The Friendship Paradox and Systematic Biases in Perceptions and Social Norms

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Abstract

The "friendship paradox" (Feld (1991)) states that, on average, people have strictly fewer friends than their friends do. This is an accounting identity that stems from the fact that people with more friends are counted as friends by more people. The point of this paper is that this over-representation of the most popular people in others' samples distorts perceptions of norms and amplifies resulting behaviors to match those perceived norms. I show that there are two things that drive people with more friends to behave more extremely than people with fewer friends and thus to distort the perceived norms. The first is that in any setting with strategic complementarities, people with more friends have more social interaction and hence more complementarity to their actions. The second is that people who have the most innate preference for a given activity form more relationships as they benefit most from the complementarities. As proven here, these two effects lead people with more friends to choose more extreme actions, which in turn feeds back via the friendship paradox to increase overall perceptions of behavior and then via complementarities to amplify average behavior. These theoretical results are consistent with the multitude of studies finding that students (from middle school through university) consistently overestimate peer consumption of alcohol, cigarettes, and drugs. This amplifies students' own behaviors, and can help explain problems with adolescent abuse of drugs and binge-drinking, as well as other behaviors. The analysis explains why policies that simply inform students of actual norms are effective in improving the accuracy of their perceptions and reducing behavior. I also discuss how these results change in cases of strategic substitutes, where individuals overestimate free-riding by peers.

JEL Classification Codes: D85, D13, L14, O12, Z13

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1 Introduction

Social norms are governed by perceptions of others. Those perceptions are heavily determined by those around us.[1] However, our friends are not a random selection from the population: *even on average* we are biased in the samples with whom we interact. This can systematically affect our actions and help explain a variety of distortions in peoples' beliefs and actions, ranging from consumption of cigarettes, alcohol, and drugs by adolescents to our propensity to donate to charities.

The distortion stems from the "friendship paradox" that was pointed out by the sociologist Scott Feld in 1991. Feld observed that peoples' friends have more friends than people do, on average. That is, the average number of friends that a typical person's friend has is higher than the average number of friends that people have in the population. This follows from the fact that a person with many friends is observed by more people than someone who has few friends, and so peoples' samples of friends end up weighting people not by their proportions in the population but instead in proportion to their popularity.

The extent of the friendship paradox varies by setting, but is present in *every* network in which there is at least one friendship involving people with different degrees (as proven in Lemma 1 below). For example, in a rural Indian village discussed below, friends have on average more than 40 percent more friends than the average villager. The friendship paradox is greatly magnified by social media: a study of Twitter behavior by Hodas, Kooti, and Lerman (2013) found that more than 98 percent of users had fewer followers than the people whom they followed: typically a user's "friends" had ten times as many followers, or more, than the user. Given the increased use of social media, especially by adolescents, the potential for biased perceptions in favor of a tiny proportion of the most popular users becomes overwhelming. Given that students' decisions of whether to engage in potentially risky behaviors, as well as how much they should study, etc., tend to be influenced by what they perceive others to be doing, students risk biasing their behaviors towards that of students who have the most connections. This is not just true of students, but anyone involved in choosing behaviors that are influenced by their perceptions of others' behaviors.

The impact of the friendship paradox is evident in a series of studies finding that students tend to over-estimate the frequency with which their peers smoke and consume alcohol and drugs, and often by substantial margins. For instance, a study covering one hundred U.S. college campuses by Perkins, Meilman, Leichliter, Cashin, and Presley (1999) found that students systematically over-estimate consumption of eleven different substances, including cigarettes, alcohol, marijuana, and a variety of other drugs. A further study by Perkins focusing on alcohol consumption compared students self-reported drinking behavior - how many drinks they had the last time they partied or socialized - to their perceptions of how many the typical student at their school the last time she or he partied or socialized. The median student (out of the more than 72000 students on the 130 colleges in the study

[1] For a broader discussion of norms and perceptions, see Han and Hirshleifer (2016).

conducted from 2000 to 2003) answered 4 drinks and a quarter of the students answered 5 drinks or more. However, even given how high these overall average numbers are, more than 70 percent of the students over-estimated the alcohol consumption of the typical student at their own school (see Table 2 in Perkins and Haines (2005)). To explain such misperceptions, we don't have to dig deeply into the psychology of the students. When students are attending parties or social events, they are interacting disproportionately with the people who attend the most parties - so students' perceptions of alcohol consumption ends up over-representing the people who attend more parties and under-representing those who attend infrequently.

This bias in observation due to the friendship paradox would not have any impact, however, unless those who have more connections end up behaving systematically differently from those with fewer connections. In order for the friendship paradox to matter, it has to be that more popular students are more likely to smoke or consume alcohol in order to bias students' estimates upwards. Indeed, there is empirical evidence for this in the context of student consumption of drugs and alcohol. For instance, Valente, Unger, and Johnson (2005), in a study of middle school students, found that each additional friendship accounted for a 5 percent increase in the probability that a student smoked. Tucker, Miles, D', Zhou, Green, and Shih (2013) found similar numbers for alcohol, finding that being named as a friend by five additional others accounted for a 30 percent increase in the likelihood that a middle school student had tried alcohol.

The point of this paper is to explore the impact of the friendship paradox on behavior and explain why we should expect more connected individuals to take systematically different actions from less connected agents, and then how this feeds back to affect overall behavior. The intuition is not difficult to see, but this is still important to explore and understand given its wide-ranging implications.

As I establish here, there are two basic forces at work. One is that people who have the most connections are most exposed to interactions with others, and thus given the interactions with others in any setting of strategic complements (or substitutes), their behaviors are most heavily influenced and most extreme. The second is that if people differ in their tastes for a given activity, then it is the people who benefit most from that activity who choose to have the most connections. So, if we endogenize the network, individuals with the highest marginal payoff from a given activity choose to have the highest degree. This further amplifies the effect, increasing the disparity of actions between high-degree and low-degree individuals. Combined, these forces lead people's most popular friends to exhibit more extreme behaviors, and via feedback through the complementarities to bias the overall behavior in the society. For instance, returning to the example above, since consuming alcohol by teenagers is in part (or largely) a social activity, the people who spend more time socializing with others would have more reason to consume alcohol at an early age, and would also tend be those who have a greater base proclivity to consume alcohol at an early age. So, students who are more often seen as friends by others being more likely to consume alcohol leads to biased samples and biased perceptions, consistent with the data, and feeds back to produce high levels of activity overall.

Before describing the formal model, let me begin with some background on the friendship paradox and a simple illustration of how it can bias behaviors in the context of an example.

1.1 The Friendship Paradox

Let us begin with a quick look at the data set from Coleman (1961) that was originally cited by Feld (1991). A portion of Coleman's data is pictured in Figure 1.[2] There are fourteen girls pictured. For nine of the girls, their friends have on average more friends than they do. Two girls have the same number of friends as their friends do on average, while only three of the girls are more popular than their friends on average. On average the girls have 2.6 friends, while on average their friends have 3.2 friends.

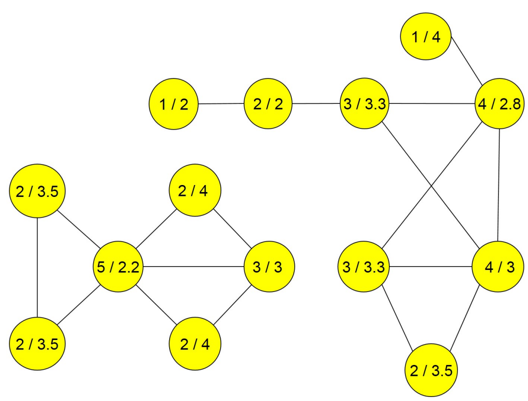


Figure 1: Data from James Coleman's (1961) study of high school friendships. Nodes are girls and links are mutual friendships. The first number listed for each girl is how many friends the girl has and the second number is the average number of friends that the girl's friends have. For instance, the girl in the lower left-hand corner has 2 friends, and those friends have 2 and 5 friends, for an average of 3.5. 9 out of 14 of the girls are less popular than their friends, 3 are more popular than their friends, and 2 are equal in popularity to their friends. The average number of friends that the girls have is 2.6, while the average number of friends that their friends have is 3.2.

To see the friendship paradox in more detail and in a larger network, consider a network of connections between households from a rural Indian village Figure 2. The full distribution of degrees and the distribution of degrees of neighbors is given in Figure 3, and we see that friends' degrees are more than forty percent larger than the average degree in the society.

The friendship paradox is easy to understand. The most popular people appear on many other peoples' friendship lists, while people with very few friends appear on relatively few peoples' lists. The following lemma provides a general statement of the friendship paradox,

[2] These are just two components of the network. The larger network not pictured here exhibits the same phenomenon: 146 girls have friends (defined mutually), and f those, 80 have fewer friends than their friends on average, while 25 have the same number as their friends, and 41 have more friends than their friends.

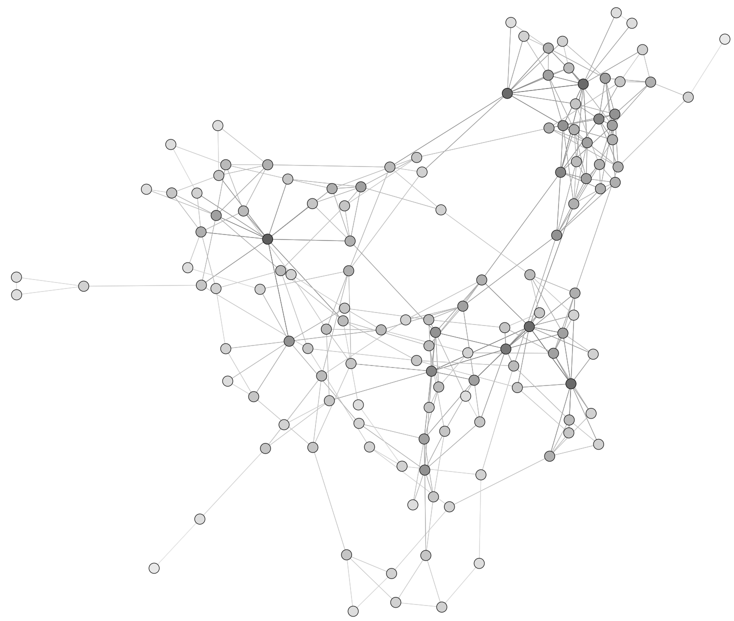
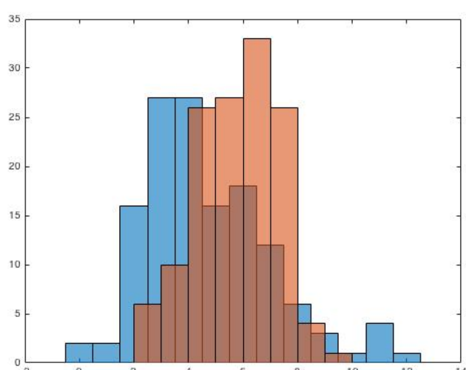
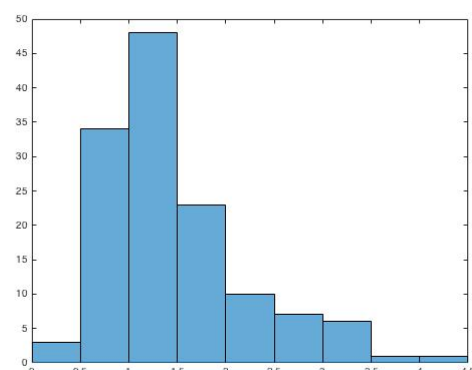


Figure 2: The Friendship Paradox illustrated in a network of 135 households from a typical rural Indian Village in the study of Banerjee, Chandrasekhar, Duflo, and Jackson (2013). Nodes are households and links are other households with whom the household exchanges favors (borrowing/lending kerosene and rice). Darker nodes have higher degree.



