Survival in Speculative Markets*

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Abstract

In a stochastic exchange economy where, due to beliefs' heterogeneity, agents engage in speculative trade, I investigate the Market Selection Hypothesis that speculation rewards agents with accurate beliefs. Assuming that markets are complete, I derive sufficient conditions for agents' survival in terms of intertemporal substitution rates and portfolio expected log-returns and use them to show that the Market Selection Hypothesis fails generically. In particular, when agents have Epstein-Zin preferences, beliefs heterogeneity may persist in the long-run or speculation may cause the agent with the most accurate beliefs to vanish. Failures occur because portfolio expected log-returns depend both on beliefs accuracy and risk preferences, through the comparison with the growth-optimal portfolio. Failures do not occur in CRRA economies because, due to the interdependence of relative risk aversion and intertemporal elasticity of substitution, portfolio returns not related to beliefs' accuracy are compensated by the component of saving that responds to uncertainty.

Keywords: Heterogeneous Beliefs; Speculation; Market Selection Hypothesis; Asset Pricing; Optimal Growth Portfolio; Epstein-Zin Preferences; Saving under Uncertainty.

JEL Classification: D53, D01, G12, G14, G11, E21

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1 Introduction

The dominant academic view of financial markets is that they facilitate hedging and risk diversification. A complementary view is that trade also occurs due to agents' disagreement about assets' return distributions. Indeed, also in standard models of financial markets such as Lucas' model or the CAPM, agents' beliefs heterogeneity makes them willing to hold different risky positions from those they would have held under pure hedging. These positions are speculative, in that they include a bet on the future realizations of assets' fundamentals. In this paper I investigate the effect of speculation on agents' relative consumption dynamics by identifying the separate roles of saving and portfolio decisions. The main result is that speculation may have long-run consequences. In particular, beliefs' heterogeneity may persist in the long-run or the agent with the most accurate beliefs may vanish.

Although speculative incentives are certainly present, a widespread position of financial economists is that speculation cannot have long-run consequences, and thus its investigation cannot help to characterize assets' returns in equilibrium. The Market Selection Hypothesis (MSH) of Friedman (1953) applied to financial markets presumes that investors with accurate beliefs can earn high returns by taking positions against investors with inaccurate beliefs. Provided markets are complete and the time horizon is long enough, these speculative positions should thus allow accurate traders to dominate the market and to bring asset prices at the fundamental value implied by their preferences and beliefs.[1]

In bounded economies with time-separable preferences, the argument is rigorously established by Sandroni (2000) and Blume and Easley (2006). Indeed, when markets are complete, each agent can trade to allocate his future consumption on the path which he believes as more likely. In equilibrium, everything else being equal, the agent with the most accurate beliefs assigns the highest likelihood to paths that are actually realized and thus holds everything in the long run. Despite the importance of the result, the exact role of portfolio and saving decisions for its validity is still unclear. Sandroni (2000) and Blume and Easley (2006) characterize an agent long-run consumption in terms of a survival index that depends on discount factor and beliefs' accuracy (as measured by the relative entropy). For log-economies the discount factor is a saving rate and beliefs' accuracy corresponds to portfolio expected log-returns, so that having accurate beliefs leads to profitable portfolio positions. Other preferences lead to different optimal saving and portfolio decisions but the survival index remains the same.[2]

[1]Note, however, that depending on agents' preferences and/or assets payoff structure vanishing traders may have a price impact, see e.g. Kogan et al. (2006, 2009) and Cvitanić and Malamud (2011).

[2]The incumbent literature finds similar results, see Section 1.1. For example, Yan (2008) shows

In this paper I shed light on the determinants of survival in speculative markets by deriving sufficient conditions for survival that depend on the separate contribution of saving and portfolio decisions. I work in a discrete-time exchange economy with intermediate consumption and complete markets.[3] I assume a finite number of agents with heterogeneous beliefs and, possibly, preferences. The main result is that speculative portfolio positions alone are not sufficient to support the MSH but, instead, can lead to its failure. In particular, since both agents' beliefs and risk preferences determine their betting positions, the average returns of a portfolio derived under correct beliefs are not necessarily higher than those of a portfolio derived under incorrect beliefs. Importantly, saving may or may not "compensate" for the under-performance of a portfolio derived under accurate beliefs. Due to the interdependence of relative risk aversion and intertemporal substitution, saving does compensate in bounded CRRA economies as those studied in Sandroni (2000) and Blume and Easley (2006) but does not compensate in more general cases, possibly leading to MSH failures.[4]

Epstein-Zin preferences, being flexible enough to disentangle intertemporal and intra-states consumption decisions, are a typical case where saving does not compensate the component of portfolio returns that is not related to beliefs' accuracy. I show that there exist parametrizations where, with full probability:

- multiple agents survive and have a positive consumption share in the long-run (**long-run heterogeneity**);
- either the agent with the most accurate beliefs vanishes or he dominates (**path dependency**);
- the agent with the most accurate beliefs vanishes (**vanishing of the accurate trader**).

Under **long-run heterogeneity**, which typically occurs when agents' Relative Risk Aversion (RRA) coefficient is larger than 1, beliefs heterogeneity is persistent and state prices keep fluctuating between agents' evaluations. The result could help explaining stock market anomalies as recently suggested in Anderson et al. (2005), Hong and Stein (2007), Cogley and Sargent (2009), Yu (2011), Bhamra and Uppal (2014), Hong and Sraer (2016), and Baker et al. (2016).

that in an unbounded economy with Constant Relative Risk Aversion (CRRA) the survival index depends on discount factor, beliefs' accuracy and the coefficient of RRA. I clarify how this extra term comes about in Section 4.2. See also footnote 5.

[3]For the sake of tractability I assume that the aggregate endowment growth rates follow and i.i.d. process, see Assumption 2.1. Sufficient conditions for survival more general endowment process can be established following the same approach.

[4]The result of Yan (2008) for unbounded CRRA economies can also be understood in terms of this compensation. See Section 4.2.

Under **vanishing of the accurate trader** or **path dependency**, the market does not (may not) select for the most accurate beliefs. Nevertheless, a single agent is rewarded in the long-run, and determines asset prices. I shall show that failures to reward the most accurate trader may occur in economies where all agents hold the same portfolio, so that only saving is relevant, as well as in economies where the saving decision is homogeneous across agents, so that only portfolios matter.

The reason why speculation may fail to validate the MSH, and even allow for long-run heterogeneity, is as follows. In Section 2, I show that the dynamics of consumption shares depends on two key quantities: the ratio between the value of next period possible consumption and current consumption, which is an intertemporal substitution rate, and the return of the portfolio that allocates next period consumption among the different states. In Section 3, I provide general sufficient conditions for an agent (or a group of agents) to survive, dominate, or vanish in terms of the comparison of log substitution rates and portfolio expected log returns.[5] In particular each agent portfolio expected log returns can be decomposed in a market risk premium, a premium for having beliefs more accurate than market beliefs,[6] and a term that depends on the comparison between his portfolio and the log-optimal portfolio derived under his beliefs. It is this last term, named Non-Log-Optimality (NLO) contribution, that makes MSH failures possible. By definition, the term is zero when the portfolio is log-optimal. The term is instead positive when beliefs and risk preferences are such that, at the prevailing market prices, the chosen portfolio is closer to the log-optimal portfolio derived under correct beliefs than the log-optimal portfolio derived under the agent beliefs.[7] In such cases it is as if the agent is using a log-optimal portfolio and has "effective" beliefs that are more accurate than his original beliefs. Effective beliefs can also be less accurate than the original beliefs, making the non-log-optimality term negative. Saving plays also a role and the comparison of agents' intertemporal substitution rates may or may not compensate for the (in)accuracy of effective beliefs.

To explicit the trade-off between saving and portfolio positions for a specific choice of risk and time preferences, in Section 4, I apply the general survival conditions of Section 3 to Epstein-Zin economies. Epstein-Zin preferences provide a natural generalization of the benchmark CRRA case, allowing intertemporal and risk preferences to be disentangled. First, in Section 4.1, I concentrate on cases where only portfolio decisions matter for agents' relative performance.[8] When

[5] The dynamics of consumption shares follows a multiplicative process, thus expected log-returns rather than expected returns determine survival.

[6] Market beliefs are defined as the beliefs of the representative agent, see Definition 2.1.

[7] The log-optimal portfolio derived using correct beliefs is the portfolio with maximal growth, see Kelly (1956) and the literature surveyed in Section 1.1.

[8] This amounts to assume that all agents employ the same saving decision in equilibrium. The latter occurs in Epstein-Zin economies where all agents have unitary IES parameter and the

all agents have log-optimal portfolios, the comparison of their returns depends only on beliefs' accuracy. The presence of agents with inaccurate beliefs implies that the agent with the most accurate beliefs has positive expected log-returns in every period. By speculating, he wins enough bets to eventually gain all the aggregate endowment. The reasoning behind the MSH is valid. Outside of the log framework, however, effective beliefs' accuracy and beliefs' accuracy differ and non-log-optimality terms become important. Moreover, effective beliefs accuracy depends on assets' equilibrium returns. As a result, given two agents, it can happen that an agent's effective beliefs are the most accurate when the returns are set by the other agent, and the other way round. In this case speculation does not support dominance of a single agent but the outcome is long-run heterogeneity. Alternatively, it could occur that one agent has the most accurate effective beliefs for all possible equilibrium asset returns, even when his beliefs are inaccurate. Depending on all agents' risk preferences and beliefs, all types of MSH failures might occur.

When saving is not homogeneous across agents, the comparison of intertemporal substitution rates plays also a role. In Sections 4.2-4.3, I study how both saving and portfolio decisions matter for survival. MSH failures, in particular long-run heterogeneity, remain possible. Bounded CRRA economies are instead special because, due to the interdependence of RRA and intertemporal elasticity of substitution (IES), the saving component due to uncertainty does compensate exactly for the difference of accuracy between beliefs and effective beliefs reflected in the NLO term (Section 4.2). However, this compensation occurs in terms of agents' relative performance. In Section 4.3, I show that it is enough that only one agent has preferences not in the CRRA class to obtain MSH failures such as long-run heterogeneity and path dependency.

The role of saving is confirmed in Section 4.4, where I analyze Epstein-Zin economies where agents' beliefs and risk preferences are such that every agent holds the market portfolio in equilibrium. Agents do not exchange speculative bets and portfolios become irrelevant.[9] I derive the intuitive result that the agent who sets the lowest interest rate when alone in the market has a higher intertemporal substitution rate in every period and thus dominates in the long-run. Beliefs, together with discount factors and IES coefficients, still play a role for long-run outcomes but only because, through the saving under uncertainty channel, they determine substitution rates. In these saving economies, the relative consumption equilibrium dynamics is deterministic and the truth plays no role. Only vanishing of the accurate trader remains possible, in accordance with Yan (2008).

same discount factors.

[9]Despite all agents earn the same returns, the contribution of beliefs' accuracy and non-log-optimality term to each agent expected log-returns does differ.

Finally, I give simple examples in Section 5 and conclude in Section 6. Appendixes A-E collect the proofs and additional material. In the next section I discuss the relation between my results and the literature.

1.1 Related Literature

Investors speculate when they take long and/or short positions that they would have not otherwise taken if they had agreed on the underlying state process.[10] A number of contributions investigate the effect of speculation on asset prices and the volume of trade (see e.g Varian, 1985, 1989; Harris and Raviv, 1993; Kandel and Pearson, 1995) or the relation between speculation and financial innovations (see e.g. Zapatero, 1998; Brock et al., 2009; Simsek, 2013). For example, Simsek (2013) decomposes agents portfolio risk as the sum the variance that remains after hedging and the variance due to speculation. The key result is that the speculative variance always increases when new assets are introduced. Here, I am instead interested in whether speculation can have long-run consequences. Indeed one could argue that financial innovation is needed to enable accurate traders to dominate by speculating against inaccurate traders. This work shows instead that, also in the idealized framework of complete markets and general intertemporal equilibrium, speculation may be a persistent feature of financial markets.

The relation between speculation and the MSH for financial economies has received increasing attention at least since the works of DeLong et al. (1990, 1991) and Blume and Easley (1992). DeLong et al. (1991) investigate whether noise traders, i.e. traders with inaccurate beliefs, might survive or even dominate against rational traders by bearing more risk. The answer is positive but the analysis is based on a partial equilibrium model. Blume and Easley (1992) study the same issue in a model where asset prices are set in equilibrium by all traders. They investigate a sequence of temporary equilibria where agents save at a constant rate and can use Arrow securities to transfer wealth across states. Controlling for the saving rate, they find that when the trader with the most accurate beliefs purchases a log-optimal portfolio, he gains all the wealth in the long run and brings asset prices to reflect his beliefs. The result provides a support for the growth optimal Kelly rule (Kelly, 1956) in equilibrium models. However, when the trader with the most accurate beliefs does not use the log-optimal rule, Blume and Easley are able to derive conditions for this trader to vanish.[11]

[10] The term speculation is also refereed to the purchase of an asset for the purpose of re-selling it at a higher price to those who value it more, see Harrison and Kreps (1978) for a formal model. See also Morris (1996) and Scheinkman and Xiong (2003).

[11] Blume and Easley (1992) work in an i.i.d. economy with Arrow securities. The studies surveyed in Evstigneev et al. (2009) propose a generalization of the Kelly rule for more complicated asset structures.

Subsequent work by Sandroni (2000) and by Blume and Easley (2006) extend the analysis to general equilibrium models with endogenous saving. Under the assumption that markets are complete, the aggregate endowment is bounded, and agents maximize an expected time-separable utility, the MSH holds: provided that all traders discount future utility at the same rate, only the trader with the most accurate beliefs dominates. The market does not select against traders whose portfolios are not log-optimal, provided that their beliefs are accurate. Vanishing of the accurate trader, can still occur but it depends on discounting future utility too much. Results are derived by solving the social planner problem and are given in terms of the comparison of a survival index that depends only on discount factors and beliefs' accuracy. Decentralized saving and portfolio positions are not explicited and their role in supporting the MSH remains unclear.

A related contribution is Yan (2008), where the MSH is investigated in a continuous-time economy where the aggregate endowment follows a Brownian motion and agents have CRRA preferences. Agents agree on the volatility of the aggregate endowment process but disagree on its drift. The findings by Sandroni (2000) and Blume and Easley (2006) are confirmed, provided that the survival index takes also into account the RRA coefficient. When the economy is growing, the agent with the lowest RRA coefficient (the highest IES) has, all else equal, a higher survival index and thus dominates in the long-run.[12]

By finding failures of the MSH even in a general equilibrium framework, my results reconcile the findings of the earlier studies by DeLong et al. (1991) and Blume and Easley (1992) with those of the later literature. Traders with inaccurate beliefs might survive, or even dominate, in equilibrium. Saving does not always offset this result. The following quote from DeLong et al. (1991) nicely summarizes my findings: noise traders (agents with inaccurate beliefs in my setting) survive when "misperceptions make them unwittingly hold portfolio closer to those that would be held by investors with log-utility" -and correct beliefs- (p. 3). In particular, I find that whether a noise trader survives, dominates, or vanishes depends on a trade-off between misperceptions and risk attitudes that can be expressed in terms of beliefs and effective beliefs' accuracy. Moreover, I explain why full dominance can occur only in market with aggregate risk, confirming a finding of Blume and Easley (1992) (Th. 5.4).[13]

For CRRA economies with intermediate consumption, I confirm the results of

[12]Other studies by Mailath and Sandroni (2003), Sandroni (2005), Jouini and Napp (2006, 2007), Cvitanić et al. (2012), Muraviev (2013), Bhamra and Uppal (2014), and Massari (2015, 2016) consider related cases. The main conclusion still holds, the only possible failure of the MSH is the vanishing of the accurate trader.

[13]Under no aggregate risk, if fair pricing holds, a trader with correct beliefs has also correct effective beliefs in the limit when he consumes most of the endowment and sets assets' returns. This holds regardless of his preferences. Thus, noise traders can never dominate almost surely.

the incumbent literature: neither long-run heterogeneity nor path dependency can occur generically. As a novelty, I write the survival index in terms of its fundamental components, intertemporal substitution rates and expected log-returns, so that a unique survival index can be given for the different cases (bounded vs unbounded economy, aggregate risk vs no aggregate risk). I also show that portfolio returns can be decomposed in the sum of a market log-return, an accuracy premium, an a non-log-optimality term. This decomposition is helpful for formulating the trade-off between beliefs' accuracy and risk-preferences in relation to an agent survival.[14]

Other influential works investigate the MSH in economies without intermediate consumption, see in particular Kogan et al. (2006, 2009) and Cvitanić and Malamud (2010, 2011). Results in the two types of economies -with or without intermediate consumption- are known to differ due to the effect of saving in the former. Having clarified that saving rules (in particular intertemporal substitution rates) are not only important per-se, but also for the fact that they might or not compensate the NLO term of portfolio expected log-returns, my work reconciles the findings of the two literature regarding the role of portfolio rules, in particular their proximity to the growth-optimal portfolio. For example, in Section 5.1, I show that the phenomenon of extinction reversal found in Cvitanić and Malamud (2010) is possible also in economies with intermediate consumption.

In analyzing the MSH in Epstein-Zin economies, this paper is related to Borovička (2015). He investigates the MSH in continuous-time exchange economies with two agents having homogeneous Epstein-Zin preferences. My results, in particular that MSH failures are possible and generic, confirm his findings. On the methodological side the two papers are, however, quite different. Borovička solves directly the central planner problem and characterizes agents (general equilibrium) optimal policies in the partial equilibrium limit of one agent being alone in the economy.[15] I study long-run survival in generality, showing the contributions of intertemporal substitution rates, beliefs' accuracy, and non-log-optimality terms in the case of general saving and portfolio decisions (Sections 2 and 3). Epstein-Zin economies become an application where the derivation of portfolio and saving decisions that are not tied together is (partly) feasible (Section 4). I use Epstein-Zin preferences because, in containing CRRA preferences as a special case, they allow to show how

[14]Sandroni (2000) and Blume and Easley (2006) work with general time-separable utilities. Here I concentrate on the special case of CRRA preferences. In Appendix B I discuss why the same link between saving and portfolio decisions should hold more in general.

[15]This strategy allows Borovička to derive market-selection outcomes for a larger region of parameters than I do here. Whether this approach is possible also in discrete-time economies and in economies with more than two agents is still an open issue. Note also that there exists parametrizations that I consider and Borovička excludes, namely when the partial equilibrium does not deliver an interior solution but the general equilibrium does.

deviations from separability destroy the exact compensation between expected log-returns and intertemporal substitution rates that holds in the separable case. My results make clear that MSH failures may occur whenever, for at least one trader, the component of expected log-returns that is not related to beliefs' accuracy does not match the saving component that responds to uncertainty. Finally, discrete time economies allow for more freedom in the modeling of the economy, in particular of the aggregate endowment process and of the degree of inaccuracy of agents.

Another related contribution is Easley and Yang (2015) where long-run relative consumption outcomes are computed in a two-agent economy with an Epstein-Zin investor and a loss-averse investor. Consistently with my results, it is shown that the loss-averse investor vanishes because his portfolio is further away from the growth-optimal one. Moreover, the loss-averse agent can survive and dominate only when he saves more than the Epstein-Zin investor.

Within the market selection literature, other studies find long-run beliefs heterogeneity. Beker and Chattopadhyay (2010) and Cogley et al. (2013) focus on two-agent economies with incomplete markets. Beker and Espino (2011) highlights the importance of learning. Cao (2013) studies an economy where markets are endogenously incomplete due to portfolio and collateral constraints. Guerdjikova and Sciubba (2015) study economies where investors are ambiguity averse. Bottazzi and Dindo (2014) and Bottazzi et al. (2015) extend the temporary equilibrium analysis of Blume and Easley (1992) to general asset structures, short-lived and long-lived respectively, and possibly incomplete markets.

2 The Economy

In this section, I introduce the exchange economy and show how, in presence of heterogeneous agents, it is possible to characterize the dynamics of equilibrium consumption and state prices directly from agents' intertemporal substitution and portfolio decisions.

Time begins at date $t = 0$ and it is indexed by $t \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. $\mathcal{S} = \{1, 2, \ldots, S\}$ is the set of states of the world, $2^{\mathcal{S}}$ is its power set, and $\Sigma = \times_{t=0}^{\infty}\mathcal{S}$ is the set of paths σ. $s_t \in \mathcal{S}$ denotes the state realized at date t and $\sigma_t = (s_0, s_1, \ldots, s_t) \in \Sigma_t$ the partial history till period t. To each partial history there corresponds a node of the uncertainty tree. $C(\sigma_t)$ is the cylinder set with base σ_t, $C(\sigma_t) = \{\sigma \in \Sigma | \, \sigma = (\sigma_t, \ldots)\}$ and $\mathcal{F}_t$ the σ-algebra generated by the cylinders, $\mathcal{F}_t = \sigma\left(\{C(\sigma_t) \, \forall \, \sigma_t \in \Sigma_t\}\right)$. $\mathcal{F}$ is the σ-algebra generated by the union of $\mathcal{F}$, $\mathcal{F} = \sigma\left(\cup_t \mathcal{F}_t\right)$. By construction $\{\mathcal{F}_t\}$ is a filtration. P is a probability measure on $(\Sigma, \{\mathcal{F}_t\})$ and $(\Sigma, \{\mathcal{F}_t\}, \mathrm{P})$ is the probability space on which I construct everything. All random variables are adapted to the filtration $\{\mathcal{F}_t\}$ and $x_t(\sigma_t)$ may be used in place of $x_t(\sigma)$. The dependence on a sequence σ, or on a partial history σ_t, is typically not explicited.

