

Uncertain Technological Change

PRELIMINARY AND INCOMPLETE

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Abstract

We develop a general equilibrium dynamic stochastic business cycle model augmented with uncertain total factor productivity shocks and investment specific technology shocks. Agents only observe noisy signals of these shocks, such as realized GDP, and update their beliefs each period about the fundamentals of the economy. We interpret this inability to perfectly observe the fundamentals of the economy as capturing informational frictions, and refer to agents' beliefs about the fundamentals as "sentiment." We estimate the model to match some important moments of the US economy and obtain direct measures of the characteristics of the unobserved technology shocks. We find that the impact of productivity shocks is generally similar to an environment without this form of uncertainty. However, a non-persistent "sentiment shock" (an erroneous signal) regarding the investment specific technology alters agents' behavior persistently, even though the underlying fundamentals governing the economy remain unchanged.

Keywords: Uncertainty, productivity shocks, investment specific technological progress, Bayesian learning, sentiment, business cycles.

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1 Introduction

Uncertainty is a key feature of macroeconomic models. Since at least Kydland and Prescott (1982), formal business cycle analysis has considered variants of the Neoclassical Growth Model where economic agents experience different kinds of stochastic shocks (to technology, preferences, policies, etc). On the other hand, a distinct type of uncertainty is essential to the *practice* of macroeconomic analysis. In particular, the state of economic fundamentals, which are crucial for making optimal choices, is not observed. As an example of this, it is de rigueur to mention in introductory macroeconomics textbooks that one problem with stabilization policy is the fiscal or monetary authority's inability to be fully informed of the state of the world at any given point in time.¹

The idea that changes in expectations about the state of the economy could be an important driver of economic fluctuations has received increasing attention in the past decade. Dating at least back to Pigou (1927) who discussed the "errors of undue optimism or undue pessimism" as a source of "industrial fluctuations," this idea regained its popularity partly as a result of the tech boom around the year 2000 and the subsequent recession². It is natural to imagine that undue optimism regarding technology levels can create a high level of investment and an economic boom. Later, when technology turns out to be less improved than previously believed, investment might decline, and a recession begins. Nonetheless, Beaudry and Portier (2004) among others find that it is difficult for business cycle models to generate a boom and bust originating from unwarranted expectations of high productivity. A common feature of these models is the reliance on an environment where expectations are developed under perfect information and certainty: at a given point in time, the state of the world is known, even if the future is not known. On the other hand, since the seminal paper by Bloom (2009), an emerging literature has studied the effects of various *types* of economic uncertainty, finding that certain forms of uncertainty can be quantitatively important for business cycle fluctuations. However, the macroeconomic impact of the notion that agents in the economy never have perfect information and therefore are always uncertain about the current state of economic fundamentals, such as technology levels, has not been fully investigated.

This paper explores business cycles in an environment in which economic agents never perfectly observe the actual values of the fundamentals of the economy. We propose a general framework with standard real business cycle model as the core, but incorporating explicitly characterized uncertainty and Bayesian updating of beliefs regarding the unobserved productivity terms. Specifically, the economy features neutral productivity shocks and investment specific productivity shocks. Although these two technologies follow known stochastic processes, we assume that, due to imperfect information, economic agents can

¹For example, Krugman and Wells (2013) emphasizes the time it takes to gather data about the state of the economy, which Parkin (2014) calls a "recognition lag." Baumol and Blinder (2012) likens stabilization policy to "a poor rifleman shooting through dense fog at an erratically moving target with an inaccurate gun and slow-moving bullets." This paper is about the "dense fog".

²See Jermann and Quadrini (2007), and Leduc and Sill (2010).

only observe noisy signals about their actual values³. This information friction partially decouples expectations from the fundamentals of the economy, such as the realizations of the productivity processes and the efficient capital stock, but instead connects them with signals and prior beliefs regarding these fundamentals, in the sense that agents learn about fundamentals by updating their prior beliefs using Bayes' rule when new information arrives⁴. One information signal is the observed output, a combination of the realized total factor productivity and the efficient capital stock. The other is a noisy indicator of the investment specific technology, combining the actual value with a disturbance affecting the accuracy of the information embodied in this signal, which is interpreted as a "sentiment" shock, capturing the market sentiment reflecting undue optimism or pessimism⁵. Therefore, the sentiment shock can be understood as the misperception of economic agents about the investment specific technology, which itself may lead to fluctuations in investment and output. Without this uncertainty, *mutatis mutandis*, this economy would generate the business cycle properties as described by Greenwood et al. (1997) and Justiniano et al. (2010). In particular, it would fail to deliver an investment boom and a subsequent recession based on any single shock, due to the lackluster of the transmission mechanism investigated in this paper.

In principle, the computation of this model is very difficult. Agents' beliefs regarding unobserved variables are themselves state variables of the model economy, and these beliefs need not have analytically tractable expressions. Similarly, the functional that maps current beliefs into future beliefs cannot be analytically characterized. Nonetheless, we develop a tractable solution strategy similar to the log-linearization methodology described in Uhlig (1999), which is able to bypass these computational challenges. We calibrate the model to match some moments of U.S. economy and obtain direct measures of the characteristics of the unobserved technology shocks. We explore the role beliefs play in characterizing the transmission mechanism of technology and sentiment shocks as driving forces of fluctuations under the uncertainty through the studies of the impulse responses of macroeconomic variables due to a one time deviation of different shocks and compare the results with those under a standard model. We find that the behavior of this economy with uncertainty and Bayesian updating follows patterns similar to those of a standard business cycle model. However, a non persistent market "sentiment" shock to the investment specific technology changes agents' behavior persistently, even though the underlying fundamentals governing the

³Besides total factor productivity, we also consider the investment specific technology. This is because in some sense this is the minimum specification required for uncertainty regarding technology to persist in the economy. If there were only neutral technology shocks, GDP would become a sufficient statistic for the unseen level of productivity. When there is uncertainty regarding investment specific productivity, it turns out that the capital stock is unknown, which implies that GDP is no longer enough for inferring the unobserved technological variables. The idea that the value of investment specific technology shock is not known is consistent with the presence of the literature that tries to measure its contribution to US growth, ranging from 20% in Hulten (1992) to 60% in Greenwood et al. (1997) and 100% in Ngai and Samaniego (2009).

⁴The learning rule in our paper is similar with that in Sargent and Williams (2005) where Bayesian updating through Kalman filter is studied. The process of signal extraction is similar with that introduced in Edge, Laubach, and Williams (2007) where agents are confused between shocks to the level or to the growth rate of the technology.

⁵See, for example, Zeno, Michael and Gernot (2013)

economy remain unchanged. For a sufficiently large sentiment shock, the macroeconomic impact can be quantitatively important. This is because agents' decisions are based on beliefs regarding the state of the world which might differ from the true state, and thus it takes a long time to correct the mistaken beliefs induced by the change of market sentiment through the gradual arrival of new signals. The response to the sentiment shock illustrates the tech boom and bust cycle such as the 2000s recession, which previous models fail to generate. This also suggests that belief twisting events may have a notable impact on the macroeconomy through the specific channel of information friction.

Our work is related to various strands in the literature. First, this paper is related to the literature on the expectations-driven business cycle. One direction of this literature has mostly emphasized news shocks about future productivity as sources of fluctuations, e.g., Beaudry and Portier (2006). Another strand studies the sentiment and business cycles. Jaimovich and Rebelo (2006, 2011) study the business cycle responses to mistakenly optimistic beliefs about productivity in the neoclassical growth model and show that incorrect beliefs can generate fluctuations but the quantitative effect on volatility is small. Angeletos and La'O (2012) proposes a theoretical model about the importance of sentiment shocks, modeled as the shocks that impact the availability of information to different individuals in the economy and the aggregate expectations on economic fundamentals. Enders et al. (2013) empirically studies the effect of optimism shocks on economic activities using a VAR model and survey data. This paper is particularly closely related to the latter strand, in the sense that the market sentiment specified here does not relate to news about future productivity or policies, but represent unwarranted optimism and pessimism that are orthogonal to the current state of the economy.

Second, this paper is also related to the literature of various forms of uncertainty and their relation with economic fluctuations. Among others, Collard et al (2009), studies four scenarios of imperfect information, and their impact on business cycles. Bloom (2009) studies a partial equilibrium model with uncertainty shocks in terms of stochastic volatility and its impact on the economy. Lorenzoni (2009) studies the model in which consumers have imperfect information about the level of aggregate productivity, featuring "noise shock" which has similar impact to aggregate demand shocks. This paper introduces another type of uncertainty, originating from the unobservability of different productivity shocks which leads to boom and bust cycles.

Finally, this paper is related to the extensive literature about importance of total factor productivity and investment specific technology for economic dynamics, including influential papers such as Kydland and Prescott (1982), Greenwood et al. (1997) and Fisher (2006). This literature focuses on the role of technology shocks in driving business cycles for the US post-war period and a general implication is that large negative shocks generate recessions, which remains controversial. Our paper, however, proposes another mechanism to explain the fluctuations, that recessions may not result from a negative shock on the technology parameters, but from incorrect beliefs on the imperfectly observed investment specific

technology.

Our paper is organized as follows. Section 2 describes the model and environment. Section 3 develops the solution strategy. Section 4 presents the data and estimation methodology. Section 5 shows the simulation results. The last section concludes and suggests direction for future research.

2 Model

We build on a standard real business cycle (RBC) framework and extend it to a general model with uncertainty and Bayesian updating of beliefs regarding the unobserved productivity values and the capital stock.

2.1 Standard RBC model

We start with a description of a simple standard RBC model with total factor productivity (TFP) shocks and investment specific technology (IST) shocks. There is no information friction.

There is a social planner whose objective is to maximize the lifetime discounted utility of a representative agent. The social planner's problem is

$$\begin{aligned}
 & \max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t) \\
 & \text{s.t.} \\
 C_t + I_t &= A_t e^{z_t} K_t^\alpha n_t^{1-\alpha} \\
 K_{t+1} &= B_t e^{q_t} I_t + (1 - \delta) K_t \\
 A_t &= e^{\gamma t} \quad B_t = e^{\gamma_q t} \\
 z_{t+1} &= \psi z_t + \varepsilon_{t+1} \quad q_{t+1} = \lambda q_t + \epsilon_{t+1}
 \end{aligned}$$

C_t is consumption, I_t is investment, K_t is the capital stock, n_t is labor input. The evolution of capital stock is affected by investment specific technology progress. A_t and B_t are respectively the deterministic trend of TFP progress and of IST progress. z_t and q_t are respectively the TFP shock and IST shock, both of which follow AR(1) processes. ε_t and ϵ_t are disturbances to TFP shock and IST shock. They are i.i.d normally distributed and not correlated. δ is the capital depreciation rate.

In order to proceed, we transform the model into one with a balanced growth path. As is standard, divide consumption, investment and output by their common constant growth factor and divide capital by its growth factor. Specifically, we divide K_t by the gross growth rate e^{g_k} to get a detrended capital stock

$k_t = \frac{K_t}{e^{gk^t}}$; we divide C_t , I_t and Y_t with the common growth factor g to get a detrended $c_t = \frac{C_t}{e^{gt}}$, $i_t = \frac{I_t}{e^{gt}}$, $y_t = \frac{Y_t}{e^{gt}}$.⁶ We assume N_t are stable.

