Money and Asset Liquidity in Frictional Capital Markets*

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Abstract

We endogenize asset liquidity and financing constraints in a dynamic general equilibrium model with search frictions on capital markets. Assets trading on frictional capital markets are only partially saleable. Liquid assets, such as fiat money, are not subject to search frictions and can be used to insure idiosyncratic investment risks. Assets traded on frictional capital markets thus carry a liquidity premium over liquid assets. We show that, in equilibrium, low asset saleability is typically associated with lower asset prices, tighter financing constraints and, hence, stronger demand for public liquidity. Lower asset liquidity feeds into real allocations, constraining real investment, consumption, and production.

One important function of financial intermediaries and markets is liquidity provision. Technological constraints, transaction costs, and information frictions may prevent private agents from contracting and trading with each other in order to channel resources from investors with excess liquidity to firms with funding needs. Frictional capital markets can, thus, severely limit firms' ability to finance idiosyncratic investment opportunities. Financial intermediaries and dealers on financial markets offer specialized services to facilitate transactions of privately-issued financial assets, thus improving liquidity provision and significantly affecting asset prices. In view of frictional private asset markets, publicly created liquidity such as fiat money, provides an alternative - albeit low-yielding - hedge against financing constraints.

Nevertheless, the macro literature rarely studies private liquidity provision together with asset pricing implications. This paper explores the macroeconomic impact of variation in liquidity provision through the financial sector. We model financial intermediation as a competitive search process, through which buying quotes and selling quotes of financial assets are matched (see e.g., Moen (1997) for competitive search and Rocheteau & Weill (2011) for a survey of search and asset liquidity). This is related to the random search framework in Cui & Radde (2016). Importantly, asset liquidity and financing constraints are generated endogenously through this search and matching process. Therefore, asset prices and liquidity vary with aggregate conditions and feed into real allocations. We show, in particular, that less liquid private capital markets are associated with stronger demand for public liquidity and tighter financing constraints, depressing real economic activities.

*The views expressed in this paper are those of the authors and do not necessarily reflect those of the European Central Bank (ECB).

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1 The Model

Consider an economy consisting of a continuum of households (with a continuum of members), firms, and financial intermediaries, each with measure one. Similar to Shi (2015), there are idiosyncratic risks related to household members' types. Time is discrete and denoted by $t = 0, 1, 2, ...$ Each period is divided into four sub-periods.

(1) *The households' decision period.* The aggregate productivity (TFP) shock A_t is realized. All members in a representative household *equally divide* the household's financial assets consisting of money and privately issued financial claims. The household instructs its members on the optimal type-specific choices to be carried out after types have been realized.

(2) *The production period.* Each member receives a status draw, becoming an entrepreneur (type i) with a probability χ and a worker (type n), otherwise. Only entrepreneurs have access to investment projects. Competitive firms rent aggregate capital stock K_t and labour N_t from households to produce final goods Y_t according to the production technology with TFP A_t: $Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$, where $\alpha \in (0, 1)$. The rental rate of capital and the wage rate are given by

$$r_t = \alpha A_t \left(K_t/N_t\right)^{\alpha-1}, \quad w_t = (1-\alpha) A_t \left(K_t/N_t\right)^{\alpha} \tag{1}$$

(3) *The investment period.* There is no insurance among household members, and they keep separated until the consumption stage. Entrepreneurs seek financing and undertake investment projects, which transform consumption goods into capital stock one-for-one. Capital markets open in which entrepreneurs offer financial claims for sale on asset search markets to finance new investment. Financial intermediaries facilitate asset transactions on the search markets by implementing a costly matching process. Because of search frictions, private financial claims are only partially liquid. Money, in *fixed supply* $\bar{B}$, is fully liquid as it can be traded on a frictionless spot market.[1]

(4) *The consumption period.* Entrepreneurs and workers reunite again in their respective households, pool all assets together, and *equally* share consumption goods across all members.

To be clear, we focus on the type of equilibrium in which there are gains from trading both private claims and money.

1.1 A Representative Household

Preferences. The household has a per-period utility $u(C_t)$ over the total household consumption C_t, where $u'(.) > 0$ and $u''(.) < 0$. The household's preference is represented by $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t)$, where $\mathbb{E}$ is the expectation operator and $\beta \in (0, 1)$ is the discount factor.

Households' Wealth. Households hold nominal and fully liquid assets (money) with nominal price P_t. In addition, physical capital (K_t), earning the rental return r_t, is owned by households and depreciates at a rate $\delta \in (0, 1)$. There is a financial claim to the future return of every unit of capital. For example, the owner of one unit of claims issued at time $t-1$ is entitled to r_t at t, $(1-\delta)r_{t+1}$ at time $t+1$, $(1-\delta)^2 r_{t+2}$ at time $t+2$, and so on. For exposition simplicity, we follow Kiyotaki & Moore (2012) and normalize the claims by

[1] We focus on the equilibrium in which this intrinsically worthless asset is valued for its liquidity service and accepted by all market participants.

the capital stock, such that they depreciate at the same rate δ, but earn a return r_{t+s} at any date $t+s$ ($\forall s \geq 0$).