The economy contains I traders and a single consumption good. Trader $i \in \mathcal{I} = \{1, 2, \ldots, I\}$ consumption in period t on path σ is $c^i_t(\sigma)$. A consumption plan is a stochastic process $\{c_t\}$ and each trader i is endowed with the particular consumption plan $\{e^i_t\}$. The aggregate endowment is $\{e_t\}$ and for all t, s, and σ_t the growth rate of the economy is

$$g_{s,t}(\sigma_t) = \frac{e_{t+1}(\sigma_{t+1})}{e_t(\sigma_t)} \quad \text{when} \quad \sigma_{t+1} = (\sigma_t, s)\,.$$

For all t and σ_t, I denote with $\hat{g}_t$ the date t vector of de-trended growth rates

$$\hat{g}_{s,t} = \frac{g_{s,t}}{\exp \mathrm{E}_{\mathrm{P}}[\log g_t]}\,. \tag{1}$$

I assume that the growth process is i.i.d..

Assumption 2.1. *For all $t \in \mathbb{N}_0$, $s \in \mathcal{S}$, and $\sigma_t \in \Sigma_t$, $g_{s,t} = g_s$ and $\mathrm{P}(C(\sigma_T)) = \prod_{t=1}^{T} \mathrm{P}_{s_t}$ for a measure $\mathrm{P} = (\mathrm{P}_1, \ldots, \mathrm{P}_S)$ on $(\mathcal{S}, 2^{\mathcal{S}})$.*

With an abuse of notation, I use P to denote both the measure on $(\Sigma, \{\mathcal{F}\})$ and on $(\mathcal{S}, 2^{\mathcal{S}})$. As it is briefly discussed in Section 5.1, the assumption is without loss of generality for market selection purposes in that MSH failures can also be obtained with more complicated growth processes.

Each agent objective is to maximize a certain utility of his consumption stream. Agents may transfer their initial endowment across time and states by trading assets in a complete market. In evaluating consumption streams $\{c^i_t\}$ agent i uses subjective beliefs, a probability measure on $(\Sigma, \{\mathcal{F}\})$. I shall assume that all agents believe that the world is i.i.d., so that beliefs on $(\Sigma, \{\mathcal{F}\})$ are generated by beliefs on $(\mathcal{S}, 2^{\mathcal{S}})$, that beliefs are absolute continue with respect to each other and the truth, and that beliefs are heterogeneous.

Assumption 2.2. *For all agents $i \in \mathcal{I}$, beliefs on $(\Sigma, \{\mathcal{F}\})$ are generated by constant beliefs Q^i on $(\mathcal{S}, 2^{\mathcal{S}})$ with $\mathrm{Q}^i_s > 0 \Leftrightarrow \mathrm{P}_s > 0$, for all s in $\mathcal{S}$. Moreover, $\mathrm{Q}^i \neq \mathrm{Q}^j$ for all i and j in $\mathcal{I}$.*

As with Assumption 2.1, the i.i.d. part of Assumption 2.2 is without loss of generality: as long as agents' disagreement persists in the long-run, MSH failures can be shown to occur also with more complicated beliefs. The absolute continuity assumption is instead needed for the existence of an equilibrium, see Appendix A.3.

Other than assuming that the market is complete I do not further specify its structure.[16] Asset prices are determined in equilibrium. I postpone the characterization of agents' consumption and asset demand and just assume for now that there exists a no-arbitrage equilibrium where all agents' consumption plans are strictly positive.[17] The notation for state prices mimic the one for conditional probabilities: for $\tau > 0$, $q_{\sigma_{t+\tau},t}$ is the price of a unit of the consumption good after partial history $\sigma_{t+\tau}$ relative to one unit of consumption in date t. Due to the no-arbitrage condition state prices satisfy:

$$q_{\sigma_{t+\tau},t} = \frac{q_{\sigma_{t+\tau},0}}{q_{\sigma_t,0}} \,.$$

Using the vector of one-period states prices, q_t, one obtains the interest rate from t to $t+1$, r_t, and the corresponding discount rate δ_t. Risk neutral probabilities (normalized state prices) are $\mathrm{Q}^0_t = r_t q_t$.

In order to investigate the consumption dynamics, I characterize agents' portfolio and saving decisions as follows. Consider agent i equilibrium consumption $\{c^i_t\}$ in two subsequent periods t and $t+1$. Since for each agent i consumption is an adapted process, for every t and every history σ_t there exists a scalar $\delta^i_t > 0$ and a vector $\alpha^i_t \in \Delta^S_+$ such that[18]

$$\begin{cases} \delta^i_t(\sigma_t) & = & \dfrac{\delta_t \sum_{s\in\mathcal{S}} \mathrm{Q}^0_{s,t}\, c^i_{t+1}(\sigma_t, s)}{c^i_t(\sigma_t)} \,, \\[2ex] \alpha_{s,t}(\sigma_t) & = & \dfrac{\mathrm{Q}^0_{s,t}\, c^i_{t+1}(\sigma_t, s)}{\sum_{s'\in\mathcal{S}} \mathrm{Q}^0_{s',t}\, c^i_{t+1}(\sigma_t, s')} \,, \quad \text{for all } s \in \mathcal{S} \,. \end{cases} \tag{2}$$

The scalar δ^i_t is the ratio between date t value of next period contingent consumption and date t consumption, agent i intertemporal substitution decision in a stochastic context, and it is thus related to how much agent i saves. The vector α^i_t gives agent i allocation decision across states, only for consumption in period $t+1$,

[16] Date $t = 0$ trading does not require agents to hold rational prices expectations but it amounts to trade infinite assets in the initial period. Sequential trading of short- or long-lived assets makes the opposite assumptions. Depending on the chosen asset structure, relevant assumptions on the budget constraint should be taken to guarantee the existence of an equilibrium. In particular, under date $t = 0$ trading no bankruptcy is allowed. Under sequential trading no bankruptcy and no Ponzi schemes are allowed, see also Araujo and Sandroni (1999).

[17] When the aggregate endowment is growing I also assume that agents are discounting the future enough so that their equilibrium value function is finite. See also Assumption A.1 in Appendix A.

[18] The fact that both δ^i_t and α^i_t are positive follows from showing that consumption is positive in equilibrium.

and it is thus related to agent i portfolio decision.[19] When agents are maximizing an objective function, the equilibrium value of agents' intertemporal substitution rates and portfolios is determined by a set of Euler equations. Rules are particularly simple when agent i maximizes the expected discounted stream of log consumption with discount factor β, resulting in $\delta_t^i = \beta$ and $\alpha_t^i = \mathrm{Q}^i$. Exploiting the analogy with the logarithmic trader we can consider δ_t^i as an *effective discount factor* and α_t^i as an *effective belief.* We shall write $\delta_t^i(\delta_t, \mathrm{Q}_t^0)$ and $\alpha_t^i(\mathrm{Q}_t^0)$ when we want to underline the dependence of saving and portfolio rules on state prices. I postpone to the next section the exact specification of both $\delta_t^i(\delta_t, \mathrm{Q}_t^0)$ and $\alpha_t^i(\mathrm{Q}_t^0)$ when agents have Epstein-Zin recursive preferences.

Given the sequence of portfolios $\{\alpha_t^i\}$ and intertemporal substitution rates $\{\delta_t^i\}$, from (2) the dynamics of agent i consumption is thus

$$c_{t+1}^i(\sigma_t, s) = \frac{\delta_t^i}{\delta_t} \frac{\alpha_{s,t}^i}{\mathrm{Q}_{s,t}^0} c_t^i(\sigma_t)\,.$$

Similarly, in terms of relative consumption $\phi_t^i = \frac{c_t^i}{e_t}$, we have:

$$\phi_{t+1}^i(\sigma_t, s) = \frac{\delta_t^i}{\delta_t} \frac{\alpha_{s,t}^i}{\mathrm{Q}_{s,t}^0\, g_s} \phi_t^i(\sigma_t)\,. \tag{3}$$

The same type of dynamics of consumption shares holds also for groups of agents provided aggregate substitution rates and portfolios are defined. Consider a subset $\mathcal{J} \subset \mathcal{I}$ with J traders and define, for every t, σ_t, $\Phi_t^{\mathcal{J}} = \sum_{j\in\mathcal{J}} \phi_t^j$ and $\phi_t^{\mathcal{J},j} = \frac{\phi_t^j}{\Phi_t^{\mathcal{J}}}$ so that $\phi_t^{\mathcal{J}} \in \Delta_+^J$. By definition $\Phi_t^{\mathcal{I}} = 1$ and $\phi_t^{\mathcal{I},i} = \phi_t^i$ with $\phi_t^{\mathcal{I}} = \phi_t \in \Delta_+^I$.

By repeating the same computation as in (2), aggregate substitution rates and portfolio of group $\mathcal{J}$ are, respectively,

$$\delta_t^{\mathcal{J}} = \sum_{j\in\mathcal{J}} \delta_t^j \phi_t^{\mathcal{J},j} \tag{4}$$

and

$$\alpha_t^{\mathcal{J}} = \sum_{j\in\mathcal{J}} \alpha_t^j \varphi_t^{\mathcal{J},j}\,, \tag{5}$$

where[20]

$$\varphi_t^{\mathcal{J},j} = \frac{\delta_t^j \phi_t^{\mathcal{J},j}}{\delta_t^{\mathcal{J}}}\,. \tag{6}$$

[19] α_t^i and δ_t^i can be interpreted as the one-period portfolio and saving decisions. The expressions of the full portfolio and saving rules, $\bar{\alpha}_t^i$ and $\bar{\delta}_t^i$, are given in Appendix A.1.

[20] By construction also $\varphi_t \in \Delta_+^J$.

In fact, it can be checked that the consumption dynamics (3) can be written also for groups as

$$\Phi^{\mathcal{J}}_{t+1}(\sigma_t, s) = \frac{\delta^{\mathcal{J}}_t}{\delta_t} \frac{\alpha^{\mathcal{J}}_{s,t}}{\mathrm{Q}^0_{s,t}\, g_s} \Phi^{\mathcal{J}}_t(\sigma_t) \quad \text{for all } s \text{ and } \sigma_t\,. \tag{7}$$

Equilibrium prices and discount rates can be found by considering (7) for the set $\mathfrak{I}$ of all the agents as follows. Using that $\Phi^{\mathfrak{I}}_t = 1$ and summing (7) over s we first find

$$\delta_t = \frac{\delta^{\mathfrak{I}}_t}{\sum_{s' \in \mathcal{S}} g_{s'}\, \mathrm{Q}^0_{s',t}}\,. \tag{8}$$

Discount rates are determined by the aggregate substitution rate and take also into account the rate of change in the aggregate endowment. Equilibrium state prices are instead fixed by the clearing condition of the Arrow security markets. The normalized value of the supply of Arrow security corresponding to state s is

$$l_s(\mathrm{Q}^0_t) = \frac{\mathrm{Q}^0_{s,t}\, e_{t+1}(\sigma_t, s)}{\sum_{s' \in \mathcal{S}} \mathrm{Q}^0_{s',t}\, e_{t+1}(\sigma_t, s')} = \frac{\mathrm{Q}^0_{s,t}\, g_s}{\sum_{s' \in \mathcal{S}} \mathrm{Q}^0_{s',t}\, g_{s'}}\,.$$

Using the above and the formula for discount rates (8) in (7) when $\mathcal{J} = \mathfrak{I}$ we find

$$l_s(\mathrm{Q}^0_t) = \alpha^{\mathfrak{I}}_{s,t}(\mathrm{Q}^0_t)\,, \quad \text{for all} \quad s \in \mathcal{S}\,. \tag{9}$$

Normalized state prices are such that the aggregate portfolio, the convex combination of each agent's portfolio, equates the normalized value of the aggregate supply. Importantly, each agent impact on the aggregate portfolio depends, other than on his relative consumption share, also on the ratio of between his intertemporal substitution rates and the economy discount factors, see (4-6). When individual demands are derived from the maximization of an objective function, equilibrium state prices incorporates all agents risk preferences and beliefs, each agent i having a contribution that depends on his weight φ^i. I provide a graphical representation of market clearing condition using a supply and demand plot in Section 5.

The system of equations (3) and (8-9) for all $i \in \mathfrak{I}$, $t \in \mathbb{N}_0$, and $\sigma_t \in \Sigma_t$ characterizes agents' relative consumption and state prices on an equilibrium path. Thus, an equilibrium allocation and supporting prices can be computed iteratively if:

C1 we know that a competitive equilibrium exists and is interior;

C2 for all t, σ_t one-period optimal portfolio and substitution decisions of all agents can be recovered from quantities (state prices, agents' beliefs and consumption) known in σ_t;

C3 the equilibrium relative consumption distribution ϕ_0 is known.

As it is shown in Appendix A and in the next section, conditions **C1-C2** do hold, for example, when agents' intertemporal substitution rates and portfolios come from the maximization of specific parametrizations of an Epstein-Zin recursive utility. Regarding condition **C3**, note that long-run properties can be characterized even when it does not hold. In fact, provided long-run outcomes of (3) and (8-9) are identified for every initial consumption distribution, also equilibrium long-run outcomes are characterized.

Before we use (7) to provide sufficient conditions for a group of agents to survive, dominate, or vanish it is useful to use aggregate portfolios $\alpha^{\mathcal{J}}$ to introduce the concept of group $\mathcal{J}$ beliefs. We assume that fair pricing holds under no aggregate risk and define group $\mathcal{J}$ beliefs $\mathrm{Q}_t^{\mathcal{J}}$ in date t as those normalized state prices that would hold in equilibrium in an economy with no aggregate risk and a representative agent who invests using the aggregate portfolio $\alpha_t^{\mathcal{J}}$.

Definition 2.1. *Given an economy with a set $\mathcal{I}$ of traders each with a sequence of substitution rules $\{\delta_t^i\}$, portfolios $\{\alpha_t^i\}$, and relative consumption process $\{\phi_t^i\}$, group $\mathcal{J} \subset \mathcal{I}$ has beliefs $\mathrm{Q}_t^{\mathcal{J}}(\sigma_t) \in \Delta_+^S$ when they solve*

$$1 = \alpha_{s,t}^{\mathcal{J}}(\mathrm{Q}_t^{\mathcal{J}}) \,, \quad \textit{for all} \quad s \in \mathcal{S} \,.$$

In particular the beliefs of the entire set of traders $\mathcal{I}$, $\mathrm{Q}_t^{\mathcal{I}}(\sigma_t)$, are the market beliefs in date-t-history-σ_t.

In a log economy where all agents have the same discount factor, group $\mathcal{J}$ beliefs are a convex combination of all agents beliefs weighted by their consumption share, in accordance with Rubinstein (1974).[21]

2.1 Epstein-Zin Economies

Saving and portfolio decisions can be explicitly derived when agents maximize specific parametrizations of a recursive utility of the Epstein-Zin type as in Epstein and Zin (1989). We assume that agent i with beliefs Q^i maximizes a utility U^i that has a recursive structure of the type

$$U_t^i = \left((1-\beta^i) c_t^{1-\rho^i} + \beta^i \left(\mathrm{E}_{\mathrm{Q}^i}[(U_{t+1}^i)^{1-\gamma^i}] \right)^{\frac{1-\rho^i}{1-\gamma^i}} \right)^{\frac{1}{1-\rho^i}} \,, \; t \in \mathbb{N}_0 \,. \qquad (10)$$

[21] By definition, if agent i preferences satisfy the fair pricing condition, then $\mathrm{Q}_t^{\mathcal{J}} = \mathrm{Q}^i$ when $\mathcal{J} = \{i\}$.

$\beta^i \in (0,1)$ is the discount factor; $\gamma^i \in (0,\infty)$ is the coefficient of Relative Risk Aversion (RRA); $\rho^i \in (0,\infty)$ is the inverse of the coefficient of Intertemporal Elasticity of Substitution (IES) on a deterministic consumption path. The utility is defined also for $\gamma^i = 1$ and $\rho^i = 1$ by taking the appropriate limits, see also Epstein and Zin (1989).

Parameters are chosen such that the utility of the aggregate endowment is finite, implying that to the recursive formulation there corresponds an utility over consumption streams, see Assumption A.1 in Appendix A.3.[22] In these cases one can use Euler equations to characterize (interior) equilibrium allocation as a function of market prices, see Appendix A.2. For generic values of the discount factor β^i, the IES coefficient ρ^i, and the RRA coefficient γ^i, Euler equations involving subsequent time periods are coupled, so that the intertemporal and saving decision depends on all future state prices and beliefs , violating **C2**. However, as I show in Proposition A.1 in Appendix A, under specific preferences parametrization, optimal decisions can be recovered from contemporaneous prices and beliefs so that **C2** holds.[23]

For agent i, in date t and history σ_t one finds:

$$\delta_t^i = \delta_t \left(\frac{\beta^i}{\delta_t}\right)^{\frac{1}{\rho^i}} \left(\sum_{s' \in S} (\mathrm{Q}^i_{s'})^{\frac{1}{\gamma^i}} (\mathrm{Q}^0_{s',t})^{1-\frac{1}{\gamma^i}}\right)^{\frac{\gamma^i}{1-\gamma^i}\frac{1-\rho^i}{\rho^i}}, \tag{11}$$

$$\alpha^i_{s,t} = \frac{(\mathrm{Q}^i_s)^{\frac{1}{\gamma^i}} (\mathrm{Q}^0_{s,t})^{1-\frac{1}{\gamma^i}}}{\sum_{s' \in S} (\mathrm{Q}^i_{s'})^{\frac{1}{\gamma^i}} (\mathrm{Q}^0_{s',t})^{1-\frac{1}{\gamma^i}}} \quad \text{for all } s \in \mathcal{S}\,. \tag{12}$$

Intertemporal rates of substitution δ_t depend also on market discount rates, the IES coefficient, and the discount factor. Portfolio decisions α_t depend on beliefs, relative state prices, and the RRA coefficient. Viewed as a function of market prices, we can define the Epstein-Zin intertemporal substitution rule, $\delta(\cdot,\cdot;\beta^i,\rho^i,\gamma^i,\mathrm{Q}^i)$ such that $\delta^i_t = \delta(\delta_t, \mathrm{Q}^0_t; \beta^i, \rho^i, \gamma^i, \mathrm{Q}^i)$, and the Epstein-Zin portfolio rule, $\alpha(\cdot;\gamma^i,\mathrm{Q}^i)$ such that $\alpha^i_{s,t} = \alpha_s(\mathrm{Q}^0_t;\gamma^i,\mathrm{Q}^i)$ for all $s \in \mathcal{S}$. The dependence of intertemporal substitution rule on beliefs and normalized state prices represents a saving under uncertainty component in that it would not be present under no aggregate risk, i.e. if $g_s = g$ for all s, and if agents shared the same beliefs, $\mathrm{Q}^i = \mathrm{Q}$ for all $i \in \mathcal{I}$. The portfolio rule is particularly simple when $\gamma = 1$, leading to $\alpha^i = \mathrm{Q}^i$. Agent i

[22] For example, when the aggregate endowment is growing, a sufficient condition is that $\rho^i > 1$ or, when $\rho^i < 1$, that the agent i discounts future expected utility fast enough.

[23] The special cases are: $\rho^i = 1$; γ^i and Q^i such that agent i holds the market portfolio in equilibrium; the CRRA limit of $\gamma^i = \rho^i$. Despite many parameter specifications are left out, these cases are enough to show that the MSH fails and to shed light on the role of saving and portfolio decisions for market selection purposes.

'bets his beliefs' as in the CRRA log-case ($\gamma = \rho = 1$). For this reason we shall call this portfolio rule the log-optimal one also outside the CRRA framework.

Using rules, the market equilibrium condition (9) implicitly set date-t-history-σ_t state prices depending on the date-t-history-σ_t consumption distribution. In Section 4, I shall use the system of equations (3) and (8-9), with rules (δ, α) for all $i \in \mathcal{I}$, in order to characterize long-run consumption distributions and prices iteratively. In the next section I provide survival results for general rules.

3 Market Selection

We are interested in studying whether, in terms of consumption, a group $\mathcal{J} \subset \mathcal{I}$ survives, vanishes, or dominates. Since aggregate consumption can be unbounded or converge to zero, we can focus on the relative consumption $\Phi_t^{\mathcal{J}} = \sum_{j \in \mathcal{J}} \phi_t^j$. Consistently with the literature define:

Definition 3.1. *Group $\mathcal{J}$ survives on σ if* $\limsup_{t\to\infty} \Phi_t^{\mathcal{J}}(\sigma) > 0$*, he vanishes when* $\lim_{t\to\infty} \Phi_t^{\mathcal{J}}(\sigma) = 0$*, he dominates when* $\lim_{t\to\infty} \Phi_t^{\mathcal{J}}(\sigma) = 1$.

I shall show that intertemporal substitution rates and portfolio expected log-returns can be used to give sufficient conditions for an agent to vanish, survive, or dominate P-a.s.. The general idea is as follows.

