theta) \leq \hat{M}_f(\Theta)$ for each $f \in F$. Then, Proposition 2 follows from proving a sequence of claims.

Claim 3. $M_\varnothing = \hat{M}_\varnothing$.

Proof. Suppose to the contrary that $M_\varnothing \neq \hat{M}_\varnothing$. Then, with their densities denoted by $m_\varnothing$ and $\hat{m}_\varnothing$, $E_\varnothing = \{\theta \in \Theta \,|\, m_\varnothing(\theta) > \hat{m}_\varnothing(\theta)\}$ must be a non-empty set of positive (Lebesgue) measure, due to the fact that $M_\varnothing(\Theta) = G(\Theta) - \sum_{f \in F} M_f(\Theta) \geq G(\Theta) - \sum_{f \in F} \hat{M}_f(\Theta) = \hat{M}_\varnothing(\Theta)$. Also, letting $\hat{E}_f = \{\theta \in \Theta \,|\, \hat{m}_f(\theta) > m_f(\theta)\}$, there must be at least one firm f for which $E_\varnothing \cap \hat{E}_f$ is a non-empty set of positive measure, since otherwise we would have $\sum_{f' \in \tilde{F}} m_{f'}(\theta) > \sum_{f' \in \tilde{F}} \hat{m}_{f'}(\theta)$ for all $\theta \in E_\varnothing$, a contradiction. Now fixing such a firm f and letting $\tilde{E} = E_\varnothing \cap \hat{E}_f$, define

$$\tilde{m}_f(\theta) = \begin{cases} \min\{m_f(\theta) + m_\varnothing(\theta), \hat{m}_f(\theta)\} & \text{if } \theta \in \tilde{E} \\ m_f(\theta) & \text{otherwise.} \end{cases}$$

and let $\tilde{M}_f$ denote the corresponding measure. Note that $\tilde{m}_f(\theta) > m_f(\theta)$ for all $\theta \in \tilde{E}$, and also that $(M_f \vee \tilde{M}_f) = \tilde{M}_f \neq M_f$ and $\tilde{M}_f \sqsubset (M_f \vee \hat{M}_f)$. Letting $M'_f = C_f(\tilde{M}_f)$, we show below that f and M'_f are a blocking coalition for M, contradicting the stability of M.

First of all, it follows from the revealed preference that $C_f(M_f \vee M'_f) = M'_f$. To show that $M'_f \neq M_f$, note first that $\hat{m}_f(\theta) > m_f(\theta), \forall \theta \in \tilde{E}$ means $(\hat{M}_f \vee M_f)(\tilde{E}) = \hat{M}_f(\tilde{E})$, so

$$R_f(M_f \vee \hat{M}_f)(\tilde{E}) = (M_f \vee \hat{M}_f)(\tilde{E}) - C_f(M_f \vee \hat{M}_f)(\tilde{E}) = \hat{M}_f(\tilde{E}) - \hat{M}_f(\tilde{E}) = 0.$$

Then, since f has a substitutable preference and $\tilde{M}_f \sqsubset (M_f \vee \hat{M}_f)$, we have $R_f(\tilde{M}_f)(\tilde{E}) = 0$, which means $M'_f(\tilde{E}) = C_f(\tilde{M}_f)(\tilde{E}) = \tilde{M}_f(\tilde{E}) \neq M_f(\tilde{E})$. It only remains to show that

$M'_f \sqsubset D^{\preceq f}(M)$. For this, note that since $\hat{M}$ is individually rational and $\hat{m}_f(\theta) > 0, \forall \theta \in \tilde{E}$, we have $f \succ_\theta \emptyset, \forall \theta \in \tilde{E}$. Given the definition of $\tilde{M}_f$, this implies that $\tilde{M}_f \sqsubset D^{\preceq f}(M)$ and thus $M'_f \sqsubset \tilde{M}_f \sqsubset D^{\preceq f}(M)$. ▌

Meanwhile, $\sum_{f \in F} M_f = \sum_{f \in F} \hat{M}_f$ since $M_\emptyset = \hat{M}_\emptyset$ as shown in the above claim. Hence, we conclude that $M_f(\Theta) = \hat{M}_f(\Theta)$ for each $f \in F$.

We then prove the next claim.

Claim 4. *For each $f \in \bar{F}$, there must be some t such that $p^t_f < \hat{p}^t_f$.*

Proof. Suppose to the contrary that $\hat{p}^t_f \leq p^t_f (< 1)$ for all $t \in T$. Since $\sum_{t \in T} M^t_f(\Theta) = M_f(\Theta) = \hat{M}_f(\Theta) = \sum_{t \in T} \hat{M}^t_f(\Theta)$ and $M_f \neq \hat{M}_f$, there must exist $t \in T$ such that the set $\{\theta \in \Theta^t | s_f(\theta) > p^t_f \geq \hat{p}^t_f \text{ and } m^t_f(\theta) > \hat{m}^t_f(\theta)\}$ has a positive measure. A contradiction then arises since, due to the fact that C_f selects all workers of type t whose scores are above the optimal cutoff $\hat{p}^t_f$ and that $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$, the measure of type $\theta \in \Theta^t$ workers selected when $\hat{M}_f \vee M_f$ is available is equal to $\hat{m}^t_f(\theta) = \max\{\hat{m}^t_f(\theta), m^t_f(\theta)\}$ for all $\theta \in \Theta^t$ with $s_f(\theta) \geq \hat{p}^t_f$, which cannot be smaller than $m^t_f(\theta)$. ▌

Claim 5. *For any $f \in \bar{F}$ and $t \in T$, if $\hat{p}^t_f = 0$, then $\hat{M}_f(\Theta^t \cap \cdot) = M_f(\Theta^t \cap \cdot)$.*

Proof. Let us first observe that for any $f \in \bar{F}$ and t, if $\hat{M}_f(\Theta^t) < M_f(\Theta^t)$, then we have $\hat{p}^t_f > p^t_f$ since, as we argued in the proof of Claim 4, the fact that $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ implies that $\hat{m}^t_f(\theta) = \max\{\hat{m}^t_f(\theta), m^t_f(\theta)\} \geq m^t_f(\theta)$ for all $\theta \in \Theta^t$ with $s_f(\theta) \geq \hat{p}^t_f$. We also know that if $\hat{M}_f(\Theta^t) < q^t_f$, then $\hat{p}^t_f = \hat{p}_f \leq \hat{p}^{t'}_f$, for all $t' \in T$. Hence, if $\hat{M}_f(\Theta^t) < M_f(\Theta^t)$ for $f \in \bar{F}$ and t, then $p^t_f < \hat{p}^t_f \leq \hat{p}^{t'}_f$, for all $t' \in T$.