Hence, each household owns a portfolio of money, private claims issued by other households, and the fraction of their own physical capital which has *not* been issued to other households. The latter has the same price and saleability as claims already issued, since both new claims to unissued capital and old claims would need to be traded on the same search market. Therefore, besides liquid assets B_t, we only need to keep track of net private claims defined as, $S_t =$ claims on others' capital + unissued capital stock.

Asset accumulation. Let S_t^j and B_t^j denote the total net private claims and money belonging to type $j \in \{i, n\}$ members at the beginning of t. As all assets are equally divided among members, we have $S_t^i = \chi S_t$, $S_t^n = (1-\chi)S_t$, $B_t^i = \chi B_t$, and $B_t^n = (1-\chi)B_t$ by the law of large numbers.

Let S_{t+1}^j and B_{t+1}^j denote the end-of-period total net private claims and money for type j. Then, we know that $S_{t+1} = S_{t+1}^i + S_{t+1}^n$ and $B_{t+1} = B_{t+1}^i + B_{t+1}^n$ as household members pool assets together at the end of t. Further, household members face two financing constraints. First, no private agent can issue money, i.e.,

$$B_{t+1}^j \geq 0 \tag{2}$$

The second constraint relates to the accumulation of private claims. For each group, the net private assets position evolves according to

$$S_{t+1}^j = (1-\delta)S_t^j + I_t^j - M_t^j \tag{3}$$

where I_t^j is investment into capital stock, and M_t^j is the quantity of private claims sold on the search market. Due to search frictions on private capital markets, only an endogenously determined fraction $\phi_t \in (0,1)$ (i.e., *asset saleability*) of new or existing assets can be issued or resold. Therefore, agents need to retain a fraction $(1-\phi_t)$ of their portfolio of existing private claims and claims issued against new investment, thus limiting the external funding for new investment Then, (3) implies the second financing constraint:

$$S_{t+1}^j \geq (1-\phi_t)\left[I_t^j + (1-\delta)S_t^j\right] \tag{4}$$

Workers' flow-of-funds constraint. All workers are the same, and the worker group does not invest ($I_t^n = 0$). For simplicity, labour supply is assumed to be fixed at $N_t = \bar{N}$. They accumulate financial assets ($M_t^i < 0$ and $B_{t+1}^n > 0$) to implement their household's intertemporal consumption smoothing purposes. Thus, neither of their financing constraints is binding. They use labour income ($w_t N_t$) and the return on private claims ($S_t^n = (1-\chi)S_t$) and money ($B_t^n = (1-\chi)B_t$) to finance consumption goods (C_t^n) and the end-of-period portfolio of private claims (S_{t+1}^n) and money (B_{t+1}^n) :

$$\begin{aligned} C_t^n + \frac{B_{t+1}^n}{P_t} + q_t^n S_{t+1}^n &= w_t N_t + \frac{(1-\chi)\,B_t}{P_t} \\ &+ \left[r + q_t^n(1-\delta)\right](1-\chi)\,S_t \end{aligned} \tag{5}$$

where private assets are purchased at the price q_t^n, while money is valued in real terms by $1/P_t$.

Entrepreneurs' flow-of-funds constraint. The entrepreneur group needs to finance new investment ($I_t^i > 0$). They can use return on private claims ($S_t^i = \chi S_t$) and money ($B_t^i = \chi B_t$) together with the issuance (or reselling) of private claims ($M_t^i = I_t^i + (1-\delta)S_t^i - S_{t+1}^i$) to finance consumption (C_t^i), new money holdings (B_{t+1}^i), and physical investment (I_t^i):

$$\begin{aligned} C_t^i + \frac{B_{t+1}^i}{P_t} + I_t^i &= q_t^i \left[I_t^i + (1-\delta)S_t^i - S_{t+1}^i \right] \\ &+ \frac{\chi B_t}{P_t} + r_t \chi S_t \end{aligned}$$

where private claims are issued or resold at the selling price q_t^i. Note q_t^i is also equal to Tobin's q: the ratio of the market value of capital to the replacement cost (i.e., unity). As long as $q_t^i > 1$, entrepreneurs will use all available resources to create new capital. We assume and later verify that $q_t^i > 1$ when $\kappa > 0$. That is, both financing constraints (2) and (4) bind, and entrepreneurs do not bring consumption goods back to their household ($C_t^i = 0$). Hence, $S_{t+1}^i = (1-\phi_t)\left[I_t^i + (1-\delta)\chi S_t\right]$ according to (4), and we can express investment as $I_t^i = \frac{S_{t+1}^i - (1-\phi_t)(1-\delta)\chi S_t}{1-\phi_t}$ and rewrite the entrepreneurs' flow-of-funds constraint to