Since the consumption dynamics (3) is a multiplicative process, the log-consumption follows an additive process and $z_t^{\mathcal{J}} = \log \frac{\Phi_t^{\mathcal{J}}}{1-\Phi_t^{\mathcal{J}}}$ is an adapted process defined on the real line. In particular the evolution of $\{z_t^{\mathcal{J}}\}$ is governed by

$$z_{t+1}^{\mathcal{J}}(\sigma_{t+1}) = z_t^{\mathcal{J}}(\sigma_t) + \epsilon_{t+1}^{\mathcal{J}}(\sigma_{t+1})\,, \tag{13}$$

where, denoting $-\mathcal{J} = \mathcal{I} \setminus \mathcal{J}$,

$$\epsilon_{t+1}^{\mathcal{J}}(\sigma_{t+1}) = \log \frac{\delta_t^{\mathcal{J}}}{\delta_t^{-\mathcal{J}}} + \log \frac{\alpha_{s,t}^{\mathcal{J}}}{\alpha_{s,t}^{-\mathcal{J}}} \quad \text{when} \quad \sigma_{t+1} = (\sigma_t, s)\,, \quad \text{for all} \quad s \in \mathcal{S}\,.$$

To compute the drift of $\{z_t^{\mathcal{J}}\}$ Denote the relative entropy of Q with respect to P, also Kullback-Leibler divergence, as

$$I_P(Q) = \sum_{s\in\mathcal{S}} \mathrm{P}_s \log \frac{\mathrm{P}_s}{\mathrm{Q}_s}\,,$$

Consistently with the incumbent literature define group $\mathcal{J}$ *generalized survival index* in node σ_t as

$$k_t^{\mathcal{J}} := \log \delta_t^{\mathcal{J}} - I_P(\alpha_t^{\mathcal{J}})\,. \tag{14}$$

In log economies $\delta^i_t = \beta^i$ and $\alpha^i_t = \mathrm{Q}^i$, so that the generalized survival index depends on discount factors and beliefs' accuracy. More in general it depends on *effective discount factors* and *effective beliefs*. This is similar to the approach that has been followed by most of the literature, where survival indexes are derived from the Euler equation of the Pareto optimal allocation problem. Here, instead, survival indexes are derived directly from saving and portfolio decisions.

Although the two types of survival indexes are different, we shall show that the sign of their difference, which determines whether relative growth rates are positive or negative, is the same. From (13) and (14) the conditional drift of the relative consumption process is

$$\mathrm{E}_{\mathrm{P}}[\epsilon^{\mathcal{J}}_{t+1}|\,\mathcal{F}_t] = k^{\mathcal{J}}_t - k^{-\mathcal{J}}_t \,. \tag{15}$$

If group $\mathcal{J}$ has an higher survival index than group $-\mathcal{J}$, the drift of the relative consumption process is in its favor and thus group $\mathcal{J}$ gains, in expectation, consumption.

The proposed survival index (14) has two advantages. First, it can be defined also for groups, rather than only for individuals. It is so because the correct unit of aggregation has been identified, named substitution rates and portfolios of next period consumption. Second, it can be decomposed into the direct effect of saving and portfolio decisions. The importance of saving is evident in the log of substitution rates. It is intuitive that if group $\mathcal{I}$ postpones consumption in date t with respect to the aggregate decision of all the other agent, then his relative consumption has a positive contribution from date t to date $t+1$. To interpret the contribution of portfolio decisions let us compute the return of the portfolio $\alpha^{\mathcal{J}}_t$. Using the relative consumption dynamics (7) one finds that the return in date state $s_t = s$ is

$$r^{\mathcal{J}}_{s,t} := \frac{\sum_{j\in\mathcal{J}} c^j_{t+1}(\sigma_t, s)}{\sum_{j\in\mathcal{J}}\sum_{s'\in\mathcal{S}} q_{s',t} c^j_{t+1}(\sigma_t, s')} = \frac{1}{\delta_t}\frac{\alpha^{\mathcal{J}}_{s,t}}{\mathrm{Q}^0_{s,t}} \,.$$

Group $\mathcal{J}$ expected log-return in date t is thus

$$\mathrm{E}_{\mathrm{P}}[\log r^{\mathcal{J}}_t|\,\mathcal{F}_t] = \log r_t + \mu^{\mathcal{J}}_t \,,$$

where

$$\mu^{\mathcal{J}}_t = I_P(Q^0_t) - I_P(\alpha^{\mathcal{J}}_t) \tag{16}$$

is the expected log-return in excess of the log risk-free rate. In view of his role for the market selection, I denote $\mu^{\mathcal{J}}_t$, as group $\mathcal{J}$ *growth premium* in period t. The growth premium can be itself decomposed into three parts by adding and subtracting two terms, the relative entropy of the group beliefs $I_P(Q^{\mathcal{J}})$, a measure of accuracy of group $\mathcal{J}$, and the relative entropy of the market beliefs $I_P(Q^{\mathcal{I}}_t)$, a measure of accuracy of market beliefs. The growth premium decomposition leads to

$$\mu_t^{\mathcal{J}} = [I_P(Q_t^0) - I_P(Q_t^{\mathtt{J}})] + [I_P(Q_t^{\mathtt{J}}) - I_P(Q_t^{\mathcal{J}})] + [I_P(Q_t^{\mathcal{J}}) - I_P(\alpha_t^{\mathcal{J}})]\,. \qquad (17)$$

The first difference is the excess (log) risk premium of an agents with market beliefs and log-preferences. By definition of market beliefs, this term is zero under no aggregate risk. The second difference is an accuracy premium that rewards (punishes) group $\mathcal{J}$ for being more (less) accurate than the market. It is zero where a group's belief coincide with market's belief (e.g. when it is alone in the economy). The third difference is a Non-Log-Optimality (NLO) term that rewards (punishes) group $\mathcal{J}$ for having effective beliefs that are more (less) accurate than beliefs $\mathrm{Q}^{\mathcal{J}}$. It is zero when all members of the group have a log-optimal portfolio, since in this case effective beliefs $\alpha^{\mathcal{J}}$ and beliefs $\mathrm{Q}^{\mathcal{J}}$ coincide. The NLO contribution measures whether group $\mathcal{J}$ agent is better-off or worse-off, in terms of expected log-returns, by using a non-log optimal rules rather than the log-optimal rule derived under its beliefs. If group $\mathcal{J}$ beliefs are correct, the NLO term is negative since $\mathcal{J}$ would have been better off using a log-optimal portfolio. Effective beliefs are less accurate. However, when group $\mathcal{J}$ beliefs are not correct, effective beliefs could be more accurate than beliefs leading to a positive NLO contribution. NLO terms measure the compensation between risk preferences and beliefs in relation to expected growth.[24]

Using growth premia, the drift of the relative log-consumption dynamics can be written as

$$\mathrm{E}_{\mathrm{P}}[\epsilon_{t+1}^{\mathcal{J}}|\mathcal{F}_t] = \log\frac{\delta_t^{\mathcal{J}}}{\delta_t^{-\mathcal{J}}} + \mu_t^{\mathcal{J}} - \mu_t^{-\mathcal{J}}. \qquad (18)$$

where

$$\mu_t^{\mathcal{J}} - \mu_t^{-\mathcal{J}} = \{I_P(Q_t^{-\mathcal{J}}) - I_P(Q_t^{\mathcal{J}})\} + \{[I_P(Q_t^{\mathcal{J}}) - I_P(\alpha_t^{\mathcal{J}})] - [I_P(Q_t^{-\mathcal{J}}) - I_P(\alpha_t^{-\mathcal{J}})]\}.$$

Other than on the comparison of substitution rates, the expected value of the relative growth depends both on the relative accuracy of groups beliefs, the group with more accurate beliefs having a higher contribution to its expected growth, and on the relative size of NLO contributions. As we shall see, all results in the paper are essentially due to the role of these NLO terms.

Two sets of results shall be derived from the difference of agents' survival indexes in (15) or (18). Using the Law of Large Numbers for uncorrelated martingales, one can state necessary or sufficient conditions in terms of survival indexes time averages.[25]

[24]See also the quote from DeLong et al. (1991) in Section 1.1.
[25]This approach is inspired by Sandroni (2000) (see e.g. his Proposition 3).

Theorem 3.1. *Under the Assumptions 2.1, 2.2, consider an economy with $\mathcal{I}$ agents where a competitive equilibrium exists and is interior. Define substitution rates $\{\delta^i\}$ and portfolios $\{\alpha^i\}$ for all $i \in \mathcal{I}$, $t \in \mathbb{N}_0$ and $\sigma_t \in \Sigma_t$ as in* (2). *Given a set $\mathcal{J} \subset \mathcal{I}$ of agents, if*

$$\text{Prob}\left\{ \lim_{T\to\infty} \sum_{t=1}^{T} \frac{k_t^{\mathcal{J}} - k_t^{-\mathcal{J}}}{T} > 0 \right\} = 1\,,$$

then group $\mathcal{J}$ dominates P-a.s. If instead

$$\text{Prob}\left\{ \lim_{T\to\infty} \sum_{t=1}^{T} \frac{k_t^{\mathcal{J}} - k_t^{-\mathcal{J}}}{T} < 0 \right\} = 1\,,$$

then group $\mathcal{J}$ vanishes.

Two problems arise when applying the theorem. First, since generalized survival indexes depend, through substitution rates and growth premia, on equilibrium prices and discount factors, these sufficient conditions can be evaluated analytically only under special assumptions. One example is homogeneous CRRA economies, where the trade-off between saving and portfolio decisions does not depend on market clearing price, see e.g. Proposition 4.2. Second, the theorem does not provide sufficient conditions for survival but only for dominance and vanishing. Assume for example that there exists an economy where it can be established that two groups have the same average survival index. It could be so for two reasons, either because dominance of one group is too slow (slower than exponential) or, rather, because both groups are doing equally well and survive. In fact, the following corollary can be established.

Corollary 3.1. *Under the Assumptions of Theorem 3.1 if both groups $\mathcal{J}$ and $-\mathcal{J}$ survive P-a.s, then their average generalized survival indexes are equal P-a.s..*

The Corollary is particularly instructive when agents have the same substitution rules, since in this case all surviving agents must have effective beliefs that are equally accurate. A specific example is in Section 4.1.

To establish sufficient conditions for survival it is enough to evaluate substitution rates and growth premia in the limit of one group consuming all the endowment. For the generic subset $\mathcal{J}$ of $\mathcal{I}$, given the relative consumption $\Phi^{\mathcal{J}} \in [0,1]$ and relative consumption distributions $\phi^{\mathcal{J}} \in \Delta^J$ and $\phi^{-\mathcal{J}} \in \Delta^{-J}$, denote

$$\delta^{\mathcal{J}}(\phi^{\mathcal{J}}, \phi^{-\mathcal{J}}, \Phi^{\mathcal{J}}) \quad \text{and} \quad \alpha^{\mathcal{J}}(\phi^{\mathcal{J}}, \phi^{-\mathcal{J}}, \Phi^{\mathcal{J}})$$

the substitution and portfolio decisions used by group $\mathcal{J}$ when market equilibrium prices and discount rates are determined by groups $\phi^{\mathcal{J}}$ and $\phi^{-\mathcal{J}}$ with relative size given by $\Phi^{\mathcal{J}}$. When group $\mathcal{J}$ has all the relative consumption we write[26]

$$\delta^{\mathcal{J}}|_{\mathcal{J}} = \delta^{\mathcal{J}}(\phi^{\mathcal{J}}, \phi^{-\mathcal{J}}, \Phi^{\mathcal{J}} = 1) \quad \text{and} \quad \alpha^{\mathcal{J}}|_{\mathcal{J}} = \alpha^{\mathcal{J}}(\phi^{\mathcal{J}}, \phi^{-\mathcal{J}}, \Phi^{\mathcal{J}} = 1)\,.$$

$k^{\mathcal{J}}|_{\mathcal{J}}$ is the corresponding generalized survival index. Similarly $\delta^{\mathcal{J}}|_{-\mathcal{J}}$ and $\alpha^{\mathcal{J}}|_{-\mathcal{J}}$ are the rules used by group $\mathcal{J}$ when $\Phi^{\mathcal{J}} = 0$. The following theorem gives sufficient conditions for survival of group $\mathcal{J}$ in terms of the sign of $k^{\mathcal{J}}|_{-\mathcal{J}} - k^{-\mathcal{J}}|_{-\mathcal{J}}$.

Theorem 3.2. *Under the same assumptions of Theorem 3.1, assume further that the process $\{z^{\mathcal{J}}\}$ is bounded* [27] *and that its conditional drift $\mathrm{E_P}[\epsilon^{\mathcal{J}}_{t+1}|z^{\mathcal{J}}_t = z\,, \mathcal{F}_t]$ is continuous in z. If*

$$\min_{\phi^{\mathcal{J}} \in \Delta^J, \phi^{-\mathcal{J}} \in \Delta^{-J}} \left\{ k^{\mathcal{J}}|_{-\mathcal{J}} - k^{-\mathcal{J}}|_{-\mathcal{J}} \right\} > 0\,,$$

then group $\mathcal{J}$ survives P-a.s.. If furthermore

$$\max_{\phi^{\mathcal{J}} \in \Delta^J, \phi^{-\mathcal{J}} \in \Delta^{-J}} \left\{ k^{\mathcal{J}}|_{\mathcal{J}} - k^{-\mathcal{J}}|_{\mathcal{J}} \right\} < 0\,,$$

then both groups survive P-a.s..

For a group of agents to survive it is sufficient to have the sum of log substitution rate and expected log returns larger than the rest of the agents, at the state prices set by the latter and for all possible distribution of consumption within groups. The result is rather intuitive. Using "limit" generalized survival indexes it is also possible to establish when a group vanishes or dominates.

Theorem 3.3. *Under the assumption of Theorem 3.2, consider a set $\mathcal{J} \subset \mathcal{I}$ of agents and assume further that the process has finite positive and negative increments.*[28] *If*

$$\min_{\phi^{\mathcal{J}} \in \Delta^J, \phi^{-\mathcal{J}} \in \Delta^{-J}} \left\{ k^{\mathcal{J}}|_{-\mathcal{J}} - k^{-\mathcal{J}}|_{-\mathcal{J}} \right\} > 0 \quad \textit{and} \quad \min_{\phi^{\mathcal{J}} \in \Delta^J, \phi^{-\mathcal{J}} \in \Delta^{-J}} \left\{ k^{\mathcal{J}}|_{\mathcal{J}} - k^{-\mathcal{J}}|_{\mathcal{J}} \right\} > 0\,,$$

then group $\mathcal{J}$ dominates P-a.s.; if

$$\max_{\phi^{\mathcal{J}} \in \Delta^J, \phi^{-\mathcal{J}} \in \Delta^{-J}} \left\{ k^{\mathcal{J}}|_{-\mathcal{J}} - k^{-\mathcal{J}}|_{-\mathcal{J}} \right\} < 0 \quad \textit{and} \quad \max_{\phi^{\mathcal{J}} \in \Delta^J, \phi^{-\mathcal{J}} \in \Delta^{-J}} \left\{ k^{\mathcal{J}}|_{\mathcal{J}} - k^{-\mathcal{J}}|_{\mathcal{J}} \right\} < 0\,,$$

[26]Note that both $\delta^{\mathcal{J}}|_{\mathcal{J}}$ and $\alpha^{\mathcal{J}}|_{\mathcal{J}}$ still depend on $\phi^{\mathcal{J}}$ and $\phi^{-\mathcal{J}}$. The dependence is omitted in order to ease notation.

[27]The theorem requires a bounded process in the sense that with full probability the realized innovation $\epsilon^{\mathcal{J}}_t$ should be bounded from above and from below. This is typically the case when agents speculative position are bounded as well. See Appendix E.

[28]The process has finite positive and negative increments when with positive probability $\varepsilon > 0$ it increases of at least ε and it decreases with at least ε. See Appendix E.

then group $\mathcal{J}$ vanishes P-a.s.; if

$$\max_{\phi^{\mathcal{J}}\in\Delta^J,\phi^{-\mathcal{J}}\in\Delta^{-J}} \left\{k^{\mathcal{J}}|_{-\mathcal{J}} - k^{-\mathcal{J}}|_{-\mathcal{J}}\right\} < 0 \quad \textit{and} \qquad \min_{\phi^{\mathcal{J}}\in\Delta^J,\phi^{-\mathcal{J}}\in\Delta^{-J}} \left\{k^{\mathcal{J}}|_{\mathcal{J}} - k^{\mathcal{J}}|_{-\mathcal{J}}\right\} > 0\,,$$

then P-a.s. either group $\mathcal{J}$ or group $-\mathcal{J}$ dominates.

Both Theorem 3.2 and Theorem 3.3 rely on applications of the martingale convergence theorem as used in Bottazzi and Dindo (2015). In economies with more than 2 agents, they provide only weak sufficient conditions as, even when one group has null relative consumption, the sign of the survival index difference is likely to depend on the exact consumption distribution within both groups. In 2-agent economies, however, groups are uniquely identified so that, provided agents' "limit" survival indexes are not identical, one of the four sign combinations of the two theorems applies: Sufficient conditions set by Theorems 3.2-3.3 become tight.

4 Selection in Epstein-Zin Economies

In this section I apply Theorems 3.2-3.3 to an exchange economy where substitution rates and portfolio decisions are derived from the maximization of an Epstein-Zin recursive utility. This allows use to consider exchange economies where, in equilibrium,

i) all agents use the same intertemporal substitution rate but hold different portfolios -so that for all $\mathcal{J}$ $\log \frac{\delta_t^{\mathcal{J}}}{\delta_t^{-\mathcal{J}}} = 0$ for all t and σ_t, but, generically, $\mu_t^{\mathcal{J}} \neq \mu_t^{-\mathcal{J}}$;

ii) at least one agent, say i, has interdependent intertemporal and risk preferences, so that $\log \delta_t^i$ and μ_t^i are tied together and have common terms;

iii) all agents hold the market portfolio but use different intertemporal substitution rates so that for all $\mathcal{J}$, t and σ_t, $\mu_t^{\mathcal{J}} = \mu_t^{-\mathcal{J}}$ but, generically, $\log \frac{\delta_t^{\mathcal{J}}}{\delta_t^{-\mathcal{J}}} \neq 0$.

Given *i*), we can analyze the property of the long-run consumption dynamics when only growth premia matter. This is the content of Section 4.1. MSH failures are generic. Due to *ii*), we can show that in CRRA economies there exists an exact compensation between the difference of growth premia and log substitution rates. As a result only vanishing of the accurate trader is possible and it is due to the saving component. This is the content of Section 4.2. However, in Section 4.3, we show that if at least one agent has not CRRA preferences all failures are still possible. Given *iii*), we can move to analyze market selection when only saving behavior matter, see Section 4.4. Again, only vanishing of the accurate trader occurs. Despite beliefs heterogeneity matter for long-run outcomes, the relative consumption dynamics is deterministic and the truth has no role.

4.1 Selection of Portfolios

When all agents have the same IES parameter $\rho^i = 1$ and discount rate $\beta^i = \beta$, agents choose the same intertemporal substitution rate in equilibrium: $\delta^i_t = \beta$ for all i, t and σ_t. As a result the comparison of generalized survival indexes as in (15) is a comparison of growth premia:[29]

$$k^{\mathcal{J}}_t - k^{-\mathcal{J}}_t = \mu^{\mathcal{J}}_t - \mu^{-\mathcal{J}}_t \,. \tag{19}$$

When i is the representative agent (homogeneous preferences and beliefs economy, hedging is the only motive behind trade), date t normalized state prices and market discount factors are given by

$$\begin{aligned} \mathrm{Q}^0_{s,t}\,|_i &= \frac{\mathrm{Q}^i_s\,\hat{g}_s^{-\gamma^i}}{\sum_{s'\in\mathcal{S}}\mathrm{Q}^i_{s'}\,\hat{g}_{s'}^{-\gamma^i}}\,, \quad \text{for all } s\,, \\ \delta_t|_i &= = \beta e^{-\,\mathrm{E_P}[\log g]}\,\frac{\sum_{s\in\mathcal{S}}\mathrm{Q}^i_s\,\hat{g}_s^{-\gamma^i}}{\sum_{s\in\mathcal{S}}\mathrm{Q}^i_s\,\hat{g}_s^{1-\gamma^i}} = \frac{\beta}{\sum_{s\in\mathcal{S}}\mathrm{Q}^0_{s,t}\,g_s}\,, \end{aligned} \tag{20}$$

where $\hat{g}$ is the vector of de-trended growth rates as in (1). Date t equilibrium saving and portfolio decisions are

$$\begin{aligned} \alpha^i_{s,t}|_i &= \frac{\mathrm{Q}^i_s\,\hat{g}_s^{1-\gamma^i}}{\sum_{s\in\mathcal{S}}\mathrm{Q}^i_{s'}\,\hat{g}_{s'}^{1-\gamma^i}} \quad \text{for all } s\,, \\ \delta^i_t|_i &= \beta\,. \end{aligned} \tag{21}$$

Equilibrium discount rates, normalized state prices, and portfolio decisions are particularly simple when there is no aggregate risk, $g_s = g$ for all $s \in \mathcal{S}$, leading to and $\mathrm{Q}^0_t = \alpha^i_t|_i = \mathrm{Q}^i$ and $\delta_t|_i = \frac{\beta}{g}$.

Under heterogeneous beliefs and, possibly, RRA coefficients, equilibrium market discount rates and normalized state prices do instead depend on the contemporaneous consumption distribution. A simplification of (9) occurs because all agents are saving at the same rate. Agents' price impacts φ and relative consumption weights ϕ coincide so that

$$\alpha^{\mathcal{I}}_t(\mathrm{Q}^0_t) = \sum_{i\in\mathcal{I}}\alpha^i_{s,t}(\mathrm{Q}^0_t)\phi^i_t \quad \text{for all}\, s \in \mathcal{S}\,,$$

while the market discount factor is

$$\delta_t = \frac{\beta}{\sum_{s\in\mathcal{S}}\mathrm{Q}^0_{s,t}\,g_s}\,.$$

It is instructive to consider the well-known case of a log-economy, $\gamma^i = \gamma = 1$ for all agents, first.