(a) Histogram: Blue = Own Degree, Red = Avg. Neighbors' Degree.



(b) Histogram: Ratio of Average Neighbors' Degree over Own Degree.

Figure 3: Comparison of own to neighbors' degrees in the network from Figure 2. The ratio of the average of neighbors' degrees compared to the average degree is 1.43.

showing that it holds in all networks. One can find a variation of this lemma in Just, Callender, and LaMar (2015).

A finite set $N = \{1, \dots, n\}$ of agents, with generic indices i, j, are members of an undirected[3] network $g \in \{0,1\}^{n \times n}$, so that g is symmetric and has 0's on the diagonals. Agent $i \in N$ has $d_i(g) = \sum_j g_{ij}$ links in the network.

LEMMA **1** *[The Friendship Paradox - General Networks]*

For any network, the average degree of neighbors at least as high as the average degree and the inequality is strict if and only if at least two linked agents have different degrees. That is, $\frac{1}{n}\sum_{i:d_i(g)>0} \frac{\sum_{j:g_{ij}>0} d_j(g)}{d_i(g)} \geq \frac{\sum_i d_i(g)}{n}$*, with strict inequality if (and only if) at least two linked agents have different degrees.*

The proof is straightforward and for completeness appears in the appendix. A stronger characterization of the magnitude of the friendship paradox appears in Lemma 2 below, as it can be derived once we give more structure to the set of networks considered. Other variations on the friendship paradox and bounds on its magnitude in specific models can be found in Jackson (2008, Section 4.2.1), Lattanzi and Singer (2015), Cao and Ross (2016).

This paradox, although easily understood, has wide-ranging implications, as we shall see.

1.2 An Example of the Impact of the Friendship Paradox

To see the implications of the friendship paradox most starkly, let us consider a simple example.

A society of agents are influenced by their friends.[4] The agents choose one of two actions, either solid and plaid. They each have a slight preference for solid or plaid and in the first period they follow those preferences. However, agents are conformists and prefer to mach the majority of others, and only follow their own preference if there were equal numbers of others in each style. They start with the choices in the upper left-hand figure, with only four people preferring solid and eight preferring plaid. If they could all see the whole group and best replied to that, then they would all choose plaid in the next period. However, instead agents actually see and react to their neighbors in the network.

We start with the four most popular agents preferring solid, as pictured in Figure 4. The remaining figures show what happens each following period under a best-reply dynamic, in each period agents best respond to the choices that they see among their neighbors in the previous period. So, Figure 5a has the best responses to Figure 4. The popular agents all see each other and some others, but a majority of whom they see are solid and so they stay with solid. Some other agents react to the popular agents and switch to solid. Iterating on this in Figures 5b to 5b, Solid cascades and becomes the unanimous choice.

[3]The paradox extends to directed networks when one considers the average in-degree of friends.

[4] To see similar examples illustrating biased estimation of opinions, see Lerman, Yan, , and Wu (2015). On can also find examples in popular blogs (e.g., see Kevin Schaul's *Washington Post* blog from Oct. 9, 2015 "A quick puzzle to tell whether you know what people are thinking").

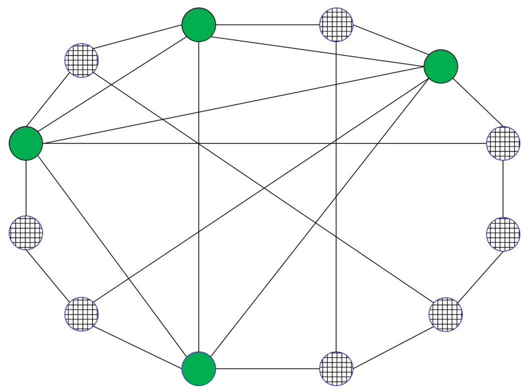
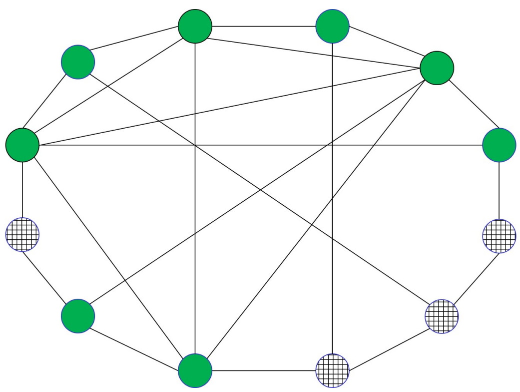
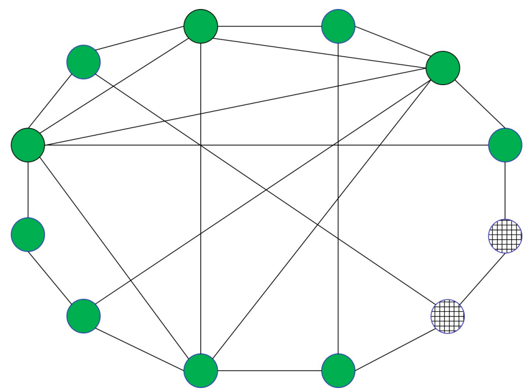


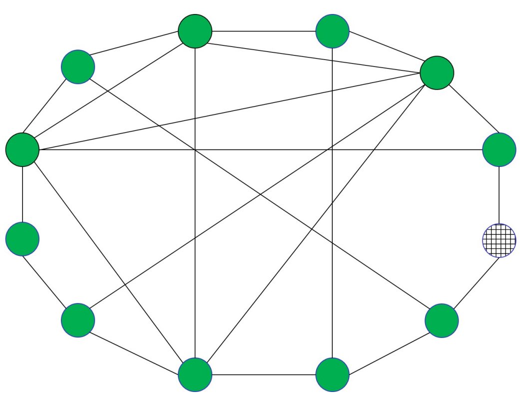
Figure 4: The four most connected agents have a base preference for solid and the eight others prefer plaid.



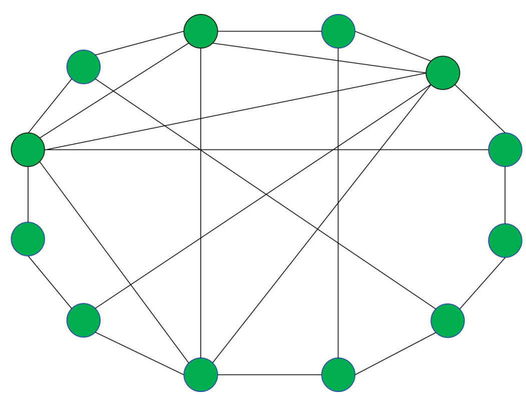
(a) Day 2, Four plaids switch to match the popular agents.



(b) Day 3, More switch.



(c) Day 4, The changes to solids continue.



(d) Day 5, All agents conform to solids.

Figure 5: Best reply dynamics: agents wish to match the behaviors of their neighbors and use their own preference to break ties. The most popular are all friends with each other and all stay with solid. Popular students are over-represented in other agents' neighborhoods and perceptions, and lead a cascade to solid.

We see the role of the friendship paradox's role by examining Figure 6, which shows agents' perceptions of the fraction preferring plaid based on their observations of their friends. Three quarters of them see at least half preferring Solid even though the actual population fraction is only one third.

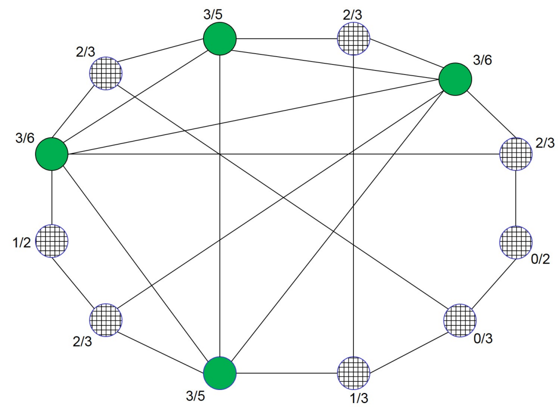


Figure 6: The Friendship paradox at work. The fractions next to the agents are their perceptions of the preferences are for solids over plaids, based on what they see among their friends in the first period. Most of them perceive a majority preference for solid, with only the few agents in the lower right perceiving a majority for plaids.

The effect in this example is consistent with a set of experiments by Kearns, Judd, Tan, and Wortman (2009). They set up a laboratory version of a committee or political party having to agree on a candidate, either red or blue. Groups of 36 subjects had to coordinate unanimously on a candidate in order to get paid. Like our solid-plaid example above, the subjects were connected in a network. They were at computer screens and could each toggle back and forth between red or blue at any instant. They could also see which color their friends in the network were supporting at any time; but they could not see the choices of any other subjects beyond their friends. Their objective was to reach a consensus within 60 seconds. If all 36 subjects ever managed to reach the same color at some instant, then the experiment ended with that being the consensus. If the subjects came to a consensus, unanimously supporting the same candidate, then they won a monetary payment. If they did not reach a consensus then they did not receive the payment. Again, similar to our solid-plaid example, the subjects had preferences over the candidates. Some subjects received a higher monetary payment if the red candidate was the consensus and others got paid more if the blue candidate was the consensus. For instance, in some treatments, a red-supporting subject got paid fifty cents if the group unanimously supported the blue candidate and a dollar fifty if the group unanimously supported the red candidate, and nothing if the group failed to reach unanimity. Thus, subjects preferred to have a consensus on their preferred candidate, but would rather reach a consensus on the other candidate than to fail to reach a consensus.

There were twenty seven runs of the experiment in which the network was set up in a manner similar to our solid-plaid example: only a small minority of the subjects were supporters of one color and vast majority of subjects were supporters of the other color.[5] The key was that the small minority of subjects preferring red were the 'popular' nodes - having many more friends in the network.

A consensus was reached in 24 out of the 27 iterations of the experiment. More importantly, *every* one of the successful groups reached a consensus that was the minority group of 'popular' subjects' preferred candidate, even though the majority preferred the other candidate. So, consistently, even when only six subjects preferred one color, and thirty preferred the other, the group still settled on the preferred color of the small group of the most popular subjects. Kearns et al. (2009) also ran some other variations of the experiment in which the networks were instead structured to be more evenly balanced with more equal numbers of supporters of red and blue, and in which the subjects had similar numbers of connections - so without a set of popular subjects. In those versions of the experiment the coordination was significantly less likely to occur: only 11 out of 27 groups managed to coordinate when the number of connections was fairly evenly balanced among subjects.