$$q_t^r S_{t+1}^i = \frac{\chi B_t}{P_t} + \left[r_t + (1-\delta)\right]\chi S_t \tag{6}$$

$$\text{where} \quad q_t^r \equiv \frac{1-\phi_t q_t^i}{1-\phi_t} < 1 \tag{7}$$

The right-hand side of (6) is total net-worth, and the left-hand side's end-of-period private assets are valued at q_t^r, the *effective* replacement cost of private assets: for every unit of new investment, a ϕ_t fraction is issued at the price q_t^i; entrepreneurs need to make a "down-payment" $(1-\phi_t q_t^i)$ and retain a fraction $(1-\phi_t)$ as inside equity claims. The lower q_t^r is, the more S_{t+1}^i entrepreneurs can bring back to the household.

Once we know S_{t+1}^i from (6), aggregate investment $I_t = I_t^i = \frac{S_{t+1}^i - (1-\phi_t)(1-\delta)\chi S_t}{1-\phi_t}$ can be backed out as

$$I_t = \frac{\left[r_t + (1-\delta)\phi_t q_t^i\right]\chi S_t + \chi B_t/P_t}{1-\phi_t q_t^i} \tag{8}$$

Noticing $(1-\delta)\phi_t q_t^i \chi S_t$ is the saleable part of old claims, we know that entrepreneurs' *liquid net-worth* can be "leveraged" with a factor $(1-\phi_t q_t^i)^{-1}$ to invest in new capital.

A household's problem. We work out a household-wide budget constraint. Let ρ_t be the ratio of the purchasing price to the effective replacement cost:

$$\rho_t \equiv \frac{q_t^n}{q_t^r} = \frac{(1-\phi_t)q_t^n}{1-\phi_t q_t^i} \tag{9}$$

Then, multiplying (6) by ρ_t and adding (5), we have

$$\begin{aligned} C_t + \frac{B_{t+1}}{P_t} + q_t^n S_{t+1} &= w_t N_t + (\chi\rho_t + 1 - \chi)\left(\frac{B_t}{P_t} + r_t S_t\right) \\ &+ \left[\chi\rho_t + (1-\chi)q_t^n\right](1-\delta)S_t \end{aligned} \tag{10}$$

where we have used the fact that $S_{t+1} = S^i_{t+1} + S^n_{t+1}$, $B_{t+1} = B^i_{t+1} + B^n_{t+1}$, and $C_t = C^i_t + C^n_t$. Let $J(S_t, B_t; \Gamma_t)$ be the value function of a household with private claims S_t and money stock B_t, given the aggregate state $\Gamma_t \equiv (K_t, A_t)$. Then, J satisfies the following Bellman equation

$$\begin{aligned} J(S_t, B_t; \Gamma_t) \;=\;& \max_{\{C_t, S_{t+1}, B_{t+1}\}} \{u(C_t) \\ &+ \beta \mathbb{E}\left[J(S_{t+1}, B_{t+1}; \Gamma_{t+1}) | \Gamma_t\right]\}, \quad s.t. \quad (10) \end{aligned}$$

1.2 Financial Intermediation and Capital Markets

Each intermediary can engage in purchasing claims from entrepreneurs, selling claims to workers, and trade claims with other intermediaries.

Search and matching. There is a measure one continuum of capital sub-markets with free entry, denoted by superscript m. Intermediaries need time and resources to search for valuable projects on each market, in which entrepreneurs post U^m_t units of sell offers backed by capital stock. Intermediaries can choose on which sub-market to post buy quotes V^m_t at a cost of κ per unit of quotes. The probabilities of filling a buy quote and a sell quote (offer) are f^m_t and ϕ^m_t, respectively. The matching technology of intermediaries is characterized by a matching function

$$M^m_t = M(U^m_t, V^m_t) = \xi \left(U^m_t\right)^\eta \left(V^m_t\right)^{1-\eta}$$

where ξ captures matching efficiency and η the matching elasticity with respect to sell quotes U_t. The matching technology endogenizes $\phi^m_t \equiv \frac{M(U^m_t, V^m_t)}{U^m_t}$ and $f^m_t \equiv \frac{M(U^m_t, V^m_t)}{V^m_t}$. Therefore,

$$f_t = \xi^{\frac{1}{1-\eta}} \phi_t^{\frac{\eta}{\eta-1}} \tag{11}$$

To maximize external funding via the capital market, entrepreneurs post quotes amounting to $U^m_t = I^i_t + (1-\delta)\chi S_t$, of which a fraction $\phi^m_t U^m_t$ is funded. ϕ^m_t is thus again asset saleability.

Selling claims. In order to sell claims to workers, sell quotes U^m_t need to be communicated by intermediaries to workers also at a cost of κ per unit of quotes. One can think of the costs as advertisement costs to investors. Before communicating and selling claims to workers, intermediaries in sub-market m can freely trade claims with non-communicated sell quotes among themselves at a competitive price q^m_t.

