

GRESHAM'S LAW OF MODEL AVERAGING

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ABSTRACT. An agent operating in a self-referential environment thinks the parameters of his model might be time-varying. In response, he estimates two models, one with time-varying parameters, and another with constant parameters. Forecasts are then based on a Bayesian Model Averaging strategy, which mixes forecasts from the two models. In reality, structural parameters are constant, but the (unknown) true model features expectational feedback, which the agent's reduced form models neglect. This feedback allows the agent's fears of parameter instability to be self-confirming. Within the context of a standard linear present value asset pricing model, we use the tools of large deviations theory to show that the agent's self-confirming beliefs about parameter instability exhibit Markov-switching dynamics between periods of tranquility and periods of instability. However, as feedback increases, the duration of the unstable state increases, and instability becomes the norm. Even though the constant parameter model would converge to the (constant parameter) Rational Expectations Equilibrium if considered in isolation, the mere presence of an unstable alternative drives it out of consideration.

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1. INTRODUCTION

Econometric model builders quickly discover their parameter estimates are unstable.¹ It's not at all clear how to respond to this. Maybe this drift is signalling model misspecification. If so, then by appropriately adapting a model's specification, parameter drift should dissipate over time. Unfortunately, evidence suggests that drift persists even when models are adapted in response to the drift. Another possibility is that the underlying environment is inherently and exogenously nonstationary, so there is simply no hope of describing economic dynamics in models with constant parameters. Clearly, this is a rather negative prognosis. Our paper considers a new possibility, one that is consistent with both the observed persistence of parameter drift, and its heteroskedastic nature. We show that in self-referential environments, where the agent's own beliefs influence the data-generating process (DGP), it is possible that persistent parameter drift becomes self-confirming. That is, parameters drift simply because agents think they *might* drift. We show that this instability can arise even in models that would have unique and determinate equilibria if parameters were known. Self-confirming volatility arises here because

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¹See, e.g., Cogley and Sargent (2005), Fernandez-Villaverde and Rubio-Ramirez (2007), and Inoue and Rossi (2011) for evidence on parameter instability in macroeconomic models. Bacchetta and van Wincoop (2013) discuss parameter instability in exchange rate models.

agents are assumed to be unaware of their own influence over the DGP, and respond to it indirectly by adapting parameter estimates.²

We consider a standard present value asset pricing model. This model relates current prices to current fundamentals, and to current expectations of future prices. Agents are assumed to be unaware of this expectational feedback. Instead, they posit reduced form models, and update coefficient estimates as needed. If agents believe parameters are constant, and update estimates accordingly using recursive Least-Squares, their beliefs will eventually converge to the true (constant parameters) Rational Expectations equilibrium (see, e.g., Evans and Honkapohja (2001) for the necessary stability conditions). On the other hand, if they are convinced parameters drift, with a constant innovation variance that is strictly positive, they will estimate parameters using the Kalman filter, and their beliefs will exhibit persistent fluctuations around the Rational Expectations equilibrium.

A large recent literature argues that these so-called ‘constant gain’ (or ‘perpetual learning’) models are useful for understanding a wide variety of dynamic economic phenomena.³ Unfortunately, several nagging questions plague this literature - Why are agents so convinced that parameters are time-varying? In terms of explaining volatility, don’t these models in a sense “assume the result”? What if agents’ beliefs were less dogmatic, and allowed for the *possibility* that parameters were constant? As in Kalai and Lehrer (1993), wouldn’t this ‘grain of truth’ property cause the constant parameter model to eventually dominate?

Our paper addresses these questions. We do so by extending an example presented by Evans, Honkapohja, Sargent, and Williams (2013). They study a standard cobweb model, in which agents consider two models. One model has constant parameters, and the other has time-varying parameters (TVP). When computing forecasts of next period’s price, agents hedge their bets by engaging in a traditional Bayesian Model Averaging strategy. That is, forecasts are just the current probability weighted average of the two models’ forecasts. Using simulations, they find that if expectational feedback is sufficiently strong, the weight on the TVP model often converges to one, even though the underlying structural model features constant parameters. As in Gresham’s Law, ‘bad models drive out good models’.⁴

We show that averaging between constant and TVP models generates a hierarchy of *four* time-scales. The data operate on a relatively fast calendar time-scale. Estimates of the TVP model evolve on a slower time-scale, determined by the innovation variance of the parameters. Estimates of the constant-parameter model evolve even slower, on a time-scale determined by the inverse of the historical sample size. Finally, the model weight evolves on a variable time-scale, but spends most of its time in the neighborhood of

²Another possibility is sometimes advanced, namely, that parameter drift is indicative of the Lucas Critique at work. This is an argument that Lucas (1976) himself made. However, as noted by Sargent (1999), the Lucas Critique (by itself) cannot explain parameter drift.

³Examples include: Sargent (1999), Cho, Williams, and Sargent (2002), Marcet and Nicolini (2003), Kasa (2004), Chakraborty and Evans (2008), and Benhabib and Dave (2014).

⁴Gresham’s Law is named for Sir Thomas Gresham, who was a financial adviser to Queen Elizabeth I. He is often credited for noting that ‘bad money drives out good money’. Not surprisingly, ‘Gresham’s Law’ is a bit of a misnomer. As DeRoover (1949) documents, it was certainly known before Gresham, with clear descriptions by Copernicus, Oresme, and even Aristophanes. There is also debate about its empirical validity (Rolnick and Weber (1986)).

either 0 or 1, where it evolves on a time-scale that is even slower than that of the constant parameters model. This hierarchy of time-scales allows us to exploit standard time-scale separation methods to simplify the analysis of the dynamics (Borkar (2008)).

Using these methods, we prove that if feedback is sufficiently strong, the weight on the TVP model converges to one. In this sense, Gresham was right; bad models do indeed drive out good models. With empirically plausible parameter values, we find that steady state asset price volatility is more than 90% higher than it would be if agents just used the constant parameters model. The intuition for why the TVP model eventually dominates is the following - When the weight on the TVP model is close to one, the world is relatively volatile (due to feedback). This makes the constant parameters model perform relatively poorly, since it is unable to track the feedback-induced time-variation in the data. Of course, the tables are somewhat turned when the weight on the TVP model is close to zero. Now the world is relatively tranquil, and the TVP model suffers from additional noise, which puts it at a competitive disadvantage. However, as long as this noise isn't too large, the TVP model can take advantage of its ability to respond to rare sequences of shocks that generate 'large deviations' in the estimates of the constant parameters model. In a sense, during tranquil times, the TVP model is lying in wait, ready to pounce on, and exploit, large deviation events. These events provide a foothold for the TVP model, which due to feedback, allows it to regain its dominance. It is tempting to speculate whether this sort of mechanism could be one factor in the lingering, long-term effects of rare events like financial crises.

The remainder of the paper is organized as follows. The next section presents our asset pricing version of the model in Evans, Honkapohja, Sargent, and Williams (2013). We first study the implications of learning with only one model, and discuss whether beliefs converge to self-confirming equilibria. We then allow the agent to consider both models simultaneously, and examine the implications of Bayesian Model Averaging. Section 3 contains our proof that the weight on the TVP model eventually converges to one. Section 4 illustrates our results with a variety of simulations. These simulations reveal that during the transition the agent occasionally switches between the two models. Section 5 discusses the robustness of the results to alternative definitions of the model space, and emphasizes the severity of Kalai and Lehrer's (1993) 'grain of truth' condition. Finally, the Conclusion discusses a few extensions and potential applications, while the Appendix collects proofs of various technical results.

2. TO BELIEVE IS TO SEE

To illustrate the basic idea, let us start with a motivating example. This example is inspired by the numerical simulations of Evans, Honkapohja, Sargent, and Williams (2013) using a cobweb model. We argue that the findings of Evans, Honkapohja, Sargent, and Williams (2013) can potentially apply to a broader class of dynamic models.

2.1. Motivating example. Consider the following workhorse asset-pricing model, in which an asset price at time t , p_t , is determined according to

$$p_t = \delta z_t + \alpha \mathbf{E}_t p_{t+1} + \sigma \epsilon_t \tag{2.1}$$

where z_t denotes observed fundamentals (e.g., dividends), and where $\alpha \in (0, 1)$ is a (constant) discount rate, which determines the strength of expectational feedback. Empirically, it is close to one. The ϵ_t shock is Gaussian white noise. Fundamentals are assumed to evolve according to the AR(1) process

$$z_t = \rho z_{t-1} + \sigma_z \epsilon_{z,t} \quad (2.2)$$

for $\rho \in (0, 1)$. The fundamentals shock, $\epsilon_{z,t}$, is Gaussian white noise, and is assumed to be orthogonal to the price shock ϵ_t . The unique stationary rational expectations equilibrium is

$$p_t = \frac{\delta}{1 - \alpha\rho} z_t + \sigma \epsilon_t. \quad (2.3)$$

Along the equilibrium path, the dynamics of p_t can only be explained by the dynamics of fundamentals, z_t . Any excess volatility of p_t over the volatility of z_t must be soaked-up by the exogenous shock ϵ_t .

It is well known that Rational Expectations versions of this kind of model cannot explain observed asset price dynamics (Shiller (1989)). Not only are prices excessively volatile, but this volatility comes in recurrent ‘waves’. Practitioners respond to this using reduced form ARCH models. Instead, we try to *explain* this persistent stochastic volatility by assuming that agents are engaged in a process of Bayesian learning. Of course, the notion that learning might help to explain asset price volatility is hardly new (see, e.g., Timmermann (1996) for an early and influential example). However, early examples were based on least-squares learning, which exhibited asymptotic convergence to the Rational Expectations Equilibrium. This would be fine if volatility appeared to dissipate over time, but as noted earlier, there is no evidence for this. In response, a more recent literature has assumed that agents use so-called *constant gain* learning, which discounts old data. This keeps learning alive. For example, Benhabib and Dave (2014) show that constant gain learning can generate persistent excess volatility, and can explain why asset prices have fat-tailed distributions even when the distribution of fundamentals is thin-tailed.

Our paper builds on the work of Benhabib and Dave (2014). The key parameter in their analysis is the update gain. Not only do they assume it is bounded away from zero, but they restrict it to be *constant*. Following Sargent and Williams (2005), they note that a constant gain can provide a good approximation to the (steady state) gain of an optimal Kalman filtering algorithm. However, they go on to show that the learning dynamics exhibit recurrent escapes from this steady state. This calls into question whether agents would in fact cling to a constant gain in the presence of such instability. Here we allow the agent to effectively employ a time-varying gain, which is not restricted to be nonzero. We do this by supposing that agents *average* between a constant gain and a decreasing/least-squares gain. Evolution of the model probability weights delivers a state-dependent gain. In some respects, our analysis resembles the gain-switching algorithm of Marcet and Nicolini (2003). However, they require the agent to commit to one or the other, whereas we permit the agent to be a Bayesian, and average between the two. Despite the fact that our specification of the gain is somewhat different, like Benhabib and Dave (2014), we rely on the theory of large deviations to provide an analytical characterization of the Markov-switching escape dynamics.

