

Household wealth and portfolio choice when tail events are salient*

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Abstract

Robust experimental evidence of expected utility violations establishes that individuals overweight utility from low probability gains and losses. These findings motivated development of rank dependent utility (RDU). We characterize optimal RDU portfolio choice when facing dynamic, binomial returns. Our calibration shows optimal terminal wealth has significant downside protection, upside exposure, and a lottery component. Optimal dynamic trades require higher risky share after good returns and, possibly, nonparticipation when returns are poor. RDU portfolios counterfactually exhibit both excessive elasticity of risky share to wealth and momentum rebalancing. Our results suggest a puzzling inconsistency between behavior inside and outside the laboratory.

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1 Introduction

The expected utility (EU) revolution in economics was set in motion by von Neumann and Morgenstern's (1944) influential book, and even today most models in finance utilize this framework.¹ An ongoing debate on the shortcomings of EU as a theory of choice, however, dates back nearly as far as the book's publication.² A major challenge to the empirical relevance of EU comes in the form of the deceptively simple paradoxes proposed by Allais (1953). In the *common ratio problem*, for example, Allais predicted that most people will prefer a certain gain of \$1 million to an 80% chance of a \$5 million gain, but the same people will also tend to prefer a 4% probability of a \$5 million gain to a 5% chance of a \$1 million gain.³ Allais' paradoxes formed the basis of later experimental work that rejected EU as a model of behavior in the lab, and deeper investigation led to the hypothesis that individuals weight outcome utilities using *decision weights* that are different from outcome probabilities. Robust experimental evidence suggests decision weights that overemphasize small probability gains and losses are common,⁴ and to model this behavior without violating stochastic dominance Quiggin (1982) introduced the concept of a *probability weighting function* that transforms the cumulative distribution of risky payouts to produce decision weights. This rank dependent utility (RDU) model generalizes EU and can be constructed to describe choices consistent with the common-ratio effect of Allais (Prelec (1998)). Preferences that incorporate probability weighting as in RDU have emerged as the dominant tool for experimental design and interpretation in laboratory settings studying choice under uncertainty.⁵

¹A growing literature explores the implications of alternative preferences for asset pricing, corporate finance, and macroeconomics. For current reviews, see Barberis (2013), Baker and Wurgler (2011), and Backus, Routledge, and Zin (2005).

²Early work questioning whether EU is a valid model for decision making even in simple settings with objective risks are Preston and Baratta (1948) and Allais (1953). See Camerer (1995) for a more comprehensive history.

³EU applied to the first choice implies $v(w+1) - 0.8v(w+5) - .2v(w) > 0$, where w is initial wealth (millions) and v is the cardinal felicity function. Linearity of EU in probability implies that the bet with the \$1 million payoff should also be selected in the second choice because $0.05v(w+1) + .95v(w) - 0.04v(w+5) - .96v(w) = \frac{1}{20}(v(w+1) - .8v(w+5) - .2v(w)) > 0$. Hence, typical behavior when faced with the common ratio problem violates EU.

⁴See, for example, MacCrimmon and Larsson (1979), Gonzalez and Wu (1999), and Abdellaoui, Vossmann, and Weber (2005). The reviews of Camerer (1995) and Starmer (2000) summarize other supporting evidence.

⁵Tversky and Kahneman (1992) utilize probability weighting similar to Quiggin's in their cumulative prospect theory (CPT), a generalization of prospect theory that accommodates choice in settings with non-binary outcomes. Yaari's (1987) dual theory of choice is a special case of RDU with a linear felicity function.

Financial economists have nonetheless been reluctant to model preferences with probability weights, perhaps owing to skepticism about whether findings from the stylized environments necessary for controlled experiments can be applied in general settings. Our paper takes an important step toward understanding the implications of probability weighting outside the lab by formulating explicit predictions for optimal asset allocation decisions of individuals with RDU.⁶ We focus on theoretical predictions that we can compare directly to recent empirical findings from new, extensive microeconomic datasets on household investment decisions, such as those constructed and investigated by Brunnermeier and Nagel (2008), Calvet, Campbell, and Sodini (2007, 2009), Calvet and Sodini (2013) and Chiappori and Paiella (2011). Our novel theoretical framework proves to be a useful tool for linking lab and field observations, allowing the direct use of experimental subject preference parameter estimates to predict quantitatively the determinants of fractional holdings of risky assets in household portfolios. Our main finding is that calibrations of preferences suggested in the prior literature to best explain experimental data imply portfolio risky shares that respond far more strongly to changes in wealth than can be justified by household level data. The experimental evidence on probability weighting is very robust, thus our paper points out a puzzling inconsistency between the lab and micro evidence.

The theory we develop applies to investors wishing to save for consumption at some future date by trading in a stock and bond. The information structure is binomial with independent and identically distributed returns, hence markets are dynamically complete. In the solution to this problem under EU an investor with constant relative risk aversion should place a constant fraction of wealth in the stock at every decision point, and the resulting optimal terminal logarithmic wealth is linear in the stock return. We solve a generalized version of this problem for investors with RDU, providing new methods for determining the optimal terminal wealth and the trading strategy that replicates it.

In the portfolio choice context, modifying the utility function to incorporate probability weights complicates the analysis considerably. Alternative portfolio weights give rise to different

⁶Our work follows prior studies of probability weighting in financial applications by Barberis and Huang (2008), Barberis (2012), Chapman and Polkovichchenko (2009), Epstein and Zin (1990), Polkovichchenko (2005), and Polkovichchenko and Zhao (2013).

rankings of state-contingent payoffs and consequently to the decision weights associated with the states. This effectively makes the investor's objective an endogenous function of the choice variables. In our binomial context, the number of possible state-payoff rankings when investing for T periods is $2^T!$, which grows at an extraordinary rate (faster than exponential), giving rise to a considerable computational challenge. Interestingly, the problem becomes more tractable when the one period stock return is calibrated to produce equi-probable payouts, in which case we show that optimal terminal wealth is nonincreasing in the state price. This observation allows us to reformulate the problem of choosing optimal terminal wealth to a constrained optimization problem with 2^T constraints, corresponding to the budget constraint and the weak ranking of consumption across states. The remaining challenge is to identify the binding constraints. We show that a separation principle allows us to solve this problem in two stages: First, using information only on state prices and probability weights, states for which optimal wealth will be distinct and states for which optimal wealth will be equal can be identified. Second, a standard portfolio choice problem is solved on a redefined state space, where all states with equal terminal wealth are merged. Exploiting the structure of the binomial model in this manner allows us to solve for optimal terminal wealth and the underlying portfolio strategy for practically relevant investing horizons with a variety of interesting parameterizations of the model.

We calibrate our model to provide advice for investing in a stock index and a bond over a 10-year horizon. Under RDU, decision weights are defined as the increments of a probability weighting function, and to be consistent with common evidence that individuals overweight rare gains and losses we model an inverse-S probability weighting function. The specific form that we choose, suggested by Prelec (1998), has broad experimental support and we parameterize this function using the estimates provided by Abdellaoui, Baillon, Placido, and Wakker (2011). The optimal return on wealth for an EU investor is linear in the return on the stock; by contrast we find that the return on wealth of the RDU investor is highly convex. Optimal terminal RDU wealth is constant in nearly all states for which the stock return is negative. On the other hand, terminal wealth is highly exposed to the stock market in states for which stock returns are high. Approximations to the optimal RDU payout can be provided by a strategy that invests a

constant fraction in the stock, combined with at-the-money puts and out-of-the-money calls on the stock. Interestingly, structured products are commonly offered that promise such returns. We also find that RDU investors desire distinct wealth levels in high return states with the same state prices, which they can achieve by creating long-short positions of Arrow-Debreu securities. This portfolio payout in high return states can be interpreted as gambling and to the best of our knowledge this optimizing behavior has not been previously identified.⁷ To provide an economically meaningful metric of the difference between the optimal EU and RDU wealth choices, we calculate that an RDU investor would be willing to forgo 9% of wealth in order to switch from the EU to the RDU payoff.

In our calibrated model, the trading strategy that replicates optimal EU wealth requires that a constant 50% of wealth be invested in stock irrespective of the historical stock returns. Implementing this policy requires the EU investor to sell stock following positive returns and buy stock following negative returns. The fraction of wealth invested in stock by the RDU investor is also about to 50% initially but then varies depending on the evolution of the stock return. The optimal risky share increases along paths with good news, achieving levels as high as 400%, and decreases along paths with bad news, reaching levels as low as zero.⁸

An active, recent empirical literature on portfolio choice provides important evidence for evaluating the plausibility of our model and, as a consequence, for the hypothesis that investor preferences incorporate probability weights. The Survey of Consumer Finances (SCF) collects disaggregated data that forms the basis of several influential studies documenting investing behavior of U.S. households. Campbell (2006) summarizes prior findings that are based on this information and provides new evidence on wealth, participation in financial markets, and asset allocation from the 2001 SCF. He documents that the poorest U.S. households hold virtually no financial assets, and even at the 80th percentile of wealth almost 20% of households hold no public equity. Strong wealth effects are also present in the cross section of asset allocation

⁷In support of this prediction, Kumar (2009) provides empirical evidence of demand for stocks with lottery features.

⁸Our baseline calibration produces nonparticipation at low wealth levels due to exit from risky positions. We show, however, that the RDU model cannot explain the occurrence of entry into risky positions when the probability weighting function has the commonly parameterized inverse-S shape.

decisions, consistent with prior findings that the wealthiest households bear significantly greater risk. Other studies utilizing the SCF show a lack of diversification in typical household portfolios (e.g., Polkvnichenko (2005)).

Panel data from a variety of sources has been instrumental for providing further insights into investor behavior. Brunnermeier and Nagel (2008) use the Panel Study of Income Dynamics survey data to investigate portfolio risky share dynamics. They find little evidence that individual household portfolio risky share responds to changes in wealth. In addition, they document strong inertia in household portfolios whereby exogenous changes in financial wealth are not offset by active trading in asset markets. In a series of papers utilizing high-resolution Swedish data, Calvet, Campbell, and Sodini (2007, 2009) and Calvet and Sodini (2013) find evidence of nonparticipation in risky asset markets, a positive elasticity of risky share with respect to wealth, lack of diversification, and active trading that partially offsets portfolio imbalances caused by past returns. Using the Survey of Household Income and Wealth from the Bank of Italy, Chiappori and Paiella (2011) find the elasticity of the risky asset share to wealth to be small. Grinblatt and Keloharju (2000, 2001) use Finnish data to show that domestic investors employ contrarian investing styles, confirming prior evidence of a disposition effect (e.g., Shefrin and Statman (1985) and Odean (1998)).

We simulate both cross-sectional and panel data from our calibrated model and focus primarily on the magnitude of the elasticity of risky share to wealth consistent with RDU investor behavior. Using regression specifications from the empirical literature, we find that estimates of elasticities in our simulated samples are counterfactually high. The upper bound on wealth elasticity in the actual data is on the order of 0.25, meaning that a 10% increase in wealth will cause an increase in the risky share from, say, 50% to 51.25%. Our simulated data produces an elasticity of about 1.5, six times the highest estimate from the field data. As another indication of the strong response of RDU investors to the return on wealth, we find that in our calibration investors optimally purchase the risky asset after good market returns and sell following poor returns, indicating a momentum style of rebalancing. This trading behavior is inconsistent with the inertia documented by Brunnermeier and Nagel (2008) and the evidence in Calvet, Camp-

bell, and Sodini (2009) that households offset about half the passive change in their asset mix with active (contrarian) trading. To reconcile the simulated and actual data requires that we practically eliminate the distinction between probabilities and decision weights in our model. We therefore document an inconsistency between the importance of probability weights in the experimental and household finance data.

The theoretical literature on the implications of RDU for portfolio choice is surprisingly recent. Polkovnichenko (2005) solves for optimal portfolios of RDU investors in a single period, three security, incomplete market setting and shows that investors choose underdiversified portfolios.⁹ An important distinction between Polkovnichenko (2005) and our paper is that our quantitative results originate purely from our preference structure and not from the particular assumptions regarding market structure. For example, he provides an analysis of nonparticipation due to RDU, a result we show does not generalize to the complete markets setting. Carlier and Dana (2011) solve for optimal consumption in a static, complete market setting when the state space admits a continuous distribution. Relative to Carlier and Dana (2011), our solution method is constructive and employs an economically intuitive concavification technique. No prior work exists in a finite state setting, and our mathematical generalization is useful for providing new economic insights. For example, we establish that investors with RDU preferences express a demand for gambling and this economic motive is not present in the continuous state setting. Probability weighting is an integral component of Tversky and Kahneman's (1992) CPT, and recent work by Barberis (2012) and Jin and Zhou (2008) examines the consequences of probability weighting in this context for gambling behavior and portfolio choice.