On each sub-market m, the cost of vacancies in order to have 1 unit of claims matched is κ / f^m_t, while the benefit amounts to $q^m_t - q^{i,m}_t$. Because of the competitive environment, we have one zero-profit condition:

$$\frac{\kappa}{f^m_t} = q^m_t - q^{i,m}_t \tag{12}$$

In addition, for each unit of claims delivered to workers, the total number of sell offers is $1/\phi^m_t$ and the communication cost is κ/ϕ^m_t , while the benefit is the price difference $q^n_t - q^m_t$. We thus have another zero-profit condition:

$$\frac{\kappa}{\phi^m_t} = q^n_t - q^m_t \tag{13}$$

In light of the two zero-profit conditions above, intermediaries are indifferent between all sub-markets and we can omit the superscript m.

Asset price. Given these features of search-matching and intermediation, each sub-market is characterized by its saleability-sell-price combination (ϕ_t, q_t^i). Accordingly, entrepreneurs choose the sub-market in which to post their sell offers, by minimizing the effective replacement cost $q_t^r = \frac{1-\phi_t q_t^i}{1-\phi_t}$, subject to the zero-profit condition (12) and the relationship between f_t and ϕ_t:

$$\min_{\{\phi_t, q_t^i\}} \quad \frac{1-\phi_t q_t^i}{1-\phi_t}, \quad s.t. \quad (11), (12)$$

Doing so maximizes the end-of-period S_{t+1}^i, according to (6). The optimal solution (see the proof in the Appendix) yields

$$q_t^i = 1 + \frac{\kappa\eta}{1-\eta}\frac{(1-\phi_t)}{f_t} \geq 1 \tag{14}$$

We thus verify that $q_t^i > 1$ when $\kappa > 0$. Using the two zero-profit conditions (12) and (13) together, we can eliminate q_t^m and obtain:

$$q_t^n - q_t^i = \kappa\left(\frac{1}{f_t} + \frac{1}{\phi_t}\right) \tag{15}$$

2 The Equilibrium with Two Types of Assets

As previously stated, we focus on the particular type of equilibrium, in which money and private claims co-exist. We have such co-existence type of equilibrium, whenever the intermediation cost κ is large enough for money to relax entrepreneurs' financing constraints, while at the same time being sufficiently small, such that the issuance of private claims remains profitable.[2]

Portfolio choices. The household's optimal portfolio choices for money and private financial assets yield two asset pricing formulae (Euler equations):

$$\mathbb{E}\left[\Delta_{t+1}\frac{\chi\rho_{t+1} + 1 - \chi}{P_{t+1}/P_t}|\Gamma_t\right] = 1 \tag{16}$$

$$\mathbb{E}\left[\Delta_{t+1}\left[\chi\rho_{t+1}r_{t+1}^{ni} + (1-\chi)r_{t+1}^{nn}\right]|\Gamma_t\right] = 1 \tag{17}$$

where $\Delta_{t+1} = \frac{\beta u'(C_{t+1})}{u'(C_t)}$ is the stochastic discount factor. $r_{t+1}^{ni} \equiv \frac{r_{t+1}+(1-\delta)}{q_t^n}$ is the return on private assets purchased at t from the point of view of a worker, who becomes an entrepreneur at $t+1$; and $r_{t+1}^{nn} \equiv \frac{r_{t+1}+(1-\delta)q_{t+1}^n}{q_t^n}$ is the corresponding return if the worker does not change type.

[2]When $\kappa \to 0$ private assets provide sufficient liquidity, as (14), (15), and (9) jointly imply that $q_t^i = q_t^n = q_t^r = \rho_t = 1$. In the absence of intermediation costs, money is, therefore, not valued as a lubricant for investment financing, such that $B_{t+1}/P_t = B_t/P_t = 0$. The efficient level of investment can be implemented by issuing private claims only and our model resembles a standard real business cycle model. For a full treatment of the conditions for the circulation of different types of financial assets, the reader is referred to Cui & Radde (2016).

Equilibrium characterization. A *recursive competitive equilibrium* is a mapping $\Gamma_t \equiv (K_t, A_t) \rightarrow \Gamma_{t+1} \equiv (K_{t+1}, A_{t+1})$, with consumption, investment, and portfolio choices $\{C_t, I_t, S_{t+1}, B_{t+1}\}$, asset market features $\{\phi_t, f_t\}$, a collection of prices $\{\rho_t, q_t^i, q_t^n, P_t, w_t, r_t\}$, and with an exogenous process for $\{A_t\}$, such that: firms' optimality conditions in (1) hold; given prices, the policy functions solve the representative household's problem, satisfying (9), (10), (16), and (17); I_t is determined by (8); the capital and money market clears: $K_{t+1} = (1-\delta) K_t + I_t$, $S_t = K_t$, $B_t = \bar{B}$; the asset search market "clears": (11), (14), and (15) hold.