2.2. Learning with a correct model. Suppose the agent knows the fundamentals process in eq. (2.2), but does not know the structural price equation in eq. (2.1). Instead, the agent postulates the following state-space model for prices

$$p_t = \beta_t z_t + \sigma \epsilon_t \quad (2.4)$$

$$\beta_t = \beta_{t-1} + \sigma_v v_t \quad (2.5)$$

where it is assumed that $\text{cov}(\epsilon, v) = 0$. Note that the Rational Expectations equilibrium is a special case of this, with $\sigma_v = 0$ and $\beta = \delta/(1 - \alpha\rho)$. For now, let's suppose the agent adopts the dogmatic prior that parameters are constant.

$$\mathcal{M}_0 : \sigma_v^2 = 0.$$

Given this belief, he estimates the unknown parameter of his model using the following Kalman filter algorithm

$$\hat{\beta}_{t+1} = \hat{\beta}_t + \left(\frac{\Sigma_t}{\sigma^2 + \Sigma_t z_t^2} \right) z_t (p_t - \hat{\beta}_t z_t) \quad (2.6)$$

$$\Sigma_{t+1} = \Sigma_t - \frac{(z_t \Sigma_t)^2}{\sigma^2 + \Sigma_t z_t^2} \quad (2.7)$$

where we adopt the common assumption that $\hat{\beta}_t$ is based on time- $(t-1)$ information, while the time- t forecast of prices, $\rho \hat{\beta}_t z_t$, can incorporate the latest z_t observation. This assumption is made to avoid simultaneity between beliefs and observations.⁵ The process, Σ_t , represents the agent's evolving estimate of the variance of $\hat{\beta}_t$. Notice that given his beliefs that parameters are constant, Σ_t converges to zero at rate t^{-1} . This makes sense. If parameters really are constant, then each new observation contributes less and less relative to the existing stock of knowledge. On the other hand, notice that during the transition, the agent's beliefs are inconsistent with the data. He *thinks* β is constant, but due to expectational feedback, his own learning causes β to be time-varying. This can be seen by substituting the agent's time- t forecast into the true model in eq. (2.1)

$$\begin{aligned} p_t &= [\delta + \rho \alpha \hat{\beta}_t] z_t + \sigma \epsilon_t \\ &= T(\hat{\beta}_t) z_t + \sigma \epsilon_t \end{aligned} \quad (2.8)$$

It's fair to say that opinions differ as to whether this inconsistency is important. As long as the T -mapping between beliefs and outcomes has the appropriate stability properties, the agent's incorrect beliefs will eventually be corrected. That is, learning-induced parameter variation eventually dissipates, and the agent eventually learns the Rational Expectations equilibrium. However, as pointed out by Bray and Savin (1986), in practice this convergence can be quite slow, and one could then reasonably ask why agents aren't able to detect the parameter variation that their own learning generates. If they do, wouldn't they want to revise their learning algorithm, and if they do, will learning still take place?⁶

⁵See Evans and Honkapohja (2001) for further discussion.

⁶McGough (2003) addresses this issue. He pushes the analysis one step back, and shows that if agents start out with a time-varying parameter learning algorithm, but have priors that this variation damps out over time, then agents can still eventually converge to a constant parameter Rational Expectations equilibrium.

In our view, this debate is largely academic, since the more serious problem with this model is that it fails to explain the data. Since learning is transitory, so is any learning induced parameter instability. Although there is some evidence in favor of a ‘Great Moderation’ in the volatility of macroeconomic aggregates (at least until the recent financial crisis!), there is little or no evidence for such moderation in asset markets. As a result, more recent work assumes agents view parameter instability as a permanent feature of the environment.

2.3. Learning with a wrong model. Now assume the agent has a different dogmatic prior. Suppose he is now convinced that parameters are time-varying, which can be expressed as the parameter restriction

$$\mathcal{M}_1 : \sigma_v^2 > 0.$$

Although this is a ‘wrong model’ from the perspective of the (unknown) Rational Expectations equilibrium, the more serious specification error here is that the agent does not even entertain the possibility that parameters might be constant. This prevents him from ever learning the Rational Expectations equilibrium (Bullard (1992)). Still, due to feedback, there is a sense in which his beliefs about parameter instability can be self-confirming, since ongoing belief revisions will produce ongoing parameter instability.

The belief that $\sigma_v^2 > 0$ produces only a minor change in the Kalman filtering algorithm in eqs. (2.6)-(2.7). We just need to replace the Riccati equation in (2.7) with the new Riccati equation

$$\Sigma_{t+1} = \Sigma_t - \frac{(z_t \Sigma_t)^2}{\sigma^2 + \Sigma_t z_t^2} + \sigma_v^2 \quad (2.9)$$

The additional σ_v^2 term causes Σ_t to now converge to a strictly positive limit, $\bar{\Sigma} > 0$. As noted by Benveniste et. al. (1990, pgs. 139-40), if we assume $\sigma_v^2 \ll \sigma^2$, which we will do in what follows, we can use the approximation $\sigma^2 + \Sigma_t z_t^2 \approx \sigma^2$ in the above formulas (Σ_t is small relative to σ^2 and scales inversely with z_t^2). The Riccati equation in (2.9) then delivers the following approximation for the steady state variance of the state, $\bar{\Sigma} \approx \sigma \cdot \sigma_v M_z^{-1/2}$, where $M_z = E(z_t^2)$ denotes the second moment of the fundamentals process. In addition, if we further assume that priors about parameter drift take the particular form, $\sigma_v^2 = \gamma^2 \sigma^2 M_z^{-1}$, then the steady state Kalman filter takes the form of the following (discounted) recursive least-squares algorithm

$$\hat{\beta}_{t+1} = \hat{\beta}_t + \gamma M_z^{-1} z_t (p_t - \hat{\beta}_t z_t) \quad (2.10)$$

where the agent’s priors about parameter instability are now captured by the so-called ‘gain’ parameter, γ . If the agent thinks parameters are more unstable, he will use a higher gain.

Constant gain learning algorithms have explained a wide variety of dynamic economic phenomena. For example, Cho, Williams, and Sargent (2002) show they potentially explain US inflation dynamics. Kasa (2004) argues they can explain recurrent currency crises. Chakraborty and Evans (2008) show they can explain observed biases in forward exchange rates, while Benhabib and Dave (2014) show they explain fat tails in asset price distributions.

An important question raised by this literature arises from the fact that the agent's model is 'wrong'. Wouldn't a smart agent eventually discover this?⁷ On the one hand, this is an easy question to answer. Since his prior dogmatically rules out the 'right' constant parameter model, there is simply no way the agent can ever detect his misspecification, even with an infinite sample. On the other hand, due to the presence of expectational feedback, a more subtle question is whether the agent's beliefs about parameter instability can become 'self-confirming' (Sargent (2008))? That is, to what extent are the random walk priors in eq. (2.5) consistent with the observed behavior of the parameters in the agent's model? Would an agent have an incentive to revise his prior in light of the data that are themselves (partially) generated by those priors?

It is useful to divide this question into two pieces, one related to the innovation variance, σ_v^2 , and the other to the random walk nature of the dynamics. As noted above, the innovation variance is reflected in the magnitude of the gain parameter. Typically the gain is treated as a free parameter, and is calibrated to match some feature of the data. However, as noted by Sargent (1999, chpt. 6), in self-referential models the gain should *not* be treated as a free parameter. It is an equilibrium object. This is because the optimal gain depends on the volatility of the data, but at the same time, the volatility of the data depends on the gain. Evidently, as in a Rational Expectation Equilibrium, we need a fixed point.

In a prescient paper, Evans and Honkapohja (1993) addressed the problem of computing this fixed point. They posed the problem as one of computing a Nash equilibrium. In particular, they ask - Suppose everyone else is using a given gain parameter, so that the data-generating process is consistent with this gain. Would an individual agent have an incentive to switch to a *different* gain? Under appropriate stability conditions, one can then compute the equilibrium gain by iterating on a best response mapping as usual. Evans and Ramey (2006) extend the work of Evans and Honkapohja (1993). They propose a Recursive Prediction Error algorithm, and show that it does a good job tracking the optimal gain in real-time. They also point out that due to forecast externalities, the Nash gain is typically Pareto suboptimal. More recently, Kostyshyna (2012) uses Kushner and Yang's (1995) adaptive gain algorithm to revisit the same hyperinflation episodes studied by Marcet and Nicolini (2003). The idea here is to recursively update the gain in exactly the same way that parameters are updated. The only difference is that now there is a constant gain governing the evolution of the parameter update gain. Kostyshyna (2012) shows that her algorithm performs better than the discrete, markov-switching algorithm of Marcet and Nicolini (2003). In sum, σ_v^2 can indeed become self-confirming, and agents can use a variety of algorithms to estimate it.

To address the second issue we need to study the dynamics of the agent's parameter estimation algorithm in eq. (2.10). After substituting in the actual price process this can be written as

$$\hat{\beta}_{t+1} = \hat{\beta}_t + \gamma M_z^{-1} z_t \left\{ [\delta + (\alpha\rho - 1)\hat{\beta}_t] z_t + \sigma\epsilon_t \right\} \quad (2.11)$$

⁷Of course, a constant gain model could be the 'right' model too, if the underlying environment features exogenously time-varying parameters. After all, it is this possibility that motivates their use in the first place. Interestingly, however, most existing applications of constant gain learning feature environments in which doubts about parameter stability are entirely in the head of the agents.