1.3 The Contribution of the Current Paper

The experiments and example described above show how the friendship paradox *can* matter. However, the example and experiment have starting behavior that is correlated with degree. What is missing is an understanding of why higher degree individuals' actions should exhibit any systematic pattern that differs from others, and how this feeds back to the society. If we had begun with higher degree individuals split evenly between solid and plaid in our example, or blue and red in the experiments, then there would have been no predictable bias in the outcome.

The contribution of this paper is to show why the friendship paradox matters by embedding it in settings in which agents' behaviors are influenced by their friends and the overall level of activity of their friends. This builds on a previous literature that has established comparative statics in games of strategic complements, including Jackson and Yariv (2005, 2007); Ballester, Calvó-Armengol, and Zenou (2006); Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010); Bergemann and Morris (2013); Bramoullé and Kranton (2014). Generally, in games with strategic complements or substitutes, higher degree individuals are exposed to more activity and are more affected. This leads them to take systematically more extreme actions, which then feeds back to increase overall activity in the network. Also, agents who benefit most from the activity choose to have more interactions, further increasing their own behavior and that of others. As we show below, these two effects lead to systematic and predictable overall distortions in the equilibria of such games played on

[5]The precise mix varied across iterations of the experiments: for instance with 6 subjects preferring red and 30 preferring blue; or 9 preferring red and 27 preferring blue, or 14 preferring red and 22 preferring blue. Which color was the minority was randomized across experiments so that some bias in peoples' intrinsic preferences over colors did not bias the results.

networks. This predictable pattern allows me to document the welfare implications of the friendship paradox.

The results that I present are as follows. The main results demonstrate the two forces described above. I start with the setting of a linear-quadratic game of strategic complements in which a closed-form solution for behavior is obtainable. There I show how the fact that agents' actions are ordered by their degree biases overall activity upwards in a game on a network compared to a benchmark society with uniform-at-random matching. Next, I endogenize the network, showing that people with greater preference for an activity choose to have higher degree, and that this leads to a further amplification of the overall activity in the society. After studying the linear-quadratic setting, I use results on monotone comparative statics to show that these two effects extend to a general class of supermodular games played on networks. I also provide results on comparative statics and welfare orderings, showing how the friendship paradox improves overall welfare in settings with positive externalities and is harmful in settings with negative-enough externalities. Finally, I examine how the results change when moving to a setting of strategic substitutes. There, higher degree agents (on a fixed network) choose lower actions when they are exposed to a higher total action by their friends. This leads agents in a network to perceive lower levels of behavior by their neighbors, compared to a random matching, as the highest degree individuals take the lowest actions. Given the substitute condition, agents respond to lower perceived actions by their neighbors by increasing their own actions. Thus, when accounting for the behavior as a function of degree and its feedback, we find that activity on a network ends up being higher in the network setting than in a benchmark with uniform-at-random matching. However, once we endogenize the network, the result in the case of strategic substitutes becomes ambiguous, as then agents with greater preference for the activity choose higher degree, but this offsets the proclivity of higher degree agents to free-ride more, leading to competing effects. Thus, while the ordering in the context of endogenous networks with strategic complements is clear and distorts behavior of all agents upwards, in the case of strategic substitutes the overall effect is ambiguous.

2 A Model and the Friendship Paradox

2.1 Agents and Random Networks

A finite set $N = \{1, \dots, n\}$ of agents, with generic indices i, j, are members of a random network. We examine interactions at an interim stage, when each agent knows his or her degree but not the full structure of the network. (For more on this perspective, see Jackson and Yariv (2005, 2007); Manshadi and Johari (2009); Galeotti et al. (2010).) In particular, agents do not know how many friends each of their friends has (or will have).[6]

[6] Indeed, there is evidence that people know little about many friends their friends have (e.g., see Friedkin (1983); Krackhardt (1987, 2014)).

Agent $i \in N$ has $d_i \in \{1, 2, \ldots\}$ links in the network.[7]

Given some i, let $P_i(d)$ be the degree distribution of the population of i's potential neighbors under a random network formation model, not conditioning on the fact that the person ends up connected to i; and suppose that this marginal distribution is the same for each of i's neighbors. Let $\mathrm{E}_i[\cdot]$ denote the expectation and $\mathrm{Var}_i[\cdot]$ be the variance associated with P_i. This allows for general degree distributions, including scale-free distributions, Poisson distributions, and hybrids. The only condition on the network formation process presumed in what follows is that any idiosyncracies (e.g., homophily, assortativity, etc.) in the distribution are already accounted for in subscripting by i, which can thus condition on i's characteristics and degree, and then the relative chance that one of i's links goes to a neighbor with degree d will be in relative proportion to that degree. The then implies that the probability that some given link of i connects to an agent who has degree d is given by

$$\widetilde{P}_i(d) = \frac{d}{\mathrm{E}_i[d]} P_i(d). \tag{1}$$

Let me emphasize the perspective here. A network has formed, or will form, and we examine a particular node i who knows its degree d_i and the distribution from which the degrees of its neighbors are drawn but not their actual degrees. The degrees of the potential neighbors are described by P_i. If we look at any one of i's links and ask what the distribution of degrees of the neighbor on that link is, then it is described by $\widetilde{P}_i(d)$.[8] This follows directly since people with higher degrees must be friends more frequently – in proportion to their degree. For instance, if half of the population has degree 2 and half has degree 1, then two thirds of the friends in the network must be of degree 2 as they are twice as likely to be linked to as the degree 1 people.

2.2 The Friendship Paradox

Let $\widetilde{\mathrm{E}}_i$ denote expectations with respect to $\widetilde{P}_i$. From (1) it follows that the expected degree of i's neighbors (the expectation of d under $\widetilde{P}_i(d)$) is

$$\widetilde{\mathrm{E}}_i[d] = \sum_d d\widetilde{P}_i(d) = \sum_d d\frac{d}{\mathrm{E}_i[d]} P_i(d) = \frac{\mathrm{E}_i[d^2]}{\mathrm{E}_i[d]} = \mathrm{E}_i[d] + \frac{\mathrm{Var}_i[d]}{\mathrm{E}_i[d]}. \tag{2}$$

This leads to the following lemma, which is a more explicit statement of the friendship paradox in the context of a random network model.

Lemma **2** *[The Friendship Paradox]*
The expected degree of a neighbor of any agent i is $\widetilde{\mathrm{E}}_i[d] = \mathrm{E}_i[d] + \frac{\mathrm{Var}_i[d]}{\mathrm{E}_i[d]}$.

[7] Agents who are isolated play no role in what follows, and so I focus on the population of agents who have at least one connection in the network, and so d_i is always positive.

[8] For more discussion of this, see Newman, Strogatz, and Watts (2001) and Section 4.2 of Jackson (2008).

In an extreme case, in which all nodes are perfectly positively assortatively matched, then $\mathrm{Var}_i[d] = 0$ and the expected degree of a node's neighbor is simply the same as its degree. However, generally there is some variation in degree across neighbors and so the paradox implies that the average degree of the population of potential neighbors will be strictly less than the average realized neighbors' degree.

In particular, if the expectations, E_i are similar across agents and we can drop the subscript, and the variance is positive (the network has some possibility of not being regular), then we get an immediate corollary that

$$\mathrm{E}[d] < \widetilde{\mathrm{E}}[d] = E[d] + \frac{\mathrm{Var}[d]}{\mathrm{E}[d]}.$$

So, the expected degree of a neighbor is the population average plus a factor which is the ratio of the variance of the distribution over the average. Moreover, all agents whose degrees are no higher than average, or in fact are even slightly above average, have strictly lower degrees than the expected degrees of their neighbors. It is only agents whose degrees are substantially above the average (by at least $\frac{\mathrm{Var}[d]}{\mathrm{E}[d]}$) who have degrees as high as their neighbors' expected degrees.

3 A Linear-Quadratic Game

Let us now analyze the impact of the friendship paradox in the context of a setting with strategic complementarities.

Before turning to the general case, let us first explore how the friendship paradox plays out in a linear quadratic game. This is a variation on the games studied by Ballester, Calvó-Armengol, and Zenou (2006); Bergemann and Morris (2013); Bramoullé and Kranton (2014); Belhaj, Bramoulle, and Deroïan (2014); de Marti and Zenou (2015).[9] The advantage of the linear-quadratic formulation is that it admits a closed-form solution and cleanly illustrates the intuition behind the general results.

Agent i has a type $\theta_i \in \Theta$, where Θ is a compact subset of $I\!R_+$. Types have a support that includes positive values, and may be correlated with degrees (as we will derive in the endogenous network case below). So, extend P_i and $\widetilde{P}_i$ denote the probability distributions that i perceives jointly over the types and degrees of her potential neighbors (unconditionally and conditional upon being linked, respectively); and so when using P_i and $\widetilde{P}_i$ in the previous section we were considering its marginal just on degrees.

Agent i chooses an action $x_i \in I\!R_+$ and gets utility described by

$$\theta_i x_i + a x_i \sum_{j \in N_i} x_j - \frac{c x_i^2}{2} + \phi \sum_{j \neq i} x_j$$

where N_i are i's neighbors in the network. The scalar $a > 0$ captures the level of complementarity of an agent's action with his or her friends' actions, $c > 0$ scales the cost of

[9] For an overview, see Jackson and Zenou (2014).

taking the action, and $\phi \in I\!R$ is a parameter that captures the extent of global externalities - either positive or negative. For instance, if x_i is a level of criminal activity then ϕ would be negative, while if x_i is a level of knowledge acquisition or human capital investment then ϕ would be positive.

Agents choose their actions simultaneously knowing their own types and degrees and expecting over the degrees of their neighbors, and so maximize

$$\theta_i x_i + a x_i \mathrm{E}_i\left[\sum_{j\in N_i} x_j\right] - \frac{c x_i^2}{2} + \phi \mathrm{E}_i\left[\sum_{j\neq i} x_j\right]. \tag{3}$$

In cases in which the random network is such that each of i's neighbors comes from the same distribution, then we can write this as

$$\theta_i x_i + a x_i d_i \widetilde{\mathrm{E}}_i\left[x_j\right] - \frac{c x_i^2}{2} + \phi \mathrm{E}_i\left[\sum_{j\neq i} x_j\right]. \tag{4}$$

Note that the first expectation is over i's neighbors, so conditions on being linked to them and hence the $\widetilde{\mathrm{E}}_i$ reflecting the friendship paradox, while the second expectation is over the whole population as it is a global externality and so is simply E_i.

I consider the Bayesian equilibria of this game.

3.1 Equilibrium

To solve for an equilibrium in closed form, consider the case in which all agents face the same degree distribution over their neighbors' types and degrees; so P_i is the same for all i. Let $\widetilde{\mathrm{E}}\left[\cdot\right]$ denote $\widetilde{\mathrm{E}}_i\left[\cdot_j\right]$, for a generic i. Agents may still differ with regards to their own realized type and degree, but their expectations over the rest of the population are similar.