After detrending and rewriting the utility function so that it is in terms of detrended variables, we get

$$\begin{aligned} & \max E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, n_t) \\ & s.t. \\ & c_t + i_t = e^{z_t} k_t^\alpha n_t^{1-\alpha} \\ & k_{t+1} e^{gk} = e^{q_t} i_t + (1 - \delta) k_t \\ & z_{t+1} = \psi z_t + \varepsilon_{t+1} \quad q_{t+1} = \rho q_t + \epsilon_{t+1} \end{aligned}$$

This is the core of our model.

2.2 The model with uncertainty and beliefs

2.2.1 Environment

As in the standard model, there is a social planner whose objective is to maximize the lifetime discounted utility of a representative agent⁷. However, the social planner cannot observe the exact values the realized TFP and IST shocks. As we shall see, this implies that she doesn't know the efficient capital stock either since the evolution of the capital stock is dependent on IST. She can observe noisy signals regarding these shocks, which convey information about the unobserved fundamentals. The optimal decision depends on the expectations of future productivity values, which change over time, and on beliefs about their current values. The true data generating processes for IST and TFP are exogenous and known to the social planner. The posterior beliefs reflect subjective perceptions embodied in the prior beliefs, along with agents' observations of the signals reflecting the stochastic processes driving the evolution of the economy. Posterior beliefs are generated according to Bayes' rule. In this paper, we consider the case of adaptive Bayesian learning in the sense that the social planner treats her current state of beliefs about the distributions of IST, TFP and the capital stock as the true ones. She does not take into account possible future updating of past beliefs when she makes current decision⁸.

⁶ g and g_k are functions of α , γ and γ_q .

⁷In the appendix 4 we describe a decentralized economy with uncertainty. Since effective capital stock is not observed by economic agents, we assume the observed quantity of capital in the economy is what the market price of capital relies on.

⁸As is shown in Cogley and Sargent (2008), given low values of the coefficient of relative risk aversion, the adaptive learning, as in our model, generates same consumption and investment choices as fully Bayesian learning (i.e., internalizing future updating), in the sense that people behave as if they make rational expectation by taking what would happen in the future into account when they make current decisions, even though they only make the optimal decision without considering

The assumption and timeline of the model is as follows:

Before observing the signals

At the beginning of each period t , the prior beliefs on the distribution of capital, TFP and IST, denoted respectively as g_t^K , g_t^Z , g_t^Q , are given⁹. They are formed before observing current period's signals. The investment decision at period $t-1$ affects the actual capital stock and output in period t . It also affects the believed capital stock in period t in a way disciplined by the evolution of beliefs outlined later. The labor decision, due to the feature of the uncertainty economy, is naturally different from that under standard RBC model. In a standard RBC model, TFP shocks are realized at the beginning of each period; after observing TFP, agents adjust labor according to the substitution rate between marginal benefit of consumption and leisure. Under our setup, since the social planner cannot observe the actual TFP and capital stock values, it is impossible to base the decision on the exact substitution rate. Instead, she relies on her beliefs about TFP and the capital stock to choose labor according to expectations regarding the substitution rates. We assume that the choice of labor is made at period t before observing signals, affecting both the utility at t and the output at period t .

Regarding the underlying processes of the fundamentals, at the beginning of each period t , TFP and IST shocks are realized, but unobserved to the social planner. The two signals are also uncovered after the production process. One signal is the realized output y_t , a function of the realized but unobserved TFP z_t , unobserved predetermined k_t and labor input n_{t-1} , following

$$y_t = e^{z_t} k_t^\alpha n_t^{1-\alpha}$$

The other signal is about the realized IST, a function of q_t and v_t . v_t is a disturbance in the signal observation, following an i.i.d normal distribution.

$$\phi_t = q_t + v_t$$

As mentioned in the introduction, this disturbance can be interpreted as sentiment shock in the literature, which reflects agents' attitude towards the economy¹⁰. For example, if agents are over optimistic due to some unwarranted information, when observing an increased value of the signal, they would believe that it is the increase of q_t that leads to the increase of ϕ_t , although what actually happened is a higher v_t but not a change in q_t . As a result, agents would react by increasing investment, which would eventually lead to recessions when they gradually learn that q had not changed. This fluctuation does not involve

that the current beliefs may be further updated in the future.

⁹For the rest of the paper, $m()$ denotes posterior beliefs, whereas $g()$ denotes prior beliefs. Capital notations such as Q, Z, K, Y means random variables, whereas small notations such as q, z, k, y means the realized values of the corresponding random variables.

¹⁰One candidate for the signal would be the relative price of investment to consumption. Details will be explained in the calibration section.

any change of fundamentals.

After observing the signals

After observing the two signals, the social planner is able to update her beliefs of the capital stock, TFP and IST following Bayes' rule, given prior beliefs:

$$m_t^{Z,Q,K}(z, q, k|y_t, \phi_t) \propto g(y_t, \phi_t|z, q, k)g_t^{Z,Q,K}(z, q, k)$$

where $m_t^{Z,Q,K}$ is the posterior joint belief on z_t, q_t and k_t , and $g_t^{Z,Q,K}$ is the prior joint belief on z_t, q_t and k_t , $g(y_t, \phi_t|z, q, k)$ is the likelihood of y_t and ϕ_t conditional on z_t, q_t and k_t ¹¹

In addition, knowing the true evolution of TFP and IST following AR(1) processes, the social planner can derive a prior belief of TFP for period $t + 1$, g_{t+1}^Z , and a prior belief of IST for period $t + 1$, g_{t+1}^Q by the law of conjugate distribution. Knowing the true evolution process of the capital stock, the social planner can also derive a prior belief of capital for period $t + 1$, g_{t+1}^K , given the choice of i_t , which is optimally chosen by the social planner expecting how these choices would affect output at period $t + 1$. The expectations of signals y_{t+1} , (the likelihood of the output conditional on prior distributions of unobserved variables and choice variables for period $t + 1$) $g(y_{t+1}|z, q, k)$. are formed according to the known production function. Similar rule applies to ϕ_{t+1} . The calculation of the updating of beliefs are demonstrated in Appendix 1.

2.2.2 Social planner's problem

As a summary, The social planner makes two optimal decisions at each period t ¹².

1. Choose n_t before observing the signals. After production, observe signals and update beliefs. We write this behavior in terms of the relation of two value functions: $W(m_t^Z, m_t^Q, m_t^K(i_{t-1})|\phi_t, y_t; n_t)$ and $V(g_t^Z, g_t^Q, g_t^K(i_{t-1}))$. $W(m_t^Z, m_t^Q, m_t^K(i_{t-1})|\phi_t, y_t; n_t)$ is the value function after observing signals and updating beliefs, while $V(g_t^Z, g_t^Q, g_t^K(i_{t-1}))$ is the value function before observing signals.

¹¹In order to separately get the posterior beliefs of z_t, q_t and k_t , we can integrate the joint belief as follows

$$m_t^Z = \int \int m_t^{Z,Q,K}(z, q, k|y_t, \phi_t) f_1(z|q, k) dq dk$$

$$m_t^Q = \int \int m_t^{Z,Q,K}(z, q, k|y_t, \phi_t) f_2(q|z, k) dz dk$$

$$m_t^K = \int \int m_t^{Z,Q,K}(z, q, k|y_t, \phi_t) f_3(k|z, q) dz dq$$

¹²It is equivalent if we write the two optimal decisions in one value function. We separate them for the purpose of emphasizing the importance of the updating of beliefs in our model.

$$V \left(g_t^Z, g_t^Q, g_t^K(i_{t-1}) \right) = \max_{n_t} \int \int W \left(m_t^Z, m_t^Q, m_t^K(i_{t-1}) | \phi_t, y_t; n_t \right) g_t^{Y,Q}(y, \phi | g_t^Z, g_t^Q, g_t^K(i_{t-1}), n_t) dy d\phi$$

2. After updating beliefs, optimally choose investment maximize the utility. This decision can be specified as a dynamic programming problem.

Given updated beliefs, the social planner chooses i_t to solve

$$\begin{aligned} W \left(m_t^Z, m_t^Q, m_t^K(i_{t-1}) | \phi_t, y_t; n_t \right) &= \max_{i_t} \left\{ u(c_t, n_t) + \beta E_t V \left(g_{t+1}^Z, g_{t+1}^Q, g_{t+1}^K(i_t) \right) \right\} \\ &\text{s.t.} \\ c_t + i_t &= y_t \\ g_{t+1}^K(k') &= \frac{e^{g_k} k' - i_t e^q}{(1 - \delta)} m_t^K \left(\frac{e^{g_k} k' - i_t e^q}{(1 - \delta)} \right) m_t^Q dq \\ g_{t+1}^Z(z') \text{ and } g_{t+1}^Q(q') &\text{ follow conjugate calculation} \end{aligned}$$

$E()$ is the rational expectation based on prior beliefs for next period. The first constraint is the budget constraint. The rest are the evolutions of beliefs. Labor and investment affects expected value of future output.

2.2.3 Optimization conditions

labor: The first order condition for n_t is

$$\begin{aligned} &\int W_y \left(m_t^Z, m_t^Q, m_t^K(i_{t-1}) | \phi_t, y_t; n_t \right) \frac{\delta g_t^Y(y, \phi | g_t^Z, g_t^K(i_{t-1}), n_t)}{\delta n_t} dy + \int \frac{\delta W \left(m_t^Z, m_t^Q, m_t^K(i_{t-1}) | \phi_t, y_t; n_t \right)}{\delta n_t} g_t^Y(y) \\ &= \int u_c(c_t, n_t) \frac{\delta g_t^Y}{\delta n_t} dy + \frac{\delta W \left(m_t^Z, m_t^Q, m_t^K(i_{t-1}) | \phi_t, y_t; n_t \right)}{\delta n_t} = 0 \end{aligned}$$

The Envelope condition for n_t is

$$\frac{\delta W \left(m_t^Z, m_t^Q, m_t^K(i_{t-1}) | \phi_t, y_t; n_t \right)}{\delta n_t} = u_n(c_t, n_t)$$

Combine them, we get the optimization condition for n_t

$$\int [u_n(c_t, n_t) + \beta u_c(c_t, n_t) \frac{\delta g_t^Y}{\delta n_t}] dy = 0$$

investment: The first order condition for i_t is

$$-u_c(c_t, n_t) + \frac{\delta \beta E_t V(g_{t+1}^Z, g_{t+1}^Q, g_{t+1}^K(i_t))}{\delta i_t} = 0$$

The Envelope condition for i_t is

$$\begin{aligned} V_i(g_t^Z, g_t^Q, g_t^K(i_{t-1}), n_{t-1}) &= \frac{\partial V(g_t^Z, g_t^Q, g_t^K(i_{t-1}), n_{t-1})}{\partial i_{t-1}} \\ &= \int W_y(m_t^Z, m_t^Q, m_t^K(i_{t-1}) | \phi_t, y_t; n_t) \frac{\delta g_t^Y(y_t | g_t^Z, g_t^Q, g_t^K(i_{t-1}), n_{t-1})}{\delta g_t^K} \frac{\delta g_t^K}{\delta i_{t-1}} dy \\ &\quad + \int \frac{\delta W(m_t^Z, m_t^Q, m_t^K(i_{t-1}) | \phi_t, y_t; n_t)}{\delta m_t^K} g_t^Y(y_t | g_t^Z, g_t^Q, g_t^K(i_{t-1}), n_{t-1}) \frac{\delta m_t^K}{\delta i_{t-1}} dy \\ &= \int [u_c(c_t, n_t) \frac{\delta g_t^Y}{\delta g_t^K} \frac{\delta g_t^K}{\delta i_{t-1}} + u_c(c_t, n_t) \frac{\delta c_t}{\delta m_t^K} \frac{\delta m_t^K}{\partial i_{t-1}} g_t^Y] dy \end{aligned}$$

Combine them, we get the optimization condition for i_t

$$-u_c(c_t, n_t) + \beta \int u_c(c_{t+1}, n_{t+1}) \left[\frac{\delta g_{t+1}^Y}{\delta g_{t+1}^K} \frac{\delta g_{t+1}^K}{\delta i_t} + \frac{\delta c_{t+1}}{\delta m_{t+1}^K} \frac{\delta m_{t+1}^K}{\delta i_t} g_{t+1}^Y \right] dy = 0$$

where $\delta(\cdot)$ denotes functional derivative and $\partial(\cdot)$ denotes regular derivative.