Let $\beta^* = \delta/(1 - \alpha\rho)$ denote the Rational Expectations equilibrium. Also let $\tau_t = t \cdot \gamma$, and then define $\hat{\beta}(\tau_t) = \hat{\beta}_t$. We can then form the piecewise-constant continuous-time interpolation, $\hat{\beta}(\tau) = \hat{\beta}(\tau_t)$ for $\tau \in [t\gamma, t\gamma + \gamma]$. Although for a fixed γ (and σ_v^2) the paths of $\hat{\beta}(\tau)$ are not continuous, they converge to the following continuous limit as $\sigma_v^2 \rightarrow 0$ (see Evans and Honkapohja (2001) for a proof)

Proposition 2.1. *As $\sigma_v^2 \rightarrow 0$, $\hat{\beta}(\tau)$ converges weakly to the following diffusion process*

$$d\hat{\beta} = -(1 - \alpha\rho)(\hat{\beta} - \beta^*)d\tau + \gamma M_z^{-1/2} \sigma dW_\tau \quad (2.12)$$

This is an Ornstein-Uhlenbeck process, which generates a stationary Gaussian distribution centered on the Rational Expectations equilibrium, β^* . Notice that the innovation variance is consistent with the agent's priors, since $\gamma^2 \sigma^2 M_z^{-1} = \sigma_v^2$. However, notice also that $d\hat{\beta}$ is autocorrelated. That is, $\hat{\beta}$ does *not* follow a random walk. Strictly speaking then, the agent's priors are misspecified. However, remember that traditional definitions of self-confirming equilibria presume that agents have access to infinite samples. In practice, agents only have access to finite samples. Given this, we can ask whether the agent could statistically reject his prior.⁸ This will be difficult when the drift in eq. (2.12) is small. This is the case when: (1) Estimates are close to the β^* , (2) Fundamentals are persistent, so that $\rho \approx 1$, and (3) Feedback is strong, so that $\alpha \approx 1$.

These results show that if the 'grain of truth' assumption fails, wrong beliefs can be quite persistent (Esponda and Pouzo (2014)). One might argue, however, that the persistence of \mathcal{M}_1 is driven entirely by the fact that the agent's beliefs fail to satisfy the grain of truth assumption. If the agent were to expand his priors to include \mathcal{M}_0 , it would eventually dominate (Kalai and Lehrer (1993)).

We claim otherwise, and demonstrate that the problem arising from the presence of misspecified models can be far more insidious. We do this by expanding upon the example presented by Evans, Honkapohja, Sargent, and Williams (2013). We show that misspecified models can survive the grain of truth assumption. The mere presence of a misspecified alternative can disrupt the learning process.

2.4. Model Averaging. Dogmatic priors (about anything) are rarely a good idea. So let's now suppose the agent hedges his bets by entertaining the possibility that parameters are constant. Forecasts are then constructed using a traditional Bayesian Model Averaging (BMA) strategy. This strategy effectively 'convexifies' the model space. If we let π_t denote the current probability assigned to \mathcal{M}_1 , the TVP model, and let $\beta_t(i)$ denote the current parameter estimate for \mathcal{M}_i , the agent's time- t forecast becomes⁹

$$E_t p_{t+1} = \rho[\pi_t \beta_t(1) + (1 - \pi_t) \beta_t(0)] z_t \quad (2.13)$$

Substituting this into the actual law of motion for prices implies that parameter estimates evolve according to

$$\beta_{t+1}(i) = \beta_t(i) + \left(\frac{\Sigma_t(i)}{\sigma^2 + \Sigma_t(i) z_t^2} \right) z_t \{ [\delta + \alpha\rho[\pi_t \beta_t(1) + (1 - \pi_t) \beta_t(0)] - \beta_t(i)] z_t + \sigma \epsilon_t \} \quad (2.14)$$

⁸In the language of Hansen and Sargent (2008), we can compute the detection error probability.

⁹To ease notation in what follows, we shall henceforth omit the hats from the parameter estimates.

Note that the only difference between the two arises from their gain sequences, $\Sigma_t(i)$, and that these two gain sequences are independent of any model averaging. Still, it appears things have become vastly more complicated. Not only are the $\beta_t(i)$'s coupled, but they appear to depend on the evolution of the model weights, π_t . Fortunately, things are not as bad as they look, thanks to a time-scale separation, and the following global asymptotic stability result, which shows that the $\beta_t(i)$'s converge to the same, unique, self-confirming equilibrium values for all values of π_t :

Proposition 2.2. *If $\alpha\rho < 1$, then as $t \rightarrow \infty$ and $\sigma_v^2 \rightarrow 0$, $\beta_t(0) \xrightarrow{w.p.1} \frac{\delta}{1-\rho\alpha}$ and $\beta_t(1) \Rightarrow \frac{\delta}{1-\rho\alpha}$ for all $\pi_t \in (0, 1)$.*

Proof. See Appendix A. □

This result is not too surprising, since both models are forecasting the same thing using the same variable. However, it is quite useful, since it implies that all the action in the dynamics lies in the evolution of the model weight, π_t .

3. MODEL AVERAGING DYNAMICS

Since the decision maker is Bayesian, he updates π_t according to Bayes rule, starting from a given prior $\pi_0 \in (0, 1)$. Since we're now assuming $\pi_0 > 0$, the agent assigns a positive weight to the REE model. His prior therefore satisfies the grain of truth condition.

Conversely, since \mathcal{M}_1 is assigned a positive probability, the agent's model is misspecified. The model is misspecified, not because it excludes a variable, but because it includes a variable which is not in the true model (i.e., σ_v^2). Normally, in situations where the data-generating process is independent of the agent's actions, this sort of practice of starting from a larger model is rather innocuous. The data will show that any variables not in the true model are insignificant. We shall now see that this is no longer the case when the data-generating process is endogenous.

3.1. Odds ratio. The agent updates his prior $\pi_t = \pi_t(1)$ that the data is generated according to \mathcal{M}_1 . After a long tedious calculation, the Bayesian updating scheme for π_t can be written as (see Evans, Honkapohja, Sargent, and Williams (2013) for a partial derivation)

$$\frac{1}{\pi_{t+1}} - 1 = \frac{A_{t+1}(0)}{A_{t+1}(1)} \left(\frac{1}{\pi_t} - 1 \right) \quad (3.15)$$

where

$$A_t(i) = \frac{1}{\sqrt{\sigma^2 + \Sigma_t(i)z_t^2}} e^{-\frac{(p_t - \rho\beta_t(i)z_t)^2}{2(\sigma^2 + \Sigma_t(i)z_t^2)}}$$

is the time- t predictive likelihood function for model \mathcal{M}_i . To study the dynamics of π_t it is useful to rewrite eq. (3.15) as follows

$$\pi_{t+1} = \pi_t + \pi_t(1 - \pi_t) \left[\frac{A_{t+1}(1)/A_{t+1}(0) - 1}{1 + \pi_t(A_{t+1}(1)/A_{t+1}(0) - 1)} \right] \quad (3.16)$$

which has the familiar form of a discrete-time replicator equation, with a stochastic, state-dependent, fitness function determined by the likelihood ratio. Equation (3.16) reveals a lot about the model averaging dynamics. First, it is clear that the boundary points

$\pi = \{0, 1\}$ are trivially stable fixed points, since they are absorbing. Second, we can also see that there could be an interior fixed point, where $\mathbb{E}(A_{t+1}(1)/A_{t+1}(0)) = 1$. Later, as part of the proof of Lemma 3.2 we shall see that this occurs when $\pi = \frac{1}{2\rho\alpha}$, which is interior if feedback is strong enough (i.e., if $\alpha > \frac{1}{2\rho}$). However, we shall also see there that this fixed point is unstable. So we know already that π_t will spend most of its time near the boundary points. This will become apparent when we turn to the simulations in Section 4. One remaining issue is whether π_t could ever become absorbed at one of the boundary points.

Proposition 3.1. *As long as the likelihoods of \mathcal{M}_0 and \mathcal{M}_1 have full support, the boundary points $\pi_t = \{0, 1\}$ are unattainable in finite time.*

Proof. This result is quite intuitive. With two full support probability distributions, you can never conclude that a history of any finite length couldn't have come from either of the distributions. Slightly more formally, if the distributions have full support, they are mutually absolutely continuous, so the likelihood ratio in eq. (3.16) is strictly bounded between 0 and some upper bound B . To see why $\pi_t < 1$ for all t , notice that $\pi_{t+1} < \pi_t + \pi_t(1 - \pi_t)M$ for some $M < 1$, since the likelihood ratio is bounded by B . Therefore, since $\pi + \pi(1 - \pi) \in [0, 1]$ for $\pi \in [0, 1]$, we have

$$\begin{aligned} \pi_{t+1} &\leq \pi_t + \pi_t(1 - \pi_t)M \\ &< \pi_t + \pi_t(1 - \pi_t) \\ &\leq 1 \end{aligned}$$

and so the result follows by induction. The argument for why $\pi_t > 0$ is completely symmetric. \square

Since the distributions here are assumed to be Gaussian, they obviously have full support, so Proposition 3.1 applies. Although the boundary points are unattainable, the replicator equation for π_t in eq. (3.16) makes it clear that π_t will spend most of its time near these boundary points, since the relationship between π_t and π_{t+1} has the familiar logit function shape, which flattens out near the boundaries. As a result, π_t evolves very slowly near the boundary points. In fact, we shall now show that it evolves even more slowly than the t^{-1} time-scale of $\beta_t(0)$. This means that when studying the dynamics of the coefficient estimates near the boundaries, we can treat π_t as fixed.

3.2. Log Odds Ratio. Let us initialize the likelihood ratio at the prior odds ratio:

$$\frac{A_0(0)}{A_0(1)} = \frac{\pi_0(0)}{\pi_0(1)}.$$

By iteration we get

$$\frac{\pi_{t+1}(0)}{\pi_{t+1}(1)} = \frac{1}{\pi_{t+1}} - 1 = \prod_{k=0}^{t+1} \frac{A_k(0)}{A_k(1)},$$

Taking logs and dividing by $(t + 1)$,

$$\frac{1}{t+1} \ln \left(\frac{1}{\pi_{t+1}} - 1 \right) = \frac{1}{t+1} \sum_{k=0}^{t+1} \ln \frac{A_k(0)}{A_k(1)}.$$

Now define the average log odds ratio, ϕ_t , as follows

$$\phi_t = \frac{1}{t} \ln \left(\frac{1}{\pi_t} - 1 \right) = \frac{1}{t} \ln \left(\frac{\pi_t(0)}{\pi_t(1)} \right)$$

which can be written recursively as the following stochastic approximation algorithm

$$\phi_t = \phi_{t-1} + \frac{1}{t} \left[\ln \frac{A_t(0)}{A_t(1)} - \phi_{t-1} \right].$$

Invoking well knowing results from stochastic approximation, we know that the asymptotic properties of ϕ_t are determined by the stability properties of the following ODE

$$\dot{\phi} = \mathbb{E} \left[\ln \frac{A_t(0)}{A_t(1)} \right] - \phi$$

which has a unique stable point

$$\phi^* = \mathbb{E} \ln \frac{A_t(0)}{A_t(1)}.$$

Note that if $\phi^* > 0$, $\pi_t \rightarrow 0$, while if $\phi^* < 0$, $\pi_t \rightarrow 1$. Thus, the focus of the ensuing analysis is to identify the sign of ϕ^* , rather than its value.