Lemma **3** *[Equilibrium Characterization]*

If $c > a\widetilde{\mathrm{E}}\left[d\right]$, then there is a unique Bayesian equilibrium to the game. It is symmetric and the associated equilibrium actions are:

$$x^{friend}(\theta_i, d_i) \;=\; \frac{\theta_i}{c} + \frac{a d_i \widetilde{\mathrm{E}}\left[\theta\right]}{c\left(c - a\widetilde{\mathrm{E}}\left[d\right]\right)}. \tag{5}$$

In what follows, let us maintain the assumption that $c > a\widetilde{\mathrm{E}}\left[d\right]$.[10]

The following comparative statics on the equilibrium as we change c, a, P are straightforward variations on results in the literature (e.g., Ballester et al. (2006); Galeotti et al. (2010)). These comparative statics offer helpful insight in the proofs of the main results that follow, as it shows how varying the distribution of degrees affects the equilibrium, which is one way of thinking about what happens due to the friendship paradox which changes degrees relative to population averages. Let $x^{friend}_{a,c,P}, U^{friend}_{a,c,P}$ denote the dependence of the equilibrium actions and utilities on the parameters of the setting.

[10]If costs are too low, then there is no equilibrium as the feedback drives best responses to be infinite.

Lemma 4 *[Comparative Statics]*

Compare two settings (a, c, P, ϕ) and (a', c', P', ϕ).[11] *An increase in local complementarities, a decrease in the cost of action, a first order stochastic dominance increase in the distribution of neighbors' degrees, or a mean-preserving spread in the degree distribution, all increase equilibrium actions of every type and the equilibrium utility of every type of agent. That is, if $a \geq a'$, $c \leq c'$ and either $\widetilde{P} \geq_{FOSD} \widetilde{P'}$ or P is a mean-preserving spread of P',*[12] *with at least one of the inequalities being strict, then $x^{friend}_{a,c,P}(\theta_i, d_i) > x^{friend}_{a',c',P'}(\theta_i, d_i)$ for all i and for every θ_i, d_i. Correspondingly, if ϕ is not too negative, then $U^{friend}_{a,c,P}(\theta_i, d_i) > U^{friend}_{a',c',P'}(\theta_i, d_i)$, with the reverse inequality if ϕ is negative enough.*

The comparative statics are intuitive. Increasing the interaction factor, decreasing the cost of actions, and increasing the spread of degrees in the society, all increase the levels of activity by agents and the feedback effects, as well as the amplification due to the friendship paradox.

3.2 The Benchmark of a Playing with the Population

To understand the impact of the friendship paradox on behavior let us compare the equilibrium behavior in a network to the equilibrium behavior in a benchmark in which, instead of being in a network, agents are just randomly matched with the population for each of their interactions.[13] Denote the equilibrium in this case by x^{bench}. This is the same as the equilibrium in (5) except that expectations are taken over the whole population with even weighting, E, rather than conditional expected degrees of neighbors by the fact that they are in a network ($\widetilde{\mathrm{E}}$). That is, instead of maximizing (4) agents maximize

$$\theta_i x_i + a x_i d_i \mathrm{E}_i\left[x_j\right] - \frac{c x_i^2}{2} + \phi \mathrm{E}_i\left[\sum_{j \neq i} x_j\right]. \tag{6}$$

and so the solution is

$$x^{bench}(\theta_i, d_i) = \frac{\theta_i}{c} + \frac{a d_i \mathrm{E}\left[\theta\right]}{c(c - a\mathrm{E}[d])}.$$

This benchmark still allows people to have different numbers of interactions, but those interactions are with other individuals chosen uniformly at random from the population (independently of his or her degree), rather than within a network. So, this is a situation in which people care about the average level of behavior in the population but still can differ in how many interactions they have.

[11] Changes in ϕ do not impact actions, only welfare.

[12] $\widetilde{P} \geq_{FOSD} \widetilde{P'}$ indicates first-order stochastic dominance. Note that this condition applies to the distribution of neighbors' degrees, and it would *not* follow from stochastic dominance of P over P' (see footnote 19 in Galeotti et al. (2010)). In contrast, the mean-preserving spread is directly on the underlying degree distributions.

[13] This benchmark would also hold in the case of a directed network in which a neighbor's degree and type is completely uncoupled from the fact that the agent is a neighbor.

3.3 The Friendship Paradox and Increased Social Norms of Behavior

First, let us consider cases in which θ_j and d_j are uncorrelated so that the expectation of a neighbor's θ_j is simply a random draw from the populations' distribution of θ's: $\widetilde{\mathrm{E}}[\theta] = \mathrm{E}[\theta]$. This allows us to separate the two effects of the degree of the agent and their preference types. In Section 3.7 we endogenize the networks in which case these become correlated.

It follows from (5) that the equilibrium behavior of agents in a network is given by

$$x^{friend}(\theta_i, d_i) = \frac{\theta_i}{c} + \frac{ad_i\mathrm{E}\left[\theta\right]}{c(c - a\widetilde{\mathrm{E}}[d])}.$$

Thus, the '*friend*' introduces the 'friendship paradox' to the equilibrium relative to the benchmark case, reflected in the $\widetilde{\mathrm{E}}$ in the denominator. Given that $\widetilde{\mathrm{E}}[d] > \mathrm{E}[d]$, the actions in of every type of agent are strictly higher in the case with network matching than in the benchmark, as summarized in the following proposition.

Proposition **1** *[The Impact of the Friendship Paradox on Behavior]*

Consider a random network model, (a, c, ϕ, P), that has a degree distribution that has a positive variance and for which $\widetilde{\mathrm{E}}[\theta_j] = \mathrm{E}[\theta_j]$.

Then, $x^{friend}(\theta_i, d_i) > x^{bench}(\theta_i, d_i)$ for all θ_i and d_i. Thus, $\widetilde{\mathrm{E}}[x^{friend}] > \mathrm{E}[x^{friend}] > \mathrm{E}[x^{bench}]$.

Proposition 1 states that equilibrium actions of all types of agents are strictly higher when they are interacting in a network and exposed to the friendship paradox, as compared to being randomly matched to the population without weighting by degree. It also states that expected neighbors' actions are even higher than the population average under the network equilibrium. This last observation is really what drives the result: neighbors in a network have higher expected degree than the population average and so are expected to take higher actions given the complementarities. Higher neighbors' actions feed back via the complementarities and raise the overall equilibrium behaviors in the network compared to the population-matching benchmark.

To get an impression of how the magnitude of the network effect varies with parameters, I have plotted the ratio of x^{friend}/x^{bench} evaluated at the average degree and type in Figure 7. The x-axis varies the background parameters, which can all be collapsed to one parameter $c/(a\mathrm{E}[d])$ capturing the relative cost of action compared to the social interaction factor and average degree. Three curves are plotted for different values of how different the average degrees of neighbors are compared to the average degree. Utility comparisons are even more extreme as actions enter those quadratically - so these curves would be squared.

3.4 Welfare Implications

The ranking of equilibrium actions has strong welfare implications: we can Pareto rank the two different scenarios, depending on the nature of the global externalities.

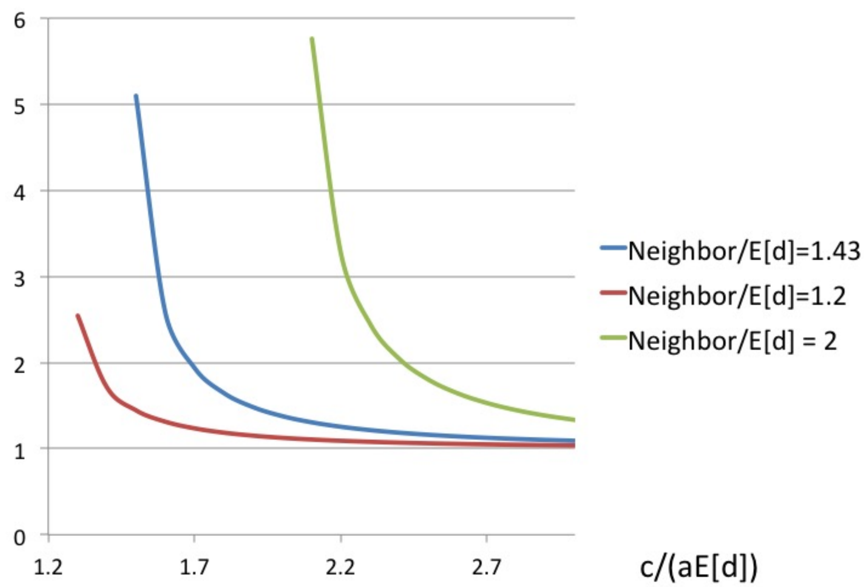


Figure 7: The ratio of x^{friend}/x^{bench} evaluated at the average degree and type. The x-axis is the value of $c/a\mathrm{E}[d]$ and the three different curves correspond to three different values of $\widetilde{\mathrm{E}}[d]/\mathrm{E}[d]$, with the 1.43 version from the Indian village data, and the other two for higher and lower ratios.

Let $U^{friend}(\theta_i, d_i)$ denote the expected utility of an agent of type θ_i, d_i under the equilibrium associated with the random network (so (4) evaluated with respect to (5)), and $U^{bench}(\theta_i, d_i)$ be the corresponding expected utility (6) under the benchmark expectations (of population matching) and the corresponding benchmark equilibrium actions.

Proposition **2** *[Strict Pareto Rankings]*

Consider a random network model (a, c, ϕ, P), that has a degree distribution that has a positive variance and for which $\widetilde{\mathrm{E}}[\theta_j] = \mathrm{E}[\theta]$. If externalities are positive or not too negative (there exists $\overline{\phi} < 0$ such that if $\phi \geq \overline{\phi}$), then the utility of every agent is higher in the network setting than in the population-matching benchmark: $U^{friend}(\theta_i, d_i) > U^{bench}(\theta_i, d_i)$ for all θ_i and d_i. If externalities are negative enough (there exists $\underline{\phi} \leq \overline{\phi} < 0$ such that if $\phi \leq \underline{\phi}$), then the inequality is reversed: $U^{friend}(\theta_i, d_i) < U^{bench}(\theta_i, d_i)$ for all θ_i and d_i.

The intuition behind the proposition is as follows. There are two forms of externalities: local ones which are positive and come through the complementarities of the actions of the agents, and global ones which could be positive or negative. In the case where both forms of externalities are positive, then having higher actions by other agents strictly increases an agent's payoff from any given level of action, and thus from the best response too. In that case, the actions of neighbors in the network setting are strictly higher than in the benchmark and so each agent of any type gets a higher utility from any possible action that she takes, and thus also when comparing best responses. The same is true if the global externalities are not too negative, as then every agent still sees a bigger positive effect from local neighbors than negative effect overall. Once, the global externality is negative enough, agents' utilities are hurt so much by others' increased actions, that even the benefits that they see from the local externality cannot offset the loss, and in that case they prefer to have lower actions of all agents and so prefer the benchmark setting to the network setting.

In the region of medium-sized negative externalities, between $\underline{\phi}$ and $\overline{\phi}$, some types of agents prefer to be in the network equilibrium and others prefer the population-matching benchmark. In that middle range, people with higher types and degrees benefit enough from the feedback due to the friendship paradox to overcome the negative externalities and prefer the network setting, while people with lower types and degrees do not and prefer the benchmark setting.

