All-Pay Auctions with Ties

Alan Gelder · Dan Kovenock · Brian Roberson

24 December 2016

Abstract We study the two-player, complete information all-pay auction in which a tie ensues if neither player outbids the other by more than a given amount. In the event of a tie, each player receives an identical fraction of the winning prize. Thus players engage in two margins of competition: losing versus tying, and tying versus winning. Two pertinent parameters are the margin required for victory and the value of tying relative to winning. We fully characterize the set of Nash equilibria for the entire parameter space. For much of the parameter space, there is a unique Nash equilibrium which is also symmetric. Equilibria typically involve randomizing over multiple disjoint intervals, so that in essence players randomize between attempting to tie and attempting to win. In equilibrium, expected bids and payoffs are non-monotonic in both the margin required for victory and the relative value of tying.

Keywords All-pay auction · contest · ties · draws · bid differential

JEL Classification C72 · D44 · D72 · D74

1 Introduction

Eighteen seasons into his playing career, baseball great Frank Robinson famously stated, "Close don't count in baseball. Close only counts in horseshoes and hand grenades" (*Time* magazine, 31 July 1973). Although being close but coming up short does not count for much in baseball, it still has value in many other contexts. Ties in the business arena can take the form of multi-source or split-award procurement contracts. A firm that clearly outshines the competition may receive the full contract; but when competitors fail to adequately distinguish themselves, the contract may be split between them.[1] Political gridlock connotes a tie in which the status quo is perpetuated

A. Gelder
Institute for Defense Analyses, Alexandria, VA 22311, USA
E-mail: abgelder@gmail.com

D. Kovenock
Economic Science Institute, Chapman University, Orange, CA 92866, USA
E-mail: kovenock@chapman.edu

B. Roberson
Department of Economics, Purdue University, West Lafayette, IN 47907, USA
E-mail: brobers@purdue.edu

[1] For example, in the first great engine war of the mid-1980's, the U.S. Air Force split the award of a $10 billion engine contract between General Electric and Pratt & Whitney. A key feature of that split-award decision was

instead of either party achieving its ideal policy. Even so, for a party that has fallen behind in numbers, achieving gridlock is a partial victory. In military conflicts, instead of one side dominating the other, more or less equally matched forces may tie in the sense of a stalemate or a standoff. Simply having the larger force, of itself, is not enough to win. Winning rather entails far and away surpassing the opponent by some critical degree.

Contests such as these where players compete by making sunk resource expenditures have long been modeled with the classic all-pay auction (Baye et al. 1996; Hillman and Riley 1989).[2] In its standard formulation, the prize is awarded to the player with the highest resource expenditure, no matter how large or small the difference in the players' expenditures may be. Ties are therefore a knife-edge event. For many contest settings, however, the magnitude of the expenditure difference matters, and a tie becomes a viable third outcome. This paper extends the two-player, complete information all-pay auction by allowing a tie to occur when the difference in expenditures falls below a specified threshold. We refer to this threshold as the *tie margin*. The introduction of a tie margin creates two distinct margins of competition: losing versus tying, and tying versus winning. As such, players are concerned with both the tie margin and the *tie prize*, or the prize value that each player receives in the event of a tie. Under the assumption that winning is preferred to tying and that tying is preferred to losing, this paper completely characterizes the set of equilibria for the entire parameter space of tie margins and tie prizes.

As in the classic all-pay auction (with a tie margin of zero), equilibrium in the all-pay auction with a tie prize and a (strictly positive) tie margin is in mixed strategies. In any mixed strategy equilibrium of the all-pay auction with ties, each bid (in the support of an equilibrium mixed strategy) faces either a losing-versus-tying margin of competition or a tying-versus-winning margin of competition. For example, any bid between zero and the size of the tie margin can at best tie an opponent's low bid. Thus, losing-versus-tying is the relevant margin of competition for bids in this range. For bids above the size of the tie margin, in any equilibrium, players' mixed strategies randomize over a set of intervals of bids and systematic gaps in such a way that each bid faces a single margin of competition: losing-versus-tying or tying-versus-winning. Introducing a strictly positive tie margin therefore results in a mixed-strategy equilibrium featuring the randomization of bids across disjoint intervals. These intervals are then further divided into sub-intervals which have one of two distinct density rates, corresponding to either the losing-versus-tying margin of competition or the tying-versus-winning margin of competition. We find that there exists a unique symmetric equilibrium for nearly all parameter configurations. For a range of parameters in which the tie prize is less than or equal to half of the winning prize, there also exist asymmetric equilibria.

In the unique symmetric mixed-strategy equilibrium, the number of disjoint intervals in the support—as well as the measure of each interval—is dependent on both the tie margin and the tie prize. For a given tie prize, the number of disjoint intervals in the support of the symmetric equilibrium (weakly) increases as the tie margin decreases. In the limit, as the tie margin approaches zero, the number of disjoint intervals becomes arbitrarily large and converges to the equilibrium of the classic all-pay auction (with a tie margin of zero), in which players continuously randomize over the entire interval of bids from zero to the value of the winning prize. As a result

the strategic uncertainty as to whether a single proposal would sufficiently dominate the competition and win the contract outright or whether the proposals would be relatively close and result in a split contract. More recently, the second great engine war over the contract to supply engines for the F-35 joint strike fighter currently features a single winner (or supplier), Pratt & Whitney. See Drewes (1987) and Amick (2005) for further details.

[2] The list of applications is widespread and includes lobbying, litigation, R&D competitions, college admissions, election campaigns, warfare, etc. Konrad (2009) and Dechenaux et al. (2015) respectively survey the theoretical and experimental literature.

of the discrete jumps between the disjoint intervals, we find that for sufficiently high tie prizes, expected bids are non-monotonic in the tie-margin. A parallel result is that expected bids are also non-monotonic in the tie prize for a range of sufficiently low tie margins.

Multiple margins of competition—along the lines of losing-versus-tying and tying-versus-winning—arise in several related applications. For example, Szech and Weinschenk (2013) build on the work of Klemperer (1987) in examining a generalized form of price competition with consumer rebates. In the simplest case, each consumer receives a rebate from exactly one firm and each firm offers a homogeneous rebate to its distinct subset of the consumers. A profile of prices then results in one of two outcomes. If there exists a firm whose price is less than its competitor's after-rebate price, then that firm wins all of the consumers. Otherwise, each firm wins its own subset of the consumers. Note that rebates generate a tie margin, and thus this environment features both a losing-versus-tying margin of competition and a tying-versus-winning margin of competition. This same issue arises in the context of price competition with tariffs as in Fisher and Wilson (1995), where tariffs generate a tie margin. It likewise occurs in price competition with segmented consumers and transportation costs as in Shilony (1977), where transportation costs across segments generate a tie margin. Also related are split-award (procurement) auctions which feature explicit rules for how awards are split, conditional on the profile of bids received. Recent examples include Chaturvedi et al. (2014) and Gong et al. (2012).

Our model is naturally connected to several variants of the all-pay auction.[3] For example, Szech (2015) extends Che and Gale's (1998) model of the all-pay auction with a common bidding cap to examine the issue of asymmetric tie-breaking rules.[4] The tie-margin is zero in that formulation, but because of the presence of a bidding cap, the choice of a tie-breaking rule (which is equivalent to a tie prize under risk neutrality) is an important determinant of equilibrium behavior. Stong (2014) identifies preliminary results for the all-pay auction with ties under incomplete information. Focusing specifically on a case where the tie margins are relatively large, he likewise identifies that players randomize their bids over disjoint intervals in equilibrium.[5] In the case of an all-pay auction in which the strategy space is discrete, ties may arise with positive probability (e.g. Bouckaert et al. 1992; Baye et al. 1994; Cohen and Sela 2007; Cohen and Schwartz 2013). With a discrete strategy space the gap between feasible bids creates similar losing-verses-tying and tying-verses-winning tradeoffs as our tie margin. Under the assumption that winning is weakly preferred to tying and that tying is weakly preferred to losing, Cohen and Sela (2007) argue that the size of the prize in the event of a tie does not affect the players' efforts. As we show in this paper, the discrete strategy space game creates qualitatively different incentives than the continuous strategy space version of the game.

[3] Other contest success functions, aside from the all-pay auction, allow for random noise to play a role, thereby creating a probabilistic relationship between a player's expenditure and his probability of winning. The possibility of ties under alternative contest success functions is discussed in more detail in Gelder et al. (2015).

[4] Che and Gale show that if bidders in an all-pay auction have asymmetric valuations for the winning prize, then an auction-designer can increase expected revenue by introducing a bidding cap that levels the playing field by reducing the stronger player's ability to outbid the weaker player. Szech (2015) then goes on to show that the auction designer can do even better, with regards to equilibrium expected expenditure, by introducing an asymmetric tie-breaking rule that favors the weaker player (instead of using a symmetric tie-breaking rule where each player wins the prize with equal probability in the event of a tie).

[5] Stong focuses on equilibria where the upper bound of the support is no more than twice the size of the tie margin (in our map of the parameter space, this corresponds to region II of Figure 1). His model has incomplete information in that players have privately known valuations of the winning prize and privately known bidding costs. The ratio of the tie prize to the winning prize is, however, common across players.

2 Model

The all-pay auction with ties, which we label:

$$APT\{\delta, \beta, v\}$$

is the one-shot, complete-information game in which two players each privately submit a non-refundable bid $x_i \geq 0$. The difference between the bids determines the outcome. A player wins a prize of $v > 0$ if his bid exceeds his opponent's by strictly more than $\delta \geq 0$. If the two bids are within δ of each other, each player receives βv where $\beta \in [0,\ 1)$. A player who is outbid by more than δ receives no prize. Thus, player i's payoff is as follows:

$$u_i(x_i,\ x_{-i}) = \begin{cases} v - x_i & \text{if } \ x_i - x_{-i}\ > \delta \\ \beta v - x_i & \text{if } \ |x_i - x_{-i}| \leq \delta \\ -x_i & \text{if } \ x_{-i} - x_i\ > \delta \end{cases} \tag{1}$$

For much of the parameter space, equilibria are in non-degenerate mixed strategies. Letting G_i be player i's bid distribution, and denoting a mass point at x within G_i as $\alpha_i(x) \in [0, 1]$, we can write player i's expected utility for a bid of x as:

$$\begin{aligned} u_i(x,\ G_{-i}) &= [G_{-i}(x-\delta) - \alpha_{-i}(x-\delta)]\, v\, + \\ &\qquad\qquad [G_{-i}(x+\delta) - G_{-i}(x-\delta) + \alpha_{-i}(x-\delta)]\, \beta v - x \\ &= G_{-i}(x+\delta)\, \beta v + [G_{-i}(x-\delta) - \alpha_{-i}(x-\delta)]\, (1-\beta) v - x \end{aligned} \tag{2}$$

The last line of Equation 2 highlights the two relevant margins of competition. A bid of x narrowly ties a bid of $x+\delta$ and is rewarded with βv, the marginal benefit for tying relative to losing. Similarly, x beats bids below $x - \delta$, and since these bids otherwise would be tied, the marginal benefit is $(1-\beta)v$.

3 Symmetric Equilibrium

For any given tie margin $\delta \geq 0$ and tie prize $\beta v \in [0,\ v)$, the all-pay auction with ties has a symmetric Nash equilibrium. As illustrated by the different regions in Figure 1, the qualitative nature of equilibria does, however, vary throughout the parameter space. This section addresses these different regions in turn and fully characterizes the set of symmetric equilibria. In all but a few special cases, there is a unique symmetric equilibrium—often with the further distinction of being the unique Nash equilibrium. Asymmetric equilibria, when they exist, are characterized in Section 5.

3.1 Prohibitively large tie margins: $\delta \geq (1-\beta)v$ where $\delta > 0$

We begin with the case of large tie margins. That is, $\delta \geq (1-\beta)v$ for $\delta > 0$, or the area depicted by region I of Figure 1. The tie margin δ is a strict lower bound on the cost of winning versus tying, and $(1-\beta)v$ is the added benefit from doing so. Therefore, when this condition holds, players are content to settle for a tie since the cost of winning exceeds the associated benefit. Not only do players tie, but they tie with bids of zero in equilibrium:

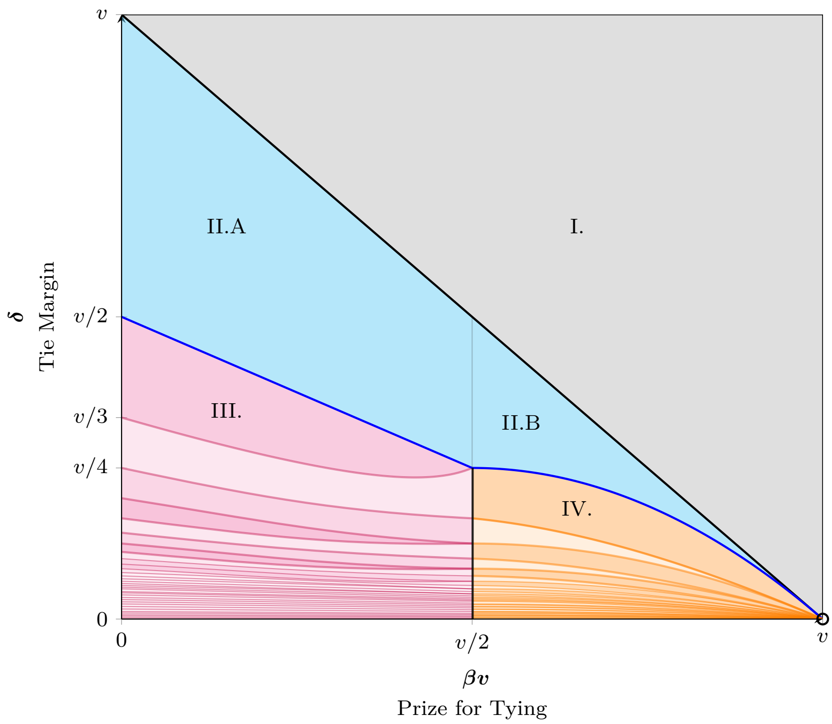


Fig. 1: Symmetric equilibrium regions in the parameter space.

Theorem 1 *If $\delta \geq (1-\beta)v$ and $\delta > 0$, there exists a unique Nash equilibrium in which each player bids* 0 *with probability one.*

A tie margin of this size is prohibitive in the sense that it completely stymies any active competition. This is the only parameter range for $APT\{\delta, \beta, v\}$ in which an equilibrium exists in pure strategies.

3.2 Limiting Boundaries: $\beta = 0$, $\delta \in (0,\ (1-\beta)v)$; and $\beta \in [0,\ 1)$, $\delta = 0$

Before proceeding to the core of our analysis in regions II through IV of Figure 1, we briefly consider equilibrium behavior along the limiting boundaries where either the tie prize or the tie margin is equal to zero. The more familiar of these cases is $\delta = 0$ and $\beta \in [0,\ 1)$, which is nothing more than the standard all-pay auction. Player's simply need to outbid their opponent. Equilibrium behavior is therefore invariant to the exact specification of the tie prize since, after all, tying is a zero probability event. Along the vertical axis of Figure 1, winning requires outbidding the other player by a strictly positive tie margin. Yet the result of tying is no different from losing. The unique equilibrium for $\delta \in (0,\ v)$ and $\beta = 0$ then entails a set of mass points, starting at 0 and placed at multiples of δ thereafter.[6] With the possible exception of the top mass point, each contains just enough mass (δ/v) so that beating it by δ exactly compensates for the cost of bidding. Thus here, as in the standard all-pay auction, the expected equilibrium payoff is zero. Formally, we have the following:

[6] The existence of this equilibrium requires a slight alteration of Equation 1: $u_i(x_i,\, x_{-i}) = v - x_i$ if $x_i - x_{-i} = \delta$. Otherwise, the incentive would be to have mass points that are slightly more than δ apart. With a continuous bidding space, however, this would not constitute an equilibrium since payoffs could be improved by placing the mass points even closer.

Theorem 2 (Boundaries) *If $\delta = 0$ and $\beta \in [0,\ 1)$, the unique Nash equilibrium coincides with the standard all-pay auction. That is, each player bids as follows:*

$$G_i(x) = \begin{cases} x/v & x \in [0,\ v] \\ 1 & x > v \end{cases}$$

If $\delta \in (0,\ v)$ and $\beta = 0$, the unique Nash equilibrium consists of each player bidding according to the following distribution:

$$G_i(x) = \begin{cases} \delta/v & x \in [0,\ \delta) \\ 2\delta/v & x \in [\delta,\ 2\delta) \\ \vdots & \vdots \\ z\delta/v & x \in [(z-1)\delta,\ z\delta) \\ \vdots & \vdots \\ \left\lfloor \frac{v}{\delta} \right\rfloor \delta/v & x \in \left[\left(\left\lfloor \frac{v}{\delta} \right\rfloor - 1 \right) \delta,\ \left\lfloor \frac{v}{\delta} \right\rfloor \delta \right) \\ 1 & x \geq \left\lfloor \frac{v}{\delta} \right\rfloor \delta \end{cases}$$

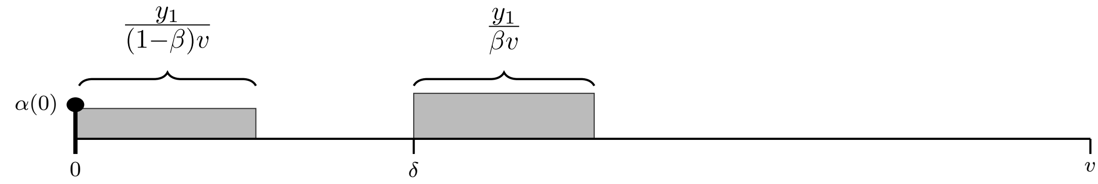
As we will see, these are indeed limiting equilibria in the sense that for a fixed tie prize, the symmetric equilibria of $APT\{\delta, \beta, v\}$ converge to the equilibrium of the standard all-pay auction as the tie margin goes to zero. Likewise, for a fixed tie margin, the symmetric equilibria of $APT\{\delta, \beta, v\}$ converge to the equilibrium outlined above with mass points spaced δ apart as the tie prize goes to zero.

3.3 Tie prize less than one-half: $\beta \in (0,\ 1/2)$ and $\delta \in (0,\ (1-\beta)v)$

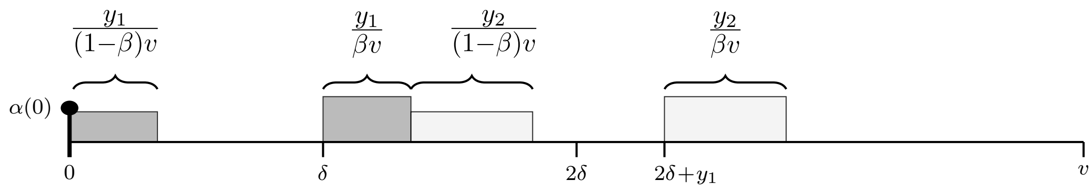
Turning to regions II.A and III of Figure 1, for any given $\beta \in (0,\ 1/2)$ and $\delta \in (0,\ (1-\beta)v)$, there is a unique symmetric equilibrium. Precise equilibrium strategies differ from one another as a function of β, δ, and v, yet each is composed of two fundamental building blocks. The first is simply a mass point at zero $\alpha(0) \in (0, 1)$. Second, the remainder of the bidding strategy consists of interval pairs, each with continuously distributed mass. Intervals within each pair are disjoint, have the same length, and have lower bounds that are exactly δ apart. Each interval also has a uniform distribution but with differing density rates: the lower interval in each pair has a density rate of $1/[(1-\beta)v]$, while the upper interval has a density rate of $1/(\beta v)$. One final characteristic is that there are no gaps between successive interval pairs—the upper bound of the upper interval of one pair is the lower bound of the lower interval of the next. Figure 2 illustrates examples of equilibrium strategies with one, two, and three pairs of intervals. The strategy with one interval pair corresponds to the equilibrium structure of region II.A in Figure 1, while the strategies with two and three interval pairs match the structure of the top two subdivisions of region III. Successive subdivisions have successively more interval pairs. Each of the strategies in Figure 2 is plotted with the same scale, which extends from zero to the value of the winning prize v. The tie margin δ, however, decreases with each additional interval pair.

By definition, for a Nash equilibrium to hold, players must receive the same expected payoff from any bid within the support of their strategy.[7] The density rates of $1/[(1-\beta)v]$ and $1/(\beta v)$

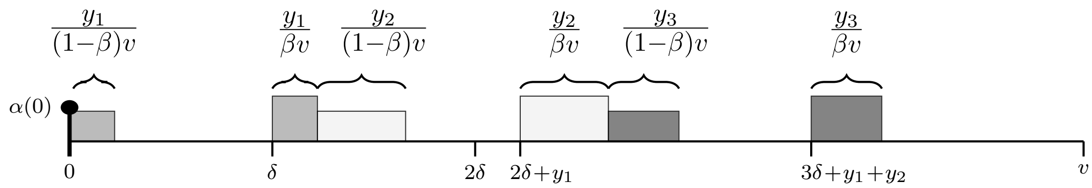
[7] With a mass point at zero, one possible exception is a bid of precisely δ. Tying instead of beating $\alpha(0) > 0$ leads to a strictly lower payoff at δ compared to bids arbitrarily close to δ from above. However, since distributional



(a) One pair of intervals, plus a mass point at zero.



(b) Two pairs of intervals, plus a mass point at zero.



(c) Three pairs of intervals, plus a mass point at zero.

Fig. 2: Density plots of the unique symmetric equilibrium strategies for various δ when $\beta < 1/2$. Each interval's mass is labeled in terms of its length y_i and density rate: $1/[(1-\beta)v]$ or $1/\beta v$.

serve to keep players indifferent between bids within an interval. For instance, if x' and x are bids in the lower interval of a given pair ($x' > x$), then the cost of going between x' and x is exactly offset by the additional mass in the upper interval that can be tied. That is, $x' - x = [(x'-x)/\beta v] \times \beta v$. As is evident from Figure 2, there is also no additional mass to beat by increasing from x to x': $G_i(x' - \delta) = G_i(x - \delta)$. Likewise, if x' and x are instead bids within the upper interval of a given pair, there is no additional mass to tie in $[x+\delta,\ x'+\delta]$, so the bidding cost is offset by mass in the lower interval that can be beat instead of tied: $x' - x = [(x'-x)/(1-\beta)v] \times (1-\beta)v$. We also need to ensure that expected payoffs remain the same across intervals. Without a gap in the support between successive interval pairs, there is a fluid transition between the upper interval of one pair and the lower interval of the next. The concern is whether expected payoffs are maintained across the gap between the lower and upper intervals of each pair. Letting x_j be an element of the lower interval of the j^{th} interval pair, the following must hold for $j \in \{1, \ldots, k\}$, where k is the number of interval pairs:

$$u_i(x_j,\ G_{-i}) = u_i(x_j + \delta,\ G_{-i}) \tag{3}$$

supports are necessarily closed sets, a bid of δ is permitted to be in the support as the endpoint of an interval. Even still, equilibrium behavior is invariant to imposing a special tie breaking rule where a bid of δ beats a bid of zero—a bid of δ occurs with zero probability either way. Other examples of select points within equilibrium supports having lower payoffs due the the presence of mass points include Osborne and Pitchik (1986) and Deneckere and Kovenock (1996).