[29] When the IES parameter ρ^i is one, the one-period decisions δ^i_t and α^i_t coincide with the 'full' saving and portfolio decisions $\bar{\delta}^i_t$ and $\bar{\alpha}^i_t$, see also Appendix A.

Log-Optimal Portfolios The relative consumption dynamics is particularly simple when $\gamma = 1$, as each agent "bets" his own beliefs. Aggregate portfolio and aggregate beliefs coincide and are a convex combination of all agents beliefs. In an economy with I agent consider the relative performance of agent i and agent j. The difference of growth premia is:

$$\mu_t^i - \mu_t^j = I_P(Q^j) - I_P(Q^i)\,. \tag{22}$$

If agent i has more accurate beliefs, then he has a larger growth premium in every period. The dynamics of the log consumption ratio of i and j has positive drift, implying that agent j vanishes with respect to agent i (equivalently, agent i dominates with respect to agent j).[30] If, moreover, agent i has the most accurate beliefs with respect to any other agent, then he also dominates against all of them in a I-agent economy. In a log-economy speculation enables the agent with the most accurate beliefs to play a favorable game of chance in every period. The result is well known. For equilibrium economies it goes back at least to Blume and Easley (1992).[31]

Non Log-optimal Portfolios When agents portfolio rule are not log-optimal, portfolio choices do not correspond to beliefs. It is still convenient to evaluate agents' portfolios through the lenses of log-optimality. Using the decomposition (17) the difference of agent i and agent j growth premia is

$$\mu_t^i - \mu_t^j = \left\{ I_P(Q^j) - I_P(Q^i) \right\} + \left\{ \left[I_P(Q^i) - I_P(\alpha_t^i) \right] - \left[I_P(Q^j) - I_P(\alpha_t^j) \right] \right\}\,.$$

The first part depends on the relative accuracy of beliefs, as in log-economies. When RRA is not 1, however, the difference of growth premia depends also on the difference of endogenously determined NLO terms.

Exploiting Theorems 3.2-3.3, it is sufficient to characterize the relative portfolio performance at the prices set by each agent in isolation to establish if an agent survives, vanishes, or dominates. Denoting the NLO contribution of agent i as ν_t^i, in the limit of agent i having all the consumption, from (12) and (20) one finds

$$\nu^i|_i - \nu^j|_i = \frac{1-\gamma^j}{\gamma^j}\left(I_P(Q^j) - I_P(Q^i) \right) + \Delta^{i,j}|_i \tag{23}$$

[30] A version of Theorem (3.1) can also be stated for the relative performance of subsets of $\mathcal{J}$ and $\mathcal{H}$ of $\mathcal{I}$ that do not form a partition of $\mathcal{I}$. In this case the process is $\{z^{\mathcal{J},\mathcal{H}} = \log \frac{\Phi^{\mathcal{J}}}{\Phi^{\mathcal{H}}}\}$ and dominance is relative to the other group.

[31] The result is straightforward when beliefs are distinct as assumed in 2.2. It is more subtle to establish when beliefs are not uniformly bounded away from each-others, for example when more agents learn the correct probabilities but with different speed of convergence. See also Sandroni (2000), Blume and Easley (2006), and Massari (2015, 2016).

where

$$\Delta^{i,j}|_i = \log \frac{\sum_{s\in\mathcal{S}} (\mathrm{Q}_s^j\, \hat{g}_s^{1-\gamma^j})^{\frac{1}{\gamma^j}} (\mathrm{Q}_s^i\, \hat{g}_s^{1-\gamma^i})^{1-\frac{1}{\gamma^j}}}{\sum_{s\in\mathcal{S}} \mathrm{Q}_s^i\, \hat{g}_s^{1-\gamma^i}} \,. \tag{24}$$

Adding the contribution due to beliefs relative accuracy, the difference of generalized survival indexes when agent i dominates can be written as

$$k^i|_i - k^j|_i = \mu^i|_i - \mu^j|_i = \frac{1}{\gamma^j}\left(I_P(Q^j) - I_P(Q^i)\right) + \Delta^{i,j}|_i \,.$$

Even if agent i has correct beliefs, $\mathrm{Q}^i = \mathrm{P}$, $\Delta^{i,j}|_i$ could still be so negative to imply a higher portfolio expected log-return for agent j at the prices determined by agent i. The same holds for $\mu^i|_j - \mu^j|_j$, which can be found by interchanging the role of agent i and j. Not only $\mu^i|_i - \mu^j|_i$ can be negative even if i has correct beliefs but, also, the signs of $\mu^i|_i - \mu^j|_i$ and $\mu^i|_j - \mu^j|_j$ can be different. The following proposition is an application of Theorems 3.2-3.3 to these Epstein-Zin 2-agent economies. I state the result by comparing the relative accuracy of beliefs with $\Delta^{i,j}|_i$ and $\Delta^{i,j}|_j$.

Proposition 4.1. *Under the Assumptions 2.1, 2.2, A.1, consider the equilibrium paths of an economy with two agents, i and j, maximizing an Epstein-Zin utility with $\rho^i = \rho^j = 1$ and $\beta^i = \beta^j = \beta$.*

i) If

$$\gamma^i \Delta^{j,i}|_j < I_P(Q^j) - I_P(Q^i) < -\gamma^j \Delta^{i,j}|_i \,,$$

then $k^i|_j - k^j|_j > 0$, $k^i|_i - k^j|_i < 0$ and both agents survive P-almost surely.

ii) If

$$-\gamma^j \Delta^{i,j}|_i < I_P(Q^j) - I_P(Q^i) < \gamma^i \Delta^{j,i}|_j \,,$$

then $k^i|_j - k^j|_j < 0$, $k^i|_i - k^j|_i > 0$ and there exists two sets Γ^+ and Γ^- with $\mathrm{P}(\Gamma^+ \cup \Gamma^-) = 1$ such that agent i dominates on σ when $\sigma \in \Gamma^+$ and agent j dominates on σ when $\sigma \in \Gamma^-$.

iii) If

$$I_P(Q^j) - I_P(Q^i) > \gamma^i \Delta^{j,i}|_j \quad \textit{and} \quad I_P(Q^j) - I_P(Q^i) > -\gamma^j \Delta^{i,j}|_i \,,$$

then $k^i|_j - k^j|_j > 0$, $k^i|_i - k^j|_i > 0$ and agent i dominates P-almost surely. Likewise, if both reversed inequalities hold, so that $k^i|_j - k^j|_j < 0$ and $k^i|_i - k^j|_i < 0$, then agent j dominates P-almost surely.

For a given relative beliefs' accuracy, the endogenous component of the NLO term can be such that both agents survive, meaning that disagreement is persistent; both dominate, but on different paths; or only one agent dominates, but not necessarily the one with most accurate beliefs. Assume trader i has less accurate beliefs than trader j. The reason behind the survival, or even dominance, of a trader i is that his NLO compensation can be larger than the corresponding compensation of agent j. Stated in different terms, the accuracy of effective beliefs, which depend also on preferences and equilibrium prices, can overturn the accuracy premium. Overall, agent i could hold a portfolio closer to the growth optimal portfolio than the portfolio held by agent j. It is enough to check for the above to happen at the prices set by either agent consuming all the endowment in one period to say whether i dominates on almost all paths, survives on almost all paths, or dominates on a set of paths with positive measure. I provide a graphical representation of all possible outcomes using a supply and demand plot in Section 5.

A restriction on the possible long-run dynamics occurs when agents have homogeneous risk preferences γ. The following corollary relies on the fact that under no-aggregate risk, or with aggregate risk and $S = 2$, the difference of NLO compensations can be ordered.[32] When $\gamma > 1$, agent i cannot have a higher NLO term at the prices determined by j than he has at his prices, thus excluding that both agents dominate on different path. When $\gamma < 1$, agent i cannot have a higher NLO term at the prices he determines than he has at the prices determined by j, thus excluding that both agents survive and that beliefs disagreement is persistent. The following corollary proves the statement.

Corollary 4.1. *Under the assumption of Proposition 4.1, assume $\gamma^i = \gamma^j = \gamma$, no aggregate risk, or aggregate risk but $S = 2$. If $\gamma > 1$, then only cases i) and iii) are possible. If instead $\gamma < 1$, then only cases ii) and iii) are possible. If otherwise $\gamma = 1$, and $I_P(Q^i) \neq I_P(Q^j)$, only case iii) is possible.*

In homogeneous risk aversion economies, for all γ, either agent could dominate almost surely. However, survival of both agents can only occur when they have less risky portfolio than log-optimal ones. This is because risk aversion implies conservative positions even in presence of non accurate beliefs. More risk averse portfolios with incorrect beliefs tend to be close to log-optimal portfolios with correct beliefs at the prices set by the other agent, thus leading to accurate effective beliefs. On the contrary, path dependency can only occur when agents hold more risky portfolio than log-optimal ones. Non accurate beliefs leads to extreme portfolios. Less risk averse portfolios with incorrect beliefs tend to be very far from the log-optimal portfolio with correct beliefs at the prices set by the other agent,

[32]The case of $S = 2$, a binomial tree economy, is the one exploited in the examples of Section 5.

thus leading to less accurate effective beliefs. Beliefs heterogeneity is persistent in the first case and transient in the second.

Finally, the next corollary is another application of Proposition 4.1 that addresses the fate of an agent with correct beliefs.

Corollary 4.2. *Under the assumption of Proposition 4.1, assume that agent i has correct beliefs,* $\mathrm{Q}^i = \mathrm{P}$.

i) If agent i has $\gamma^i = 1$, then he dominates P*-almost surely*

ii) If the economy has no aggregate risk, then either i dominates P*-almost surely or case ii) of Proposition 4.1 can occur.*

iii) Otherwise, any of the cases of Proposition 4.1 can occur.

Since the log-optimal portfolio derived under correct beliefs guarantees the highest growth-premium for all prices, an agent who use this portfolio dominates almost surely.[33] When instead the agent with correct beliefs does not use the log-optimal rule, anything can happen. Not only can he vanish, but there are also cases where he is not the only survivor and beliefs heterogeneity is persistent. However, agents' co-existence can never occur when there is no-aggregate risk. The reason is that in such an economy, if agent i has correct beliefs the equilibrium portfolio he holds in the limit of having all the consumption is also log-optimal (both imply fair pricing under no aggregate risk), leading to correct effective beliefs $\alpha^i|_i = \mathrm{P}$. As a result $\mu^i|_i - \mu^j|_i$ is always positive and by having a higher growth premium at the returns he sets, both long-run heterogeneity and almost sure vanishing never occur.[34]

4.2 Selection in CRRA Economies

I turn to analyze the outcome of selection when agents not only hold different portfolios but also differ in how they transfer consumption intertemporally. I start with CRRA economies, for which it is known that only one agent dominates generically, see Sandroni (2000) and Blume and Easley (2006) for bounded economies and Yan (2008) for unbounded economies. I illustrate how all their results emerge in terms of substitution and portfolio decisions, and how they can be generalized.

[33]The result is well known at least since Kelly (1956). It was first extended to economies where prices are set in equilibrium by Theorem 5.1 of Blume and Easley (1992). See also the discussion after Proposition 1 in Sandroni (2000).

[34]When $\rho = 1$ the economy is equivalent to one where saving is exogenously fixed to β and portfolio are chosen myopically. With this respect the possibility that an agent with correct beliefs vanishes P-almost surely in economies with aggregate risk is equivalent to Theorem 5.4 of Blume and Easley (1992). All other MSH failures are new.

Throughout this section I assume that for all $i \in \mathcal{I}$ the RRA coefficient γ^i and the IES coefficient ρ^i coincide, leading to substitution and portfolio decisions that are optimal for a CRRA agent with RRA coefficient γ^i.

When agent i is the representative agent (homogeneous preferences and beliefs), the joint solution of (9) and (11-12) leads to

$$
\begin{aligned}
\mathrm{Q}^0_{s,t}\,|_i &= \frac{\mathrm{Q}^i_s\,\hat{g}_s^{-\gamma^i}}{\sum_{s'\in\mathcal{S}}\mathrm{Q}^i_{s'}\,\hat{g}_{s'}^{-\gamma^i}}\,, \quad \text{for all } s\,,\\
\delta_t|_i &= \beta^i e^{-\gamma^i\,\mathrm{E_P}[\log g]}\sum_{s\in\mathcal{S}}\mathrm{Q}^i_s\,\hat{g}_s^{-\gamma^i} = \frac{\beta e^{-\gamma^i\,\mathrm{E_P}[\log g]}}{\sum_{s\in\mathcal{S}}\mathrm{Q}^0_{s,t}\,\hat{g}_s^{\gamma^i}}\,.
\end{aligned}
\tag{25}
$$

Agent i equilibrium saving and portfolio decisions are

$$
\begin{aligned}
\delta^i_t|_i &= \beta^i e^{(1-\gamma^i)\,\mathrm{E_P}[\log g]}\sum_{s\in\mathcal{S}}\mathrm{Q}^i_s\,\hat{g}_s^{1-\gamma^i}\,,\\
\alpha^i_{s,t}|_i &= \frac{\mathrm{Q}^i_s\,\hat{g}_s^{1-\gamma^i}}{\sum_{s'\in\mathcal{S}}\mathrm{Q}^i_{s'}\,\hat{g}_{s'}^{1-\gamma^i}}\,, \quad \text{for all } s\,.
\end{aligned}
\tag{26}
$$

The situation is different when agents have heterogeneous beliefs. I consider economies with homogeneous preferences first.

4.2.1 Homogeneous γ

When agents have heterogeneous beliefs and discount factors, but have the same RRA coefficient, the difference of agent i and agent j NLO terms computed at the generic set of prices Q^0_t is

$$
\nu^i_t - \nu^j_t = \left(\frac{1-\gamma}{\gamma}\right)\big(I_P(Q^j) - I_P(Q^i)\big) + \log\frac{\sum_{s\in\mathcal{S}}(\mathrm{Q}^j_s)^{\frac{1}{\gamma}}(\mathrm{Q}^0_{s,t})^{1-\frac{1}{\gamma}}}{\sum_{s\in\mathcal{S}}(\mathrm{Q}^i_s)^{\frac{1}{\gamma}}(\mathrm{Q}^0_{s,t})^{1-\frac{1}{\gamma}}}\,.
$$

As we have shown in the previous section, for a given relative beliefs' accuracy there could be prices for which agent i has higher growth premium and prices where the opposite occurs. Preferences play a role. (The result applies also here since, for a given γ, CRRA portfolio decisions and Epstein-Zin portfolio decisions coincide.)

Given the difference of generalized survival indexes in (15), in order to establish long-run outcomes we should complement the analysis of portfolio decisions with the analysis of saving. Give the CRRA intertemporal substitution rules in (11), the log-ratio of agent i to agent j substitution rate is

$$
\log\frac{\delta^i_t}{\delta^j_t} = \frac{1}{\gamma}\log\frac{\beta^i}{\beta^j} + \log\frac{\sum_{s\in\mathcal{S}}(\mathrm{Q}^i_s)^{\frac{1}{\gamma}}(\mathrm{Q}^0_{s,t})^{1-\frac{1}{\gamma}}}{\sum_{s\in\mathcal{S}}(\mathrm{Q}^j_s)^{\frac{1}{\gamma}}(\mathrm{Q}^0_{s,t})^{1-\frac{1}{\gamma}}}\,.
$$

Other than by the discount factor and IES coefficient $1/\gamma$, the comparison of agents' substitution rates ratio depends also on beliefs Q^i, Q^j and normalized state prices Q^0. This last terms reflect how agents adjust their saving in speculative markets. As a function of Q^0,

$$\sum_{s\in\mathcal{S}}(\mathrm{Q}^i_s)^{\frac{1}{\gamma}}(\mathrm{Q}^0_{s,t})^{1-\frac{1}{\gamma}}$$

has a maximum of 1 in Q^i when $\gamma > 1$ and a minimum of 1 in Q^i when $\gamma < 1$. When $\gamma < 1$ an agent postpones consumption from one period to the next whenever normalized state prices do not coincides with his beliefs. The opposite occurs when $\gamma > 1$.

Importantly, the comparison of the saving under uncertainty terms depends on normalized state prices and individual beliefs through a term that matches exactly the price dependent part of the NLO term. Given that the two terms off-set each others, the difference of generalized survival indexes is determined, for all t and σ_t, only by (exogenously given) discount factors and beliefs:

$$k^i_t - k^j_t = \frac{1}{\gamma}\left(\log\frac{\beta^i}{\beta^j} + I_P(Q^j) - I_P(Q^i)\right) .$$

CRRA economies with homogeneous preferences behave, as market selection is concerned, as log-economies: controlling for discount factors only beliefs' accuracy matters.[35]

We have explained why the comparison of generalized survival indexes can be given in terms of the survival index defined in Blume and Easley (2006),

$$k^i_{BE} = \log\beta^i - I_P(Q^i) .$$

In fact, although k_t and k_{BE} differ, it is the sign of $k^i_t - k^j_t$ that matters and, as we have just showed, this sign is equal to the sign of $k^i_{BE} - k^j_{BE}$ for all t and σ_t. Note however that the RRA coefficient still matters for the relative consumption dynamics in that it determines the speed of convergence.[36] We recover the following result as an application of Theorem 3.1:[37,38]

Proposition 4.2. *Under the Assumptions 2.1, 2.2, A.1, consider the equilibrium paths of a CRRA economy with I agents where $\gamma^i = \gamma$ for all $i \in \mathcal{I}$. If there exists*

[35] In fact, in all these economies market selection works as Bayesian learning on the set of agents beliefs, see Massari (2016).

[36] Note that the speed of convergence has direct effects on long-run survival in large economies, see Massari (2015)

[37] See Section 3.1 of Blume and Easley (2006) for the same result in an economy with bounded aggregate endowment. The result holds also when the aggregate endowment is not bounded and the growth process is i.i.d., as shown by Yan (2008).

[38] When discount factors and beliefs are such that survival indexes are equal results are more subtle, see Blume and Easley (2009). These cases are however non-generic.

an agent, say i, such that

$$k^i_{BE} > k^j_{BE} \quad \textit{for all } j \neq i\,,$$

then for all t, σ_t, $k^i_t - k^j_t > 0$ for all $j \neq i$, and agent i dominates P*-almost surely. In particular, if agent i is the only agent with correct beliefs, $\mathrm{Q}^i = \mathrm{P}$, and if he has $\beta^i \geq \beta^j$ for all $j \neq i$, then he dominates* P*-almost surely.*

If the agent with most accurate beliefs has also the largest discount factors, then he dominates. Note, however, that the dominance is due by the aggregate effect of saving and portfolio decision. Whereas in log-economies, the agent with the most accurate beliefs dominates due to his portfolio, in non-log economies portfolios may not always reward the agent with most accurate beliefs. Whether growth premia favor the agent with the most correct beliefs depends on both beliefs and risk preferences of all the other agents, as we have pointed out in Section 4.1. Dominance still occur but the differentiated saving is crucial to the result. In other words, despite log-economies and CRRA non-log economies are equivalent in terms of the long-run outcome of the relative consumption process, economically how those long-run outcomes are achieved is rather different. I provide a specific example in Section 5.

Note also that the same result of Proposition 4.2 can be established also for more general aggregate endowment processes than we assume in Assumption 2.1, such as an unbounded economy with non i.i.d. growth. In fact, despite the growth process influences both equilibrium growth premia and log substitution rates, we have just shown that its impact drops out in their sum.

4.2.2 Heterogeneous γ

When preferences are heterogeneous the difference of log substitution rates and log-optimality premia depends on state prices and, thus, on agents' consumption distribution. However, it is still possible to characterize the long-run dynamics based only on (exogenous) agents' characteristics by computing the relative effect of saving and portfolio decisions in the limit of one agent having most of consumption in one period.

As in Section 4.1, I concentrate on two-agent economies, and denote agents with i and j. In the limit of agent i having all of the consumption the difference of portfolio log-optimality premia coincides with (23). The ratio of intertemporal substitutions is instead given by

$$\frac{\delta^i}{\delta^j}\Big|_i = \left(\frac{\beta^i}{\beta^j} e^{(\gamma^j - \gamma^i)\,\mathrm{E}_{\mathrm{P}}[\log g]}\right)^{\frac{1}{\gamma^j}} \frac{\sum_{s\in\mathcal{S}} \mathrm{Q}^i_s\, \hat{g}_s^{1-\gamma^i}}{\sum_{s\in\mathcal{S}} (\mathrm{Q}^j_s\, \hat{g}_s^{1-\gamma^j})^{\frac{1}{\gamma^j}} (\mathrm{Q}^i_s\, \hat{g}_s^{1-\gamma^i})^{1-\frac{1}{\gamma^j}}}\,. \tag{27}$$

As with homogeneous preferences, the price dependent component of the NLO terms in (24) and the saving component that incorporates beliefs heterogeneity cancel out. As a result

$$k^i|_i - k^j|_i = \frac{1}{\gamma_j}\left(\log\frac{\beta^i}{\beta^j} + I_P(Q^j) - I_P(Q^i) + (\gamma^j - \gamma^i)\,\mathrm{E}_{\mathrm{P}}[\log g]\right) . \tag{28}$$

In particular, asymptotic drifts depend on beliefs only through their accuracy. In this case, however, the drift depends also on the expected log growth rate of the economy. Defining the modified survival index

$$k^i_Y = \log\beta^i - I_P(Q^i) - \gamma^i\,\mathrm{E}_{\mathrm{P}}[\log g] \tag{29}$$

one obtains

$$k^i|_i - k^j|_i = \frac{1}{\gamma^j}\left(k^i_Y - k^j_Y\right) \quad \text{and} \quad k^i|_j - k^j|_j = \frac{1}{\gamma^i}\left(k^i_Y - k^j_Y\right) .$$

The modified survival index, which is the equivalent to the one established by Yan (2008) for continuous-time economies, takes into account discount factors, beliefs, and IES/RRA coefficients γ. The latter matters for survival when $\mathrm{E}_{\mathrm{P}}[\log g] \neq 0$ because it influences an agent's substitution rate, as it is evident from (27). When $\mathrm{E}_{\mathrm{P}}[\log g]$ is positive, a large IES (low γ) denotes a high propensity to transfer consumption to future dates and it is thus advantageous for survival.