For the labor, the marginal cost of labor equals to the perceived marginal benefit of labor, the benefit that one additional unit of labor can create more output; for the investment, the marginal cost of investment today in terms of consumption, equals to the perceived marginal benefit of capital tomorrow, both from the increase of the output and the increase of the capital stock due to an additional unit of investment.

2.3 Implications of uncertainty

Before continuing to solve the model, we compare the optimal conditions of our model with the standard model outlined at the beginning. This can shed some light on how the uncertainty might affect economic agents' behavior.

The optimal conditions under a standard RBC model are

$$u_n(c_t, n_t) + u_c(c_t, n_t) \frac{\partial Y_t}{\partial n_t} = 0$$

and

$$-u_c(c_t, n_t) + \beta \int u_c(c_{t+1}, n_{t+1}) \left[\frac{\partial Y_{t+1}}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial i_t} + \frac{\partial c_{t+1}}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial i_t} \right] dy = 0$$

Whereas under our setup with uncertainty, we have

$$u_n(c_t, n_t) + \beta \int u_c(c_t, n_t) \frac{\delta g_t^Y}{\delta n_t} dy = 0$$

and

$$-u_c(c_t, n_t) + \beta \int u_c(c_{t+1}, n_{t+1}) \left[\frac{\delta g_{t+1}^Y}{\delta g_{t+1}^K} \frac{\delta g_{t+1}^K}{\delta i_t} + \frac{\delta c_{t+1}}{\delta m_{t+1}^K} \frac{\delta m_{t+1}^K}{\delta i_t} g_t^Y \right] dy = 0 = 0$$

For the investment, it is clear that the introduction of uncertainty influences the way how investment would affect future output. When there is no uncertainty, increase of investment (decrease of consumption) today would increase the capital stock, which would increase the future output and decrease the future real interest rate in a certain way. However, when there is uncertainty regarding the values of the technology shocks and capital stock, effects of increase of investment today are uncertain, in the sense that it is possible that the realized output tomorrow is very different from what has been expected, due to incorrect beliefs about the capital stock and the technology values at the moment of decision making. Therefore, the effects of investment is not only determined by the fundamentals such as technologies and preferences, but also affected by the accuracy of beliefs, which in turn depends on factors that are not captured in the standard model, but are the key in our model, such as the information frictions and market sentiment. This could shed some light on the controversial prediction from the standard RBC model that in order for a recession to happen, there has to be a technological regress. Our model, instead, implies that when the market sentiment is unfavorable for a technological progress, there could be a recession too, even if there is no change in economic fundamentals. This implication shares some similarity with the prediction of the literature which emphasizes the importance of animal spirits and the existence of self-fulfilling equilibrium. Similarly, regarding the decision of labor, the return to labor is certain in the standard model whereas under our setup, the substitution rate between consumption and leisure isn't. Since the decision of labor is disconnected with the realized productivity values at least for some periods before beliefs are unwound, but coupled with perceived productivity values, the correlation between wage and the productivity process might be weaker. Therefore this model could potentially generate a comovement of labor-related macroeconomic variables that is more consistent with the data than that predicted by

standard RBC model.

3 Solution method

As shown in the literature, it is generally difficult to solve problems where state variables contain continuous functions. For example, Krusell and Smith in their (1998) influential paper used approximate aggregation to deal with this issue. They showed that under some conditions, second order moments of distributions of state variables do not affect the dynamics of the model. Therefore a characterization of the mean of the state variables suffices for the solution of the model. By contrast, we will demonstrate in this section that both first moments and second moments matter for the dynamics of our model, and our solution method is designed to capture this feature.

Under our setup, there are beliefs regarding the IST, TFP and capital that serve as state variables. To calculate the evolution of beliefs, a nonlinear filter, such as Particles filter are needed. Moreover, for the Euler equations, for example, we need to calculate the effect of a change in belief of capital on the expected output due to the change of investment. This is a derivative of a function with respect to the distribution of capital, another continuous function. Therefore, the method for functional derivative is needed, which would lead to a complicated and non tractable solution. In this paper, we develop a methodology to overcome the complication of functional derivatives and nonlinear filters while keeping the main structure of the model. The method is based on a first order log-linearization approximation using Taylor expansion, which enables to consider the evolution of beliefs of linearized variables rather than that of the original variables. Through the log-linearization, we are able to transform the Bayesian learning process to be linear and Gaussian under some initial conditions, and to implement the calculation of the linear Gaussian Markov process through the Kalman filter. In this way, the evolution of beliefs can be parameterized so that only mean and variance matters for the decision, rather than the entire distribution. The state variables are no longer functions of distributions, but key variables that characterize the distributions.

For the rest of this section, suppose:

1. M_t^K, M_t^Z, M_t^Q are respectively the prior means of capital, TFP and IST respectively for period t , given at the beginning of period t ; $\Sigma_t^K, \Sigma_t^Z, \Sigma_t^Q$ are respectively the prior variances of capital, TFP and IST respectively for period t .
2. $\mu_t^K, \mu_t^Z, \mu_t^Q$ are respectively the posterior means of capital, TFP and IST at period t ; $\sigma_t^K, \sigma_t^Z, \sigma_t^Q$ are respectively the posterior variances of capital, TFP and IST.
3. X is the true value of the variable X , and be_X is the prior belief of the variable X . So $M_t^Z = be_z_t, M_t^Q = be_q_t, M_t^{\tilde{K}} = be_k_t$

Later it will be shown that the Bayesian updating conditional variances of the unobservables converge to constants quickly, and thereafter the evolution of variables vary only with the beliefs of the means of

capital, TFP and IST after the initial updating periods¹³. We will focus on the economic implications of the model when beliefs of these conditional variances are stable, so that the model can be further simplified and become tractable.

3.1 Transformation of variables

The equations that are needed for calculating the evolution of beliefs are:

The signal for TFP and capital (production function)

$$y_t = e^{z_t} k_t^\alpha n_t^{1-\alpha}$$

The signal for IST

$$\phi_t = q_t + v_t$$

The evolution of capital

$$k_t e^{g_k} = e^{q_{t-1}} i_{t-1} + (1 - \delta) k_{t-1}$$

Define $X_t = \bar{X} e^{\tilde{X}_t}$, where $\tilde{X}_t = \log X_t - \log \bar{X}$, and \bar{X} is the steady state of X . Then we can rewrite the above functions in a linear system as

$$\tilde{y}_t = z_t + \alpha \tilde{k}_t + (1 - \alpha) \tilde{n}_t$$

$$\tilde{\phi}_t = q_t + v_t$$

$$\tilde{k}_t = \frac{(1 - \delta)}{e^{g_k}} \tilde{k}_{t-1} + \frac{\bar{i}}{\bar{k} e^{g_k}} \tilde{i}_{t-1} + \frac{\bar{i}}{\bar{k} e^{g_k}} q_{t-1}$$

The evolution of TFP and IST are the same as before

In addition, we assume:

The initial TFP z_{-1} is a random Gaussian variable, independent of the noise processes, with $z_{-1} \sim N(M_0^Z, \Sigma_0^Z)$

The initial IST q_{-1} is a random Gaussian variable, independent of the noise processes, with $q_{-1} \sim N(M_0^Q, \Sigma_0^Q)$

¹³The dynamics of the economy is still dependent on the constant values of these conditional variances covariances.

The initial capital \tilde{k}_{-1} is a random Gaussian variable, independent of the noise processes, with $\tilde{k}_{-1} \sim N(M_0^{\tilde{K}}, \Sigma_0^{\tilde{K}})$

In this way, all unknown random processes will become Gaussian processes, and again, means and variances is sufficient for the characterization of the model.

3.2 Evolution of linearized beliefs

At period t , the social planner observes the signals regarding current period TFP z_t , IST q_t , and the capital stock \tilde{k}_t . The signals are the output, y_t and signal ϕ_t

Given $M_t^Q, \Sigma_t^Q, M_t^Z, \Sigma_t^Z, M_t^{\tilde{K}}, \Sigma_t^{\tilde{K}}, i_{t-1}, n_t, y_t$ and ϕ_t , the social planner derives posterior beliefs of z_t, q_t and \tilde{k}_t using the two signals observed. Then, given the posterior beliefs of z_t, q_t and \tilde{k}_t , the social planner derives a prior belief of z_{t+1}, q_{t+1} and \tilde{k}_{t+1} for the next period, using

$$\tilde{k}_{t+1} = \frac{(1-\delta)}{e^{gk}} \tilde{k}_t + \frac{\bar{i}}{k e^{gk}} \tilde{y}_t + \frac{\bar{i}}{\bar{k} e^{gk}} q_t$$

and

$$\begin{aligned} z_{t+1} &= \psi z_t + \varepsilon_t, \varepsilon_t \sim N(\mu_\varepsilon, \sigma_\varepsilon^2) \\ q_{t+1} &= \rho q_t + \epsilon_t, \epsilon_t \sim N(\mu_\epsilon, \sigma_\epsilon^2) \end{aligned}$$

The calculation of the prior and posterior beliefs are shown in Appendix 2. The key methodology is the state space representation of the system of equations and the Kalman filter. We have the evolution of beliefs (of means of variables) as follows¹⁴:

$$be_{z_{t+1}} \equiv M_{t+1}^Z = \psi[be_{z_t} + AA(\tilde{y}_t - (1-\alpha)\tilde{n}_t - be_{z_t} - \alpha be_{\tilde{k}_t}) + BB(\tilde{\phi}_t - be_{q_t})]$$

$$be_{q_{t+1}} \equiv M_{t+1}^Q = \rho[be_{q_t} + EE(\tilde{y}_t - (1-\alpha)\tilde{n}_t - be_{z_t} - \alpha be_{\tilde{k}_t}) + FF(\tilde{\phi}_t - be_{q_t})]$$