3.3. Benchmark time scale. We use the sample average time scale, $1/t$, as the benchmark: $\forall \tau > 0$, we can find the unique integer satisfying

$$\sum_{k=1}^{K-1} \frac{1}{k} < \tau < \sum_{k=1}^K \frac{1}{k}.$$

Let $m(\tau) = K$ and define

$$t_K = \sum_{k=1}^K \frac{1}{k}$$

Therefore, $t_K \rightarrow \infty$ as $K \rightarrow \infty$. We are interested in the sample paths over the tail interval $[t_K, t_K + \tau)$ where we initialize $\phi_{t_K} = \phi_0$, when letting $K \rightarrow \infty$. Since $\sigma_v^2 > 0$, $\beta_t(1)$ evolves at the speed of a constant gain algorithm

$$\lim_{t \rightarrow \infty} t |\beta_t(1) - \beta_{t-1}(1)| = \infty,$$

while $\beta_t(0)$ evolves at the speed of a decreasing gain algorithm so that it evolves on the same time scale as ϕ_t ,

$$0 < \lim_{t \rightarrow \infty} t |\beta_t(0) - \beta_{t-1}(0)| < \infty.$$

3.4. Evolution of π_t . The evolution speed of π_t is state dependent. If π_t is in the interior of $[0, 1]$, it evolves at much faster rate than if it's near the boundary. To make things more difficult, π_t does not have a simple recursive form as do $\beta_t(i)$ ($i = 1, 2$) and ϕ_t . In principle, we have to consider different cases, depending upon the speed of π_t . As it turns out, every case follows more or less the same logic, which greatly simplifies the analysis.

Let Π be the collection of all sample paths of $\{\pi_t\}$, endowed with a probability distribution. Consider

$$\Pi_0 = \{ \{ \pi_t \} \mid \text{there is no subsequence converging to 0 or 1} \}.$$

Lemma 3.2. Π_0 is a null set.

Proof. See Appendix B □

Therefore, without loss of generality, we can assume that $\pi = \{0, 1\}$ are the only limit points of $\{\pi_t\}$. After renumbering a convergent subsequence, suppose $\pi_t \rightarrow 1$. Following the same reasoning as in the proof of Lemma 3.2, we can prove that

$$\beta_t(0) \rightarrow \frac{\delta}{1 - \alpha\rho}$$

with probability 1, and

$$\beta_t(1) \rightarrow \frac{\delta}{1 - \alpha\rho}$$

weakly.

If α is significantly larger than $1/2\rho$, and π_t close to 1, then a simple calculation shows that

$$\mathbf{E} \ln \frac{A_{t+1}(0)}{A_{t+1}(1)} < 0$$

which implies that

$$\phi_t \rightarrow \mathbf{E} \ln \frac{A_{t+1}(0)}{A_{t+1}(1)} < 0.$$

with probability 1. Given $\beta_t(0) = \beta_t(1) = \delta/(1 - \alpha\rho)$, $\pi_t \rightarrow 1$ with probability 1, proving that $\pi_t = 1$ is locally attracting.

Similarly, we can show that if π_t is close to 0, then

$$\mathbf{E} \ln \frac{A_t(0)}{A_t(1)} > 0.$$

Following the same argument, we prove that $\pi_t = 0$ is locally attracting.

In general, the speed of evolution of π_t compared to $\beta_t(i)$ ($i = 1, 2$) is difficult to compute. But, in the neighborhood of the boundaries, we can show that π_t evolves on an even slower time scale than $\beta_t(0)$.

Lemma 3.3. Suppose that $\phi^* \neq 0$. Then,

$$\lim_{t \rightarrow \infty} t(\pi_t - \pi_{t-1}) = 0.$$

Proof. See Appendix C □

Lemma 3.3 asserts that in the neighborhood of the boundaries, the hierarchy of time scales is such that $\beta_t(1)$ evolves at a faster time scale than $\beta_t(0)$, while $\beta_t(0)$, which evolves on the same time scale as ϕ_t , evolves at a faster time scale than π_t . This time scale hierarchy greatly facilitates the analysis of escape dynamics.

3.5. Escape Dynamics. We have three endogenous variables $(\pi_t, \beta_t(0), \beta_t(1))$, which converge to one of the two locally stable points: $(0, \delta/(1 - \alpha\rho), \delta/(1 - \alpha\rho))$ or $(1, \delta/(1 - \alpha\rho), \delta/(1 - \alpha\rho))$. Let us identify a specific stable point by the value of π_t at the stable point. Similarly, let \mathcal{D}_0 be the domain of attraction to $\pi_t = 0$, and \mathcal{D}_1 be the domain of attraction to $\pi_t = 1$. To calculate the relative duration times of $(\pi_t, \beta_t(0), \beta_t(1))$ around each locally attractive boundary point, we need to compute the following ratio

$$\lim_{\sigma_v^2 \rightarrow 0} \lim_{t \rightarrow \infty} \frac{\mathbb{P}\left(\exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_0 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(1, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho}\right)\right)}{\mathbb{P}\left(\exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_1 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(0, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho}\right)\right)}$$

We continue to use the time scale of $\beta_t(0)$ as the benchmark, which is slower than that of $\beta_t(1)$ but faster than that of π_t . We shall demonstrate that at this time scale, no escape can occur from the neighborhood of $\pi_t = 1$, whereas the escape probability from the neighborhood of $\pi_t = 0$ is of the order e^{-tr^*} for some positive $r^* < \infty$. Thus, we prove that we should observe escapes from $\pi_t = 0$ much more frequently than escapes from $\pi_t = 1$.

Proposition 3.4. *Let \mathcal{N}_1 be a small neighborhood near the boundary $\pi_t = 1$, and let $\mu_t(\pi_t \in \mathcal{N}_1)$ be the empirical occupancy measure of π_t in \mathcal{N}_1 . Then as $t \rightarrow \infty$ and $t \cdot \sigma_v$ remains bounded, $\mu_t(\pi_t \in \mathcal{N}_1) \rightarrow 1$.*

Proof. The domains of attraction can be computed by comparing mean-squared errors. A simple calculation shows that

$$\mathcal{D}_0 = \left\{ (\pi, \beta(0), \beta(1)) \mid \left(\beta(0) - \frac{\delta}{1-\alpha\rho} \right)^2 < (1 - 2\alpha\rho\pi)\sigma_\xi^2 \left(\frac{1-\alpha\rho\pi}{1-\alpha\rho} \right)^2 \right\}.$$

Recall that we are calculating the domain of attraction according to the time scale of $\beta_t(0)$. Thus, $\beta_t(1)$ does not show up in \mathcal{D}_0 , because it is already distributed around $\beta_t(0)$. Note that

$$1 - 2\alpha\rho\pi > 0$$

must hold, in order to have $(\pi, \beta(0), \beta(1)) \in \mathcal{D}_0$. Thus, in order to enter \mathcal{D}_0 , it is essential that

$$\pi_t < \frac{1}{2\alpha\rho}.$$

In order to calculate the probability distribution of $(\pi_t, \beta_t(0), \beta_t(1))$ in the long run as $t \rightarrow \infty$ and $\sigma_v^2 \rightarrow 0$, we must compare probabilities of switching from one domain of attraction to another domain of attraction. We first compute

$$\mathbb{P}\left(\exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_0 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(1, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho}\right)\right).$$

Since π_t evolves at a slower time scale than $\beta_t(0)$,

$$\begin{aligned} & \mathbb{P} \left(\exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_0 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(1, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho} \right) \right) \\ &= \mathbb{P} \left(\exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_0 \mid \pi_t = 1, (\pi_0, \beta_0(0), \beta_0(1)) = \left(1, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho} \right) \right) \\ &\leq \mathbb{P} \left(\pi_t \leq \frac{1}{2\alpha\rho} \mid \pi_t = 1, (\pi_0, \beta_0(0), \beta_0(1)) = \left(1, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho} \right) \right) = 0. \end{aligned}$$

Thus, we do not observe any escape from $\pi_t = 1$ on the $\beta_t(0)$ time scale. If escape occurs, it will happen at the time scale of π_t , which is slower than that of $\beta_t(0)$. This result explains why π_t appears to be “stuck” at $\pi_t = 1$, once it reaches a small neighborhood of $\pi = 1$,

Next, let us compute the escape probability from $\pi_t = 0$.

$$\mathbb{P} \left((\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_1 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(0, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho} \right) \right).$$

Since $\beta_t(0)$ evolves on a faster time scale than π_t ,

$$\mathbb{P} ((\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_1 \mid \pi_t = 0).$$

Note that \mathcal{D}_0 is a narrow “cone” in the space of $(\beta(0), \pi)$, with its apex at $(\beta(0), \pi) = \left(\frac{\delta}{1-\alpha\rho}, \frac{1}{2\alpha\rho} \right)$ and its base along the line $\pi = 0$, where $\beta(0)$ is in $\left[\frac{\delta}{1-\alpha\rho} - \frac{\sigma_\xi}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho} + \frac{\sigma_\xi}{1-\alpha\rho} \right]$. Figure 1 plots \mathcal{D}_0 for the baseline parameter values used in the following simulations.