Fix now any $f \in \bar{F}$ and $t \in T$ for which $\hat{p}^t_f = 0$. Since it means $\hat{p}^t_f \leq p^t_f$, we must have $\hat{M}_f(\Theta^t) \geq M_f(\Theta^t)$ according to the above argument. If, in addition, $\hat{M}_f(\Theta^t) > M_f(\Theta^t)$, then the fact that $\hat{M}_f(\Theta) = M_f(\Theta)$ implies that there must exist t' such that $\hat{M}_f(\Theta^{t'}) < M_f(\Theta^{t'})$. This means that $p^{t'}_f < \hat{p}^{t'}_f \leq \hat{p}^t_f = 0$, a contradiction. Hence, $\hat{M}_f(\Theta^t) = M_f(\Theta^t)$.

Given $\hat{p}^t_f = 0$ (i.e. the lowest possible score), we must have $\max\{\hat{m}^t_f(\theta), m^t_f(\theta)\} = \hat{m}^t_f(\theta)$ for all $\theta \in \Theta^t$. In order that $\hat{M}_f(\Theta^t) = M_f(\Theta^t)$, we must then have $\hat{m}^t_f(\theta) = m^t_f(\theta)$ for (almost) all $\theta \in \Theta^t$, which leads to the desired conclusion. ▌

Claim 6. *For any $t \in T$, if there is some $f \in \bar{F}$ such that $\hat{p}^t_f > p^t_f$, then we must have $\hat{p}^t_{f'} > 0, \forall f' \in \bar{F}$.*

Proof. Fix a firm $f \in \bar{F}$ with $\hat{p}^t_f > p^t_f$. Suppose to the contrary that the set $\bar{F}_0 = \{f' \in \bar{F} | \hat{p}^t_{f'} = 0\}$ is nonempty, and note that $f \notin \bar{F}_0$. Then, let us define $\bar{F}_+ = \bar{F} \backslash \bar{F}_0$ and consider the set

$$\{\theta \in \Theta | f \succ_\theta f'', \forall f'' \neq f, s_f(\theta) \in (p^t_f, \hat{p}^t_f), \text{ and } s_{f'}(\theta) < \hat{p}^t_{f'} \forall f' \in \bar{F}_+ \setminus \{f\}\}.$$

Since M is stable, all worker types in this set must be matched with f under M, which implies that they cannot be matched with any firm in $\tilde{F}\backslash\bar{F}$ under $\hat{M}$ since $\hat{M}_f = M_f$ for each $f \in F\backslash\bar{F}$ by assumption and also since $\hat{M}_\varnothing = M_\varnothing$ by Claim 3. Moreover, these workers cannot be matched with any firm $f' \in \bar{F}_+$ under $\hat{M}$ since their scores are below $\hat{p}^t_{f'}$. It thus follows that they must be matched with firms in $\bar{F}_0$ under $\hat{M}$ while being matched with $f \notin \bar{F}_0$ under M, which contradicts Claim 5. ▌

Claim 7. *Rich preferences hold.*

Proof. Fix any $f \in \bar{F}$ and $t \in T$ (given by Claim 4) such that $p^t_f < \hat{p}^t_f$, and let

$$\tilde{\Theta}^t_f := \{\theta \in \Theta | f \succ_\theta f'', \forall f'' \neq f, s_f(\theta) \in (p^t_f, \hat{p}^t_f), \text{ and } s_{f'}(\theta) < \hat{p}^t_{f'} \forall f' \in \bar{F} \setminus \{f\}\}$$

be a set of ethnic type t workers who prefer f to all other firms and have scores that will make them employable at f under M but not under $\hat{M}$ and not employable at all other firms in $\bar{F}$ under $\hat{M}$. Let $M' := \sum_{t \in T} G(\tilde{\Theta}^t_f \cap \cdot)$ denote the measure of these workers. The full support assumption and the fact (given by Claim 6) that $\hat{p}^t_{f'} > 0, \forall f' \in \bar{F}$ implies that $M'(\Theta) > 0$.

We show that these workers are not employed by any firm in $\bar{F}$ under either $\hat{M}$ or M. It is easy to see that these workers are not employed by any firm in $\bar{F}$ under $\hat{M}$ since their scores are below the cutoffs of these firms at $\hat{M}$. Since $\sum_{f \in F} M_f = \sum_{f \in F} \hat{M}_f$, and since $M_f = \hat{M}_f$ for each $f \in F \setminus \bar{F}$, we must have $\sum_{f \in \bar{F}} M_f = \sum_{f \in \bar{F}} \hat{M}_f$. It thus follows that these workers are not employed by firms in $\bar{F}$ under matching M either.