Asset pricing contributions that incorporate probability weighting are more numerous. Epstein and Zin (1990) study risk premia in an endowment economy with a representative agent whose preferences are recursive. Atemporal preference in their utility specification is RDU rather than CRRA, and this device elevates risk premia. Chapman and Polkovnichenko (2009) also solve a consumption based asset pricing model with heterogenous RDU investors and show that,

⁹In a related setting, Shefrin and Statman (2000) model portfolio choices of investors with Yaari (1987) preferences modified to accommodate Roy's (1952) *safety first* constraint and show the optimality of undiversified portfolios.

due to endogenous nonparticipation by one investor type, asset prices and dynamics are inconsistent with those produced by a representative RDU agent. Within the extensive literature on asset pricing and portfolio choice under CPT few models incorporate probability weighting. Notable exceptions are Benartzi and Thaler (1995) and Barberis and Huang (2008). We do not contribute directly to the asset pricing literature but our results on portfolio choice problem are useful for building intuition for general equilibrium results.

Our paper is conceptually similar to many other papers which quantitatively evaluate the impact of nonstandard preferences on portfolio choice. The literature includes, Ang, Bekaert, and Liu (2005) who study portfolio choice under disappointment aversion, Barberis and Xiong (2009) who show when the disposition effect can be induced by loss aversion, Barberis and Xiong (2012) and Ingersoll and Jin (2013) who analyze the impact of realization utility on portfolio choice.

2 The Portfolio Choice Model with Probability Weighting Functions

In this section, we formally describe the optimal portfolio choice problem of an RDU investor.

2.1 Securities and Information Structure

We model trade in a bond and stock in a frictionless market at T dates, $t = 0, \dots, T - 1$. The riskfree bond produces the constant return $R = e^r$ each period. Stock returns ρ_t between dates $t - 1$ and t are independent Bernoulli distributed, where

$$\rho_t = \begin{cases} u = e^{\mu + \sqrt{\frac{p}{1-p}}\sigma} & \text{with probability } p \\ d = e^{\mu - \sqrt{\frac{1-p}{p}}\sigma} & \text{with probability } 1 - p. \end{cases} \quad (1)$$

This specification implies that the mean and standard deviation of logarithmic one-period stock returns are μ and σ . Trading strategies with risky share π_{t-1} of wealth in stock produce portfolio

returns $R_t = \pi_{t-1}(\rho_t - R) + R$.

Markets are dynamically complete in this setting, and our return processes allow for the creation at each date of two short-lived Arrow-Debreu (AD) securities. One pays a unit of wealth when the stock return is $\rho_t = u$ and zero otherwise, with price denoted ξ_u , and the other pays one minus this payout, with price denoted ξ_d . These securities can be replicated using the bond and the stock and their no-arbitrage prices are given by

$$\xi_u = \frac{R - d}{R(u - d)}, \quad \xi_d = \frac{u - R}{R(u - d)}. \quad (2)$$

The state space Ω is the set of $M = 2^T$ T -tuples $\omega = (\omega_1, \dots, \omega_T)$ with $\omega_t \in \{u, d\}$. We associate each state with a unique index m chosen from the set $\mathcal{M} = \{1, \dots, M\}$. Define the random variable k_m that assigns to each state m the number of down returns in the state. The probability of occurrence of state m is then given by $p_m = p^{T-k_m}(1-p)^{k_m}$. At time-0, long-lived AD claims on each of the terminal states can be created from sequences of short-lived AD securities. To preclude arbitrage, these prices must be given by

$$\psi_m = \xi_u^{T-k_m} \xi_d^{k_m}. \quad (3)$$

2.2 Investor Preferences

We consider preferences over state contingent date- T wealth $X : \mathcal{M} \rightarrow \mathbb{R}$. We associate to each occurrence of the terminal wealth X_m the power cardinal felicity function $v(X_m) = \frac{X_m^{1-\gamma}}{1-\gamma}$, where $\gamma > 0$. We choose the power form of felicity to eliminate direct dependence of risky share on wealth levels in portfolio choice decisions.

We first present the standard definition of RDU which operates on the $N \leq M$ distinct outcomes $x_1 > x_2 > \dots > x_N$ of terminal wealth X (e.g., Wakker (2010), Ch. 6). In our finite state setting, the cumulative distribution function (CDF) associated with X is a non-decreasing step function F . The value of the decumulative distribution function (DDF) $1 - F(x_i)$ can be interpreted as the rank of outcome x_i , where low values of $1 - F(x_i)$ indicate highly

ranked outcomes x_i . To complete the RDU specification, a non-decreasing probability weighting function H is defined which operates on the DDF, normalized without loss of generality so that $H(0) = 0$ and $H(1) = 1$.¹⁰ Decision weights w_i are calculated from increments of the probability weighting function

$$w_i = \begin{cases} H(1 - F(x_{i+1})) - H(1 - F(x_i)) & \text{for } i = 1, \dots, N-1 \\ 1 - H(1 - F(x_N)) & \text{for } i = N. \end{cases} \quad (4)$$

Utility of risky terminal wealth is then given by

$$U(X) = \sum_{i=1}^N w_i v(x_i). \quad (5)$$

This formulation makes it clear that RDU is law invariant, meaning that utility depends only on the CDF of terminal wealth. For example, in a setting with equally probable states, the distribution of terminal wealth Y constructed by permuting the state-contingent wealth levels X_m of terminal wealth X produces two distinct wealth allocations with the same CDF. Equations (4) and (5) imply $U(X) = U(Y)$.

For the portfolio choice problem, in which the budget constraint acts directly on the state-contingent payouts X_m but only indirectly on the outcomes x_i , it is convenient to derive an equivalent representation of RDU preference. Consider any of the $M!$ orderings of the states $\theta : \mathcal{M} \rightarrow \mathcal{M}$ and the associated permutation of the states $\lambda = \theta^{-1}$. Using this notation, θ_m denotes the (weak) rank of state m and λ_j denotes the state with unique rank j . Define

$$h_j = \begin{cases} H\left(\sum_{i=1}^j p_{\lambda_i}\right) - H\left(\sum_{i=1}^{j-1} p_{\lambda_i}\right) & \text{for } j = 2, \dots, M \\ H(p_{\lambda_1}) & \text{for } j = 1, \end{cases} \quad (6)$$

where, for example, $\sum_{i=1}^{j-1} p_{\lambda_i}$ is the probability of states with strictly higher rank than state λ_j . The utility associated with any terminal wealth allocation satisfying $X_{\lambda_1} \geq \dots \geq X_{\lambda_M}$ is then

¹⁰This is possible because the cardinal felicity function is unique only up to an affine transform.

given by

$$U(X) = \sum_{m=1}^M h_{\theta_m} v(X_m). \quad (7)$$

The equivalence of equations (5) and (7) when $X_{\lambda_1} > \dots > X_{\lambda_M}$ follows by defining $x_i = X_{\lambda_i}$ and verifying from equations (4) and (6) that $w_i = h_i$ for $i = 1, \dots, M$. The equivalence is more general, however, and applies to arbitrary ranked wealth.¹¹

2.3 The Portfolio Choice Problem

We formally state the portfolio choice problem by first characterizing the optimal terminal wealth.

Problem 1. *The agent chooses ordering θ (equivalently, permutation λ) and terminal wealth X to solve*

$$\sup_{\theta, X} \sum_{m=1}^M h_{\theta_m} v(X_m) \quad (8)$$

subject to the budget constraint

$$X_0 = \sum_{m=1}^M \psi_m X_m, \quad (9)$$

and the $M - 1$ constraints ensuring wealth obeys the ranking implied by the ordering θ :

$$X_{\lambda_1} \geq \dots \geq X_{\lambda_M}. \quad (10)$$

This formulation of the RDU agent's optimal choice breaks the problem into stages. First, a possible ordering θ of states is considered. This defines a particular objective function (8) that applies to any terminal wealth obeying the constraints (10). The agent selects the best choice X^θ among candidate solutions that also satisfy the budget constraint (9). The global optimal choice is the best of the candidate solutions X^θ among all of the $M!$ possible orderings.

¹¹To see this, assume a sequence of states with $X_{\lambda_{j-1}} > X_{\lambda_j} = \dots = X_{\lambda_{j+n}} > X_{\lambda_{j+n+1}}$. Summing the terms $h_j + \dots + h_{j+n}$ gives $H(1 - F(X_{\lambda_{j+n+1}})) - H(1 - F(X_{\lambda_j}))$. There exists an index i such that $x_i = X_{\lambda_j}$ and $x_{i+1} = X_{\lambda_{j+n+1}}$, with an associated decision weight $w_i = H(1 - F(x_{i+1})) - H(1 - F(x_i))$. Mapping to decision weights in such a manner for states with equal wealth allows us to establish that the two utility representations in equations (5) and (7) are equivalent.

The dependence of the objective function (8) on the ordering θ does not arise when investors have EU preferences. To understand this, note that the probability weighting function H is the identity function under EU. Equation (6) then simplifies to $h_j = p_{\lambda_j}$, meaning that the decision weight associated with the outcome of rank j equals the probability of the state λ_j . Since the outcome in state λ_j is X_{λ_j} , the utility associated with terminal wealth satisfying the ordering (10) is then given by

$$\sum_{j=1}^M p_{\lambda_j} v(X_{\lambda_j}) = \sum_{m=1}^M p_m v(X_m). \quad (11)$$

The ordering θ therefore has no impact on the objective function (8). Since the order can be arbitrarily chosen, the constraints (10) are irrelevant and Problem 1 collapses to the standard EU portfolio choice problem.

To determine the optimal portfolio at every date t , we apply standard replicating techniques. This defines a unique adapted trading strategy π_t for $t = 0, \dots, T - 1$. We consider only the case where the agent can pre-commit to a trading strategy formulated at the initial date or, equivalently, to a buy-and-hold a portfolio of long-lived AD securities.

3 A Theoretical Characterization of the Optimal RDU Portfolio

The RDU portfolio choice problem is intractable except when the number of dates T is very small (i.e., $T \leq 2$). In order to make progress when T is large, in the remainder of the paper we adopt the assumption that $p = \frac{1}{2}$. We do not view this assumption as empirically limiting since distributions typically proposed for risky returns can be approximated by appropriate sequences of uniformly distributed single period returns.¹² We also do not consider the assumption limiting for the purposes of the theory, since it is usually possible to transform a model with a non-uniform distribution into a model with a uniform distribution by allowing side-bets.¹³ As we show in

¹²For example, it is common to model independent single period logarithmic returns having mean μ and standard deviation σ using equation (1) with $p = \frac{1}{2}$.

¹³For example, a one-period model with two states with probabilities $1/3$ and $2/3$ can be transformed into an equi-probable three-state model by allowing investors to undertake a binary, uniformly distributed, fair bet in the higher probability state.

this section, by exploiting the structure of the uniform model we can circumvent the difficulties associated with the brute force approach to solving Problem 1.

We assume that the stock investment provides a risk premium, $\mathbb{E}(\rho_t) > R$, implying that $\xi_d > \xi_u$. It is also convenient to, without loss of generality, consider an indexation of the states with the property that state prices form a non-decreasing sequence:¹⁴

$$\psi_1 \leq \cdots \leq \psi_M. \quad (12)$$

3.1 The Expected Utility Benchmark

As shown in Section 2.3, Problem 1 is a standard constrained optimization problem when the investor's preferences can be represented by EU. The solution to this problem is well understood, and we provide it as a benchmark against which to compare the optimal RDU solution.

Proposition 1. *Optimal state-contingent terminal wealth X for an expected utility investor is given by*

$$X_m = \left(\frac{\xi_u}{\xi_d} \right)^{\frac{k_m}{\gamma}} \bar{X}, \quad (13)$$

where

$$\bar{X} = X_0 \left(\xi_u^T \sum_{n=0}^T \binom{T}{n} \left(\frac{\xi_u}{\xi_d} \right)^{\frac{1-\gamma}{\gamma} n} \right)^{-1} \quad (14)$$

is the highest consumption level, achieved only in state $m = 1$. The optimal portfolio is constant over time and the optimal risky share is

$$\pi^* = \frac{R}{u - R} \frac{\left(\frac{\xi_u}{\xi_d} \right)^{-1/\gamma} - 1}{1 + \left(\frac{\xi_u}{\xi_d} \right)^{1-1/\gamma}}. \quad (15)$$

Equation (13) shows that the terminal wealth of EU investors is a decreasing function of the

¹⁴It is possible to do this as follows: The state for which $\omega_t = u$ for all t is assigned the index $m = 1$ and the state for which $\omega_t = d$ for all t is assigned the index M . Beginning with the subset of states for which only one down stock return occurs, randomly assign the indices from $m = 2$ to $m = 1 + n_1$, where $n_1 = \binom{T}{1}$ is the number of states in that subset. Next, we consider the subset of states with two down returns and randomly assign indices $m = 2 + n_1$ to $m = 1 + n_1 + n_2$ where $n_2 = \binom{T}{2}$. We proceed in this manner until all of the states are assigned.

number of down returns k_m in state m . This implies that the optimal ordering of states is the identity $\theta_m = m$ and that it is optimal to equalize wealth across states if and only if they share a common state price.