We denote the length of each interval in the j^{th} pair as y_j. With just one interval pair ($k{=}1$), as in the top panel of Figure 2, the expected utilities for bids of x_1 and $x_1+\delta$ are:

$$u_i(x_1,\ G_{-i}) = \left[\alpha(0) + \frac{y_1}{(1-\beta)v} + \frac{x_1}{\beta v}\right]\beta v - x_1$$

$$u_i(x_1+\delta,\ G_{-i}) = \left[\alpha(0) + \frac{x_1}{(1-\beta)v}\right]v + \left[\frac{y_1-x_1}{(1-\beta)v} + \frac{y_1}{\beta v}\right]\beta v - x_1 - \delta$$

So $u_i(x_1,\ G_{-i}) = u_i(x_1+\delta,\ G_{-i})$ implies that:

$$\delta = \alpha(0)(1-\beta)v + y_1 \tag{4}$$

If there are two or more interval pairs ($k \geq 2$), then $u_i(x_1+\delta,\ G_{-i})$ has the additional term of $y_2\,[\beta/(1-\beta)]$ since a bid of $x_1+\delta$ also ties the lower interval of the second interval pair. In which case, $u_i(x_1,\ G_{-i}) = u_i(x_1+\delta,\ G_{-i})$ becomes:

$$\delta = \alpha(0)(1-\beta)v + y_1 + y_2\left(\frac{\beta}{1-\beta}\right) \tag{5}$$

Equation 3 yields similar expressions for x_j when $j \geq 2$. The bids of x_j and $x_j+\delta$ both beat any mass in the first $j-2$ interval pairs, as well as mass in the lower interval of the $(j-1)^{th}$ pair. The only relevant intervals are the upper interval of the $(j-1)^{th}$ pair, the lower interval of the $(j+1)^{th}$ pair (if $j<k$), and both intervals in the j^{th} pair. For $j \in \{2,\ldots,k-1\}$, Equation 3 implies:

$$\delta = y_{j-1}\left(\frac{1-\beta}{\beta}\right) + y_j + y_{j+1}\left(\frac{\beta}{1-\beta}\right) \tag{6}$$

Finally, for $j=k$, we have:

$$\delta = y_{k-1}\left(\frac{1-\beta}{\beta}\right) + y_k \tag{7}$$

The set of Equations 5, 6, and 7 (or only Equation 4 for case where $k=1$) form a system of k equation with $k+1$ unknowns: $(\alpha(0),\ y_1,\ldots,\ y_k)$. We close this system by requiring that the total mass in the distribution sum to one:

$$1 = \alpha(0) + \sum_{j=1}^{k} \frac{y_j}{(1-\beta)\beta v} \tag{8}$$

Together with equation 8, we refer to this system of $k+1$ equations collectively as System ($\star$). As is shown in the appendix, for any $k \geq 1$, this system uniquely defines values for $\alpha(0)$ and $y_1,\ldots,\ y_k$ (see Proposition 1). In a symmetric equilibrium, each of these values must also be strictly positive (if one of the values were zero, then players could not be indifferent between the upper and lower intervals of an adjoining pair). Although payoff equivalence within the support is given by System ($\star$), we must also check for potential deviations.

We can quickly rule out the profitability of placing bids within the gaps between intervals. For instance, when there is only one pair of intervals as in panel (a) of Figure 2, the upper bound of the lower interval y_1 ties the upper bound of the upper interval $\delta + y_1$, so there is no additional mass to tie by bidding in the gap $(y_1,\ \delta)$. Neither is there any mass to beat since all bids within this gap are below the tie margin δ. Therefore, given the added cost, bids within the gap are strictly dominated. The argument is similar when there are two or more pairs of intervals, as in panels (b) and (c) of Figure 2. Bids within $(y_1,\ \delta)$ still cannot beat any mass, and although these bids may tie the lower interval of the second pair, the density rate of $1/[(1-\beta)v]$ over $(\delta+y_1,\ \delta+y_1+y_2]$ is not enough to compensate for the cost of bidding. For the gap between the lower and upper intervals of the j^{th} pair, $(\underline{o}_j,\ \overline{o}_j)$, where $j \geq 2$, a player's expected payoff is monotonically decreasing over the first part of the gap. That is, over $(\underline{o}_j,\ \underline{o}_j + \delta - y_j - y_{j-1}]$, where there is no mass to beat that $\underline{o}_j$ does not already beat, and any additional mass to tie that $\underline{o}_j$ does not already tie has the lower density rate of $1/[(1-\beta)v]$. Payoffs over the remainder of the gap, however, are monotonically increasing since the expected benefit of beating mass in the upper interval of the $(j-1)^{th}$ pair surpassing the cost (i.e. $(1-\beta)v/(\beta v) > 1$ for $\beta < 1/2$). A deviating player would therefore want to increase his bid to $\overline{o}_j$, which is in the support, and beat the $(j-1)^{th}$ pair entirely.

Ultimately, the number of interval pairs within an equilibrium is pinned down by whether it is profitable to outbid the support. The point to check is a bid of exactly δ above the maximal element in the support. Such a bid is a sure win. Bidding any higher is strictly dominated due to the cost, and bids that are lower either entail a higher cost for tying and beating the same amount of mass as the maximal element in the support, or they beat some—but not all—of the topmost interval; and at a density rate of $1/(\beta v)$, there is a higher payoff for beating the entire interval. In equilibrium, there must be enough interval pairs so that the cost of guaranteeing a sure win is sufficiently high relative to the payoff achieved from the costless bid of zero. Although the exact number of interval pairs k is implicitly defined, there is a finite set of possible values it can take. The upper bound of that set $\lfloor v/\delta \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function, is based on the fact that intervals within a pair have lower bounds that are spaced δ apart, and a bid of zero strictly dominates bids greater than v. An explicit characterization of equilibrium is given below.

Theorem 3 *For $\delta \in (0,\ (1-\beta)v)$ and $\beta \in (0,\ 1/2)$, the game $APT\{\delta, \beta, v\}$ has a unique symmetric equilibrium. Let $\boldsymbol{x}^\star = (\alpha, y_1, \ldots, y_z, \ldots, y_k)$ be the unique solution to System ($\star$), where $k \in \{1, \ldots, \lfloor v/\delta \rfloor\}$ is implicitly defined as the largest integer for which all elements of $\boldsymbol{x}^\star$ are strictly positive. Each player's bid distribution G_i is then defined as follows:*

$$G_i(x) = \begin{cases} \alpha + \dfrac{x}{(1-\beta)v} & x \in [0,\ y_1) \\ \alpha + \dfrac{y_1}{(1-\beta)v} & x \in [y_1,\ \delta) \\ \alpha + \dfrac{y_1}{(1-\beta)v} + \dfrac{x-\delta}{\beta v} & x \in [\delta,\ \delta + y_1) \\ \alpha + \dfrac{y_1}{(1-\beta)\beta v} + \dfrac{x - \delta - y_1}{(1-\beta)v} & x \in [\delta + y_1,\ \delta + y_1 + y_2) \\ \alpha + \dfrac{y_1}{(1-\beta)\beta v} + \dfrac{y_2}{(1-\beta)v} & x \in [\delta + y_1 + y_2,\ 2\delta + y_1) \\ \alpha + \dfrac{y_1}{(1-\beta)\beta v} + \dfrac{y_2}{(1-\beta)v} + \dfrac{x - 2\delta - y_1}{\beta v} & x \in [2\delta + y_1,\ 2\delta + y_1 + y_2) \\ \vdots & \vdots \\ \alpha + \dfrac{\sum_{j=1}^{z-1} y_j}{(1-\beta)\beta v} + \dfrac{x - (z-1)\delta - \sum_{j=1}^{z-1} y_j}{(1-\beta)v} & x \in [(z-1)\delta + \sum_{j=1}^{z-1} y_j, \\ & \quad (z-1)\delta + \sum_{j=1}^{z} y_j) \\ \alpha + \dfrac{\sum_{j=1}^{z-1} y_j}{(1-\beta)\beta v} + \dfrac{y_z}{(1-\beta)v} & x \in [(z-1)\delta + \sum_{j=1}^{z} y_j, \\ & \quad z\delta + \sum_{j=1}^{z-1} y_j) \\ \alpha + \dfrac{\sum_{j=1}^{z-1} y_j}{(1-\beta)\beta v} + \dfrac{y_z}{(1-\beta)v} + \dfrac{x - z\delta - \sum_{j=1}^{z-1} y_j}{\beta v} & x \in [z\delta + \sum_{j=1}^{z-1} y_j, \\ & \quad z\delta + \sum_{j=1}^{z} y_j) \end{cases}$$

⋮	⋮
1	$x > k\delta + \sum_{j=1}^{k} y_j$

Our exposition thus far has focused on arguing that G_i is indeed an equilibrium. Theorem 3 extends well beyond that, however, in asserting that G_i is in fact the unique symmetric equilibrium. The necessary conditions for establishing uniqueness are presented in the appendix as a series of lemmata, each successively pinning down the precise form that an equilibrium strategy must take. The first of these lemmata rules out the possibility of a mass point at any bid greater than zero (Lemma 1). The following two identify $1/[(1-\beta)v]$ and $1/(\beta v)$ as the only permissible density rates for continuously distributed mass in an equilibrium distribution (Lemmata 2 and 3). In the process of identifying these density rates, it becomes clear that equilibrium supports must be composed of interval pairs with systematic gaps (Corollary 1). Specifically, whenever one player's support has an interval with a density rate of $1/[(1-\beta)v]$, the other player's must have no mass δ below it and an interval with a density rate of $1/(\beta v)$ placed δ above it. Likewise, every interval with a density rate of $1/(\beta v)$ must necessarily be balanced by the other player placing no mass δ above it and an interval of density rate $1/[(1-\beta)v]$ at δ below it. Beyond that, it is unclear up to this point how these interval pairs fit together and whether they are symmetric across players.

The next lemma, the first to exclusively focus on $\beta \in (0,\ 1/2)$, goes a long way in clarifying how the interval pairs fit together.[8] It does so by specifying three properties that must hold for any mass that is placed δ above a gap in the other player's distribution (i.e. regions where the cost differential between any two bids must be fully offset by tying additional mass rather than beating it). In particular, mass in any such region must be connected, have a density rate of $1/[(1-\beta)v]$, and have a lower bound that is precisely δ above the start of the gap in the other player's distribution (Lemma 4). A major implication is that interval pairs cannot overlap. That is, the upper interval of one pair must always fall below the lower interval of the next. Another critical linchpin for piecing together the equilibrium structure is that any interval with a density rate of $1/[(1-\beta)v]$ must immediately follow either a mass point or an interval with a density rate of $1/(\beta v)$. Thus there cannot be any gaps between subsequent interval pairs. This result follows by combining the previous lemma with an additional lemma that eliminates the possibility that an interval of density rate $1/[(1-\beta)v]$ could immediately follow a gap of δ or more (Lemma 5). Any equilibrium, symmetric or asymmetric, must satisfy these properties. The culmination of these results is Proposition 1, which identifies that there is indeed a unique symmetric equilibrium.

Several distinct patterns appear in the equilibrium manifold by varying the tie prize βv and the tie margin δ—especially with regard to the number and width of the interval pairs. Fixing β at 0.4 and the winning prize v at 100, Figure 3 portrays the support of the unique symmetric equilibrium when $\delta \in (0,\ (1-\beta)v)$. For any given δ on the vertical axis, the gray and blue shaded areas of the horizontal cross section at that point form the equilibrium support. For instance, the three examples of equilibria in Figure 2 are drawn to scale so that they correspond to the three horizontal red lines in Figure 3. The top red line at $\delta = 33.33$ (or $v/3$) involves one pair of intervals: $[0,\ 17.78]$ and $[33.33,\ 51.11]$, with respective density rates of 1/60 and 1/40 (alternatively, $1/[(1-\beta)v]$ and $1/\beta v$). The second red line at $\delta = 25$ ($v/4$) has two pairs of intervals, the first pair in gray ($[0,\ 8.67]$ and $[25,\ 33.67]$), and the second in blue ($[33.67,\ 45.67]$ and $[58.67,\ 70.67]$). Then at $\delta = 20$ ($v/5$) there are three interval pairs, marked gray, blue, and gray again ($[0,\ 4.44]$ and

[8] Lemmata 1 and 2 hold for all $\beta \in (0,\ 1)$ and $\delta \in (0,\ (1-\beta)v)$, while Lemma 3 and Corollary 1 hold over the same range but exclude $\beta = 1/2$.

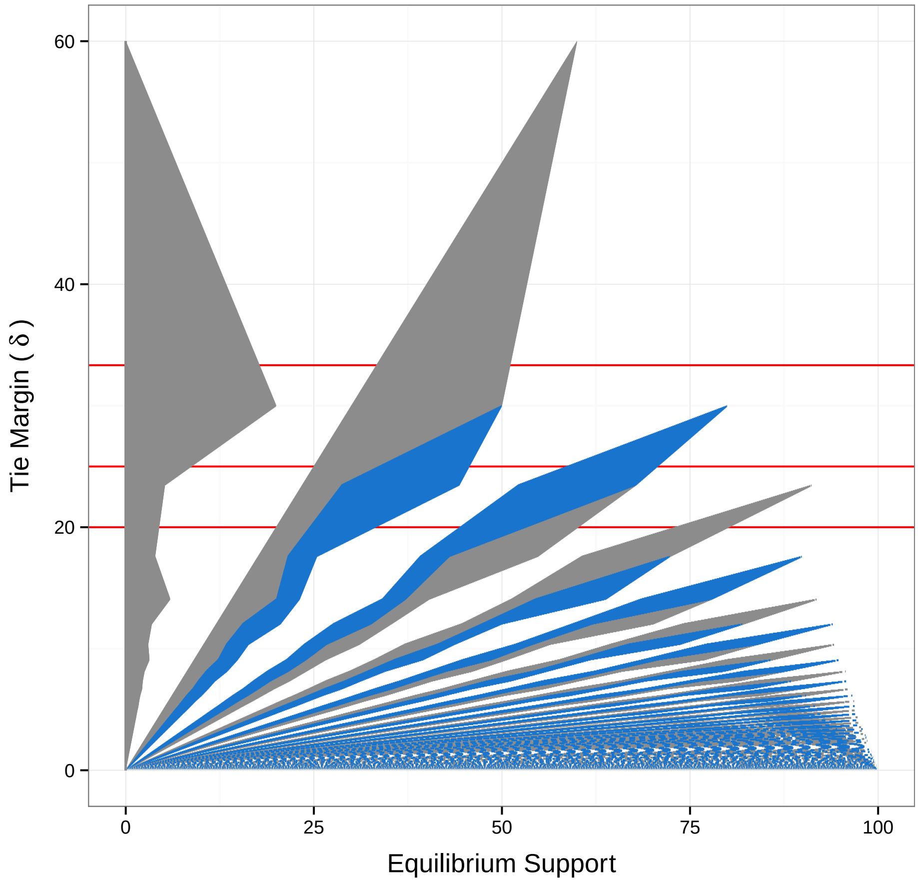


Fig. 3: Support of the unique symmetric equilibrium: $\beta = 0.4$, $v = 100$, and $\delta \in (0,\ (1-\beta)v)$. The three horizontal red lines correspond to the equilibria in Figure 2 for $\delta \in \{v/3,\ v/4,\ v/5\}$.

$[20,\ 24.44]$; $[24.44,\ 33.14]$ and $[44.44,\ 53.14]$; $[53.14,\ 60.10]$ and $[73.14,\ 80.10]$). Successive interval pairs arise at the point where it would otherwise become profitable to outbid the upper bound of the support by δ. In the uppermost pair, the width of the intervals always increases as δ decreases. The reverse is true for the second highest pair. Thereafter the effect is non-monotonic—the width sometimes increasing, sometimes decreasing. The net effect, however, is that as δ decreases, the intervals and their corresponding gaps become increasingly fine and gradually fill the full range of bids between zero and the winning prize v. Finally, as the tie margin δ approach zero, the equilibrium converges to that of the standard all pay auction in which players uniformly randomize between zero and v at the rate of $1/v$. Although for any strictly positive δ there are gaps in the support, the average density rate over any measurable subset of $[0,\ v]$ converges to $1/v$.

In addition to varying δ, we can also examine the equilibrium manifold for different sizes of the tie prize. Figure 4 contains eight panels showing the equilibrium support for $\delta \in (0,\ (1-\beta)v)$ when $\beta \in \{0.1, 0.3, 0.45, 0.49999, 0.50001, 0.55, 0.7, 0.9\}$ and $v = 100$. The tie margin is once again on the vertical axis, and the equilibrium support for a particular δ is the shaded portion of the horizontal cross section. The scale of the vertical axis varies from panel to panel with changes in $(1-\beta)v$, but for comparing the width of intervals, the scale of the horizontal axis is constant across all eight panels. In particular, focusing on the first four panels where $\beta < 1/2$, the width of the widest interval is increasing in β. The maximal width of an interval ranges from 5 in the first panel with $\beta = 0.1$ to just shy of 25 in the fourth panel with $\beta = 0.49999$ (the maximum occurring at $\delta = (1-\beta)v/2$ in each case). When a tie has little value, the narrow intervals reflect a strategy of randomizing almost discretely over distinct bidding levels. Indeed, as β approaches 0, bids are increasingly concentrated near each multiple of δ, so that the distribution converges to the limiting case in Theorem 2. As a tie becomes more valuable, however, the intervals widen and players randomize not only across but also within the different bidding levels. Another closely related feature of the panels with $\beta < 1/2$ is that the width of the intervals in the bottommost pair alternately grow to δ or shrink to zero as β increases to 1/2. This sawtooth pattern is discernible with $\beta = 0.4$ in Figure 3, but it becomes especially prominent in the third and fourth panels of Figure 4 when $\beta = 0.45$ and 0.49999. Notably, by taking on this sawtooth pattern, the bottommost interval pair is able to make a continuous transition from the equilibrium structure where $\beta < 1/2$ to the otherwise qualitatively different structure for $\beta > 1/2$.

3.4 Tie prize greater than one-half: $\beta \in (1/2,\ 1)$ and $\delta \in (0,\ (1-\beta)v)$

We next consider symmetric equilibria for a tie prize greater than one-half the winning prize. Specifically, we are turning to regions II.B and IV of Figure 1 where $\beta \in (1/2,\ 1)$ and $\delta \in (0,\ (1-\beta)v)$. The increased desirability of tying leads to strikingly different patterns in the structure of equilibria, as the last four panels of Figure 4 can attest. These differences remain minimal in region II.B where $\delta \in ((1-\beta)\beta v,\ (1-\beta)v)$. There, like in region II.A, the unique Nash equilibrium takes the form depicted in panel (a) of Figure 2; the sole caveat being that with $\beta > 1/2$, the density rate of $1/[(1-\beta)v]$ over the bottom interval now exceeds the density rate of $1/\beta v$ over the top interval. Real differences to the structure of equilibria only come into play in region IV with $\delta \in (0, (1-\beta)\beta v]$.

Instead of tightly packed intervals with gaps that are never more than δ apart as we saw for $\beta < 1/2$, equilibria with $\beta > 1/2$ are primarily built of intervals that are 2δ in length. Moreover, as the tie margin δ decreases in size, these equilibria are composed of successively more intervals of length-2δ. Two other components of the equilibrium strategies are a mass point at zero, and an occasional pair of intervals with lower bounds at zero and δ. This structure is visible from the supports in Figure 4 where the intervals of length-2δ are represented by the purple and green regions—the change in colors based on the presence or absence of the periodically appearing pair of intervals in gray. Besides just looking at the supports, Figure 5 plots four examples of the actual equilibrium strategies. These four examples correspond to the equilibrium structure in the top four subdivisions of region IV (see Figure 1). Each length-2δ interval has a density rate of $1/[(1-\beta)v]$ over the bottom half and $1/(\beta v)$ over the top half. Figure 5 also shows that these density rates apply to the pair of intervals at zero and δ. As with $\beta < 1/2$, the tandem density rates of $1/[(1-\beta)v]$ and $1/(\beta v)$ allow players to remain indifferent over bids within each interval. The larger issue is maintaining indifference across intervals.

Consecutive intervals of length-2δ are spaced so that their lower bounds, ϕ_j and ϕ_{j+1}, are always $\delta/[(1-\beta)\beta]$ apart. This spacing satisfies $u_i(\phi_j,\ G_{-i}) = u_i(\phi_{j+1},\ G_{-i})$ and also ensures that the bids between the length-2δ intervals are strictly dominated. Expected payoffs monotonically decrease over $(\phi_j + 2\delta,\ \phi_{j+1} - \delta]$ since there is no additional mass to tie, and beating mass that was previously tied cannot compensate for the cost of bidding (i.e. $(1-\beta)/\beta < 1$). Conversely, for bids in $[\phi_{j+1} - \delta,\ \phi_{j+1})$, tying mass at a rate of $1/[(1-\beta)v]$ more than covers the bidding cost (i.e. $\beta/(1-\beta) > 1$), and expected payoffs continue to rise until ϕ_{j+1}. These same arguments also rule out the possibility of a profitable deviation in $[\phi_1 - \delta,\ \phi_1)$ or in $(\phi_p + 2\delta,\ \phi_p + 3\delta]$, where p is the total number of length-2δ intervals. Furthermore, for bids above $\phi_p + 3\delta$, winning is already

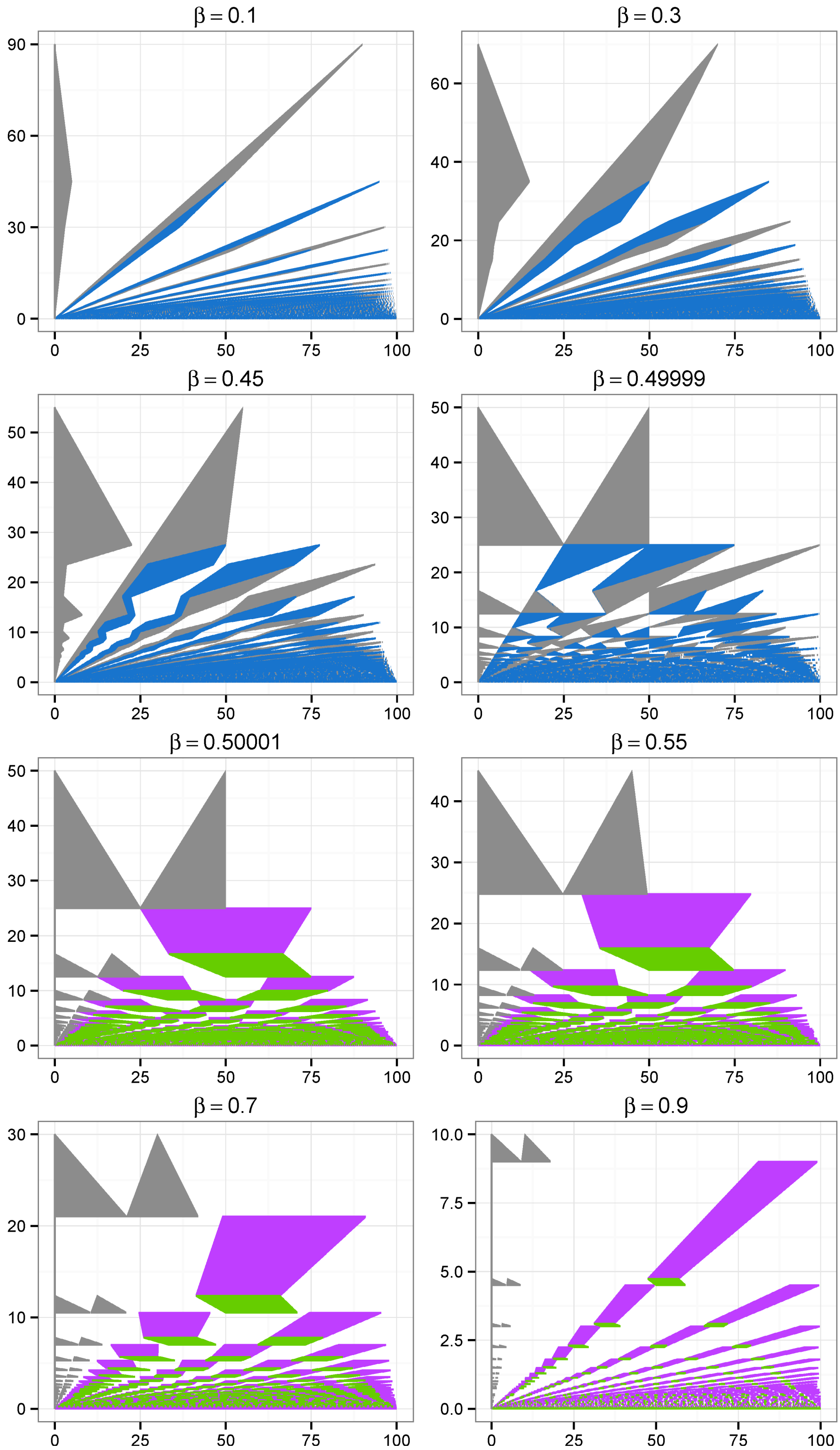
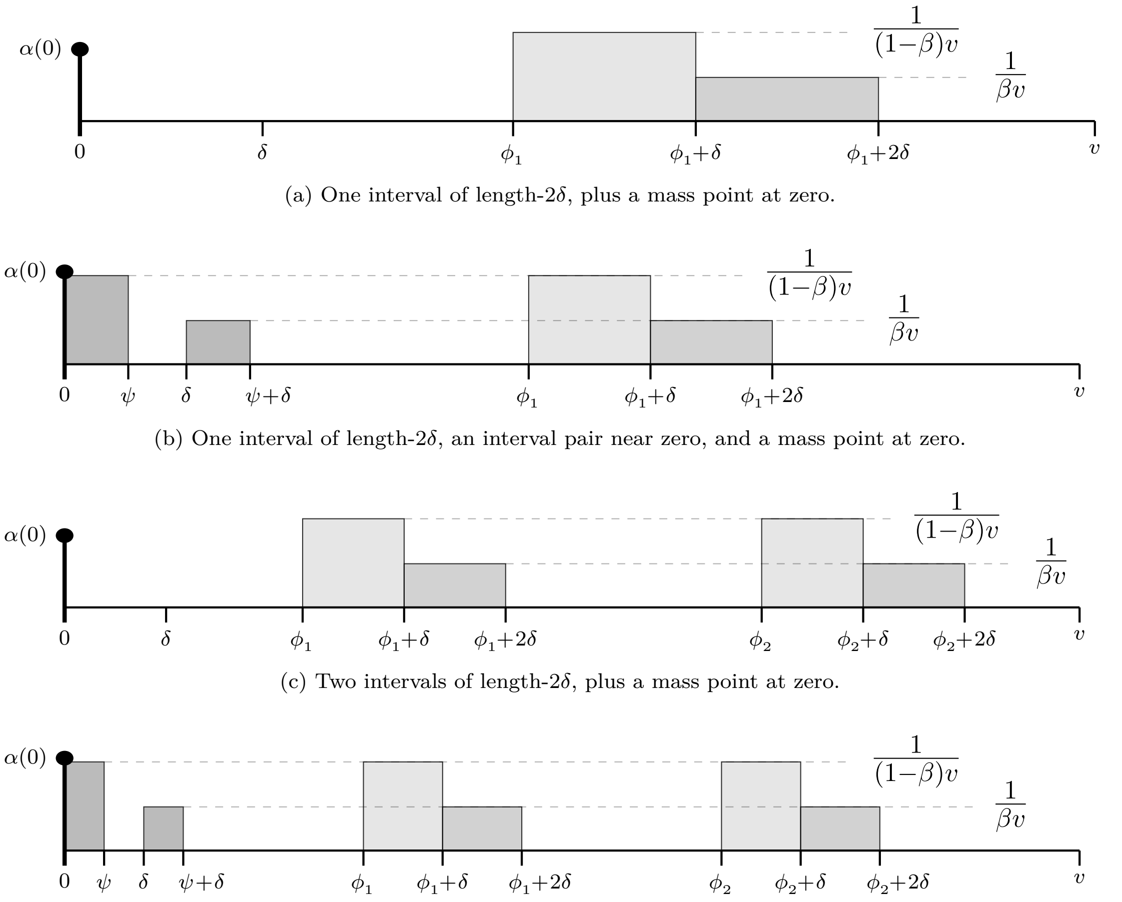


Fig. 4: Support of the unique symmetric equilibrium for various β when $v = 100$. The vertical axis plots $\delta \in (0, (1-\beta)v)$. Shaded regions on the horizontal axis for a given δ mark the support.