Next, note that the above argument implies $M' \sqsubset \hat{M}^f_{\bar{F}}$. Since $\hat{p}^t_f > p^t_f$, firm f will wish to replace some of its workers with these workers under M. Hence, $M_f \neq C_f(M_f + \hat{M}^f_{\bar{F}})$, so the rich preferences property follows. ▌

E Proof of Theorem 7

Let Γ be the limit continuum economy which the sequence $(\Gamma^q)_q$ converges to. For any population G, fix a sequence (G^q) of finite-economy populations such that $G^q \xrightarrow{w^*} G$. Let $\Theta^q = \{\theta^q_1, \theta^q_2, \ldots, \theta^q_{\bar{q}}\} \subset \Theta$ be the support for G^q.[41] We say that X^q is **feasible in** Γ^q if, for each $\theta \in \Theta^q$, $X^q(\theta)$ is a multiple of $1/q$ and $X^q \sqsubset G^q$. We first prove a couple of preliminary results.

[41]Note that we allow for possibility that there are more than one worker of the same type even in finite economies, so $\bar{q}$ may be strictly smaller than q.

Lemma 4. *For any $r > 0$, there is a finite number of open balls, $B_1, \ldots, B_L$ that have radius smaller than r with a boundary of zero measure (i.e. $G(\partial B_\ell) = 0, \forall \ell$) and cover Θ.*

Proof. Let $B(\theta, r) = \{\theta' \in \Theta \,|\, \|\theta' - \theta\| < r\}$ and $S(\theta, r) = \{\theta' \in \Theta \,|\, \|\theta' - \theta\| = r\}$, where $\|\cdot\|$ is a metric for the space Θ. For all $\theta \in \Theta$ and $r > 0$, there must be some $r_\theta \in (0, r)$ such that $G(S(\theta, r_\theta)) = 0$,[42] which means that $\partial B(\theta, r_\theta) = S(\theta, r_\theta)$ has a zero measure. Consider now a collection $\{B(\theta, r_\theta) \,|\, \theta \in \Theta\}$ of open balls that covers Θ. The compactness of Θ then guarantees the existence of a finite cover, as desired. ▌

Lemma 5. *Suppose $G^q \xrightarrow{w^*} G$ and $X \sqsubset G$. Then, there exists a sequence $\{X^q\}$ such that X^q is feasible in Γ^q and $X^q \xrightarrow{w^*} X$.*

Proof. Consider a decreasing sequence $(\epsilon_k)_k$ of real numbers converging to 0. Then, according to Lemma 4, we can find a finite cover $\{B_\ell^k\}_{\ell=1,\ldots,L_k}$ for each k such that for each ℓ, B_ℓ^k has a radius smaller than ϵ_k and $G(\partial B_\ell^k) = 0$. For each k, define $A_1^k = B_1^k$ and $A_\ell^k = B_\ell^k \setminus (\cup_{\ell'=1}^{\ell-1} B_{\ell'}^k)$ for each $\ell \geq 2$. So, for each k, $\{A_\ell^k\}$ constitutes a partition of Θ. It is straightforward to see that $G(\partial A_\ell^k) = 0, \forall \ell$, since $G(\partial B_\ell^k) = 0, \forall \ell$. Given this and $G^q \xrightarrow{w^*} G$, condition (e) of Theorem 13 implies that for each k, there exists sufficiently large q, denoted q_k, such that for all $q \geq q_k$

$$\frac{1}{q} < \frac{\epsilon_k}{L_k} \text{ and } |G(A_\ell^k) - G^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}, \forall \ell = 1, \ldots, L_k. \tag{21}$$

Let us choose $(q_k)_k$ to be a sequence that strictly increases with k.

Now we can construct X^q as follows: (i) $X^q \sqsubset G^q$ and (ii) for each k and $q \in \{q_k, \ldots, q_{k+1} - 1\}$,

$$X^q(A_\ell^k) = \max\left\{\frac{m}{q} \,\middle|\, m \in \mathbb{N} \text{ and } \frac{m}{q} \leq \min\{X(A_\ell^k), G^q(A_\ell^k)\}\right\} \text{ for each } \ell = 1, \ldots, L_k.$$

We now show that for all k and $q \in \{q_k, \ldots, q_{k+1} - 1\}$, we have

$$|X(A_\ell^k) - X^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}.$$

To see this, consider first the case $X(A_\ell^k) < G^q(A_\ell^k)$. Then, by definition of X^q and (21), we have $0 \leq X(A_\ell^k) - X^q(A_\ell^k) < \frac{1}{q} < \frac{\epsilon_k}{L_k}$. In case $X(A_\ell^k) \geq G^q(A_\ell^k)$, we have $X^q(A_\ell^k) = G^q(A_\ell^k) \leq X(A_\ell^k) \leq G(A_\ell^k)$, which implies by (21)

$$|X(A_\ell^k) - X^q(A_\ell^k)| \leq |G(A_\ell^k) - G^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}.$$

[42] To see this, note first that $B(\theta, r) = \cup_{\tilde{r} \in [0,r)} S(\theta, \tilde{r})$ and $G(B(\theta, r)) < \infty$. Then, $G(S(\theta, \tilde{r})) > 0$ for at most countably many $\tilde{r}$'s, since otherwise the set $R_n \equiv \{\tilde{r} \in [0, r) \,|\, G(S(\theta, \tilde{r})) \geq 1/n\}$ has to be infinite for at least one n, which yields $G(B(\theta, r)) \geq G(\cup_{\tilde{r} \in R_n} S(\theta, \tilde{r})) \geq \frac{\infty}{n}$, a contradiction.

We are now ready to prove that $X^q \xrightarrow{w^*} X$. We do so by invoking (b) of Theorem 13, according to which $X^q \xrightarrow{w^*} X$ if and only if $|\int hdX^q - \int hdX| \to 0$ as $q \to \infty$, for any uniformly continuous function $h \in C_u(\Theta)$.

Hence, to begin, fix any $h \in C_u(\Theta)$, and fix any $\epsilon > 0$. Next we define for each k and $q \in \{q_k, \dots, q_{k+1} - 1\}$

$$\bar{h}_\ell^{q,k} \equiv \frac{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta)h(\theta)}{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta)} = \frac{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta)h(\theta)}{X^q(A_\ell^k)}$$

if $X^q(A_\ell^k) > 0$, and if $X^q(A_\ell^k) = 0$, then define $\bar{h}_\ell^{q,k} \equiv h(\theta)$ for some arbitrarily chosen $\theta \in A_\ell^k$.