3.2 The RDU Solution with Arbitrary Probability Weights

We now solve Problem 1 in the general case, assuming a nonlinear probability weighting function. The following proposition, which is closely related to a result of Dybvig (1988), dramatically simplifies this task.

Proposition 2. *Any optimal terminal wealth solving Problem 1 must be nonincreasing in the state index*

$$m_1 < m_2 \implies X_{m_1} \geq X_{m_2}.$$

State prices are nondecreasing in the state index m , therefore this proposition implies that states with higher state prices have (weakly) less terminal wealth than states with lower state prices. To understand why this is true, suppose that the state price ψ_{m_1} is lower than the state price ψ_{m_2} but wealth in state m_1 is lower than wealth in state m_2 . Switching wealth between states produces a new terminal wealth that is less costly but has the same distribution. RDU preferences are law invariant, therefore this change has no effect on the investor's utility and allows the investor to purchase additional AD securities.

Proposition 2 allows us to restrict our attention to the ordering $\theta_m = m$ when solving Problem 1. Each state can then be assigned the following pre-determined decision weight

$$h_m = H\left(\frac{m}{M}\right) - H\left(\frac{m-1}{M}\right),$$

and solving Problem 1 is equivalent to solving the following problem:

Problem 2. *The agent chooses random terminal wealth X to solve*

$$\sup_X \sum_{m=1}^M h_m v(X_m), \quad (16)$$

subject to the budget constraint

$$X_0 \leq \sum_{m=1}^M \psi_m X_m, \quad (17)$$

and the $M - 1$ constraints ensuring the state-by-state wealth rank:

$$X_1 \geq \cdots \geq X_M. \quad (18)$$

Unlike Problem 1, this new problem has only one objective function (16) to consider. Formally, it is a portfolio choice problem with $M - 1$ nonstandard constraints that ensure greater wealth in states with lower state prices.

We now focus our attention on the challenge of determining which of the inequalities in (18) bind. To start, we construct securities which pay one unit of wealth in all states up to and including state m for $m = 1, \dots, M$. These securities are natural to consider for our problem because portfolios of these securities with nonnegative weights will satisfy the constraints (18). The no-arbitrage price of these claims is given by $\phi_m = \sum_{i=1}^m \psi_i$, and for convenience we also define $\phi_0 = 0$. Notice that ϕ_M is the bond price R^{-T} . We denote the set of these prices by $\Phi = \{\phi_0, \dots, \phi_M\}$. It is natural to also consider the decision weights associated with adding a constant amount of wealth to any state up to and including state m . These decision weights $H_m = \sum_{i=1}^m h_i$ are identical to the probability weights $H_m = H(\frac{m}{M})$. We additionally let $H_0 = 0$ and define the set $\mathcal{H} = \{H_0, \dots, H_M\}$. Notice that $H_M = 1$.

We define the function $G : \Phi \rightarrow \mathcal{H}$ by $G(\phi_m) = H_m$. For $m > 0$, at any point (ϕ_m, H_m) the “left slope” of the function is

$$\frac{H_m - H_{m-1}}{\phi_m - \phi_{m-1}} = \frac{h_m}{\psi_m}.$$

The nonlinearity of the function H and the convexity of the function ϕ may produce a sequence of

left slopes that is nonmonotonic, thus the function G may not be concave. The concavification of the function G is the smallest concave function $G^* : \mathcal{M} \rightarrow [0, 1]$ that dominates it. This concavification can be fully characterized using the vertices of G^* , i.e., the minimal set of points on the graph of G^* that, along with linear interpolation between adjacent vertices, can be used to construct G^* . Let the set of $N + 1 \leq M + 1$ vertices be denoted $\mathcal{G}^* = \{(\phi_i, H_i) : i \in \mathcal{M}_V = \{m_0 = 0, m_1, \dots, m_N = M\}\}$, where $0 < m_1 < \dots < m_{N-1} < M$ is an increasing sequence.

It is important to notice that concavity of the function G^* implies that the left slopes

$$\frac{H_{m_i} - H_{m_{i-1}}}{\phi_{m_i} - \phi_{m_{i-1}}} = \frac{\sum_{m=m_{i-1}+1}^{m_i} h_j}{\sum_{m=m_{i-1}+1}^{m_i} \psi_j}. \quad (19)$$

are descending along the sequence \mathcal{M}_V . Moreover, because G^* is the unique smallest concave function dominating the function G , adding any other point (ϕ_m, H_m) , $m \in \mathcal{M} \setminus \mathcal{M}_V$ to the graph of G^* cannot generate another sequence of slopes that descends at every point.

The vertices of the graph of G^* can be used to form a partition of the state space \mathcal{M} : $\mathcal{S}^* = \{S_1^*, \dots, S_N^*\}$ where $S_i^* = \{m_{i-1} + 1, \dots, m_i\}$ for $i = 1, \dots, N$. The following proposition shows that with this partition, the solution to Problem 2 can be found by solving a standard portfolio choice problem in which only the budget constraint is binding.

Proposition 3. *Consider the solution x_1, \dots, x_N to the unconstrained optimization problem*

$$\sup_{x_1, \dots, x_N} \sum_{i=1}^N h_i^* v(x_i), \quad (20)$$

subject to the budget constraint

$$X_0 \leq \sum_{i=1}^N \psi_i^* x_i, \quad (21)$$

where $h_i^ = \sum_{m \in S_i^*} h_m$ and where $\psi_i^* = \sum_{m \in S_i^*} \psi_m$. The solution satisfies $x_1 > x_2 > \dots > x_N$, and the terminal wealth X defined by $X_m = x_i$ for any $m \in S_i^*$ is a solution to Problem 2.*

This proposition formalizes a two-step procedure for determining optimal terminal wealth for an RDU investor. In the first step, no information regarding the felicity function v is required,

and the partition \mathcal{S}^* is determined based solely on the properties of the state prices and the probability weighting function. The partition defines sets of states where wealth is equalized within each set and is distinct across sets. In the second step, standard portfolio choice procedures apply to determine the optimal allocation of wealth across the events of the partition, with decision weights h_i^* playing the role of probability of the event S_i and the prices ψ_i^* playing the role of state prices in the budget constraint.

For an EU investor, equation (13) in Proposition 1 shows that the optimal partition consists of $T + 1$ sets S_i^* each containing all states with a given state price $\psi = \xi_u^{T-i}\xi_d^i$ for $i = 0, \dots, T$. For general probability weighting functions, the sets S_i^* may contain states with the same state prices, states with different state prices, or a single state. When S_i^* contains states with different state prices, terminal wealth will be identical for all states in that set. This “packing” of states gives the appearance of extreme risk aversion at certain wealth levels and causes nonparticipation in the stock market as wealth evolves towards this region. On the other hand, when S_i^* contains single states, states with the same state price will have different terminal wealth levels. This “unpacking” of states gives the appearance of risk seeking at certain wealth levels and gives rise to gambling-like behavior. This behavior occurs regardless of the felicity function. The simultaneous purchase of insurance and participation in lotteries does not occur for EU investors. It is well-known that RDU preferences can produce such behavior and our formulation of the optimal wealth choice clarifies the mechanism underlying these optimal choices.

It is instructive to decompose optimal terminal wealth into two components. Let $X = Y + X^g$ where $Y = \mathbb{E}(X|k)$ is the expected terminal wealth conditional on number of down movements k in a stock return history, and $X^g = X - Y$ is the residual. The component X^g is initially mean zero and costless.¹⁵ We thus interpret X^g as a collection of fair gambles that are optimally added by the RDU investor to the component of wealth Y . Since adding X^g to Y results in a second-order stochastically dominated payoff, any EU investor would choose to set $X^g = 0$. Optimal choice of second-order stochastically dominated payoffs by RDU investors has not been

¹⁵Recall that since $\psi(T)$ is a function of k , $\mathbb{E}[\psi(T)X^g] = \mathbb{E}[\mathbb{E}(\psi(T)X^g|k)] = \mathbb{E}[\psi(T)\mathbb{E}(X^g|k)] = 0$ where the final equality follows from $\mathbb{E}(X^g|k) = 0$.

previously pointed out, to the best of our knowledge, and might explain demand for fair lotteries that otherwise seems irrational.

3.3 Implications for Dynamic Portfolio Choice

Optimal terminal wealth X can be produced by the appropriate purchases of AD securities, but this static implementation has limited empirical relevance. We now provide some general properties of the unique dynamic replicating strategy for terminal wealth X , which is formally a process of shares of stock $\Delta^X(t)$ and units of bond $B^X(t)$ for $t = 0, \dots, T - 1$.

Proposition 3 implies that if G^* is linear then optimal terminal wealth is a constant and when G^* is nonlinear at least two distinct sets of states exist for which optimal terminal wealth is different. The monotonicity result in Proposition 2 shows that whenever wealth is distinct across states, states with lower state prices will optimally have higher wealth. A direct consequence of this monotonicity is that Δ^X is non-negative, as formalized in the following proposition.

Proposition 4. *In the dynamic replicating strategy for X , it is never optimal to short the stock (i.e., $\Delta^X(t) \geq 0$ for all t). Nonparticipation at the initial date $\Delta^X(0) = 0$ occurs only when the function G^* is linear.*

RDU has been associated with “first-order risk aversion” and, as a consequence, nonparticipation in equity markets (e.g., Epstein and Zin (1990)). This proposition shows that non-participation by RDU investors is a special case. In fact, as we show in the next section, common probability weighting functions and state price specifications interact to form a nonlinear function G^* that induces positive initial demand for stock. Nonetheless, we will show in the next section that even when G^* is nonlinear, the replicating strategy may involve selling all stock along certain paths.

We finally consider the cross-section of portfolio holdings within a population of identical RDU investors. We assume each investor independently chooses identically distributed terminal wealth X_i , where the component of wealth associated with gambling X_i^g does not necessarily

agree state-by-state with the gambling component X_j^g of investor $j \neq i$. The following proposition shows that when investors independently implement portfolio trading strategies that replicate optimal terminal wealth, the gambling components have no impact on aggregate terminal wealth or aggregate stock holdings.

Proposition 5. *If investors independently choose terminal wealth, aggregate terminal wealth is given by $X = Y$ and the dynamics of aggregate holdings of stock are given by Δ^Y , the process for the share of stock in the replicating strategy for Y . At any date t , aggregate risky share is equal to $\pi(t) = \Delta^Y(t)/Y(t)$.*

This proposition shows that in a large cross-section, the share of aggregate wealth invested in stock is exactly the share associated with the replicating strategy for the non-gambling part of optimal wealth. Equivalently, the wealth-weighted portfolio holdings of all investors is given by $\Delta^Y(t)/Y(t)$. This result allows formulation of quantitatively relevant empirical predictions for dynamic trading strategies in large cross-sections, such as those we explore in the next section.

4 RDU Household Behavior in a Calibrated Model

To quantify wealth and risky share dynamics, we calibrate our RDU model and solve the optimal investment problem. We then simulate household wealth and portfolio dynamics to produce a synthetic panel dataset from which we generate empirically relevant moments that we compare to estimates from Brunnermeier and Nagel (2008), Calvet and Sodini (2013), and other prior literature.

4.1 Calibration

Our solution technique can be applied in a broad variety of applications. Important settings to develop quantitative guidelines for asset allocation are saving for retirement, a dependent's education, or the purchase of a first home. All of these applications require a significant fraction of wealth to be appropriately invested. We consider an investor with a 10 year investing horizon

who rebalances his portfolio annually, which we feel is relevant for many applications. This specification produces $M = 2^{10}$ states, significantly more than could be managed without our theoretical insights. We have confirmed that the qualitative conclusions from our calibration are relevant for other horizons and rebalancing frequencies.

To proceed in this setting requires that we specify the investor's subjective belief, represented by a probability distribution of future stock and bond returns. Although a large literature documents errors in judgment in settings such as ours (see, e.g., Camerer (1995)), we make the strong but common assumption that subjective beliefs are consistent with learning based on historical data (see, e.g., Wachter and Yogo (2010)). We choose an annual riskfree bond return of $r = 0.25\%$, roughly equal to the current yield on one-year T-bills.¹⁶ Guided by Fama and French (2002), who estimate an equity risk premium of 4.32%, substantially lower than the historical average excess market return they compute (7.43%), we set the stock risk premium $\mathbb{E}(\rho_t - e^r) = 5\%$. The volatility of the stock, $\sigma = 0.20$, is chosen to match the historical standard deviation of the CRSP value weighted index (e.g., Bansal and Yaron (2004)).

A variety of functional forms describing probability weighting have been proposed in the prior literature. The inverse-S probability weighting function, which gives rise to the common ratio and common consequence behavior proposed by Allais (1953), is a robust finding, including in studies where it is not necessary to parameterize the functional form of the preference representation (Wu and Gonzales (1996, 1999), Abdellaoui (2000)). Probability weights are known to be “source dependent,” whereby the departure from the identity function is less extreme the more knowledgeable is the decision maker about the source of uncertainty (Heath and Tversky (1991)). Abdellaoui, Baillon, Placido, and Wakker (2011) estimate probability weights for subjects presented with prospects from sources with known probabilities (drawing from an urn of known composition) and unknown probabilities (draws from an urn of unknown composition, a day's temperature in Paris, or the change on a given day of the CAC40 French stock index). To parameterize our weighting function for choice in our setting with subjective rather than objective beliefs, we make use of their findings from experiments with unknown probabilities.