¹⁴ $\begin{pmatrix} AA & BB \\ CC & DD \\ EE & FF \end{pmatrix}$ is a function of model parameters and conditional covariances of q, v and \tilde{k} . As is shown in the appendix

2, the conditional covariance matrices do not depend on the measurement \tilde{y}_t and $\tilde{\phi}_t$. They can therefore be computed in advance, given variances of the noise. Eventually and quickly they converge to some constants. We will use the converged values of $\begin{pmatrix} AA_t & BB_t \\ CC_t & DD_t \\ EE_t & FF_t \end{pmatrix}$ for the calculation and simulation.

$$\begin{aligned}
be_k_{t+1} &\equiv M_{t+1}^{\tilde{K}} = \frac{(1-\delta)}{e^{g_k}} [be_k_t + CC(\tilde{y}_t - (1-\alpha)\tilde{n}_t - be_z_t - \alpha be_k_t) + DD(\tilde{\phi}_t - be_q_t)] \\
&\quad + \frac{\bar{i}}{\bar{k}e^{g_k}} [be_q_t + EE(\tilde{y}_t - (1-\alpha)\tilde{n}_t - be_z_t - \alpha be_k_t) + FF(\tilde{\phi}_t - be_q_t)] \\
&\quad + \frac{\bar{i}}{\bar{k}e^{g_k}} \tilde{i}_t
\end{aligned}$$

This result shows that the prior beliefs regarding TFP, IST and the capital stock for the next period, which influences agents' optimal choices of investment and labor at this period, are influenced by the prior beliefs regarding TFP, IST and the capital stock for current period, and the realized signals which are determined by the realized but unobserved shocks $\varepsilon_t, \epsilon_t$ and v_t , the stochastic process of the unobserved shocks, as well as the choices of \tilde{i}_t , and \tilde{n}_t . Therefore, as mentioned before, the behavior of agents are affected by the accuracy of their initial expectations and the learning process (filtering information) on the current state of the economy.

3.3 Maximization problem

Without loss of generality, we combine the two stages value functions in last section into a one stage dynamic problem by not explicitly including the learning and updating process, given that we already characterize this process above. Notice that, however, the decisions of n_t is made before observing signals, whereas the decision of i_t are made after observing signals. The objective function of the social planner becomes

$$\begin{aligned}
V(g_t^Z, g_t^Q, g_t^K(i_{t-1})) &= E_t \{ \max_{n_t, i_t} u(c_t, n_t) + \beta be_V(g_{t+1}^Z, g_{t+1}^Q, g_{t+1}^K(i_t)) \} \\
s.t. y_t &= c_t + i_t \\
&\text{and evolution of beliefs}
\end{aligned}$$

where $V()$ is the value function before observing signals and updating beliefs, dependent on the beliefs regarding the unobserved productivity values and capital stock at period t , whereas $be_V()$ is the expected value function for the next period, dependent on the expected productivity values and capital stock for period $t+1$.

Optimal condition for i_t

$$-u_c(y_t - i_t, n_t) + \beta E_t [u_c(y_{t+1} - i_{t+1}, n_{t+1}) (\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + \frac{1-\delta}{e^{g_{t+1}}} \frac{e^{\frac{q_{t+1}}{\rho}}}{e^{g_k}})] = 0$$

Optimal condition for n_t

$$E_t[u_n(y_t - i_t, n_t) + u_c(y_t - i_t, n_t)(1 - \alpha)e^{z_t}k_t^\alpha n_t^{-\alpha}] = 0$$

After linearization of the Bayesian learning process, these conditions still characterizes the same decision procedure as the original model but are similar in the structure with those under the standard RBC model. The model becomes tractable and implementable for the simulation and estimation.

4 Estimation

4.1 Equations characterizing the equilibrium

To proceed with the numerical analysis, values must be assigned to the parameters in our model. We summarize the key equations that characterize our model as follows, with the corresponding linearized system of equations outlined in the Appendix 3.

Equations regarding the update of beliefs:

$$\begin{aligned} be_z_{t+1} &= \psi[be_z_t + AA(z_t + \alpha\tilde{k}_t - be_z_t - \alpha be_k_t) + BB(\phi_t - be_q_t)] \\ be_q_{t+1} &= \rho[be_q_t + EE(z_t + \alpha\tilde{k}_t - be_z_t - \alpha be_k_t) + FF(\phi_t - be_q_t)] \\ be_k_{t+1} &= \frac{(1-\delta)}{e^{gk}}[be_k_t + CC(z_t + \alpha\tilde{k}_t - be_z_t - \alpha be_k_t) + DD(\phi_t - be_q_t)] + \\ &\quad + \frac{\bar{i}}{ke^{gk}}[be_q_t + EE(z_t + \alpha\tilde{k}_t - be_z_t - \alpha be_k_t) + FF(\phi_t - be_q_t)] + \\ &\quad + \frac{\bar{i}}{ke^{gk}}\tilde{i}_t \end{aligned}$$

Equations regarding the fundamentals of the economy:

$$\begin{aligned} y_t &= e^{z_t}k_t^\alpha n_t^{1-\alpha} \\ k_t e^{gk} &= e^{q_t-1}i_{t-1} + (1 - \delta)k_{t-1} \\ \phi_t &= q_t + v_t \\ z_t &= \psi z_{t-1} + \varepsilon_t \\ q_t &= \rho q_{t-1} + \epsilon_t \end{aligned}$$

Equations regarding the optimization conditions:

$$\begin{aligned} -u_c(y_t - i_t, n_t) + \beta E_t[u_c(y_{t+1} - i_{t+1}, n_{t+1})(\alpha e^{z_{t+1}}k_{t+1}^{\alpha-1}n_t^{1-\alpha} + \frac{1-\delta}{e^{q_{t+1}}})\frac{e^{\frac{q_{t+1}}{\rho}}}{e^{gk}}] &= 0 \\ E_t[u_n(y_t - i_t, n_t) + u_c(y_t - i_t, n_t)(1 - \alpha)e^{z_t}k_t^\alpha n_t^{-\alpha}] &= 0 \end{aligned}$$

There are in total 10 equations, and 10 unknown endogenous variables:

Belief variables: be_z, be_q, be_k

Realized shocks: z, q

Endogenous variables: k, y, n, i, ϕ

Notice that the linearized system we get as the solution is very different from the standard linearized models where certainty equivalence holds. As pointed out in Lorenzoni (2009), under our setup with

information frictions, the covariance matrices of the shocks and of the prediction errors not only represent the volatility of stochastic shocks, but also affect the Bayesian updating process and thus the decisions of economic agents. Therefore both means and variances of the unobservables matter for the solution of the model.

4.2 Calibration strategy

For the standard parameters in this model, we follow parameterization of the literature. The utility function parameters are set to $\beta = 0.95$, $A = 1.75$ as in Aruoba et al. (2006). The technology parameters are set to $\alpha = 0.33$, $\delta = 0.1$ and the detrending parameters $g_k = 1.035$ as in Greenwood et al. (1997).

The rest of the parameters are unique for our model, characterizing the evolution of beliefs and productivity shocks. They are the parameters representing the conditional covariances of the prediction errors AA, BB, CC, DD, EE, FF , and representing the stochastic shock processes $\psi, \rho, \sigma_v, \sigma_\epsilon, \sigma_\varepsilon$. In order to jointly estimate these parameters, including the variance of productivity shocks, we use an unobserved components model with the Kalman filter technique. In this model, ϕ and y are treated as observed measurement variables, whereas z_t, q_t , and \tilde{k}_t are unobserved state variables. Conceptually, this specification captures the same idea of imperfect information and uncertainty as in the theoretical framework.

The state space representation of the unobserved components model includes:

Measurement equations:

$$\begin{pmatrix} \tilde{y}_t - (1 - \alpha)\tilde{n}_t \\ \tilde{\phi}_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \end{pmatrix} * \begin{pmatrix} z_t \\ q_t \\ \tilde{k}_t \end{pmatrix} + \begin{pmatrix} 0 \\ v_t \end{pmatrix}$$

State equations:

$$\begin{pmatrix} z_t \\ q_t \\ \tilde{k}_t \end{pmatrix} = \begin{pmatrix} \psi & 0 & 0 \\ 0 & \rho & 0 \\ 0 & \frac{\bar{i}}{ke^{g_k}} & \frac{(1-\delta)}{e^{g_k}} \end{pmatrix} * \begin{pmatrix} z_{t-1} \\ q_{t-1} \\ \tilde{k}_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{\bar{i}}{ke^{g_k}} \end{pmatrix} \tilde{i}_{t-1} + \begin{pmatrix} \varepsilon_t & 0 & 0 \\ 0 & \epsilon_t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

As shown in the appendix 2, $\begin{pmatrix} AA & BB \\ CC & DD \\ EE & FF \end{pmatrix}$ can be calculated based on the conditional covariance

matrices of the prediction errors, which are by-products of the estimation.

We do the estimation by matching the measurement variables of the Gaussian process from our model with the observed annual data: y, ϕ, n and i . All the original data are extracted from National Income and Product Accounts (NIPA) dataset, between 1954 and 2013. y corresponds to the real GDP. The real series for GDP are obtained by dividing the nominal GDP by the chain-weighted deflator for consumption

of nondurable and services. i corresponds to the real investment. We define investment as the sum of consumers' expenditures on durables and gross private domestic investment. We construct the real series for investment in the same manner as the real GDP. n corresponds to labor input. We measure the labor input by the hours of all persons in the non-farm business sector divided by population. Finally, we use the relative price of capital goods to consumption goods as the signal ϕ of IST q ¹⁵. The relative price corresponds to the ratio of the chain weighted deflators for investment and consumption as defined above. Hence, for investment we rely on deflators for consumption of durable goods and private investment¹⁶.

The calibration results are shown in the following table.

calibration outside the model	$\beta = 0.95, A = 1.75, \alpha = 0.33, \delta_0 = 0.1, g_k = 1.035$
calibration inside the model	$\psi = 0.82, \rho = 0.55, \sigma_v = 0.04, \sigma_\epsilon = 0.07, \sigma_\epsilon = 0.02$
	$AA = 1.16, BB = -0.15, CC = -0.5, DD = 1.5, EE = -0.48, FF = 0.46$

5 Numerical analysis

In this part, we conduct some numerical analyses and investigate how shocks on TFP, IST and sentiment affect the dynamics of an economy. Especially, we propose a possible mechanism that could shed some light on the tech boom and bust cycle which has happened around the year 2000, which cannot be explained by a standard business cycle model.¹⁷

5.1 Uncertainty and fluctuations

In order to illustrate the influences of different shocks in an "uncertain" world, we do the quantitative experiments by calculating the impulse responses of variables to each individual shock. At period 0, the economy is assumed to be initially in a correct steady state where all variables are in the steady state value of zero and the believed output, capital stock and productivity values are accurate and identical to the

¹⁵As pointed out in Justiniano et al. (2011), in the short run, the relative price of investment can be seen as determined mainly by a combination of investment specific shock and market behavior. Here we assume that market behavior is generally reflecting market sentiment-economic agents' attitude towards market condition.