Note $C_u(\Theta)$ is endowed with the sup norm $\|\cdot\|_\infty$ and $\|h\|_\infty$ is finite for any $h \in C_u(\Theta)$. Hence, there exists $K \in \mathbb{N}$ sufficiently large such that, for all $k > K$ and $q \in \{q_k, \dots, q_{k+1} - 1\}$, we have $\|h\|_\infty \epsilon_k < \epsilon/2$, and

$$\sum_{\ell=1}^{L_k} \sup_{\theta \in A_\ell^k} |\bar{h}_\ell^{q,k} - h(\theta)| X(A_\ell^k) < \frac{\epsilon}{2}, \forall \ell = 1, \dots, L_k, \tag{22}$$

which is possible since h is uniformly continuous, $A_\ell^k \subset B_\ell^k$, and B_ℓ^k has a radius smaller than ϵ_k with ϵ_k converging to 0 as $k \to \infty$.

Then, for any $q > Q := q_K$, there exists $k > K$ with $q \in \{q_k, \dots, q_{k+1} - 1\}$ such that

$$\begin{aligned}
&\left| \int hdX^q - \int hdX \right| \\
=&\left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} X^q(A_\ell^k) - \int hdX \right| \\
\leq&\left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} (X^q(A_\ell^k) - X(A_\ell^k)) \right| + \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} X(A_\ell^k) - \int hdX \right| \\
\leq&\sum_{\ell=1}^{L_k} \|h\|_\infty |X^q(A_\ell^k) - X(A_\ell^k)| + \left| \sum_{\ell=1}^{L_k} \int \bar{h}_\ell^{q,k} \mathbb{1}_{A_\ell^k} dX - \sum_{\ell=1}^{L_k} \int h \mathbb{1}_{A_\ell^k} dX \right| \\
\leq&\|h\|_\infty \epsilon_k + \sum_{\ell=1}^{L_k} \sup_{\theta \in A_\ell^k} |\bar{h}_\ell^{q,k} - h(\theta)| X(A_\ell^k) \\
\leq&\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

where the third inequality follows from (21) while the fourth from (22). ▮

By Theorem 2, there exists a stable matching M in the continuum economy. Now, construct M^q by the following procedure.

1. Define $M^q_{f_1}$ as the X^q in Lemma 5 with respect to $X := M_{f_1}$.

2. Define $M^q_{f_2}$ as the X^q in Lemma 5 with respect to $X := M_{f_2}$, while replacing G with $G - M_{f_1}$, and G^q with $G^q - M^q_{f_1}$. (This is possible since $G^q - M^q_{f_1} \xrightarrow{w^*} G - M_{f_1}$.)

3. Generally, for any $k \in \{1, 2, \ldots, n\}$, inductively define $M^q_{f_k}$ as the X^q in Lemma 5 with respect to $X := M_{f_k}$, while replacing G with $G - \sum_{k'<k} M_{f_{k'}}$, and G^q with $G^q - \sum_{k'<k} M^q_{f_{k'}}$.

Noting that the number of firms is finite, we have $M^q \xrightarrow{w^*} M$. Thus, by continuity of firms' utility functions, for any $f \in F$ and $\epsilon > 0$,

$$u_f(M^q_f) > u_f(M_f) - \frac{\epsilon}{2}, \tag{23}$$

for any sufficiently large q. Let $D^{\preceq f}(M^q)$ be the subpopulation of workers in economy Γ^q who weakly prefer f to their match in M^q.[43] Since $M^q \xrightarrow{w^*} M$, we have $D^{\preceq f}(M^q) \xrightarrow{w^*} D^{\preceq f}(M)$.[44] Let $\tilde{M}^q_f = C_f(D^{\preceq f}(M^q))$. In words, $\tilde{M}^q_f$ is the most profitable block of M^q for f in the continuum economy, that is, the optimal deviation in a situation where the current matching is M^q, but the firm can deviate to any subpopulation, not just a discrete distribution. Then the above-mentioned property that $D^{\preceq f}(M^q) \xrightarrow{w^*} D^{\preceq f}(M)$ and continuity of C_f imply that $\tilde{M}^q_f \xrightarrow{w^*} M_f$. Thus, by continuity of firms' utility functions,

$$u_f(\tilde{M}^q_f) < u_f(M_f) + \frac{\epsilon}{2}, \tag{24}$$

for any sufficiently large q. Let M'_f be the most profitable block of M^q for f in economy Γ^q. Then M'_f is the optimal deviation facing the same population G^q and matching M^q as when computing $\tilde{M}_f$ but with an additional constraint that the deviation is feasible in Γ^q (multiples of $1/q$), so $u_f(M'_f) \leq u_f(\tilde{M}_f)$. This and inequality (24) imply

$$u_f(M'_f) < u_f(M_f) + \frac{\epsilon}{2}. \tag{25}$$

Combining inequalities (23) and (25), we obtain $u_f(M'_f) < u_f(M^q_f) + \epsilon$, completing the proof.

[43] To be precise, $D^{\preceq f}(M^q)$ is given as in (1) with G and X being replaced by G^q and M^q, respectively.

[44] This convergence can be shown using an argument similar to that which we have used to establish the continuity of Ψ in the proof of Lemma 3.