¹⁶The exact choice of the riskfree rate has little impact on the results we present.

Prelec (1998) axiomatizes choice consistent with the common ratio effect and derives a preference representation with the weighting function

$$H(t) = \exp(-\beta[-\ln(t)]^\alpha), \quad (22)$$

where the parameter α controls whether the function H is inverse-S shaped ($\alpha < 1$) or S shaped ($\alpha > 1$) and the parameter β controls the location of the inflexion point separating the convex region of H from the concave region of H . Abdellaoui, Baillon, Placido, and Wakker (2011) show that the parameters $\alpha = 0.65$ and $\beta = 1$ lead to the best fit to the aggregated data from their study of individuals choosing among draws from an urn of unknown composition. We utilize the Prelec weighting function with these parameter estimates as our best candidate, based on the prior literature, for a weighting function in our setting.

Expected utility investors with a risk aversion parameter of $\gamma = 2.4$ choose to hold 50% of their wealth in stock under our calibration of stock and bond returns. This risky share is consistent with data from Brunnermeier and Nagel (2008) and Calvet and Sodini (2013) on fractional holdings of liquid assets (cash, bonds, stocks and mutual funds) in risky assets. To estimate the parameter γ for the RDU investor, notice that probability weighting functions interact with the parameter γ to induce higher or lower aversion to risks. To control for this interaction, we determine certainty equivalents of EU and RDU investors to a 100% one-year investment in the stock. These certainty equivalents are equal when $\gamma = 1.5$ in the RDU felicity function, therefore in the results that follow, we set $\gamma = 2.4$ for the EU investor and $\gamma = 1.5$ for the RDU investor.¹⁷

¹⁷Abdellaoui, Baillon, Placido, and Wakker (2011) estimate that their experimental data from small-stakes gambles are best approximated when their felicity function is approximately linear in gains. The finding of risk neutrality in such settings is to be expected when subjects exhibit declining marginal utility of wealth (Rabin (2000)). Our felicity function operates on wealth, not gains, and we choose to rely on evidence from the field data, where the impact of declining marginal utility of wealth gives rise to measurable effects on investor choice, to calibrate the parameter γ .

4.2 Optimal Household Investment and Wealth

We follow the two-step procedure outlined in Section 3.2 to solve the portfolio choice problem. In the first step, we undertake the concavification procedure to identify optimal groups of states within which terminal wealth will be equal. Figure 1 shows the function G (blue points), mapping the cumulative state prices Φ to probability weights \mathcal{H} , and the concavification of this function G^* (solid red line). Vertices of G^* , three examples of which are indicated in the figure, define the optimal partition \mathcal{S}^* of the state space, and within each subset S_i^* of this partition the RDU investor's terminal wealth will be equalized. States with high state prices are assigned high values of the cumulative state price density and are thus located on the right in the figure. The concavification G^* is above G when state prices are high, indicating that optimal terminal wealth is equalized between these states. For example, wealth for all states associated with points on the graph G between the vertices A and B will be identical, as will wealth for states associated with the points between the vertices B and C. Unlike for the expected utility solution, the vertices A and B group states for which optimal wealth will be equal even though state prices are different. On the other hand, the two functions coincide when state prices are low, in the left region of the figure, indicating distinct wealth allocations among these states. The vertices of G^* in this region identify states for which state prices are equal but optimal terminal wealth is distinct, again contrasting with the expected utility solution in which wealth is a strictly decreasing function of the state price.

With the optimal partition of states in hand, we now proceed to the second step of the solution technique in which we apply Proposition 3 to solve for optimal terminal wealth using standard techniques for portfolio choice on a state space redefined by the partition \mathcal{S}^* . Figure 2 plots the relationships between optimal terminal wealth for an RDU investor (green '+') or an EU investor (red '*') and logarithmic 10-year stock returns. Consistent with Proposition 1, we observe that logarithmic payoffs chosen by the EU investor are upward sloping and linear in the logarithmic stock return. By comparison, the logarithmic payoff of an RDU investor is distinctly convex in the logarithmic stock return. When stock returns are low, wealth levels

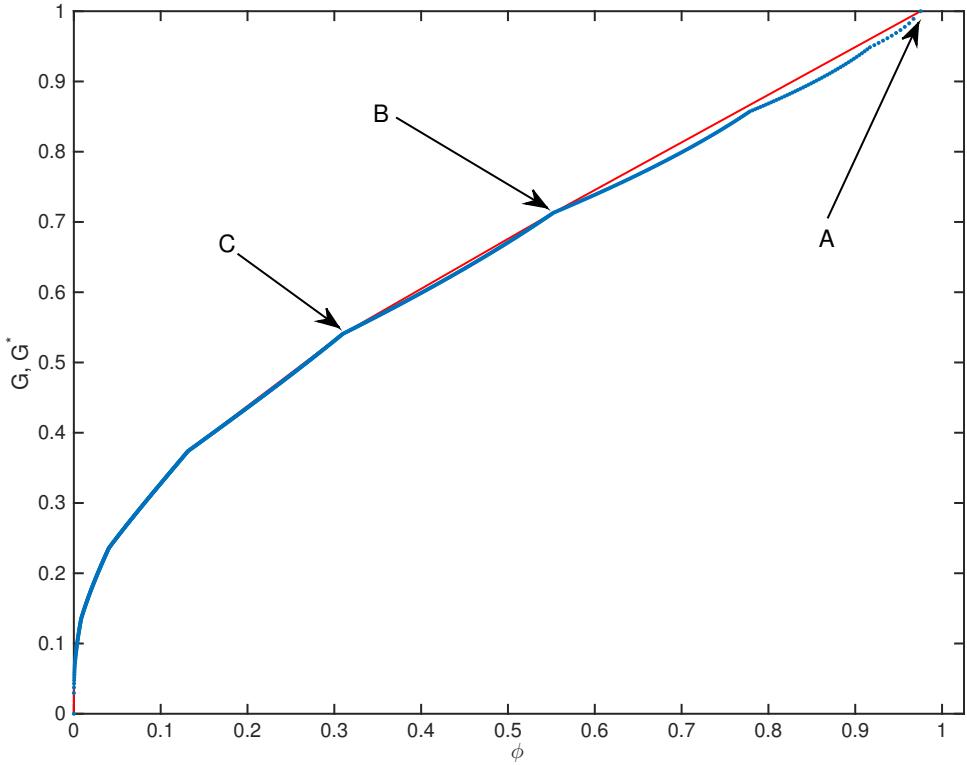


Figure 1: This figure presents the functions G and G^* which are used in the first stage of our solution technique to partition the state space into sets that optimally contain constant but distinct terminal wealth.

are higher than those chosen by the EU agent and are independent of the stock return. This constant payoff can be obtained by purchasing downside protection using a put option on the stock. The figure shows that under our baseline calibration, downside protection is present in almost all losing states. When stock returns are high, wealth levels are higher than those chosen by the EU agent. This performance-sensitive payoff can be achieved by purchasing out-of-the-money call options on the stock and can produce high wealth levels relative to those chosen by the EU agent. For example, at the highest stock return level, produced when no down return materializes during the 10 year investing horizon, the RDU agent's wealth increases by a factor of more than $e^4 \approx 55$ times, whereas the EU agent's payoff increases by about $e^{1.3} \approx 4$ times. To achieve maximum utility, Figure 2 shows that the RDU investor alters the optimal portfolio of the EU investor by selling AD securities from states that produce mid-range stock returns and buying AD securities in states that produce extreme stock returns. This demand for

downside protection and upside opportunities is a robust implication of the inverse-S probability weighting function which leads to U-shaped decision weights. The RDU investor places more emphasis on winning and losing states when constructing the optimal portfolio than is justified by the probabilities. Interestingly, the optimal terminal wealth could be produced by structured products commonly offered by financial institutions to retail investors that offer market exposure with limited loss.¹⁸

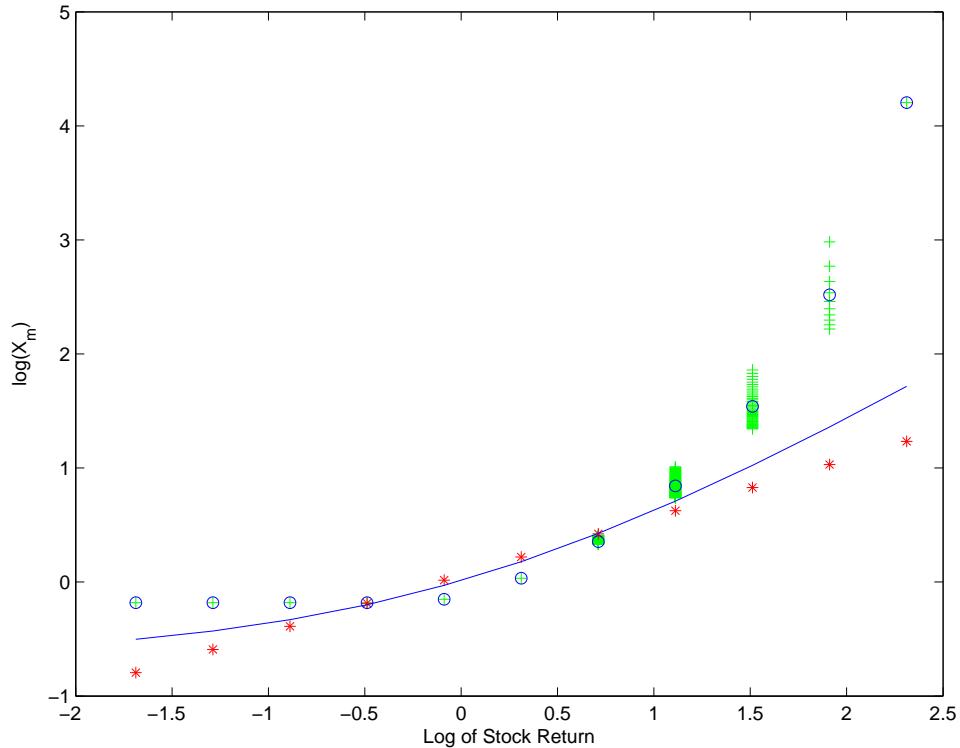


Figure 2: The logarithm of optimal terminal wealth versus the logarithm of the terminal stock return, for EU (red ‘*’) and RDEU (green ‘+’) investors. Average terminal wealth of the RDEU investor, conditional on the T period stock return, are also identified (blue ‘o’).

Figure 2 also shows that, unlike for an EU investor, stock returns do not uniquely determine terminal RDU wealth. This can be seen in the figure for high stock returns, where single realizations of the terminal stock price may be associated with multiple payoffs from the RDU portfolio. For example, at the second-highest logarithmic stock return level of about 2, achieved

¹⁸J.P.Morgan Chase currently offers a set of products that provide various participation levels (leverage) when an index increases in value while simultaneously providing a downside buffer as limited insurance against a drop in the value of the index. See, for example, their “5yr SPX Contingent Buffer Return Enhanced Note, at least 100% participation, 50.00% European protection, Uncapped” product (CUSIP 48125UBC5).

in the 10 states $m = 2, 3, \dots, 11$ for which 9 out of 10 one-period returns are “up,” the logarithmic payoff of the RDU portfolio varies from about 2.2 to 3.0. This implies that optimal RDU wealth depends not only on the terminal stock price, as is the case for the optimal wealth of the EU investor, but also on the exact sequence of 10 single period stock returns. To understand this result, consider a portfolio that produces a constant payoff \bar{x} in states $m = 2, \dots, 11$. Such a payoff can be achieved by holding an equal amount of each of the 10 state AD securities. Note that AD security prices depend only on the number of down returns in a state, thus each of these AD securities has the same price. Suppose that the RDU investor uses two of these securities to transfer a small amount of wealth from state $m = 3$ to state $m = 2$. This results in an increase in utility of approximately $(h_2 - h_3)v'(\bar{x})$ per unit of wealth transferred, and strict concavity of G^* in this region, as is the case under our baseline calibration, implies that the difference $h_2 - h_3 > 0$. This implies that the RDU investor will prefer an non-constant payout across these states, which can be interpreted as a preference to gamble with the proceeds \bar{x} within these states.