¹⁶In the state space equations, all variables are defined as the log deviation from the steady state. Therefore, before estimating, we need to transform the observed data to the percentage deviation from the trend, so that the resulted data have the same conceptual meaning with $\tilde{y}, \tilde{\phi}, \tilde{n}$ and \tilde{i} .

¹⁷In order to study the responses of macroeconomic variables to different shocks, we first solve the model and get policy functions for the endogenous variables. We express endogenous variables, such as output, consumption, investment, etc., as functions of the state variables. The equations used for the calculation are shown in appendix 3. After we get a log-linearized system of equations characterizing the equilibrium of the model, we continue to solve for the recursive equilibrium law of motion of endogenous variables via the method of undetermined coefficients, as introduced in Uhlig (1999), given the parameter values calibrated in the last section.

actual output, capital stock and productivity values¹⁸. The responses following the shocks are interpreted as the percentage deviation from the steady state, consistent with our specification. Variables, except for beliefs, respond at current period when there is a current shock. The responses of beliefs on technologies, however, reflect changes of the posterior beliefs on current technologies or the prior beliefs for the next period¹⁹.

5.1.1 Total factor productivity shock

Figure 1 depicts the effects on economic variables after one standard deviation shock on TFP. Solid lines show the responses under our framework, whereas dotted lines under standard framework. Both models, with and without information frictions, generate similar persistent responses of macroeconomic variables. When there is a positive shock on TFP, given capital and labor input, which are assumed already determined in the last period, there are immediate increases in output, investment, consumption and labor supply. For example, output increases more than 1% from the steady state. The effect on labor, however, is small. The reason why uncertainty doesn't seem to influence the pattern of the behavior of the economy is because: 1. We assume the economy starts at a correctly perceived steady state. Therefore at least at the beginning, agents' beliefs coincide with the true productivity values and capital stock. 2. Even though agents cannot observe the actual realized values of the TFP shock, their beliefs regarding its value when a shock hits are rather accurate, for there are no other disturbances to the production process and thus agents can derive the shock accurately from the signal, given an accurate belief on the initial capital stock. Since agents choose consumption and investment conditional on their beliefs, the initial accurate beliefs guarantee to some extent that agents behave as optimally as if they can observe the actual current productivity value. Moreover, it turns out that agents' believed TFP values also evolve similarly with the actual TFP series. This implies that, on average, agents' Bayesian learning and updating are surprisingly accurate and close to what is actually happening, if their initial beliefs are accurate. Therefore, a typical TFP shock generates similar persistency of fluctuations under uncertain environment and standard environment, unless the initial beliefs are deviated from the actual values.

From a quantitative point of view, however, the uncertainty does seem to lead to some difference in the magnitude of agents' behavior facing a TFP shock. Even though agents derive the initial shock accurately, they, however, don't acknowledge the fact that their beliefs are accurate. Because agents don't recognize the fact that the actual productivity values are expected, they not only correctly believe there is an increase in current TFP when observing signals, they also wrongly believe there is a slight change in the IST for

¹⁸When doing this analysis, standing outside the studied world, we assume and therefore know that agents' beliefs of economic variables coincide with their true values. The economic agents, however, are not aware of this. Their beliefs, right or wrong, can only become gradually unwound when they receive more signals in the subsequent periods.

¹⁹As shown earlier, there is a linear relation between posterior beliefs on current period technologies and prior beliefs on the same technologies for the next period. Therefore the evolution of these beliefs follow the same dynamics.

the current and subsequent periods, which would not happen if they can observe the productivity values. In the second period, since the believed IST starts to rise, and agents' choices depend on the believed productivity values rather than the actual productivity values, agents invest a little more compared to the standard framework where they know exactly there is no change in IST. As shown in figure 1, the subsequent responses in investment and thus output are stronger in our model with uncertainty than those without uncertainty.

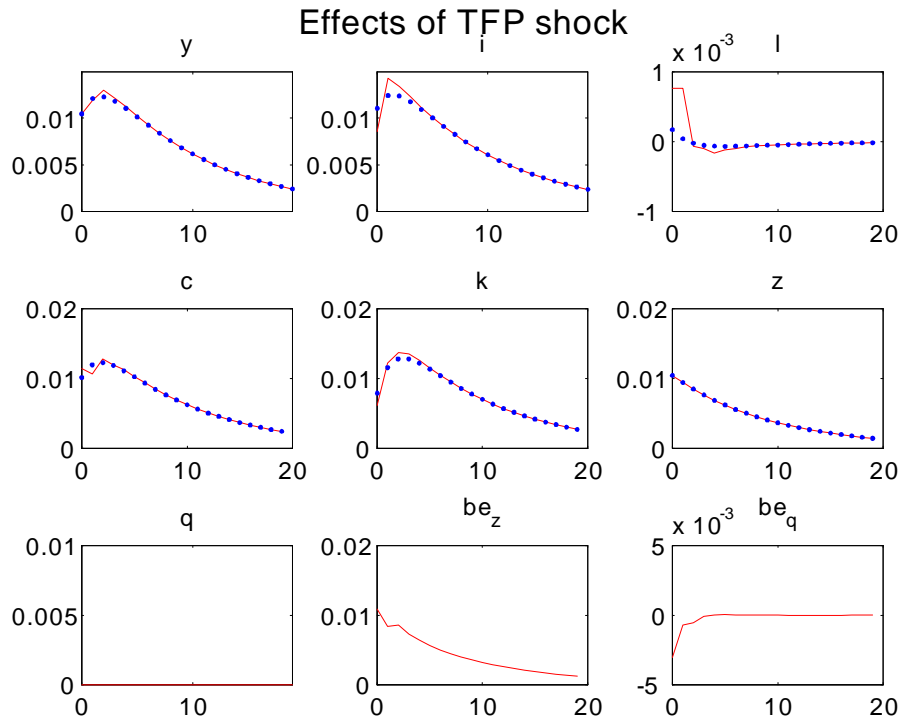


Figure 1: Effects of TFP shock under "uncertain" and "certain" world

5.1.2 Investment specific technology shock

In Figure 2, the solid lines depict the effects on economic variables after one standard deviation shock on IST in our framework, whereas the dotted lines for the standard framework. Similarly, the impulse responses show the evolution of economic variables are similar with and without uncertainty, for the same reason explained above on the TFP shock: the initial beliefs of TFP and capital coincide with the their actual values and the evolution of beliefs regarding IST follows similar path with that of the actual IST. However, the beliefs on IST are different from the actual values not surprisingly, due to the fact that agents can only observe a noisy signal for IST, whereas they observe a non-noisy signal for TFP in the sense that no other disturbances influence the beliefs on TFP but market sentiment affects the beliefs

on IST. As shown in figure 2, the believed IST is smaller than the actual IST, different from the TFP situation where the believed TFP is almost the same as the actual TFP. As a result, investment under uncertainty is smaller than without uncertainty, because, again, agents' decisions are influenced by the believed productivity values rather than the realized values. A possible explanation for this under-reaction is that, when agents observe the increased signal ϕ which indeed totally reflects the increased IST, they might interpret it as a combination of the change in the disturbance and in the IST value. Consequently, they tend to behave conservatively which leads to a slightly lower investment for each period over the subsequent time. As a result, the output and consumption are also slightly lower. This result implies that information frictions could affect to some extent the level of investment and output in the transition period of an economy, leading to a potential loss of economic efficiency.

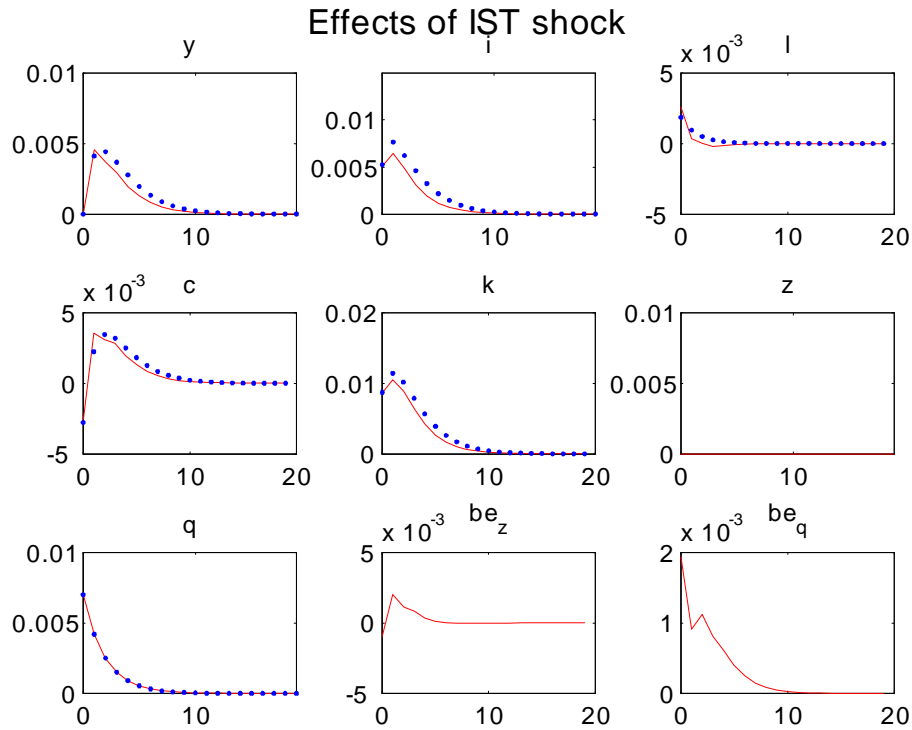


Figure 2: Effects of IST shock under "uncertain" and "certain" world

5.2 Sentiment shock and boom-bust cycles

Our framework with information frictions and uncertainty can also shed some light on the scenario where a tech boom and bust occurs. Figure 3 shows the effects on economic variables after a one standard deviation (positive) shock on the market sentiment, an example of undue optimism. Following an optimism shock on the investment technology, investment increases immediately. Intuitively, agents know that the observed

signal is a combination of IST and disturbance, but contribute the increase of ϕ to at least some increase in IST, even though the actual series remain unchanged. Due to the increase in the believed value of IST, agents consequently invest more. In the next period, an increase of the capital stock increases the output which leads to an increase of consumption and decrease of labor input. Wealth effect dominates in this case. At the mean time, agents observe that ϕ returns to the steady state level, and thus they conjecture that their beliefs of increase in IST for the last period are likely to be wrong. Therefore, they adjust their previous beliefs regarding both IST and TFP: their perceived IST declines and their perceived TFP rises. As a result, investment decreases to below the steady state level. Since both labor input and investment decline, the subsequent output also declines. This is similar as what happens during a tech boom and bust cycle. An increase in market sentiment, for example, an optimistic expectation of the influence of a new technology may increase investment. Later, when agents find out gradually that it is just a tech bubble and no fundamental productivity actually advances, they adjust their beliefs and decisions to the opposite direction which might create a bust.