F Proof for Section 8

Proof of Theorem 11. **(Proof of the first part):** Suppose M is a stable matching in $\mathcal{X}^{|\Omega|}$. We prove that $X = (D^{\preceq\omega}(M))_{\omega\in\Omega}$ is a fixed point of T. Let us first show that for each $\omega \in \Omega$, $X_\omega \in \mathcal{X}$. It is clear that as each M_ω is countably additive, so is $M_\omega(\Theta_P \cap \cdot)$, which implies that $X_\omega(\cdot) = D^{\preceq\omega}(M)(\cdot) = \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap \cdot)$ is also countably additive. It is also clear that since $(M_\omega)_{\omega\in\Omega}$ is a matching, $X_\omega \sqsubset G$. Thus, we have $X_\omega \in \mathcal{X}$.

We next claim that $M_f = C_f(X_f)$ for all $f \in \tilde{F}$. This is immediate for $f = \emptyset$ since $M_\emptyset = X_\emptyset = C_\emptyset(X_\emptyset)$. To prove the claim for $f \neq \emptyset$, suppose for a contradiction that $M_f \neq C_f(X_f)$, and let us denote $\widetilde{M}_f = C_f(X_f)$. Note that due to the restriction that $C_f(X_f) \in \mathcal{Y}_f(X_f)$, we have $\widetilde{M}_f^{\preceq} \sqsubset X_f$ and thus $(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq}) \sqsubset X_f$. Then, by the revealed preference, we have $\widetilde{M}_f = C_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq})$, which means that M is not stable since $\widetilde{M}_f^{\preceq} \sqsubset X_f = D^{\preceq f}(M)$, yielding the desired contradiction.

We next prove $X = T(X)$. The fact that $M_\omega = C_\omega(X_{f(\omega)}), \forall\omega \in \Omega$ means that $X_\omega - M_\omega = R_\omega(X_{f(\omega)}), \forall\omega \in \Omega$. Then, for each $\omega \in \Omega$ and $E \in \Sigma$, we obtain

$$\begin{aligned}
\sum_{P\in\mathcal{P}} R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap E) &= \sum_{P\in\mathcal{P}} \Big(X_{\omega_-^P}(\Theta_P \cap E) - M_{\omega_-^P}(\Theta_P \cap E) \Big) \\
&= \sum_{P\in\mathcal{P}} \left(\sum_{\omega'\in\Omega:\omega'\preceq_P\omega_-^P} M_{\omega'}(\Theta_P \cap E) - M_{\omega_-^P}(\Theta_P \cap E) \right) \\
&= \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega:\omega'\prec_P\omega} M_{\omega'}(\Theta_P \cap E) = X_\omega(E),
\end{aligned}$$

where the second and fourth equality follows from the definition of $X_{\omega_-^P}$ and X_ω, respectively, while the third from the fact that ω_-^P is an immediate predecessor of ω. The above equation holds for every firm $\omega \in \Omega$, we conclude that $X \in T(X)$, i.e. X is a fixed point of T.

(Proof of the second part): Let us first introduce some notations. Let ω_+^P denote an **immediate successor** of $\omega \in \Omega$ at $P \in \mathcal{P}$: that is, $\omega_+^P \prec_P \omega$, and for any $\omega' \prec_P \omega$, $\omega' \preceq_P \omega_+^P$. Also, let $X_{\omega_+^P}(\Theta_P \cap \cdot) \equiv 0$ for any contract ω that is ranked last at P. Note that for any $\omega, \tilde{\omega} \in \Omega$, $\omega = \tilde{\omega}_-^P$ if and only if $\tilde{\omega} = \omega_+^P$.

Suppose now that $X = (X_\omega)_{\omega\in\Omega} \in \mathcal{X}^{|\Omega|}$ is a fixed point of T. For each contract $\omega \in \Omega$ and $E \in \Sigma$, define

$$M_\omega(E) = X_\omega(E) - \sum_{P\in\mathcal{P}} X_{\omega_+^P}(\Theta_P \cap E). \tag{26}$$

We first verify that for each $\omega \in \Omega$, $M_\omega \in \mathcal{X}$. First, it is clear that for each $\omega \in \Omega$, as both $X_\omega(\cdot)$ and $X_{\omega_+^P}(\Theta_P \cap \cdot)$ are countably additive, so is M_ω. It is also clear that for each $\omega \in \Omega$, $M_\omega \sqsubset X_\omega$.

Let us next show that for all $\omega \in \Omega$, $P \in \mathcal{P}$, and $E \in \Sigma$,

$$X_\omega(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E), \tag{27}$$

which means that $X_\omega = D^{\preceq \omega}(M)$. To do so, consider first a contract ω that is ranked last at P. By (26) and the fact that $X_{\omega_+^P}(\Theta_P \cap \cdot) \equiv 0$, we have $M_\omega(\Theta_P \cap E) = X_\omega(\Theta_P \cap E)$. Hence, (27) holds for such ω. Consider now any $\omega \in \Omega$ which is not ranked last, and assume for an inductive argument that (27) holds true for ω_+^P, so $X_{\omega_+^P}(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega_+^P} M_{\omega'}(\Theta_P \cap E)$. Then, by (26), we have

$$\begin{aligned} X_\omega(\Theta_P \cap E) = M_\omega(\Theta_P \cap E) + X_{\omega_+^P}(\Theta_P \cap E) &= M_\omega(\Theta_P \cap E) + \sum_{\omega' \in \Omega : \omega' \preceq_P \omega_+^P} M_{\omega'}(\Theta_P \cap E) \\ &= \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E), \end{aligned}$$

as desired.

To show that $M = (M_\omega)_{\omega \in \Omega}$ is a matching, let ω be a top-ranked contract at P. Then, the definition of T and the fact that X is a fixed point of T imply that for any $E \in \Sigma$,

$$G(\Theta_P \cap E) = X_\omega(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E) = \sum_{\omega' \in \Omega} M_{\omega'}(\Theta_P \cap E),$$

where the second equality follows from (27). Since the above equation holds for every $P \in \mathcal{P}$, M is a matching.