Similarly to Proposition 4.1, it is only the 'asymptotic' drift, whose sign is equal to the difference of modified survival indexes k_Y, that determines who dominates.[39] Generically, no long-run heterogeneity of beliefs is possible.[40]

Proposition 4.3. *Under the Assumptions 2.1, 2.2, A.1, consider the equilibrium paths of a CRRA economy with two agents, i and j. If*

$$k^i_Y > k^j_Y ,$$

then $k^i|_j - k^j|_j > 0$, $k^i|_i - k^j|_i > 0$, and agent i dominates. In particular, if agent j has correct beliefs and agent i has non correct beliefs but

$$I_P(Q^i) < \log\frac{\beta^i}{\beta^j} + (\gamma^j - \gamma^i)\,\mathrm{E}_{\mathrm{P}}[\log g] ,$$

then agent j vanishes.

[39] Also in these cases however, preferences, in particular those of the negligible agent, still matter for the speed of convergence.

[40] Previous works establish the result also in I-agent economies, see Sandroni (2000), Blume and Easley (2006), Yan (2008), or even in economies with a continuous of agents, see Massari (2015).

As with homogeneous preferences economies, the derivation of the survival index clarifies the relative importance of portfolio and saving decisions. As with Proposition 4.2, the proposition can be generalized for economies where the growth process is not i.i.d.. In this case the survival index becomes time dependent and the relative importance of having a lower IES or having accurate beliefs changes over time. For example, provided discounts rate are equal, only one of the two factors matters for survival in the limit cases of $\mathrm{E_P}[\log g(t)]$ converging to zero or diverging (provided the equilibrium is still well defined).

I appendix B I discuss the applicability of these results to other expected time-separable utilities.

4.3 CRRA and (IES$= 1$, RRA$\neq 1$) Economies

In an economy where not all agents have CRRA preferences, the compensation between saving under uncertainty and NLO terms should not occur. As a result all MSH failures should be possible. I show that this is indeed the case by studying an economy with a CRRA agent, agent i, and an Epstein-Zin agent with IES 1, agent j. In particular, I shall exploit the results from the previous section and compare agent j to a CRRA investors with same RRA coefficient and beliefs, an investor that would hold the same one period portfolio α, but who uses a different substitution rule. The portfolio analysis is thus the same as for CRRA economies, implying that $\nu^i|_i - \nu^j|_i$ is as in (23), and we can focus the attention on changes in substitution rates.

We need to distinguish the case of agent i, the CRRA agent, dominating from the case of agent j, the Epstein-Zin agent, dominating. When agent i dominates, he sets the market discount rate. Agent j differs from a corresponding CRRA agent with $IES = 1/\gamma^j$ in that his substitution rate is $\delta^j|_i = \beta^j$ instead of

$$\delta^{(j,CRRA)}|_i = (\beta^j)^{\frac{1}{\gamma^j}} (\beta^i)^{1-\frac{1}{\gamma^j}} \sum_{s\in\mathcal{S}} (\mathrm{Q}^j_s)^{\frac{1}{\gamma^j}} (\mathrm{Q}^i_s)^{1-\frac{1}{\gamma^j}} g_s^{\frac{\gamma^i}{\gamma^j}(1-\gamma^j)} \tag{30}$$

which is obtained from (11) with $\rho^j = \gamma^j$ and market discount factor δ set by agent i. As a consequence

$$\frac{\delta^i}{\delta^j}|_i = \delta^{i,(j,CRRA)}|_i \,\frac{\delta^{(j,CRRA)}|_i}{\beta^j}\,.$$

Expliciting $\delta^{(j,CRRA)}|_i$ and using the difference of survival indexes found for the CRRA case one finds

$$k^i|_i - k^j|_i = \frac{1}{\gamma^j}(k^i_Y - k^j_Y) + \left(1 - \frac{1}{\gamma^j}\right)\log\frac{\beta^i}{\beta^j} + \log \sum_{s\in\mathcal{S}} (\mathrm{Q}^j_s)^{\frac{1}{\gamma^j}} (\mathrm{Q}^i_s)^{1-\frac{1}{\gamma^j}} g_s^{\frac{\gamma^i}{\gamma^j}(1-\gamma^j)}\,. \tag{31}$$

The relative performance of agent i and j is not only governed by the differences of k_Y survival indexes, as in the CRRA case. Provided $\gamma^j \neq 1$, otherwise we would be back in a CRRA economy, there is an extra term, due to saving, that matters for the survival of agent j. Even when agent i has a higher modified survival index than agent j, the ordering of generalized survival indexes might be different. In fact agent j can still survive provided he postpone consumption more than he would have done as a CRRA agent with $IES = 1/\gamma^j$. The above equation gives the precise trade-off.

The other case of interest is when the Epstein-Zin agent, j, dominates. With respect to a CRRA economy there are two differences. As when i dominates, agent j saves at a different rate then if he had a CRRA substitution rule with $IES = 1/\gamma^j$. Now, however, also agent i substitutes at a different rate because the equilibrium interest rate imposed by agent j is not as in the corresponding CRRA economy. As a result:

$$\frac{\delta^i}{\delta^j}|_j = \delta^{i,(j,CRRA)}|_{(j,CRRA)} \frac{\delta^i|_j}{\delta^i|_{(j,CRRA)}} \frac{\delta^{(j,CRRA)}|_{(j,CRRA)}}{\delta^j|_j} \,,$$

After some simplifications,

$$\frac{\delta^i}{\delta^j}|_j = \delta^{i,(j,CRRA)}|_{(j,CRRA)} \left(\sum_{s \in \mathcal{S}} \mathrm{Q}_s^j \, g_s^{1-\gamma^j} \right)^{\frac{1}{\gamma^i}} .$$

Exploiting what we know for CRRA economies the difference of generalized survival indexes becomes

$$k^i|_j - k^j|_j = \frac{1}{\gamma^i}(k_Y^i - k_Y^j) + \frac{1}{\gamma^i} \log \sum_{s \in \mathcal{S}} \mathrm{Q}_s^j \, g_s^{1-\gamma^j} \tag{32}$$

The ordering of survival indexes k_Y is not enough to determine long-run outcomes. There is an extra term due to the fact that the Epstein-Zin agent has not $IES = 1/\gamma^j$, so that both the amount he saves and the discount rate he imposes differ from the corresponding CRRA economy.

Based on the sign of (31) and (32) it is still possible to characterize long run outcomes along the lines of Proposition 4.1. I particular I shall concentrate on the case when both agents survive, so that beliefs heterogeneity is persistent and consumption keeps fluctuating. For simplicity, I assume that the economy has a constant aggregate endowment, $g_s = 1$ for all $s \in \mathcal{S}$. When agent j dominates, he sets as the same equilibrium discount factor $\delta_t = \beta^j$ as a CRRA agent with $IES = 1/\gamma^j$, and also saves at the same rate β^j. As a result agents' relative performance can be given in terms of survival indexes as in a CRRA economy and (32) becomes

$$k^i|_j - k^j|_j = \frac{1}{\gamma^i}(k_{BE}^i - k_{BE}^j) \,.$$

However, when agent i dominates, agent j substitutes differently than he would have done under CRRA preferences with $IES = 1/\gamma^j$, as can be seen from (31) with $g_s = 1$ for all s. Survival of both agents is established in the next corollary.

Corollary 4.3. *Under the Assumptions 2.1, 2.2, A.1, consider the equilibrium paths of a two-agent exchange economy, with a CRRA agent, i, and an Epstein-Zin agent with $IES = 1$, j, and assume that the aggregate endowment is constant. If*

$$0 < k^i_{BE} - k^j_{BE} < (\gamma^j - 1)\log\frac{\beta^j}{\beta^i} - \gamma^j \log \sum_{s\in\mathcal{S}} (\mathrm{Q}^j_s)^{\frac{1}{\gamma^j}} (\mathrm{Q}^i_s)^{1-\frac{1}{\gamma^j}}\,,$$

then $k^i|_j - k^j|_j > 0$, $k^i|_i - k^j|_i < 0$, and both agents survive P*-almost surely.*

Provided beliefs are heterogeneous, the last term of the inequality is positive when $\beta^j \geq \beta^i$ and $\gamma^j > 1$, so that $IES = 1 > 1/\gamma^j$. The result confirms that it is the differentiated saving of the Epstein-Zin agent j with respect to the corresponding CRRA agent that, in not balancing exactly the term coming from the portfolio NLO term, might keep him alive even when his modified survival index is lower than that of agent i.

4.4 Selection of Intertemporal Substitution Rates

In Section 4.1, we have seen that speculation that results only in different portfolios could generate MSH failures. Here I address the opposite issue, that is, whether the differentiated saving decisions that are due to different preferences and beliefs could lead to market selection failures when growth premia do not play a role.

In order to answer this question, I investigate the outcome of market selection when all agents hold the same portfolio in equilibrium (the market portfolio) so that only saving matters. It turns out that the constraint imposed by the equal portfolio requirement, see Assumption 4.1 below, is such that the 'ordering' of substitution rates is stable. An intuitive result holds: the agent who fixes the highest market discount rate when alone in the market is also the one who saves the most for all possible equilibrium prices, and thus dominates in the long run.

Agents hold the same portfolio when they agree on normalized prices, or

$$\mathrm{Q}^0_s = \frac{\mathrm{Q}^i_s\, g_s^{\gamma^i}}{\sum_{s'\in\mathcal{S}} \mathrm{Q}^i_{s'}\, g_{s'}^{\gamma^i}} \quad \text{for all } s \in \mathcal{S}\,, \tag{33}$$

for all $i \in \mathcal{I}$. The condition can always be met in the sense that, given a set of normalized state prices Q^0, for any RRA coefficient γ there exists beliefs Q^i, namely

$$\mathrm{Q}^i_s = \frac{\mathrm{Q}^0_s\, g_s^{-\gamma^i}}{\sum_{s'\in\mathcal{S}} \mathrm{Q}^0_s\, g_s^{-\gamma^i}} \quad \text{for all } s \in \mathcal{S}\,,$$

such that (33) holds. We can thus assume

Assumption 4.1. *There exists a vector* $\mathrm{Q} \in \Delta_+^S$ *such that for all* $i \in \mathcal{I}$ *beliefs* Q^i *and RRA coefficients* γ^i *it holds*

$$\frac{\mathrm{Q}_s^i \, g_s^{\gamma^i}}{\sum_{s' \in \mathcal{S}} \mathrm{Q}_{s'}^i \, g_{s'}^{\gamma^i}} = \mathrm{Q}_s \quad \textit{for all } s \in \mathcal{S}\,. \tag{34}$$

If the aggregate endowment were not risky, or if it were risky but all agents had the same risk preferences γ, each agent holding the market portfolio would only occur under homogeneous beliefs. However the combination of a risky aggregate endowment and heterogeneous risk preferences is such that agents could still hold the same portfolio in equilibrium even when they have heterogeneous beliefs. The case is non-generic, perturbing the belief of an agent would break (34), but it serves the purpose of analyzing selection of substitution rates in stochastic economies.

When all agents hold the same portfolio in equilibrium, long-run outcomes are only determined by the comparison of their substitution rates in (11). When the initial allocation is such that each agent i starts with a fraction ϕ^i of the aggregate endowment, agents exchange claims on the aggregate endowment to transfer their consumption across dates. Agents with a long position are saving more than agents with a short position and are thus gaining consumption in relative terms. Speculative motives and risk sharing motives inter-act in such a way that agents are not betting.

The next proposition establishes that whether the fact that an agent has a long or short position can be established by comparing the discount rate that they would set when alone in the market. Whether an agent dominates or vanishes thus depends on the comparison of these single-agent economy rates. At this purpose I derive $\delta|_i$, the equilibrium rate when i is the representative agent. Simple computations lead to

$$\delta|_i = \beta^i e^{-\rho^i \, \mathrm{E}_{\mathrm{P}}[\log g]} \left(\sum_{s \in \mathcal{S}} \mathrm{Q}_s^i \, \hat{g}_s^{-\gamma^i} \right) \left(\sum_{s \in \mathcal{S}} \mathrm{Q}_s^i \, \hat{g}_s^{1-\gamma^i} \right)^{\frac{\gamma^i - \rho^i}{1-\gamma^i}} . \tag{35}$$

The role of the IES coefficient ρ^i in setting discount rates stands out.

Proposition 4.4. *Under the Assumptions 2.2, 4.1, A.1, consider the equilibrium paths of an economy with I agents maximizing Epstein-Zin preferences.*

i) if for all $j \neq i$

$$\delta|_i > \delta|_j \,,$$

then for all t*,* σ_t*, and* P*,* $k_t^i - k_t^j > 0$ *for all* $j \neq i$*, and* i *dominates on all* $\sigma \in \Sigma$*;*

ii) if there exist a $j \neq i$ such that

$$\delta|_i < \delta|_j \,,$$

then for all t, σ_t, and P, *$k_t^i - k_t^j < 0$, and i vanishes on all $\sigma \in \Sigma$.*

The proof relies on showing that for each pair of agents i and j, the ordering of market discount rates $\delta|_i$ and $\delta|_j$ implies a stable ordering of substitution rates δ_t^i and δ_t^i for all equilibrium discount factors δ_t.

It is important to note that although beliefs do matter, in that through the saving under uncertainty channel they induce higher or lower substitution rates, the truth does not matter in these economies. Agents transfer consumption only across time and not across states. No bets are exchanged. The relative consumption dynamics is deterministic and sufficient conditions *i)* and *ii)* imply a stable order of survival indexes for all measures P.[41] Dominance and vanishing hold on every path σ.

Proposition 4.4 establishes only sufficient conditions in that it could happen that two agents define the same maximal discount rate in equilibrium, yet save differently. These situations are even less generic than Assumption 4.1.

Proposition 4.4 can be combined with Proposition 4.3 in the case of CRRA preferences, $\gamma^i = \rho^i$ for all $i \in \mathcal{I}$. Under Assumption 4.1, modified survival indexes k_Y reflect only a differentiated saving behavior and dominance occurs universally, the truth has no role. Indeed, although the survival index k_Y seems to depend on P, the constraint imposed on beliefs by (34) is such that survival indexes computed under different P are all equal. The agent with the highest survival index dominates on all path $\sigma \in \Sigma$.

Proposition 4.5. *Under the Assumptions 2.1, 2.2, 4.1, A.1, consider the equilibrium paths of a CRRA economy. Survival indexes $\{k_Y^i,\, i \in \mathcal{I}\}$ do not depend on the truth* P*. The agent with the highest survival index k_Y dominates on all $\sigma \in \Sigma$.*

The comparison of rates $\delta|_i$ is also particularly simple in an Epstein-Zin economy without aggregate risk leading to the following corollary.

Corollary 4.4. *Under the assumptions of Proposition 4.4, assume further that there is no aggregate risk, $g_s = g$ for all $s \in \mathcal{S}$. If for all $j \neq i$*

$$\beta^i g^{-\rho^i} > \beta^j g^{-\rho^j} \,,$$

then i dominates on all $\sigma \in \Sigma$.

[41] As long as agents' beliefs are i.i.d. and all hold the market portfolio, i.e. Assumptions 2.2 and 4.1 respectively, Assumption 2.1 can be relaxed and the statement holds for any P on $(\Sigma, \{\mathcal{F}\})$.

Under no aggregate risk, the saving economy 'survival index' $\delta|_i$ can be simplified to $\beta^i g^{-\rho^i}$, which expresses the trade off between discount factors and IES coefficient. As in deterministic economies, controlling for discount factors, when $g > 1$ the agent with highest IES (lowest ρ) dominates. The opposite result holds when $g < 1$.

Under aggregate risk, although the comparison of equilibrium discount factors can still be simplified due to Assumption 4.1, its implication for preferences, discount factors, and beliefs is not straightforward. The following Corollary analyzes 'growing' 2-agent economies.

Corollary 4.5. *Under the assumptions of Proposition 4.4, consider only two agents, i and j, with $\beta^i = \beta^j$ and $\rho^i < 1 < \rho^j$. If $\log \mathrm{E}_{\mathrm{Q}^0}[\hat{g}] + \mathrm{E}_P[\log g] \geq 0$, then agent i dominates surely.*

Controlling for discount factors, in an growing economies where market beliefs are not too pessimistic, having a IES larger than 1 is sufficient for dominating against an agent with IES lower than 1, irrespectively of risk preferences.

5 Examples

In this section I shall consider simple illustrative examples of two-state, $S = 2$, speculative economies. The advantage of working with only two states is that equilibrium substitution rates, portfolios, and state prices have a convenient graphical representation in a 2 dimensional plot. Due to normalizations, only the first component of state prices needs to be tracked, the same holds for portfolios. Dropping time indexes to simplify the notation, Q^0 shall be the normalized price of state 1, Q^i the probability assigned by agent i to the realization of state 1, $\delta^i(\delta, \mathrm{Q}^0)$ the intertemporal substitution rule, and $\alpha^i(\mathrm{Q}^0)$ the portfolio that allocates next period consumption.

We concentrate on portfolio rules first. Figure 1 illustrates two examples of CRRA portfolio rules, showing how normalized equilibrium prices are determined by their aggregation. In the left panel, there is no aggregate risk. In the right panel, there is aggregate risk. In both cases, according to the market equilibrium equation (9), the equilibrium price Q^0 is found at the intersection of aggregate portfolio, the convex combination of rules $\alpha^1(\mathrm{Q}^0)$ and $\alpha^2(\mathrm{Q}^0)$ with the first component of the normalized supply, $l_1(\mathrm{Q}^0)$. Under no aggregate risk, left panel, (normalized) state prices and market beliefs coincide. Under aggregate risk, right panel, (normalized) state prices and market beliefs do not coincide. The latter are still given by the interception of the aggregate demand with the diagonal of the plot. In a dynamic economy agents' price impacts $\{\varphi\}$ are given both by relative consumption and substitution rates, as in (6).

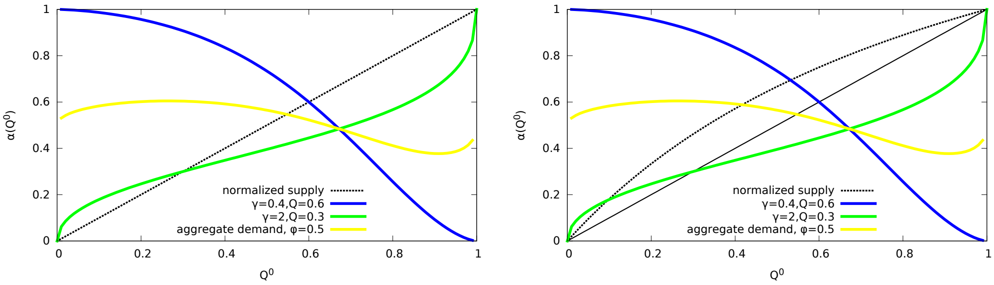

Figure 1: Market equilibrium with CRRA rules. In homogeneous (heterogeneous) economies clearing prices are determined by the intersection of a rule (a convex combination of rules) with the normalized supply. Left panel: no-aggregate risk. Right panel: aggregate risk, $g_1 = 2g_2$.

On the same plot one can also visualize the stability conditions. It is convenient to assume that the two states are equally likely, $\mathrm{P} = (1/2, 1/2)$. In this case the relative entropy $I_P(Q)$ becomes symmetric around its minimum $\mathrm{Q} = 1/2$, $I_P(Q) = I_P(1-Q)$. As a result portfolio premia can be evaluated using the euclidean distance of their first components: given $\alpha^i, \alpha^j \in (0,1)$, $I_P(\alpha^i) \gtreqless I_P(\alpha^j)$ if and only if $|\alpha^i - 1/2| \gtreqless |\alpha^j - 1/2|$.[42]

In Figure 2, I add a graphical representation of the growth premium and of its decomposition. I assume that agent i is the representative investor, so that his beliefs are market beliefs, and:

$$\mu^i = I_P(\mathrm{Q}^0) - I_P(\alpha^i(\mathrm{Q}^0)) = \left[I_P(\mathrm{Q}^0) - I_P(Q^i)\right] + \nu^i \,.$$

In the panel the solution of $\alpha^i(\mathrm{Q}) = \mathrm{Q}$ is equal to agent i belief Q^i. The horizontal line P represents the true probability that state 1 is realized. The vertical euclidean distance between the two horizontal lines P and Q^i is proportional to $I_P(Q^i)$. The full expected return μ^i computed at the equilibrium price Q^0 can thus be visualized as the difference of the distances of $\mathrm{Q}^0 = 0.3$ and $\alpha^i(\mathrm{Q}^0)$ from P. When, as in the plot, $\alpha^i(\mathrm{Q}^0)$ is further away than Q^0 the growth premium is negative. Although beliefs are more accurate than prices, i.e. $I_P(Q^0) - I_P(Q^i) > 0$, by non using a log-optimal portfolio agent i has a negative NLO term η^i, as can be visualized by the difference of the distances of Q^i and $\alpha^i(\mathrm{Q}^0)$ from P. Effective beliefs are less accurate than beliefs. As we shall see also the opposite might occur.