3 Asset Liquidity and the Macroeconomy

To spell out the impact of search frictions on asset liquidity, we focus on the long-run equilibrium featuring both money and private assets.

Liquidity premium. When money is valued by investors, it relaxes their financing constraints. To see this, consider the asset pricing formula for money (16). In the steady state, this conditions implies that $[\chi\rho + 1 - \chi] P_t/P_{t+1} = \beta^{-1}$. As money is in fixed supply, $P_t/P_{t+1} = 1$ in the steady state. Therefore, $\rho = \rho^* \equiv 1 + \frac{\beta^{-1}-1}{\chi} > 1$, or by definition, $q^n/q^r = \rho^* > 1$. This means that the cost of private claims for workers exceeds that of entrepreneurs, as the latter cannot issue as many private assets as they would desire and remain financially constrained.

In view of binding financing constraints, investors will demand a higher return from holding only partially resaleable private assets relative to money. As a result, a positive liquidity premium emerges between the returns on private assets and money, defined as

$$\Delta_t^{LP} \equiv \mathbb{E}_t \left[\chi r_{t+1}^{ni} + (1-\chi) r_{t+1}^{nn}\right] - \mathbb{E}_t \left[\frac{P_t}{P_{t+1}}\right]$$

Proposition 1:
Suppose $\kappa > 0$ and private claims and money co-exist. Then, $r/q^n > \delta$ and money provides a liquidity service in a neighborhood around the steady state. The steady state liquidity premium amounts to $\Delta^{LP} = (1 - (\rho^*)^{-1}) (r/q^n - \delta) (1-\chi) > 0$.

Proof. See the proof in the Appendix. □

The proof basically uses the two asset pricing formulae (16) and (17), which imply that the two assets earn the same return after adjusting ρ. Only if $\kappa = 0$ and thus $\rho = 1$, the liquidity premium is zero. With valued money in fixed supply, we must have $\rho = \rho^* > 1$, and the equilibrium thus features a liquidity premium.

Asset saleability and prices. The liquidity of private assets depends both on their price from the point of view of constrained entrepreneurs and their physical saleability. In fact, both dimensions are related as the steady state asset price q^i is a function of asset saleability ϕ. Specifically, we can rewrite (14) by using (11) as

$$q^i = 1 + \frac{\kappa\eta\xi^{\frac{1}{\eta-1}}}{1-\eta} \phi^{\frac{\eta}{1-\eta}} (1-\phi) \tag{18}$$

Further, since money is valued in the co-existence type of equilibrium, $\rho = \rho^*$ is uniquely pinned down from the Euler equation (16) as discussed above. Using the definition of $\rho = q^n/q^r$, we know that

$$(1-\phi)q^n - \rho(1-\phi q^i) = 0 \tag{19}$$

where q^i satisfies (18) and $q^n = q^i + \kappa\left[\phi^{-1} + \xi^{\frac{1}{\eta-1}}\phi^{\frac{\eta}{1-\eta}}\right]$ from (15). Then, condition (19) determines the steady state asset saleability ϕ, as q^n and q^i are only functions of ϕ.

Importantly, (19) could admit multiple solutions of ϕ. For instances, if $\eta = \frac{1}{2}$, (19) becomes a quartic equation of the unknown ϕ (see the Appendix for derivation). Then, multiple values of $\phi \in (0,1)$ could solve (19), implying multiple co-existence equilibria. Intuitively, the competitive asset search process may generate multiplicity, as coordination between sellers and buyers on different sub-markets of private claims may lead to different outcomes. As a comparison, the random search framework of Cui & Radde (2016) only features a unique co-existence equilibrium.

Whether a lower level of the steady-state equilibrium saleability ϕ is associated with a higher or a lower asset price q^i depends on the relative strength of asset supply and asset demand effects: On the one hand, a lower level of ϕ implies tighter financing constraints and less supply relative to demand on the asset search market. As intermediaries have to offer more attractive conditions to attract scarce supply, this should be reflected in a higher equilibrium asset price. On the other hand, lower equilibrium asset saleability implies that private assets are less effective investments to hedge future funding needs, which would reduce demand, increase the equilibrium liquidity premium, and compress the asset price.

Which effect dominates depends on the parameters of the economy. Notice that $dq^i/d\phi = \frac{\kappa\eta\xi^{\frac{1}{\eta-1}}}{(1-\eta)^2}\phi^{\frac{\eta}{1-\eta}}\left(\frac{\eta}{\phi}-1\right)$, we obtain a sufficient condition for q^i and ϕ to positively co-move:

Proposition 2:
Suppose there are multiple values of asset saleability $\phi \in (0,1)$ that solve (19). If $\phi < \eta$, for all values of ϕ, asset price q^i drops when ϕ falls.