(a) One interval of length-2δ, plus a mass point at zero.

(b) One interval of length-2δ, an interval pair near zero, and a mass point at zero.

(c) Two intervals of length-2δ, plus a mass point at zero.

(d) Two intervals of length-2δ, an interval pair near zero, and a mass point at zero.

Fig. 5: Density plots of symmetric equilibrium strategies for various δ when $\beta > 1/2$. Equilibria contain intervals of length-2δ, a mass point at zero, and, periodically, an interval pair near zero.

assured but the bidding cost continues to rise, so these cannot be profitable either. The higher value of tying precludes the incentive to outbid the distribution by δ.

The lower bound of the first length-2δ interval ϕ_1 is defined to satisfy the indifference condition $u_i(0,\ G_{-i}) = u_i(\phi_1,\ G_{-i})$. These expected utilities are, however, contingent on the presence or absence of the interval pair at zero and δ. When absent, $u_i(0,\ G_{-i}) = \alpha(0)\beta v$ and $u_i(\phi_1,\ G_{-i}) = \alpha(0)v + [\delta/(1-\beta)v]\,\beta v - \phi_1$, so $\phi_1 = \alpha(0)(1-\beta)v + [\delta\beta/(1-\beta)]$. The size of the mass point at zero is then simply the remainder after each of the length-2δ intervals. Specifically, for p intervals of length-2δ, $\alpha(0) = 1 - [\delta p/(1-\beta)\beta v]$. Transitioning from the absence to the presence of the interval pair at zero and δ stems from monitoring a particularly critical potential deviation. Bidding in $(0,\ \min\{\delta,\ \phi_1 - \delta\}]$ is never profitable since it entails a higher bidding cost without any additional mass to tie or beat. However, if $\delta < \phi_1 - \delta$, it may become profitable to bid immediately above δ so as to beat the mass point at zero. Since a bid of δ technically ties a bid of zero, and since a deviating player would want to bid on the extreme low end of $(\delta,\ \phi_1 - \delta)$ to reduce bidding costs, we denote the supremum of player i's expected utility as δ is approached from above by $u_i(\overline{\delta},\ G_{-i})$. (This notation is repeatedly used in the proofs in the appendix.) In equilibrium, we must have $u_i(0,\ G_{-i}) \geq u_i(\overline{\delta},\ G_{-i})$, so:

$$\delta \geq \alpha(0)(1-\beta)v \quad \Rightarrow \quad \delta \geq \frac{(1-\beta)\beta v}{p+\beta} \tag{9}$$

For a given p, as soon as Equation 9 fails to hold, the interval pair at zero and δ becomes part of the equilibrium. With the presence of the interval pair, $u_i(0,\ G_{-i}) = u_i(\phi_1,\ G_{-i})$ then implies the following:

$$\phi_1 = \alpha(0)(1-\beta)v + \psi\left(\frac{1+\beta}{\beta}\right) + \delta\left(\frac{\beta}{1-\beta}\right)$$

We can fully specify ϕ_1, $\alpha(0)$, and ψ with the use of two additional equations. The first is the constraint that the total mass must sum to one:

$$1 = \alpha(0) + \frac{\psi}{(1-\beta)\beta v} + \frac{\delta p}{(1-\beta)\beta v}$$

The second, $\delta = \alpha(0)(1-\beta)v + \psi$, comes from equating expected utilities across the intervals at δ and zero.[9] Combining these yields the following:

$$\phi_1 = v - \frac{\delta p}{(1-\beta)\beta}; \quad \alpha(0) = \frac{\delta(1+p) - (1-\beta)\beta v}{(1-\beta)^2 v}; \quad \psi = \beta v - \delta\left(\frac{p+\beta}{1-\beta}\right) \tag{10}$$

The positivity of ψ is based on reversing the inequality in Equation 9. Additionally, the constraint that $\alpha(0) \geq 0$ coincides with the constraint that $\psi \leq \delta$. Each is satisfied so long as:

$$\delta \geq \frac{(1-\beta)\beta v}{1+p} \tag{11}$$

When Equation 11 holds with equality, $\alpha(0) = 0$ and $\psi = \delta$, so the interval pair at zero and δ becomes yet another length-2δ interval. Without a mass point at zero to anchor expected payoffs, there are a multiplicity of symmetric equilibria, and the lower bound of the support may assume any value in $[0,\ \beta^2 v/(1+p)]$ (here, p does not include the newly formed length-2δ interval). Expected payoffs for these equilibria fall from $\delta\beta/(1-\beta)$ to 0 as the lower bound increases from 0 to $\beta^2 v/(1+p)$. These two extremes for the lower bound can be seen in Figure 4. For instance, in the panel with $\beta = 0.7$, the first length-2δ interval is formed at $\delta = 21$. As δ approaches 21 from above (i.e. $p = 0$), the length-2δ interval is marked in gray and has a lower bound of zero; whereas, from below, it is marked in purple and has a lower bound at $\beta^2 v/(1+p) = (0.7)^2 \times 100 = 49$. Then at $\delta = 10.5$, a second length-2δ interval comes into being. When approached from above (i.e. $p = 1$), the two length-2δ intervals have lower bounds of 0 and $50 = \delta/[(1-\beta)\beta]$, colored gray and green. From below and marked in purple, the lower bounds are $24.5 = (0.7)^2 \times 100/2$ and 74.5. Hence, transitioning from the presence to the absence of an interval pair at zero and δ involves jumping from one extreme to the other of the set of equilibria at $\delta = (1-\beta)\beta v/(1+p)$. We can formally characterize the equilibria as follows:

Theorem 4 *Let $\delta \in (0,\ (1-\beta)v)$ and $\beta \in (1/2, 1)$. Also, define $p = \lfloor (1-\beta)\beta v/\delta \rfloor$ where $\lfloor \cdot \rfloor$ is the floor function. If $\delta \neq (1-\beta)\beta v/(1+p)$, there exists a unique Nash equilibrium which is also symmetric. The characterization is as follows. If $p > 1$, let $\phi_z - \phi_{z-1} = \delta/(1-\beta)\beta$ for*

[9] This is equivalent to Equation 4 with y_1 replaced by ψ where the equivalence holds because a bid of $\psi + \delta$ does not tie a bid of ϕ_1. We can also note that there are no profitable deviations in $(\psi,\ \delta)$ or in $(\delta + \psi,\ \phi_1 - \delta]$ since there is no additional mass to tie or beat.

$z \in \{2, \ldots, p\}$. For $p = 0$, define $H(x) = 1$. Otherwise, define $H : [\phi_1,\ \infty) \to [\xi,\ 1]$ as:

$$H(x) = \begin{cases} \xi + \dfrac{x - \phi_1}{(1-\beta)v} & x \in [\phi_1,\ \phi_1 + \delta) \\ \xi + \dfrac{\delta}{(1-\beta)v} + \dfrac{x - \phi_1 - \delta}{\beta v} & x \in [\phi_1 + \delta,\ \phi_1 + 2\delta) \\ \xi + \dfrac{\delta}{(1-\beta)\beta v} & x \in [\phi_1 + 2\delta,\ \phi_2) \\ \vdots & \vdots \\ \xi + \dfrac{(z-1)\delta}{(1-\beta)\beta v} + \dfrac{x - \phi_z}{(1-\beta)v} & x \in [\phi_z,\ \phi_z + \delta) \\ \xi + \dfrac{(z-1)\delta}{(1-\beta)\beta v} + \dfrac{\delta}{(1-\beta)v} + \dfrac{x - \phi_z - \delta}{\beta v} & x \in [\phi_z + \delta,\ \phi_z + 2\delta) \\ \xi + \dfrac{z\delta}{(1-\beta)\beta v} & x \in [\phi_z + 2\delta,\ \phi_{z+1}) \\ \vdots & \vdots \\ 1 & x > \phi_p + 2\delta \end{cases}$$

For $p \geq 1$, if $\delta \in [\,(1-\beta)\beta v/(p+\beta),\ (1-\beta)\beta v/p\,)$, let $\phi_1 = \alpha(0)(1-\beta)v + [\delta\beta/(1-\beta)]$ and $\alpha(0) = \xi = 1 - [\delta p/(1-\beta)\beta v]$. Each player's equilibrium strategy is:

$$G_i(x) = \begin{cases} \alpha(0) & x \in [0,\ \phi_1) \\ H(x) & x \geq \phi_1 \end{cases} \tag{12}$$

If $\delta \in (\,(1-\beta)\beta v/(1+p),\ (1-\beta)\beta v/(p+\beta)\,)$, then ϕ_1, $\alpha(0)$, and ψ are defined by Equation 10. With $\xi = \alpha(0) + [\psi/(1-\beta)\beta v]$, each player has the following equilibrium strategy:

$$G_i(x) = \begin{cases} \alpha(0) + \dfrac{x}{(1-\beta)v} & x \in [0,\ \psi) \\ \alpha(0) + \dfrac{\psi}{(1-\beta)v} & x \in [\psi,\ \delta) \\ \alpha(0) + \dfrac{\psi}{(1-\beta)v} + \dfrac{x-\delta}{\beta v} & x \in [\delta,\ \delta + \psi) \\ \alpha(0) + \dfrac{\psi}{(1-\beta)\beta v} & x \in [\delta + \psi,\ \phi_1) \\ H(x) & x \geq \phi_1 \end{cases} \tag{13}$$

Finally, for $p \geq 1$, if $\delta = (1-\beta)\beta v/p$, there is a continuum of symmetric Nash equilibrium (which also constitutes the full set of Nash equilibrium). Let $\ell \in [0,\ \beta^2 v/(1+p)]$; $\phi_1 = \ell + [\delta/(1-\beta)\beta]$; and $\xi = \delta/(1-\beta)\beta v$. The complete set of symmetric equilibria is characterized by the following strategy:

$$G_i(x) = \begin{cases} \dfrac{x - \ell}{(1-\beta)v} & x \in [\ell,\ \ell + \delta) \\ \dfrac{\delta}{(1-\beta)v} + \dfrac{x - \ell - \delta}{\beta v} & x \in [\ell + \delta,\ \ell + 2\delta) \\ \dfrac{\delta}{(1-\beta)\beta v} & x \in [\ell + 2\delta,\ \phi_1) \\ H(x) & x \geq \phi_1 \end{cases}$$

Once again, this is much more than a statement of existence. Theorem 4 fully characterizes the set of Nash equilibria for all $\beta \in (1/2,\ 1)$ and $\delta \in (0,\ (1-\beta)v)$. Moreover, aside from a measure-zero set (i.e. $\delta = (1-\beta)\beta v/(1+p)$), the equilibrium specified is unique. In terms of necessary conditions, Lemmata 1 through 3 and Corollary 1 again apply. Four additional lemmata are then required to isolate the exact equilibrium structure. The first specifies that there must be a space of at least 2δ between any two portions of the support that have a density rate of $1/[(1-\beta)v]$ (Lemma 6). Given the structure that is already in place for coupling intervals of $1/[(1-\beta)v]$ in one player's support with intervals of $1/(\beta v)$ in the other's (see Corollary 1), limiting the proximity of consecutive intervals of a particular density rate is quite powerful. Building on this, Lemma 7 identifies that if either player's support has a gap of at least δ, which is immediately followed by a density rate of $1/[(1-\beta)v]$, then the remainder of each player's distribution is made up of intervals of length-2δ (although the lower bounds and density rates of these intervals may differ across players). The existence of such a gap in a player's support is clarified by Lemma 8, which also delineates how mass must be distributed below the gap. Symmetry is then finally established by Lemma 9.

3.5 Tie prize equal to one-half: $\beta = 1/2,\ \delta \in (0,\ (1-\beta)v)$

We conclude our characterization with a sketch of equilibria when β takes on the rather unique value of 1/2. Equilibria thus far have been typified by the density rates of $1/\beta v$ and $1/[(1-\beta)v]$. Moreover, the uniqueness results in Theorems 3 and 4 are largely dependent on these density rates being distinct from one another. Since these density rates are equal at $\beta = 1/2$, uniqueness is easily lost. A significant exception where uniqueness is maintained is $\delta \in (v/4,\ (1-\beta)v)$ with $\beta = 1/2$. This is the thin gray line separating regions II.A and II.B in Figure 1, and the equilibrium falls right in step: a mass point at zero, followed by randomization at the rate of $v/2$ over a lower and an upper interval.[10] Given the smooth transition for these values of δ between the panels where $\beta = 0.49999$ and $\beta = 0.50001$ in Figure 4, uniqueness is only natural. These panels also illustrate that for smaller values of δ, the bottom portion of the distribution is unique. There is either a mass point at zero with a pair of intervals beginning at 0 and δ, or an isolated mass point at zero.[11] The remainder of the distribution is then characterized by the following property:[12]

$\mathscr{P}$. G_i contains $p = \lfloor (1-\beta)\beta v/\delta \rfloor$ intervals of length 4δ, each with a total mass of $4\delta/v$. These length-4δ intervals begin at $\overline{x}_i$, with the bottom of one interval being the top of the next. The bottom quarter of each length-4δ interval contains no mass; the third quarter (from the bottom) has $2\delta/v$, which is uniformly distributed at the rate of $2/v$; and the distribution of the remaining $2\delta/v$ over the second and top quarter is subject to the following constraints:

i. The mass is continuously distributed.
ii. For any x in the top quarter, $g_i(x) + g_i(x-2\delta) = 2/v$ (where $g_i(x)$ is the density rate of G_i at x).
iii. For any x in the higher of two adjacent length-4δ intervals, $g_i(x) = g_i(x-4\delta)$ (i.e. density rates are the same across length-4δ intervals).

Although there tends to be a rich multiplicity of equilibria, the bottom of the distribution gives further structure to $\mathscr{P}$. In particular, except for the case where $\delta = v/4p$, the set of equilibria for

[10] Formally, the equilibrium distribution is equivalent to the one in Theorem 3 with $k = 1$. It is also equivalent to Equation 13 in Theorem 4 with p=0. The necessary conditions for this equilibrium are established by part B of Lemma 12.

[11] In particular, the portion of the distribution before $H(x)$ in Equations 12 and 13 of Theorem 4 is the bottom portion of any equilibrium distribution when $\beta = 1/2$ and $\delta \neq (1-\beta)\beta v/p$ (where $p = \lfloor (1-\beta)\beta v/\delta \rfloor$, as defined in Theorem 4). Further details on the bottom of the distribution can be found in Lemma 12.

[12] This property comprises the entire distribution for the case where $\delta = (1-\beta)\beta v/p$.

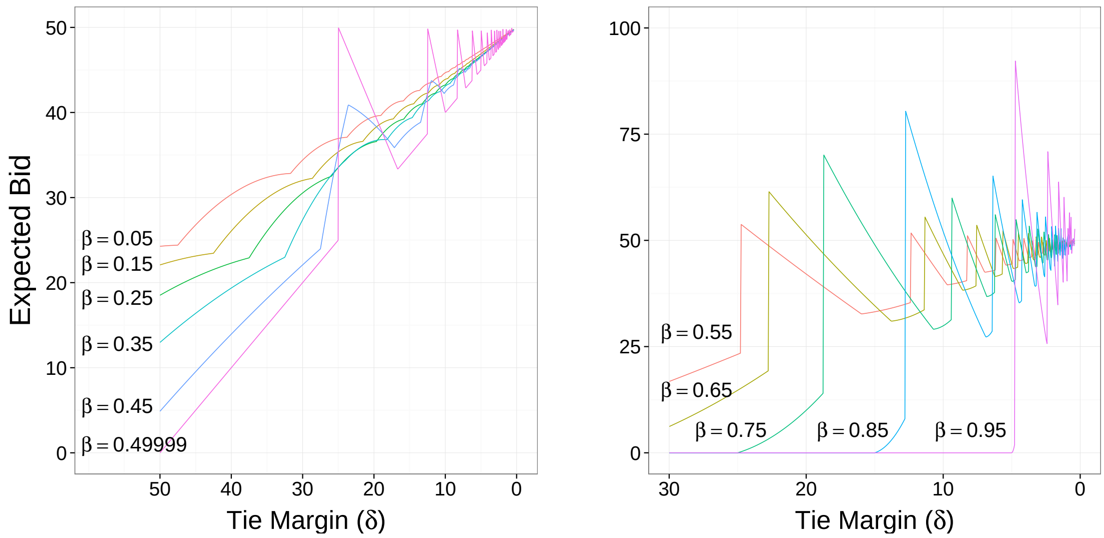


Fig. 6: Expected equilibrium bids by tie margin and tie prize; $v = 100$.

any given δ are payoff equivalent.[13] This structure also allows a unique equilibrium to emerge at $\delta = v/(4p+2)$. On Figure 4, this corresponds to the pinnacle of the gray triangles at 0 and δ, and indeed there is a fluid transition here between the panels where $\beta = 0.49999$ and $\beta = 0.50001$ (for $\beta > 1/2$, this is also the point where the color changes from purple to green).

4 Comparative Statics

The characterization of equilibrium naturally leads to predictions about how aggressively players compete on average. We can particularly address the question of whether players tend to be more or less aggressive for various fluctuations in the size of the tie prize and the tie margin. From the perspective of a policy maker or contest organizer who is trying to achieve some overall level of competition—be it high or low—this issue is paramount. Following the equilibrium characterization in Theorems 3 and 4, Figure 6 plots a player's expected bid (vertical axis) in terms of both tie margins (horizontal axis) and tie prizes (different lines within each graph). Fixing the prize for winning again at $v = 100$, the left panel shows six values of $\beta < 1/2$, while five values of $\beta > 1/2$ are represented in the right panel (axes have differing scales across panels). Several features here are worth highlighting, the first of which we state as a formal result.

Theorem 5 *For sufficiently low tie margins (roughly $\delta < 0.2625 \times v$), expected equilibrium bids are non-monotonic in the tie prize. Likewise, for sufficiently high tie prizes (roughly $\beta > 0.3347$), expected equilibrium bids are non-monotonic in the tie margin.*

Cutoffs here are based on numerical calculations. In terms of Figure 6, the first part of the theorem refers to lines that cross, while the second part addresses non-monotonicity within a given line. Not only is there non-monotonicity in each dimension, but the magnitudes vary considerably. Fixing β, the largest oscillation is always the first. Thereafter, oscillations occur more rapidly as δ continues to decrease, but the crest-to-trough distances becomes successively smaller. Increasing β leads to more volatile oscillations in the sense that the crest-to-trough distances are greater

[13] When $\delta \in [v/(4p+2),\ v/4p)$ the expected payoff is $(v/2) - 2\delta p$, whereas the expected payoff is δ for $\delta \in (v/(4p+4),\ v/(4p+2))$.

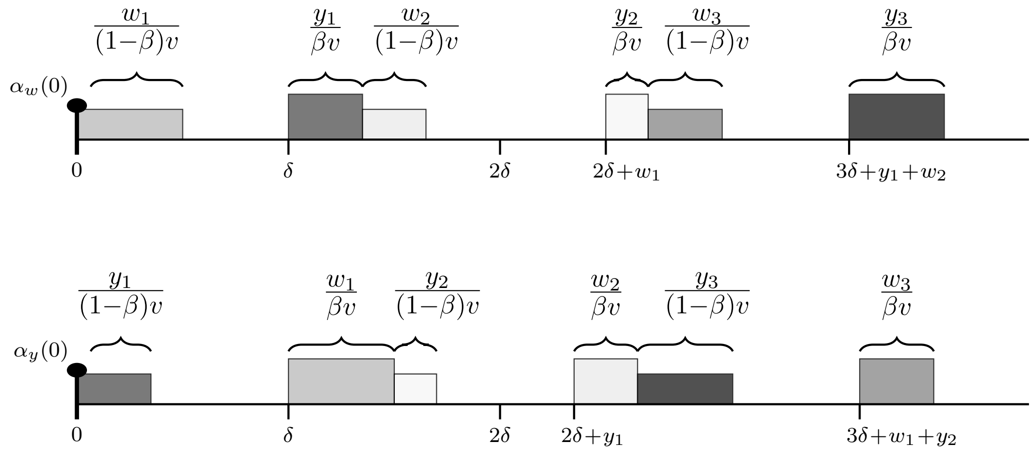


Fig. 7: Density plot of strategies for player w (Top) and player y (Bottom) when $\beta < 1/2$. Each interval's mass is labeled in terms of its density rate ($1/[(1-\beta)v]$ or $1/\beta v$), and length (y_i or w_i).

over even shorter wavelengths. The angle of ascent also grows so that each oscillation begins with a shear vertical jump once $\beta \geq 1/2$. These jumps correspond to the parameter values for which there is a continuum of equilibria, and every expected bid within the jump is attainable by some equilibrium. Another thing to note is the convergence of expected bids to $v/2$, the expected bid in the standard all-pay auction, as the tie margin decreases to zero. Although the paths to convergence differ considerably between the two panels. On the left, with $\beta < 1/2$, convergence is entirely from below; whereas on the right, with $\beta > 1/2$, the oscillations extend both far above and below $v/2$, so that $v/2$ only becomes a focal point as the oscillations vanish in size.

5 Asymmetric Equilibria

The occurrence of asymmetric equilibria is limited to select portions of the parameter space, solely arising when $\beta \leq 1/2$.[14] We have seen that Lemmata 1 through 5 form a set of necessary conditions which must hold in any equilibrium, either symmetric or asymmetric, when $\beta \in (0,\, 1/2)$ and $\delta \in (0,\, (1-\beta)v)$ (see the paragraphs following Theorem 3). Taken together, these lemmata specify that an equilibrium must have the form depicted in Figure 7. Namely, mass points at zero $\alpha_w(0)$, $\alpha_y(0) \in [0, 1)$, followed by interval pairs with the familiar density rates of $[1/(1-\beta)v]$ and $1/(\beta v)$ — the length of the lower interval in one player's distribution matching the length of the upper interval in the other player's distribution. We label the length of the successive $1/[(1-\beta)v]$ segments for player w as $w_1, w_2, \ldots, w_k \geq 0$, and likewise for player y as $y_1, y_2, \ldots, y_k \geq 0$. When each player's mass point does indeed have positive mass and when each of the interval pairs has a positive length, then the equilibrium is necessarily symmetric (see Proposition 1). Asymmetric equilibria arise when an interval pair has a length of zero or a mass point has no mass.