The terminal wealth of each investor can be decomposed into two economically distinct parts: 1) a component for which wealth is constant when conditioning on the terminal stock price, chosen to achieve an optimal risk/reward tradeoff, and 2) a component due to demand for gambling using fair bets, present because the RDU investor overweights small probability winning outcomes. To visualize this decomposition, Figure 2 plots the conditional wealth levels (blue ‘o’), and deviation of optimal terminal wealth from this first component of terminal wealth (i.e, the difference between the green ‘+’ and the blue ‘o’) represents the gambling component. The figure shows that the gambling component of wealth is only relevant following high stock returns. If we assume a continuum of investors with identical initial wealth and preferences, the first component of wealth will be identical in the cross section. The gambling component, however, depends on how investors match states and gambling outcomes and is not uniquely defined. For example, among the 10 states with only one down stock return, labeled $m = 2, \dots, 11$, the utility of an investor that chooses a gamble resulting in lower wealth in state $m = 2$ and higher wealth in state $m = 3$ is identical to the utility of an investor with an otherwise identical terminal

wealth but permutes these two state-contingent wealth levels. This creates endogenous heterogeneity in portfolio holdings, an empirically relevant implication of our model.¹⁹ We choose to focus on implications for average per-capital risky share, possibly conditioned on observable quantities like wealth and age, because this quantity is directly comparable to values reported in existing large-scale empirical studies. For a continuum of investors who are equally likely to choose any one of the gambling portfolios, we show in Proposition 5 that trading associated with the gambling component of terminal wealth has no impact on aggregate or average per-capita risky share. We therefore focus our analysis on the stock holdings used to implement the first component of terminal wealth.

We compute that our calibrated model implies an RDU investor would be willing to pay approximately 9% of initial wealth in order to exchange EU optimal terminal wealth for RDU optimal terminal wealth. This is an economically significant amount that provides a meaningful metric for the distance between the RDU and EU wealth profiles presented in Figure 2. The value implies that introduction of a structured product that approximates the RDU optimal terminal wealth could charge fees of up to 9% and still attract investors who would otherwise hold EU optimal wealth.

The blue curve in Figure 2 represents the terminal payout that results from a completely passive strategy in which the investor initially purchase \$0.50 of stock and bond and then does no further trading. The terminal wealth produced by this strategy is intermediate between the optimal EU and RDU payoffs. Interestingly, the RDU investor would prefer this portfolio strategy to the constant weight portfolio: we calculate that our RDU investor would only pay about 5% of initial wealth in order to swap the passive payout for the optimal payout. This finding suggests that in our calibrated model, RDU preferences provide a partial rationale for infrequent portfolio rebalancing.

Figure 3 presents a graphical description of the unique dynamic trading strategy in stocks

¹⁹This is fundamentally due to the fact that multiple state-conditional terminal wealth allocations produce the same payoff distribution and, due to law invariance of RDU preference, produce the same utility. Although it is possible to further explore this implication, which to the best of our knowledge has not been studied empirically, we restrict our attention to other previously reported empirical regularities.

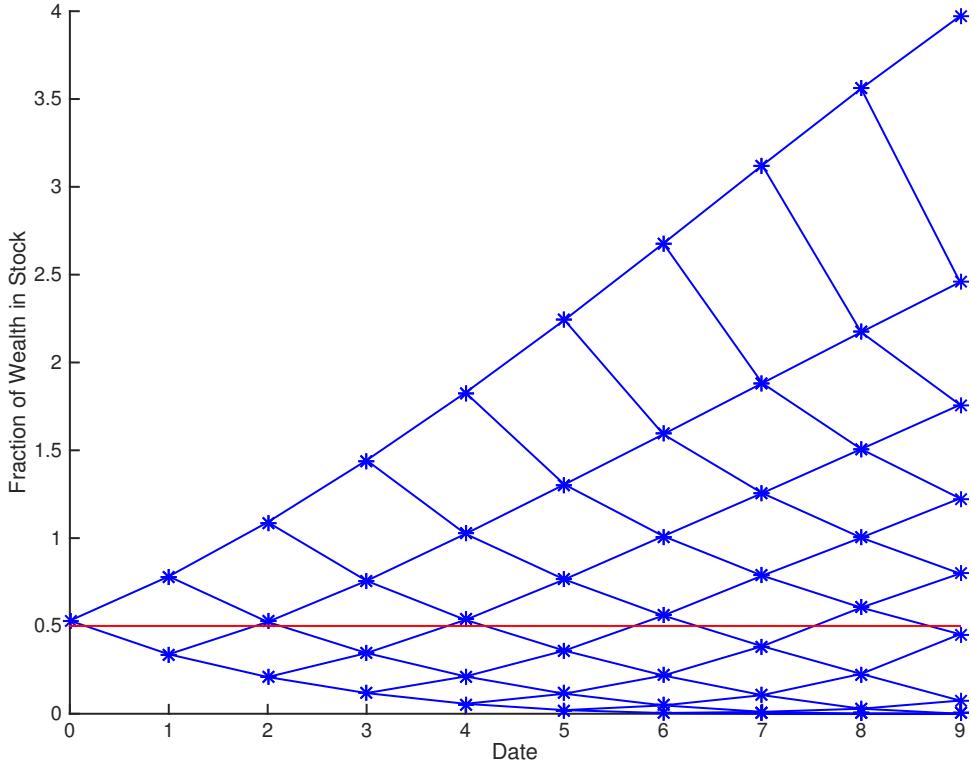


Figure 3: The figure depicts the optimal portfolio weight π along the nine periods binomial tree assuming that the terminal wealth is equal to the average of outcomes conditional on the terminal stock price. This portfolio will produce the optimal level of utility when combined with a pre-committed randomization strategy at each terminal node. The red line horizontal line at $\pi = 0.50$ depicts the optimal EU portfolio, which is the solution in the special case of RDU with a linear probability weighting function H .

and bonds, with annual rebalancing, that replicates the first component of the optimal terminal wealth (i.e., the mean optimal terminal wealth conditional on the stock return). Each point represents the portfolio share in stock as a function of time and the history of stock returns prior to that date. This trading strategy is contrasted with the optimal risky share of an EU investor (red horizontal line). The figure shows that initial risky share is approximately equal for RDU and EU investors, not surprising given our calibration strategy for the parameter γ . Unlike the expected utility case where the optimal portfolio share in stock is constant, however, the portfolio share in stock of the RDU investor is history dependent. Following poor markets returns, investors hold a lower fraction of wealth in stock whereas following good market returns, investors increase their proportional stock holdings. Non-participation can occur following poor

stock performance. For example, following five consecutive down return years, investor stock holdings are negligible, a strategy required to provide constant payoffs in the states where the 10-year stock return is low. Following good stock performance, achieving convexity in terminal wealth requires substantial increases in stock holding. For example, following five up stock returns investors choose to place about 220% of their wealth in stock.

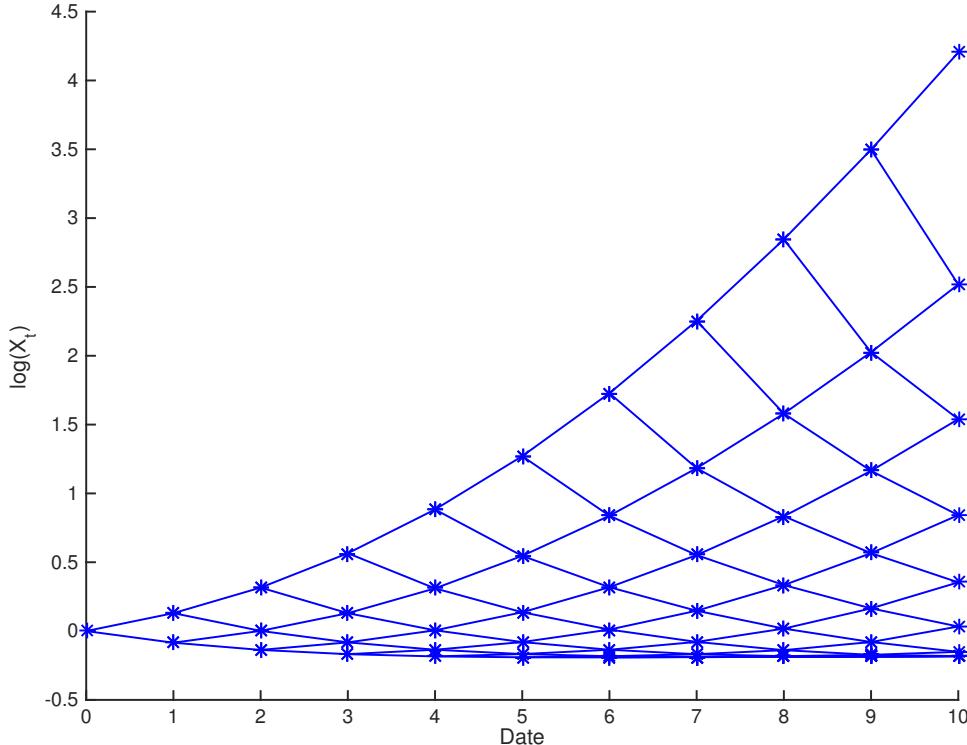


Figure 4: The figure depicts the evolution of wealth along the nine periods binomial tree assuming that the terminal wealth is equal to the average of outcomes conditional on the terminal stock price.

Figure 4 shows the process for average RDU investor wealth resulting from the optimal trading strategy. By construction, the endpoints in the wealth tree correspond to the blue ‘o’ state-contingent wealths in Figure 2. Comparing Figures 3 and 4 shows that a wealth effect should be apparent from a time series of individual investor wealth and risky share, a result that we explore in detail below.

4.3 The Empirical Relevance of the RDU Calibration

Empirical research on household portfolio choice examines the determinants of stock market participation, the risky share conditional on participation, and portfolio rebalancing. In this subsection, we compare moments of data simulated from our calibrated model to this prior evidence.

We utilize an overlapping generations framework to provide simulated data. Every year $\{0, \dots, 9\}$ cohorts of identical investors initiate the optimal dynamic trading strategy presented in Figure 3. We assume that each generation has identical initial wealth that we normalize to $X_0 = 1$. This normalization is without loss of generality for portfolio choice because with the power felicity function, portfolio share dynamics are independent of wealth. Ten generations exist as of date-nine, with the youngest generation just beginning their investing program and the oldest generation entering the last year of their investing timespan. We utilize this framework to generate both cross-sectional datasets, for example by sampling wealth and risky share from each generation as of date-nine, and panel datasets, for example by sampling from each generation at dates six to nine.

Calvet, Campbell, and Sodini (2007, 2009) and Calvet and Sodini (2013) utilize a Swedish panel spanning the end of the calendar years 1999-2002. We use these dates to produce an annual time series of market returns from 1993-2002, coding $\rho_t = u, d$ according to whether the market return exceeds the median return, conditional on the level of the riskfree return. The resulting sequence, using US market returns, is $(u, d, u, u, u, u, u, d, d, d)$. This return sequence is used to create a time-series of wealth and risky share for the oldest generation $i = 0$ by sampling from the appropriate subset of nodes at the first ten dates from Figures 3 and 4. For each younger generation i , we discard the first $i \leq 10$ elements of the market return sequence and sampling from the relevant subset of nodes at the first $10 - i$ dates. Our balanced panel samples from the last four-year history of each generation (dates 6-9 for the oldest generation, 5-8 for the next oldest, etc.), thus each cohort plays the role of an individual in our dynamic panel sample. To simulate a twin dataset comparable to Calvet and Sodini (2013), we form all

possible pairings of cohorts and treat each such pair as a twin. This dataset is consistent with each twin independently, with probability 1/10, beginning his or her investing program at date 0 – 9.

4.3.1 Implications of RDU in the Cross-section

Cross-sectional studies provide strong evidence of a positive correlation between wealth and risky share.²⁰ Figure 5, which summarizes information from our 2002 cross-section (solid line) in manner that can be directly compared to Figure 1 of Calvet, Campbell, and Sodini (2007) (dashed line), shows that our simulated data is qualitatively consistent with a wealth effect. The figure also shows that although the riskfree holdings are generally lower in the simulated data, the difference in holdings of the richest and poorest households is comparable to that of Swedish households in 2002. This result must be interpreted with caution, however, since the cross-sectional variation of wealth in the simulated data is counterfactually low, implying that wealth variation has a much stronger impact on risky share in the simulated data than in the Swedish data.²¹

The prior literature proposes a number of explanations for the cross-sectional evidence that the portfolio share in the riskfree asset decreases when wealth increases. Theoretical explanations rely on an explicit dependence of risk aversion on wealth as captured by the felicity function $v(X_T)$, (e.g., as in a constant absolute risk aversion preference specification) or on a dependence of utility on other state variables (e.g., as in subsistence and habit models). Our calibrated model shows that investors with rank dependent utility optimally increase their risky share when wealth increases and that this behavior can produce cross-sectional data in which wealth effects will be detected.

²⁰See, for example, Cohn, Lewellen, Lease, and Schlarbaum (1975), Morin and Suarez (1983), Blake (1996), and Carroll (2000).

²¹Figure A.1 in the appendix of Calvet, Campbell, and Sodini (2007) shows that the 80th percentile household has approximately 1,000 times the wealth of the 20th percentile household. The comparable figure in our simulated data is 1.3. Wealth dispersion in the Swedish data is due to economic forces that are not present in our model. The only force at play for wealth distribution in our model is due to the fact that different cohorts experience different return histories. Our results show that the difference in portfolio holdings alone are not sufficient to produce the wealth dispersion that we observe in the Swedish sample.