Intuitively, when ϕ is sufficiently small, the low hedging value of private assets is associated with a high sensitivity of demand from workers to market conditions. As a result, the demand effect dominates the supply effect, such that a lower level ϕ is associated with a higher liquidity premium and a lower q^i. One should note that the dominance of demand effect crucially depends on the endogenous search and matching.

The macroeconomy. When both ϕ and q^i fall, aggregate investment in (8) falls because both liquid net-worth and leverage drop. That is, an economy with particularly illiquid asset markets simultaneously features a low asset price and tighter financing constraints - and thus less investment. This result illustrates the effect of asset illiquidity via financing constraints on real allocations.

Aggregate disturbances, such as shocks to TFP, could cause the economy to switch between long-run equilibria with more or less liquid financial markets. Frictional capital markets may, thus, not only amplify macroeconomic dynamics, but also generate large macroeconomic swings between different steady states.

4 A Numerical Example

The following example highlights that coordination problems on financial markets strongly impact asset liquidity and portfolio allocations between private and public financial assets, thus significantly affecting real economic activities.

Let $\beta = 0.99$, $\delta = 0.025$, and $\alpha = 0.33$ as in a standard calibration for a quarterly macro model. Following Shi (2015) and interpreting χ as the fraction of firms that has investment opportunities each quarter, we set $\chi = 0.056$ in-line with Doms & Dunne (1998). We also let $\eta = 0.5$.

Finally, set $\xi = 0.2$ and κ =0.01 such that the "leverage factor" $(1 - \phi q^i)^{-1}$ is 1.35 and annualized liquidity premium amounts to 50 basis points with the highest ϕ. Then,

$$\phi_1 = 0.0764 \quad \text{and} \quad \phi_2 = 0.2464$$

solve (19). That is, private financial markets are active, but exhibit different degrees of efficiency. The corresponding (real) values of money B/P are 0.2941 and 0.0374 (a drop of 87.29%), i.e., agents in the equilibrium with highly efficient private financial markets value public liquidity less.

Notice that in these two equilibria satisfies $\phi < \eta$, such that we know from Proposition 2 that the equilibrium q^i will be higher if ϕ is higher. In fact, when steady state saleability increases from ϕ_1 to ϕ_2, asset price q^i increases from 1.03 to 1.06. In the mean time, liquidity premium decreases 5 basis points. Also note that $\phi < \frac{1}{2}$ seems empirically plausible according to Del Negro *et al.* (2011), as otherwise all claims will have a turn-over rate of more than 50% within a quarter.

By affecting asset liquidity, participation decisions in the financial market can, thus, have a strong impact on firms' financing constraints, output and capital accumulation. For example, when steady state saleability switches from ϕ_1 to ϕ_2, investment, consumption, and output increase by 6.54%, 2.94%, and 2.11%, respectively. The details of calculation are available in the Appendix.

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Appendices

A Equilibrium Conditions

Assuming both private claims and money co-exist. Define $L_t \equiv B_t/P_{t-1}$ and inflation $\Pi_t \equiv P_t/P_{t-1}$. We also substitute S_t by $S_t = K_t$. Then, we solve (K_{t+1},L_{t+1}, C_t, I_t, r_t, Π_t, q_t, q^i_t, q^n_t, ρ_t, ϕ_t, f_t), given (K_t, L_t, A_t) with the following equilibrium conditions collected from the main text:

$$r_t = A_t \alpha (K_t/\bar{N})^{\alpha-1} \tag{20}$$

$$1 = \beta \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} \frac{\chi\rho_{t+1} + 1 - \chi}{\Pi_{t+1}} \right] \tag{21}$$

$$1 = \beta \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} \frac{[(\chi\rho_{t+1} + 1 - \chi) r_{t+1} + [\chi\rho_{t+1} + (1-\chi) q^n]\,(1-\delta)]}{q^n_t} \right] \tag{22}$$

$$I_t = \frac{\left[r_t + (1-\delta)\phi_t q^i_t\right] \chi K_t + \chi L_t/\Pi_t}{1 - \phi_t q^i_t}. \tag{23}$$

$$q^i = 1 + \kappa \left[1 + \frac{\eta}{1-\eta} \frac{(1-\phi)}{f} \right] \tag{24}$$

$$q^n_t - q^i_t = \kappa \left[\phi^{-1} + f^{-1} \right] \tag{25}$$

$$f_t = \xi^{\frac{1}{1-\eta}} \phi_t^{\frac{\eta}{\eta-1}} \tag{26}$$

$$\rho_t = \frac{q^n_t}{q^r_t} = \frac{(1-\phi_t) q^n_t}{1 - \phi_t q^i_t} \tag{27}$$

$$C_t + L_{t+1} + q^n_t K_{t+1} = \left[\frac{(1-\alpha)}{\alpha} + \chi\rho_t + 1 - \chi \right] r_t K_t \tag{28}$$

$$+ \left[\chi\rho_t + (1-\chi) q^n_t\right] K_t + \left[\chi\rho_t + 1 - \chi\right] \frac{L_t}{\Pi_t} \tag{29}$$

$$K_{t+1} = (1-\delta) K_t + I_t \tag{30}$$

$$L_{t+1} = \frac{L_t}{\Pi_t} \tag{31}$$

where we have used the fact that $w_t N_t = \frac{(1-\alpha) r_t K_t}{\alpha}$ in the household budget constraint (28). The co-existence of money and private claims is equivalent to that (16), (17), $L_t \geq 0$, and $\phi_t > 0$ all satisfy.