Let us now fix any $\omega \in \Omega_f$ and show that $M_\omega = C_\omega(X_{f(\omega)})$, which is equivalent to showing $X_\omega - M_\omega = R_\omega(X_{f(\omega)})$. Recall that $X_{\omega_+^P}(\Theta_P \cap E) = R_\omega(X_{f(\omega)})(\Theta_P \cap E)$ for all $P \in \mathcal{P}$ and $E \in \Sigma$. Then, (26) implies

$$X_\omega(\cdot) - M_\omega(\cdot) = \sum_{P \in \mathcal{P}} X_{\omega_+^P}(\Theta_P \cap \cdot) = \sum_{P \in \mathcal{P}} R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot) = R_\omega(X_{f(\omega)})(\cdot),$$

as desired.

We now prove that $(M_\omega)_{\omega \in \Omega}$ is stable. To prove the first part of Condition 1 of Definition 12, note first that $M_{\omega_\emptyset} = C_{\omega_\emptyset}(X_\emptyset) = X_{\omega_\emptyset}$. Then, for every $P \in \mathcal{P}$,

$$\sum_{\omega : \omega \prec_P \omega_\emptyset} M_\omega(\Theta_P) = \sum_{\omega : \omega \preceq_P \omega_\emptyset} M_\omega(\Theta_P) - M_{\omega_\emptyset}(\Theta_P) = X_{\omega_\emptyset}(\Theta_P) - M_{\omega_\emptyset}(\Theta_P) = 0,$$

where the middle equality follows from (27). The above equation means that for each $\omega \prec_P \omega_\emptyset$, we have $M_\omega(\Theta_P) = 0$, as desired. The second part of Condition 1 of Definition 12 (i.e. $M_f = C_\omega(M_f^{\preceq})$) follows from the revealed preference property since $M_f = C_f(X_f)$ and also since (27) means $M_f^{\preceq\omega} \sqsubset X_\omega$ for each $\omega \in \Omega_f$ or $M_f^{\preceq} \sqsubset X_f$.

It only remains to check Condition 2 of Definition 12. Suppose for a contradiction that it fails. Then, there exist f and $\widetilde{M}_f$ such that

$$M_f \neq \widetilde{M}_f = C_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq}) \text{ and } \widetilde{M}_f^{\preceq} \sqsubset D^{\preceq f}(M). \tag{28}$$

Since then $(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq}) \sqsubset D^{\preceq f}(M) = X_f$ and $M_f = C_f(X_f)$, the revealed preference property implies $M_f = C_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq})$, contradicting (28). We have thus proven that M is stable. ▌

G Proofs for Section 7

First, we establish the following result:

Lemma 6. *For any $X \sqsubset G$, $\tilde{C}_f(X)$ is nonempty and a singleton set. Also, $\tilde{C}_f$ satisfies the revealed preference property.*

Proof. We first establish that for X, $\tilde{C}_f(X)$ is a singleton set. To do so, for any $X \in \mathcal{X}$, $f \in F$, $k \in I_f$, and $\alpha^k \in [0,1]$, define $\zeta_f^k(\alpha^k) := \sum_{\theta \in S_f^k} \min\{X(\theta), \alpha^k G(\theta)\}$. From now on, we assume $C_f(X) \neq \{X\}$ since, if $C_f(X) = \{X\}$, then we have $\tilde{C}_f(X) = \{X\}$, a singleton set as desired. We show that there exists a unique $\hat{\alpha}^k$ satisfying $\zeta_f^k(\hat{\alpha}^k) = \Gamma_f^k(X)$, which means that $\tilde{C}_f(X)$ is a singleton set. First, we must have $\hat{\alpha}^k < \max_{\theta \in S_f^k} X(\theta)$ since otherwise $\zeta_f^k(\hat{\alpha}^k) = \sum_{\theta \in S_f^k} X(\theta) > \Gamma_f^k(X)$ (which follows from the assumption that $C_f(z) \neq \{X\}$ and thus, for any $X' \in C_f(X)$, $X' \sqsubset X$ and $X' \neq X$). Next, observe that $\zeta_f^k(\cdot)$ is strictly increasing in the range $[0, \max_{\theta \in S_f^k} \frac{X(\theta)}{G(\theta)})$. Then, the continuity of ζ_f^k, along with the fact that $\zeta_f^k(0) = 0$ and $\zeta_f^k(\max_{\theta \in S_f^k} \frac{X(\theta)}{G(\theta)}) > \Gamma_f^k(X)$, implies that there is a unique $\hat{\alpha}^k \in [0, \max_{\theta \in S_f^k} \frac{X(\theta)}{G(\theta)})$ satisfying $\zeta_f^k(\hat{\alpha}^k) = \Gamma_f^k(X)$.

To show the revealed preference property, consider any $X, X', X'' \in \mathcal{X}$ such that $\tilde{C}_f(X) = \{X'\}$ and $X' \sqsubset X'' \sqsubset X$. Since we already know that $C_f(\cdot)$ satisfies the revealed preference property, we have $X' \in C_f(X'')$. It suffices to show that $X' \in B_f(X'')$, since it means $\tilde{C}_f(X'') = \{X'\}$, from which the revealed preference property follows. To do so, note that $X' \in B_f(X)$ means that $X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\}$ for each k and $\theta \in S_f^k$. Then, since $X(\theta) \geq X''(\theta) \geq X'(\theta)$ and $\alpha^k G(\theta) \geq X'(\theta)$, we have

$$X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\} \geq \min\{X''(\theta), \alpha^k G(\theta)\} \geq X'(\theta),$$

so $X'(\theta) = \min\{X''(\theta), \alpha^k G(\theta)\}$ as desired. ▌

Proof of Theorem 9. **("only if" part)** Consider a strongly stable matching M in the time share model $(G, F, \mathcal{P}_\Theta, C_F)$. We show that M is stable in the time share model $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$. First, it is clear that M is individually rational for workers. To show that the same is true for firms, i.e., $M_f \in \tilde{C}_f(M_f)$ for each $f \in F$, note first that we have $M_f \in C_f(M_f)$, since M is strongly stable and thus stable. That $M_f \in B_f(M_f)$ follows from setting $\alpha^k = 1$, since then $M_f(\theta) \le \alpha^k G(\theta) = G(\theta), \forall \theta \in \Theta_f^k$ so $\min\{M_f(\theta), \alpha^k G(\theta)\} = M_f(\theta), \forall \theta \in \Theta_f^k$.