Zeroing out an interval pair or a mass point is not without consequence. Each segment with a density rate of $[1/(1-\beta)v]$ must be immediately preceded by some other mass (see Lemma 5 and the discussion following its proof in Appendix B). So if one player's mass point does not have any mass, their first interval must have a length of zero (i.e. if $\alpha_w(0) = 0$ then $w_1 = 0$). Similarly, when one interval pair has a length of zero, the next adjoining one must be zeroed out as well (so

[14] For $\beta = 1/2$, asymmetric equilibria occur along parts of the line where $\delta \in (0,\, v/4]$. As noted previously, such equilibria are governed by Property $\mathscr{P}$. Our focus in this section is $\beta < 1/2$.

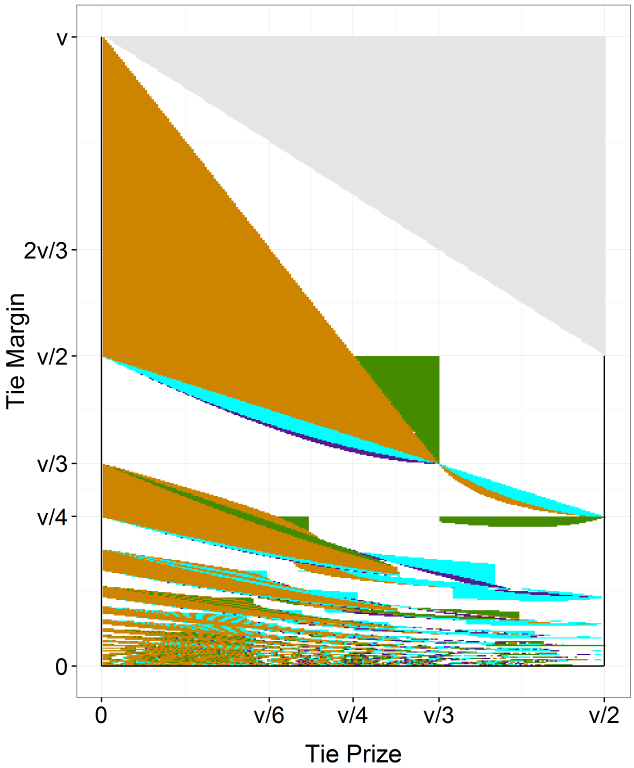


Fig. 8: Regions where asymmetric equilibria exist for $\beta \in (0,\ 1/2)$ and $\delta \in (0,\ (1-\beta)v)$. Adjoining regions with different forms of asymmetric equilibria are colored differently.

if $w_j = 0$ then $y_{j+1} = 0$, and if $y_j = 0$ then $w_{j+1} = 0$ for all $j \geq 1$). Exhaustively checking for all forms of asymmetric equilibria is therefore an exercise in varying the combination of interval pairs which are set to zero (potentially including one player's mass point). This process is described in Algorithm 1 in Appendix E.

Omitting a certain combination of interval pairs may form an equilibrium in one part of the parameter space, while another part of the parameter space may require a different combination. Figure 8 portrays the portions of the parameter space (with $\beta < 1/2$) where any form of asymmetric equilibrium exists. The changing colors represent the transition from one form of equilibrium to another between adjoining regions that each have asymmetric equilibria. For instance, in the triangle formed by $(0,\ v/2)$, $(0,\ v)$, and $(v/3,\ v/3)$ (i.e. by $\delta \in [(1-\beta)v/2,\ (1-2\beta)v]$ for $\beta \in (0,\ 1/3]$), there is only one positive mass point and one interval pair. Player y specifically has a mass point $\alpha_y(0) > 0$ and randomizes over $[0,\ y_1]$, while player w solely randomizes over $[\delta,\ \delta + y_1]$ (the assignment of players to the roles of y and w is of course arbitrary and the roles and could be switched).[15] The adjoining triangle to the right, spanning $(v/4,\ v/2)$, $(v/3,\ v/2)$, and $(v/3,\ v/3)$, is similar except that, in addition to $\alpha_y(0) > 0$ and $y_1 > 0$, player w also has a strictly positive mass point at zero: $\alpha_w(0) > 0$. Immediately below the first triangle, there are also two crescent-shaped regions (with endpoints of $(0,\ v/2)$ and $(v/3,\ v/3)$). Each of these regions involve two interval pairs with positive lengths ($y_1,\ w_2 > 0$). Player y thus randomizes between the antipodes of either very low or very high bids, while player w targets bids in the middle.[16]

[15] Solving for this equilibrium, $y_1 = \beta v$ and $\alpha_y(0) = 1 - [\beta/(1-\beta)]$.

[16] Player y randomizes over $[0,\ \delta]$ at $1/[(1-\beta)v]$ and over $[2\delta + y_1,\ 2\delta + y_1 + w_2]$ at $1/(\beta v)$. Player w randomizes over $[\delta,\ \delta + y_1]$ at $1/(\beta v)$ and over $[\delta + y_1,\ \delta + y_1 + w_2]$ at the rate of $1/[(1-\beta)v]$.

The difference between these two crescents is that player w has a mass point at zero in the lower crescent, but not in the upper one.

Following suit with the symmetric case, as the tie margin δ decreases, the number of interval pairs included in each asymmetric equilibrium progressively increases.[17] Most notably, this leads again to a convergence result in which the average density rate over any measurable subset of $[0, v]$ approaches $1/v$ (the standard all-pay auction rate) as δ goes to zero. It is also worth mentioning that for $\delta < v/4$, the regions distinguishing one form of asymmetric equilibrium from another occasionally overlap so that a given point may have multiple distinct forms of asymmetric equilibria.

6 Conclusion

We theoretically characterize equilibrium behavior in the two-player, complete information, all-pay auction with ties. Unlike the typical knife-edge formulation of players tying if they submit precisely the same bid, a tie occurs in our environment if neither player outbids the other by more than a predetermined amount. This expands the win-loss paradigm so that ties become a viable third outcome. Such an expansion is vital for studying contests where the distinction of winning may not be granted to any player; and when it is granted, it requires a quantum difference in performance (as is the case in military standoffs, political gridlock, and split-award procurement contracts to name a few). A pervasive feature of equilibrium strategies here is that players randomize over multiple disjoint intervals. Even when asymmetric equilibria exist, this feature continues to persist. Intuitively, players randomize their costly bids between attempting to win and attempting to tie—the spacing between the disjoint intervals reflecting the required threshold for winning.

There are several natural extensions to this study of ties. Just as the standard all-pay auction has been used as a building block for studying a variety of dynamic contests (including elimination tournaments, "best-of" multi-battle competitions, tug-of-wars, and hybrids of these), the all-pay auction with ties can similarly be used as a basis for more expansive dynamic architectures. A pertinent caveat to our model is that it is fundamentally symmetric. Yet players may have asymmetric valuations over either of their two margins of competition: the relative value of winning compared to tying, or the relative value of tying compared to losing. A further caveat is that the threshold for winning remains constant over all ranges of bids. Winning requirements could, however, be much more nuanced based on the intensity and level of the competition. The competition's very nature may even endogenously determine the size of the threshold. In addition to these, the all-pay auction with ties may be extended along any of the numerous dimensions in which the standard all-pay auction has been studied.

[17] Three additional examples from Figure 8 will suffice. Of the two crescents with endpoints of $(v/3, v/3)$ and $(v/2, v/4)$, the upper one has positive values for $\alpha_y(0)$, $\alpha_w(0)$, y_1, w_1, and y_2, while the lower crescent also has a positive value for w_3. Another crescent between $(v/3, v/4)$ and $(v/2, v/4)$ has positive values for $\alpha_y(0)$, $\alpha_w(0)$, y_1, w_1, y_2, w_2, and y_3 (omitting w_3).

Appendices

A General Necessary Conditions

Here we present the necessary conditions for equilibrium. Our first result limits the occurrence of a mass point to a bid of zero. If a mass point were to occur elsewhere, the opponent could profit by either barely tying it from below or slightly beating it from above, which in turn provides an incentive for moving the mass point. A mass point at zero is different, however, because it cannot be undercut.

Lemma 1 *Let $\delta \in (0,\ (1-\beta)v\,)$ and $\beta > 0$. A positive mass point may only occur at zero in any equilibrium.*

Proof Suppose that G_{-i} includes a mass point at $x^* > 0$ of size $\alpha_{-i}(x^*) \in (0,1)$. For any $k \in (0,\delta)$, unless G_i either has some mass in $[x^*+\delta-k,\ x^*+\delta]$ which x^* can tie or some mass in $[x^*-\delta-k,\ x^*-\delta)$ which x^* can beat, then it is profitable for player $-i$ to move x^* down since this reduces the cost of bidding but maintains the probability of a tie and a win. Suppose that G_i does indeed contain mass in $[x^*+\delta-k,\ x^*+\delta]$ or $[x^*-\delta-k,\ x^*-\delta)$ for any arbitrarily small k. We will show that there exists a k such that placing mass in either interval is not optimal for player i. In which case, a mass point at x^* cannot be a best response for player $-i$. We begin by showing that for sufficiently small λ, player i strictly prefers a bid of $x^*+\delta+\lambda$ to a bid of $x^*+\delta-\mu$, where $\lambda > \mu \geq 0$. From Equation 2, we have:

$$\begin{aligned} u_i(x^*{+}\delta{+}\lambda,\ G_{-i}) - u_i(x^*{+}\delta{-}\mu,\ G_{-i}) &= [G_{-i}(x^*{+}2\delta{+}\lambda) - G_{-i}(x^*{+}2\delta{-}\mu)]\beta v \\ &+ [G_{-i}(x^*{+}\lambda) - G_{-i}(x^*{-}\mu) - \alpha_{-i}(x^*{+}\lambda) + \alpha_{-i}(x^*{-}\mu)](1-\beta)v - \lambda - \mu \end{aligned}$$

By definition, $G_{-i}(x^*{+}\lambda) - G_{-i}(x^*{-}\mu) \geq \alpha_{-i}(x^*)$. Let y^* be the next mass point in G_{-i}, if such a mass point exists. That is, $y^* = \min\{x \in \mathrm{supp}(G_{-i}) \mid x > x^* \text{ and } \alpha(x) > 0\}$. If y^* does not exist, then $\alpha_{-i}(x^*{+}\lambda) = 0$. Otherwise, if $x^*{+}\lambda < y^*$, then we still have $\alpha_{-i}(x^*{+}\lambda) = 0$. Hence, for sufficiently small λ:

$$u_i(x^*{+}\delta{+}\lambda,\ G_{-i}) - u_i(x^*{+}\delta{-}\mu,\ G_{-i}) \ \geq\ \alpha_{-i}(x^*)(1-\beta)v - \lambda - \mu$$

Since $\lambda > \mu \geq 0$, then $u_i(x^*+\delta+\lambda,\ G_{-i}) - u_i(x^*+\delta-\mu,\ G_{-i}) > 0$ holds whenever $\lambda + \mu < \alpha_{-i}(x^*)(1-\beta)v$; or rather, whenever $\lambda < \alpha_{-i}(x^*)(1-\beta)(v/2)$ (and with the further restriction that $\lambda < y^* - x^*$ if y^* exists). So for any k that is less than such a λ, player i strictly prefers a bid of $x^*+\delta+\lambda$ to any bid in $[x^*+\delta-k,\ x^*+\delta]$. Since bids below zero are not possible, this completes the proof for $x^* \in (0,\ \delta]$. For $x^* > \delta$, we must also rule out the possibility of mass in $[x^*-\delta-k,\ x^*-\delta)$. Similar to the previous argument, we will show that player i strictly prefers a bid of $x^*-\delta$ to a bid of $x^*-\delta-\gamma$ where $\gamma > 0$:

$$\begin{aligned} u_i(x^*{-}\delta,\ G_{-i}) - u_i(x^*{-}\delta{-}\gamma,\ G_{-i}) &= [G_{-i}(x^*) - G_{-i}(x^*{-}\gamma)]\beta v \\ &+ [G_{-i}(x^*{-}2\delta) - G_{-i}(x^*{-}2\delta{-}\gamma) - \alpha_{-i}(x^*{-}2\delta) + \alpha_{-i}(x^*{-}2\delta{-}\gamma)](1-\beta)v - \gamma \\ &\geq\ \alpha_{-i}(x^*)\beta v - \alpha_{-i}(x^*-2\delta)(1-\beta)v - \gamma \end{aligned}$$

The inequality follows from observing that $G_{-i}(x^*) - G_{-i}(x^*{-}\gamma) \geq \alpha_{-i}(x^*)$ and that the omitted terms are weakly positive. Then if $\gamma < \alpha_{-i}(x^*)\beta v - \alpha_{-i}(x^*-2\delta)(1-\beta)v$, player i profits from moving any mass in $[x^*-\delta-\gamma,\ x^*-\delta)$ up to $x^*-\delta$. Moreover, if $\alpha_{-i}(x^*)\beta > \alpha_{-i}(x^*-2\delta)(1-\beta)$, then such a γ exists, which in turn implies that there is a k that meets our requirements (in particular, any $k < \min\{\gamma,\ \lambda\}$ where γ and λ satisfy the bounds specified above). Since $\alpha_{-i}(x^*-2\delta)$ necessarily equals zero for $x^* \in (0,\ 2\delta)$, mass points in $(0,\ 2\delta)$ are not a best response. Now suppose that $\alpha_{-i}(x^*)\beta \leq \alpha_{-i}(x^*-2\delta)(1-\beta)$ where $x^* \geq 2\delta$. Following the above argument, $x^*-2\delta$ can only be sustained as a mass point in equilibrium if $\alpha_{-i}(x^*-2\delta)\beta \leq \alpha_{-i}(x^*-4\delta)(1-\beta)$. More generally, a mass point in equilibrium at $x^*-2q\delta$ requires that $\alpha_{-i}(x^*-2q\delta)\beta \leq \alpha_{-i}(x^*-2[q+1]\delta)(1-\beta)$ for $q \in \{0, \ldots, \lfloor x^*/2\delta \rfloor - 1\}$. However, since there are no mass points in $(0,\ 2\delta)$, then provided that x^* is not evenly divisible by 2δ, there cannot be a mass point at $x^*-2q\delta$. The final case to consider is a sequence of mass points at 0, 2δ, 4δ, etc. It suffices to show that a mass point at 2δ is not a best response; any successive mass points would then fail to hold in equilibrium. For $\alpha_{-i}(2\delta) > 0$ to be sustained in equilibrium, G_i must contain mass in a neighborhood immediately below δ. This, however, cannot be. Since $\alpha_{-i}(2\delta) > 0$ requires that $\alpha_{-i}(0) > 0$, player i strictly prefers a bid slightly above δ to a bid of $\delta - c$ where $c < \alpha_{-i}(0)(1-\beta)v$. Specifically, $u_i(\overline{\delta},\ G_{-i}) - u_i(\delta - c,\ G_{-i}) = [G_{-i}(2\delta) - G_{-i}(2\delta - c)]\beta v + \alpha_{-i}(0)(1-\beta)v - c > 0$ for $c < \alpha_{-i}(0)(1-\beta)v$.[18] Therefore, $k < \min\{c,\ \lambda\}$ with c and λ meeting their respective bounds satisfies our requirements, so a mass point at 2δ is not optimal. □

With mass points limited to zero, the next result stems from the indifference condition that must hold when players are randomizing between multiple bids. That is, $u_i(x,\ G_{-i}) = u_i(y,\ G_{-i})$ for $x, y \in \text{supp}(G_i)$ where $x > y$. Using Lemma 1 and Equation 2, we can restate this indifference condition as:

$$[G_{-i}(x+\delta) - G_{-i}(y+\delta)]\,\beta v + [G_{-i}(x-\delta) - G_{-i}(y-\delta)]\,(1-\beta)v = x - y \tag{14}$$

The added cost of the higher bid must either be compensated by tying mass in $[y+\delta,\ x+\delta]$ or beating mass in $[y-\delta,\ x-\delta]$. Notably, the absence of mass in either of these intervals pins down the necessary distribution over the other. This principle is formalized in the following lemma.

Lemma 2 *For $\delta \in (0,\ (1-\beta)v\,)$ and $\beta > 0$, let G_i and G_{-i} be equilibrium distributions for players i and $-i$.*

A. *Let $b \geq 0$ satisfy $G_{-i}(b) - \alpha_{-i}(b) = G_{-i}(b-c) - \alpha_{-i}(b-c)$ for $c \in (0,\ \delta]$. If the subset $(\underline{a},\ \overline{a}] \subseteq (b+\delta-c,\ b+\delta]$ is in the support of G_i, then $(\underline{a}+\delta,\ \overline{a}+\delta]$ is in the support of G_{-i}. Moreover, the distribution over $(\underline{a}+\delta,\ \overline{a}+\delta]$ in G_{-i} is uniform at the rate of $1/(\beta v)$.*

B. *Let $b > \delta$ satisfy $G_{-i}(b) = G_{-i}(b+c)$ for $c \in (0,\ \delta]$. If the subset $(\underline{a},\ \overline{a}] \subseteq (b-\delta,\ b-\delta+c]$ is in the support of G_i, then $(\underline{a}-\delta,\ \overline{a}-\delta]$ is in the support of G_{-i}. Additionally, the distribution over $(\underline{a}-\delta,\ \overline{a}-\delta]$ in G_{-i} is uniform at the rate of $1/[(1-\beta)v]$.*

Proof Part A. To show the first claim, suppose that the subset $(\underline{a}+\delta,\ \overline{a}+\delta]$ is not in the support of G_{-i}. Since G_{-i} also has no mass in $[b-c,\ b)$, player i strictly prefers a bid of $\underline{a}$ to any $k \in (\underline{a},\ \overline{a}]$. This is because the probability of winning or tying remains unchanged but the cost of bidding is higher (i.e. $u_i(\underline{a},\ G_{-i}) - u_i(k,\ G_{-i}) = k - \underline{a} > 0$). Hence, $(\underline{a},\ \overline{a}]$ cannot be in the support of G_i. For the second claim, suppose that $(\underline{a},\ \overline{a}]$ is in the support of G_i. Consequently, $(\underline{a}+\delta,\ \overline{a}+\delta]$ is then in the support of G_{-i}. Let $x, y \in (\underline{a},\ \overline{a}]$ where $x > y$. By payoff equivalence, $u_i(x,\ G_{-i}) = u_i(y,\ G_{-i})$, which then implies that $G_{-i}(x+\delta) - G_{-i}(y+\delta) = (1/\beta v)(x-y)$. Since this equation holds for any x and y, including values which are arbitrarily close, the result then follows.

[18] The notation $u_i(\overline{\delta},\ G_{-i})$ is defined on p. 14 as the limit of player i's expected utility as δ is approached from above.

Part B. If $(\underline{a}-\delta,\ \overline{a}-\delta]$ is not in the support of G_{-i}, then since G_{-i} has no mass in $(b,\ b+c]$, player i strictly prefers a bid of $\underline{a}$ to any $k \in (\underline{a},\ \overline{a}]$ (i.e. the probability of a win and a tie remains the same, but k has a higher bidding cost than $\underline{a}$). So $(\underline{a},\ \overline{a}]$ is not in the support of G_i. For the next part, suppose that $(\underline{a},\ \overline{a}]$ is in the support of G_i. By payoff equivalence, we have $u_i(x,\ G_{-i}) = u_i(y,\ G_{-i})$ for any $x, y \in (\underline{a},\ \overline{a}]$ such that $x > y$. From this equality, $G_{-i}(x-\delta) - G_{-i}(y-\delta) - \alpha_{-i}(x-\delta) + \alpha_{-i}(y-\delta) = (x-y)/[(1-\beta)v]$. Since mass points may only occur at zero in equilibrium (see Lemma 1), and since this equation holds for any x and y which are arbitrarily close, the result then follows. □

Lemma 2 is particularly applicable at the upper and lower bound of a distribution. If $\overline{x}_i$ is the upper bound of G_i, then any mass in G_{-i} over $[\overline{x}_i-\delta,\ \overline{x}_i]$ must be balanced by mass in G_i, shifted down by δ in $[\overline{x}_i-2\delta,\ \overline{x}_i-\delta]$, with a density rate of $1/[(1-\beta)v]$. Precisely δ below that, if there is more mass in G_{-i}, Equation 14 can again be used to identify the necessary density rate for mass in G_i over $[\overline{x}_i-4\delta,\ \overline{x}_i-3\delta]$. This process continues—iteratively moving down the distribution—and a similar process holds for moving up the distribution. However, the necessary density rates can only hold in both directions if $\beta = 1/2$. We therefore obtain the following result.

Lemma 3 *Let $\delta \in (0,\ (1-\beta)v\,)$ and $\beta > 0$. In any equilibrium, if $\beta \neq 1/2$, then all continuously distributed mass must be uniform at a rate of either $1/(\beta v)$ or $1/[(1-\beta)v]$.*

Proof Let $[\underline{s}_0,\ \overline{s}_0] \in \operatorname{supp}(G_i)$ be given such that $G_{-i}(\underline{s}_0-\delta) = G_{-i}(\overline{s}_0-\delta)$ if $\overline{s}_0 > \delta$. By Lemma 2.A, $[\underline{s}_0+\delta,\ \overline{s}_0+\delta] \in \operatorname{supp}(G_{-i})$ with uniformly distributed mass at the rate of $1/(\beta v)$. Likewise, by Lemma 2.B, if $G_i(\underline{s}_0+2\delta) = G_i(\overline{s}_0+2\delta)$, then the distribution over $[\underline{s}_0,\ \overline{s}_0]$ in G_i is uniform at the rate of $1/[(1-\beta)v]$. Suppose instead that $G_i(\underline{s}_0+2\delta) < G_i(\overline{s}_0+2\delta)$. Let $[\underline{s}_1,\ \overline{s}_1] \subseteq [\underline{s}_0+2\delta,\ \overline{s}_0+2\delta]$ such that $[\underline{s}_1,\ \overline{s}_1] \in \operatorname{supp}(G_i)$ (by Lemma 1, such an interval must exist since mass points may only occur at zero). We first show that for $\beta < 1/2$ that $[\underline{s}_1,\ \overline{s}_1] \notin \operatorname{supp}(G_i)$. For any $x, y \in [\underline{s}_1,\ \overline{s}_1]$, since $G_{-i}(x-\delta) - G_{-i}(y-\delta) = (x-y)/\beta v$, Equation 14 implies that:

$$G_{-i}(x+\delta) - G_{-i}(y+\delta) = (x-y)\left(\frac{2\beta-1}{\beta^2 v}\right) \tag{15}$$

Since this holds for x and y which are arbitrarily close, the distribution over $[\underline{s}_1+\delta,\ \overline{s}_1+\delta]$ must be uniform at a rate of $(2\beta-1)/(\beta^2 v)$. For $\beta < 1/2$, this contradicts the monotonicity of G_{-i}, and so $[\underline{s}_1,\ \overline{s}_1] \notin \operatorname{supp}(G_i)$. This completes the proof for $\beta < 1/2$. For $\beta = 1/2$, there is simply no mass in $[\underline{s}_1+\delta,\ \overline{s}_1+\delta]$. We therefore turn to the case where $\beta > 1/2$.