This graphical analysis can be used to illustrate the finding of Corollary 4.1. The left panel of Figure 3 shows an example where agents are more risk adverse

[42] If $\mathrm{P} \neq (1/2, 1/2)$ one should simply re-scale the vertical axis to compare 'left' and 'right' portfolio deviations from the truth.

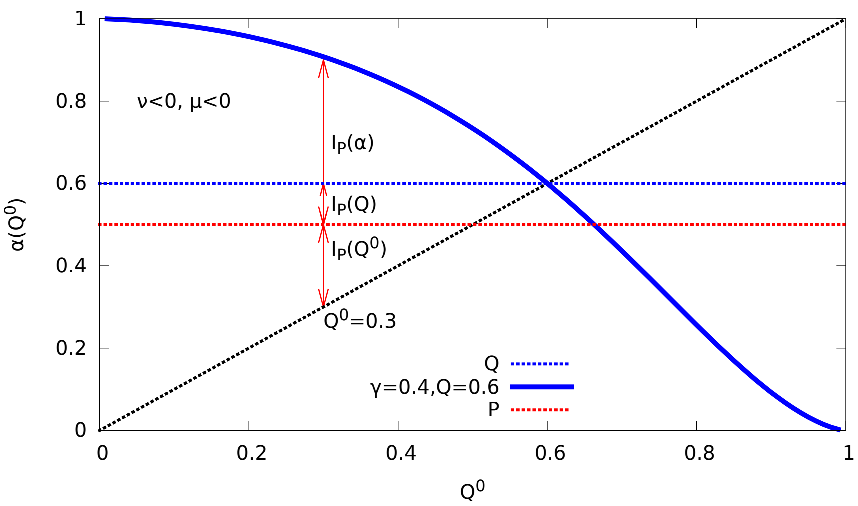

Figure 2: Expected log-returns decomposition agent i is the representative agent and $\mathrm{Q}^0 = 0.3$.

than the log agent, $\gamma^i = \gamma^j > 1$. Assume that agent i beliefs are more accurate than agent j beliefs. If both agents had log-optimal portfolios then agent i would have positive expected log-returns in every period. However by using less risky portfolios each agent receives a particularly high NLO term when state prices coincide with the other agent belief. As a result, notwithstanding that agent j has less accurate beliefs, there exist state prices where agent j has a higher expected log-return. The right panel of Figure 3 shows the same example when $\gamma^i = \gamma^j < 1$. In this case each agent receives a particularly low NLO compensation term when prices are close to the beliefs of the other agent.

In Epstein-Zin economies where agents have the same $\rho = 1$ and the same β, only growth premia matter for survival. Portfolios as in the left panels are associated to long-run heterogeneity whereas portfolios as in the right panel are associated to dominance depending on the initial conditions, case i) and ii) of Proposition 4.1 respectively.

These considerations can be used to illustrate the findings of Corollary 4.2 on the possible failures to dominate of an agent who knows the truth. Assuming aggregate risk, in Figure 4 I analyze the case of a RRA coefficient γ larger than 1. In the left panel I plot portfolio rules. Only two rules at the time should be considered, the one of the agent who knows the truth, agent one with $\mathrm{Q}^1 = \mathrm{P}$, and the rule of agent 2 having inaccurate beliefs. Three degrees of inaccuracy are considered. Given that state 1 is the good state, $(g_1 = 2g_2)$, in case k agent 2 is

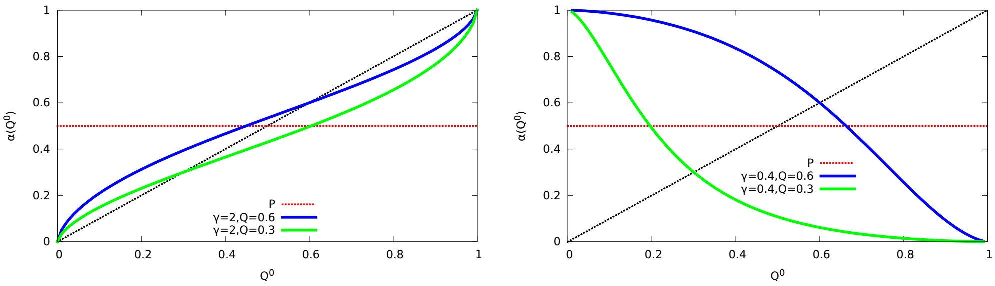

Figure 3: Comparison of growth premia in a market without aggregate risk and two agents with the same RRA coefficient γ and heterogeneous beliefs. Left panel: portfolio rules when $\gamma > 1$. Right panel: portfolio rules when $\gamma < 1$.

very optimist, in case j agent 2 is optimist, in case i agent 2 is pessimist. In the right panel, I plot the difference of expected log-returns (and thus growth premia) in the limit of either agent 1 or agent 2 is dominating, $\mu^1|_1 - \mu^2|_1$ and $\mu^1|_2 - \mu^2|_2$ respectively, as a function of the beliefs of agent 2. The three cases i, j, k, are identified on the horizontal axis. In this panel the comparison with relative beliefs' accuracy is also presented. If beliefs' accuracy were the only source of portfolio expected log-returns agent 1 would have positive expected log-returns larger than those of agent 2 and thus dominate in the long-run.

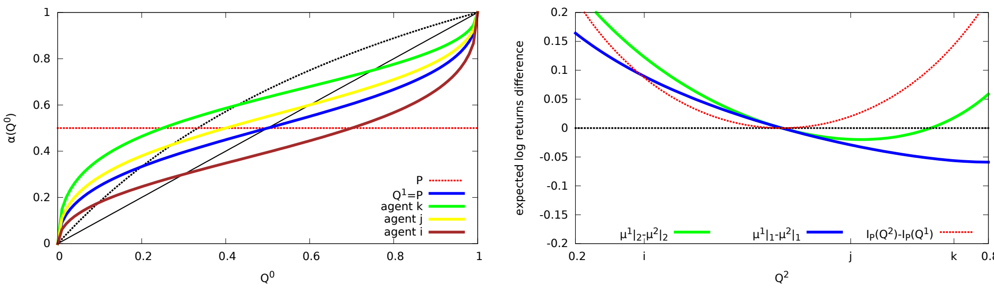

Figure 4: Comparison of growth premia in a market with aggregate risk ($g_1 = 2g_2$) when one agent has correct beliefs when $\gamma = 2$. Left panel: portfolio rules. Right panel: limit portfolio expected log returns as a function of agent 2 beliefs Q^2.

However, also NLO terms play a role and determine the sign combinations of asymptotic drifts. From these signs and Theorems 3.2-3.3, we can infer that knowing the truth leads to dominance (against pessimists as agent i), vanishing (against optimists as agent j), or survival of both (against extreme optimists as agent k). The plot confirms the intuition in DeLong et al. (1991) about the trade-off between optimism and risk aversion but clarifies that depending on the degree of optimism both dominance of the noise trader or survival of both agents is possible. The reason is that when agents' RRA is higher than $\gamma = 1$, a noise trader with optimistic beliefs might have a portfolio that is closer to the log-optimal portfolio derived under the truth than the portfolio derived using correct beliefs. When optimism is mild (as for agent j) the latter observation holds for all possible equilibrium prices and the optimistic trader dominates. When the optimism is strong (as for agent k) there are prices where it is the agent with correct beliefs that has a higher growth premium. Since each agent has a higher growth premium at the prices set by the other agent, both survive. Trading never settles and state prices keep fluctuating between the evaluation of the rational trade and that of the noise trader. An equilibrium path of normalized state prices and of relative consumption shares is shown in Figure 5. State prices keep fluctuating between the two agents evaluation and never settle down.

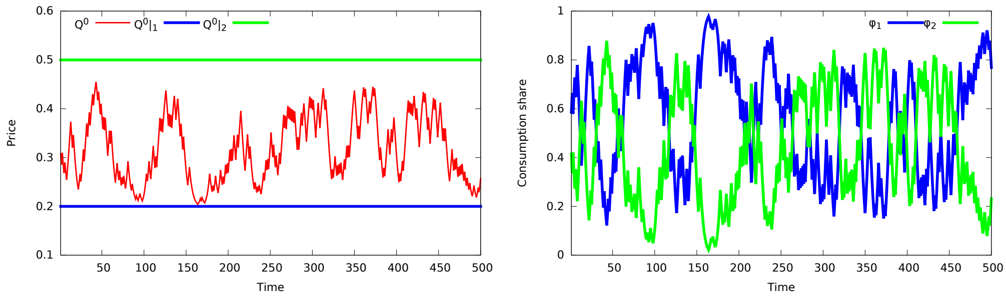

Figure 5: Equilibrium path of state $s = 1$ normalized prices (left panel) and agents' relative consumption shares (right panel) when $g_1 = 2g_2$, $\rho^1 = \rho^2 = 1$, $\beta^1 = \beta^2 = \beta \in (0,1)$, $\gamma^1 = \gamma^2 = 2$, $\mathrm{Q}^1 = \mathrm{P} = (1/2, 1/2)$, $\mathrm{Q}^2 = (4/5, 1/5)$.

As established in Corollary 4.2, when $\gamma > 1$ path dependency never occurs. Figure 6 shows instead portfolio rules and asymptotic drifts for the case $\gamma < 1$. In this case the possible outcomes are dominance, vanishing, or path-dependency. Given the trade-off between risk aversion and degree of optimism/pessimism, here the irrational agent is chosen optimist (agent k), pessimist (agent j), or very pessimist (agent i). The rational agent dominates against an optimist (as agent k), vanishes against a pessimist (as agent j), and might dominate or vanish, depending on the realization σ, against an extreme pessimist (as agent i).

As explained in Section 4.2, in CRRA economies the relative size of portfolios growth premia is exactly as the one I have just discussed for these Epstein-Zin economies. However, in CRRA economies, the component of intertemporal substitution rates that incorporates beliefs heterogeneity compensates for the inaccuracy

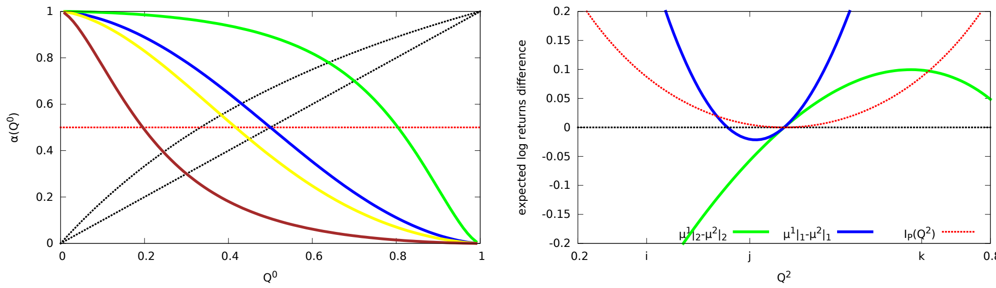

Figure 6: Comparison of growth premia in a market with aggregate risk ($g_1 = 2g_2$) when one agent has correct beliefs when $\gamma = 0.4$. Left panel: portfolio rules. Right panel: limit portfolio expected log returns as a function of agent 2 beliefs Q^2.

of effective beliefs. In Figure 7, I plot the log-ratio between the intertemporal rate of substitution of the agent with correct beliefs (agent 1) and the intertemporal rate of substitution of agent 2 as a function of his beliefs Q^2. In the left panel $\gamma > 1$. In the right panel $\gamma < 1$. In all cases the combined effect of saving and portfolio returns is such that the aggregate effect if proportional to the difference of the relative entropy of beliefs, $I_P(Q^2) - I_P(Q^1)$, leading to the dominance of the accurate trader. For example when $\gamma > 1$ and the accurate agent is trading with an optimist he always saves more enough to counterbalance the under-performance of his portfolio (see e.g. agent j and k in the left panel). A similar effect occurs when $\gamma < 1$. Importantly, note that saving does not always go in favor of the correct agent and in many cases the contribution of portfolios is still crucial. Whether the correct agent dominates due to his portfolios, to saving, or to the sum of the two needs to be judged case by case.

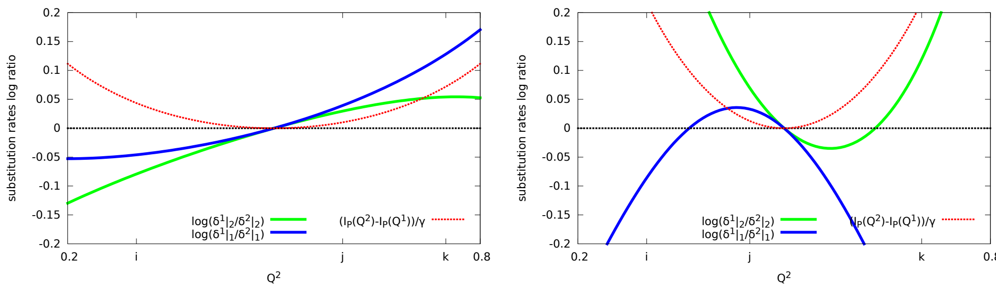

Figure 7: Ratio of intertemporal substitution rates of the agent with correct beliefs and agent 2 as a function of his beliefs Q^2. Left panel: $\gamma = \rho = 2$. Right panel: $\gamma = \rho = 0.4$.

5.1 Generalizations

All the above examples are in two-agent economies. When more than two agents are trading in the same market there is no conceptual difficulty, growth premia and substitution rates are still determining long-run outcomes. The limitation is technical in that growth premia need to be evaluated in the limit of one group of agents consuming all the aggregate endowment, and in these limits state prices depend on all remaining agents' consumption distribution. Using Theorems 3.2-3.3, sufficient conditions similar to those of Proposition 4.1 are established but are far from being tight. In Figure 8, I present three examples of three-agent economies. In the left panel, under no-aggregate risk, agent i survives against the combination (j, k), because the growth premium of i is larger than the growth premium of (j, k) for all prices determined by (j, k). However, agent i does not dominate and both i and (j, k) survive. Under aggregate risk, instead, agent i dominates. In the right panel, agent i vanishes.

These three agents economies shed light on the phenomenon of extinction reversal as shown in Cvitanić and Malamud (2010). Consider agent j and agent k in isolation. Since for all equilibrium prices agent j has higher growth premia than agent k, then he dominates almost surely. Assume now that a log-optimal agent with correct beliefs is also trading in the market. His portfolio rule coincides with the line denoted as P. By dominating, this third agent moves equilibrium prices to $\mathrm{Q}^0 = \mathrm{P}$. At these prices, growth premia of agent k become larger than those of agent j so that extinction reversal occurs. Generalized survival indexes, depending on endogenous quantities, might change their order.

The same weakening of the sufficient conditions applies also to non i.i.d. economies. When the growth rate g follows a generic process, or when beliefs are not i.i.d., equilibrium prices computed under the assumption that an agent, or a group of agents, consumes the aggregate endowment become a random variable. Growth premia should thus be compared for all the relevant possible histories of the process. Only when inequalities hold in all these cases, they are sufficient to characterize long-run outcomes.

Finally note that the same approach used in this paper can be extended beyond Epstein-Zin economies whenever an equilibrium path of prices and consumption distribution can be shown to exist, and date t one-period portfolio and substitution rules depend on information up to t. The system of equations (3) and (8-9) can be used to characterize consumption and state prices in the long-run, and thus to address the MSH. The recursive preferences proposed in Weil (1993) combining a unitary IES parameter and a constant ARA portfolio choice are a possible case.

Alternatively, one can consider temporary equilibrium model of sequential trading where agents are not assumed to have rational expectations on future prices. Assume that each agent decides how much wealth to save and how to allocate

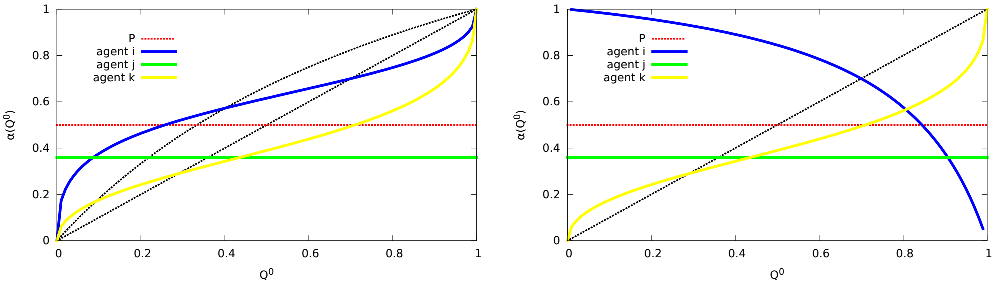

Figure 8: Three-agent economies. Left panel: agent i survives but does not dominate under no-aggregate risk, and dominates under aggregate risk. Right panel: agent i vanishes.

the saved wealth to the purchase of Arrow securities by using (adapted) rules $\bar{\delta}_t^i$ and $\bar{\alpha}_t^i$ that depend only on the information available till time $t-1$. If future prices and current prices are involved, expectations should be computed given the information up to $t-1$. Then, one period saving and portfolio rules δ_t^i and α_t^i, can be easily derived from the wealth dynamics as

$$\begin{aligned} \delta_t^i &= \frac{\bar{\delta}_t^i}{1-\bar{\delta}_t^i} \sum_{s\in\mathcal{S}} (1-\bar{\delta}_{t+1}^i(\sigma_{t-1},s))\bar{\alpha}_{s,t}^i \\ \alpha_{s,t}^i &= \frac{(1-\bar{\delta}_{t+1}^i(\sigma_{t-1},s))\bar{\alpha}_{s,t}^i}{\sum_{s'\in\mathcal{S}}(1-\bar{\delta}_{t+1}^i(\sigma_{t-1},s'))\bar{\alpha}_{s',t}^i}\,, \quad \text{for all } s\in\mathcal{S}\,. \end{aligned} \tag{36}$$

One period substitution and portfolio rules depend only on the information up to period t. Exchange economies where some agents maximize a CRRA utility and have rational price expectations while other agents use 'behavioral rules' of this sort can also be analyzed. Whether an agent vanishes, survives, or dominates is determined the by comparison of log substitution rates and growth premia as established in Theorems 3.2-3.3.

6 Conclusion

This paper explains why in dynamic stochastic exchange economies where agents have heterogeneous beliefs, speculation may not support the Market Selection Hypothesis. The result is established by characterizing long-run outcomes of agents' relative consumption process in terms of the comparison of agents' log substitution rates and portfolio growth premia. The latter are shown to depend on the market log-risk premium, an accuracy premium, and a non-log-optimality term.

In the special case of log-economies, provided discount factors are equal, comparison of portfolio growth premia depend only on agents' relative accuracy. Portfolio speculative positions favor the agent with correct beliefs. However, outside the log-utility framework, the growth premium depends both on beliefs' accuracy and on the comparison of an agent's portfolio choice with the corresponding log-optimal portfolio. This last term, named the Non-Log-Optimality (NLO) contribution, leads to generic failures of Market Selection Hypothesis. In an Epstein-Zin economy where all agents use the same intertemporal substitution rates, three types of failures are identified: multiple agents survive a.s., leading to heterogeneity of beliefs also in the long run; the agent with accurate beliefs vanishes on some paths and dominates on others; the agent with accurate beliefs vanishes a.s.. The failures are shown to be robust to cases where agents use different intertemporal substitution rates. CRRA economies are instead special because, due to interdependence of intertemporal and risk preferences, the response to beliefs heterogeneity incorporated in intertemporal substitution rates and NLO terms compensate each-others. The only long-run outcome is the dominance of a unique agent, so that the only possible MSH failure is the vanishing of the agent with the most accurate beliefs. However, also in CRRA economies, the relative importance of saving and portfolio decisions for long-run survival depends on all agents' preferences and beliefs.

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A General Equilibrium with Epstein-Zin Agents

A.1 Saving and portfolio decisions

Saving and portfolio decisions, $\bar{\delta}^i_t$ and $\bar{\alpha}^i_t$ respectively, can be computed starting from intertemporal substitution rates and one-period portfolio decisions, δ^i_t and α^i_t respectively. Agent i wealth (total net worth) after history σ_t is

$$w^i_t = c^i_t + \sum_{T>0,\, \sigma_{t+T}} q_{\sigma_{t+T},t}\, c^i_{t+T}\,,$$

where σ_{t+T} takes values in $\Sigma_{t+T}(\sigma_t)$, the subset of Σ_{t+T} whose elements have a common history σ_t. Iterating the consumption dynamics (3) in the expression for w^i_t above one finds

$$w^i_t(\sigma_t) = c^i_t(\sigma_t)\left(1 + \delta^i_t(\sigma_t) + \delta^i_t(\sigma_t) \sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta^i_{t+\tau}(\sigma_{t+\tau})\alpha^i_{s_{t+\tau},t+\tau-1}(\sigma_{t+\tau-1})\right)\,.$$

Defining agent i date t saving decision as $\bar{\delta}^i_t$ such that $c^i_t = w^i_t(1-\bar{\delta}^i_t)$ we find

$$\bar{\delta}^i_t = \delta^i_t \frac{1 + \sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta^i_{t+\tau}\alpha^i_{s_{t+\tau},t+\tau-1}}{1 + \delta^i_t + \delta^i_t \sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta^i_{t+\tau}\alpha^i_{s_{t+\tau},t+\tau-1}}\,. \tag{37}$$

When an interior equilibrium is well defined $\bar{\delta}^i_t \in (0,1)$.