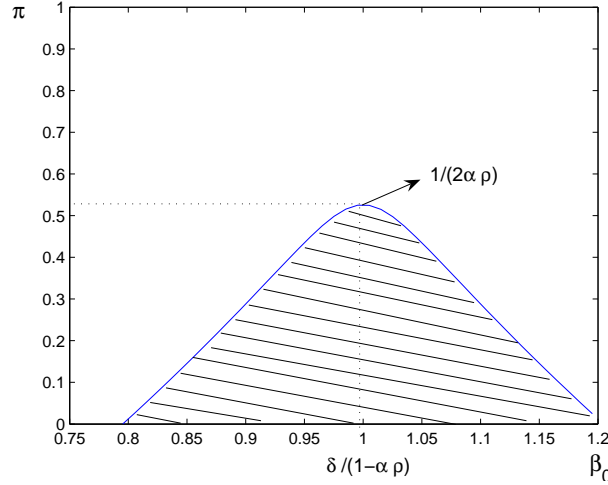


FIGURE 1. Domain of Attraction for $\pi = 0$

If $(\beta_t(0), \pi)$ is outside this interval, $(\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_1$, which is then driven by the mean dynamics to another locally stable outcome where $\pi_t = 1$. Thus,

$$\mathbb{P} ((\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_1 \mid \pi_t = 0) \geq \mathbb{P} \left(\beta_t(0) \notin \left[\frac{\delta}{1-\alpha\rho} - \frac{\sigma_\xi}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho} + \frac{\sigma_\xi}{1-\alpha\rho} \right] \mid \pi_t = 0 \right).$$

Since $\beta_t(0)$ evolves according to (2.6), which satisfies the regularity conditions in Dupuis and Kushner (1989), we know that $\beta_t(0)$ has a finite but strictly positive large deviations rate function, which means that the right hand side is of order e^{-tr^*} , where $0 < r^* < \infty$. Therefore,

$$\lim_{\sigma_v^2 \rightarrow 0} \lim_{t \rightarrow \infty} \frac{\mathbb{P} \left((\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_0 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(1, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho} \right) \right)}{\mathbb{P} \left((\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_1 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(0, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho} \right) \right)} = 0$$

as claimed. \square

This result shows that the TVP model asymptotically dominates. It will be used ‘almost always’. This is because it is better able to react to the volatility that it itself creates! Although \mathcal{M}_1 is misspecified and Pareto suboptimal relative to the Rational Expectations equilibrium, in practice this equilibrium must be *learned* via some adaptive process, and what our result shows is that this learning process can be subverted by the mere presence of misspecified alternatives, even when the correctly specified model would converge if considered in isolation. This result therefore echoes the conclusions of Sargent (1993), who notes that adaptive learning models often need a lot of ‘prompting’ before they converge. Elimination of misspecified alternatives can be interpreted as a form of prompting. Our result also offers an interesting counterpoint to Kalai and Lehrer (1993). Despite the apparent satisfaction of their ‘grain of truth’ condition (since $\pi_0 < 1$), Bayesian learning converges to the ‘wrong’ model. We discuss this apparent contradiction in more detail in Section 5.

One should keep in mind that, like all asymptotic results, the empirical relevance of this result is certainly open to question. Although e^{-tr^*} remains positive for all t , it becomes a *very* small number, very quickly. This means that \mathcal{M}_0 can survive for a *long* time, especially if it has already withstood the test of time. In fact, since the escape probability from \mathcal{M}_0 declines rapidly with respect to calendar time, if \mathcal{M}_1 is going to dominate on any sort of realistic time scale, it must do so relatively quickly. However, we can only address these finite sample issues via simulations, to which we now turn.

4. SIMULATIONS

As noted in Section 2, the present-value asset pricing model in eqs. (2.1)-(2.2) has been subjected to a lot of previous empirical work, mostly with negative results. Perhaps its biggest problem is its failure to generate sufficient volatility (Shiller (1989)). Our results suggest that this negative assessment could be premature. To examine this possibility, we calibrate the model using parameter values that have been used in the past, and see whether this can generate the sort of self-confirming volatility that our analysis suggests is possible.

Most of the parameters are easy to calibrate. We know observed fundamentals are persistent, so we set $\rho = .99$. Remember, the agent is assumed to know this. Similarly, we know discount factors are close to 1, so we set the feedback parameter to $\alpha = .96$. Since δ depends on units, we just normalize it by setting $\delta = (1 - \alpha\rho)$. This implies the self-confirming equilibrium value, $\beta = 1.0$. In principle, the innovations variances, (σ^2, σ_z^2) , could be calibrated to match those of observed assets prices and fundamentals. However,

since what really matters is the comparison between actual and predicted volatility, we follow Evans, Honkapohja, Sargent, and Williams (2013) and just normalize them to unity ($\sigma^2 = \sigma_z^2 = 1$). That leaves one remaining free parameter, σ_v^2 . Of course, this is a crucial parameter, since it determines the agent's prior beliefs about parameter instability. If it's too big, then the TVP model will be at a big disadvantage during tranquil times, and will therefore have a difficult time displacing the constant parameter model. On the other hand, if it's too small, self-confirming volatility will be empirically irrelevant.

Figures 2-5 report typical simulations for three alternative values, $\sigma_v^2 = (.0005, .0001, .00001)$.

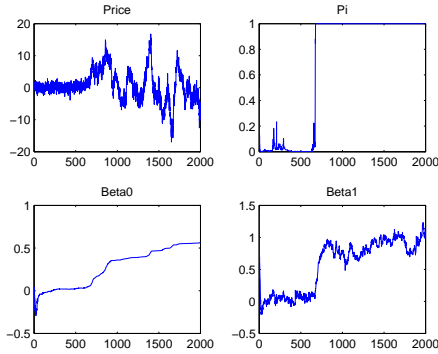


FIGURE 2. $\sigma_v^2 = .0005$

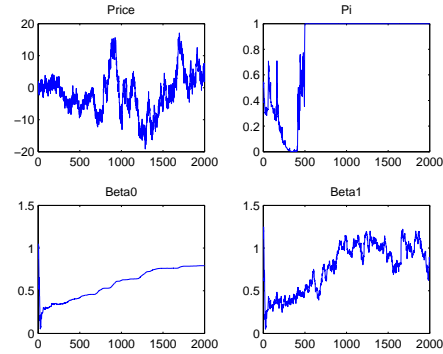


FIGURE 3. $\sigma_v^2 = .0005$

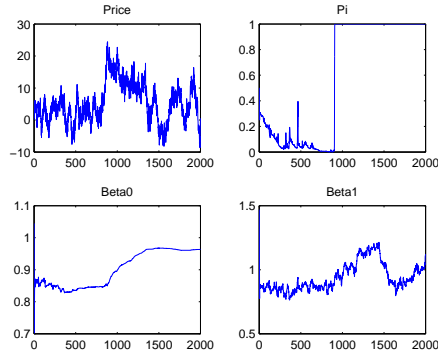


FIGURE 4. $\sigma_v^2 = .0001$

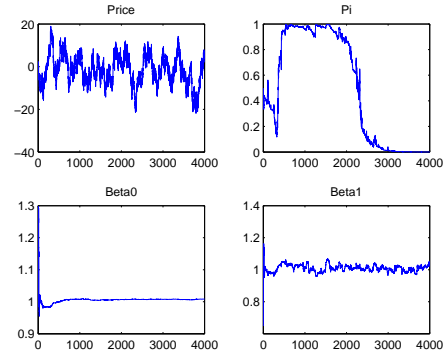


FIGURE 5. $\sigma_v^2 = .00001$

Figures 2-3 are for the value $\sigma_v^2 = .0005$. When this is the case, steady-state price volatility is 93.3% higher when $\pi = 1$ than when $\pi = 0$, which is quite significant, although less than the excess volatility detected by Shiller (1989). The higher price volatility when $\pi = 1$ is apparent. The implied steady-state gain associated with this value of σ_v^2 is $\gamma = .07$, which is quite typical of values used in prior empirical work. These figures also illustrate a typical feature of the sample paths when σ_v^2 is relatively high, i.e., convergence to one or the other boundaries occurs relatively quickly, usually by around $T = 500$.

Figures 4-5 use smaller values of σ_v^2 . Generally speaking, smaller values of σ_v^2 delay convergence. In Figure 4, where $\sigma_v^2 = .0001$, convergence to $\pi = 1$ once again takes place, but now its price volatility implications are not quite so dramatic. Volatility is only 41.7% higher when $\pi = 1$. Once again, the implied steady-state gain ($\gamma = .03$) is typical of values used in empirical work. Figure 5 uses a still smaller prior variance, $\sigma_v^2 = .00001$. Now the two models do not differ by much. Steady-state price volatility is only 13% higher when $\pi = 1$. Notice that because the two models are so similar, it becomes easier to escape the $\pi = 1$ equilibrium. Since the TVP world is not that volatile, a constant parameter model does not do that badly.¹⁰

The one feature that is perhaps not accurately portrayed by these figures is the fact that on empirically relevant time-scales convergence to either boundary can occur. This fact was emphasized by Evans, Honkapohja, Sargent, and Williams (2013). Although our previous results imply that eventually the $\pi = 1$ equilibrium will dominate, our simulations indicate that the $\pi = 0$ equilibrium can persist for a *long* time. For example, we conducted 10,000 simulations, each of length $T = 2000$, and counted the proportion of times convergence to $\pi = 1$ occurred for various values of σ_v^2 . As above, the simulations were initialized at $\pi = 0.5$, with small random perturbations of the coefficients around their self-confirming equilibrium values. Figure 6 displays the results,

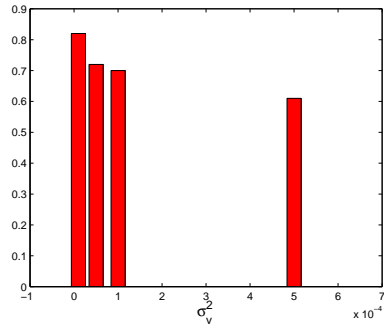


FIGURE 6. Prob of Convergence to $\pi = 1$

Not surprisingly, the probability of convergence to $\pi = 1$ declines with σ_v^2 . For $\sigma_v^2 = .1 \times 10^{-4}$, convergence occurs more than 80% of the time, whereas for the benchmark value used above, $\sigma_v^2 = 5 \times 10^{-4}$, convergence to $\pi = 1$ occurs only about 60% of the time.¹¹ As our above analysis makes clear, however, one must exercise some caution when interpreting results like these. In Figure 6, ‘convergence’ was simply defined as the value of π at the end of each simulation run (i.e., at $T = 2000$). According to our large deviations results, however, *eventually* π will escape from 1. What our theory actually predicts is that as the sample length becomes infinitely long, the *proportion of time* spent near $\pi = 1$ goes to unity. It doesn’t imply that π never returns to 0. Although escapes from $\pi = 1$ are more likely to occur relatively early in the game, before $\beta_t(0)$ has settled down, as

¹⁰Notice that Figure 5 uses a $T = 4000$ sample length, while Figures 2-4 use $T = 2000$. As σ_v^2 decreases, things evolve more slowly, so it becomes necessary to expand the simulation length.

¹¹For $\sigma_v^2 = 0$ convergence should occur 50% of the time.

σ_v^2 decreases, escapes can occur relatively late as well. Figure 5 nicely illustrates this possibility. Hence, the results in Figure 6 are merely meant to convey the possibility that on realistic time-scales, ‘convergence’ to either $\pi = 1$ or to $\pi = 0$ can occur.