Suppose for contradiction that $G_i(\underline{s}_1+2\delta) = G_i(\overline{s}_1+2\delta)$. By Lemma 2.B, mass over the interval $[\underline{s}_1,\ \overline{s}_1]$ in G_i must be uniform with a density rate of $1/[(1-\beta)v]$. Using Equation 14, the indifference condition $u_{-i}(x,\ G_i) = u_{-i}(y,\ G_i)$, with $x, y \in [\underline{s}_1-\delta,\ \overline{s}_1-\delta] \subseteq [\underline{s}_0+\delta,\ \overline{s}_0+\delta]$ and $x > y$ implies:[19]

$$G_i(x-\delta) - G_i(y-\delta) = (x-y)\left(\frac{1-2\beta}{(1-\beta)^2 v}\right)$$

This is a contradiction for $\beta > 1/2$, and it is reached if any subset of $[\underline{s}_1+2\delta,\ \overline{s}_1+2\delta]$ is not in the support of G_i. Thus, if $[\underline{s}_1+\delta,\ \overline{s}_1+\delta] \in \operatorname{supp}(G_{-i})$, it must be that $[\underline{s}_1+2\delta,\ \overline{s}_1+2\delta] \in \operatorname{supp}(G_i)$. This argument holds more generally. Letting $[\underline{s}_1+(\ell-2)\delta,\ \overline{s}_1+(\ell-2)\delta]$ and $[\underline{s}_1+\ell\delta,\ \overline{s}_1+\ell\delta] \in \operatorname{supp}(G_{-i})$ for any $\ell \in \mathbb{N}$, and supposing that $[\underline{s}_1+(\ell+1)\delta,\ \overline{s}_1+(\ell+1)\delta] \notin \operatorname{supp}(G_i)$, then the density rates over $[\underline{s}_1+(\ell-1)\delta,\ \overline{s}_1+(\ell-1)\delta]$ and $[\underline{s}_1+(\ell-3)\delta,\ \overline{s}_1+(\ell-3)\delta]$ in G_i are $1/[(1-\beta)v]$ and $(1-2\beta)/[(1-\beta)^2 v]$. A similar statement holds if $[\underline{c},\ \overline{c}] \notin \operatorname{supp}(G_i)$ for any $[\underline{c},\ \overline{c}] \subseteq [\underline{s}_1+(\ell+1)\delta,\ \overline{s}_1+(\ell+1)\delta]$. Hence,

[19] Recall that Equation 14 is written in terms of player i's indifference: $u_i(x,\ G_{-i}) = u_i(y,\ G_{-i})$. So the corresponding version of Equation 14 for player $-i$ replaces each G_{-i} with G_i.

by contradiction, $[\underline{s}_1 + (\ell+1)\delta,\ \overline{s}_1 + (\ell+1)\delta] \in \text{supp}(G_i)$. Mass over intervals that are 2δ apart in G_{-i} requires that G_i has mass in an even higher interval. However, as we will show next, this in turn requires that G_{-i} has mass in yet a higher interval.

Again drawing on Equation 14, in order for player i to be indifferent between $x, y \in [\underline{s}_1 + (\ell-1)\delta,\ \overline{s}_1 + (\ell-1)\delta]$, where $x > y$, then:

$$G_{-i}(x+\delta) - G_{-i}(y+\delta) = \left(\frac{x-y}{\beta v}\right) - [G_{-i}(x-\delta) - G_{-i}(y-\delta)]\left(\frac{1-\beta}{\beta}\right)$$

For $\ell = 1$, the specific value of $G_{-i}(x+\delta) - G_{-i}(y+\delta)$ is given by Equation 15. This value then becomes $G_{-i}(x-\delta) - G_{-i}(y-\delta)$ for $\ell = 2$. Iterating, we obtain the following general form for $\ell \geq 1$:[20]

$$G_{-i}(x+\delta) - G_{-i}(y+\delta) = (x-y)\left(\frac{1}{\beta^{\ell+1}v}\right)\sum_{j=0}^{\ell}(-1)^{\ell-j}\beta^j(1-\beta)^{\ell-j} \qquad (16)$$

The positivity of Equation 16 for $\beta > 1/2$ can be seen by doing a pairwise summation of right-hand side terms (i.e. sum $j = \ell$ with $j = \ell - 1$; $j = \ell - 2$ with $j = \ell - 3$; etc.). Thus, we have:

$$\sum_{j=0}^{\ell}(-1)^{\ell-j}\beta^j(1-\beta)^{\ell-j} = (1-\beta)^\ell\,\mathcal{I}(\ell) + (2\beta-1)\sum_{j=0}^{\lfloor(\ell-1)/2\rfloor}\beta^{\ell-1-2j}(1-\beta)^{2j} > 0$$

where $\mathcal{I}(\ell)$ is an indicator function equal to 1 if ℓ is even and 0 otherwise, and $\lfloor\cdot\rfloor$ is the floor function. Since Equation 16 is strictly positive for any $x, y \in [\underline{s}_1 + (\ell-1)\delta,\ \overline{s}_1 + (\ell-1)\delta]$, then $[\underline{s}_1 + \ell\delta,\ \overline{s}_1 + \ell\delta] \in \text{supp}(G_{-i})$.

The escalating supports of G_i and G_{-i} ultimately rise above v where bids are strictly dominated, contradicting the initial supposition that G_{-i} have mass over intervals that are 2δ apart (i.e. there is no pair of intervals $[\underline{s}_1+(\ell-2)\delta,\ \overline{s}_1+(\ell-2)\delta]$ and $[\underline{s}_1+\ell\delta,\ \overline{s}_1+\ell\delta]$ that are both in the support of G_{-i} for any $\ell \in \mathbb{N}$). In particular, $[\underline{s}_1 + \delta,\ \overline{s}_1 + \delta] \notin \text{supp}(G_{-i})$, so $[\underline{s}_1,\ \overline{s}_1] \notin \text{supp}(G_i)$. Therefore, $[\underline{s}_0,\ \overline{s}_0]$ has a density rate of $1/[(1-\beta)v]$ in G_i. □

The density rates of $1/(\beta v)$ and $1/[(1-\beta)v]$ have intuitive appeal since βv is the marginal value of tying relative to losing and $(1-\beta)v$ is the marginal value of winning relative to tying. In isolating these density rates, we also derive the following corollary result:

Corollary 1 *Let $\delta \in (0,\ (1-\beta)v\,)$, $\beta > 0$, and $\beta \neq 1/2$. For any $\overline{z} > \underline{z} \geq 0$ such that $\overline{z} - \underline{z} \leq \delta$, in equilibrium the interval $[\underline{z},\ \overline{z}]$ has a density rate of $1/[(1-\beta)v]$ in G_i if and only if $[\underline{z}+\delta,\ \overline{z}+\delta]$ has a density rate of $1/(\beta v)$ in G_{-i}. In which case, $G_{-i}(\underline{z}-\delta) = G_{-i}(\overline{z}-\delta)$, and $G_i(\underline{z}+2\delta) = G_i(\overline{z}+2\delta)$.*

B Proofs Specific to $\beta < 1/2$

Lemma 4 *Let $\delta \in (0,\ (1-\beta)v\,)$ and $\beta < 1/2$. In any equilibrium, any continuously distributed mass over $[0,\ \delta]$ must be connected, have a lower bound of zero and a density rate of $1/[(1-\beta)v]$. Similarly, if $p, q \in \text{supp}(G_{-i})$ such that $p < q$ and $G_{-i}(p) = G_{-i}(q)$, then any continuously distributed mass in G_i over $[p+\delta,\ q+\delta]$ must also be connected, have a lower bound of $p+\delta$, and have a density rate of $1/[(1-\beta)v]$.*

[20] This pattern is easier to see by writing $2\beta - 1$ as $\beta - (1-\beta)$ for $\ell = 1$; and then $\beta^2 - \beta(1-\beta) + (1-\beta)^2$ for $\ell = 2$, etc.

Proof For $z \in (0,\ \delta)$, suppose there exists $[z,\ z+b] \in \text{supp}(G_i)$ such that $G_i(z) = G_i(z-c)$, where $b > 0$ and $c \in (0,\ z]$. In equilibrium, $u_i(z,\ G_{-i}) \geq u_i(z-c,\ G_{-i})$, and so:

$$[G_{-i}(z+\delta) - G_{-i}(z+\delta-c)]\beta v \geq c$$

By Corollary 1, since $G_i(z) = G_i(z-c)$, then any mass in G_{-i} over $[z+\delta-c,\ z+\delta]$ must have a density rate of $1/[(1-\beta)v]$. So for some $s \in [0,\ c]$, we have:

$$[G_{-i}(z+\delta) - G_{-i}(z+\delta-c)]\beta v = s\beta/(1-\beta) \geq c$$

However, this cannot hold for $\beta < 1/2$. The same argument holds for $z \in [p+\delta,\ q+\delta]$ and $c \in (p+\delta,\ z]$, where p and q are defined as in the statement of the lemma. □

Lemma 5 *Let* $\delta \in (0,\ (1-\beta)v\ )$ *and* $\beta < 1/2$. *In equilibrium, there does not exist* $\underline{z} \geq 0$ *such that* $G_i(\underline{z}-\delta) = G_i(\underline{z})$ *and* $[\underline{z},\ \overline{z}]$ *has a density rate of* $1/[(1-\beta)v]$ *in* G_i *for* $\overline{z} > \underline{z}$.

Proof Suppose to the contrary that in equilibrium there exists a $\underline{z}$ such that $G_i(\underline{z}-\delta) = G_i(\underline{z})$ and $[\underline{z},\ \overline{z}]$ has a density rate of $1/[(1-\beta)v]$ in G_i. By Corollary 1, $[\underline{z}+\delta,\ \overline{z}+\delta]$ has a density rate of $1/(\beta v)$ in G_{-i}. In equilibrium, $u_{-i}(\underline{z}+\delta,\ G_i) \geq u_{-i}(\underline{z},\ G_i)$, which can only be satisfied if $[\underline{z}+\delta,\ \underline{z}+2\delta]$ has a density rate of $1/(\beta v)$ in G_i (since $G_i(\underline{z}-\delta) = G_i(\underline{z})$ and $\beta < 1/2$). So now $\underline{z}+2\delta \in \text{supp}(G_i)$. We reach a contradiction in that equilibrium requires $u_i(\underline{z}+2\delta,\ G_{-i}) \geq u_i(\overline{z}+2\delta,\ G_{-i})$, but $u_i(\underline{z}+2\delta,\ G_{-i}) - u_i(\overline{z}+2\delta,\ G_{-i}) \leq (\overline{z}-\underline{z}) - (\overline{z}-\underline{z})[(1-\beta)v/\beta v] < 0$. □

Since every $1/[(1-\beta)v]$ segment must be preceded by some other mass, equilibrium requires that at least one player must have a mass point at zero. Moreover, that same player must also have some mass in $[0,\ \delta]$ that is connected, with a density rate of $1/[(1-\beta)v]$, and a lower bound of zero (i.e. the properties in Lemma 4). If neither player's distribution began this way, Corollary 1, Lemma 4, and Lemma 5 would prohibit the placing of any continuously distributed mass in either player's distribution. Assuming that at least one player's distribution does indeed comply, these results dictate the pattern for placing any further mass. Labeling the distributions G_w and G_y, suppose that G_w has a mass point $\alpha_w(0) \in (0,\ 1)$ and a $1/[(1-\beta)v]$ segment in $[0,\ \delta]$ of length $w_1 > 0$. So by Corollary 1, G_y has a $1/(\beta v)$ segment of length w_1 spanning $[\delta,\ \delta+w_1]$. Drawing on Lemma 4, the gap in G_w following w_1 implies that any mass in G_y in the region above $\delta+w_1$ must be connected, with a density rate of $1/[(1-\beta)v]$, and a lower bound of $\delta+w_1$. If G_y does indeed have a $1/[(1-\beta)v]$ segment here, say of length y_2, then δ above that, G_w has a $1/(\beta v)$ segment of length y_2. Every $1/[(1-\beta)v]$ segment must follow in the immediate wake of a mass point at zero or a $1/(\beta v)$ segment, and the occurrence of $1/(\beta v)$ segments is wholly determined by $1/[(1-\beta)v]$ segments. Corollary 1 further limits the combined length of adjoining $1/(\beta v)$ and $1/[(1-\beta)v]$ segments to no more than δ. Any equilibrium must therefore be of the form depicted in Figure 7. Mass points $\alpha_w(0)$ and $\alpha_y(0) \in [0,1)$ are followed by alternating $1/[(1-\beta)v]$ and $1/(\beta v)$ segments. We again label the length of the successive $1/[(1-\beta)v]$ segments for player w as $w_1, w_2, \ldots, w_k \geq 0$, and likewise for player y as $y_1, y_2, \ldots, y_k \geq 0$. If indeed some $w_j = 0$ then $y_{j+1} = 0$, and like dominoes, $w_{j+2} = 0$, $y_{j+3} = 0$, etc. The principle again being that a $1/[(1-\beta)v]$ segment can only follow a $1/(\beta v)$ segment or a mass point at zero. Symmetric equilibria are therefore restricted to the case where all y_j and w_j are strictly positive (and, by implication, $\alpha_w(0),\ \alpha_y(0) > 0$). As Proposition 1 states, there is a unique equilibrium in which these are all positive and that equilibrium is symmetric.

Proposition 1 *Let* $\delta \in (0,\ (1-\beta)v\ )$ *and* $\beta < 1/2$. *For any* $k \in \mathbb{N}$, *there is a unique equilibrium satisfying the constraint that* $y_1,\ w_1,\ \ldots,\ y_k,\ w_k$ *are all strictly positive. Moreover, that equilibrium is symmetric* (*i.e.* $y_1 = w_1,\ \ldots,\ y_k = w_k$ *and* $\alpha_y(0) = \alpha_w(0)$).

Proof In equilibrium, all points within the support must have the same expected payoff. This property must particularly hold at each break in each player's support. Given the constraint that $y_1,\ w_1,\ \ldots,\ y_k,\ w_k$ are all strictly positive, each player's support has k breaks. For player w:

$$\begin{aligned}
&u_w(\overline{\delta},\ G_y) = u_w(w_1,\ G_y)\\
&u_w(2\delta + w_1,\ G_y) = u_w(\delta + y_1 + w_2,\ G_y)\\
&u_w(3\delta + y_1 + w_2,\ G_y) = u_w(2\delta + w_1 + y_2 + w_3,\ G_y)\\
&u_w(4\delta + w_1 + y_2 + w_3,\ G_y) = u_w(3\delta + y_1 + w_2 + y_3 + w_4,\ G_y)
\end{aligned}$$

In general, for an even integer q:

$$u_w\left(q\delta + \sum_{j\geq 1,\, odd}^{q-1} w_j + \sum_{j\geq 2,\, even}^{q-2} y_j,\ \ G_y\right) = u_w\left((q-1)\delta + \sum_{j\geq 1,\, odd}^{q-1} y_j + \sum_{j\geq 2,\, even}^{q} w_j,\ \ G_y\right)$$

$$u_w\left((q+1)\delta + \sum_{j\geq 1,\, odd}^{q-1} y_j + \sum_{j\geq 2,\, even}^{q-2} w_j,\ \ G_y\right) = u_w\left(q\delta + \sum_{j\geq 1,\, odd}^{q-1} w_j + \sum_{j\geq 2,\, even}^{q} y_j,\ \ G_y\right)$$

Corresponding equations for player y merely reverse the roles of all w_j and y_j. Across players, this system has a total of $2k$ equations with $2k + 2$ unknowns (i.e. $\alpha_y(0),\ \alpha_w(0),\ y_1,\ w_1,\ \ldots,\ y_k,\ w_k$). We close the system by requiring the total mass in each distribution to sum to one:

$$1 = \alpha_w(0) + \sum_{j=1}^{k} \frac{w_j}{(1-\beta)v} + \frac{y_j}{\beta v}$$

$$1 = \alpha_y(0) + \sum_{j=1}^{k} \frac{y_j}{(1-\beta)v} + \frac{w_j}{\beta v}$$

These mass constraints and the system of indifference conditions can be written in matrix form, where each row in the matrix represents an equation. The matrix for the case of $k = 4$ is as follows:

$$\begin{pmatrix}
-1 & 1 & 0 & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v}\\
-1 & 0 & 1 & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} & \frac{1}{(1-\beta)v}\\
-\delta & (1-\beta)v & 0 & 0 & 1 & \frac{\beta}{1-\beta} & 0 & 0 & 0 & 0 & 0\\
-\delta & 0 & (1-\beta)v & 1 & 0 & 0 & \frac{\beta}{1-\beta} & 0 & 0 & 0 & 0\\
-\delta & 0 & 0 & 1 & \frac{1-2\beta}{\beta} & 0 & 1 & \frac{\beta}{1-\beta} & 0 & 0 & 0\\
-\delta & 0 & 0 & \frac{1-2\beta}{\beta} & 1 & 1 & 0 & 0 & \frac{\beta}{1-\beta} & 0 & 0\\
-\delta & 0 & 0 & -1 & 1 & 1 & \frac{1-2\beta}{\beta} & 0 & 1 & \frac{\beta}{1-\beta} & 0\\
-\delta & 0 & 0 & 1 & -1 & \frac{1-2\beta}{\beta} & 1 & 1 & 0 & 0 & \frac{\beta}{1-\beta}\\
-\delta & 0 & 0 & 1 & -1 & -1 & 1 & 1 & \frac{1-2\beta}{\beta} & 0 & 1\\
-\delta & 0 & 0 & -1 & 1 & 1 & -1 & \frac{1-2\beta}{\beta} & 1 & 1 & 0
\end{pmatrix}
\begin{pmatrix} 1 \\ \alpha_y(0) \\ \alpha_w(0) \\ y_1 \\ w_1 \\ y_2 \\ w_2 \\ y_3 \\ w_3 \\ y_4 \\ w_4 \end{pmatrix} = \mathbf{0}$$

The top two rows are the mass constraints for players y and w. The third and fourth rows correspond to $u_w(\overline{\delta},\ G_y) = u_w(w_1,\ G_y)$ and $u_y(\overline{\delta},\ G_w) = u_y(y_1,\ G_w)$; the fifth and sixth to $u_w(2\delta + w_1,\ G_y) = u_w(\delta + y_1 + w_2,\ G_y)$ and $u_y(2\delta + y_1,\ G_w) = u_y(\delta + w_1 + y_2,\ G_w)$. Each successive pair of rows correspond to the indifference conditions over the next jump in each player's support. The above matrix with $k = 4$ is also useful for visualizing the general form for an arbitrary k. We denote the system for a given k by $\boldsymbol{A_k x = 0}$. The matrices $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, and $\boldsymbol{A_3}$ are all partitions of the above matrix $\boldsymbol{A_4}$. For $\boldsymbol{A_1}$, it is the partition formed by the first four rows and five columns of $\boldsymbol{A_4}$. The matrix $\boldsymbol{A_2}$ comprises the first six rows and seven columns, and the first eight rows and nine columns form the matrix $\boldsymbol{A_3}$. In general, $\boldsymbol{A_k}$ has $r_k = 2k + 2$ rows and $c_k = 2k + 3$ columns. The matrix $\boldsymbol{A_{k-1}}$ constitutes the first $r_k - 2$ rows and $c_k - 2$ columns of $\boldsymbol{A_k}$. Elements in the last two rows and columns of $\boldsymbol{A_k}$ fit a clearly defined pattern. For $k \geq 2$, the first three elements of rows $r_k - 1$ and r_k are $\{\delta, 0, 0\}$. The fourth and fifth elements are shown in the matrix for $k \leq 4$. For $k \geq 4$, the fourth and fifth element of row $r_k - 1$ is the fourth and fifth element of row $r_k - 2$; and for row r_k it is the fourth and fifth elements of row $r_k - 3$. Elements six through c_k of rows $r_k - 1$ and r_k are the same as elements four through $c_k - 2$ of rows $r_k - 3$ and $r_k - 2$. Besides rows $r_k - 1$ and r_k, the only nonzero elements of columns $c_k - 1$ and c_k are in rows 1, 2, $r_k - 3$, and $r_k - 2$. The last two elements of rows 1 and 2 follow the established pattern for the mass constraints: $\{1/[(1-\beta)v],\ 1/(\beta v)\}$ and $\{1/(\beta v),\ 1/[(1-\beta)v]\}$. For rows $r_k - 3$ and $r_k - 2$, the last two elements are $\{\beta/(1-\beta),\ 0\}$ and $\{0,\ \beta/(1-\beta)\}$.

We obtain our uniqueness result by showing that the r_k rows of $\boldsymbol{A_k}$ are linearly independent for any $k \in \mathbb{N}$. The r_k rows are linearly independent if and only if the system $\boldsymbol{A_k x = 0}$ has a unique solution. We proceed by induction. Table 2 shows the unique solution to the system when $k \in \{1, 2, 3\}$, and it is also relevant that the unique solution is symmetric in each case. Now suppose that for $k \geq 3$ that the r_k rows of $\boldsymbol{A_k}$ are linearly independent. In the matrix $\boldsymbol{A_{k+1}}$, rows 1 through $r_{k+1} - 4$ are still linearly independent. This follows because the last two columns of rows 3 through $r_{k+1} - 4$ only contain zeros, and rows 1 and 2 are always linearly independent of each other because of their second and third elements. Since rows 1, 2, $r_{k+1} - 3$, and r_{k+1} are the only rows with nonzero elements in column $c_{k+1} - 1$, and since rows 1, 2, $r_{k+1} - 2$, and $r_{k+1} - 1$ are the only rows with nonzero elements in column c_k, then if each of these groups of four rows are linearly independent, then all r_{k+1} rows are linearly independent. For the first group (rows 1, 2, $r_{k+1} - 3$, and r_{k+1}), linear independence can be seen by looking at columns 2, 3, $c_{k+1} - 2$, and $c_{k+1} - 1$:

$$\begin{pmatrix} 1 & 0 & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} \\ 0 & 1 & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} \\ 0 & 0 & 1 & \frac{\beta}{1-\beta} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Linear independence for the second group (rows 1, 2, $r_{k+1} - 2$, and $r_{k+1} - 1$) can be seen from columns 2, 3, $c_{k+1} - 3$, and c_{k+1}:

$$\begin{pmatrix} 1 & 0 & \frac{1}{(1-\beta)v} & \frac{1}{\beta v} \\ 0 & 1 & \frac{1}{\beta v} & \frac{1}{(1-\beta)v} \\ 0 & 0 & 1 & \frac{\beta}{1-\beta} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Thus the r_{k+1} rows of $\boldsymbol{A_{k+1}}$ are linearly independent, so $\boldsymbol{A_{k+1} x = 0}$ has a unique solution. We next demonstrate that the unique solution is symmetric. That is, $\alpha_y(0) = \alpha_w(0)$ and $y_1 = w_1, \ldots, y_k = w_k$. As we stated earlier, for $k \in \{1, 2, 3\}$, the unique solution to $\boldsymbol{A_k x = 0}$ is indeed

Table 2: Unique solution to the system $\boldsymbol{A_k x}=\boldsymbol{0}$ for $k \in \{1,\, 2,\, 3\}$.