To show next that there is no blocking coalition, we first prove the following claim:

Claim 8. *Fix any strongly stable matching M in the time share model $(G, F, \mathcal{P}_\Theta, C_F)$ and let $X_f = D^{\preceq f}(M)$. For each firm $f \in F$, there is some $\alpha^k \in [0,1]$ for each $k \in I_f$ such that $M_f(\theta) = \min\{X_f(\theta), \alpha^k G(\theta)\}, \forall \theta \in \Theta_f^k$.*

Proof. It suffices to show that for any $k \in I_f$ and $\theta, \theta' \in \Theta_f^k$, if $M_f(\theta) < X_f(\theta)$ and $M_f(\theta') < X_f(\theta')$, then $\frac{M_f(\theta)}{G(\theta)} = \frac{M_f(\theta')}{G(\theta')}$. Note first that

$$X_f(\theta) = D^{\preceq f}(M)(\theta) = \sum_{f' \in \tilde{F}: f' \preceq_\theta f} M_{f'}(\theta) = M_f(\theta) + \sum_{f' \in \tilde{F}: f' \prec_\theta f} M_{f'}(\theta).$$

If $M_f(\theta) < X_f(\theta)$ and $M_f(\theta') < X_f(\theta')$, then we have $\sum_{f' \in \tilde{F}: f' \prec_\theta f} M_{f'}(\theta) > 0$ and $\sum_{f' \in \tilde{F}: f' \prec_{\theta'} f} M_{f'}(\theta') > 0$. Given this, the strong stability implies that $\frac{M_f(\theta)}{G(\theta)} = \frac{M_f(\theta')}{G(\theta')}$. ▌

Since M is stable in the time share model $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$, we have $M_f \in C_f(X_f)$ for each $f \in F$ with $X_f = D^{\preceq f}(M)$. Then, for any $M'_f \sqsubset X_f$, we have $M_f \in C_f(M'_f \vee M_f)$ due to the fact that $M_f \in C_f(X_f)$, $(M'_f \vee M_f) \sqsubset X_f$, and C_f satisfies the revealed preference. The condition of no blocking coalition—i.e., Condition 2 of Definition 4—will then hold if $M_f \in B_f(M'_f \vee M_f)$, since it means $M_f \in C_f(M'_f \vee M_f) \cap B_f(M'_f \vee M_f) = \tilde{C}_f(M'_f \vee M_f)$. For this, for each k, we choose α^k as in Claim 8 and show that for all $\theta \in \Theta_f^k$, $M_f(\theta) = \min\{\max\{M'_f(\theta), M_f(\theta)\}, \alpha^k G(\theta)\}$. Let us first consider the case where $M_f(\theta) = X_f(\theta) < \alpha^k G(\theta)$. Since $M'_f(\theta) \le X_f(\theta)$, we have $\max\{M'_f(\theta), M_f(\theta)\} = M_f(\theta) < \alpha^k G(\theta)$ and thus $M_f(\theta) = \min\{\max\{M'_f(\theta), M_f(\theta)\}, \alpha^k G(\theta)\}$. For the other case where $M_f(\theta) = \alpha^k G(\theta) \le X_f(\theta)$, observe that $M_f(\theta) = \alpha^k G(\theta) = \min\{\max\{M'_f(\theta), M_f(\theta)\}, \alpha^k G(\theta)\}$ since $\max\{M'_f(\theta), M_f(\theta)\} \ge M_f(\theta) = \alpha^k G(\theta)$.

("if" part) Consider a stable matching $M = (M_f)_{f \in \tilde{F}}$ in $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ and let $X_f = D^{\preceq f}(M)$ for each $f \in \tilde{F}$. To show that M is strongly stable in the time share model $(G, F, \mathcal{P}_\Theta, C_F)$, we first show that it is stable (i.e. Condition (i) of Definition 11). It is straightforward, thus omitted, to check the individual rationality. To check the condition of no blocking coalition, suppose to the contrary that there is a blocking pair f and M'_f, which means that $M'_f \sqsubset X_f$, $M'_f \in C_f(M'_f \vee M_f)$, and $M_f \notin C_f(M'_f \vee M_f)$. Given this, by Lemma 6, there exists $\tilde{M}_f$ such that $\tilde{C}_f(M'_f \vee M_f) = \{\tilde{M}_f\}$. First, by the revealed preference property of $\tilde{C}_f$ and the fact that $\tilde{M}_f \sqsubset (\tilde{M}_f \vee M_f) \sqsubset (M'_f \vee M_f)$, we have $\tilde{M}_f \in \tilde{C}_f(\tilde{M}_f \vee M_f)$ and $M_f \notin \tilde{C}_f(\tilde{M}_f \vee M_f)$. Second, since $M_f \sqsubset X_f$ and $M'_f \sqsubset X_f$, we have $\tilde{M}_f \sqsubset (M'_f \vee M_f) \sqsubset X_f$. In sum, f and $\tilde{M}_f$ form a blocking pair in $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$, which is a contradiction.