The fact that convergence to either $\pi = 0$ or $\pi = 1$ can occur on relevant time-scales is interesting, since it suggests that whether we live in a tranquil or volatile economy is somewhat random and history-dependent. It also highlights a potentially adverse long-term effect of ‘large deviation’ events, like financial crises. Although being alert to the possibility of financial crises is probably a good thing on net, if it makes individuals living in a less than fully understood self-referential environment more reactive, it could create its own problems.

5. DISCUSSION

The results thus far cast doubt on the ability of agents to adaptively learn Rational Expectations Equilibria. Here we discuss the robustness of these results. We do so by considering two modifications of the model space. The first narrows the domain of \mathcal{M}_0 by only considering a *single* constant parameter model, i.e., the rational expectations equilibrium in eq. (2.3). The second convexifies the model space directly, by allowing the agent to consider any arbitrary value of σ_v^2 , rather than just convex combinations of two extreme values, as was done in the previous model averaging exercise.

5.1. Grain of truth? Kalai and Lehrer (1993) showed that Bayesian learning converges to Nash equilibrium in repeated games as long as players’ priors concerning their rivals’ strategies contain a ‘grain of truth’. That is, as long as priors and the induced distribution of play are mutually absolutely continuous. Although here there is only a single agent, and no strategic interaction, Kalai and Lehrer’s (1993) result is still of some interest, since as usual we can interpret model uncertainty as reflecting the unknown choice of a fictitious player called ‘nature’. From this perspective, the above results may seem puzzling. After all, \mathcal{M}_0 appears to be correctly specified, since it contains the rational expectations equilibrium

$$\beta^* = \frac{\delta}{1 - \rho\alpha}$$

as an element. However, as was noted in Section 2.2, all constant parameter models are misspecified during the transition, as they neglect feedback induced parameter variation. It is only in the limit that they become correctly specified. This may seem like a technicality, but it is central to our Gresham’s Law result. If the agent only entertains constant parameter models, then learning does indeed converge to the Rational Expectations Equilibrium (assuming the usual E-stability conditions). However, we showed above that if the agent expands the initial model class only slightly, and in a very natural way, to consider TVP alternatives, then the usual convergence and E-stability results evaporate. In a sense, TVP models can exploit the transition dynamics, and eventually drive out the constant parameter model.

Given this result, it is natural to ask the following question - What if we shut down the misspecified transition dynamics by endowing the agent with more a priori knowledge? Suppose he *knows* that the true model is either the *single* constant parameter model in eq. (2.3), or a TVP alternative, which he adaptively updates as usual. He then bases his

forecasts on a Bayesian model averaging strategy as before. Narrowing the initial model space in this way produces the following result,

Proposition 5.1. *Suppose that \mathcal{M}_0 consists of only the rational expectations equilibrium, and assume $\alpha\rho < 1$. For given $\sigma_v^2 > 0$, the Bayesian learning dynamics of π_t converge on finite time intervals to either 1 or 0, but over infinite horizons we have*

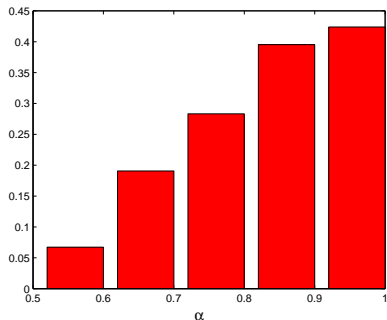
$$\lim_{\sigma_v \rightarrow 0} \lim_{t \rightarrow \infty} \pi_t = 0.$$

Proof. See Appendix D □

Interestingly, this is precisely the opposite of our Gresham's Law result! Now Kalai-Lehrer (1993) seems to 'work', and the agent eventually learns the true model. To see why this happens, refer back to Figure 1, which depicts the basin of attraction for $\pi = 0$. Before, when the constant parameter model had to be estimated, escapes to $\pi = 1$ could occur 'horizontally', as estimates of $\beta_t(0)$ fluctuated. Indeed, since we saw that $\beta_t(0)$ evolves much faster than π_t in the neighborhood of $\pi = 0$, this is in fact the way escapes occur (at least with very high probability). Now, however, with only a single constant parameter model on the table, escapes to $\pi = 1$ must occur 'vertically'. This is much more difficult, since π_t evolves so slowly. Somewhat ironically, although the size of the basin of attraction shrinks to just a vertical line at $\beta_t(0) = \delta/(1 - \alpha\rho)$, its durability increases.

Although Proposition 5.1 predicts the asymptotic distribution of π_t is degenerate, it does make predictions about finite sample distributions as well. The constant parameter model ($\pi = 0$) now dominates because its basin of attraction is now larger. Escapes from each model occur along a vertical line determined by the (identical) self-confirming values of the coefficients (i.e., $\beta(i) = \delta/(1 - \alpha\rho)$). The threshold value of π determining the boundary between the two basins of attraction is $\pi^* = 1/(2\alpha\rho)$. Note that $\pi^* > 1/2$ as long as $\alpha\rho < 1$. Hence, the basin of attraction of \mathcal{M}_0 is larger, and so it asymptotically dominates. This result also suggests, however, that the rate at which this occurs depends on the strength of expectational feedback, i.e., on the magnitude of α . As α increases, the boundary of the basin of attraction moves away from $\pi = 1$, making it harder to escape from the TVP model. Therefore, on finite time intervals, we should find that the proportion of sample paths that appear to converge to $\pi = 1$ increases as α increases.

Figure 7 corroborates this prediction. We conducted 1200 simulations, each of length $T = 40,000$ periods, for five different values of the feedback parameter, ranging from $\alpha = .56$ to $\alpha = .96$ in increments of 0.1. We set $\sigma_v^2 = .0001$ in all simulations. (All other parameter values are the same as in the benchmark calibration discussed in the previous section). Figure 7 reports the proportion of the 1200 simulation runs that end at $\pi = 1$ when $T = 40,000$. Our large deviations analysis suggests that this proportion should increase with α , and that is precisely what we find. Note, however, that the proportion is always less than 0.5. Remember that in this asset-pricing context, empirically plausible values of α are close to one. So even if the constant parameter model eventually prevails, the TVP model can persist for a *long* time with reasonably high probability.

FIGURE 7. Prob of Convergence to $\pi = 1$

In our view, Proposition 5.1 should offer little consolation to those wishing to base Rational Expectations Equilibria on adaptive learning foundations, for two reasons: (1) As emphasized by Nachbar (1997) and Young (2004), Bayesian learning only works here because the agent’s prior is very informative.¹² He must commit himself to ruling out a priori many reasonable alternatives (i.e., any constant parameter model other than the measure zero Rational Expectations Equilibrium). Where does this knowledge come from? The whole idea behind learning is that if agents are initially open-minded, experience will allow them to discard incorrect beliefs, and to eventually discover ‘the truth’. In contrast, for learning to work here, agents must be initially closed-minded in just the right way. (2) Given the usual adaptive learning convergence results, one might naively suspect that the convergence to $\pi = 0$ observed in Proposition 5.1 occurs on a Law of Large Numbers t^{-1} time-scale. Unfortunately, this is not the case. Since $\pi_t = 1$ is locally stable, escapes to $\pi_t = 0$ occur on an exponentially long large deviations time scale. This time-scale is orders of magnitude longer than the time-scale governing convergence to $\pi = 1$ in our earlier Gresham’s Law result.

5.2. Convexification. Normally, with exogenous data, it would make no difference whether a parameter known to lie in some interval is estimated by mixing between the two extremes, or by estimating it directly. With endogenous data, however, this could make a difference. What if the agent convexified the model space by estimating σ_v^2 directly, via some sort of nonlinear adaptive filtering algorithm (e.g., Mehra (1972)), or perhaps by estimating a time-varying gain instead, via an adaptive step-size algorithm (Kushner and Yang (1995))? Although $\pi = 1$ is locally stable against nonlocal alternative models, would it also be stable against local alternatives?

In this case, there is no model averaging. There is just *one* model, with σ_v^2 viewed as an unknown parameter to be estimated. To address the stability question we exploit the connection discussed in section 2.3 between σ_v^2 and the steady-state gain, γ . Because the data are endogenous, we must employ the macroeconomist’s ‘big K , little k ’ trick, which in our case we refer to as ‘big Γ , little γ ’. That is, our stability question can be posed as follows: Given that data are generated according to the aggregate gain parameter Γ , would an individual agent have an incentive to use a different gain, γ ? If not, then $\gamma = \Gamma$

¹²In the language of Sargent (1993), we have provided the agent with a lot of ‘prompting’.

is a Nash equilibrium gain, and the associated $\sigma_v^2 > 0$ represents self-confirming parameter instability. The stability question can then be addressed by checking the (local) stability of the best response map, $\gamma = B(\Gamma)$, at the self-confirming equilibrium.

To simplify the analysis, we consider a special case, where $z_t = 1$ (i.e., $\rho = 1$ and $\sigma_z = 0$). The true model becomes

$$p_t = \delta + \alpha \mathbf{E}_t p_{t+1} + \sigma \epsilon_t \quad (5.17)$$

and the agent's perceived model becomes

$$p_t = \beta_t + \sigma \epsilon_t \quad (5.18)$$

$$\beta_t = \beta_{t-1} + \sigma_v v_t \quad (5.19)$$

where σ_v is now considered to be an unknown parameter. Note that if $\sigma_v^2 > 0$, the agent's model is misspecified. As in Sargent (1999), the agent uses a random walk to approximate a constant mean. Equations (5.18)-(5.19) represent an example of Muth's (1960) 'random walk plus noise' model, in which constant gain updating is optimal. To see this, write p_t as the following ARMA(1,1) process

$$p_t = p_{t-1} + \varepsilon_t - (1 - \Gamma)\varepsilon_{t-1} \quad \Gamma = \frac{\sqrt{4s + s^2} - s}{2} \quad \sigma_\varepsilon^2 = \frac{\sigma^2}{1 - \Gamma} \quad (5.20)$$

where $s = \sigma_v^2/\sigma^2$ is the signal-to-noise ratio. Muth (1960) showed that optimal price forecasts, $\mathbf{E}_t p_{t+1} \equiv \hat{p}_{t+1}$, evolve according to the constant gain algorithm

$$\hat{p}_{t+1} = \hat{p}_t + \Gamma(p_t - \hat{p}_t) \quad (5.21)$$

This implies that the optimal forecast of next period's price is just a geometrically distributed average of current and past prices,

$$\hat{p}_{t+1} = \left(\frac{\Gamma}{1 - (1 - \Gamma)L} \right) p_t \quad (5.22)$$