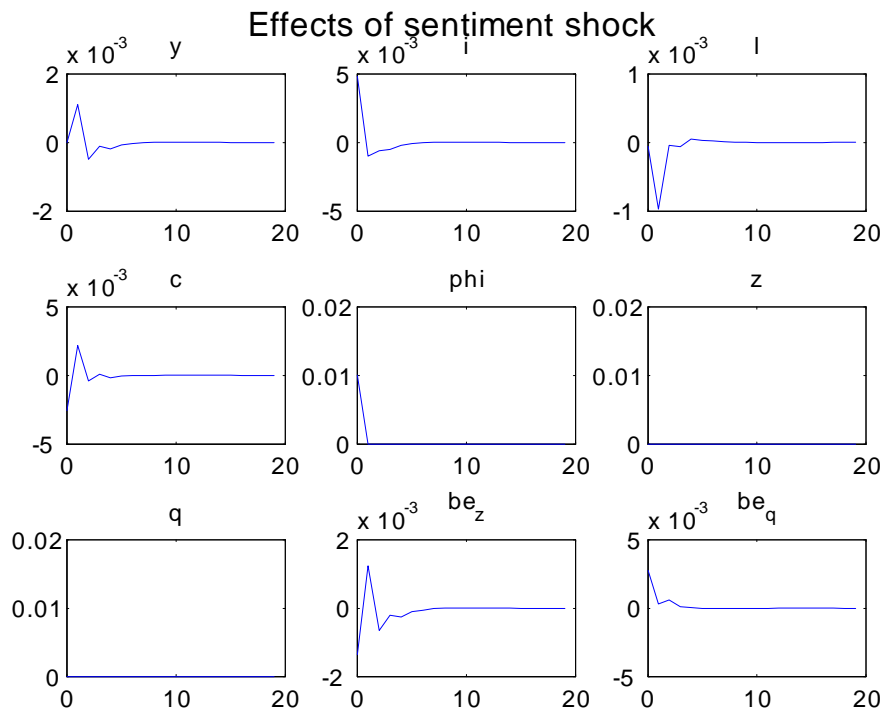


Figure 3: Effects of sentiment shock under "uncertain" world

The persistence of the response of investment due to this sentiment shock is not strong. This is because the assumption that signals for IST are not persistent, in the sense that after one period deviation, it returns to the steady state immediately, happening in the second period. This is a meaningful indicator

for agents who are aware of the process of the signal. However, in a tech boom and bust cycle such as the one around 2000, it takes years for the signals to uncover the actual levels of the productivity, and therefore a persistent misperception of beliefs is more likely, which would lead to more persistent responses of investment and output. Nonetheless, this model successfully generates the tech boom and bust cycle, which a standard business cycle model fails to produce.

Compared to the impulse responses due to the TFP shock, it is not surprising that on average, the fluctuations due to the sentiment shock is smaller. One possible reason is that, the experiment we do relies on the values calibrated earlier, where we use the annual data between 1954 and 2013 for the estimation. Over this period, the average standard deviation of the sentiment shock is relatively small. Nevertheless, it still generates sizable effects on economic variables. It is likely that during some periods, such as the Great Recession when market sentiment changed dramatically, the volatility of this shock is much bigger, and then the influence on the economic dynamics would be more significant. This proposes another mechanism to explain economic fluctuations, that some recessions may not result from large negative shocks on technologies, but from incorrect beliefs regarding these technologies due to the errors agents make when they filter information from the observed signals, and one possible source of the incorrect beliefs stems from the market attitude towards the unobserved economic fundamentals.

5.3 A test of sentiment shocks

We have shown above that sentiment shock is a potential source of a boom and bust cycle, which the classical productivity shocks cannot generate. As shown in the literature, under standard RBC model with only one type of productivity (TFP) and imperfect information, the key difference between sentiment shock and fundamental productivity shock is that, when sentiment shock happens, agents tend to over react; when fundamental shock happens, they go to the opposite direction. Lorenzoni (2009) uses a bivariate semi-structural VAR to identify technology and non technology noise shocks. Enders et al. (2013) uses a VAR to identify optimism shocks by combining long run restrictions with the short run properties of technology and optimism shocks. Both of them use the survey data from the Survey of Professional Forecasters and found that a technology shock results in a positive forecast error, which is the discrepancy between the realized output and the expected output for the same period given all available information, and an optimism shock results in a negative forecast error. However, as mentioned in Lorenzoni (2009), the result could be different under richer identification strategies. In our paper, when there is a sentiment shock, agents' behavior is not only affected by the confusion originating from the noisy signal which is a combination of IST and the sentiment, but also affected by the way how they update beliefs on the TFP shocks through Bayesian learning process.

For illustration, figure 4 plots the forecast error predicted by our model, generated by TFP shock, IST shock and sentiment shock respectively. Intuitively, when there is an increase in TFP, since agents can

learn it quickly from the production process as explained before, the only sizable forecast error is due to the fact that initially the believed output is in the steady state of zero while the realized output reflects the increase in TFP, which leads to a deviation from the steady state and therefore a forecast error. What are more interesting are the forecast errors due to IST shock and sentiment shock. When there is an increase in IST, which is supposed to affect the effective capital stock in the following period, agents observe a higher value of signal ϕ . Since agents behave conservatively, the realized output in the following period is higher than the believed output, resulting in a positive forecast error. In the subsequent periods, as shown in figure 2, since the believed evolution of IST becomes very similar with that of the true IST, the forecast error reduces. The responses due to TFP and IST are consistent with the predictions from Lorenzoni (2009) and Enders et al. (2013). However, we didn't get an immediate decrease in the forecast error due to a sentiment shock; but it increases initially. This different prediction from our model is due to the fact that, when sentiment shock hits, agents not only expect an increase in IST, but also a decrease in TFP. Therefore, the sentiment shock on the investment technology in our model behaves as an optimism shock on IST, and a pessimism shock on TFP. The latter has a dominant effect and leads to a lower expected output than the realized output, for in reality no change ever happens to TFP. In the following period, when the believed TFP increases, the economy behaves as if there is an optimism shock on TFP and thus generates a negative forecast error. This dynamics fits very well with the empirical evidence regarding optimism/pessimism shocks studied in Enders et al. (2013), which implies that our model is capable to capture the transmission mechanism of influence of sentiment shock and could shed some light on the ongoing debate regarding the role of "animal spirits" or "market sentiment" as a driving source of business cycles.

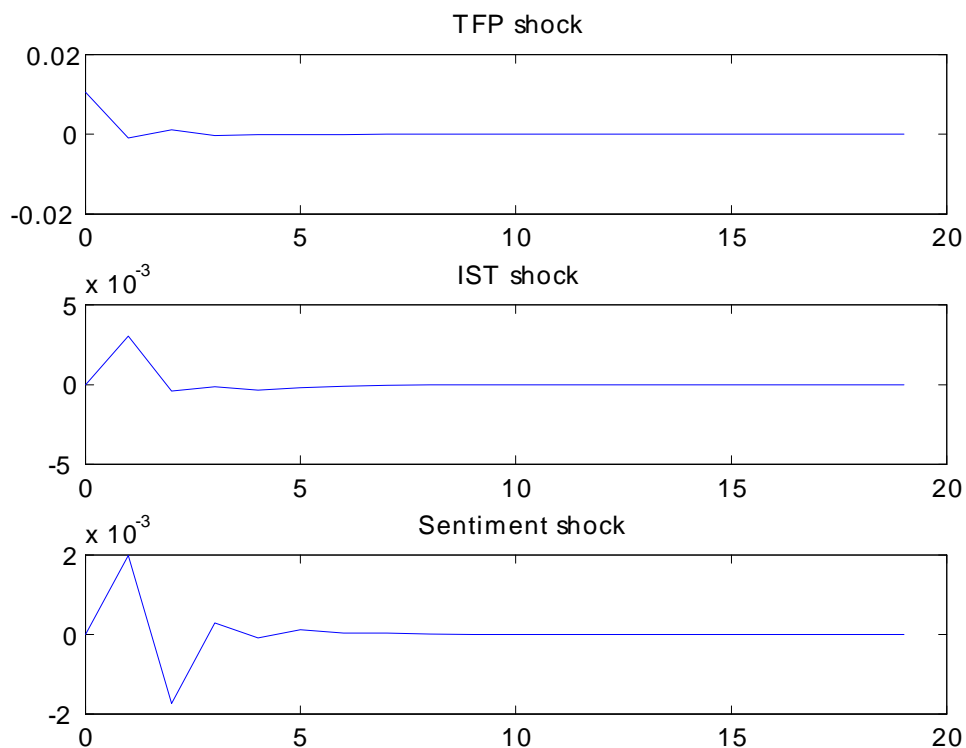


Figure 4: Forecast error due to different shocks.

6 Conclusion

In this paper we propose a general dynamic stochastic model with uncertain productivity values and Bayesian learning to study the effects of technology shocks and sentiment shock on macroeconomic fluctuations. We build the model on a standard real business cycle framework, but augment it with an environment where agents cannot fully observe the realized technology shocks in the economy; Instead, they have prior beliefs on these values, receive noisy signals about the shocks, and update prior beliefs to get posterior beliefs according to Bayes' rule. They make decisions of consumption, investment and labor based on their optimally updated beliefs rather than the realized shocks. The model generates similar patterns of responses of macroeconomic variables to the total factor productivity shocks and investment specific technology shocks. However, it shows some insightful predictions on the business cycle fluctuations. One implication is that the uncertainty world with information friction doesn't generate more volatile movement of variables, but the magnitude of the responses of these variables to shocks tend to be different. Furthermore, a non persistent market sentiment shock on the investment specific technology is likely to change agents' behavior persistently, even though the underlying fundamentals governing the

economy remain unchanged. This is because the sentiment shock not only affects the beliefs on investment specific technology, but also believed neutral productivity, and the latter can change agents' behavior as well. The responses to the sentiment shock mirror tech boom and bust cycles such as the recession in US around the year 2000, which standard RBC models fail to generate.

This general framework can also be used to study the effects of twisted beliefs on the economy. The idea of twisted beliefs are detailed in a seminal paper by Cecchetti, Lam and Mark (2000) who showed that if agents' expectations about the stochastic shocks that drive the economy are "twisted", an otherwise standard asset pricing model could match asset pricing facts more plausible. In our simulation we assume that economy initially is in an untwisted steady state in the sense that the perceived technologies, output and capital are the same as their corresponding fundamentals. It would be interesting to study the impact of twisted beliefs on macroeconomic variables by assuming the economy starts with believed capital or techniques that differ from the true ones. This exercise will help to shed some light on the business cycles where uncertainty serves as stimulus, such as the recent financial crisis.

Finally, our model keeps the essential elements of a medium scale DSGE framework while not including all the factors. We can simply introduce other features to the model such as the utilization rate and adjustment cost of the capital, and recalibrate/estimate the model using quarterly data. In particular, the study of a decentralized framework where the relative price of investment is endogenously determined in the uncertainty world would be very interesting. This can provide us with some insight about the measurement of the investment specific technology and contributes to the debate of this topic. On the other hand, we can also conduct an empirical test on the impacts of sentiment shocks. As stressed by Blanchard et al. (2009), it is only possible and feasible to estimate sentiment shocks if one resorts to fully specified general equilibrium models. We will use an identification strategy that is a combination of the structural implications from our general equilibrium framework and the feature of the dataset we will use, the Survey of Professional Forecasters. We think this exercise would provide other plausible implications for business cycle mechanisms.