$\boldsymbol{k=1}$	$\alpha_y(0)=\alpha_w(0)$	$[\delta-(1-\beta)\beta v]\ /\ [(1-\beta)^2 v]$
	$y_1=w_1$	$[(1-\beta)\beta v-\beta\delta]\ /\ (1-\beta)$
$\boldsymbol{k=2}$	$\alpha_y(0)=\alpha_w(0)$	$(1-2\beta)\delta\ /\ [(1-\beta)^2 v]$
	$y_1=w_1$	$\left[(1-2\beta^2)\beta\delta-(1-\beta)^2\beta^2 v\right]\ /\ [(1-\beta)(1-2\beta)]$
	$y_2=w_2$	$([1-\beta]v-2\delta)(1-\beta)\beta\ /\ (1-2\beta)$
$\boldsymbol{k=3}$	$\alpha_y(0)=\alpha_w(0)$	$\dfrac{\delta-(7\beta^2-7\beta+6)(1-\beta)\beta\delta+(1-\beta)^3\beta^3 v}{(1-\beta)^3(1-3\beta+2\beta^2+\beta^3)v}$
	$y_1=w_1$	$(1-2\beta)\beta\delta\ /\ (1-\beta)^2$
	$y_2=w_2$	$\left[([1-3\beta]\beta+1)\beta\delta-(1-\beta)^2\beta v\right]\ /\ [1-3\beta+2\beta^2+\beta^3]$
	$y_3=w_3$	$\left[([6-2\beta]\beta-3)\beta\delta+(1-\beta)^3\beta v\right]\ /\ [1-3\beta+2\beta^2+\beta^3]$

symmetric (see Table 2). With symmetry, the system $\boldsymbol{A_k x}=\boldsymbol{0}$ collapses to the following system, which we call $\boldsymbol{B_k x}=\boldsymbol{0}$:

$$\begin{pmatrix} -1 & 1 & \frac{1}{(1-\beta)\beta v} & \frac{1}{(1-\beta)\beta v} & \frac{1}{(1-\beta)\beta v} & \frac{1}{(1-\beta)\beta v} & \cdots & \frac{1}{(1-\beta)\beta v} \\ -\delta & (1-\beta)v & 1 & \frac{\beta}{1-\beta} & 0 & 0 & \cdots & 0 \\ -\delta & 0 & \frac{1-\beta}{\beta} & 1 & \frac{\beta}{1-\beta} & 0 & \cdots & 0 \\ -\delta & 0 & 0 & \frac{1-\beta}{\beta} & 1 & \frac{\beta}{1-\beta} & & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \ddots & \vdots \\ -\delta & 0 & 0 & 0 & 0 & \frac{1-\beta}{\beta} & 1 & \frac{\beta}{1-\beta} \\ -\delta & 0 & 0 & 0 & 0 & 0 & \frac{1-\beta}{\beta} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_y(0) \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_k \end{pmatrix} = \boldsymbol{0}$$

By construction, if $\boldsymbol{B_k x}=\boldsymbol{0}$ has a unique solution, then it must coincide with the unique solution of $\boldsymbol{A_k x}=\boldsymbol{0}$. The matrix $\boldsymbol{B_k}$ has $k+1$ rows and $k+2$ columns, and $\boldsymbol{B_{k-1}}$ makes up the first k rows and $k+1$ columns of $\boldsymbol{B_k}$. The last column only has three nonzero elements: $1/[(1-\beta)\beta v]$ in the first row, $\beta/(1-\beta)$ in the second to last row, and 1 in the last row. The only other nonzero elements of the last row are the first and second to last elements: $-\delta$ and $(1-\beta)/\beta$. Since the linear independence of $\boldsymbol{B_{k-1}}$ guarantees that the first k rows of $\boldsymbol{B_k}$ are still linearly independent, we just need to check that the last row is also linearly independent. Since the last row is only one of three rows with a nonzero element in the last column (the other two rows being the first and second to last), it suffices to check that these three rows are linearly independent. This can be seen from the above matrix. Therefore, $\boldsymbol{B_k}$ has a unique solution. The sufficient conditions for equilibria when $\delta \in (0,\ (1-\beta)v\,)$ and $\beta < 1/2$ are established in Section 3.3. □

C Proofs Specific to $\beta > 1/2$

Lemma 6 *Let $\delta \in (0,\ (1-\beta)v\,)$ and $\beta > 1/2$. In any equilibrium, the subset of G_i that has a density rate of $1/[(1-\beta)v]$ in $[z,\ z+2\delta]$ for $z \geq 0$ is connected.*

Proof Suppose to the contrary that for some $z \geq 0$ that the subset of G_i with a density rate of $1/[(1-\beta)v]$ in $[z,\ z+2\delta]$ is disconnected. That is, G_i contains at least two intervals, $[a_1,\, b_1]$,

$[a_2,\, b_2] \subset [z,\ z+2\delta]$ where $b_1 < a_2$, with density rates of $1/[(1-\beta)v]$; furthermore, the density rates in a neighborhood immediately above b_1 and immediately below a_2 differ from $1/[(1-\beta)v]$. By Corollary 1, the intervals $[a_1+\delta,\, b_1+\delta]$ and $[a_2+\delta,\, b_2+\delta]$ have a density rate of $1/(\beta v)$ in G_{-i}. So in equilibrium, $u_{-i}(b_1+\delta,\ G_i) = u_{-i}(a_2+\delta,\ G_i)$:

$$[G_i(a_2+2\delta) - G_i(b_1+2\delta)]\,\beta v + [G_i(a_2) - G_i(b_1)]\,(1-\beta)v = a_2 - b_1$$

This can be rewritten using Lemma 3 as:

$$\left[\frac{r}{(1-\beta)v} + \frac{s}{\beta v}\right]\beta v + \left[\frac{m}{(1-\beta)v} + \frac{n}{\beta v}\right](1-\beta)v =$$
$$r\left(\frac{\beta}{1-\beta}\right) + s + m + n\left(\frac{1-\beta}{\beta}\right) = a_2 - b_1 \tag{17}$$

Here, r and s denote the respective lengths of the support of G_i in $[b_1{+}2\delta,\ a_2{+}2\delta]$ with density rates of $1/[(1-\beta)v]$ and $1/(\beta v)$; m and n and defined similarly for $[b_1,\ a_2]$. Suppose for contradiction that $r = 0$. Using Corollary 1, $[b_1+\delta,\, a_2+\delta]$ in G_{-i} must contain a portion of length s with density rate $1/[(1-\beta)v]$, a portion of length m with density rate of $1/(\beta v)$, and a portion of length n with no mass. Thus,

$$s + m + n \leq a_2 - b_1 \tag{18}$$

With $r = 0$ and $\beta > 1/2$, Equations 17 and 18 can only be jointly satisfied if $n = 0$ and if $s + m = a_2 - b_1$. Our next step is to show that if an interval $[\underline{c},\, \overline{c}]$ has a density rate of $1/[(1-\beta)v]$ in G_i (where the density rates differ in neighborhoods immediately above $\overline{c}$ and immediately below $\underline{c}$), then G_i has no mass in the interval $(\,\overline{c} - (\overline{c}-\underline{c})\beta/(1-\beta),\ \overline{c}\,]$. This follows from observing that $u_{-i}(\overline{c}-\delta) > u_{-i}(x-\delta)$ for $x \in (\,\overline{c} - (\overline{c}-\underline{c})\beta/(1-\beta),\ \overline{c}\,]$ when $\beta > 1/2$. Specifically,

$$[G_i(\overline{c}) - G_i(x)]\,\beta v + [G_i(\overline{c}-2\delta) - G_i(x-2\delta)]\,(1-\beta)v - \overline{c} + x$$
$$\geq (\overline{c} - \underline{c})\left(\frac{\beta}{1-\beta}\right) - \overline{c} + x > 0$$

Thus, for an interval of length ℓ, at most $[(1-\beta)/\beta]\ell$ of the interval may have a density rate of $1/[(1-\beta)v]$; in which case, the bottom $[(2\beta-1)/\beta]\ell$ of the interval contains no mass. Hence, $m < (a_2 - b_1)[(1-\beta)/\beta]$, with a strict inequality since there is a space with no mass below a_2. Likewise, $s < (a_2 - b_1 - m)[(1-\beta)/\beta]$, and so for $\beta > 1/2$:

$$m + s < \left(\frac{1-\beta}{\beta}\right)\left(\frac{3\beta-1}{\beta}\right)(a_2 - b_1) < a_2 - b_1$$

Therefore, it must be that $r > 0$. So there exists at least one interval $[a_3,\, b_3] \subset [b_1{+}2\delta,\ a_2{+}2\delta]$ in G_i with a density rate of $1/[(1-\beta)v]$ (with different density rates immediately above b_3 and below a_3). Since $[a_2,\, b_2]$ and $[a_3,\, b_3]$ are within a 2δ interval (i.e. $b_3 - a_2 \leq 2\delta$), then the same argument we just used for $[a_1,\, b_1]$ and $[a_2,\, b_2]$ implies that there is an interval $[a_4,\, b_4] \subset [b_2+2\delta,\ a_3+2\delta]$ in G_i with a density rate of $1/[(1-\beta)v]$. In general, $[a_k,\, b_k] \subset [b_{k-2}+2\delta,\ a_{k-1}+2\delta]$ is in G_i and has a density rate of $1/[(1-\beta)v]$, where $k \in \{3, 4, \ldots\}$. The sequence of a_k is then unbounded ($a_{k+2} - a_k > 2\delta$), rising to bids that are strictly dominated. This contradicts our original supposition that the subset of G_i with a density rate of $1/[(1-\beta)v]$ in $[z,\ z+2\delta]$ is disconnected. □

We can now largely piece together what an equilibrium strategy must look like. Lemma 7 all but characterizes the nature of equilibria when $\beta > 1/2$. It states that if there is an interval of length δ or greater that is not part of a player's support, and it is immediately followed by an interval with a density rate of $1/[(1-\beta)v]$, then each player's distribution over all higher bids must be of a specific form. Namely, the remaining portion of each player's support is made up of connected intervals of length 2δ with a density rate of $1/[(1-\beta)v]$ over a lower portion and $1/(\beta v)$ over the remainder.

Lemma 7 *Let $\delta \in (0,\ (1-\beta)v\,)$ and $\beta > 1/2$. Let $\phi_1 \geq 0$ be the lowest bid such that $G_i(\phi_1 - \delta) = G_i(\phi_1)$ and $[\phi_1,\ \phi_1 + c]$ has a density rate of $1/[(1-\beta)v]$ in G_i for $c \in (0,\ \delta]$. If ϕ_1 exists, then in equilibrium, any portion of the support weakly greater than $\phi_1 + c - \delta$ in G_i and G_{-i} must have a distribution of the following form. For $j \in \{1, \ldots, k\}$, $k \in \mathbb{N}$, let ϕ_j and φ_j satisfy $\phi_{j+1} \geq \phi_j + \varphi_j + 3\delta$, $\varphi_{j+1} \in [\max\{\phi_j + 3\delta - \phi_{j+1},\ -\delta\},\ \delta]$, and $\varphi_1 \in [c - \delta,\ \delta]$. Then G_i has a density rate of $1/[(1-\beta)v]$ over $[\phi_j,\ \phi_j + \varphi_j + \delta]$ and $1/(\beta v)$ over $[\phi_j + \varphi_j + \delta,\ \phi_j + 2\delta]$; and G_{-i} has a density rate of $1/[(1-\beta)v]$ over $[\phi_j + \varphi_j,\ \phi_j + \delta]$ and $1/(\beta v)$ over $[\phi_j + \delta,\ \phi_j + \varphi_j + 2\delta]$. All other bids weakly greater than $\phi_1 + c - \delta$ are not in the support of G_i or G_{-i}.*

Proof Since $[\phi_1,\ \phi_1 + c]$ has a density rate of $1/[(1-\beta)v]$ in G_i, then by Corollary 1, $G_{-i}(\phi_1 - \delta) = G_{-i}(\phi_1 + c - \delta)$ and $[\phi_1 + \delta,\ \phi_1 + c + \delta]$ has a density rate of $1/(\beta v)$ in G_{-i}. Since $\phi_1 + \delta \in \text{supp}(G_{-i})$, then in equilibrium, $u_{-i}(\phi_1 + \delta,\ G_i) \geq u_{-i}(\phi_1,\ G_i)$. Using the density rates permitted by Lemma 3 and recalling that $G_i(\phi_1 - \delta) = G_i(\phi_1)$, we have:

$$\begin{aligned} u_{-i}(\phi_1 + \delta,\ G_i) - u_{-i}(\phi_1,\ G_i) &= [G_i(\phi_1 + 2\delta) - G_i(\phi_1 + \delta)]\,\beta v - \delta \\ &= \left[\frac{\tau}{\beta v} + \frac{\mu}{(1-\beta)v}\right]\beta v - \delta \geq 0 \end{aligned}$$

Here, τ and μ are the lengths of the support over $[\phi_1 + \delta,\ \phi_1 + 2\delta]$ in G_i that have density rates of $1/(\beta v)$ and $1/[(1-\beta)v]$, respectively. So $\tau + \mu \leq \delta$. Suppose $\mu = 0$. Then $\tau = \delta$ (or rather $[\phi_1 + \delta,\ \phi_1 + 2\delta]$ has a density rate of $1/(\beta v)$ in G_i), so Corollary 1 implies that $G_{-i}(\phi_1 + 2\delta) = G_{-i}(\phi_1 + 3\delta)$ and $[\phi_1,\ \phi_1 + \delta]$ has a density rate of $1/[(1-\beta)v]$ in G_{-i}. Since $\phi_1 + c$ and $\phi_1 + \delta \in \text{supp}(G_i)$, $u_i(\phi_1 + c,\ G_{-i}) = u_i(\phi_1 + \delta,\ G_{-i})$, so:

$$[G_{-i}(\phi_1 + 2\delta) - G_{-i}(\phi_1 + c + \delta)]\beta v + [G_{-i}(\phi_1) - G_{-i}(\phi_1 + c - \delta)](1-\beta)v = \delta - c$$

By Lemma 6, any mass in G_{-i} over $[\phi_1 + c + \delta,\ \phi_1 + 2\delta]$ must have a density rate of $1/(\beta v)$ (since any mass with a density rate of $1/[(1-\beta)v]$ would be disconnected from $[\phi_1,\ \phi_1 + \delta]$). The same is true for mass in G_i over $[\phi_1 + c - 2\delta,\ \phi_1 - \delta]$, so any mass in G_{-i} over $[\phi_1 + c - \delta,\ \phi_1]$ necessarily has a density rate of $1/[(1-\beta)v]$. Thus, for $q, r \in [0,\ \delta - c]$, we have:

$$\left[\frac{q}{\beta v}\right]\beta v + \left[\frac{r}{(1-\beta)v}\right](1-\beta)v = \delta - c$$

Lemma 6 is satisfied if G_{-i} has a density rate of $1/[(1-\beta)v]$ over $[\phi_1 - r,\ \phi_1 + \delta]$ and $1/(\beta v)$ over $[\phi_1 + \delta,\ \phi_1 + \delta + c + q]$. Substituting $q = \delta - c - r$, the second interval becomes $[\phi_1 + \delta,\ \phi_1 - r + 2\delta]$. Using Corollary 1, G_i has a density rate of $1/[(1-\beta)v]$ over $[\phi_1,\ \phi_1 - r + \delta]$ and $1/(\beta v)$ over $[\phi_1 - r + \delta,\ \phi_1 + 2\delta]$.

Now suppose instead that $\mu > 0$. Lemma 6 implies that the interval $[\phi_1,\ \phi_1 + \mu + \delta]$ has a density rate of $1/[(1-\beta)v]$ in G_i. Then, by Corollary 1, $[\phi_1 + \delta,\ \phi_1 + \mu + 2\delta]$ has a density rate of $1/(\beta v)$ in G_{-i}. Since $\phi_1 + \delta \in \text{supp}(G_{-i})$, $u_{-i}(\phi_1 + \delta,\ G_i) \geq u_{-i}(\phi_1 + \mu,\ G_i)$, or rather:

$$[G_i(\phi_1 + 2\delta) - G_i(\phi_1 + \delta + \mu)]\,\beta v \geq \delta - \mu$$

A density rate of $1/[(1-\beta)v]$ is not permitted in G_i over $[\phi_1 + \delta + \mu,\ \phi_1 + 2\delta]$, so the entire interval must instead have a density rate of $1/(\beta v)$. In which case, G_{-i} has a density rate of $1/[(1-\beta)v]$ over $[\phi_1 + \mu,\ \phi_1 + \delta]$. Combining the results for the cases where $\mu = 0$ and $\mu > 0$, G_i has a density rate of $1/[(1-\beta)v]$ over $[\phi_1,\ \phi_1+\varphi_1+\delta]$ and $1/(\beta v)$ over $[\phi_1+\varphi_1+\delta,\ \phi_1+2\delta]$; and G_{-i} has a density rate of $1/[(1-\beta)v]$ over $[\phi_1+\varphi_1,\ \phi_1+\delta]$ and $1/(\beta v)$ over $[\phi_1+\delta,\ \phi_1+\varphi_1+2\delta]$, where $\varphi_1 \in [c-\delta,\ \delta]$.

We next must show that this pattern holds for the remainder of the distribution. To do so, we first note that by Corollary 1, $G_i(\phi_1+2\delta) = G_i(\phi_1+\varphi_1+3\delta)$ and $G_{-i}(\phi_1+\varphi_1+2\delta) = G_{-i}(\phi_1+3\delta)$. Either $\phi_1+\varphi_1+3\delta-(\phi_1+2\delta) \geq \delta$ or $\phi_1+3\delta-(\phi_1+\varphi_1+2\delta) \geq \delta$. Let $\widehat{x}$ be the next element in either support. That is, $\widehat{x} = \min\{x \in \text{supp}(G_i)\bigcup\text{supp}(G_{-i}) \mid x \geq \min\{\phi_1+3\delta,\ \phi_1+\varphi_1+3\delta\}\}$. If $\widehat{x}$ does not exist, then we are done. Otherwise, $[\widehat{x},\ \widehat{x}+e]$ has a density rate of $1/[(1-\beta)v]$ in either G_i or G_{-i} for $e \in (0,\ \delta]$. If $\widehat{x} \geq \max\{\phi_1+3\delta,\ \phi_1+\varphi_1+3\delta\}$, then the conditions of the lemma are again satisfied, so the pattern continues (this includes the case where $\varphi = 0$). For the case where $\min\{\phi_1+3\delta,\ \phi_1+\varphi_1+3\delta\} \leq \widehat{x} < \max\{\phi_1+3\delta,\ \phi_1+\varphi_1+3\delta\}$, suppose without loss of generality that $\varphi_1 > 0$. Then $\widehat{x} \in \text{supp}(G_{-i})$. Since $u_{-i}(\widehat{x},\ G_i) = u_{-i}(\phi_1+\varphi_1+2\delta,\ G_i)$, then:

$$(\delta-\varphi_1)\left(\frac{1-\beta}{\beta}\right) + [G_i(\widehat{x}+\delta)-G_i(\widehat{x}-\delta)]\beta v \ = \ \widehat{x}-(\phi_1+\varphi_1+2\delta) \ \geq \ \delta-\varphi_1$$

Since $\beta > 1/2$, the equation can only be satisfied if $G_i(\widehat{x}+\delta)-G_i(\widehat{x}-\delta) > 0$. This mass must be in $[\phi_1+\varphi_1+3\delta,\ \widehat{x}+\delta]$ with a density rate of $1/[(1-\beta)v]$, and so the conditions of the lemma are satisfied yet again.[21] The pattern thus continues as long as there is any remaining mass to place in G_i or G_{-i}. □

Lemma 8 *Let* $\delta \in (0,\ (1-\beta)v\,)$ *and* $\beta > 1/2$. *If* $\max\{G_i(2\delta),\ G_{-i}(2\delta)\} < 1$ *in equilibrium, then* ϕ_1 *exists. Moreover, any mass in either player's distribution below* $\min\{\phi_1,\ \phi_1+\varphi_1\}$ *is limited to a mass point at zero, a lower interval at a density rate of* $1/[(1-\beta)v]$, *and an upper interval at a density rate of* $1/\beta v$. *The mass points and lower intervals may be distributed according to one of three forms, with the upper intervals following Corollary* 1*:*

A. $\alpha_i(0) > 0$ *and* $\alpha_{-i}(0) > 0$. *Lower intervals begin at zero and have length* $\psi_k \in [0,\ \delta)$ *for* $k \in \{i,\ -i\}$. *This is also the only possible equilibrium form when* ϕ_1 *does not exist.*
B. $\alpha_i(0) > 0$ *and* $\alpha_{-i}(0) \geq 0$. *Player i's lower interval begins at zero and has length* $\psi_i \in [0,\ 2\delta]$. *Player* $-i$ *has no lower interval.*
C. $\alpha_i(0) > 0$ *and* $\alpha_{-i}(0) = 0$. *Player i's lower interval begins at zero and has length* $\psi_i \in [\delta,\ 2\delta)$. *Player* $-i$*'s lower interval begins at* $\psi_i-\delta$ *and has length* $\psi_{-i} \in (0,\ 2\delta-\psi_i)$.

Proof The existence of ϕ_1 when $\max\{G_i(2\delta),\ G_{-i}(2\delta)\} < 1$ is trivial if $\alpha_i(0) = \alpha_{-i}(0) = 0$. It is likewise trivial if $G_i(\delta) = G_i(0)$ and $G_{-i}(\delta) = G_{-i}(0)$. We must therefore consider the cases where at least one player has a mass point at zero and at least one player has strictly positive mass in $(0,\ \delta]$. By Lemma 3 and Corollary 1, any mass in $(0,\ \delta]$ must have a density rate of $1/[(1-\beta)v]$. For $k \in \{i,\ -i\}$, let $[\mu_k,\ M_k]$ denote the lowest interval in G_k with a density rate of $1/[(1-\beta)v]$. Without loss of generality with respect to players, there are four cases.

Case 1: $\alpha_i(0) > 0$, $\alpha_{-i}(0) > 0$, $\mu_i < \delta$, $\mu_{-i} < \delta$. Since $\alpha_i(0),\ \alpha_{-i}(0) > 0$, then $M_i,\ M_{-i} < \delta$. Otherwise, elements of the support in $[\mu_k,\ \delta]$ would have a strictly lower expected payoff than elements in $(\delta,\ M_k]$. If $\mu_i > 0$, then to maintain payoff equivalence with zero, the other player's support must have mass over $[\delta,\ \mu_i+\delta]$ which can be tied. But by Lemma 6 and Corollary 1, neither player has mass in $[\delta,\ \mu_k+\delta]$. So $\mu_i = \mu_{-i} = 0$. They also imply that player i has no mass in $[M_i+\delta,\ M_{-i}+2\delta]$ and player $-i$ has no mass in $[M_{-i}+\delta,\ M_i+2\delta]$. The length of at

[21] Even if $\varphi_1 = \delta$, it is always the case that $\widehat{x}-(\phi_1+\varphi_1+2\delta) > 0$. This follows because, as we saw in the proof of Lemma 6, when $\beta > 1/2$, an interval with a density rate of $1/[(1-\beta)v]$ cannot immediately follow an interval with a density rate of $1/(\beta v)$.

least one of these intervals is weakly greater than δ. So if $G_i(\delta + M_{-i}) < 1$ or $G_{-i}(\delta + M_i) < 1$, then ϕ_1 exists, and any mass below $\min\{\phi_1,\ \phi_1 + \varphi_1\}$ is distributed according to the first form in Lemma 8.

Case 2: $\alpha_i(0) > 0$, $\alpha_{-i}(0) \geq 0$, $\mu_i < \delta$, $\mu_{-i} \geq \delta$. With $\mu_{-i} \geq \delta$, there are two subcases: $[\mu_{-i},\ M_{-i}] \subseteq [\delta,\ \mu_i + \delta]$ and $\mu_{-i} \geq M_i + \delta$. (Corollary 1 prohibits anything else.) Suppose first that $[\mu_{-i},\ M_{-i}] \subseteq [\delta,\ \mu_i + \delta]$. Therefore, $\mu_i > 0$. For player $-i$, $\mu_{-i} > \delta$ is strictly dominated by $\mu_{-i} = \delta$ since $G_i(0) = G_i(\mu_{-i} - \delta)$ and $G_i(2\delta) = G_i(\mu_{-i} + \delta)$ (see Lemma 6 and Corollary 1). In equilibrium, $u_i(0,\ G_{-i}) = u_i(\mu_i,\ G_{-i})$, so:

$$[G_{-i}(\mu_i + \delta) - G_{-i}(\delta)]\,\beta v = \mu_i \quad \Rightarrow \quad \left(\frac{M_{-i} - \mu_{-i}}{(1-\beta)v}\right)\beta v = \mu_i$$

Since $\beta > 1/2$, the last equality implies that $M_{-i} - \mu_{-i} < \mu_i$. However, with $\mu_{-i} = \delta$ and $M_{-i} < \mu_i + \delta$, player i could profitably deviate by shifting μ_i down to $M_{-i} - \delta$. So this is not an equilibrium. For the second subcase, suppose now that $\mu_{-i} \geq M_i + \delta$. By Lemma 6 and Corollary 1, $G_i(M_i + 2\delta) = G_i(M_i)$, so ϕ_1 exists if $G_i(M_i) < 1$. Also, $\mu_{-i} \geq M_i + \delta$ means that $G_{-i}(\delta) = G_{-i}(\mu_i + \delta)$, so we can only have $u_i(0,\ G_{-i}) = u_i(\mu_i,\ G_{-i})$ if $\mu_i = 0$. If $\alpha_{-i}(0) > 0$, then $M_i < \delta$; otherwise, Corollary 1 sets the upper bound of M_i at 2δ. Thus any mass below $\min\{\phi_1,\ \phi_1 + \varphi_1\}$ is distributed according to the second form in Lemma 8. Although it is possible for $M_i + \delta > 2\delta$ when $\alpha_{-i}(0) = 0$, we still have $G_{-i}(M_i + \delta) < 1$. A contradiction arises if $G_{-i}(M_i + \delta) = 1$, since $G_i(M_i)$ would also equal one by Corollary 1. But both cannot equal one since G_i has more mass:

$$\frac{M_i - \mu_i}{(1-\beta)v} + \alpha_i(0) > \frac{M_i - \mu_i}{\beta v}$$

Case 3: $\alpha_i(0) > 0$, $\alpha_{-i}(0) = 0$, $\mu_i < \delta$, $\mu_{-i} < \delta$. By Lemma 6 and Corollary 1, player i has no mass in $[M_{-i} + \delta,\ M_i + 2\delta]$, and player $-i$ has no mass in $[M_i + \delta,\ M_{-i} + 2\delta]$. At least one of these intervals has a length weakly greater than δ, so if $G_i(M_{-i} + \delta) < 1$ or $G_{-i}(M_i + \delta) < 1$, then ϕ_1 exists. Any mass below $\min\{\phi_1,\ \phi_1 + \varphi_1\}$ must be distributed according to the third form in Lemma 8. This is seen by demonstrating that $M_{-i} < \delta$, $M_i \geq \delta$, $\mu_i = 0$, $\mu_{-i} = M_i - \delta$, and $M_i \geq \delta$. Since $\alpha_i(0) > 0$ and $\mu_{-i} < \delta$, then $M_{-i} < \delta$. The result that $\mu_i = 0$ follows because $M_{-i} < \delta$ and $G_{-i}(\delta) = G_{-i}(\mu_i + \delta)$ (see Lemma 6 and Corollary 1). Without additional mass to tie, $\mu_i > 0$ is strictly dominated. Similarly, $\mu_{-i} = \max\{0,\ M_i - \delta\}$ since $G_i(M_i) = G_i(\mu_{-i} + \delta)$. We can pin down μ_{-i} further. Since $\alpha_{-i}(0) = 0$, unless $M_i \geq \delta$, player i would have a strictly lower payoff over $[\mu_{-i} + \delta,\ M_{-i} + \delta]$ than over $[0,\ M_i]$. Thus $\mu_{-i} = M_i - \delta$ and $M_i \geq \delta$. It remains to show that $G_{-i}(M_i + \delta) = 1$ is not an equilibrium. If indeed $G_{-i}(M_i + \delta) = 1$, then the total mass in G_i and G_{-i} is described as follows:

$$G_i: \qquad 1 = \alpha_i(0) + \frac{M_i}{(1-\beta)v} + \frac{M_{-i} - \mu_{-i}}{\beta v} \tag{19}$$

$$G_{-i}: \qquad 1 = \frac{M_{-i} - \mu_{-i}}{(1-\beta)v} + \frac{M_i}{\beta v} \tag{20}$$

Since $\delta,\ \mu_{-i} \in \text{supp}(G_{-i})$, and since $\mu_{-i} = M_i - \delta$, equilibrium requires that $u_{-i}(\overline{\delta},\ G_i) = u_{-i}(M_i - \delta,\ G_i)$. This can be written as:

$$\alpha_i(0)(1-\beta)v + (M_{-i} - \mu_{-i}) = 2\delta - M_i \tag{21}$$

Rearranging Equation 20 and combining Equations 19 and 21, we obtain the following:

$$M_i = \beta v \left[1 - \frac{M_{-i} - \mu_{-i}}{(1-\beta)v}\right] \quad \text{where} \quad M_{-i} - \mu_{-i} = \frac{[2\delta - (1-\beta)v]\beta}{2\beta - 1}$$

Using these expressions, $M_i > \delta$ is equivalent to $(1-\beta)\beta v > \delta$, which in turn implies that $M_{-i} - \mu_{-i} > (1-\beta)\beta v$. So Equation 21 can only be satisfied if $\alpha_i(0) < 0$, which obviously cannot hold in equilibrium.