Substituting this into the true model in eq. (5.17) yields the actual price process as a function of aggregate beliefs

$$\begin{aligned} p_t &= \frac{\delta}{1 - \alpha} + \left(\frac{1 - (1 - \Gamma)L}{1 - \frac{(1 - \Gamma)L}{1 - \alpha\Gamma}} \right) \frac{\epsilon_t}{1 - \alpha\Gamma} \\ &\equiv \bar{p} + f(L; \Gamma)\tilde{\epsilon}_t \end{aligned} \quad (5.23)$$

Now for the 'big Γ , little γ ' trick. Suppose prices evolve according eq. (5.23), and that an individual agent has the perceived model

$$\begin{aligned} p_t &= \frac{1 - (1 - \gamma)L}{1 - L} u_t \\ &\equiv h(L; \gamma)u_t \end{aligned} \quad (5.24)$$

What would be the agent's *optimal* gain? The solution of this problem defines a best response map, $\gamma = B(\Gamma)$, and a fixed point of this mapping, $\gamma = B(\gamma)$, defines a Nash equilibrium gain. Note that the agent's model is misspecified, since it omits the constant that appears in the actual prices process in eq. (5.23). The agent needs to use γ to compromise between tracking the dynamics generated by $\Gamma > 0$, and fitting the omitted

constant, \bar{p} . This compromise is optimally resolved by minimizing the Kullback-Leibler (KLIC) distance between equations (5.23) and (5.24)¹³

$$\begin{aligned} \gamma^* = B(\Gamma) &= \operatorname{argmin}_{\gamma} \left\{ \mathbb{E}[h(L; \gamma)^{-1}(\bar{p} + f(L; \Gamma)\tilde{\epsilon}_t)]^2 \right\} \\ &= \operatorname{argmin}_{\gamma} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [\log H(\omega; \gamma) + \sigma_{\tilde{\epsilon}}^2 H(\omega; \gamma)^{-1} F(\omega; \Gamma) + \bar{p}^2 H(0)^{-1}] d\omega \right\} \end{aligned}$$

where $F(\omega) = f(e^{-i\omega})f(e^{i\omega})$ and $H(\omega) = h(e^{-i\omega})h(e^{i\omega})$ are the spectral densities of $f(L)$ in eq. (5.23) and $h(L)$ in eq. (5.24). Although this problem cannot be solved with pencil and paper, it is easily solved numerically. Figure 8 plots the best response map using the same benchmark parameter values as before (except, of course, $\rho = 1$ now)¹⁴

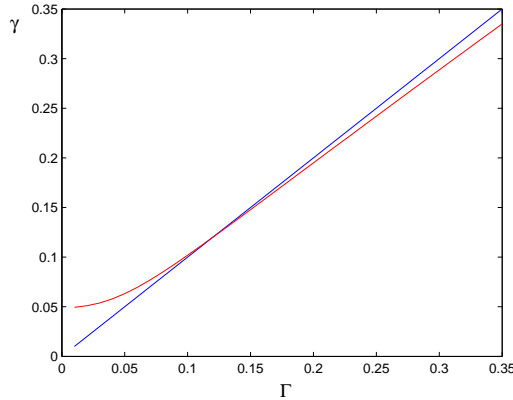


FIGURE 8. Best Response Mapping $\gamma = B(\Gamma)$

Not surprisingly, the agent's optimal gain increases when the external environment becomes more volatile, i.e., as Γ increases. What is more interesting is that the slope of the best response mapping is less than one. This means the equilibrium gain is *stable*. If agents believe that parameters are unstable, no single agent can do better by thinking they are less unstable. Figure 8 suggests that the best response map intersects the 45 degree line somewhere in the interval $(.10, .15)$. This suggests that the value of σ_v^2 used for the benchmark TVP model in section 4 was a little too small, since it implied a steady-state gain of .072.

¹³See Sargent (1999, chpt. 6) for another example of this problem.

¹⁴Note, the unit root in the perceived model in eq. (5.24) implies that its spectral density is not well defined. (It is infinite at $\omega = 0$). In the numerical calculations, we approximate by setting $(1 - L) = (1 - \eta L)$, where $\eta = .995$. This means that our frequency domain objective is ill-equipped to find the degenerate fixed point where $\gamma = \Gamma = 0$. When this is the case, the true model exhibits i.i.d fluctuations around a mean of $\delta/(1 - \alpha)$, while the agent's perceived model exhibits i.i.d fluctuations around a mean of zero. The only difference between these two processes occurs at frequency zero, which is only being approximated here.

6. CONCLUSION

Parameter instability is a fact of life for applied econometricians. This paper has proposed one explanation for why this might be. We show that if econometric models are used in a less than fully understood self-referential environment, parameter instability can become a self-confirming equilibrium. Parameter estimates are unstable simply because model-builders think they *might* be unstable.

Clearly, this sort of volatility trap is an undesirable state of affairs, which raises questions about how it could be avoided. There are two main possibilities. First, not surprisingly, better theory would produce better outcomes. The agents here suffer bad outcomes because they do not fully understand their environment. If they knew the true model in eq. (2.1), they would know that data are endogenous, and would avoid reacting to their own shadows. They would simply estimate a constant parameters reduced form model. A second, and arguably more realistic possibility, is to devise econometric procedures that are more robust to misspecified endogeneity. In Cho and Kasa (2015), we argue that in this sort of environment, model selection might actually be preferable to model averaging. If agents selected either a constant or TVP model based on sequential application of a specification or hypothesis test, the constant parameter model would prevail, as it would no longer have to compete with the TVP model.

APPENDIX A. PROOF OF PROPOSITION 2.2

For any $\sigma_v > 0$, we know from our analysis in sections 2.2 and 2.3 that $\beta_t(1)$ evolves ‘faster’ than $\beta_t(0)$. We want to exploit this time-scale separation when deriving asymptotic approximations. To do this, we assume $\sigma_v \rightarrow 0$ more slowly than $t^{-1} \rightarrow 0$, so that $t \cdot \sigma_v \rightarrow \infty$. In other words, we assume $\sigma_v = O(t^{-(1-\delta)})$ for some $\delta > 0$. Given this, we can derive the following mean ODE for $\beta(0)$ and the following diffusion for $\beta(1)$ (we do not provide the details here, since they are standard. See, e.g., Evans and Honkapohja (2001))

$$\dot{\beta}(0) = \delta + \alpha\rho[\pi\beta(1) + (1 - \pi)\beta(0)] - \beta(0) \quad (\text{A.25})$$

$$d\beta(1) = -(1 - \alpha\rho\pi)[\beta(1) - \bar{\beta}(1)]dt + \sigma_v dW \quad (\text{A.26})$$

where

$$\bar{\beta}(1) = \frac{\delta + \alpha\rho(1 - \pi)\beta(0)}{1 - \alpha\rho\pi} \quad (\text{A.27})$$

is the long-run mean of $\beta(1)$. Note that it depends on $\beta(0)$. Also note that this system is globally stable as long as $\alpha\rho < 1$. Now, since $\sigma_v = O(t^{-(1-\delta)})$, we can assume that $\beta(1)$ has converged to its long-run mean for any given value $\beta(0)$. Therefore, we can simply substitute the long-run mean in eq. (A.27) into (A.25) to derive the following autonomous ODE for $\beta(0)$

$$\dot{\beta}(0) = (1 - \alpha\rho\pi)^{-1}[\delta - (1 - \alpha\rho)\beta(0)] \quad (\text{A.28})$$

Note that this converges to $\delta/(1 - \alpha\rho)$ for all $\pi \in (0, 1)$. Finally, if substitute $\beta(0) = \delta/(1 - \alpha\rho)$ into (A.27) we find $\beta(1) = \delta/(1 - \alpha\rho)$ also, again for all $\pi \in (0, 1)$. \square

APPENDIX B. PROOF OF LEMMA 3.2

Fix a sequence $\{\pi_t\}$ in Π_0 . Since the sequence is a subset of a compact set, it has a convergent subsequence. After renumbering the subsequence, let us assume that

$$\lim_{t \rightarrow \infty} \pi_t = \pi^* \in (0, 1)$$

since $\{\pi_t\} \in \Pi_0$. Depending upon the rate of convergence (or the time scale according to which π_t converges to π^*), we have to treat π_t has already converged to π^* .¹⁵

We only prove the case in which $\pi_t \rightarrow \pi^*$ according to the fastest time scale, in particular, faster than the time scale of $\beta_t(1)$. Proofs for the remaining cases follow the same logic.

Since π_t evolves according to the fastest time scale, assume that

$$\pi_t = \pi^*.$$

Under the assumption of Gaussian distributions,

$$\ln \frac{A_t(0)}{A_t(1)} = -\frac{(p_t - \rho\beta_t(0)z_t)^2}{2(\sigma^2 + \Sigma_t(0)z_t^2)} + \frac{(p_t - \rho\beta_t(1)z_t)^2}{2(\sigma^2 + \Sigma_t(1)z_t^2)} + \frac{1}{2} \ln \left[\frac{\sigma^2 + \Sigma_t(1)z_t^2}{\sigma^2 + \Sigma_t(0)z_t^2} \right]. \quad (\text{B.29})$$

Since the first two terms are normalized Gaussian variables,

$$\mathbf{E} \ln \frac{A_t(0)}{A_t(1)} = \mathbf{E} \frac{1}{2} \ln \left[\frac{\sigma^2 + \Sigma_t(1)z_t^2}{\sigma^2 + \Sigma_t(0)z_t^2} \right].$$

Recall (2.6), and note that $\Sigma_t(0) \rightarrow 0$. On the other hand, $\Sigma_t(1)$ is uniformly bounded away from 0, as $t \rightarrow \infty$, and the lower bound converges to 0, as $\sigma_v^2 \rightarrow 0$. Thus, $\beta_t(1)$ evolves on a faster time scale than $\beta_t(0)$. In calculating the limit value of (B.29), we first let $\beta_t(1)$ reach its own ‘limit’, and then let $\beta_t(0)$ go to its own limit point.