7 References

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8 Appendix

8.1 Appendix 1: Calculation of update of beliefs

For the IST and TFP, given their posteriors beliefs, and their known true processes of evolution that follow AR(1) processes:

$$q_{t+1} = \rho q_t + \epsilon_{t+1}$$

$$z_{t+1} = \psi z_t + \varepsilon_{t+1}$$

We can derive the prior belief of TFP for period $t + 1$, g_{t+1}^Z , and the prior belief of IST for period $t + 1$, g_{t+1}^Q by conjugate distribution calculation.

For the capital, given the posterior belief, and the known true process of evolution:

$$k_{t+1}e^{gk} = e^{q_t}i_t + (1 - \delta)k_t$$

We can calculate the cumulative density function for a random variable $K_{t+1} = \frac{(1-\delta)K_t + e^{Q_t}i_t}{e^{gk}} \leq k'$ as

$$\begin{aligned} F_{t+1}^K(k'|m_t^K, m_t^q) &= Prob\left(\frac{(1-\delta)K_t + e^{Q_t}i_t}{e^{gk}} \leq k'\right) \\ &= \int \int_{\frac{k(1-\delta) + i_t e^q}{e^{gk}} \leq k'} m_t^K m_t^q dk dq \\ &\quad \text{where } k \text{ is the realized value of } K_t, q \text{ is the realized value of } Q_t \\ &= \int_{q=0}^{\infty} \int_{k=0}^{\frac{e^{gk}k' - i_t e^q}{(1-\delta)}} m_t^K m_t^Q dk dq \\ &= \int_{k=0}^{\infty} \int_{q=0}^{\log\left(\frac{e^{gk}k' - k(1-\delta)}{i_t}\right)} m_t^K m_t^Q dk dq \end{aligned}$$

and therefore the prior belief of capital for period $t + 1$, g_{t+1}^K , the probability density function is

$$\begin{aligned} g_{t+1}^K(k') &= \int_{k=0}^{\infty} \frac{e^{gk}}{e^{gk}k' - k(1-\delta)} m_t^K m_t^Q \left(\log\left(\frac{e^{gk}k' - k(1-\delta)}{i_t}\right)\right) dk \\ &= \frac{e^{gk}}{(1-\delta)} \int_{q=0}^{\infty} m_t^K \left(\frac{e^{gk}k' - i_t e^q}{(1-\delta)}\right) m_t^Q dq \end{aligned}$$

Meantime, we can also derive an expectation for the next period signals (the likelihood of the signals

conditional on prior distributions of unobserved variables, $g(y_{t+1}, \phi_{t+1}|z, q, k)$). That is, for next period output, the cumulative density function is

$$\begin{aligned}
F_{t+1}^Y(y'|g_{t+1}^Z, g_{t+1}^K, n') &= Prob(e^{Z_{t+1}} K_{t+1}^\alpha n_{t+1}^{1-\alpha} \leq y') \\
&= \int \int_{e^{z'} k'^\alpha n'^{1-\alpha} \leq y'} g_{t+1}^K g_{t+1}^Z dk' dz' \\
&= \int_{z'=0}^{\infty} \int_{k'=0}^{\left(\frac{y'}{e^{z'} n'^{1-\alpha}}\right)^{1/\alpha}} g_{t+1}^K g_{t+1}^Z dk' dz' \\
&= \int_{k'=0}^{\infty} \int_{z'=0}^{\log\left(\frac{y'}{k'^\alpha n'^{1-\alpha}}\right)} g_{t+1}^K g_{t+1}^Z dk' dz'
\end{aligned}$$

and therefore the prior belief of output for period $t + 1$, g_{t+1}^Y , the probability density function is

$$\begin{aligned}
g_{t+1}^Y(y') &= \frac{1}{\alpha} y'^{\frac{1}{\alpha}-1} n'^{\frac{\alpha-1}{\alpha}} \int_{z'=0}^{\infty} \left(\frac{1}{e^{z'}}\right)^{1/\alpha} g_{t+1}^K \left(\left[\frac{y'}{e^{z'} n'^{1-\alpha}}\right]^{1/\alpha}\right) g_{t+1}^Z dz' \\
&= \int_{k'=0}^{\infty} \frac{k'^\alpha n'^{1-\alpha}}{y'} g_{t+1}^K g_{t+1}^Z \left(\log\left(\frac{y'}{k'^\alpha n'^{1-\alpha}}\right)\right) dk'
\end{aligned}$$

8.2 Appendix 2: Calculation of evolution of linearized beliefs

The calculation can be done through the Kalman filter as follows.

Measurement update: Since $\tilde{y}_t = z_t + \alpha \tilde{k}_t + (1 - \alpha) \tilde{n}_t$, $\tilde{\phi}_t = q_t + v_t$, the conditional vector

$$\begin{pmatrix} z_t \\ \tilde{k}_t \\ q_t \\ \tilde{y}_t - (1 - \alpha) \tilde{n}_t \\ \tilde{\phi}_t \end{pmatrix} | Y_{t-1}, \Phi_{t-1}, \tilde{n}_t$$

is Gaussian, with mean and variance:

$$\left[\begin{array}{c} \begin{pmatrix} M_t^Z \\ M_t^{\tilde{K}} \\ M_t^Q \end{pmatrix} \\ \begin{pmatrix} M_t^Z + \alpha M_t^{\tilde{K}} \\ M_t^Q \end{pmatrix} \end{array} \right],$$

$$\left[\begin{array}{c} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{pmatrix}, \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{pmatrix}, \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v \end{pmatrix} \end{array} \right]$$

To compute $\begin{pmatrix} z_t \\ \tilde{k}_t \\ q_t \end{pmatrix} | Y_t, \Phi_t, \tilde{n}_t = \begin{pmatrix} z_t \\ \tilde{k}_t \\ q_t \end{pmatrix} | y_t, Y_{t-1}, \phi_t, \Phi_{t-1}, \tilde{n}_t$, which is the posterior belief on z_t, \tilde{k}_t and q_t , we apply the formula for conditional expectation of Gaussian random variables, with everything preconditioned on Y_t and Φ_t . It follows that $\begin{pmatrix} z_t \\ \tilde{k}_t \\ q_t \end{pmatrix} | Y_t, \Phi_t, \tilde{n}_t$ is Gaussian, with mean

$$\begin{aligned}
\begin{pmatrix} \mu_t^Z \\ \mu_t^{\tilde{K}} \\ \mu_t^Q \end{pmatrix} &= E \left(\begin{bmatrix} z_t \\ \tilde{k}_t \\ q_t \end{bmatrix} \middle| Y_t, \Phi_t, \tilde{n}_t \right) \\
&= \begin{bmatrix} M_t^Z \\ M_t^{\tilde{K}} \\ M_t^Q \end{bmatrix} + \begin{bmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_{t,t}^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ 0 & 1 \end{bmatrix} \\
&\quad \left[\begin{pmatrix} 1 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_{t,t}^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v \end{pmatrix} \right]^{-1} \\
&\quad \begin{bmatrix} \tilde{y}_t - (1 - \alpha)\tilde{n}_t - M_t^Z - \alpha M_t^{\tilde{K}} \\ \tilde{\phi}_t - M_t^Q \end{bmatrix} \\
&= \begin{bmatrix} M_t^Z \\ M_t^{\tilde{K}} \\ M_t^Q \end{bmatrix} + \begin{pmatrix} AA_t & BB_t \\ CC_t & DD_t \\ EE_t & FF_t \end{pmatrix} \begin{bmatrix} \tilde{y}_t - (1 - \alpha)\tilde{n}_t - M_t^Z - \alpha M_t^{\tilde{K}} \\ \tilde{\phi}_t - M_t^Q \end{bmatrix}
\end{aligned}$$

and covariance

$$\begin{aligned}
\begin{pmatrix} \sigma_t^Z & \sigma_t^{\tilde{K},Z} & \sigma_t^{Z,Q} \\ \sigma_t^{\tilde{K},Z} & \sigma_t^{\tilde{K}} & \sigma_t^{\tilde{K},Q} \\ \sigma_t^{Z,Q} & \sigma_t^{\tilde{K},Q} & \sigma_t^Q \end{pmatrix} &= cov \left(\begin{bmatrix} z_t \\ \tilde{k}_t \\ q_t \end{bmatrix} \middle| Y_t, \Phi_t, \tilde{n}_t \right) \\
&= \begin{bmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_{t,t}^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{bmatrix} - \begin{bmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_{t,t}^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ 0 & 1 \end{bmatrix} \\
&\quad \left[\begin{pmatrix} 1 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_{t,t}^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v \end{pmatrix} \right]^{-1} \\
&\quad \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_{t,t}^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{bmatrix}
\end{aligned}$$

where

$$\begin{pmatrix} AA_t & BB_t \\ CC_t & DD_t \\ EE_t & FF_t \end{pmatrix} = \begin{bmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_{t,t}^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{pmatrix} 1 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{\tilde{K},Z} & \Sigma_t^{Z,Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K}} & \Sigma_t^{\tilde{K},Q} \\ \Sigma_{t,t}^{Z,Q} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^Q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v \end{pmatrix} \right]^{-1}$$

$\begin{pmatrix} AA_t & BB_t \\ CC_t & DD_t \\ EE_t & FF_t \end{pmatrix}$ converges to a constant matrix very quickly as explained later. Therefore we will use

the converged values of $\begin{pmatrix} AA & BB \\ CC & DD \\ EE & FF \end{pmatrix}$ for the calculation and simulation.