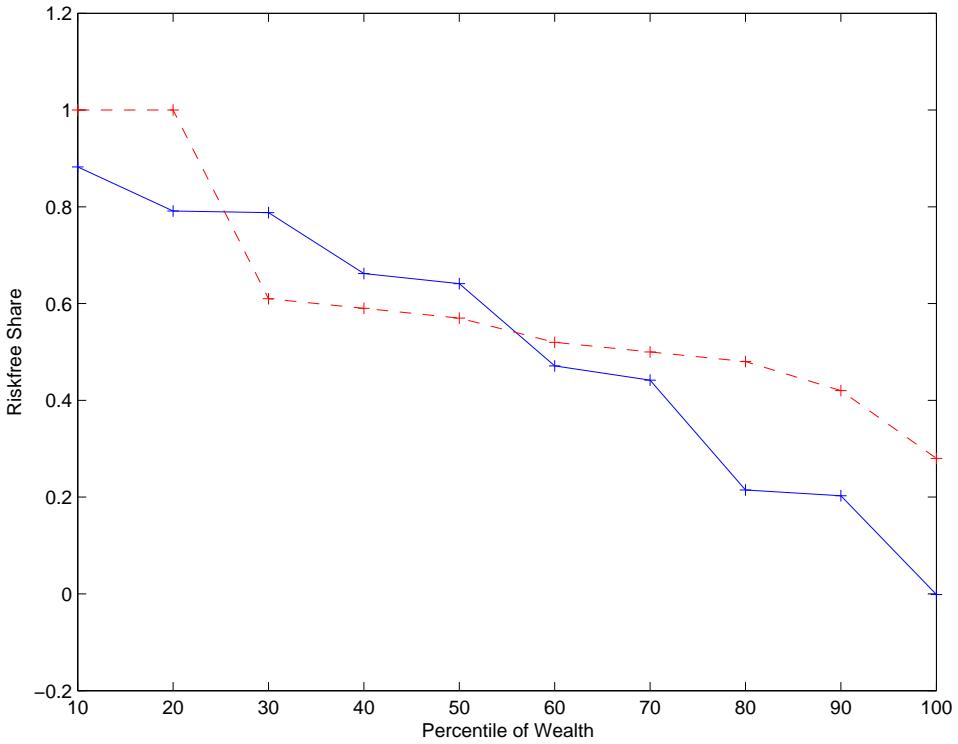


Figure 5: The figure illustrates the correlation between wealth and riskfree share. The blue line maps the percentile of wealth to the riskfree share in the simulated sample of households. The red line maps the percentile of wealth to riskfree share for the Swedish Data (reported from Calvet, Campbell, and Sodini (2007)).

4.3.2 Implications of RDU in Panel Data

The evidence from the cross-section is difficult to interpret when considering how wealth and risky share are endogenously determined. Several explanations for between-individual correlation have been considered in settings where there is no within-individual wealth effect: Transaction costs make the participation decision non-trivial, and if this selection mechanism causes wealth to be positively correlated with the participation decision then the data may give rise to the appearance of higher risky shares among the wealthy even though there is no variation in risky share when conditioning on participation (Tracy and Schneider (2001), Vissing-Jorgensen (2002)). Heterogeneity in risk aversion among individuals with CRRA preferences can give rise to reverse causality, since when more risk tolerant investors hold more stock in rising markets, they will be relatively more wealthy following positive return realizations (Chiappori and Paiella

(2011)).²² The risk characteristics of non-financial wealth, including real estate, entrepreneurial wealth, and labor income, may lead investors to select a mix of liquid assets, such as stocks and bonds, that give the appearance of a wealth effect due to unobservable variation in the composition of non-financial wealth (Heaton and Lucas (2000)).

In light of these challenges, recent empirical studies utilize panel data and instrumentation to explore determinants of portfolio choice (see, for example, Brunnermeier and Nagel (2008), Calvet, Campbell, and Sodini (2007, 2009), Chiappori and Paiella (2011), and Calvet and Sodini (2013)). Panel data allows the study of within-individual variation in wealth and risky share using a variety of empirical techniques. We will consider two empirical specifications. First, the dynamic panel model accounts for unobservable variation due to individual and time effects:

$$y_{it} = a_i + a_t + \beta x_{it} + \varepsilon_{it}, \quad (23)$$

where $x_{it} = \ln(X_{it})$ is the logarithm of wealth of cohort i at date t and the independent variable is either the corresponding portfolio share $y_{it} = \pi_{it}$, as in Brunnermeier and Nagel (2008), or its logarithm $y_{it} = \ln(\pi_{it})$, as in Calvet, Campbell, and Sodini (2009). Calvet and Sodini (2013) create a panel of twin investors from their Swedish dataset and estimate the model

$$\ln(\pi_{1kt}) = a_k + a_t + \beta x_{1kt} + \varepsilon_{1kt} \quad (24)$$

$$\ln(\pi_{2kt}) = a_k + a_t + \beta x_{2kt} + \varepsilon_{2kt}, \quad (25)$$

where the index k identifies a twin pair. The regression coefficient on wealth can be identified by differencing the twin observations and running a linear regression with a time fixed effects.

We now quantify the wealth effect in the context of our RDU model. Figure 6 plots the logarithm of risky share and wealth realizations for each individual in our simulated dataset. The figure confirms that risky share and wealth are positively correlated in the pooled sample. In addition, the relationship is strongly concave, indicating that the slope, which corresponds

²²See, however, Carroll (2000) who argues that other features of the cross-section are inconsistent with this source of endogeneity.

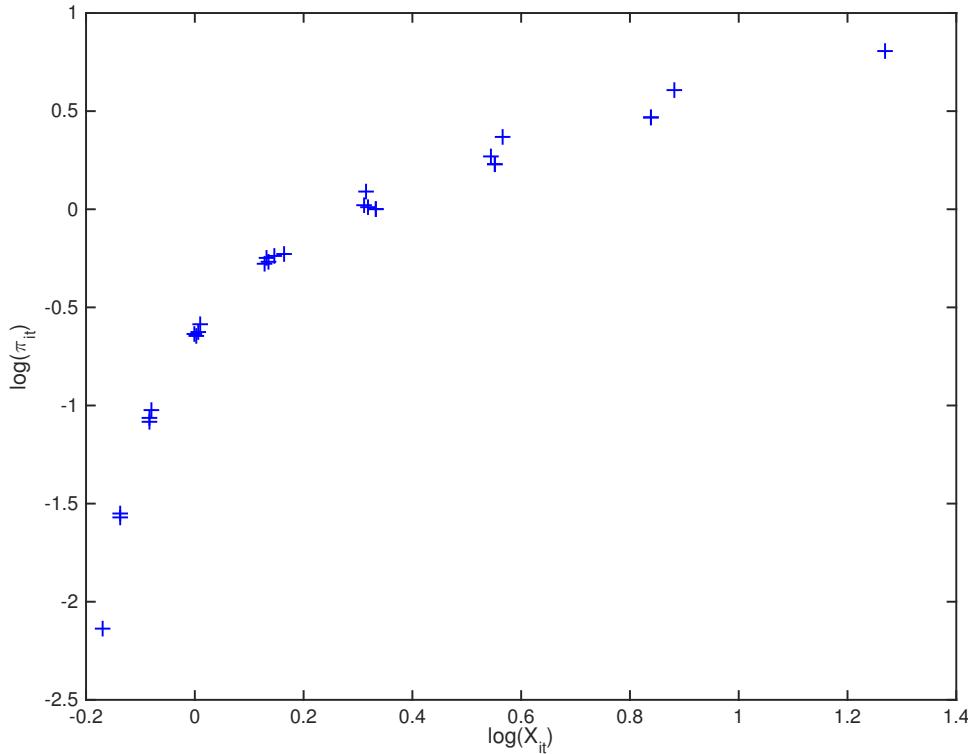


Figure 6: Each sign '+' represents the log wealth and the log risky share for a cohort at a given time during the four years from which we sample our balanced panel.

to the wealth elasticities of risky share, decreases markedly when wealth increases. This feature of our simulated data is qualitatively consistent with the findings reported in Table V of Calvet and Sodini (2013) who document that in subsamples of individuals from the lowest and highest wealth quartiles, the risky share elasticities decrease from 0.289 to 0.101.

Table 1 presents estimates of the sensitivity of portfolio risky share to wealth, both from our simulated data and from the recent empirical literature (Brunnermeier and Nagel (2008), Calvet, Campbell, and Sodini (2009), Chiappori and Paiella (2011), and Calvet and Sodini (2013)). Panel A presents results from regressions run in data on levels (e.g., estimating equation (23) using fixed effects) and Panel B presents results from regressions of first differences, in time for each individual, of the dependent and independent variables (i.e., by estimating the time differenced equation (23)). In Panel A, the row labeled “Simulated” shows that portfolios of RDU investors in the simulated data respond strongly to changes in wealth. For example, the coefficient from the pooled regression (1) indicates that a 10% increase in wealth leads to a

14.57% increase in portfolio share (e.g., a risky share of 50% would increase to 57%). The finding of a strong elasticity is robust to specifications that add individual effects (2) and twin effects (3). The top row in Panel B provides results from the simulated data when accounting for individual fixed effects by first differencing the data. Regression (4) confirms the findings in Panel A, showing that a 10% increase in wealth leads to an increase in risky share of 0.1522 (e.g., a risky share of 50% would increase to 65%). The elasticity is negative in regression (6). This result follows from an endogeneity in the simulated data created by the strong concavity of the wealth-risky share relationship (Figure 6), as we now explain. First, note that the elasticity of the risky share decreases when wealth increases. Second, in the simulated data, where a sequence of three down returns are realized, changes in log wealth are negatively correlated with the level of wealth.²³ The net effect of these correlations with log wealth produces a strong negative bias in the estimated coefficient in regression (6). The concavity of the relationship between risky share (in levels) and log wealth is much weaker, thus this effect does not create a bias in regression specification (4). The positive coefficient in regression (2) of Panel A, which provides an alternative estimate to the slope coefficient in equation (23) using standard fixed effects estimation and would be comparable to the coefficient in regression (6) if the underlying relation were linear, confirms that in our simulated data increases in wealth lead to higher portfolio shares even when accounting for individual fixed effects.

Table 1 shows that the RDU model produces counterfactually high portfolio share elasticities. For example, Panel A shows that even the highest estimate from the actual data, extracted from Table II of Calvet and Sodini (2013), indicates only a 2.31% increase in risky share when wealth increases by 10%. Measurement error would have to account for an enormous fraction of the variance in wealth or wealth changes in order to produce a downward bias sufficient to explain the results in the table.²⁴ We can reconcile the findings from the simulated data in Panel

²³This follows from the fact that 1) the risky share is small, on average, when wealth is low, and 2) the decrease in log wealth is smaller when the risky share is low.

²⁴Denoting the ratio of unbiased to biased regression coefficients by β^U/β^B and the measurement error by ϵ_x , standard results imply $\beta^U/\beta^B = 1 + \text{var}(\epsilon_x)/\text{var}(x)$. Comparing simulated coefficients to empirical estimates in Table 1 then indicates that the variance of measurement error must be at least six times the variance of the level or change in log wealth. This amount of measurement error is not consistent with empirical findings from the prior literature that explicitly account for measurement error by instrumenting for the dependent variables.

Table 1. RISKY SHARE ELASTICITIES WITH RESPECT TO WEALTH.

		A. Regressions in Levels ($\log \pi$)		
		(1)	(2)	(3)
Simulated		1.457	1.398	1.639
Actual		(CS II) 0.231* (CP 2) 0.036* (CP 5) 0.114* (CP 7) 0.070*		(CS II) 0.223*
Fixed Effects	Individual	n	y	n
	Twin	n	n	y
B. Regressions in First Differences				
		$\Delta\pi$		$\Delta \log \pi$
		OLS (4)	IV (5)	OLS (6)
Simulated		1.522		-0.067
Actual		(CCS A5) -0.100* (BN 4) 0.017 (BN 5) -0.103*	(CCS A5) 0.32* (BN 4) -0.136 (BN 5) -0.355*	(CP 3) 0.017 (CP 5) 0.077* (CP 7) 0.047*
				(CS IA.XXXII) 0.225* (CP 3) 0.019 (BN A.2) 0.017 (BN A.3) -0.165*

A by setting $\alpha = 0.96$ (coefficients not reported). To completely eliminate the wealth effect, consistent with the empirical evidence in Brunnermeier and Nagel (2008) and Chiappori and Paiella (2011) (see Panel B of Table 1), would require that we remove probability weights from the preference specification (i.e., set $\alpha = 1$). Table 1 thus illustrates that generalizing behavior based on measurements of probability weights in the lab is problematic in the portfolio choice context since the inverse-S probability weighting elicited from the lab experiments exhibits more curvature than is required to match risky share wealth elasticities measured in the field data.