B Steady State

In steady state, $\Pi = 1$ and we know from the Euler equation for money (16) that

$$\rho = \rho^* \equiv 1 + \frac{\beta^{-1} - 1}{\chi} \tag{32}$$

Using the definition of ρ in (27), we know that

$$(1-\phi) q^n - \rho(1 - \phi q^i) = 0 \tag{33}$$

where

$$q^i = 1 + \frac{\kappa\eta\xi^{\frac{1}{\eta-1}}}{1-\eta}(1-\phi)\phi^{\frac{\eta}{1-\eta}} \tag{34}$$

$$q^n = q^i + \kappa\left[\phi^{-1} + \xi^{\frac{1}{\eta-1}}\phi^{\frac{\eta}{1-\eta}}\right] \tag{35}$$

Then, we know that ϕ is a solution to equation (33), which only depends on parameters β, χ, κ, and η. Once we know ϕ, we can solve $f = \xi^{\frac{1}{1-\eta}}\phi^{\frac{\eta}{\eta-1}}$, q^n , and q^i.

So far, we obtain all relevant variables in the financial market. Now, we solve for real variables. Using (20), we know that the marginal product of capital and the level of capital stock are

$$r = \delta q^n + \left[\beta^{-1} - 1 + \chi\right](q^n - 1) \tag{36}$$

$$K = \left(\frac{r}{A\alpha}\right)^{\frac{1}{\alpha-1}} \tag{37}$$

We then use the investment equation (23) and the relationship $I = \delta K$ from (30) to derive the real value of liquidity

$$L = \left[\delta\left[1 - \phi q^i\right]/\chi - \left(r + (1-\delta)\phi q^i\right)\right]K \tag{38}$$

Our assumption is correct if $L \geq 0$. Finally, consumption C can be solved directly from the household's budget constraint (28).

$$C = \left[\frac{(1-\alpha)}{\alpha} + \chi\rho + 1 - \chi\right] rK + \chi(\rho - 1)K + \chi(\rho - 1)L \tag{39}$$

C Proofs

We follow the notation in the main text.

Proof to optimal (ϕ, q^i) *pairs.* Since $\frac{\kappa}{f} = q - q^i$, then $1 - \phi q^i = \frac{\kappa\phi}{f} + 1 - \phi q^n$ and

$$q^r = \frac{1 - \phi q^i}{1-\phi} = \frac{\frac{\kappa\phi}{f} + 1 - \phi q}{1-\phi}$$

To minimize q^r, we have the first-order condition with respect to ϕ as

$$\frac{dq^r(\phi)}{d\phi} = \frac{(1-\phi)\left[\frac{\kappa}{(1-\eta)f} - q\right] + \left[\frac{\kappa\phi}{f} + 1 - \phi q\right]}{(1-\phi)^2} = 0 \tag{40}$$

where we have used the fact that $f = \xi^{\frac{1}{1-\eta}}\phi^{\frac{\eta}{\eta-1}}$. Rearranging, we have

$$q = 1 + \kappa\left[\frac{1-\phi}{(1-\eta)f} + \frac{\phi}{f}\right] \tag{41}$$

which implies that

$$q^i = 1 + \frac{\kappa\eta(1-\phi)}{(1-\eta)f}$$

as shown in the main text. Given q^n, ϕ can be backed out from $\frac{\kappa}{f} = q^n - q^i$ and $f = \xi^{\frac{1}{1-\eta}}\phi^{\frac{\eta}{\eta-1}}$.

Finally, we check the second-order condition to ensure minimization. Using (40), we can express

$$\frac{dq^r(\phi)}{d\phi} = \frac{\kappa}{(1-\eta)}\frac{1}{(1-\phi)f} + \frac{1-q}{(1-\phi)^2} + \frac{\kappa\phi}{f(1-\phi)^2}$$

Then, we use again $f = \xi^{\frac{1}{1-\eta}} \phi^{\frac{\eta}{\eta-1}}$ to derive

$$\frac{d^2 q^r(\phi)}{d\phi^2} = \frac{\kappa}{(1-\eta)} \frac{\frac{\eta}{1-\eta}\frac{1-\phi}{\phi f} + \frac{1}{f}}{(1-\phi)^2} + \frac{2(1-q)}{(1-\phi)^3} + \kappa \frac{\frac{(1-\phi)^2}{(1-\eta)f} + 2\theta(1-\phi)}{(1-\phi)^4}$$