From the saving decision $\bar{\delta}^i_t$ and the consumption dynamics (3) it is possible to derive the wealth dynamics

$$w^i_{t+1} = \bar{\delta}^i_t \frac{\bar{\alpha}^i_{s,t}}{q_{s,t}} w^i_t \quad \text{on} \quad (\sigma_t, s)\,, \tag{38}$$

where we have defined $\bar{\alpha}^i_t$ as

$$\bar{\alpha}^i_{s,t}(\sigma_t) = \alpha^i_{s,t}(\sigma_t)\frac{\delta^i_t(\sigma_t)}{\bar{\delta}^i_t(\sigma_t)}\frac{1-\bar{\delta}^i_t(\sigma_t)}{1-\bar{\delta}^i_{t+1}(\sigma_t,s)}\,. \tag{39}$$

or, in terms of one-period portfolios and substitution rates,

$$\bar{\alpha}^i_{s,t} = \alpha^i_{s,t}\, \frac{1 + \delta^i_{t+1}(\sigma_t,s) + \delta^i_{t+1}(\sigma_t,s) \sum_{T>0,\, \sigma_{t+1+T}} \prod_{\tau=1}^{T} \delta^i_{t+1+\tau}(\sigma_{t+1+\tau})\alpha^i_{s_{t+1+\tau},t+\tau}(\sigma_{t+\tau})}{1 + \sum_{T>0,\, \sigma_{t+T}} \prod_{\tau=1}^{T} \delta^i_{t+\tau}(\sigma_{t+\tau})\alpha^i_{s_{t+\tau},t+\tau-1}(\sigma_{t+\tau-1})}\,.$$

$\bar{\alpha}^i_t$ is agent i 'full' portfolio decision in node σ_t in that it specifies, how to allocate saved wealth $\bar{\delta}^i_t w^i_t$ among future states. It can be easily checked that $\bar{\alpha}^i_t \in \Delta^s_+$ and that the date $t+1$ return of the full portfolio in state s is $\frac{\bar{\alpha}_{s,t}}{q_{s,t}}$.

Although full and one-period decisions differ, there exists a limit under which they coincide. The following result characterizes when it is the case.

Lemma A.1. *If in equilibrium agent i intertemporal substitution rates are history-independent (deterministic),*

$$\delta^i_t(\sigma_t) = \delta^i_t(\sigma'_t) \quad \textit{for all} \quad t, \sigma_t, \sigma'_t$$

then

$$\bar{\alpha}^i_t = \alpha^i_t$$

If, moreover, intertemporal substitution rates are time-independent,

$$\delta^i_t = \delta^i_{t+1} = \delta^i \quad \textit{for all} \quad t,$$

then

$$\bar{\delta}^i_t = \delta^i \, .$$

Proof. Provided intertemporal substitution rates are history-idependent, i.e. deterministic, the ratio

$$\frac{\delta^i_t(\sigma_t)}{\bar{\delta}^i_t(\sigma_t)} \frac{1 - \bar{\delta}^i_t(\sigma_t)}{1 - \bar{\delta}^i_{t+1}(\sigma_t, s)}$$

on the left hand side of (39) is one for all $s \in \mathcal{S}$, proving the first part of the Lemma. If substitution rates are also constant, the sum of the geometric series of compounded rates can be computed leading to $\bar{\delta}^i_t = \delta^i$. □

A.2 Portfolio and saving decisions under Epstein-Zin preferences

In agent agent i maximizes a recursive utility of the type (10) one can use the first order conditions, see e.g. Epstein and Zin (1991), to characterize equilibrium allocation and prices. In terms of saving and portfolio decisions one finds

$$\frac{\mathrm{Q}^i_s}{q_{s,t}} (\beta^i)^{\frac{1-\gamma^i}{1-\rho^i}} \left(\frac{\delta^i_t \alpha^i_{s,t}}{q_{s,t}} \right)^{-\rho^i \frac{1-\gamma^i}{1-\rho^i}} \left(\frac{\bar{\alpha}^i_{s,t}}{q_{s,t}} \right)^{\frac{\rho^i - \gamma^i}{1-\rho^i}} = 1 \quad \text{for all} \quad s,\, t,\, \sigma_t \, , \tag{40}$$

where we have used that $\frac{\bar{\alpha}^i_{s,t}}{q_{s,t}}$ is the return of agent full portfolio. Unless we are in the CRRA case, $\gamma^i = \rho^i$, the full portfolio $\bar{\alpha}^i_t$ enters in the first order condition. Since $\bar{\alpha}^i_t$ depends on all future one-period substitution and portfolio decisions, all first order conditions are coupled. However when Lemma A.1 applies, $\bar{\alpha}^i_t$ and α^i_t coincide so that (40) can be solved to find one-period optimal decisions in terms of beliefs, preferences, and market prices.

We have the following

Proposition A.1. *If in equilibrium agent i intertemporal substitution rates are history-independent, then for all t and σ_t, it holds*

$$
\begin{aligned}
\delta^i_t &= \delta_t \left(\frac{\beta^i}{\delta_t}\right)^{\frac{1}{\rho^i}} \left(\sum_{s'\in S} (\mathrm{Q}^i_{s'})^{\frac{1}{\gamma^i}} (\mathrm{Q}^0_{s',t})^{1-\frac{1}{\gamma^i}}\right)^{\frac{\gamma^i}{\rho^i}\frac{1-\rho^i}{1-\gamma^i}}, && (41)\\
\alpha^i_{s,t} &= \frac{(\mathrm{Q}^i_s)^{\frac{1}{\gamma^i}} (\mathrm{Q}^0_{s,t})^{1-\frac{1}{\gamma^i}}}{\sum_{s'\in S} (\mathrm{Q}^i_{s'})^{\frac{1}{\gamma^i}} (\mathrm{Q}^0_{s',t})^{1-\frac{1}{\gamma^i}}}, \quad s = 1, \dots, S\,. && (42)
\end{aligned}
$$

Moreover, $\bar{\alpha}^i_t = \alpha^i_t$.

Proof. The result follows from the application of Lemma A.1, which allows to use α^i_t in place of $\bar{\alpha}^i_t$, and from the direct solution of (40) in terms of α^i_t and δ^i_t. □

When substitution rates are history-independent both substitution and portfolio decisions depend only on contemporaneous market prices and rates, as in an expected utility framework. The functions $\delta^i(\cdot,\cdot)$ and $\alpha^i(\cdot)$ such that $\delta^i_t = \delta^i(\delta_t, \mathrm{Q}^0_t)$ and $\alpha^i_t = \alpha^i(\mathrm{Q}^0_t)$ are, respectively, the intertemporal substitution rule and one-period portfolio rule of agent i.

Although the proposition does not say when equilibria are such that the one-period substitution decision coming from these rules is history, or time, independent, one can judge directly from the functional form in (41). This is the content of the next two corollaries. The first illustrates the well-known case of simple saving rules when the IES parameter is 1.

Corollary A.1. *If agent i has $\rho^i = 1$, then for all t and all σ_t $\bar{\delta}^i_t = \delta^i_t = \beta^i$ and $\bar{\alpha}^i_t = \alpha^i_t = \alpha^i(\mathrm{Q}^0_t)$.*

Proof. Other than from direct substitution of $\rho = 1$ in (41), the result can be established starting from the Euler equation of the recursive formulation limit, see Epstein and Zin (1991). □

The corollary applies also when growth rates are not i.i.d.. Instead the next result applies only when the economy is i.i.d., both in beliefs and growth rates, see Assumptions 2.1-2.2.

Corollary A.2. *In an economy where Assumptions 2.1-2.2 hold, if all agents hold the market portfolio, then for all t and σ_t agent i one-period substitution decisions are as in* (41) *with*

$$
\mathrm{Q}^0_{s,t} = \left(\frac{\mathrm{Q}^i_s}{g_s^{\gamma^i}}\right)\left(\sum_{s'\in \mathcal{S}} \frac{\mathrm{Q}^i_{s'}}{g_{s'}^{\gamma^i}}\right)^{-1}, \quad s = 1, \dots, S\,. \tag{43}
$$

Proof. Under Assumptions 2.1-2.2 beliefs and growth rates are i.i.d. so that, when all agents hold the market portfolio, state prices as well as one-period portfolio decisions do not depend on time and states. One-period substitution rules still depend on time, through the market discount rate, but do no depend on partial histories as the dynamics of discount rates is deterministic. As a result Lemma A.1 applies and Proposition A.1 holds. □

A.3 General equilibrium

Given an economy with a set $\mathcal{I}$ of Epstein-Zin agents, consumption paths $\{c_t^i\}$ for all i, normalized states prices $\{\mathrm{Q}_t^0\}$, and market discount rates $\{\delta_t\}$ generated by (3-9) with rules as in (41-42) are an equilibrium of the exchange economy for a given initial allocation $\{c_0^i \text{ for all } i\}$ provided that: *i*) an interior equilibrium is shown to exist, otherwise the system (3-9) might have no solutions; *ii*) agents value function are finite in equilibrium, so that recursive preferences are well defined and Euler equations are sufficient, see also Epstein and Zin (1989) and Ma (1993).

Regarding *i*), under time-0 trading the existence of an equilibrium follows from Peleg and Yaari (1970), provided the recursive formulation of utility gives a well define utility over consumption streams and provided strict desirability holds. Both require finiteness of the value functions, that is *ii*). Since Epstein-Zin preferences are dynamically consistent, as long as markets are (dynamically) complete and an equilibrium exists, time-0 trading and sequential trading achieve the same equilibrium allocations. Depending on the chosen asset structure, different assumptions on the budget constraint are necessary to guarantee the existence of an equilibrium: under date $t = 0$ trading no bankruptcy is allowed, under sequential trading no bankruptcy and no Ponzi schemes are allowed, see also Araujo and Sandroni (1999). When an equilibrium exists, it must be interior: agents consumption is positive on all paths σ due to the fact that for every t consumption in t and expected value of date $t+1$ utility are evaluated via a CES aggregator with a finite elasticity of substitution equal to $1/\rho$. As a result, it is never optimal to have zero consumption. Regarding *ii*) a sufficient condition is that each agent value function is finite when he consumes all the aggregate endowment along the paths of maximal and minimal growth, that is, assuming that there is no uncertainty in the economy. To see why, name s^+ the state of maximal growth and s^- the state of minimal growth. Agent i utility on the path $\{e_t^s\} = \{e_0, g_s e_0, g_s^2 e_0, \ldots\}$, with s either s^+ or s^-, can be easily computed from (10) as

$$U_0^i = \left((1-\beta) \sum_{t=0}^{\infty} e_0 \left(\beta^i g_s^{1-\rho^i} \right)^t \right)^{\frac{1}{1-\rho^i}} .$$

The time zero utility is finite provided that

$$\beta^i g_s^{1-\rho^i} < 1 \quad s = s^+, s^- \tag{44}$$

Note also that since the path of maximum and minimum growth are certain, and no agent consumes all the aggregate endowment, then

$$\{e_t^{s-}\} \leq \{c_t^i\} \leq \{e_t^{s+}\}$$

(inequalities for sequences are valid component by component). Adding that preferences are monotone, the latter implies

$$U_0^i(\{c_t^i\}) \in (\infty, \infty)$$

for all the feasible allocations $\{c_t^i\}$, provided that agent i preferences satisfy the bound (44).

The argument is concluded by assuming that for each agent $i \in \mathcal{I}$ discount factors β^i and IES coefficients ρ^i are such that both inequalities (44) hold. We have the following.

Assumption A.1. *For every agents $i \in \mathcal{I}$, the discount factor β^i and the IES parameter ρ^i are such that* (44) *in both the maximum and minimum growth state.*

Finally, I have not excluded the possibility that multiple equilibria exist. As long as each equilibrium obeys (3-9), the market selection results derived from growth premia and intertemporal rates of substitutions apply.

B Time-separability beyond the CRRA case

Sandroni (2000) and Blume and Easley (2006) show that discount factors and beliefs determine long-run survival for all economies where preferences are represented by an expected time-separable utility with Bernoulli utility $u(c)$ satisfying $u'(c) > 0$, $u''(c) < 0$, and $\lim_{c \to 0} u'(c) = +\infty$, provided that the aggregate endowment is bounded from above and from below. Does the the same trade-off between portfolio log-returns and log substitution ratios hold also when u is not of the CRRA type? Under the same assumptions on u, the marginal utility $f_i(c^i) = du^i(c^i)/dc^i$ is a strictly decreasing positive function unbounded from above with well defined inverse $f_i^{-1}(\cdot)$. Solving the Euler equations leads to the following portfolio and substitution rule:

$$\begin{aligned} \delta_t^i &= \sum_{s' \in \mathcal{S}} f_i^{-1}\left(\frac{\mathrm{Q}^0_{s',t}}{\beta^i \, \mathrm{Q}^i_{s',t}} f_i\left(c_t^i\right)\right) \frac{\mathrm{Q}^0_{s',t}}{c_t^i}\,, \\ \alpha^i_{s,t} &= \frac{f_i^{-1}\left(\frac{\mathrm{Q}^0_{s,t}}{\beta^i \, \mathrm{Q}^i_{s,t}} f_i\left(c_t^i\right)\right) \frac{\mathrm{Q}^0_{s,t}}{c_t^i}}{\delta_t^i}\,. \end{aligned}$$

Although it is difficult to use the former to characterize long-run consumption, it is evident that the two are related. Moreover, given the decomposition of a portfolio growth premium, f_i determines only the NLO term. As with CRRA preference, in order for the asymptotic relative 'ranking' not to depend on normalized state prices, NLO terms and differences of log substitution rates should compensate each-other.

C Other recursive preferences

In this Section I derive intertemporal and portfolio decisions for the recursive preferences used in Weil (1993).

D Proofs of Section 3

Proof of Theorem 3.1 and of Corollary 3.1 The Theorem 3.1 is a direct application of the Law of Large Numbers for uncorrelated martingales, see also Proposition 1 in Sandroni (2000) for a similar application.

Consider the additive process $z^{\mathfrak{J}}_t$ with innovation $\epsilon^{\mathfrak{J}}_t$. The process $\{Z_t\}$ with

$$Z_t = \epsilon^{\mathfrak{J}}_t - \mathrm{E}[\epsilon^{\mathfrak{J}}_t | \mathcal{F}_{t-1}]$$

is a uncorrelated martingale with zero expected value so that, by the LLN for uncorrelated martingales,

$$\lim_{T\to\infty} \frac{\sum_{t=1}^{T} Z_t}{T} = 0 \quad \text{P-almost surely}\,.$$

By assumption

$$\lim_{T\to\infty} \frac{\sum_{t=1}^{T} \mathrm{E}[\epsilon^{\mathfrak{J}}_t | \mathcal{F}_{t-1}]}{T} > 0 \quad \text{P-almost surely}\,.$$

The latter implies

$$\lim_{T\to\infty} z^{\mathfrak{J}}_T = +\infty\,.$$

Given the proof of the Theorem above the Corollary can be proved by contradiction.

Proof of Proposition 4.1 Given a filtered probability space $(\mathrm{P}, \Sigma, \Im)$ and a real process $\{x_t\}$ defined on $(\Sigma, \Im)$, adapted to the filtration $\{\Im_t\}$, Bottazzi and Dindo (2015) prove the following theorems, which rely on the Martingale Convergence Theorem and owe to Lamperti (1960).

Theorem D.1. *Consider a finite increments process x_t with $|x_{t+1} - x_t| < B$ P-a.s.. If there exist $M > B$ and $\epsilon > 0$ such that,* P*-a.s.,* $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] < x - \epsilon$ *for all $x > M$ and* $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ *for all $x < -M$, then there exists a real interval $L = (a, b)$ such that for any t it is* $\mathrm{Prob}\{x_{t'} \in L \text{ for some } t' > t\} = 1$.

Theorem D.2. *Consider a finite increments process x_t with $|x_{t+1} - x_t| < B$ P-a.s.. and such that for all t* $\mathrm{Prob}\{x_{t+1} - x_t > \gamma|\Im_t\} > \gamma$ *for some $\gamma > 0$. If there exist $M > B$ and $\epsilon > 0$ such that,* P*-a.s.,* $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ *for all $x > M$ and* $\mathrm{E}[x_{t+1}|x_t = y, \Im_t] > x + \epsilon$ *for all $x < -M$, then* $\mathrm{Prob}\{\lim_{t\to\infty} x_t = +\infty\} = 1$.

Theorem D.3. *Consider a finite increments process x_t with $|x_{t+1} - x_t| < B$ P-a.s.. and such that for all t* $\mathrm{Prob}\{x_{t+1} - x_t > \gamma|\Im_t\} > \gamma$ *and* $\mathrm{Prob}\{x_{t+1} - x_t < -\gamma|\Im_t\} > \gamma$ *for some $\gamma > 0$. If there exist $M > B$ and $\epsilon > 0$ such that,* P*-a.s.,* $\mathrm{E}[x_{t+1}|x_t = x, \Im_t] > x + \epsilon$ *for all $x > M$ and* $\mathrm{E}[x_{t+1}|x_t = y, \Im_t] < x - \epsilon$ *for all $x < -M$, then there exists two sets of initial conditions, Γ^+ and Γ^- with $\Gamma^+ \cup \Gamma^- = \mathbb{R}$, such that $\lim_{t\to\infty} x_t = +\infty$ if $x_0 \in \Gamma^+$ and $\lim_{t\to\infty} x_t = -\infty$ if $x_0 \in \Gamma^-$.*

Theorem 3.2 are a direct application of the three theorems above having $x_t = z_t^{\jmath}$. By the permanence of sign theorem, continuity of the conditional drift in z guarantess that the sign of the drift of the process in the limit of one group dominating is equal to the the sign of the drift in a properly chosen neighborhood around it.

E Proofs of Section 4

Proof of Proposition 4.1 Statement $i)$ of the Proposition follows from Theorem D.1, provided we prove that the log consumption ratio $x_t = z_t^{i,j}$ has finite increments B. In fact, by continuity of the conditional drift[43], there exists an $M > B$ such that the drift hypothesis of Theorem D.1 are satisfied when the limit of the drift is positive for $z \to -\infty$ and negative for $z \to +\infty$. The latter follows from by the assumed inequality on relative beliefs accuracy and NLO terms, as explained in the text above the proposition. Using the same argument statements $iii)$ and $ii)$ follow from Theorem D.2 and Theorem D.3, respectively, provided we prove that the log consumption ratio $z_t^{i,j}$ has finite increments and a finite probability to jump of at least a given step. This is the content of the following lemma.

[43]The continuity of the conditional drift follows from the continuity CRRA rules α^i and α^j seen as a function of z, which in turn follows from the existence of continuous maps $q_s(\phi^i)$ in the neighborhood of $\phi^i = 1$ and $\phi^i = 0$ (due to the local uniqueness of homogeneous economy equilibria).