4.3.3 Portfolio Rebalancing

In order to determine the trading activity associated with the optimal RDU portfolio, we calculate the difference between active and passive portfolio weights for all stock return histories. Specifically, following Calvet, Campbell, and Sodini (2009), we define the passive weight as

$$\pi_t^P = \frac{\pi_{t-1}\rho_t}{\pi_{t-1}\rho_t + (1 - \pi_{t-1})R}. \quad (26)$$

The difference $\pi_t - \pi_t^P$ between the active and passive weights provides information on trading activity; positive differences indicate that investors must purchase stock in order to rebalance to optimal weights after a return realization and negative differences imply stock sales. Figure 7 plots the active-passive difference as a function of lagged stock return for every possible stock return history generated by the portfolio weights in Figure 3. The figure shows that for our baseline calibration, RDU investors purchase stock following positive returns and sell following negative returns. This implies that RDU investors implement a momentum-style investing strategy, opposite to the contrarian investing strategy required to rebalance to constant weights employed by EU investors. Our model can be recalibrated so that the RDU investor undertakes more contrarian trades by increasing the parameter α and flattening the probability weighting function. This will have the effect of also reducing the wealth effect illustrated in Table 1.

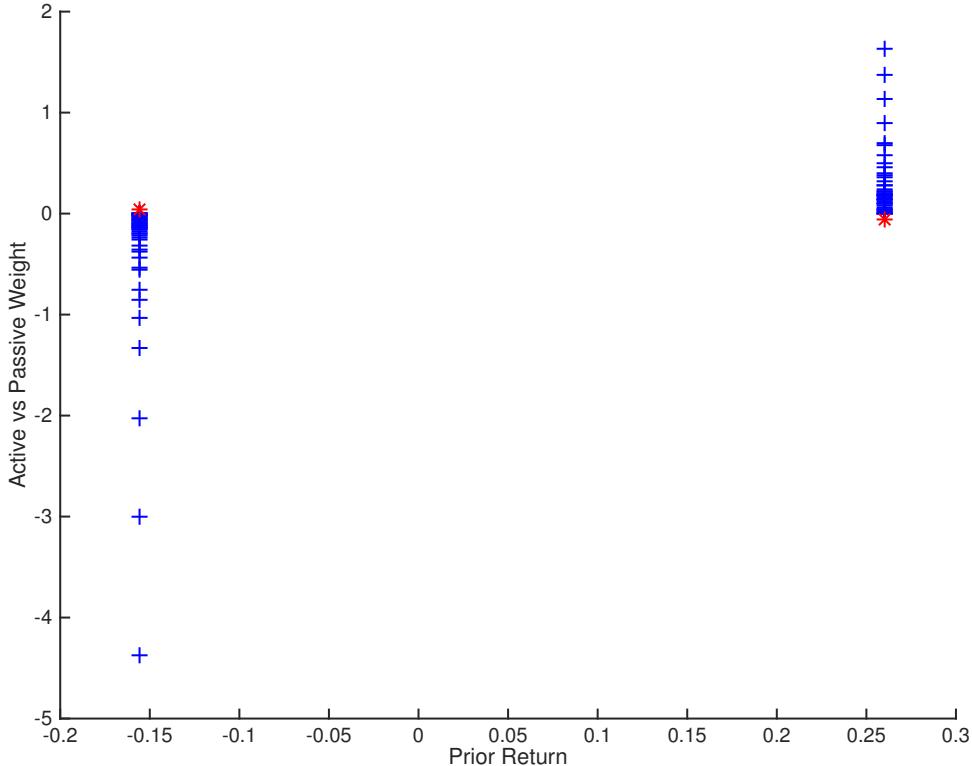


Figure 7: The figure illustrates the difference between active and passive portfolio weights at each node in the binomial tree as a function of past one-period return. RDU rebalancing (blue '+') is consistent with momentum-style investing whereas EU rebalancing (red '*') is contrarian.

Empirical studies of portfolio rebalancing by households documents a strong effect of pas-

sive returns on portfolio active share. Calvet, Campbell, and Sodini (2009) find that Swedish investors trades are contrarian, whereby approximately half of passive risky share changes are offset through active rebalancing. Brunnermeier and Nagel (2008) find that US investor portfolio weights are almost entirely explained by passive returns, indicating no active trade. The momentum-style trading of RDU investors in our calibrated model, which is to be expected given the excessive wealth effects documented in Section 4.3.2, is not found in either of these empirical studies. This again suggests that extrapolating evidence on probability weighting from the experimental setting predicts investing behavior that is inconsistent with the field data.

5 Conclusion

Our paper provides a new methodology for determining optimal asset allocation of investors with RDU that leads to precise and novel empirical predictions. Existing empirical evidence of household investing behavior confirms many of implications of a calibrated version of the model. The optimal dynamic trading strategies of RDU investors gives rise to higher participation in stock investments and higher risky share along paths where investors relatively rich. This dynamic portfolio gives rise to a convex relationship between stock return and terminal wealth, and there is anecdotal evidence of demand for structured products that produce such returns. Investors also optimally demand lottery-like payoffs which have been shown to be an important component of household wealth. Cross sectional data from such investors drawn from different cohorts has the potential to reconcile with empirical evidence. Our calibrated model produces momentum-style rebalancing trades, however, inconsistent with evidence on contrarian-style household rebalancing. Our model is also not able to produce portfolio dynamics in which investors switch from a safe to risky portfolio.

Our calibrated model makes several empirically relevant predictions. In a cross-section of investors from different birth cohorts: 1) Wealth and risky share are positively correlated, and 2) A fraction of investors may hold no stock. Our model also predicts that risky share for all RDU investors will increase following good stock returns and that RDU investors implement

trading strategies consistent with a momentum style. As mentioned previously, investors who do not wish to trade dynamically, perhaps because of transaction costs, will find structured products that offer downside protection and upside market exposure to be attractive relative to structured products that implement constant market exposure. The model implies demand for gambling conditional on the terminal stock price or, alternatively, heterogeneity in portfolio holdings among individual investors to produce differentiated payouts among states with the same stock return.

It is, perhaps, unfair to attribute the successes and failures of the model to our assumptions regarding preferences alone. Several auxiliary assumptions are present in the benchmark models of investing behavior: Dynamically complete markets with independent one-period Bernoulli distributed return, perfectly tradeable investor wealth, frictionless trade, and power felicity. Our model has purposefully been constructed to allow us to understand the pure effect of deviating from the traditional EU-based portfolio choice model along only the preference dimension. Our solution technique can be easily modified to accommodate relaxation of many of the other assumptions as long as markets remain complete. For example, if expected stock returns are predictable, the prices of contingent claims against terminal wealth will change but our two-stage solution technique still produce the exact form of optimal wealth and replicating portfolio strategies. Our paper therefore provides a framework for understanding RDU portfolio choice in more elaborate settings that more closely approximate reality.

One critical assumption underlying our model is that investors pre-commit to their trading strategy. This is not an issue when investors have EU, since the independence axiom implies dynamic consistency. RDU relaxes independence and a byproduct is that preferences are dynamically inconsistent. An alternative approach to the one we adopt is to solve the problem by backward recursion in which the decision maker's current choice takes as given future decisions (Strotz (1955)).²⁵ Either assumption might be viewed as valid and our paper does not address the possibility that empirical regularities may be better explained by combining RDU with a

²⁵For recent work dealing with this issue, which is at the forefront of research on non-standard preference, see Barberis (2012).

different dynamic preference structure. We speculate that even in this framework, however, the methods and intuition we have developed will be useful for moving forward.

Appendix

Proof of Proposition 1 The formulas for X_m follow from direct application of standard constrained convex optimization techniques. The portfolio weight formula follows from the explicit calculation of the stock share in the replicating portfolio. See, for example, Dybvig and Rogers (1997).

Proof of Proposition 2 We begin the proof with the following Lemma.

Lemma 1. *Consider terminal wealth X satisfying the budget constraint (9) with equality. Assume that wealth in state m_1 is strictly less than wealth in state m_2 for two state indices $m_1 < m_2$. Suppose that, in addition, the state price in state m_1 is strictly lower than the state price in state m_2 : $\psi_{m_1} < \psi_{m_2}$. Then there exists an alternative terminal wealth \hat{X} that provides the same level of utility at strictly lower cost. This terminal wealth can be obtained by exchanging wealth between the states m_1 and m_2 :*

$$\begin{aligned}\hat{X}_{m_1} &= X_{m_2} \\ \hat{X}_{m_2} &= X_{m_1} \\ \hat{X}_m &= X_m \text{ otherwise.}\end{aligned}$$

Furthermore, there exists a terminal wealth \tilde{X} with the same cost as X that provides strictly higher utility: $U(\tilde{X}) > U(X)$.

Proof of Lemma: Given that the distribution of states is uniform, switching outcomes of the terminal wealth X across state does not affect the distribution of X . Law invariance of RDU implies that $U(\hat{X}) = U(X)$. The cost of terminal wealth X is strictly larger than the cost of terminal wealth \hat{X} :

$$\Gamma = \sum_{m=1}^M \psi_m X_m - \sum_{m=1}^M \psi_m \hat{X}_m = (\psi_{m_1} - \psi_{m_2})(X_{m_1} - X_{m_2}) > 0$$

which completes the proof of the first statement in the proposition.

Consider the terminal wealth $\tilde{X} = \hat{X} + \Gamma R^T$. Its cost is

$$\begin{aligned} \sum_{m=1}^M \psi_m \hat{X}_m + \sum_{m=1}^M \psi_m \Gamma R^T &= X_0 - \Gamma + \Gamma \\ &= X_0 \end{aligned}$$

and since $\tilde{X}_m > \hat{X}_m$ for every state m , $U(\tilde{X}) > U(\hat{X}) = U(X)$. \square

We now complete the proof of the proposition. State prices are weakly increasing in the index m . Proposition 2 implies a contradiction to the optimality of wealth plan X if $X_{m_1} < X_{m_2}$ and $\psi_{m_1} < \psi_{m_2}$. If $\psi_{m_1} = \psi_{m_2}$ then exchanging wealth in states m_1 and m_2 will not affect the cost or the distribution (hence utility) of the allocation. \square

Proof of Proposition 3 Consider an optimal terminal wealth X solving Problem 2. This wealth allocation induces a partition $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of contiguous states, where the realization of wealth is constant within each set: i.e., there exists a sequence of distinct wealths $\{x_1, \dots, x_n\}$ such that $X_m = x_i$ for all $m \in S_i$. Proposition 2 implies that this sequence must satisfy $x_1 > \dots > x_n$, since for all i , every set $m \in S_i$ has a lower index than states in the set S_{i+1} . In order to offset the incentives to shift small increments of wealth across the events S_i , marginal felicity across sets in \mathcal{S} must satisfy

$$\frac{v'(x_i)}{v'(x_{i+1})} = \left(\frac{x_i}{x_{i+1}} \right)^{-\gamma} = \frac{\bar{h}_{i+1}/\bar{\psi}_{i+1}}{\bar{h}_i/\bar{\psi}_i} \quad (27)$$

where $\bar{h}_i = \sum_{m \in S_i} h_m$ and $\bar{\psi}_i = \sum_{m \in S_i} \psi_m$. To be consistent with a decreasing sequence of wealth realizations $x_i > x_{i+1}$, the first-order conditions (27) imply

$$\frac{\bar{h}_{i+1}}{\bar{\psi}_{i+1}} < \frac{\bar{h}_i}{\bar{\psi}_i} \quad (28)$$

for $i = 1, \dots, n-1$.

We now give a geometric characterization of inequality (28). Let m_i denote the maximal element of set S_i for $i = 1, \dots, n$. The collection of these points, augmented with the element

$\{0\}$, defines the set $\mathcal{M}^S = \{0, m_1, \dots, m_n = M\}$. Define the function $G^S : \Phi \rightarrow [0, 1]$ by $G^S(\phi_{m_i}) = H_{m_i}$ for all $m_i \in \mathcal{M}^S$. For points in $\mathcal{M} \setminus \mathcal{M}^S$ the function G^S is defined by linearly interpolating between consecutive points of the set $\mathcal{G}^S = \{(\phi_m, H_m) : m \in \mathcal{M}^S\}$. For any two consecutive points of the set \mathcal{G}^S characterized by the indices m_i and m_{i+1} , the left slope of the function G^S is

$$\frac{H_{m_{i+1}} - H_{m_i}}{\phi_{m_{i+1}} - \phi_{m_i}} = \frac{\sum_{m=m_i+1}^{m_{i+1}} h_m}{\sum_{m=m_i+1}^{m_{i+1}} \psi_m} = \frac{\sum_{m \in S_{i+1}} h_m}{\sum_{m \in S_{i+1}} \psi_m} = \frac{\bar{h}_{i+1}}{\bar{\psi}_{i+1}}. \quad (29)$$

Inequalities (28) says that the left slope of the function G^S form a decreasing sequence and this implies that the function G^S is concave. The next lemma shows that in fact, the function G^S is the concavification of the function G . This result concludes the proof because the proposition characterizes the optimal policies by making use of the unique concavification of the function G .