Time update: Recall that $z_{t+1} = \psi z_t + \varepsilon_{t+1}$. $q_{t+1} = \rho q_t + \epsilon_{t+1}$, $\tilde{k}_{t+1} = \frac{(1-\delta)}{e^{g_k}} \tilde{k}_t + \frac{\bar{i}}{ke^{g_k}} \tilde{i}_t + \frac{\bar{i}}{ke^{g_k}} q_t$. Furthermore, z_{t+1} and ε_{t+1} are independent, and q_{t+1} and ϵ_{t+1} are independent given Y_t, Φ_t . Therefore, posterior beliefs of the means are

$$\begin{bmatrix} M_{t+1}^Z \\ M_{t+1}^{\tilde{K}} \\ M_{t+1}^Q \end{bmatrix} = E \left(\begin{bmatrix} z_{t+1} \\ \tilde{k}_{t+1} \\ q_{t+1} \end{bmatrix} \middle| Y_t, \Phi_t, \tilde{n}_t \right) = \begin{bmatrix} \psi & 0 & 0 \\ 0 & \frac{(1-\delta)}{e^{g_k}} & \frac{\bar{i}}{ke^{g_k}} \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} \mu_t^Z \\ \mu_t^{\tilde{K}} \\ \mu_t^Q \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\bar{i}}{ke^{g_k}} \tilde{i}_t \\ 0 \end{bmatrix}$$

posterior beliefs of the covariances are

$$\begin{bmatrix} \Sigma_{t+1}^Z & \Sigma_{t+1}^{\tilde{K},Z} & \Sigma_{t+1}^{Z,Q} \\ \Sigma_{t+1}^{\tilde{K},Z} & \Sigma_{t+1}^{\tilde{K}} & \Sigma_{t+1}^{\tilde{K},Q} \\ \Sigma_{t+1}^{Z,Q} & \Sigma_{t+1}^{\tilde{K},Q} & \Sigma_{t+1}^Q \end{bmatrix} = cov \left(\begin{bmatrix} z_{t+1} \\ \tilde{k}_{t+1} \\ q_{t+1} \end{bmatrix} \middle| Y_t, \Phi_t, \tilde{n}_t \right)$$

$$= \begin{bmatrix} \psi & 0 & 0 \\ 0 & \frac{(1-\delta)}{e^{g_k}} & \frac{\bar{i}}{ke^{g_k}} \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} \sigma_t^Z & \sigma_t^{\tilde{K},Z} & \sigma_t^{Z,Q} \\ \sigma_t^{\tilde{K},Z} & \sigma_t^{\tilde{K}} & \sigma_t^{\tilde{K},Q} \\ \sigma_t^{Z,Q} & \sigma_t^{\tilde{K},Q} & \sigma_t^Q \end{bmatrix} \begin{bmatrix} \psi & 0 & 0 \\ 0 & \frac{(1-\delta)}{e^{g_k}} & 0 \\ 0 & \frac{\bar{i}}{ke^{g_k}} & \rho \end{bmatrix} + \begin{bmatrix} \sigma_\varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_\epsilon \end{bmatrix}$$

The conditional covariance matrices do not depend on the measurement \tilde{y}_t and $\tilde{\phi}_t$. They can therefore be computed in advance, given the noise variances. Eventually and quickly they converge to some constants.

We can simplify this process, by plugging in the posterior beliefs to the next period prior beliefs, and

we can get

$$\begin{aligned} be_z_{t+1} &\equiv M_{t+1}^Z = \psi \mu_t^Z \\ &= \psi [M_t^Z + AA(\tilde{y}_t - (1 - \alpha)\tilde{n}_t - M_t^Z - \alpha M_t^{\tilde{K}}) + BB(\tilde{\phi}_t - M_t^Q)] \end{aligned}$$

$$\begin{aligned} be_q_{t+1} &\equiv M_{t+1}^Q = \rho \mu_t^Q \\ &= \rho [M_t^Q + EE(\tilde{y}_t - (1 - \alpha)\tilde{n}_t - M_t^Z - \alpha M_t^{\tilde{K}}) + FF(\tilde{\phi}_t - M_t^Q)] \end{aligned}$$

$$\begin{aligned} be_k_{t+1} &\equiv M_{t+1}^{\tilde{K}} = \frac{(1 - \delta)}{e^{g_k}} \mu_t^{\tilde{K}} + \frac{\bar{i}}{\bar{k}e^{g_k}} \mu_t^Q + \frac{\bar{i}}{\bar{k}e^{g_k}} \tilde{i}_t \\ &= \frac{(1 - \delta)}{e^{g_k}} [M_t^{\tilde{K}} + CC(\tilde{y}_t - (1 - \alpha)\tilde{n}_t - M_t^Z - \alpha M_t^{\tilde{K}}) + DD(\tilde{\phi}_t - M_t^Q)] \\ &\quad + \frac{\bar{i}}{\bar{k}e^{g_k}} [M_t^Q + EE(\tilde{y}_t - (1 - \alpha)\tilde{n}_t - M_t^Z - \alpha M_t^{\tilde{K}}) + FF(\tilde{\phi}_t - M_t^Q)] \\ &\quad + \frac{\bar{i}}{\bar{k}e^{g_k}} \tilde{i}_t \end{aligned}$$

These beliefs play an important rule in agents' decision problem.

8.3 Appendix 3: Linearized equations characterizing the equilibrium

Since the evolution of beliefs are in log-linearized form, it will be convenient if the FOCs and budget constraints are also in log-linearized form. Thus we transform the equations before doing the simulation into the following linearized system of equations, after assuming the utility function follows $U = \log c_t - An_t$:

$$\begin{aligned} be_z_{t+1} &= \psi [be_z_t + AA(z_t + \alpha \tilde{k}_t - be_z_t - \alpha be_k_t) + BB(\tilde{\phi}_t - be_q_t)] \\ be_q_{t+1} &= \rho [be_q_t + EE(z_t + \alpha \tilde{k}_t - be_z_t - \alpha be_k_t) + FF(\tilde{\phi}_t - be_q_t)] \\ be_k_{t+1} &= \frac{(1 - \delta)}{e^{g_k}} [be_k_t + CC(z_t + \alpha \tilde{k}_t - be_z_t - \alpha be_k_t) + DD(\tilde{\phi}_t - be_q_t)] + \\ &\quad + \frac{\bar{i}}{\bar{k}e^{g_k}} [be_q_t + EE(z_t + \alpha \tilde{k}_t - be_z_t - \alpha be_k_t) + FF(\tilde{\phi}_t - be_q_t)] + \frac{\bar{i}}{\bar{k}e^{g_k}} \tilde{i}_t \\ \tilde{\phi}_t &= q_t + v_t \\ z_t &= \psi z_{t-1} + \varepsilon_t \\ q_t &= \rho q_{t-1} + \epsilon_t \\ \tilde{y}_t &= z_t + \alpha \tilde{k}_t + (1 - \alpha)\tilde{n}_t \\ \tilde{k}_t &= \frac{(1 - \delta)}{e^{g_k}} \tilde{k}_{t-1} + \frac{\bar{i}}{\bar{k}e^{g_k}} \tilde{i}_{t-1} + \frac{\bar{i}}{\bar{k}e^{g_k}} q_{t-1} \\ be_y_t &= be_z_t + \alpha be_k_t + (1 - \alpha)\tilde{n}_t \end{aligned}$$

$$\begin{aligned}
\tilde{y}_t &= \frac{\bar{c}}{\bar{y}}\tilde{c}_t + \frac{\bar{i}}{\bar{y}}\tilde{i}_t \\
be_y_t &= \frac{\bar{c}}{\bar{y}}be_c_t + \frac{\bar{i}}{\bar{y}}\tilde{i}_t \\
0 &= \tilde{c}_t - be_c_{t+1} + be_r_{t+1} + \frac{be_q_{t+1}}{\rho} - be_q_{t+1} \\
0 &= \tilde{c}_t + be_w_t \\
be_r_{t+1} &= (1 - \beta(1 - \delta))(be_z_{t+1} + be_q_{t+1} + (\alpha - 1)be_k_{t+1} + (1 - \alpha)\tilde{n}_{t+1}) \\
be_w_t &= be_z_t + \alpha be_k_t - \alpha\tilde{n}_t
\end{aligned}$$

8.4 Appendix 4: Decentralized economy

In this framework of a decentralized economy, households and perfectly competitive firms are assumed not be able to observe TFP, IST and the effective capital. Households own certain quantity of capital which is observable, but the evolution of it is not known nor a necessary information set for households' choices of investment. On the other hand, the effective capital that matters for how much output can be generated is not observed due to the unobservability of IST, even though the evolution of it is known and determines households' choices of investment. Moreover, it is common knowledge that both the quantity of the observed capital and the unobserved effective capital are positively related to investment

At the beginning of each period t , before observing signals, households rent capital and provide labor to firms. Households and firms acknowledge that the realized output y_t will depend on the realized but unobserved TFP z_t , unobserved effective capital k_t , and labor n_t chosen at the beginning of period t , but not necessarily on the observed quantity of capital qu_k_t . qu_k_t is a measure of the capital stock, for example, in physical units, and the number of efficient units is known to be embodied in them even if that number is not known. The relationship between k_t and qu_k_t can be expressed using a function $qu_k_t = f_t(k_t)$. The functional form is changing over time and not known to the public, but it is known that a higher k is associated with a higher qu_k , i.e., $f_t(\cdot)$ is a monotonically increasing function with unknown specific form.

The capital rent r_t and wage w_t is paid after households and firms' observing y_t . w_t and r_t are paid based on the contribution of labor and effective capital to production. They are exogenous to each firm and household when they make decisions, but are determined in the equilibrium.

Household and firms share the same Bayesian learning process, in the sense that they observe the same signals, have the same information set and update their beliefs regarding economic fundamentals based on the same learning mechanism. The problem of households and firms are as follows:

8.4.1 Firms

At each period t , a continuum of perfectly competitive firms hire labor and rent capital from households to produce. Firms pay for the input factors after receiving the output. A representative firm's problem is:

$$\max_{n_t, qu_k_t} E_t(y_t - r_t qu_k_t - w_t n_t) = E_t(e^{z_t} k_t^\alpha n_t^{1-\alpha} - r_t qu_k_t - w_t n_t)$$

In equilibrium, after producing and receiving the output, firms pay households at the *pre-agreed* price for the observed capital and labor :

$$r_t = \alpha y_t / (qu_k_t)$$

$$w_t = (1 - \alpha) y_t / n_t$$

The functional form of these rates are known to households; the realized values are not known.

8.4.2 Households

Household makes the following decisions:

1. Before observing signals, optimally choose labor.
2. After observing signals, optimally update beliefs using Bayes' rule.
3. After updating beliefs, optimally choose investment.

$$\begin{aligned} V(g_t^Z, g_t^Q, g_t^K(i_{t-1})) &= E_t \left\{ \max_{n_t, i_t} u(c_t, n_t) + \beta e_{-} V(g_{t+1}^Z, g_{t+1}^Q, g_{t+1}^K(i_t)) \right\} \\ & \text{s.t.} \\ c_t + i_t &= r_t qu_k_t + w_t n_t = y_t \\ & \text{and evolution of beliefs} \end{aligned}$$

The first constraint is the budget constraint. The lefthand side is the expenditure of the household, which includes consumption and investment. The righthand side is the realized income of the household, composed of the income from renting capital and supplying labor.

8.4.3 Equilibrium

Firms' optimization conditions:

$$r_t = \alpha y_t / (qu_k_t)$$

$$w_t = (1 - \alpha) y_t / n_t$$

Households' optimization conditions are as before:

Optimal condition for i_t

$$-u_c(y_t - i_t, n_t) + \beta E_t[u_c(y_{t+1} - i_{t+1}, n_{t+1})(\alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + \frac{1-\delta}{e^{g_{t+1}}} \frac{e^{\frac{g_{t+1}}{\rho}}}{e^{g_k}})] = 0$$

Optimal condition for n_t

$$E_t[u_n(y_t - i_t, n_t) + u_c(y_t - i_t, n_t)(1 - \alpha)e^{z_t} k_t^\alpha n_t^{-\alpha}] = 0$$

The equilibrium is the same as the that of the social planner's problem in the main text. However, with the setup of a decentralized economy, our model can be extended to a more rich framework. We elaborate and extend this idea in another paper where we have a new Keynesian core.