Generally, neither setting has fully Pareto efficient actions, since agents are only maximizing their own utilities and not taking into account the externalities that their actions have on others. Nonetheless, this shows that the friendship paradox has strong welfare implications compared to what would happen without networked interactions. In cases such as investing in education or human capital (e.g., studying), which have positive externalities, the fact that people may base their choices off of popular individuals who have more incentives to invest in human capital is welfare-enhancing. In contrast, in cases such as delinquent behaviors among teens which have substantial negative externalities, the friendship paradox decreases welfare.

3.5 Increased Inequality

The friendship paradox also increases the inequality in actions and welfare among the population. This is captured in the following proposition (see also Proposition 8 in the appendix).

Proposition **3** *[Increased Inequality]*

Consider a random network model (a, c, ϕ, P)*, that has a degree distribution that has a positive variance and for which* $\widetilde{\mathrm{E}}[\theta_j] = \mathrm{E}[\theta]$*. Consider any* i *and two different degrees* $d_i > d_i'$*. Then*

$$x^{friend}(\theta_i, d_i) - x^{friend}(\theta_i, d_i') > x^{bench}(\theta_i, d_i) - x^{bench}(\theta_i, d_i')$$

and

$$U^{friend}(\theta_i, d_i) - U^{friend}(\theta_i, d_i') > U^{bench}(\theta_i, d_i) - U^{bench}(\theta_i, d_i')$$

for all θ_i*.*

The proposition states that the friendship paradox increases the inequality in actions and payoffs between more and less popular/central (higher versus lower degree) individuals. The intuition behind this proposition is that agents benefit from the interaction with their neighbors, and higher degree people enjoy greater interactive effects. Since the friendship paradox produces larger expected actions of neighbors, this increases the difference in the local externality experienced by higher versus lower degree individuals, which affects both actions and payoffs in the same direction. The difference in global externalities is the same regardless of degree, so those are inconsequential.

The same comparison follows if instead of comparing the friendship actions or utility to the benchmark, one compares settings with more social interactions (e.g., a first order stochastic dominance shift in $\widetilde{P}$ or an increase in a or lower c) to one with less social interactions. This is shown as Proposition 8 in the appendix.

This means that as a society experiences technological changes that enable greater social interaction, then for behaviors that involve complementarities there will be an increase in inequality in behavior and welfare between agents who have more interactions and those who have fewer. This provides a very different lens into increased inequality than other discussions of inequality in the literature.

3.6 Misperceptions and the Friendship Paradox

The above comparison was between a case in which people maximize their utility from interacting with their friends to settings in which they maximize their utility from interacting with the population average.

Another interesting case to consider, which may apply to many social settings, is one in which people really care about the population average, but they mistakenly react to their friends - so they believe that their friends are representative and do not understand the friendship paradox.

Thus, they best respond to their friends and play x^{friend} but their utility is really relative to the population average (6). Let us call the resulting expected utility in this case $U^{misperc}$.

Proposition 4 *[Strict Pareto Rankings with Misperceptions]*

Consider a random network model (a, c, ϕ, P), that has a degree distribution that has a positive variance and for which $\widetilde{\mathrm{E}}[\theta_j] = \mathrm{E}[\theta]$.

First,

$$U^{friend}(\theta_i, d_i) > U^{misper}(\theta_i, d_i)$$

for all θ_i and d_i, regardless of ϕ.

Next, if global externalities are positive enough (there exists $\overline{\phi} > 0$ such that if $\phi \geq \overline{\phi}$), then:

$$U^{misper}(\theta_i, d_i) > U^{bench}(\theta_i, d_i)$$

for all θ_i and d_i. If global externalities are negative enough (there exists $\underline{\phi} \leq \overline{\phi} < 0$ such that if $\phi \leq \underline{\phi}$), then:

$$U^{bench}(\theta_i, d_i) > U^{misper}(\theta_i, d_i)$$

for all θ_i and d_i.

U^{misper} is always worse than U^{friend} since the global externalities are the same and one loses some of the local interaction effect. The more interesting comparison is with the benchmark case. Here the inflation of behavior helps in cases of positive-enough global externalities and improves welfare, with the opposite conclusion in the case of negative-enough externalities. The comparison is ambiguous near 0 global externalities, since others' actions are higher leading to higher complementarity effects (which are positive), but people are not best-responding given that they are misperceiving the behaviors of others.

3.7 Endogenous Interactions and Further Amplifications of Behavior

We have seen that complementarities lead agents with higher degrees to take higher actions, which feed back to lead to further increase the actions of all agents given that higher degree individuals have a disproportionate impact on others' behaviors. This is one effect of networks and the friendship paradox on behavior.

I now outline how a second effect amplifies this first effect. People with higher tastes for the action – agents with higher θ_i's – benefit more from the interactions with others, and thus prefer to have a higher degree. Thus, when we model network formation we see a positive correlation between θ_i and d_i. For instance, people who get more enjoyment from some interactive behavior (especially in a social context, for instance adolescents with a predisposition for drugs or alcohol) will prefer to interact more. This selection effect further increases the behavior of agents with high degrees, as they benefit not only from the increased complementarity that accompanies their high degrees, but they also tend to have higher base propensities for high behavior to begin with. This amplifies the feedback and thus behaviors throughout the population.

To see how this works, I now endogenize the network and derive the relationship between θ_i and d_i rather than assuming that they are independent.

Consider a game in which agents choose $d_i \in \{0, 1, \ldots, n-1\}$ in a first stage (as a function of their θ_i's), and then choose x_i in a second stage. Also, consider the case in which the choices of d_i's are private, so that agents do not observe others' choices (given that they do not see their neighbors' degrees). The game in which those choices are public has similar results but seems less natural, and this formulation allows the use of Bayesian equilibrium rather than perfect Bayesian; but there do not appear to be any interesting strategic differences between the games.

Forming relationships is costly, and in order to obtain a closed-form solution, consider a quadratic cost function of the form $C(d_i) = \frac{kd_i^2}{2}$ for $k > 0$.

Let the distribution over types be i.i.d. across agents and be atomless with compact support. This ensures that any symmetric equilibrium is essentially in pure strategies, as at most a set of measure 0 of types ever have multi-valued best responses.[14] The extension to

[14]Given that the set of agents who ever are indifferent are of measure 0, any equilibrium that involves mixing has a counter-part in which the indifferent agents do not mix (and their decisions do not affect any other agents' best responses given their negligible measure) and the equilibrium still results in the same

distributions with atoms is straightforward since degree choices are still nondecreasing (in the sense of first order stochastic dominance) and each type mixes over at most two adjacent degrees.

Let $d^{endog}(\theta_i)$ be the (Bayesian) equilibrium degree choice of an agent of type θ_i and $x^{endog}(\theta_i, d^{endog}(\theta_i))$ be the resulting equilibrium actions in the second period, in the overall endogenous-network equilibrium. [15]

I do not require that the agents' choices of d_i be compatible with a feasible network across all agents. For instance, if $n = 3$ and $d_1 = 1$ while $d_2 = d_3 = 2$ then there is no network that can give all of the agents their preferred degrees at the same time. For large n the probability that any agent misses their preferred degree by even one link goes to 0, presuming some simple bounds on preferred degrees, in standard random network models such as the configuration model. Thus, I ignore this effect and allow agents to unilaterally decide on their degree, but the model could be extended to use the configuration model and account for agents' expectations that their degree is not exactly realized. This would prevent solving for the equilibrium in closed form, but would still lead to qualitatively similar results.

Agents maximize their expected utility, anticipating equilibrium choices of the other agents. The degree and action choices must solve

$$\max_{d_i, x_i} \; \theta_i x_i + a x_i d_i \widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right] - \frac{c x_i^2}{2} - \phi(n-1)\mathrm{E}\left[x^{endog}(\theta_j, d_j(\theta_j))\right] - \frac{k d_i^2}{2}, \quad (7)$$

where expectations are relative to the equilibrium distribution of others' choices.

Symmetric pure-strategy equilibria exist, and as there can be multiple equilibria I provide results that hold for any symmetric pure-strategy equilibrium with at least two degrees.[16,17]

PROPOSITION **5** *[Endogenous Network Amplifications]*

Let $P^{endog}(d)$ be the endogenous equilibrium degree distribution associated with a symmetric pure-strategy equilibrium and suppose that it involves at least two degrees and that $c > a(n-1)$.[18] *Then each agent chooses a degree which is a nondecreasing function of the*

distributions over degrees and actions.

[15] The notation indicating the dependence of x^{endog} on $d^{endog}(\theta_i)$ is redundant as they are both tied down in equilibrium as a function of θ_i, but this notation will also be useful in comparing actions to the benchmark.

[16] The game is not quite supermodular, as actions depend both on types and degree. For example, if one increases the degree that a low type chooses, then that increases the probability that a friend is of a low type, which can decrease expected neighbors' actions. Nonetheless, best responses are still monotone, and essentially unique and can be taken as step functions with at most n values, and so can be taken to be compact in the weak topology, and then existence is easy to establish from standard arguments.

[17] Here the equilibrium may not be unique. For instance, there always exists an equilibrium in which all agents choose $d_i = 0$ given that they expect all others do as well. But the characterization here applies to equilibria in which some agents choose a positive degree. Such equilibria exist by standard arguments when restricting degree choices to be positive, and then such equilibria remain an equilibrium without that restriction for k that are not too overwhelming.

[18] This last condition is stronger than is needed, which is that $c > a\widetilde{\mathrm{E}}[d]$, but this is a sufficient condition and independent of the endogenous degrees.

agent's type. Moreover, for all θ_i for which $d^{endog}(\theta_i) > 0$:

$$x^{endog}(\theta_i, d^{endog}(\theta_i)) > x^{friend}(\theta_i, d^{endog}(\theta_i)) > x^{bench}(\theta_i, d^{endog}(\theta_i)).$$

Proposition 5 distinguishes the two effects that we have been discussing. First, the bias of having higher degree people be neighbors more often leads them to have more influence, and their natural tendency to prefer higher actions given their higher rate of interaction leads to higher behaviors by agents of all types. Second, higher-type agents benefit more from having higher degrees leading to a positive correlation between degree and type, further increasing high-degree agents' actions and further increasing the behaviors of all agents.

Given these rankings of actions for each type and degree, it follows directly that the average rankings follow the same rule. There is also an extension of the welfare result, Proposition 2, to this case: with externalities that are positive or not too negative, all types of agents prefer the endogenous equilibrium to the friendship paradox equilibrium without correlation between types and degrees, to the benchmark case of playing with population averages; while with very negative externalities the ranking is exactly reversed.

PROPOSITION **6** *[Strict Pareto Rankings with Endogenous Networks]*

Consider a random network model (a, c, ϕ), and let $P^{endog}(d)$ be the endogenous equilibrium degree distribution and suppose that it involves at least two degrees and that $c > a(n-1)$ and $k > a(n-1)$. If externalities are positive or not too negative (there exists $\overline{\phi} < 0$ such that if $\phi \geq \overline{\phi}$), then

$$U^{endog}(\theta_i, d^{endog}(\theta_i)) > U^{friend}(\theta_i, d^{endog}(\theta_i)) > U^{bench}(\theta_i, d^{endog}(\theta_i))$$

for all θ_i. If externalities are negative enough (there exists $\underline{\phi} < \overline{\phi} < 0$ such that if $\phi \leq \underline{\phi}$), then the inequality is reversed.