Lemma 2. *The function G^S is the concavification of the function G : $G^S = G^*$.*

Proof. First, we show that G^S dominates G . Assume that there exists $\hat{m} \in \mathcal{M}$ such that $G^S(\phi_{\hat{m}}) < G(\phi_{\hat{m}})$. Notice that since by construction G^S and G coincide on \mathcal{M}^S , we must have $\hat{m} \in \mathcal{M} \setminus \mathcal{M}^S$. Assume for example that \hat{m} is an element of the set S_j for a given $j \in \{1, \dots, n\}$. Define the function \hat{G}^S mapping Φ into $[0, 1]$ by linearly interpolating between consecutive points of the set $\{(\phi_m, H_m) : m \in \mathcal{M}^S \cup \{\hat{m}\}\}$. The function \hat{G}^S coincides with the function G^S on all the events $(S_i)_{i \neq j}$. We also have $\hat{G}^S(\phi_{m_{j-1}}) = G^S(\phi_{m_{j-1}})$ and $\hat{G}^S(\phi_{m_j}) = G^S(\phi_{m_j})$ and $\hat{G}^S(\phi_{\hat{m}}) = G(\phi_{\hat{m}}) > G^S(\phi_{\hat{m}})$. Since $\phi_{m_{j-1}} < \phi_{\hat{m}} < \phi_{m_j}$ and G^S is linear on the interval $[\phi_{m_{j-1}}, \phi_{m_j}]$, the function \hat{G}^S must be concave at the point $\phi_{\hat{m}}$, that is

$$\frac{H_{\hat{m}} - H_{m_{j-1}}}{\phi_{\hat{m}} - \phi_{m_{j-1}}} > \frac{H_{m_j} - H_{\hat{m}}}{\phi_{m_j} - \phi_{\hat{m}}}. \quad (30)$$

We will proceed by contradiction and construct a wealth process that satisfies the budget constraint and generates a larger utility than the wealth process X . Construct the wealth process \hat{X} as follow

$$\hat{X}_m = \begin{cases} X_m & \text{for } m \in \mathcal{M} \setminus S_j \\ x_j + \eta & \text{for } m = m_{j-1} + 1, \dots, \hat{m} \\ x_j - \varepsilon & \text{for } m = \hat{m} + 1, \dots, m_j \end{cases}$$

where $\varepsilon > 0$ is a small number and where η is set so that the wealth \hat{X} satisfies the budget constraint (17) with equality, that is

$$\eta = \varepsilon \frac{\phi_{m_j} - \phi_{\hat{m}}}{\phi_{\hat{m}} - \phi_{m_{j-1}}}. \quad (31)$$

If the marginal change ε is small enough and when $i \neq j$, the decision weight attributed to each outcome x_i when calculating the utility of the wealth \hat{X} is identical to the decision weight attributed to the same outcome when calculating the utility of the wealth X . Therefore the difference in utilities can be summarized

$$U(\hat{X}) - U(X) = (H_{\hat{m}} - H_{m_{j-1}})v(x_j + \eta) + (H_{m_j} - H_{\hat{m}})v(x_j - \varepsilon) - (H_{m_j} - H_{m_{j-1}})v(x_j)$$

which can be approximated for small ε by

$$U(\hat{X}) - U(X) = v'(x_j) ((H_{\hat{m}} - H_{m_{j-1}})\eta - (H_{m_j} - H_{\hat{m}})\varepsilon)$$

and using the expression for (31), we get

$$U(\hat{X}) - U(X) = \varepsilon v'(x_j)(\phi_{m_j} - \phi_{\hat{m}}) \left(\frac{H_{\hat{m}} - H_{m_{j-1}}}{\phi_{\hat{m}} - \phi_{m_j}} - \frac{H_{m_j} - H_{\hat{m}}}{\phi_{m_j} - \phi_{\hat{m}}} \right) > 0$$

where the last inequality follows from (30). We thus conclude that G^S dominates G .

We now show that G^S is the smallest concave function that dominates G . To this end, assume to the contrary that there exists a concave function \tilde{G}^S satisfying

$$G(\phi_m) \leq \tilde{G}^S(\phi_m) \leq G^S(\phi_m)$$

for $m = 0, \dots, M$. Notice that $G(\phi_m) = \tilde{G}^S(\phi_m) = G^S(\phi_m)$ for all $m \in \mathcal{M}^S$. In particular, for any $j = 1, \dots, n$, we have $\tilde{G}^S(\phi_{m_{j-1}}) = G^S(\phi_{m_{j-1}})$ and $\tilde{G}^S(\phi_{m_j}) = G^S(\phi_{m_j})$. If there exist $\hat{m} \in \{m_{j-1} + 1, \dots, m_j - 1\}$ such that $\tilde{G}^S(\phi_{\hat{m}}) < G^S(\phi_{\hat{m}})$ then the function \tilde{G}^S cannot be concave: On the interval $[\phi_{m_{j-1}}, \phi_{m_j}]$ the function \tilde{G}^S is a continuous piecewise linear function and dominated from above by the linear function G^S and so it must admit at least one point at which it is convex. \square

\square

Proof of Proposition 4: Let the initial stock price be $P(0) = 1$, in which case the stock price at date- t is $P(t) = \rho_1 \rho_2 \cdots \rho_t$. The bond price is similarly initialized to $B(0) = 1$ and at date- t must be $B(t) = R^t$. Let $\mathcal{M}_{(\mathcal{F}_t, u)}$ (resp. $\mathcal{M}_{(\mathcal{F}_t, d)}$) denote the subset of \mathcal{M} that contains all the states $m = (\rho_1, \dots, \rho_t, u, \omega_{t+2}, \dots, \omega_T)$ (resp. $m' = (\rho_1, \dots, \rho_t, d, \omega_{t+2}, \dots, \omega_T)$). The independence of one-period returns implies that the conditional state price for state m at date $t + 1$ is $\psi_{t+1, T, m} = \xi_u^{T-t-1-k_m(t+1)} \xi_d^{k_m(t+1)}$, where $k_m(t+1)$ is the number of down returns in the subsequence $(\omega_{t+2}, \dots, \omega_T)$ of state m . For terminal wealth X , the replicating strategy (Δ^X, B^X) must solve

$$\begin{aligned}\Delta^X(t)P_t u + B^X(t)R^{t+1} &= \sum_{\mathcal{M}_{(\mathcal{F}_t, u)}} \psi_{t+1, T, m} X_m \\ \Delta^X(t)P_t d + B^X(t)R^{t+1} &= \sum_{\mathcal{M}_{(\mathcal{F}_t, d)}} \psi_{t+1, T, m} X_m,\end{aligned}$$

for $t = 0, \dots, T-1$. The final two equations imply

$$\Delta^X(t)P_t(u-d) = \sum_{\mathcal{M}_{(\mathcal{F}_t, u)}} \psi_{t+1, T, m} X_m - \sum_{\mathcal{M}_{(\mathcal{F}_t, d)}} \psi_{t+1, T, m} X_m, \quad (32)$$

showing that long stock positions in the replicating portfolio are necessary whenever wealth is higher following an “up” stock return than following a “down” stock return.

Notice that the two sets $\mathcal{M}_{(\mathcal{F}_t, u)}$ and $\mathcal{M}_{(\mathcal{F}_t, d)}$ have the same cardinality and that each state indexed by m in the set $\mathcal{M}_{(\mathcal{F}_t, u)}$ can be matched with a unique state m' in the set $\mathcal{M}_{(\mathcal{F}_t, d)}$ that

differs only by the value of ω_{t+1} . Equation (3) implies that $\psi_m < \psi_{m'}$, and from Proposition 2, which shows that optimal wealth is nonincreasing in state price, we can infer that $X_{m'} = X_m - \varepsilon_m$ where $\varepsilon_m \geq 0$. Finally, note that independence of one-period stock returns implies that the conditional state prices for payoffs in states m and m' are equal. This implies

$$\sum_{\mathcal{M}(\mathcal{F}_t, u)} \psi_{t+1, T, m} X_m - \sum_{\mathcal{M}(\mathcal{F}_t, d)} \psi_{t+1, T, m} X_m = \sum_{\mathcal{M}(\mathcal{F}_t, u)} \psi_{t+1, T, m} \varepsilon_m \geq 0.$$

We now prove the second statement of the proposition. When \bar{G}^* is linear, the partition of states is the singleton \mathcal{M} , and by Proposition 3 terminal wealth will be optimally equalized across all states. The only way to duplicate such a payoff when bond returns are riskless, as assumed, is to invest all wealth in the bond. \square

Proof of Proposition 5: We consider sequences of economies where initial wealth X_0 is split evenly among I identical investors. Applying Proposition 3, let the optimal aggregate wealth in a single investor economy be denoted $X = Y + X^g$ and define the following family of random variables associated with X^g : $X^{g,k} = X^g \mathbb{1}_{k_\omega=k}$ for $k = 0, \dots, T$. Each of these random variables is zero everywhere except on a finite number of states $m_k, \dots, m_k + n_k$ where the state price is $\psi = \xi_u^{T-k} \xi_d^k$. Assume that each investor in the I -investor economy is endowed with initial wealth X_0/I . We define the random variables $X_i^{g,k}$ in the following way:

$$X_{i,m}^{g,k} = \begin{cases} X_{\theta_k(m)}^{g,k} & \text{if } k(m) = k \\ 0 & \text{if } k(m) \neq k, \end{cases} \quad (33)$$

where for each k , θ_k is a permutation of the set $m_k, \dots, m_k + n_k$. The terminal wealth $X_i = Y/I + X_i^g/I$, where $X_i^g = \sum_k X_i^{g,k}$, has the same distribution as X/I and by the law invariance of RDU and the absence of wealth effects implied by the power felicity v , X_i must be optimal for investor i .²⁶

Assume every investor independently applies the previous procedure to choose their optimal

²⁶Equality in distribution of X and X_i follows because X_i is constructed from X by permuting the $X^{g,k}$ across equally likely states conditional on each occurrence of k_ω .

terminal wealth. In any terminal state with $k(m) = k$, the distribution across individuals i of $X_i^{g,k}$ will equal the distribution of $X^{g,k}$ across states with $k(\omega) = k$, a consequence of the uniform distribution of m . In particular, the mean of this cross-sectional distribution is zero.

Aggregate investment is given by

$$\sum_i X_i = \sum_i Y/I + \sum_i X_i^g/I \quad (34)$$

$$= Y + \sum_i \sum_k X_i^{g,k}/I \quad (35)$$

$$= Y + \sum_k \sum_i X_i^{g,k}/I. \quad (36)$$

By the strong law of large numbers, as the number of investors increases to infinity, $\sum_i X_i^{g,k}/I \rightarrow 0$ and aggregate wealth approaches $X = Y$.

We now prove the second statement in the proposition. Denote the replicating strategies for each permutation X_i of X by (Δ_i, B_i) . Since the distributions of X_i across investors are independent and uniform with finite support, the distributions of the supporting dynamic strategies (Δ_i, B_i) across individual investors are also independent and uniform with finite support. By the strong law of large numbers, $\sum_i \Delta_i$ must converge to $\mathbb{E}(\Delta_i)$ and $\sum_i B_i$ must converge to $\mathbb{E}(B_i)$. The trading strategy $(\sum_i \Delta_i, \sum_i B_i) \rightarrow (\mathbb{E}(\Delta_i), \mathbb{E}(B))$ must replicate $\sum_i X_i \rightarrow Y$, and by uniqueness of the replicating strategy for Y , $(\mathbb{E}(\Delta_i), \mathbb{E}(B)) = (\Delta^Y, B^Y)$. \square

time	t
horizon	T
bond return (log, gross)	r, R
stock return (gross)	$\rho_t, t = 1, \dots, T$
realized stock returns	u, d
probability of stock return u	p
mean log stock return	μ
std dev of log stock return	σ
portfolio return	$R_t, t = 0, \dots, T - 1$
risky share	$\pi_t, t = 0, \dots, T - 1$
short-lived AD prices	ξ_u, ξ_d
state space	Ω or \mathcal{M}
state	ω or m
number of states	M
number of down returns in state m	k_m
long-lived AD prices	ψ_m
terminal wealth	X
felicity of wealth X_m	$v(X_m)$
power utility parameter	γ
wealth outcome	x_i
number of distinct wealth outcomes	N
CDF of terminal wealth	F
probability weighting function	H
payoff rank	j
decision weight	w_i and h_j
utility of wealth X	$U(X)$
state rank and permutation	θ, λ
Prelec parameters	α, β
cumulative state price	$\phi_m \in \Phi$
cumulative decision weight	H_V
set of probability weights	\mathcal{H}
weight vs state price function	G
graph of G	\mathcal{G}
concavification of G	G^*
states index of vertices of G^*	\mathcal{M}_V
optimal partition of Ω	\mathcal{S}^*
elements of \mathcal{S}^*	S_i^*
price of payoff in S_i^*	ψ_i^*
decision weight for S_i^*	h_i^*
optimal payoff components	Y, X^g
optimal trading strategy	(Δ^i, B^i)
free variables	i, n

Table of notation

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