Case 4: $\alpha_i(0) > 0$, $\alpha_{-i}(0) > 0$, $\mu_i \geq \delta$, $\mu_{-i} < \delta$. Since $\alpha_i(0) > 0$ and $\mu_{-i} < \delta$, then $M_{-i} < \delta$. In equilibrium, $u_i(0,\ G_{-i}) = u_i(\mu_{-i} + \delta,\ G_{-i})$, which can be rewritten as:

$$[G_{-i}(\mu_{-i} + 2\delta) - G_{-i}(\mu_{-i} + \delta)]\,\beta v = \mu_{-i} + \delta \tag{22}$$

With $\mu_i \geq \delta$ and $M_{-i} < \delta$, then by Lemma 6 and Corollary 1, player $-i$ can only have mass in $[\mu_{-i} + \delta,\ \mu_{-i} + 2\delta]$ if $[\mu_i,\ M_i] \subseteq [\delta,\ \mu_{-i} + \delta]$. Then Equation 22 becomes:

$$\left(\frac{M_i - \mu_i}{\beta v}\right)\beta v = \mu_{-i} + \delta \quad \Rightarrow \quad M_i - \mu_i = \mu_{-i} + \delta$$

However, given the bounds of $[\mu_i,\ M_i] \subseteq [\delta,\ \mu_{-i} + \delta]$, this is a contradiction.

When ϕ_1 does not exist: We will show that Parts B and C of Lemma 8 cannot hold if ϕ_1 does not exist. Beginning with Part B, in order for player i's total mass of $\alpha_i(0) + [\psi_i/(1-\beta)v]$ and player $-i$'s total mass of $\alpha_{-i}(0) + [\psi_i/(\beta v)]$ to each equal one, we must have $\alpha_{-i}(0) > 0$. With this positive mass point, preventing a jump in player i's expected payoff near δ requires that $\psi_i < 0$. The two mass constraints and player $-i$'s indifference condition between 0 and δ imply that $\alpha_i(0) = \delta/[(1-\beta)v]$; $\alpha_{-i}(0) = [\delta - (1-2\beta)v]/\beta v$ and $\psi_i = (1-\beta)v - \delta$. However, so long as $\delta < (1-\beta)v$, player i could profitably deviate with a bid of δ. For Part C, the two mass constraints and player $-i$'s indifference condition between $\psi_i - \delta$ and δ imply that $\alpha_i(0) = 2[\delta - (1-\beta)\beta v]/[(1-\beta)^2 v]$; $\psi_i = \delta + [(3\beta - 1)((1-\beta)\beta v - \delta)/(1-\beta)(2\beta - 1)]$; and $\psi_{-i} = [2\delta - (1-\beta)v]\beta/(2\beta - 1)$. Note, however, that the conditions in Part C for $\alpha_i(0) > 0$ and $\psi_i \in [\delta,\, 2\delta)$ cannot be jointly satisfied. (The closest case is $\delta = (1-\beta)\beta v$ so that $\alpha_i(0) = 0$ and $\psi_i = \psi_{-i} = \delta$, but then ϕ_1 would exist.) $\square$

Lemma 9 *Let $\delta \in (0,\ (1-\beta)v)$ and $\beta > 1/2$. If $\max\{G_i(2\delta),\ G_{-i}(2\delta)\} < 1$, the equilibrium must be symmetric. In particular, $\alpha_i(0) = \alpha_{-i}(0)$, $\psi_i = \psi_{-i}$, and $\varphi_j = 0$ for $j \in \{1, \ldots, k\}$.*

Proof From Lemmata 7 and 8, mass in G_i and G_{-i} have the following forms:

$$1 = \alpha_i(0) + \frac{\psi_i}{(1-\beta)v} + \frac{\psi_{-i}}{\beta v} + \frac{\delta + \varphi_1}{(1-\beta)v} + \frac{\delta - \varphi_1}{\beta v} + \cdots + \frac{\delta + \varphi_k}{(1-\beta)v} + \frac{\delta - \varphi_k}{\beta v}$$

$$1 = \alpha_{-i}(0) + \frac{\psi_{-i}}{(1-\beta)v} + \frac{\psi_i}{\beta v} + \frac{\delta - \varphi_1}{(1-\beta)v} + \frac{\delta + \varphi_1}{\beta v} + \cdots + \frac{\delta - \varphi_k}{(1-\beta)v} + \frac{\delta + \varphi_k}{\beta v}$$

Combining these two equations yields:

$$\alpha_{-i}(0) - \alpha_i(0) = \left[\frac{4\beta - 2}{(1-\beta)\beta v}\right]\Big(\varphi_1 + \varphi_2 + \cdots + \varphi_k\Big) - \left[\frac{2\beta - 1}{(1-\beta)\beta v}\right](\psi_{-i} - \psi_i) \tag{23}$$

For $j \in \{1, \ldots, k-1\}$ for $k \geq 2$, $u_i(\phi_{j+1}) = u_i(\phi_j)$ and $u_{-i}(\phi_{j+1} + \varphi_{j+1}) = u_{-i}(\phi_j + \varphi_j)$ respectively imply the following:

$$\phi_{j+1} - \phi_j = \frac{\delta}{(1-\beta)\beta} + \left(\frac{1-\beta}{\beta}\right)\varphi_j - \left(\frac{\beta}{1-\beta}\right)\varphi_{j+1}$$

$$\phi_{j+1} - \phi_j = \frac{\delta}{(1-\beta)\beta} + \left(\frac{2\beta - 1}{\beta}\right)\varphi_j - \left(\frac{2\beta - 1}{1-\beta}\right)\varphi_{j+1}$$

We then obtain the following from the above two equations:

$$\varphi_{j+1} = \left[\frac{2 - 3\beta}{3\beta - 1}\right]\left(\frac{1-\beta}{\beta}\right)\varphi_j \tag{24}$$

Since Equation 24 takes the form $\varphi_{j+1} = H\varphi_j$, we can write all ϕ_j in terms of ϕ_1. Of particular note:

$$(\varphi_1 + \varphi_2 + \cdots + \varphi_k) \;=\; \varphi_1\left[1 + H + H^2 + \cdots + H^{k-1}\right] \;=\; \varphi_1\left[\frac{1-H^k}{1-H}\right]$$

So Equation 23 becomes:

$$[\alpha_{-i}(0) - \alpha_i(0)]\,(1-\beta)v = \left[\frac{1-H^k}{1-H}\right]\left[\frac{4\beta{-}2}{\beta}\right]\varphi_1 - \left[\frac{2\beta{-}1}{\beta}\right](\psi_{-i} - \psi_i) \tag{25}$$

If $\alpha_i(0) = \alpha_{-i}(0)$ and $\psi_i = \psi_{-i}$, then Equation 25 simplifies to:

$$0 = \left[\frac{1-H^k}{1-H}\right]\left[\frac{4\beta-2}{(1{-}\beta)\beta v}\right]\varphi_1$$

This can only be satisfied by $\varphi_1 = 0$ since $1 = H^k$ and $4\beta = 2$ both require $\beta = 1/2$. So by Equation 24, $\varphi_j = 0$ for $j \in \{1, \ldots, k\}$.

Now suppose that either $\alpha_i(0) \neq \alpha_{-i}(0)$ or $\psi_i \neq \psi_{-i}$. Based on Lemma 8, if there is any mass below ϕ_1, then at least one player has a strictly positive mass point at zero. Without loss of generality, assume this is player i. Then $u_i(0,\ G_{-i}) = u_i(\phi_1,\ G_i)$. For player $-i$, we have $u_{-i}(0,\ G_{-i}) \leq u_{-i}(\phi_1{+}\varphi_1,\ G_i)$, which holds with strict equality in equilibrium whenever $\alpha_{-i}(0) > 0$. These imply the following:

$$\begin{aligned}
\phi_1 \;&=\; \alpha_{-i}(0)(1-\beta)v + \psi_{-i} + \left(\frac{\psi_i}{\beta}\right) + (\delta - \varphi_1)\left(\frac{\beta}{1-\beta}\right)\\
\phi_1 \;&\leq\; \alpha_i(0)(1-\beta)v + \psi_i + \left(\frac{\psi_{-i}}{\beta}\right) + (\delta + \varphi_1)\left(\frac{\beta}{1-\beta}\right) - \varphi_1
\end{aligned}$$

Combined, we have:

$$[\alpha_{-i}(0) - \alpha_i(0)]\,(1-\beta)v \;\leq\; (\psi_{-i} - \psi_i)\left(\frac{1-\beta}{\beta}\right) + \left(\frac{3\beta-1}{1-\beta}\right)\varphi_1 \tag{26}$$

Together, Equations 25 and 26 imply the following, which holds with strict equality whenever $\alpha_{-i}(0) > 0$:

$$(\psi_{-i} - \psi_i) \geq (3\beta - 1)\left[(1 - H^k) - \left(\frac{1}{1-\beta}\right)\right]\varphi_1 \tag{27}$$

Based on the three forms of mass below ϕ_1 in Lemma 8, and with $\alpha_i(0) > 0$ and $\alpha_{-i}(0) \geq 0$, we have $\psi_i > 0$ and $\psi_{-i} \geq 0$. So $u_{-i}(\overline{\delta},\ G_i) = u_{-i}(\phi_1 + \varphi_1,\ G_i)$ and $u_i(\overline{\delta},\ G_{-i}) \leq u_i(\phi_1,\ G_{-i})$; the latter holds with strict equality if $\psi_{-i} > 0$. These can be rewritten as follows:

$$\begin{aligned}
\phi_1 \;&=\; \psi_i + \psi_{-i}\left(\frac{1-\beta}{\beta}\right) + \delta\left(\frac{1}{1-\beta}\right) + \varphi_1\left(\frac{2\beta-1}{1-\beta}\right)\\
\phi_1 \;&\leq\; \psi_{-i} + \psi_i\left(\frac{1-\beta}{\beta}\right) + \delta\left(\frac{1}{1-\beta}\right) - \varphi_1\left(\frac{\beta}{1-\beta}\right)
\end{aligned}$$

Combining them yields:

$$(\psi_{-i} - \psi_i) \geq \left(\frac{3\beta-1}{2\beta-1}\right)\left(\frac{\beta}{1-\beta}\right)\varphi_1 \tag{28}$$

We will show that Equations 27 and 28 can only jointly hold if $\psi_{-i} = \psi_i$ and $\varphi_1 = 0$. Suppose instead that $\psi_{-i} \neq \psi_i$, and without loss of generality, suppose that $\psi_{-i} < \psi_i$. With $\beta \in (1/2,\ 1)$, the coefficient on φ_1 in Equation 28 is strictly positive. So since the left-hand side of Equation 28 is negative, it must be that $\varphi_1 < 0$. Next note that the coefficient on φ_1 in Equation 27 is strictly negative for $\beta \in (1/2,\ 1)$. Showing that this coefficient is negative is equivalent to showing that $H^k(1-\beta) + \beta > 0$, which holds for $\beta \in (1/2,\ 1)$ since the minimal value of H^k over this range is

$4\sqrt{3}-7 \approx -0.0718$ (the minimum is obtained at $\beta = (3+\sqrt{3})/6 \approx 0.7887$ and $k = 1$). With a left-hand side that is negative, and a coefficient on φ_1 that is also negative, Equation 28 can only hold if $\varphi_1 > 0$. Hence, we have a contradiction. Since the coefficients on φ_1 in Equations 27 and 28 are nonzero for $\beta \in (1/2,\ 1)$, these equations can only hold simultaneously if $\psi_{-i} = \psi_i$ and $\varphi_1 = 0$. If this is the case, then by Equation 24, $\varphi_j = 0$ for $j \in \{1, \dots, k\}$, and by Equation 25, $\alpha_i(0) = \alpha_{-i}(0)$. The equilibrium must therefore be symmetric. □

D Proofs Specific to $\beta = 1/2$

Lemma 10 *Let $\beta = 1/2$ and $\delta \in (0,\ v/4]$. If G_i and G_{-i} are equilibrium strategies for players i and $-i$, then $u_i(a,\ G_{-i}) = u_i(a-2\delta,\ G_{-i})$ for all $a \in \mathrm{supp}(G_{-i})$ such that $a \geq 2\delta$. Equivalently, $[G_{-i}(a+\delta) - G_{-i}(\max\{a-3\delta,\ 0\})] = 4\delta/v$.*

Proof We begin by showing that for any $\kappa > 0$, $[G_{-i}(\kappa+4\delta) - G_{-i}(\kappa)] \leq 4\delta/v$. This is done by construction. Suppose that any mass in G_{-i} over $[\kappa,\ \kappa+4\delta]$ has the maximal density rate of $2/v$ (any higher density rate would violate Equation 14). This is possible if $G_{-i}(\kappa+4\delta) = G_{-i}(\kappa+6\delta)$ and $G_{-i}(\max\{\kappa-2\delta,\ 0\}) = G_{-i}(\kappa)$ so that any mass in G_i over $[\kappa-\delta,\ \kappa+\delta]$ and $[\kappa+3\delta,\ \kappa+5\delta]$ is entirely balanced by the mass in G_{-i} over $[\kappa,\ \kappa+4\delta]$. To allow for the largest amount of $[\kappa,\ \kappa+4\delta]$ to be covered at the density rate of $2/v$, we further suppose that $G_i(\kappa+\delta) = G_i(\kappa+3\delta)$, since any mass in G_i over $[\kappa+\delta,\ \kappa+3\delta]$ would necessitate a lower (perhaps zero) density rate in G_{-i} over some portion of $[\kappa,\ \kappa+4\delta]$. In equilibrium, it must be that $\kappa+\delta$ and $\kappa+3\delta \in \mathrm{supp}(G_i)$. Otherwise, if $e-d > 2\delta$ where $d = \max\{\mathrm{supp}(G_i) \cap [\kappa-\delta,\ \kappa+\delta]\}$ and $e = \min\{\mathrm{supp}(G_i) \cap [\kappa+3\delta,\ \kappa+5\delta]\}$, then we would have $d+\delta$ and $e-\delta \in \mathrm{supp}(G_{-i})$, but $u_{-i}(d+\delta,\ G_i) > u_{-i}(e-\delta,\ G_i)$ (that is, the winning and tying probability would remain the same for bids of $d+\delta$ and $e-\delta$, but the cost of effort would differ). From Equation 14, $u_i(\kappa+\delta,\ G_{-i}) = u_i(\kappa+3\delta,\ G_{-i})$ implies that $[G_{-i}(\kappa+4\delta) - G_{-i}(\kappa)] = 4\delta/v$, which is the desired upper bound. With this property in hand, the main result follows quickly. Since G_i and G_{-i} are equilibrium strategies, and since $a \in \mathrm{supp}(G_{-i})$, it cannot be the case that $u_i(a,\ G_{-i}) < u_i(a-2\delta,\ G_{-i})$. For the purpose of contradiction, suppose that $u_i(a,\ G_{-i}) > u_i(a-2\delta,\ G_{-i})$ for some a. Since $u_i(a,\ G_{-i}) = [G_{-i}(a+\delta) + G_{-i}(a-\delta)](v/2) - a$ and $u_i(a-2\delta,\ G_{-i}) = [G_{-i}(a) + G_{-i}(a-3\delta)](v/2) - (a-2\delta)$, then $u_i(a,\ G_{-i}) > u_i(a-2\delta,\ G_{-i})$ implies that $[G_{-i}(a+\delta) - G_{-i}(a-3\delta)] > 4\delta/v$. This, however, is a contradiction, and so $u_i(a,\ G_{-i}) = u_i(a-2\delta,\ G_{-i})$. □

Lemma 11 *Let $\beta = 1/2$ and $\delta \in (0,\ v/4]$. Property $\mathscr{P}$ must hold in any equilibrium.*

Proof For added clarity, we will refer to G_i and G_{-i} and G_w and G_y. We will also denote $\overline{w}_1 = \max\{x \in \mathrm{supp}(G_w)\}$ and $\overline{y}_1 = \max\{x \in \mathrm{supp}(G_y)\}$. Without loss of generality, assume that $\overline{w}_1 \geq \overline{y}_1$; so $\overline{y}_1 \in [\overline{w}_1-\delta,\ \overline{w}_1]$. Our first step is to show that $G_w(\overline{w}_1-3\delta) = G_w(\overline{w}_1-4\delta)$ and that $G_y(\overline{w}_1-3\delta) = G_y(\overline{w}_1-4\delta)$. By Lemma 10, since $\overline{w}_1 \in \mathrm{supp}(G_w)$ and $G_y(\overline{w}_1+\delta) = G_y(\overline{y}_1)$, then:

$$[G_y(\overline{y}_1) - G_y(\overline{w}_1-3\delta)] = 4\delta/v$$

Also, since $\overline{y}_1-\delta \in \mathrm{supp}(G_w)$, Lemma 10 implies that:

$$[G_y(\overline{y}_1) - G_y(\overline{y}_1-4\delta)] = 4\delta/v$$

Hence, $G_y(\overline{w}_1-3\delta) = G_y(\overline{y}_1-4\delta)$, and by a similar argument, $G_w(\overline{y}_1-3\delta) = G_w(\overline{w}_1-4\delta)$. If $\overline{w}_1 = \overline{y}_1$, then we are done. Otherwise, for $\overline{w}_1 > \overline{y}_1$, we must still show that $G_w(\overline{w}_1-3\delta) = G_w(\overline{y}_1-3\delta)$. Suppose instead that $G_w(\overline{w}_1-3\delta) > G_w(\overline{y}_1-3\delta)$. Denote $\underline{w} = \min\{x \in \mathrm{supp}(G_w) \mid x \geq \overline{y}_1-3\delta\}$.

Since $\overline{w}_1 - \delta \in \text{supp}(G_y)$, then by Lemma 10, $u_y(\overline{w}_1 - 3\delta,\ G_w) = u_y(\overline{w}_1 - \delta,\ G_w)$. Hence, in equilibrium, $u_y(\overline{w}_1 - 3\delta) \geq u_w(\underline{w})$, or rather:

$$[G_w(\overline{w}_1 - 2\delta) - G_w(\underline{w} + \delta)]v/2 \ + \ [G_w(\overline{w}_1 - 4\delta) - G_w(\underline{w} - \delta)]v/2 \ \geq \ \overline{w}_1 - 3\delta - \underline{w}$$

Since G_w has at least some mass immediately below $\overline{w}_1$ in $[\underline{w} + 3\delta,\ \overline{w}_1]$, any mass 2δ below that in G_w necessarily has a density rate less than $2/v$ (that is, $[G_w(\overline{w}_1 - 2\delta) - G_w(\underline{w} + \delta)]v/2 \ < \ \overline{w}_1 - 3\delta - \underline{w}$; otherwise Equation 14 is not satisfied for all mass in G_y over $[\underline{w} + 2\delta,\ \overline{w}_1 - \delta]$). Hence, it must be that $G_w(\overline{w}_1 - 4\delta) - G_w(\underline{w} - \delta) > 0$. For any $\ell \in \text{supp}(G_w) \cap [\underline{w} - \delta,\ \overline{w}_1 - 4\delta]$, we have $\ell - \delta \in \text{supp}(G_y)$, and so $u_y(\ell - 3\delta,\ G_y) = u_y(\ell - \delta,\ G_y)$ (see Lemmata 2 and 10). Moreover, to satisfy Equation 14 for all $\ell - \delta$, the density rates in G_w over $[\underline{w} - \delta,\ \overline{w}_1 - 4\delta]$ and those 2δ below it must sum to $2/v$. At least a portion of the mass in G_w over $[\underline{w} - 3\delta,\ \overline{w}_1 - 6\delta]$ must therefore have a density rate less than $2/v$. However, for $u_y(\ell - 3\delta,\ G_y) = u_y(\ell - \delta,\ G_y)$ to hold, G_w must either have a density rate of $2/v$ over the entirety of $[\underline{w} - 3\delta,\ \overline{w}_1 - 6\delta]$ or there must be a positive density rate over the $\ell - 4\delta$ in G_w. That is, $G_w(\overline{w}_1 - 8\delta) - G_w(\underline{w} - 4\delta) > 0$. Since the density rate over all the $\ell - \delta$ in G_y is $2/v$, the density rate over the $\ell - 3\delta$ in G_y is 0, and so $\ell - 5\delta \in \text{supp}(G_y)$. The argument then repeats. Ultimately, however, there is a contradiction: at some point the bottom of the distribution is reached, so payoffs can no longer be sustained by additional mass δ below. We therefore have the desired result that $G_w(\overline{w}_1 - 3\delta) = G_w(\overline{w}_1 - 4\delta)$ and $G_y(\overline{w}_1 - 3\delta) = G_y(\overline{w}_1 - 4\delta)$.