Note that it is important for these comparisons to make them relative to the degree distribution P^{endog} as that allows for expectations to be compared across the settings, and otherwise it is not clear which degree distribution to use for comparison.

4 General Games with Complementarities

I now show that the results above extend to general network games with strategic complements.[19] Omitted definitions here are standard from the literature on supermodular games (e.g., see Milgrom and Shannon (1994); Van Zandt and Vives (2007)).

[19]For more background on these and related games, see Jackson and Yariv (2005, 2007); Sundararajan (2007); Jackson (2008); Manshadi and Johari (2009); Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010).

Each agent i chooses a strategy from a set X_i, which is a compact metric lattice with associated partial order $\geq_i$. For each i, let θ_i lie in a partially ordered set Θ. The utility of agent i with degree d_i of type θ_i when other agents play actions x_{-i} is given by

$$u_i(x_i; (x_j)_{j \in N_i}, \theta_i, d_i),$$

where N_i is the realized set of neighbors of i.[20]

Following Van Zandt and Vives (2007), let us say that u_i has a smooth dimension if $X_i = X_{i1} \times X_{i2}$ in which X_{i1} is a compact interval of $I\!R$ and X_{i2} is a complete lattice, u_i is continuously differentiable in x_{i1}, and $\partial u_i / \partial x_{i1}$ is strictly increasing in θ_i, d_i.

Agents choose actions as a function of their types θ_i, d_i. Let there be some given measure on types θ in the population denoted μ. Let agents view their neighbors' degrees as independent across neighbors and independent of the types. Given are distributions on the degrees of agents other than i in the population P_i, and associated $\widetilde{P}_i$ defined by (1), which could be functions of (θ_i, d_i). Let $x_i^{friend}(\theta_i, d_i)$ and $x_i^{bench}(\theta_i, d_i)$ denote Bayesian equilibrium strategies corresponding to beliefs over neighbors' types and degrees defined by $\mu \times \widetilde{P}_i$ and $\mu \times P_i$, respectively.

Proposition 7 *[Network Distortions on Behavior: General Games with Strategic Complements]*

Consider a game for which u_i is continuous, bounded, and supermodular in x_i, and satisfies increasing differences in $(x_i; (x_j)_{j \in N_i}, \theta_i, d_i)$, for each i. Let P_i have weight on at least two degrees and P_i and $\widetilde{P}_i$ be monotone functions of θ_i, d_i.[21] *Then maximal Bayesian equilibria, x_i^{friend} and x_i^{bench} exist and are nondecreasing in θ_i, d_i. Moreover,*

$$x_i^{friend}(\theta_i, d_i) \geq_i x_i^{bench}(\theta_i, d_i) \quad \textit{for all } i \textit{ and } \theta_i, d_i.$$

If for each i, u_i has a smooth dimension on which $x_{i1}^{friend}(\theta_i, d_i)$ and $x_{i1}^{bench}(\theta_i, d_i)$ are interior for all (θ_i, d_i), then

$$x_{i1}^{friend}(\theta_i, d_i) >_i x_{i1}^{bench}(\theta_i, d_i) \quad \textit{for all } i \textit{ and } \theta_i, d_i.$$

Proposition 5 extends to the general case as well, presuming that u_i satisfies increasing differences in $(x_i, d_i; (x_j)_{j \in N_i}, \theta_i)$, presuming that there are nontrivial degree distributions in equilibrium.

We also have an immediate corollary that if local externalities are positive (so that u_i is increasing in $(x_j)_{j \in N_i}$), then the expected utility of the equilibria are ordered in the same way as the actions, while if local externalities are negative (u_i is decreasing in $(x_j)_{j \in N_i}$) then the welfare ordering is the reverse of the actions. Incorporating global externalities then requires

[20] We could also allow for global externalities as a function of x_{-i}. For simplicity, I drop that notation, but the results below apply directly, just with the additional notation, since N_i is defined to be the set of other agents whose actions interact strategically with i's action.

[21] Thus, if $\theta_i', d_i' \geq \theta_i, d_i$, then $P_i' \geq P_i$ and $\widetilde{P}_i' \geq \widetilde{P}_i$ in the sense of first order stochastic dominance.

a comparison between local and global effects, which do not change the results if they move in the same direction, but may lead to ambiguous effects if they conflict in direction, and then the statement requires a large enough negative global externality to reverse the welfare ordering.

The results surrounding the endogenous network also extend, as under appropriate monotonicity conditions higher types prefer higher degrees and higher actions. However, the game of degree choices is not supermodular. If a lower preference type increases its degree, then that becomes relatively more frequent as a neighbor, and can partly crowd out higher types being in a neighborhood in expectation, and so can lower the expected actions of a neighbor. This does not overturn the logic of the analysis form before - as degree will still be a non-decreasing function of type and so are actions. It just means that the techniques from supermodular games cannot be used in the proofs, and one needs to argue directly based on the monotonicity of strategy choices (and existence comes from continuity and compactness). Such direct arguments are used in the proof of Proposition 5, as even the linear-quadratic setting is not supermodular when endogenizing degree. So one can simply mimic the logic of that proof to extend the endogenous degree choice to more general utility formulations.

5 Public Goods and Strategic Substitutes

The results above concern games of strategic complements. That is a case of fundamental interest since many interactions fall into that category. Games with strategic substitutes also apply to many settings, such as those in which agents share tasks or contribute to local public goods. Let me briefly discuss how the results change in the case of substitutes.

With strategic substitutes the interaction of incentives between agents is reversed compared that under strategic complements. In a game of local strategic substitutes, i's utility is again described by a function of the form

$$u_i(x_i, (x_j)_{j\in N_i}, \theta_i, d_i),$$

in which we maintain the same assumptions as in the case of complementarities before, *except* that we reverse the direction of how $(x_j)_{j\in N_i}$ and d_i affect changes in utility with regards to changes in x_i. In particular, in this case u_i satisfies increasing differences in $(x_i; (-x_j)_{j\in N_i}, \theta_i, -d_i)$.[22] So, agents prefer to take lower actions if they have more neighbors and/or those neighbors take higher actions. It is still possible that agents have utility that increases in $(x_j)_{j\in N_i}$ and d_i, but the incentives to choose a higher x_i decreases as an agent sees more activity by others in their neighborhood. This applies to standard local public goods games.

In such a setting, using similar arguments to those behind the results above, with a sign reversal, it follows that $x^*(\theta_i, d_i)$ is nondecreasing in θ_i and *nonincreasing* in d_i. This then

[22]The sign on θ_i is not reversed, as this still captures an agent's personal predisposition for the behavior.

also provides implications for the friendship paradox. When matched with higher degree neighbors (presuming independence between θs and ds), one expects less activity from those neighbors than when matched with lower degree neighbors. Thus, the network setting leads agents to expect less action by their neighbors than in the benchmark population matching setting, and so this ultimately leads agents to *increase* their own actions in response.

To see how this works in more detail, let us consider a canonical example. The example is that of a best-shot public goods game (e.g., see Galeotti et al. (2010)).

In this setting, each agent chooses an action $x_i \in \{0, 1\}$ - whether to provide a local public good. Providing the good costs $c > 0$. The agent's payoff is then the max of the actions in his or her neighborhood, including her own action. In particular, the payoff is

$$\theta_i I_{\left[x_i+\sum_{j\in N_i} x_j>0\right]} - cx_i,$$

where I is the indicator function. This applies to settings in which if one agent invests in the public good then all of his friends can share in the value of the good. Examples include completing a task, or buying a book that can be lent to friends, or acquiring information that can be shared with the friends (but for simplicity does not transfer multiple hops). Each agent would prefer that a neighbor provide the good, but would rather provide the good if no neighbor does.

Let $\widetilde{\pi}_i$ denote the probability agent i perceives that any given one of her neighbors will provide the public good. Then agent i prefers providing the public good if

$$\theta_i - c \geq \theta_i[1 - (1 - \widetilde{\pi}_i)^{d_i}]$$

or (presuming that $\widetilde{\pi}_i < 0$)

$$\theta_i \geq \frac{c}{(1 - \widetilde{\pi}_i)^{d_i}}.$$

Thus, presuming that $\widetilde{\pi}_i < 0$, there is a threshold

$$t_i(d_i) = \frac{c}{(1 - \widetilde{\pi}_i)^{d_i}} > 0$$

for which the agent's best response is to provide the public good ($x_i = 1$) if $\theta_i > t_i(d_i)$ and not to provide the public good ($x_i = 0$) if $\theta_i < t_i(d_i)$.[23] Note that $t_i(d_i)$ is increasing in d_i.

When the distribution of neighbors' degrees and types is the same across agents, the probability that a random neighbor will provide the public good in a symmetric equilibrium, denoted by $\widetilde{\pi}$, is then

$$\widetilde{\pi} = \sum_d Pr\left[\theta > t(d)\right] \widetilde{P}(d).$$

In equilibrium this must solve

$$\widetilde{\pi} = \sum_d Pr\left[\theta > \frac{c}{(1 - \widetilde{\pi})^d}\right] \widetilde{P}(d).$$

[23]In a case in which θ_i has an atomless distribution, the indifferent case is negligible and otherwise there may be some mixing at the precise threshold of $t_i(d_i)$.

Given that the right hand side is decreasing in $\widetilde{\pi}$ and is positive when $\widetilde{\pi} = 0$ (presuming that c lies in the support of θ), this has a unique solution, associated with the unique symmetric equilibrium.

Next, note that first order stochastic dominance shifts in $\widetilde{P}(d)$ lead the right hand side to decrease for every value of $\widetilde{\pi}$ and so the equilibrium value of $\widetilde{\pi}$ must decrease.[24]

This in turn, leads to a lower value of $t(d)$, since it is increasing in $\widetilde{\pi}$.

Thus, we see that $t^{bench}(d_i) > t^{friend}(d_i)$ for every d_i, and so the thresholds are lower under the friendship paradox. This means that there is more public goods provision by all degrees of agents under the friendship paradox, but this comes from the reaction to an overall lower expected probability that a random neighbor provides the public good under the friendship paradox.

Note, however, that the expected utility of an agent of any given degree generally tends to go down in the network setting compared to the random matching setting, since agents are matched with agents of higher degrees and expect less activity from their neighbors overall. Thus, even though the network setting incentivizes more activity by agents, this is because they are more frequently matched with high degree agents who tend to free-ride more on the action. This leads to lower expected utilities by each type of agent and overall.

Although I have analyzed the case of the best-shot public goods game, it is clear that the reasoning applies to more general games, similarly to the way that the linear-quadratic results generalized in the previous section.

In games of strategic substitutes, endogenizing the network leads to ambiguous effects on overall actions and welfare. In most such settings, people who have higher payoffs from the activity also tend to benefit from having higher degree (presuming that there is some marginal gain from neighbors' provision of the public good on top of one's own provision).[25] This leads to an overall ambiguous effect as agents' high type pushes them to take higher actions but their increased endogenous degree tends to reduce their actions - and the overall effect depends on the parametric specification. Thus, while the results from the strategic complements in terms of comparisons on a fixed network have (reversed) analogs in the case of strategic substitutes, the case in which the network is endogenized does not extend. This means that the overall impact of the friendship paradox in the case of strategic substitutes can be ambiguous and will depend on details of the preferences - whether the individual incentives to provide the good or the local externalities dominate.