Let $p_t^e(i)$ be the period- t price forecast by model i ,

$$p_t^e(1) = \rho\beta_t(1)z_t.$$

Since

$$p_t = \alpha\rho[(1 - \pi_t)\beta_t(0) + \pi_t\beta_t(1)]z_t + \delta z_t + \sigma\epsilon_t,$$

¹⁵If π_t evolves at a slower time scale than $\beta_t(0)$, then we fix π_t while investigating the asymptotic properties of $\beta_t(0)$. As it turns out, we obtain the same conclusion for all cases.

the forecast error of model 1 is

$$p_t - p_t^e(1) = [\alpha\rho(1 - \pi_t)\beta_t(0) + (\alpha\rho\pi_t - 1)\beta_t(1) + \delta]z_t + \sigma\epsilon_t.$$

Since $\beta_t(1)$ evolves according to (2.6),

$$\lim_{t \rightarrow \infty} \mathbf{E} [\alpha\rho(1 - \pi_t)\beta_t(0) + (\alpha\rho\pi_t - 1)\beta_t(1) + \delta] = 0$$

in any limit point of the Bayesian learning dynamics.¹⁶ Since $\beta_t(1)$ evolves at a faster rate than $\beta_t(0)$, we can treat $\beta_t(0)$ as a constant. Since $\pi_t = \pi^s$, we treat π_t as constant also.¹⁷ Define

$$\bar{\beta}(1) = \lim_{t \rightarrow 0} \mathbf{E}\beta_t(1)$$

whose value is conditioned on π_t and $\beta_t(0)$. Since

$$\lim_{\Sigma_t(1)} \lim_{t \rightarrow 0} [\alpha\rho(1 - \pi_t)\beta_t(0) + (\alpha\rho\pi_t - 1)\bar{\beta}(1) + \delta] + \mathbf{E}(\alpha\rho\pi_t - 1)(\beta_t(1) - \bar{\beta}(1)) = 0.$$

Thus, as we found in the proof of Proposition 2.2,

$$\bar{\beta}(1) = \mathbf{E} \frac{\alpha\rho(1 - \pi_t)\beta_t(0) + \delta}{1 - \alpha\rho\pi_t}.$$

Define the deviation from the long-run mean as

$$\xi_t = \beta_t(1) - \bar{\beta}(1).$$

Model 1's mean-squared forecast error is then

$$\lim_{t \rightarrow 0} \mathbf{E}(p_t - p_t^e(1))^2 = \lim_{t \rightarrow 0} \mathbf{E}z_t^2(\alpha\rho\pi_t - 1)^2\sigma_\xi^2 + \sigma^2$$

Note that

$$\lim_{\sigma_v^2 \rightarrow 0} \sigma_\xi^2 = 0.$$

To investigate the asymptotic properties of $\beta_t(0)$, let us write

$$\beta_t(1) = \frac{\alpha\rho(1 - \pi_t)\beta_t(0) + \delta}{1 - \alpha\rho\pi_t} + \xi_t$$

Then, we can write Model 0's forecast error as

$$p_t - p_t^e(0) = z_t \left[-\frac{1 - \alpha\rho}{1 - \alpha\rho\pi_t} \left(\beta_t(0) - \frac{\delta}{1 - \alpha\rho} \right) + \alpha\rho\pi_t\xi_t \right] + \sigma\epsilon_t.$$

Since $\beta_t(0)$ evolves according to (2.6)

$$\lim_{t \rightarrow \infty} \beta_t(0) = \frac{\delta}{1 - \alpha\rho}$$

with probability 1. Thus, the mean-squared forecast error satisfies

$$\lim_{t \rightarrow \infty} \mathbf{E}(p_t - p_t^e(0))^2 = \lim_{t \rightarrow \infty} \mathbf{E}z_t^2\sigma_\xi^2(\alpha\rho\pi_t)^2 + \sigma^2$$

Thus, once again as in the proof of Proposition 2.2, in the long run

$$\lim_{t \rightarrow 0} \beta_t(1) = \frac{\delta}{1 - \alpha\rho}$$

in distribution, as $\Sigma_t(1) \rightarrow 0$ or equivalently, $\sigma_v^2 \rightarrow 0$. Note that

$$\lim_{t \rightarrow \infty} \frac{\mathbf{E}(p_t - p_t^e(0))^2}{\mathbf{E}(p_t - p_t^e(1))^2} > 1 \tag{B.30}$$

if and only if

$$\lim_{t \rightarrow \infty} \left(\frac{\alpha\rho\pi_t}{1 - \alpha\rho\pi_t} \right)^2 > 1.$$

¹⁶Existence is implied by the tightness of the underlying space.

¹⁷If π_t evolves on a slower time scale than $\beta_t(1)$, we treat π_t as a constant, while investigating the asymptotic properties of $\beta_t(1)$.

Now, notice that

$$\frac{\alpha\rho\pi_t}{1-\alpha\rho\pi_t} < 1$$

if and only if

$$\alpha\rho\pi_t < \frac{1}{2}.$$

Hence, if (B.30) holds for some $t \geq 1$, then it holds again for $t+1$, and vice versa. Thus, π_t continues to increase or decrease, if the inequality holds in either direction. Recall that $\pi^* = \lim_{t \rightarrow \infty} \pi_t$. Convergence to π^* can occur only if (B.30) holds with equality for all $t \geq 1$, which is a zero probability event. We conclude that $\pi^* \in (0, 1)$ occurs with probability 0. \square

APPENDIX C. PROOF OF LEMMA 3.3

A simple calculation shows

$$t(\pi_t - \pi_{t-1}) = \frac{t(e^{(t-1)\phi_{t-1}} - e^{t\phi_t})}{(1 + e^{t\phi_t})(1 + e^{(t-1)\phi_{t-1}})}.$$

As $t \rightarrow \infty$, we know $\phi_t \rightarrow \phi^*$ with probability 1. We also know $t(\phi_t - \phi_{t-1})$ is uniformly bounded. Hence, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} t(\pi_t - \pi_{t-1}) &= \lim_{t \rightarrow \infty} \frac{t(e^{-\phi^*} - 1)e^{t\phi^*}}{(1 + e^{t\phi^*})(1 + e^{(t-1)\phi^*})} \\ &= (e^{-\phi^*} - 1) \lim_{t \rightarrow \infty} \frac{t}{(1 + e^{-t\phi^*})(1 + e^{t\phi^*}e^{-\phi^*})} \end{aligned}$$

Finally, notice that for both $\phi^* > 0$ and $\phi^* < 0$ the denominator converges to ∞ faster than the numerator. \square

APPENDIX D. PROOF OF PROPOSITION 5.1

Recall that

$$\phi_t = \frac{1}{t} \sum_{k=1}^t \log \frac{A_k(0)}{A_k(1)}$$

where $\beta_t(0)$ is updated according to (2.6). Now let us fix $\beta_t(0) = \beta^*$, and define the corresponding value of ϕ_t as ϕ_t^e . Note that since $\beta_t(0) \rightarrow \beta^*$,

$$\phi^* = \mathbb{E} \log \frac{A_t(0)}{A_t(1)}$$

is defined for $\beta_t(0) = \beta^*$. Following the same logic as in the text,

$$\phi_t^e = \phi_{t-1}^e + \frac{1}{t} (\phi^* - \phi_{t-1}^e)$$

and

$$\phi_t^e \rightarrow \phi^*$$

with probability 1, as $t \rightarrow \infty$.

One can easily verify that $\phi^* < 0$ only if $\lim_{t \rightarrow \infty} \pi_t = 1$, and $\phi^* > 0$ only if $\lim_{t \rightarrow \infty} \pi_t = 0$. To differentiate the two locally stable points of ϕ_t^e , let us write $\phi_+^* > 0$ and $\phi_-^* < 0$ for the positive and negative locally stable points of ϕ_t^e . Note that the domain of attraction for ϕ_-^* is \mathcal{D}_1 , and similarly, the domain of attraction for ϕ_+^* is \mathcal{D}_0 . A simple calculation shows that

$$\phi_+^* + \phi_-^* > 0.$$

That is, ϕ_+^* is further away from the boundary of its domain of attraction than ϕ_-^* .

We need to calculate

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}\left(\exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_0 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(1, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho}\right)\right)}{\mathbb{P}\left(\exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_1 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(0, \frac{\delta}{1-\alpha\rho}, \frac{\delta}{1-\alpha\rho}\right)\right)}. \quad (\text{D.31})$$

If the limit vanishes, the transition from \mathcal{D}_0 to \mathcal{D}_1 dominates the transition from \mathcal{D}_1 to \mathcal{D}_0 , which implies that as $\sigma_v^2 \rightarrow 0$, the probability is massed at the stable point in \mathcal{D}_1 , i.e., $\pi = 1$. On the other hand, if the limit explodes, then the probability mass of π_t converges to 1 at $\pi_t = 0$.

Note

$$\begin{aligned} \mathbf{P} \left(\exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_0 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(1, \frac{\delta}{1 - \alpha\rho}, \frac{\delta}{1 - \alpha\rho} \right) \right) \\ = \mathbf{P}(\exists t, \phi_t^e > 0 \mid \phi_0^e = \phi^* < 0) \end{aligned}$$

and

$$\begin{aligned} \mathbf{P} \left(\exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in \mathcal{D}_1 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(0, \frac{\delta}{1 - \alpha\rho}, \frac{\delta}{1 - \alpha\rho} \right) \right) \\ = \mathbf{P}(\exists t, \phi_t^e < 0 \mid \phi_0^e = \phi^* > 0). \end{aligned}$$

The right hand side of both equations can be approximated as

$$\lim_{t \rightarrow \infty} -\frac{1}{t} \log \mathbf{P}(\exists t, \phi_t^e < 0 \mid \phi_0^e = \phi_+^* > 0) = r_0^*$$

and

$$\lim_{t \rightarrow \infty} -\frac{1}{t} \log \mathbf{P}(\exists t, \phi_t^e > 0 \mid \phi_0^e = \phi_-^* < 0) = r_1^*,$$

where r_0^* and r_1^* are the values of the potential function of ODE at the boundary of the domain of the attraction (i.e., $\phi^e = 0$). Note that (D.31) explodes if $r_0^* > r_1^*$, from which the conclusion of the proposition follows.

Since the dynamics of ϕ_t^e is approximated by a one-dimensional ODE

$$\dot{\phi}^e = \phi^* - \phi^e,$$

the potential function exists. Following Example 3.1 on page 121 of Freidlin and Wentzell (1998), we know that the potential function is

$$U(\tau) = - \int_0^\tau \mathbf{E} \log \phi^* - \phi^e(s) ds$$

where τ is the first exit time from the domain of the attraction with $\phi^e(0) = \phi^*$. Let us write $U(\tau)|_{\phi_+^* > 0}$ and $U(\tau)|_{\phi_-^* < 0}$ for the potential function around the neighborhood of $\pi = 0$ and $\pi = 1$, respectively. We apply the same convention to other variables.

Since we assume that $\rho\alpha < 1$, one can easily show that

$$\phi_+^* + \phi_-^* > 0.$$

Thus,

$$r_0^* = U(\tau)|_{\phi_+^* > 0} > U(\tau)|_{\phi_+^* < 0} = r_1^* > 0,$$

as desired. \square

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