Since G_w and G_y each have a gap of at least δ, then this argument also applies to mass below these gaps. Let $\overline{w}_2 = \max\{x \in \text{supp}(G_w) \mid x \leq \overline{w}_1 - 4\delta\}$, $\overline{y}_2 = \max\{x \in \text{supp}(G_y) \mid x \leq \overline{y}_1 - 4\delta\}$, and $m_2 = \max\{\overline{w}_2,\ \overline{y}_2\}$. Using the same argument, $G_w(m_2 - 3\delta) = G_w(m_2 - 4\delta)$ and $G_y(m_2 - 3\delta) = G_y(m_2 - 4\delta)$. Or more generally, for $\overline{w}_z = \max\{x \in \text{supp}(G_w) \mid x \leq \overline{w}_{z-1} - 4\delta\}$, $\overline{y}_z = \max\{x \in \text{supp}(G_y) \mid x \leq \overline{y}_{z-1} - 4\delta\}$, and $m_z = \max\{\overline{w}_z,\ \overline{y}_z\}$, where $z \in \{2, 3, \ldots\}$, then $G_w(m_z - 3\delta) = G_w(m_z - 4\delta)$ and $G_y(m_z - 3\delta) = G_y(m_z - 4\delta)$. Moreover,

$$[G_w(m_z) - G_w(m_z - 3\delta)] = [G_y(m_z) - G_y(m_z - 3\delta)] = 4\delta/v$$

To satisfy the constraint that all mass must sum to one, G_w and G_y each have $p \equiv \lfloor v/4\delta \rfloor$ such intervals. That is, p intervals of length 4δ, each having a total mass of $4\delta/v$, none of which is in the bottom δ (the remaining $1 - [4\delta p/v]$ is then at the bottom of the distribution; more on this later). Placing a mass of $4\delta/v$ within 3δ, with no mass δ above or below, requires that G_w and G_y each have $2\delta/v$ over $[m_z - 2\delta,\ m_z - \delta]$ and $2\delta/v$ over $[m_z - 3\delta,\ m_z - 2\delta] \bigcup [m_z - \delta,\ m_z]$ for $z \in \{1, \ldots, p\}$. Mass over $[m_z - 2\delta,\ m_z - \delta]$ must be at a density rate of $2/v$, while the density rates at x and $x - 2\delta$ for $x \in [m_z - \delta,\ m_z]$ must sum to $2/v$ (see Lemmata 2 and Equation 14).

We can also state how successive length 4δ intervals fit together. Since $[m_z - 2\delta,\ m_z - \delta]$ is in the support of G_w and G_y, then by Lemma 10, players are indifferent between any bid in $[m_z - 2\delta,\ m_z - \delta]$ and any bid in $[m_z - 3\delta,\ m_z - 4\delta]$. In particular, $u_y(m_z - 3\delta,\ G_w) = u_y(m_z - 4\delta,\ G_w)$ and $u_w(m_z - 3\delta,\ G_y) = u_w(m_z - 4\delta,\ G_y)$ respectively imply:

$$\begin{aligned} [G_w(m_z - 2\delta) - G_w(m_z - 5\delta)] &= 2\delta/v \\ [G_y(m_z - 2\delta) - G_y(m_z - 5\delta)] &= 2\delta/v \end{aligned} \tag{29}$$

So G_w and G_y each have $2\delta/v$ over $[m_z - 5\delta,\ m_z - 3\delta] \bigcup [m_z - 5\delta,\ m_z - 3\delta]$, and the density rates at $x \in [m_z - 3\delta,\ m_z - 5\delta]$ and $x - 2\delta$ must sum to $2/v$ to support the expected payoffs in $[m_z - 3\delta,\ m_z - 4\delta]$. Consequently, players are also indifferent between bids in $[m_z - 3\delta,\ m_z - 4\delta]$ and bids in $[m_z - 4\delta,\ m_z - 5\delta]$ (if $x \in [m_z - 3\delta,\ m_z - 5\delta]$ is in the player's support, then the indifference comes from Lemma 10; if not, then the indifference comes from $x-2\delta$ being in the player's support). We therefore have $u_y(m_z-3\delta,\ G_w)=u_y(m_z-5\delta,\ G_w)$ and $u_w(m_z-3\delta,\ G_y)=u_w(m_z-5\delta,\ G_y)$, and so $[G_w(m_z-2\delta)-G_w(m_z-6\delta)]=4\delta/v$ and $[G_y(m_z-2\delta)-G_y(m_z-6\delta)]=4\delta/v$. Combined with Equation 29, $[G_w(m_z-5\delta)-G_w(m_z-6\delta)]=2\delta/v$ and $[G_y(m_z-5\delta)-G_y(m_z-6\delta)]=2\delta/v$, which can only hold if the mass is distributed at a rate of $2/v$. Finally, $u_y(m_z-4\delta,\ G_w)=u_y(m_z-6\delta,\ G_w)$ and $u_y(m_z-5\delta,\ G_w)=u_y(m_z-7\delta,\ G_w)$ give us $G_w(m_z-7\delta)=G_w(m_z-8\delta)$; the corresponding equations for u_w yield $G_y(m_z-7\delta)=G_y(m_z-8\delta)$. □

Lemma 12 *Let $\beta=1/2$ and $\delta\in(0,\ v/4]$. Also, let $p=\lfloor v/4\delta\rfloor$ be the number of length-4δ intervals with the properties specified by $\mathscr{P}$. In any equilibrium, the top of the p^{th} such interval (or bottommost interval) is in $[2\delta,\ 3\delta]$ if $\delta=v/4p$; in $(3\delta,\ 4\delta]$ if $\delta\in[v/(4p+2),\ v/4p)$; and at $(v/2)-\delta(2p-4)$ if $\delta\in(v/(4p+4),\ v/(4p+2))$. Below the p^{th} interval, a total mass of $1-[4\delta p/v]$ is distributed as follows:*

A. *If $\delta\in[v/(4p+2),\ v/4p)$, the remaining $1-[4\delta p/v]$ is at zero, neither player has mass in $(0,\ \delta)$, and all equilibria have an expected payoff of $(v/2)-2\delta p$. For $\delta=v/(4p+2)$, the equilibrium is unique: there is no mass in $(0,\ 2\delta)$ and the top of the p^{th} interval is at 4δ.*

B. *If $\delta\in(v/(4p+4),\ v/(4p+2))$, each player has a mass point at zero of $[4\delta(p+1)/v]-1$, a uniform density rate of $2/v$ over the intervals $[0,\ v/2-\delta(2p+1)]$ and $[\delta,\ v/2-2\delta p]$, and an expected payoff of δ. This also holds for $\delta\in(v/4,\ v/2)$ (i.e. $p=0$).*

Proof Following the notation from the proof of Lemma 11, let $m_p=m_1-4\delta(p-1)$, where $m_1=\max\{\overline{w}_1,\ \overline{y}_1\}$. That is, m_p is the top of the p^{th} length-4δ interval (specifically for the player whose support contains the highest element; alternatively, $m_p=\max\{\overline{w}_p,\ \overline{y}_p\}$). By Lemma 11, these p intervals satisfy $\mathscr{P}$, and so the remaining mass of $1-[4\delta p/v]$ must be distributed below them at the bottom of the distribution. There are two bounds that we can quickly place on m_p. First, $m_p\geq 2\delta$; otherwise, with a maximal density rate of $2/v$ it is not possible to have $4\delta/v$ of continuously distributed mass. Second, $m_p<6\delta$, or it would be possible to have $4\delta/v$ of continuously distributed mass below $m_p-4\delta$. We will show that $m_p\in[2\delta,\ 3\delta]$ when $\delta=v/4p$; $m_p\in(3\delta,\ 4\delta]$ when $\delta\in[v/(4p+2),\ v/4p)$; and $m_p\in(5\delta,\ 6\delta)$ when $\delta\in(v/(4p+4),\ v/(4p+2))$. This covers the complete range of δ for any given p. Furthermore, we will show how the remaining $1-[4\delta p/v]$ is distributed, as well as a uniqueness result for $\delta=v/(4p+2)$.

If $m_p\in[2\delta,\ 4\delta]$, then since $4\delta/v$ is distributed over $(0,\ m_p]$, Lemma 10 requires that the remaining mass of $1-[4\delta p/v]$ be at zero. If $m_p\in[2\delta,\ 3\delta)$, then each player's support contains a neighborhood above and below δ with a density rate of $2/v$. However, if the opponent has a strictly positive mass point at zero, placing mass immediately below δ cannot hold in equilibrium. So for $m_p\in[2\delta,\ 3\delta)$, we must have $1-[4\delta p/v]=0$, or equivalently, $\delta=v/4p$. We can also quickly show that $m_p\notin(3\delta,\ 4\delta]$ when $\delta=v/4p$. Without a mass point at zero, each player's distribution must have a density rate of $2/v$ over $[\delta,\ 2\delta]$; otherwise, players could obtain a higher expected payoff by bidding zero. Maintaining even the minimum expected payoff of zero entails randomizing at the rate of $2/v$ over $[\delta,\ 3\delta]$, but then the mass of $4\delta/v$ is used up. So $m_p\notin(3\delta,\ 4\delta]$ when $\delta=v/4p$.

Therefore, if $m_p\in(3\delta,\ 4\delta]$, there must be a strictly positive mass point at zero of $1-[4\delta p/v]$. And if one player is granted that privilege, they both must be. So $\min\{\overline{w}_p,\ \overline{y}_p\}\in(3\delta,\ 4\delta]$, and 2δ is in each player's support at a density rate of $2/v$. Equating $u_w(2\delta,\ G_y)=u_w(0,\ G_y)$ and $u_y(2\delta,\ G_w)=u_y(0,\ G_w)$, we obtain:

$$G_w(3\delta)=G_y(3\delta)=4\delta/v \tag{30}$$

At least $2\delta/v$ of this $4\delta/v$ is distributed at a density rate of $2/v$ over $[\overline{w}_p - 2\delta,\ \overline{w}_p - \delta]$ or $[\overline{y}_p - 2\delta,\ \overline{y}_p - \delta]$. So the mass point of $1 - [4\delta p/v]$ must be weakly less than $2\delta/v$. This implies that $\delta \geq v/(4p+2)$. We already showed that this range of m_p does not hold for $\delta = v/4p$, and so $m_p \in (3\delta,\ 4\delta]$ implies that $\delta \in [v/(4p+2),\ v/4p)$. Equation 30 also rules out the possibility of a strictly positive mass point when $m_p = 3\delta$ (there is already $4\delta/v$ in $(0,\ 3\delta]$, so there is no room for a mass point).

Next suppose that $m_p \in (4\delta,\ 5\delta]$. We will show that this cannot hold in equilibrium. Without loss of generality, assume that $m_p = \overline{w}_p$. As a property of $\mathscr{P}$, there is no mass in G_w over $[\overline{w}_p - 4\delta,\ \overline{w}_p - 3\delta]$ and no mass in G_y over $(\max\{\overline{y}_p - 4\delta,\ 0\},\ \overline{y}_p - 3\delta]$. We begin by showing that G_y has no mass in $(0,\ \delta]$. If $\overline{y}_p \in (\overline{w}_p - \delta,\ 4\delta]$, then the remaining mass of $1 - [4\delta p/v]$ is at zero and G_y has no mass in $(0,\ \delta]$. If instead $\overline{y}_p \in (4\delta,\ \overline{w}_p]$, then by Lemma 2, since G_w has no mass in $(\delta,\ \overline{w}_p - 3\delta]$, then G_y has no mass in $(0,\ \overline{w}_p - 4\delta]$. There is also no mass in G_y over $[\overline{w}_p - 4\delta,\ \delta]$ since this is a subset of $[\overline{y}_p - 4\delta,\ \overline{y}_p - 3\delta]$. Thus, G_y has no mass in $(0,\ \delta]$. We next note that $u_w(0,\ G_y) = u_w(\overline{\delta},\ G_y)$. Since 0 is in the support of G_w, we must have $u_w(0,\ G_y) \geq u_w(\overline{\delta},\ G_y)$. But if $u_w(0,\ G_y) > u_w(\overline{\delta},\ G_y)$, then we would also have $u_w(0,\ G_y) > u_w(x',\ G_y)$ where $x' = \min\{x \in \text{supp}(G_w) \mid x \geq \overline{w}_p - 3\delta\}$. With no mass in G_y over $(0,\ \delta]$, the maximal density rate of $2/v$ over $[2\delta,\ x' + \delta]$ can compensate for the added bidding cost between δ and x', but no more. Thus, since we must have $u_w(0,\ G_y) = u_w(\overline{\delta},\ G_y)$ and since $\overline{w}_p - 3\delta > \delta$, then we must also have a density rate of $2/v$ in G_w over $[2\delta,\ \overline{w}_p - 2\delta]$. However, this implies that there is no mass in G_w over $[4\delta,\ \overline{w}_p]$, which contradicts $\overline{w}_p \in (4\delta,\ 5\delta]$.

Finally, if $m_p \in (5\delta,\ 6\delta)$, then $m_p - 4\delta \in (\delta,\ 2\delta)$. We again assume that $m_p = \overline{w}_p$. Following $\mathscr{P}$, there is no mass in G_w over $[\overline{w}_p - 4\delta,\ \overline{w}_p - 3\delta]$, and consequently, by Lemma 2, no mass in G_y over $[\overline{w}_p - 5\delta,\ \delta]$. Likewise, there is no mass in G_y over $[\overline{y}_p - 4\delta,\ \overline{y}_p - 3\delta]$, and no mass in G_w over $[\overline{y}_p - 5\delta,\ \delta]$.[22] Applying parts A and B of Lemma 2, any continuously distributed mass over $[0,\ \overline{y}_p - 5\delta] \bigcup [\delta,\ \overline{w}_p - 4\delta]$ in G_w and over $[0,\ \overline{w}_p - 5\delta] \bigcup [\delta,\ \overline{y}_p - 4\delta]$ in G_y must have a density rate of $2/v$. It follows then that G_w and G_y have the same amount of continuously distributed mass, and so $\alpha_w(0) = \alpha_y(0) \equiv \alpha(0)$. To compensate for the respective gaps in G_w and G_y over $[\overline{y}_p - 5\delta,\ \delta]$ and $[\overline{w}_p - 5\delta,\ \delta]$, we need $\alpha(0) > 0$.[23] Since a bid of $\overline{w}_p - 4\delta$ for player w and of $\overline{y}_p - 4\delta$ for player y have the same expected payoffs as a bid 2δ above that or any other bid in their support (see Lemmata 10 and 11), it must be that $u_w(\overline{w}_p - 4\delta,\ G_y) \geq u_w(\overline{\delta},\ G_y)$ and $u_y(\overline{y}_p - 4\delta,\ G_w) \geq u_y(\overline{\delta},\ G_w)$. These can only be satisfied if G_y has a density rate of $2/v$ over $[0, \overline{w}_p - 5\delta]$ and if G_w has a density rate of $2/v$ over $[0, \overline{y}_p - 5\delta]$. By Lemma 2, these in turn imply a density rate of $2/v$ over $[\delta,\ \overline{w}_p - 4\delta]$ in G_w and $[\delta,\ \overline{y}_p - 4\delta]$ in G_y. With a common mass point, each player will only be indifferent between these two intervals if $\overline{w}_p = \overline{y}_p$. From $u_w(\overline{w}_p - 4\delta,\ G_y) = u_w(0,\ G_y)$, we can identify $\overline{w}_p = (v/2) - 2\delta(p-2)$, which pins down $\alpha(0) = [4\delta(p+1)/v] - 1$.[24] We can establish bounds on δ from $u_w(\overline{w}_p - 4\delta,\ G_y) = u_w(\overline{w}_p - 3\delta,\ G_y)$, since this implies $G_y(\overline{w}_p - 5\delta) = 2\delta/v$. Thus, the remaining mass of $1 - (4\delta/v) \in (2\delta/v,\ 4\delta/v)$. Equivalently, $\delta \in (v/(4p+4),\ v/(4p+2))$. This argument also applies to the case of $\delta \in (v/4,\ v/2)$.

[22] The claim that G_w has no mass in $[\overline{y}_p - 5\delta,\ \delta]$ is contingent on $\overline{y}_p \in (5\delta,\ \overline{w}_p]$. We can quickly rule out the possibility of $\overline{y}_p \in (\overline{w}_p - \delta,\ 5\delta]$: the lack of mass in G_y over $[\overline{y}_p - 4\delta,\ \overline{y}_p - 3\delta]$ would preclude mass in G_w over $(0,\ \overline{y}_p - 4\delta)$ (see Lemma 2). Since there is also no mass in G_w over $[2\delta,\ \overline{w}_p - 3\delta]$, there would be no way for player y to recover the bidding cost between δ and $\overline{w}_p - 4\delta$ (i.e. the next element in G_y above $\overline{w}_p - 4\delta$ would need to be compensated by more than the maximal density rate of $2/v$).

[23] If $\alpha(0) = 0$, at most one player could have continuously distributed mass below δ, but that player would then have a sizable gap before there was any more mass in the other player's distribution to tie or beat. So that cannot be an equilibrium.

[24] The mass point is the remainder of $1 - (4\delta p/v)$ after subtracting $[(v/2) - 2\delta(p-2) - 5\delta] \times (2/v) \times 2$.

Algorithm 1 Asymmetric Equilibria: $\beta \in (0,\ 1/2)$, $\delta \in (0,\ (1-\beta)v)$

for all $z \in \{0,\ 1,\ 2,\ \ldots,\ \lfloor v/\delta \rfloor - 1\}$ **do** ▷ Vary first zeroed element
  Set to zero: $w_z,\ y_{z+1},\ w_{z+2},\ y_{z+3}, \ldots,\ w_{\lfloor v/\delta \rfloor -1},\ y_{\lfloor v/\delta \rfloor}$ (if $z = 0$, set to zero: $a_w(0),\ w_1,\ y_2,\ \ldots,\ w_{\lfloor v/\delta \rfloor -1},\ y_{\lfloor v/\delta \rfloor}$).
  for all $p \in \{\max\{1,\ z\}, \ldots,\ \lfloor v/\delta \rfloor\}$ **do** ▷ Vary upper bound
    Set to zero: $w_j,\ y_j$ for all $j > p$.
    Solve system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ (one indifference equation for each jump in each player's distribution, plus constraints for mass summing to one; $\boldsymbol{x}$: sizes of nonzero mass points and lengths of nonzero $w_i,\ y_i$).
    verify whether $\boldsymbol{x}^* = \boldsymbol{A}\backslash\boldsymbol{b}$ is an equilibrium:
    • Not an equilibrium if any elements of $\boldsymbol{x}^*$ are not strictly positive.
    • If $z = 0$, not an equilibrium if $u_w(0,\ G_y) > u_w(\delta,\ G_y)$.
    • Not an equilibrium if $u_y(\zeta + \delta,\ G_w) > u_y(0,\ G_w)$ where ζ is where the $1/[(1-\beta)]v$ segment of w_z would have begun ($\zeta = 0$ if $z = 0$).
    • Not an equilibrium if the player with the smaller upper bound can do better by bidding δ above the other player's upper bound.
    • Otherwise, $\boldsymbol{x}^*$ constitutes an equilibrium.

Having established the various bounds for δ for each potential value of m_p, we conclude with a uniqueness result at $\delta = v/(4p+2)$. As we have already shown, there is a mass point at zero of $1-(4\delta p/v)$ and there is no mass in $(0,\ \delta)$. Also, as we argued earlier in the paragraph covering $m_p \in (4\delta,\ 5\delta]$, we must also have $u_i(0,\ G_{-i}) = u_i(\overline{\delta},\ G_{-i})$ (or else the expected payoff of elements in the support above δ would be less than the expected payoff at zero). From this equality we obtain $G_{-i}(2\delta) - G_{-i}(\delta) = [\delta(4p+2)/v] - 1$, which equals zero when $\delta = v/(4p+2)$. Hence, there is no mass in $(0,\ 2\delta)$, so the $4\delta/v$ in the p^{th} interval must be distributed over $[2\delta,\ 4\delta]$ at a rate of $2/v$. The rest of the distribution follows from $\mathscr{P}$. □

E Asymmetric Equilibria

For $\delta \in (0,\ (1-\beta)v)$ and $\beta \in (0,\ 1/2)$, Algorithm 1 identifies the complete set of asymmetric equilibria for the game $APT\{\delta, \beta, v\}$ (the labels are the same as in Figure 7). With players arbitrarily assigned as player w or player y, the algorithm systematically varies the first $1/[(1-\beta)v]$ segment in player w's distribution to omit, as well as the uppermost interval pair in the two distributions. Then for each combination of omitted interval pairs, there are at most four conditions that must be checked to verify the existence of an asymmetric equilibrium. First, the system of equations formed from the indifference conditions between the intervals in each player's distribution needs to produce strictly positive lengths for each of the non-omitted interval pairs and strictly positive mass for the non-excluded mass points (the system of equations also includes two equations which specify that the mass in each player's distribution must sum to one). Second, if player w's mass point was excluded, player w cannot profitably deviate by bidding zero.

All other profitable deviations are captured by the third and fourth conditions. Third, bidding δ above the first omitted $1/[(1-\beta)v]$ segment in player w's distribution cannot be profitable for player y. Below this point, bids within gaps in either player's distribution can be ruled out by arguments similar to those for the symmetric case (see the paragraphs leading up to Theorem 3). Above this point, the gaps are so large that bidding within a gap does not adequately increase the amount of mass a player is tying or beating. Precisely at this point, however, player y beats all of the $1/(\beta v)$ segment that is δ below it, and so the expected payoff rises to a peak—the only peak in this gap. It therefore suffices to check that this peak is not too high. The fourth condition similarly pertains to a peak. As in the symmetric case, it merely specifies that outbidding the opponent's distribution by δ cannot be profitable. Here, however, one player's upper bound is a $1/[(1-\beta)v]$ segment while the other's is a $1/(\beta v)$ segment, which is already δ above the first. So we simply need to verify that the player with the smaller upper bound cannot profit by outbidding their opponent's distribution and winning with certainty.

References

Amick, K.G.: The next great engine war analysis and recommendations for managing the Joint Strike Fighter Engine competition. Naval Postgraduate School, Monterey, California, PhD Thesis (2005)

Baye, M.R., Kovenock, D., de Vries, C.G.: The solution to the Tullock rent-seeking game when R>2: Mixed-strategy equilibria and mean dissipation rates. Public Choice **81**(3–4), 363–380. (1994)

Baye, M.R., Kovenock D., de Vries C.G.: The all-pay auction with complete information. Econ. Theory **8**(2), 291–305 (1996)

Bouckaert, J., Degryse, H., de Vries, C.G.: Veilingen waarbij iedereen betaalt en toch iets wint. Tijdschrift voor economie en management **37**(4), 375–393 (1992)

Chaturvedi, A., Beil, D.R., Martinez-de-Albéniz, V.: Split-award auctions for supplier retention. Manage. Sci. **60**(7), 1719–1737 (2014)

Che, Y.K., Gale, I.L.: Caps on political lobbying. Am. Econ Rev. **88**(3), 643–651 (1998)

Cohen, C., Schwartz, M.: Minimizing Shirking in Auctions and Tournaments. Theor. Econ. Lett. **3**(04), 197–201 (2013)

Cohen, C., Sela, A.: Contests with ties. BE J. Theor. Econ. **7**(1) (2007)

Dechenaux, E., Kovenock, D., Sheremeta, R.M.: A survey of experimental research on contests, all-pay auctions and tournaments. Exp. Econ. **18**(4), 609–669 (2015)

Deneckere, R.J., Kovenock, D.: Bertrand-Edgeworth duopoly with unit cost asymmetry. Econ. Theory **8**(1), 1–25 (1996)

Drewes, R.W.: The Air Force and the Great Engine War. National Defense University Press, Washington, DC (1987)

Fisher, E.O'N., Wilson, C.A.: Price competition between two international firms facing tariffs. Int. J. Ind. Organ. **13**(1), 67-87 (1995)

Gelder, A., Kovenock, D., Sheremeta, R.: Behavior in All-Pay Auctions with Ties. Chapman University, Economic Science Institute Working Paper No. 15–22. SSRN No. 2683568 (2015)

Gong, J., Li, J., McAfee, R.P.: Split-award contracts with investment. J. Pub. Econ. **96**, 188–197 (2012)

Hillman, A.L., Riley, J.G.: Politically contestable rents and transfers. Econ. Polit. **1**(1), 17–39 (1989)

Klemperer, P.: Markets with consumer switching costs. Q. J. Econ. **102**(2), 375–394 (1987)

Konrad K.A.: Strategy and dynamics in contests. Oxford University Press, New York (2009)

Osborne, M.J., Pitchik, C.: Price competition in a capacity-constrained duopoly. J. Econ. Theory **38**(2), 238–260 (1986)

Shilony, Y.: Mixed pricing in oligopoly. J. Econ. Theory **14**(2), 373–388 (1977)

Stong, S.: All-pay auctions with ties. In: Contests: Uncertainty and budgets, pp. 64–78. Univ. Iowa, PhD Thesis (2014)

Szech, N.: Tie-breaks and bid-caps in all-pay auctions. Game. Econ. Behav. **92**, 138–149 (2015)

Szech, N., Weinschenk, P.: Rebates in a Bertrand game. J. Math. Econ. **49**(2), 124–133 (2013)