[24] Higher values of d lead to higher values of $\frac{c}{(1-\widetilde{\pi})^d}$, which lead to lower values of $Pr\left[\theta_j > \frac{c}{(1-\widetilde{\pi})^d}\right]$.

[25] In the case of the best-shot public goods game, the endogenous network formation game becomes degenerate. Any agent who intends to provide the public good in that game gets no additional value from having neighbors. Thus, the only agents who would choose to pay to have connections would be those planning not to provide the public good - but then they would not want to have connections in that case. To have a nontrivial game in which anyone forms connections, agents have to be endowed with some base degree. In that case, in equilibrium, only the lowest degree agents would provide the public good. Those agents actually turn out to be the higher θ agents in this particular game.

6 Concluding Remarks

'Popular' individuals disproportionately impact the perceptions in a society. If popular individuals tended to act the same as others this would not systematically bias peoples' perceptions of norms or the norms themselves. However, as we have shown, there are two ways in which popular individuals and their behaviors differ from others. First, they have more interactions and that leads them to act more extremely for any behavior that involves strategic complementarities. Second, people who are more predisposed to like a certain behavior will also seek to have more interactions involving that behavior, amplifying the effect. As shown in this paper, these two distortions both lead to increases in perceptions of behavior and ultimately feed back to increase the overall behavior in a society.

Depending on the nature of the externalities of an activity the effect of the friendship paradox can be good or bad. For instance, these results help us to understand student' systematic over-estimation of their peers' delinquencies that involve social interaction. Thus, this helps explain why drug and alcohol problems are pervasive in high school and college environments. Interestingly, the friendship paradox can also be Pareto improving in settings with positive externalities. It is worth noting that these distortions can be even further exacerbated by social media, where distortions in the number of interactions can be even more extreme and in which what is posted or communicated is also biased towards behaviors that are social in nature.

Understanding the friendship paradox's role in the formation of social norms has policy implications.[26] It sheds light on the importance of role models and information access in improving norms. Our analysis explains why programs known as 'social norm marketing' have been successful. In such programs, one simply informs people of the true population behavior. A first instance of this was use by Northern Illinois University (see Haines and Spear (1996)). The study found no improvement due to a traditional educational intervention in which they taught students about dangers of alcohol and emphasized that it was ok to abstain; but then when they used a new program of informing students of the actual (reported) rates of binge drinking they found significantly improved perceptions of others' rates of binge drinking and reduced binge drinking overall .[27] Since then many other social norm marketing programs have been used and studied, including a study by DeJong et al. (2006) that involved 18 universities with controls, and reached similar conclusions. Social norm marketing has been tried in a variety of settings, for instance in improving perceptions of others' behavior and decreasing the incidence of drinking and driving in a controlled trial in Montana (Perkins, Linkenbach, Lewis, and Neighbors (2010)). Other variants on such

[26] This fact has not been lost on marketers and is also an important driver of identifying most-at-risk individuals, for instance using snowball sampling to identify people most at risk for HIV. Taking advantage of the visibility of friends can also help in fostering adoption of new programs (e.g., Kim et al. (2015)).

[27] In the base period 43 percent of the students reported binge drinking. In that same survey just over 69 percent of the students perceived binge drinking as the 'norm'. At the end of the study of the new program, just over 34 percent of students reported binge drinking and 51 percent perceived it as the norm.

programs that our analysis explains are ones that target the highest consuming students, providing them with information about how their behavior ranks compared to the rest of the population (e.g., see Agostinelli et al. (1995)). Our analysis explains why providing information of actual norms should improve perceptions and norms in any settings with complementarities and overall negative externalities, in which people really care about how their behavior matches with the overall population and not just their friends, but base their perceptions of the norm on their own experiences.

Note that our analysis also providesinsight regarding situations in which agents hide behaviors - where we can think of the action above to be either to avoid a behavior or hide it. If agents are worried about reputation, then agents who have more interactions might be more likely to hide that they undertake some behavior. This leads people to underestimate a behavior, and their lowered perceptions of the norm can lead to more hiding of the behavior. It would be interesting to explore the implications of such results for perceived norms and openness of behavior, for instance, of homosexuality in some societies - and policies such as 'don't ask don't tell'.

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Appendix

Proof of Lemma 1: Consider a network g. The average degree of agents in the network is

$$\frac{\sum_i d_i(g)}{n} = \frac{1}{n} \sum_{i<j:g_{ij}=1} 2.$$

The average degree of neighbors is

$$\frac{1}{n} \sum_{i:d_i(g)>0} \frac{\sum_{j:g_{ij}=1} d_j(g)}{d_i(g)} = \frac{1}{n} \sum_{i<j:g_{ij}=1} \frac{d_j(g)}{d_i(g)} + \frac{d_i(g)}{d_j(g)}.$$

Thus, it suffices to show that

$$\frac{d_j(g)}{d_i(g)} + \frac{d_i(g)}{d_j(g)} \geq 2$$

and that the inequality is strict if and only if $d_i(g) \neq d_j(g)$. Note that

$$\frac{d_j(g)}{d_i(g)} + \frac{d_i(g)}{d_j(g)} - 2 = \frac{(d_j(g) - d_i(g))^2}{d_i(g)d_j(g)}.$$

The right hand side of the above equation is nonnegative, and positive if and only if $d_j(g) \neq d_i(g)$. ∎

Proof of Lemma 3: From the first order condition of maximizing (3), it follows that the best response of i as a function of i's type and degree is

$$x_i(\theta_i, d_i) = \frac{\theta_i}{c} + \frac{a\mathrm{E}_i\left[\sum_{j\in N_i} x_j\right]}{c}.$$

and so

$$x_i(\theta_i, d_i) = \frac{\theta_i}{c} + \frac{a d_i \widetilde{\mathrm{E}}_i\left[x_j\right]}{c}.$$

Taking expectations of both sides of the above expression for $x_i(\theta_i, d_i)$ with respect to $\widetilde{\mathrm{E}}$ yields

$$\widetilde{\mathrm{E}}\left[x\right] = \frac{\widetilde{\mathrm{E}}\left[\theta\right]}{c} + \frac{a\widetilde{\mathrm{E}}\left[d\right]\widetilde{\mathrm{E}}\left[x\right]}{c}.$$

Thus,

$$\widetilde{\mathrm{E}}\left[x\right] = \frac{\widetilde{\mathrm{E}}\left[\theta\right]}{c - a\widetilde{\mathrm{E}}\left[d\right]}.$$

Substituting the above expression into the solution for $x_i(\theta_i, d_i)$ leads to the following characterization of equilibrium (when P_i's are the same for all i),

$$x^{friend}(\theta_i, d_i) = \frac{\theta_i}{c} + \frac{a d_i \widetilde{\mathrm{E}}\left[x^{friend}\right]}{c} = \frac{\theta_i}{c} + \frac{a d_i \widetilde{\mathrm{E}}\left[\theta\right]}{c\left(c - a\widetilde{\mathrm{E}}\left[d\right]\right)},$$

as claimed in the lemma. ▮

Proof of Lemma 4: Recall that

$$x^{friend}(\theta_i, d_i) = \frac{\theta_i}{c} + \frac{ad_i\mathrm{E}\left[\theta\right]}{c(c - a\widetilde{\mathrm{E}}[d])}.$$

This is increasing in a, decreasing in c (under the maintained assumptions that on $c > a\widetilde{\mathrm{E}}[d]$). Note that $\widetilde{\mathrm{E}}[d]$ is increasing as we take a first order stochastic dominance shift in $\widetilde{P}$, and also as we take a mean preserving spread of P since $\widetilde{\mathrm{E}}[d] = \frac{\mathrm{E}[d^2]}{\mathrm{E}[d]}$. The comparative statics in actions follow directly.

From (4) we know that

$$U^{friend} = \theta_i x^{friend}(\theta_i, d_i) + a x^{friend}(\theta_i, d_i) d_i \widetilde{\mathrm{E}}_i\left[x^{friend}(\theta_j, d_j)\right] - \frac{c x^{friend}(\theta_i, d_i)^2}{2} + \phi \mathrm{E}_i\left[\sum_{j\neq i} x^{friend}(\theta_j, d_j)\right].$$

So, suppose that we start at a', c', P' and the corresponding x^{friend}. Keeping i's actions fixed change to a, c, P. From the above comparative statics in actions it follows that $\widetilde{\mathrm{E}}_i\left[x^{friend}(\theta_j, d_j)\right]$ and $\mathrm{E}_i\left[\sum_{j\neq i} x^{friend}(\theta_j, d_j)\right]$ both increase with a, decrease with c, and increase as we take a first order stochastic dominance shift in $\widetilde{P}$, and also as we take a mean preserving spread of P. Thus, for a nonnegative ϕ, we end up with a strict increase in the resulting U. Now, adjust x^{friend} to be the best response to a, c, P and utility can only increase. So, overall payoffs have gone up for all types. Given that this is a strict inequality when ϕ is 0, across all types and degrees in a compact set, and utilities are continuous in ϕ, this also holds for some negative ϕ's, establishing the first welfare comparison of the lemma.

Next, note that the equilibrium actions are independent of ϕ. Given that Θ is compact and degrees are bounded by $n-1$, and utility and actions are continuous in types, there is a maximum gain in

$$\theta_i x^{friend}(\theta_i, d_i) + a x^{friend}(\theta_i, d_i) d_i \widetilde{\mathrm{E}}_i\left[x^{friend}(\theta_j, d_j)\right] - \frac{c x^{friend}(\theta_i, d_i)^2}{2}$$

due to the change from a', c', P' to a, c, P. Call this X (which we know is positive from above, as it corresponds to $\phi = 0$). There is also a change in $\mathrm{E}_i\left[\sum_{j\neq i} x^{friend}(\theta_j, d_j)\right]$ which is some $Y > 0$. Then provided $X + \phi Y < 0$, then the welfare comparison will be negative. So, setting $\underline{\phi} < -X/Y$ completes the proof. ▮

Proof of Proposition 1: Recalling from (2) that

$$\widetilde{\mathrm{E}}\left[d\right] = \sum_{d>0} d\frac{P(d)d}{\mathrm{E}[d]} = \frac{\mathrm{E}[d^2]}{\mathrm{E}[d]},$$

it follows from Lemma 3, and the fact that the expectations over types of neighbors is the same as the unconditional expectation, that the equilibrium actions are

$$x^{friend}(\theta_i, d_i) = \frac{\theta_i}{c} + \frac{ad_i\mathrm{E}\left[\theta\right]}{c(c - a\frac{\mathrm{E}[d^2]}{\mathrm{E}[d]})}. \tag{8}$$