Plug in the expression of q from (41):

$$\begin{aligned}
\frac{d^2 q^r(\phi)}{d\phi^2} &= \frac{\kappa}{(1-\eta)} \frac{\frac{\eta}{1-\eta}\frac{1-\phi}{\phi f} + \frac{1}{f}}{(1-\phi)^2} - \kappa \frac{2\theta + \frac{2(1-\phi)}{(1-\eta)f}}{(1-\phi)^3} + \kappa \frac{\frac{(1-\phi)^2}{(1-\eta)f} + 2\theta(1-\phi)}{(1-\phi)^4} \\
&= \frac{\kappa}{(1-\eta)} \frac{\frac{\eta}{1-\eta}\frac{1-\phi}{\phi f} + \frac{1}{f}}{(1-\phi)^2} - \kappa \frac{\frac{1}{(1-\eta)f}}{(1-\phi)^2} \\
&= \frac{\kappa\eta}{(1-\eta)^2} \frac{1}{\phi(1-\phi)f} > 0
\end{aligned}$$

The first-order condition is thus verified to be sufficient and necessary.

Proof to Proposition 1: If private claims and money co-exist, we already know that $\rho = \rho^* = 1 + \frac{\beta^{-1}-1}{\chi} > 1$ (derived from the Euler equation for money (21)). The steady state liquidity premium can be written as

$$\begin{aligned}
\Delta^{LP} &= \chi r^{ni} + \frac{(1-\chi)r^{nn}}{\rho} + (1-\chi)\left(1-\rho^{-1}\right) r^{nn} - 1 \\
&= \rho^{-1}\beta^{-1} - 1 + (1-\chi)\left(1-\rho^{-1}\right) r^{nn} \\
&= \rho^{-1}(\chi\rho + 1 - \chi) - 1 + (1-\chi)\left(1-\rho^{-1}\right) r^{nn} \\
&= (1-\chi)(1-\rho^{-1})\left(r^{nn} - 1\right) \\
&= (1-\chi)(1-\rho^{-1})\left(r/q^n - \delta\right)
\end{aligned}$$

where we use (22) in the second equality.

Further, using (36) obtained from the Euler equation for private claims (22), we have

$$\begin{aligned}
\frac{r}{q^n} &= \delta + \left[\beta^{-1} - 1 + \chi\right]\left[1 - \frac{1}{q^n}\right] \\
&= \delta + \chi\rho\left[1 - \frac{1}{q^n}\right] > \delta
\end{aligned}$$

as $q^n > q^i > 1$ when $\kappa > 0$. Then, the liquidity premium

$$\begin{aligned}
\Delta^{LP} &= \left(1-\rho^{-1}\right)\left(r/q^n - \delta\right)(1-\chi) \\
&= (\rho - 1)\,\chi(1-\chi)\left[1 - \frac{1}{q^n}\right] > 0, \qquad (42)
\end{aligned}$$

which is an increasing function of ρ and q^n.

D A Numerical Example

We calibrate the model by setting $\beta = 0.99$, $\delta = 0.025$, and $\alpha = 0.33$ as in a standard calibration for a quarterly model. We set $\chi = 0.056$ as discussed in the main text. In the financial market, $\eta = 1/2$, $\xi = 0.20$, and $\kappa = 0.01$.

With $\eta = \frac{1}{2}$, we can substitute q^i and q^n to have a quartic equation. To see this, by using the relationship between q^i and q^n, we have

$$q^i\left[(\rho^* - 1)\phi + 1\right]\phi = \left[\frac{\kappa}{\xi^2}\phi^3 - \frac{\kappa}{\xi^2}\phi^2 + (\rho^* + \kappa)\phi - \kappa\right]$$

Using $q^i = 1 + \frac{\kappa}{\xi^2}\phi - \frac{\kappa}{\xi^2}\phi^2$, we obtain

$$a\phi^4 + b\phi^3 + c\phi^2 + d\phi + e = 0 \tag{43}$$

where the coefficients are

$$a = \frac{\kappa(\rho^* - 1)}{\xi^2}, \quad b = \frac{\kappa(\rho^* - 3)}{\xi^2}$$

$$c = -\left(\frac{2\kappa}{\xi^2} + \rho^* - 1\right), \quad d = \rho^* - 1 + \kappa, \quad e = -\kappa$$

Therefore, ϕ solves the quartic equation (43).

(32) implies that $\rho^* = 1.1871$. Solving (43) in $(0, 1)$, we obtain two roots:

$$\phi_1 = 0.0764 \quad \text{and} \quad \phi_2 = 0.2464$$

Then, we can compute q^i and q^n from (34) and (35). With the knowledge of q^n, we can compute r, K, L, C from (36), (37), (38), and (39). Finally, output $Y = rK/\alpha$ because of the Cobb-Douglas production function.