Lemma E.1. *Under the assumption of Proposition 4.1 the log relative consumption process $z_t^{i,j} = x_t$ has finite increments, that is, there exists a $B > 0$ such that*

$$|x_{t+1} - x_t| < B \quad \mathrm{P}-\textit{almost surely}.$$

Moreover if one of the sufficient conditions i) to iii) hold the process has a finite probability of jumping of at least a given step, that is, there exists a $\gamma > 0$ such that

$$\operatorname{Prob}\{x_{t+1} - x_t > \gamma | \Im_t\} > \gamma \quad \textit{and} \quad \operatorname{Prob}\{x_{t+1} - x_t < -\gamma | \Im_t\} > \gamma\,.$$

Proof. The process $z_t^{i,j} = x_t$ has innovation

$$\epsilon_{s,t+1}^{i,j} = \log \frac{\alpha_s^i(q_t)}{\alpha_s^j(q_t)} \quad \text{on}\,(\sigma_t, s)\,,$$

where α are as in (12) with $q_t = \mathrm{Q}_t^*$. For each s and t, equilibrium prices $q_{s,t}$ are in the interval $(\min\{q_s^j, q_s^i\}, \max\{q_s^j, q_s^i\})$, where q^i (q^j) is the vector of state prices when agent i (j) dominates. Name $\Omega = \times_{s\in\mathcal{S}} [\min\{q_s^j, q_s^i\}, \max\{q_s^j, q_s^i\}]$. Regarding the finite increment requirement, note that for each s there exists a maximum innovation given by

$$\epsilon_s = \max\left\{\left|\log\frac{\alpha_s^i(q_t)}{\alpha_s^j(q_t)}\right| \text{for}\, q \in \Omega\right\}\,.$$

Choosing $B > \max\{\epsilon_s\,, s \in \mathcal{S}\}$ suffices for the requirement. Turning to the existence of γ, such that jumps of at least γ occur with probability at least γ, note that if $q_s^i = q_s^j$ for all s then $q_t = q^i = q^j$ for all t and, from the market clearing equation (9) $\alpha^i(q_t) = \alpha^j(q_t)$ for all t. It follows that not only $\epsilon_t^{i,j} = 0$ for all t but also $\mu_t^i - \mu_t^j = 0$ for all t so that none of the drift conditions i) to iii) can be satisfied. As a result we exclude that $q^i \neq q^j$ and equilibrium prices q_t belong to the interior of Ω. To conclude the proof note that in equilibrium there are no arbitrages, as a result for all t and $q_t \in \Omega$ there exists at least an s and an s' such that

$$\epsilon_{s,t}^{i,j} > 0 \quad \text{and} \quad \epsilon_{s',t}^{i,j} < 0$$

(Otherwise the zero-price portfolio $\alpha^i(q_t) - \alpha^j(q_t)$, or $\alpha^j(q_t) - \alpha^i(q_t)$, would be an arbitrage). For every $q \in \Omega$ let $\epsilon^+(q)$ the maximum of such jumps (the upper envelope of $\epsilon_s(q)$ for all s) and $\epsilon^-(q)$ the lowest of such jumps. The two functions are continuous in q and Ω is compact so they have a maximum and a minimum. Moreover, since by the non arbitrage argument $\epsilon^+(q) > 0$ and $\epsilon^-(q) < 0$, the minimum of $\epsilon^+(q)$ is positive, $\epsilon^{+-} > 0$, and the maximum of $\epsilon^-(q)$ is negative, $\epsilon^{-+} < 0$. Choosing

$$\gamma = \min\{\epsilon^{+-}, |\epsilon^{-+}|, \mathrm{P}_s\; s \in \mathcal{S}\}$$

finishes the proof. □

Proof of Corollary 4.1 Define

$$\mu^{i,j}|_h = \mu^i_h - \mu^j_h \quad \text{and} \quad \nu^{i,j}|_h = \nu^i_h - \nu^j_h \,.$$

Under no-aggregate risk, $e_{s,t} = e$ for all $s \in \mathcal{S}$ and $t \in \mathbb{N}_0$. Computing the difference $\nu^{i,j}|_i - \nu^{i,j}|_j$ with rules as in (12) , gives

$$\nu^{i,j}|_i - \nu^{i,j}|_j = \log \sum_{s\in\mathcal{S}} (\mathrm{Q}^j)^{\frac{1}{\gamma}} (\mathrm{Q}^i)^{1-\frac{1}{\gamma}} + \log \sum_{s\in\mathcal{S}} (\mathrm{Q}^i)^{\frac{1}{\gamma}} (\mathrm{Q}^j)^{1-\frac{1}{\gamma}} \,.$$

For x in the simplex Δ^S, the function

$$f(x;\mathrm{Q}) = \sum_{s\in\mathcal{S}} (x)^{\frac{1}{\gamma}} (\mathrm{Q})^{1-\frac{1}{\gamma}}$$

is convex with a minimum equal to 1 in $x = \mathrm{Q}$ when $\gamma \in (0,1)$, it is concave with a maximum equal to 1 in $x = \mathrm{Q}$ when $\gamma \in (1,\infty)$. As a result, when $\gamma > 1$, $\mu^{i,j}|_j > \mu^{i,j}|_i$ so that

$$\mu^{i,j}|_i > 0 \Rightarrow \mu^{i,j}|_j > 0 \,.$$

When $\gamma \in (0,1)$ $\mu^{i,j}|_j < \mu^{i,j}|_i$ so that

$$\mu^{i,j}|_i < 0 \Rightarrow \mu^{i,j}|_j < 0 \,.$$

The two sign implications together with Proposition 4.1 prove the statement.

Consider now the aggregate risk case with $S = 2$. Since both state prices and beliefs belong to the simplex, growth premia can be seen as a function of one variable only. Focusing on state $s = 1$, e.g. the state with highest aggregate endowment growth, name $q \in (0,1)$ the state price and $\mathrm{Q}^i, \mathrm{Q}^j \in (0,1)$ agents beliefs, w.l.o.g. $\mathrm{Q}^j < \mathrm{Q}^i$. A CRRA portfolio rule (12) is thus a real function $\alpha(q;\mathrm{Q}) : (0,1) \times (0,1) \to (0,1)$. The function is increasing in q when $\gamma \in (1,\infty)$ and decreasing when $\gamma \in (0,1)$. It is always increasing in Q. Denote q^i as the state price that clears the market when i is the representative agent. From (20) when $S = 2$

$$q^i = \frac{\mathrm{Q}^i}{\mathrm{Q}^i + (1-\mathrm{Q}^i)\left(\frac{g_1}{g_2}\right)^{\gamma}}$$

and similarly for q^j. Since $g_1 > g_2$ and $\mathrm{Q}^i > \mathrm{Q}^j$, then $q^i > q^j$ for all $\gamma \in (0,\infty)$.

When $\gamma > 1$,

$$\mu^{i,j}|_i > 0 \Rightarrow \mu^{i,j}|_j > 0$$

proves the statement together with Proposition 4.1. Since $\alpha(q;\mathrm{Q})$ is increasing in Q

$$\alpha^j(q^j) < \alpha^i(q^j) \quad \text{and} \quad \alpha^j(q^i) < \alpha^i(q^i) \,.$$

Moreover since $\alpha(q;\mathrm{Q})$ is increasing also in q

$$\alpha^j(q^j) < \alpha^j(q^i) \quad \text{and} \quad \alpha^i(q^j) < \alpha^i(q^i)\,.$$

Growth premia depend of relative entropies of the form $I_P(\alpha)$. The function $I_P(x)$ is defined on $(0,1)$, is convex, and has a minimum equal to zero in $x = P$. Assume by absurd that $\mu^{i,j}|_i > 0$ and $\mu^{i,j}|_j < 0$, then in must hold that

$$\alpha^j(q^i) < P < \alpha^i(q^j)$$

as all the other cases would result in a different signs combinations. Since $P < \alpha^i(q^j)$ and $\alpha^i(q^j) < \alpha^i(q^i)$ then

$$I_P(\alpha^i(q^i)) > I_P(\alpha^i(q^j))\,.$$

$\mu^{i,j}|_i > 0$ instead implies

$$I_P(\alpha^j(q^i)) > I_P(\alpha^i(q^i))\,,$$

and $\mu^{i,j}|_j < 0$ implies

$$I_P(\alpha^i(q^j)) > I_P(\alpha^j(q^j))\,.$$

The last three inequalities imply

$$I_P(\alpha^j(q^i)) > I_P(\alpha^j(q^j))$$

which is absurd given the fact that $P > \alpha^j(q^i) > \alpha^j(q^j)$.

The proof is similar for $\gamma \in (0,1)$. Now it is

$$\mu^{i,j}|_i < 0 \Rightarrow \mu^{i,j}|_j < 0$$

that proves the statement together with Proposition 4.1. $\alpha(q;\mathrm{Q})$ is still increasing in Q but it is now decreasing in q

$$\alpha^j(q^j) > \alpha^j(q^i) \quad \text{and} \quad \alpha^i(q^j) > \alpha^i(q^i)\,.$$

Assume by absurd that $\mu^{i,j}|_i < 0$ and $\mu^{i,j}|_j > 0$, then it must hold that

$$\alpha^i(q^i) < P < \alpha^j(q^j)$$

Since $P > \alpha^i(q^i)$ and $\alpha^i(q^j) > \alpha^i(q^i)$ then

$$I_P(\alpha^i(q^j)) > I_P(\alpha^i(q^i))\,.$$

$\mu^{i,j}|_i < 0$ instead implies

$$I_P(\alpha^i(q^i)) > I_P(\alpha^j(q^i))\,,$$

and $\mu^{i,j}|_j > 0$ implies

$$I_P(\alpha^j(q^j)) > I_P(\alpha^i(q^j))\,.$$

The last three inequalities imply

$$I_P(\alpha^j(q^j)) > I_P(\alpha^j(q^i))$$

which is absurd given the fact that $P < \alpha^j(q^j) < \alpha^j(q^j)$.

Proof of Corollary 4.2 Since when agent i has $\gamma^i = 1$ and correct beliefs his growth premium is maximum for all equilibrium returns, then both $(\mu^i|_i - \mu^j|_i)$ and $(\mu^i|_j - \mu^j|_j)$ are positive and statement i) follows from case iii) of Proposition 4.1. Regarding ii) since under no aggregate risk normalized state prices are equal to beliefs in the limit of an agent consuming all the aggregate endowment, irrespectively from his risk preferences, then an agent with correct beliefs has a maximal growth premium when he dominates. It follows that if i has correct beliefs, then $(\mu^i|_j - \mu^j|_j) > 0$. Applying Proposition 4.1 either case ii) or iii), with him dominating P-almost surely, are possible. Regarding the statement iii) of the corollary, examples of preferences and beliefs such an agent with correct beliefs dominates, vanishes, or survives are given in Section 5 Figures 4-6.

Proof of Proposition 4.2 Given two agents i and j, w.l.o.g. $k^i_{BE} > k^j_{BE}$, so that the process $z^{i,j}_t = \log \frac{\phi^i_t}{\phi^j_t}$ has drift

$$\mathrm{E}[\epsilon^{i,j}_{t+1} | \mathcal{F}_t \, s.t. \, z^{i,j}_t = z] = \frac{1}{\gamma}(k^i_{BE} - k^j_{BE}) > 0$$

for every log consumption ratio $z \in (-\infty, +\infty)$. By direct application of Theorem 3.1 i dominates a.s. with respect to agent j.

Finally if $k^i > k^j$ for all $j \neq i$, then each j vanishes against i. Then also all agents (but i) vanish against i,

$$\limsup_{t \to \infty} \frac{\sum_{j \neq i} \phi^j_t}{\phi^i_t} = \limsup_{t \to \infty} \frac{1 - \phi^i_t}{\phi^i_t} = 0,$$

so that i dominates. When only i has correct beliefs, provided $\beta^i \geq \beta^j$ for all $j \neq i$, then also $k^i > k^j$ for all $j \neq i$ and the same result follows.

Note at last that although the relative log-consumption dynamics depends, through market equilibrium prices, on the growth process g_t its drift does not. It follows that the same result holds for any growth process g_t (provide the economic equilibrium is well defined).

Proof of Proposition 4.3 The relative consumption process $\{z^{i,j}_t\}$ has innovation $\epsilon^{i,j}_{s,t+1}$. As shown in the main text the relative size of the survival indexes K_Y determines the sign of $\mathrm{E}[\epsilon^{i,j}_{t+1} | \mathcal{F}_t \, s.t. \, z^{i,j}_t = z]$ in the limit of $z \to \pm\infty$. The proof follows by applying Theorem D.2 along the same line of the proof of Proposition 4.1.

In particular, we have to show that i) when both limit conditional drifts are positive there exists a $\gamma > 0$ such that $\mathrm{Prob}\{\epsilon^{i,j}_{t+1} > \gamma\} > \gamma$ and, vice-versa, when limits conditional drifts are negative there exists a $\gamma > 0$ such that $\text{Prob}\{\epsilon^{i,j}_{t+1} < -\gamma\} > \gamma$; *ii*) there exists a B such that $\text{Prob}\{|\epsilon^{i,j}_{t+1}| < B\} = 1$.

To prove *i*), for every s define

$$f_s(q_s) = \epsilon_{s,t+1} = \log \frac{(\beta^i\,\mathrm{Q}^i_s)^{\frac{1}{\gamma^i}}}{(\beta^j\,\mathrm{Q}^j)^{\frac{1}{\gamma^j}}} q_s^{\frac{1}{\gamma^j}-\frac{1}{\gamma^i}}$$

as the innovation when both i and j use CRRA saving and portfolio rules. When i dominates $q_s = q^i_s = \beta^i\,\mathrm{Q}^i_s\,g_s^{-\gamma^i}$ and

$$f_s(q^i_s) = \frac{1}{\gamma^j}\log\left(\frac{\beta^i\,\mathrm{Q}^i_s}{\beta^j\,\mathrm{Q}^j}\right) g_s^{\gamma^j-\gamma^i}\,.$$

Likewise when j dominates

$$f_s(q^j_s) = \frac{1}{\gamma^i}\log\left(\frac{\beta^i\,\mathrm{Q}^i_s}{\beta^j\,\mathrm{Q}^j}\right) g_s^{\gamma^j-\gamma^i}\,.$$

Since $f_s(q^i_s)$ and $f_s(q^j_s)$ differ only for a constant of proportionality they are either both positive or both are negative. Note also that $f_s(q_s)$ is monotone, either increasing or decreasing depending on the relative size of γ^i and γ^j. All this together and the fact that $q_s \in (q^i_s, q^j)$ (w.l.o.g. $q^i_s < q^j_s$) implies that

$$f_s(q_s) > \min\{f_s(q^i_s), f_s(q^j_s)\}. \quad \text{for all} \quad q_s \in (q^i_s, q^j)\,. \tag{45}$$

When the process is such that

$$\lim_{z\to\pm\infty} \mathrm{E}[\epsilon^{i,j}_{t+1}|\mathcal{F}_t\, s.t.\, z^{i,j}_t = z] > 0$$

then there exists at least one s such that $f_s(q^j) > 0$ and $f_s(q^i) > 0$. By (45)

$$f_s(q_s) > \min\{f_s(q^i_s), f_s(q^j_s)\}.$$

for all $q_s \in (q^i_s, q^j)$. Naming $\gamma = \min\{\min\{f_s(q^i_s), f_s(q^j_s)\}, \mathrm{P}_s\ s \in \mathcal{S}\}$ proves that

$$\text{Prob}\{\epsilon^{i,j}_{t+1} > \gamma\} > \gamma\,.$$

When the conditional drift is negative at the borders, the proof follows the same lines with $\gamma = \min\{|\max\{f_s(q^i_s), f_s(q^j_s)\}|, \mathrm{P}_s\ s \in \mathcal{S}\}$.

To prove *ii*) we need to show that there exists a B such that

$$|f_s(q_s)| < B$$

for every s. Given the properties of f_s exploited to prove point *i*) it also

$$|f_s(q_s)| < \max\{|f_s(q^i_s)|, |f_s(q^j_s)|\}.$$

Choosing $B > \max\{|f_s(q^i_s)|, |f_s(q^j_s), s \in \mathcal{S}\}$ proves the result.

Proof of Corollary 4.3 The corollary follows from the application of Theorem D.1. The theorem has two hypothesis: that the log relative consumption process has finite increments, and that the drift at $\pm\infty$ points to the center. This second hypothesis holds provided parameters are as specified, as it is proved in the main text immediately before the Corollary. I turn to show that also the finite increment hypothesis holds. Name q^i and $\delta|_i$ the normalized state price vector and market discount rate set by agent i when he consumes all the aggregate endowment. In a two-agent economy state prices are in the interior of the set $\Omega_q = \times_{s\in\mathcal{S}}[\min\{q^j_s, q^i_s\}, \max\{q^j_s, q^i_s\}]$ and market discount rate in the interior of $\Omega_\delta = [\min\{\delta|_j, \delta|_i\}, \max\{\delta|_j, \delta|_i\}]$. Given the continuity of one-period substitution and portfolio rules of both agents, for each s there exists a maximum innovation given by

$$\epsilon_s = \max\left\{\left|\log\frac{\delta^i(\delta,q)\alpha^i_s(q)}{\delta^j(\delta,q)\alpha^j_s(q)}\right| \text{for } q\in\Omega_q\,,\delta\in\Omega_\delta\right\}\,.$$

Choosing $B > \max\{\epsilon_s\,, s\in\mathcal{S}\}$ suffices for the requirement.

Proof of Proposition 4.4 Under Assumption 4.1 all agents $i \in \mathcal{I}$ hold the market portfolio so that $\alpha^i = \alpha^j$ for all i and j. As a result for any couple (i,j) the log relative consumption dynamics $z^{i,j}_t$ derived from (3) is deterministic and has innovation equal to

$$\epsilon^{i,j}_{t+1} = \log\frac{\delta^i_t}{\delta^j_t}\,.$$

Substitution rates are given by (11). Market rates δ_t are set by

$$\delta_t = \frac{\sum_{i\in\mathcal{I}}\delta^i_t\phi^i_t}{\sum_{s\in\mathcal{S}}\mathrm{Q}^0_s g_s}\,. \tag{46}$$

where Q^0 is the set of normalized state price that supports all agents holding the market portfolio. Defining for each agent i

$$k^i = \frac{(\beta^i)^{\frac{1}{\rho^i}}}{\sum_{s\in\mathcal{S}}\mathrm{Q}^0_s g_s}\left(\sum_{s\in S}(\mathrm{Q}^i_s)^{\frac{1}{\gamma^i}}(\mathrm{Q}^0_s)^{1-\frac{1}{\gamma^i}}\right)^{\frac{\gamma^i}{1-\gamma^i}\frac{1-\rho^i}{\rho^i}}\,,$$

which does not depend on time given that beliefs and market equilibrium prices are i.i.d., equation (46) becomes

$$1 = \sum_{i\in\mathcal{I}}\left(\frac{1}{\delta_t}\right)^{\frac{1}{\rho^i}} k^i\phi^i_t = \sum_{i\in\mathcal{I}} f^i(\delta_t)\phi^i_t\,,$$

where f^i is defined appropriately. Since $\rho^i > 0$ for all i, each function f^i is decreasing in δ_t. Moreover it holds $f^i(\delta|_i) = 1$, where $\delta|_i$ is the interest rate set by i when he has all the aggregate endowment:

$$\delta|_i = \frac{\beta^i}{\sum_{s\in\mathcal{S}} \mathrm{Q}^0_s\, g_s} \left(\sum_{s\in S} \mathrm{Q}^i_s\, g_s^{1-\gamma^i} \right)^{\frac{1-\rho^i}{1-\gamma^i}} .$$

It follows that for each i

$$f^i(\delta) \gtreqless 1 \;\Leftrightarrow\; \delta \lesseqgtr \delta|_i \, .$$

As a result, for each t

$$\delta_t \in (\min\{\delta|_i,\, i \in \mathcal{I}\}, \max\{\delta|_i,\, i \in \mathcal{I}\})\, .$$

Moreover for each t and j

$$f^j(q_t) < \max\{\delta|_i\, i \in \mathcal{I}\}$$

if $\delta|_j < \max\{\delta|_i\, i \in \mathcal{I}\}$, and

$$f^j(q_t) > \max\{\delta|_i\, i \in \mathcal{I}\}$$

if $\delta|_j = \max\{\delta|_i\, i \in \mathcal{I}\}$. Since by construction $\delta^i_t = \delta_t f^i(\delta_t)$ we have proved that if i defines the maximal rate $\delta|_i$, then his substitution rate is higher than that of all other agents and dominates. Dominance is sure because the dynamics of market discount rates and substitution rates is deterministic. As a result, the same relative consumption dynamics occurs for all path σ. In the same way, if i does not define the maximum rate $\delta|_i$, then he vanishes.

Proof of Proposition 4.5 The result follows by noticing that under Assumption 4.1

$$\frac{\mathrm{Q}^i_s\, g_s^{-\gamma^i}}{\mathrm{Q}^j_s\, g_s^{-\gamma^j}} = \frac{\sum_{s'\in\mathcal{S}} \mathrm{Q}^i_{s'}\, g_{s'}^{-\gamma^i}}{\sum_{s'\in\mathcal{S}} \mathrm{Q}^i_{s'}\, g_{s'}^{-\gamma^i}} \quad \text{for all} \quad s \in \mathcal{S}\, ,$$

so that survival indexes k_Y do not depend on P, and by applying Proposition 4.4.

Proof of Corollary 4.4 Under no aggregate risk, Assumption 4.1 implies that for all $i, j \in \mathcal{I}$ $\mathrm{Q}^i = \mathrm{Q}^j = \mathrm{Q}^0$. From simple computation it holds

$$\delta|_i \lesseqgtr \delta|_j \;\Leftrightarrow\; \frac{\beta^i}{\beta^j}\, g^{\rho^j-\rho^i} \lesseqgtr 1\, .$$

The above and Proposition 4.4 prove the statement.

Proof of Corollary 4.5 Under Assumption 4.1, beliefs and normalized state prices are such that when $\beta^i = \beta^j$

$$\frac{\delta|_i}{\delta|_j} = (\sum_{s\in\mathcal{S}} \mathrm{Q}^0_s\, g_s)^{\rho^j-\rho^i} \frac{\left(\sum_{s\in S}(\mathrm{Q}^i_s)^{\frac{1}{\gamma^i}}(\mathrm{Q}^0_s)^{1-\frac{1}{\gamma^i}}\right)^{\gamma^i\frac{1-\rho^i}{1-\gamma^i}}}{\left(\sum_{s\in S}(\mathrm{Q}^j_s)^{\frac{1}{\gamma^j}}(\mathrm{Q}^0_s)^{1-\frac{1}{\gamma^j}}\right)^{\gamma^j\frac{1-\rho^j}{1-\gamma^j}}}.$$

As a function of beliefs Q

$$\left(\sum_{s\in S}(\mathrm{Q}_s)^{\frac{1}{\gamma}}(\mathrm{Q}^0_s)^{1-\frac{1}{\gamma}}\right)^{\gamma\frac{1-\rho}{1-\gamma}}$$

has a stationary point in $\mathrm{Q} = \mathrm{Q}^0$ where it is equal to one. Moreover it is convex when $\rho \in (0, 1)$ and concave when $\rho > 1$. It follows that, provided $\sum_{s\in\mathcal{S}} \mathrm{Q}^0_s\, g_s \geq 1$, $\rho^i < 1 < \rho^j$ implies that the ratio $\delta|_i/\delta|_j > 1$. Applying Proposition 4.4 concludes the proof.