The first part of the proposition, that $x^{friend}(\theta_i, d_i) > x^{bench}(\theta_i, d_i)$ for all θ_i and d_i, then follows from comparing our expression for $x^{friend}(\theta_i, d_i)$ above to that of $x^{bench}(\theta_i, d_i)$, and noting that the only change is in the denominator with a comparison between $\frac{\mathrm{E}[d^2]}{\mathrm{E}[d]}$ and $\mathrm{E}[d]$. The first claim then follows directly since whenever P has a positive variance, then

$$\frac{\mathrm{E}[d^2]}{\mathrm{E}[d]} = \frac{\mathrm{Var}[d] + \mathrm{E}[d]^2}{\mathrm{E}[d]} = \frac{\mathrm{Var}[d]}{\mathrm{E}[d]} + \mathrm{E}[d] > \mathrm{E}[d].$$

This also then implies that $\mathrm{E}[x^{friend}] > \mathrm{E}[x^{bench}]$, since these are ordered pointwise. The fact that $\widetilde{\mathrm{E}}[x^{friend}] > \mathrm{E}[x^{friend}]$ follows from the fact that $\widetilde{P}$ strictly first order stochastically dominates P and x^{friend} is increasing in d_i, which completes the proof. ∎

Proof of Proposition 2: This follows from proof of Lemma 4, noting that difference the x^{bench}, U^{bench} and x^{friend}, U^{friend} just corresponds to a change in the use of P versus $\widetilde{P}$, which is a strict mean preserving spread. ∎

Proof of Proposition 3: This follows from proof of Proof of Proposition 8, below, noting that difference the x^{bench}, U^{bench} and x^{friend}, U^{friend} just corresponds to a change in the use of P versus $\widetilde{P}$, which is a strict mean preserving spread. ∎

Proposition 8 *[Increased Inequality, Part II]*

Compare two settings (a, c, P, ϕ) and (a', c', P', ϕ).[28] *An increase in local complementarities, a decrease in the cost of action, a first order stochastic dominance increase in the distribution of neighbors' degrees, or a mean-preserving spread in the degree distribution, all increase equilibrium actions of every type and the equilibrium utility of every type of agent. That is, if $a \geq a'$, $c \leq c'$ and either $\widetilde{P} \geq_{FOSD} \widetilde{P'}$ or P is a mean-preserving spread of P', with at least one of the inequalities being strict, then*

$$x^{friend}_{a,c,P}(\theta_i, d_i) - x^{friend}_{a,c,P}(\theta_i, d_i') > x^{friend}_{a',c',P'}(\theta_i, d_i) - x^{friend}_{a',c',P'}(\theta_i, d_i')$$

and

$$U^{friend}_{a,c,P}(\theta_i, d_i) - U^{friend}_{a,c,P}(\theta_i, d_i') > U^{friend}_{a',c',P'}(\theta_i, d_i) - U^{friend}_{a',c',P'}(\theta_i, d_i')$$

for all i and θ_i and $d_i > d_i'$.

Proof of Proposition 8: To see the first claim, note that

$$x^{friend}_{a,c,P}(\theta_i, d_i) - x^{friend}_{a,c,P}(\theta_i, d_i') = \frac{a(d_i - d_i')\mathrm{E}\left[\theta\right]}{c(c - a\widetilde{\mathrm{E}}[d])}.$$

This is increasing in a, decreasing in c, and increasing in $\widetilde{\mathrm{E}}[d] = \frac{\mathrm{E}[d^2]}{\mathrm{E}[d]}$ which increases when either $\widetilde{P} \geq_{FOSD} \widetilde{P'}$ or P is a mean-preserving spread of P' (at least one strict). This establishes the first part of the result.

[28] Changes in ϕ do not impact actions, only welfare.

Note that substituting $x^{friend}(\theta_i, d_i)$ into (4) it follows that

$$U^{friend}(\theta_i, d_i) = \frac{\left(\theta_i + ad_i\widetilde{\mathrm{E}}_i\left[x^{friend}(\theta_j, d_j)\right]\right)^2}{2c} + \phi\mathrm{E}_i\left[\sum_{j\neq i} x^{friend}(\theta_j, d_j)\right].$$

Thus,

$$U^{friend}(\theta_i, d_i) - U^{friend}(\theta_i, d_i') = \frac{\left(\theta_i + ad_i\widetilde{\mathrm{E}}_i\left[x^{friend}(\theta_j, d_j)\right]\right)^2 - \left(\theta_i + ad_i'\widetilde{\mathrm{E}}_i\left[x^{friend}(\theta_j, d_j)\right]\right)^2}{2c}$$

Notice that this expression is increasing in a and $\widetilde{\mathrm{E}}_i\left[x^{friend}(\theta_j, d_j)\right]$ and decreasing in c. Given that $\widetilde{\mathrm{E}}_i\left[x^{friend}(\theta_j, d_j)\right]$ increases as we make the claimed changes from a', c', P' to a, c, P, the result then follows. ∎

Proof of Proposition 5: Consider a pure strategy symmetric equilibrium in which agents put positive probability on a positive degree - so that there is some interaction.

First note that in any such symmetric equilibrium, it must be that since agents are replying when choosing degrees and action levels, their action levels must also be best replies to the other action levels holding their degree choices fixed. Thus, taking the first order conditions of (7) with respect to x_i the equilibrium x_i's satisfy

$$x^{endog}(\theta_i, d(\theta_i)) = \frac{\theta_i}{c} + \frac{ad(\theta_i)\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d(\theta_j))\right]}{c} = \frac{\theta_i}{c} + \frac{ad(\theta_i)\widetilde{\mathrm{E}}\left[\theta_j\right]}{c(c - a\widetilde{\mathrm{E}}[d(\theta_j)])}. \tag{9}$$

The second equality follows from solving for the equilibrium values as in Lemma 3, taking the $d(\theta)$ choices as given, with the only difference being that now the numerator has the expectation $\widetilde{\mathrm{E}}\left[\theta_j\right]$ (rather than $\mathrm{E}\left[\theta_j\right]$), which conditions on the fact that the degree of neighbors now correlates with their degrees.

Next, consider some i and let us examine the best response choices of d_i, knowing that these must be best responses anticipating that actions will be according to x^{endog}.

Let us first consider the choice of an agent as if he or she were maximizing a continuous random variable. Taking the first order conditions of (7) with respect to d_i (and invoking the Envelope Theorem with respect to $x^{endog}(\theta_i, d_i)$ as a function of d_i) for the maximization of i's expected utility leads to

$$ax^{endog}(\theta_i, d_i)\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right] - kd_i = 0.$$

The second derivative of the expected utility is

$$\frac{\partial x^{endog}(\theta_i, d_i)}{\partial d_i} a\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right] - k.$$

From (9) it follows that $\frac{\partial x^{endog}(\theta_i, d_i)}{\partial d_i} = \frac{a\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right]}{c}$ and so the second derivative is

$$\frac{a^2\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right]^2}{c} - k,$$

which is negative by assumption (since $ck > a^2(n-1)^2 \geq a^2\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right]^2$) .

Thus, the expected utility is strictly concave in d_i and has a maximum at a point solving

$$kd_i = ax^{endog}(\theta_i, d_i)\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right].$$

Then, substituting for $x^{endog}(\theta_i, d_i)$ and solving for d_i, the optimal degree ignoring integer constraints is:

$$d_i^* = \frac{\theta_i}{c}\left[\frac{ca\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right]}{ck - a\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right]^2}\right].$$

Given the strict concavity of the expected utility function, the maximizing integer choice for d^{endog} must put probability only on either highest integer that does not exceed d_i^* or the lowest one that is not smaller than d_i^*.

Next, note that (from (7)

$$\frac{\partial^2 EU_i(\theta_i, d_i)}{\partial\theta_i \partial d_i} = a\frac{\partial x^{endog}(\theta_i, d_i)}{\partial\theta_i}\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right] = a\widetilde{\mathrm{E}}\left[x^{endog}(\theta_j, d_j(\theta_j))\right]/c > 0.$$

This implies, together with the strict concavity of utility in d_i, that if the some θ_i weakly prefers d_i to some lower d_i', then any higher type strictly prefers the higher degree. This implies that the the optimal $d^{endog}(\cdot)$ is nondecreasing, and that at most a set of measure 0 of types will be indifferent between two degrees, and so the strategy can be taken to be pure.

The comparison between $x^{friend}(\theta_i, d^{endog}(\theta_i))$ and $x^{bench}(\theta_i, d^{endog}(\theta_i))$ follows from Proposition 1, just substituting in the induced equilibrium degree distribution.

To make the comparison between $x^{friend}(\theta_i, d^{endog}(\theta_i))$ and $x^{endog}(\theta_i, d^{endog}(\theta_i))$, note that the only difference is that

$$x_i^{endog} = \frac{\theta_i}{c} + \frac{ad_i\widetilde{\mathrm{E}}\left[\theta\right]}{c(c - a\frac{\mathrm{E}[d^2]}{\mathrm{E}[d]})}$$

while

$$x_i^{friend} = \frac{\theta_i}{c} + \frac{ad_i\mathrm{E}\left[\theta\right]}{c(c - a\frac{\mathrm{E}[d^2]}{\mathrm{E}[d]})},$$

and so it boils down to a comparison between $\widetilde{\mathrm{E}}\left[\theta\right]$ and $\mathrm{E}\left[\theta\right]$. Note that

$$\widetilde{\mathrm{E}}\left[\theta\right] = \sum_d \mathrm{E}[\theta|d^{endog}(\theta) = d]\frac{P(d)d}{\mathrm{E}[d]} > \sum_d \mathrm{E}[\theta|d^{endog}(\theta) = d]P(d) = \mathrm{E}\left[\theta\right],$$

whenever there are at least two degrees chosen in equilibrium. This follows from the fact that $d^{endog}(\theta)$ is nondecreasing in θ and pure, and so $\mathrm{E}[\theta|d^{endog}(\theta) = d]$ is increasing in d, together with the fact that $\frac{P(d)d}{\mathrm{E}[d]}$ strictly first order stochastically dominates $P(d)$, which completes the proof. ▮

Proof of Proposition 6: The comparisons between U^{friend} and U^{bench} follow from Proposition 2. To compare U^{endog} to U^{friend}, one follows a parallel argument except now the additional fact that d^{endog} is nondecreasing and the ordering between x^{endog} and x^{friend} leads the expectations of others' actions to be higher under the endogenous equilibrium. ∎

Proof of Proposition 7: The existence and monotonicity of greatest equilibria, in both the network and population matching cases, follows from Proposition 14 in Van Zandt and Vives (2007). The ordering between actions follows from the fact that $\widetilde{P}_i$ strictly first order stochastically dominates P_i (given that P_i has weight on at least two degrees) and Proposition 16 in Van Zandt and Vives (2007), as we can view the only difference between x_i^{friend} and x_i^{bench} as a change in the distributions over neighbors' degrees. The distribution over other agents' types is unchanged, given the independence, and then to see the first order stochastic dominance, note that $\widetilde{P}_i(d)/P_i(d) = d/\mathrm{E}[d]$ which is strictly increasing in d. The strict ordering of actions in the case with a smooth dimension and all interior actions then follows from their Corollary 17.[29] ∎

[29]Note that their proof extends to the case in which the stochastic dominance is strict only on one dimension of agents' types, here degrees, and that dimension drives a strict increase in the derivative of utility with respect to the smooth dimension of actions.