To show that Condition (ii) of Definition 11 also holds, suppose not. Then, there must be some f, k, and $\theta, \theta' \in \Theta^k_f$ for whom $\frac{M_f(\theta)}{G(\theta)} < \frac{M_f(\theta')}{G(\theta')}$ and $\sum_{f':f' \prec_\theta f} M_{f'}(\theta) > 0$. Fixing any such f, k, and θ, let f' be any firm such that $f' \prec_\theta f$ and $M_{f'}(\theta) > 0$. Letting $\Theta' = \arg\max_{\tilde{\theta} \in \Theta^k_f} \frac{M_f(\tilde{\theta})}{G(\tilde{\theta})}$, note that $\theta \notin \Theta'$. We define a matching for firm f as follows: for each $\tilde{\theta} \in \Theta$,

$$M'_f(\tilde{\theta}) = \begin{cases} M_f(\tilde{\theta}) + \epsilon M_{f'}(\tilde{\theta}) & \text{if } \tilde{\theta} = \theta \\ \epsilon' M_f(\tilde{\theta}) & \text{if } \tilde{\theta} \in \Theta' \\ M_f(\tilde{\theta}) & \text{otherwise} \end{cases},$$

where $\epsilon, \epsilon' \in (0,1)$ are chosen to satisfy $\frac{M'_f(\tilde{\theta})}{G(\tilde{\theta})} < \frac{M'_f(\theta')}{G(\theta')}, \forall \theta' \in \Theta', \tilde{\theta} \notin \Theta'$ and

$$M'_f(\theta) + \sum_{\theta' \in \Theta'} M'_f(\theta') = M_f(\theta) + \sum_{\theta' \in \Theta'} M_f(\theta'). \tag{29}$$

Note first that $M'_f \sqsubset X_f$, since $f' \prec_\theta f$ and thus $M'_f \sqsubset (M_f(\tilde{\theta}) + M_{f'}(\tilde{\theta}) 1_{\{\tilde{\theta}=\theta\}})_{\tilde{\theta} \in \Theta} \sqsubset D^{\preceq f}(M) = X_f$. Let us show next that $M'_f \in \tilde{C}_f(M'_f \vee M_f)$. The fact that $M_f \in \tilde{C}_f(X_f)$ (due to the stability of M) and $M_f \vee M'_f \sqsubset X_f$, implies $M_f \in \tilde{C}_f(M_f \vee M'_f)$ by the revealed preference. This means $M_f \in C_f(M_f \vee M'_f)$, which can be combined with (29) to yield

$$\sum_{\theta \in \Theta^k_f} M'_f(\theta) = \sum_{\theta \in \Theta^k_f} M_f(\theta) = \Gamma^k_f(M_f \vee M'_f), \forall k \in I_f$$

and thus $M'_f \in C_f(M_f \vee M'_f)$. To show $M'_f \in B_f(M_f \vee M'_f)$, we set $\alpha^k = \max_{\tilde{\theta} \in \Theta^k_f} \frac{M'_f(\tilde{\theta})}{G(\tilde{\theta})}$ and observe that for all $\theta' \in \Theta'$, $\min\{(M_f \vee M'_f)(\theta'), \alpha^k G(\theta')\} = \min\{M_f(\theta'), \alpha^k G(\theta')\} = \alpha^k G(\theta') = M'_f(\theta')$ while for all $\tilde{\theta} \notin \Theta'$, $\min\{(M_f \vee M'_f)(\tilde{\theta}), \alpha^k G(\tilde{\theta})\} = \min\{M'_f(\tilde{\theta}), \alpha^k G(\tilde{\theta})\} = M'_f(\tilde{\theta})$, which implies $M'_f \in B_f(M_f \vee M'_f)$. Therefore, $M'_f \in \tilde{C}_f(M_f \vee M'_f)$. A contradiction follows since the fact that $\tilde{C}_f$ is single-valued and $M'_f \neq M_f$ implies $M_f \notin \tilde{C}_f(M'_f \vee M_f)$. ▌

Proof of Theorem 10. By Theorem 9 and the Kakutani-Fan-Glicksberg fixed point theorem, it suffices to show that $\tilde{C}_f$ is closed- and convex-valued, and upper hemicontinuous.

The convexity and closed-valuedness of $\tilde{C}_f(X)$ for any $X \sqsubset G$ follow directly from the fact that $\tilde{C}_f(X)$ is a singleton set. In the rest of the proof, we prove the upper hemicontinuity. By 16.25 Theorem of Aliprantis and Border (2006), the intersection of a family of closed-valued upper hemicontinuous correspondences, one of which is also compact-valued, is upper hemicontinuous. Since we already know that $C_f(\cdot)$ is closed- and compact-valued, and upper hemicontinuous, we only need to prove that $B_f(\cdot)$ is upper hemicontinuous (given that its closed-valuedness has been proved).[45] To do so, consider sequences $(X^\ell)_{\ell\in\mathbb{N}}$ and $(\tilde{X}^\ell)_{\ell\in\mathbb{N}}$ with $\tilde{X}^\ell \in B_f(X^\ell), \forall \ell$, converging to X and $\tilde{X}$, respectively. So, for each $k \in I_f$, there is a sequence $(\alpha^k_\ell)_{\ell\in\mathbb{N}}$ such that $\tilde{X}^\ell(\theta) = \min\{X^\ell(\theta), \alpha^k_\ell G(\theta)\}, \forall \theta \in \Theta^k_f$. For each k, let α^k be a limit to which a subsequence of the sequence $(\alpha^k_\ell)_{\ell\in\mathbb{N}}$ converges. We claim that $\tilde{X}(\theta) = \min\{X(\theta), \alpha^k G(\theta)\}, \forall \theta \in \Theta^k_f$. If $\tilde{X}(\theta) > \min\{X(\theta), \alpha^k G(\theta)\}$, then one can find sufficiently large ℓ to make $\tilde{X}^\ell(\theta)$, $X^\ell(\theta)$, and α^k_ℓ close to $\tilde{X}(\theta)$, $X(\theta)$, and α^k, respectively, so that $\tilde{X}^\ell(\theta) > \min\{X^\ell(\theta), \alpha^k_\ell G(\theta)\}$, which is a contradiction. The same argument applies to the case with $\tilde{X}(\theta) < \min\{X(\theta), \alpha^k G(\theta)\}$. ▮

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