WORKING PAPER: COMMENTS WELCOME

Mortality Risk, Insurance, and the Value of Life*

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Abstract. Public health insurance and public annuity programs account for about half of federal spending and large bodies of economic research. However, they have so far been viewed as unconnected. We generalize the widely used economic theory of life-extension to explore the close connections between these two sets of policies. Our framework, which introduces incomplete annuitization and stochastic mortality into the conventional economic theory of life-extension, generates several novel findings. First, greater annuitization alters the willingness to pay for life-extension, commonly called the value of a statistical life (VSL). Second, shocks to mortality risk increase VSL for a consumer who is incompletely annuitized. Finally, we employ our framework to introduce a more general concept, the value of a statistical illness (VSI), which quantifies an individual's willingness to pay to avoid an increase in the risk of acquiring an illness that affects her mortality rate.

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I. INTRODUCTION

The economic analysis of risks to life and health has made enormous contributions to both academic discussions and public policy. Economists have used the standard tools of life-cycle consumption theory to propose a transparent framework that measures the value of mortality risk-reduction, the value of quality of life improvements, and the value of a statistical life-year (Arthur 1981; Rosen 1988; Murphy and Topel 2006). All these concepts and quantities now play central roles in public policy discussions surrounding investments in medical care, public safety, workplace safety, environmental hazards, and countless other arenas.

For analytical convenience, the standard framework has typically assumed complete annuitization and deterministic mortality risk. These assumptions sacrificed little generality in the analysis of public policies that save lives in the aggregate. However, they hamper the model's predictive power in several ways when studying individual behavior and the relationships between alternative mechanisms for risk-reduction. In addition, they gloss over policy-relevant relationships between the demand for life-extension and the structure of the annuity market, and cannot meaningfully distinguish between preventive care and therapeutic care.

Complete annuity markets better shield an individual against mortality risk. By the same logic, an incompletely annuitized consumer will have greater incentive to avoid or mitigate a sudden shock to mortality risk. A very simple example illustrates the intuition. Imagine a retiree with $120,000 in wealth, no bequest motive, and a flat optimal consumption profile. Suppose further that her ex ante life expectancy is 4 years, and that she is equally likely to live 3, 4, or 5 years. If she is fully annuitized, her consumption remains flat at $30,000 annually, regardless of what she later learns about her mortality risk. Now suppose she cannot annuitize any of her wealth. If she suddenly learns that she will live for only 3 years, she will accelerate her spending to $40,000 annually and thus increase the value of each life-year. This example demonstrates that mortality shocks can increase the value of life-extension when annuity markets are incomplete.[1] It also demonstrates that annuitization can decrease the demand for life-extension. We show that this basic intuition generalizes to models with stochastic mortality and incomplete annuity markets of various kinds.

In contrast to the complete annuitization model, the value of life-extension is predicted to vary with the size of mortality shocks when consumers are incompletely annuitized. Thus, the value of a statistical life-year may be higher for an individual diagnosed with a more fatal illness, and vice-versa. This insight, which is consistent with data on how consumers view the value of life-extension (Nord et al. 1995; Green and Gerard 2009; Linley and Hughes 2013), implies that the value of life-extension varies systematically across diseases. It also implies that the value of treatment technologies, which are used after an illness occurs, will be higher than the value of preventive technologies, even when both increase life expectancy by identical amounts. These insights ought to change the way health insurers assess the value of various medical technologies.

Finally, our framework takes the more realistic perspective that an individual faces uncertainty over his future mortality risk. This stochastic mortality assumption produces additional insights. The standard model quantifies the value of statistical lives, but it has little to say about the continuum of health

[1] Conceptually, this argument is most closely related to Philipson et al (2010), who implicitly assume that incompletely annuitized individuals will need to spend down their resources at the end of life.

events that precede death. Our framework lends itself naturally to a more general concept, the value of a statistical illness (VSI), which quantifies an individual's willingness to pay to avoid an increase in the risk of acquiring an illness that affects her mortality rate.

Our study connects the vast literature on the value of life (Arthur 1981; Murphy and Topel 2006; Rosen 1988; Hall and Jones 2007) with the literature on life-cycle consumption models that goes back to Yaari (1965). It is well known that annuitization provides substantial value by insulating individuals from consumption risk. We show that it also greatly increases the value of statistical life at older ages.[2] Our results suggest that more attention should be paid to the public finance interactions between pension and healthcare systems.

Section II reviews the predictions of the conventional model for the returns to life-extension and demonstrates how relaxing the perfect annuity assumption implies diminishing returns to life-extension. It also develops additional subsidiary implications of this framework. Section III presents empirical analysis that: (1) quantifies how health shocks change the value of statistical life when annuity markets are incomplete; (2) computes the effect of annuitization on the value of longevity; and (3) demonstrates why and to what extent treatment is more valuable than prevention. Section IV concludes.

II. THE VALUE OF LIFE WHEN MORTALITY IS DETERMINISTIC

Consider an individual who faces a mortality risk. We are interested in analyzing the value of a marginal reduction in this risk.[3] We first quantify the value of mortality risk-reduction for an individual who is fully annuitized. We then repeat the exercise for an individual who lacks access to annuity markets, and compare our findings. We assume initially that mortality is deterministic, which allows us to illustrate the basic insights of the paper in a setting familiar to the literature on the value of life. Section III extends the model to accommodate stochastic mortality and show that our conclusions are enriched but qualitatively unchanged.

II.A. The fully annuitized value of life

Let $c(t)$ be consumption at time t, W_0 be baseline wealth, $m(t)$ be exogenously determined income, ρ be the rate of time preference, and r be the rate of interest. Finally, define $q(t)$ as health-related quality of life at time t. Since it sacrifices little generality in our application, we take the life-cycle quality of life profile $q(t)$ as exogenous. As needed, one can consider any relevant quality of life profile in concert with a given profile of mortality. The maximum lifespan of a consumer is T, and her mortality rate at any point in time is given by $\mu(t)$, where $0 \leq t \leq T$. The probability that a consumer will be alive at time t is:

$$S(t) = \exp\left[-\int_0^t \mu(s)ds\right]$$

[2] Reichling and Smetters (2015) show that stochastic mortality and correlated medical costs can explain the puzzling observation that many households do not sufficiently annuitize their wealth. They take the positive correlation between health shocks and medical spending as a given. Our study sheds light on *why* these two phenomena are positively correlated.

[3] We focus on improvements in longevity but allow for improvements in quality of life as well.

At time period $t = 0$, the consumer fully annuitizes. We assume that annuitization is actuarially fair.

The consumer's maximization problem is:

$$\max_{c(t)} \int_0^T e^{-\rho t} S(t) u(c(t), q(t)) dt$$

$$s.t. \int_0^T e^{-rt} S(t) c(t) dt \leq W_0 + \int_0^T e^{-rt} S(t) m(t) dt$$

The consumer's utility function, $u(c(t), q(t))$, depends on both consumption and health-related quality of life. We assume $u(\cdot)$ is strictly increasing, concave, and twice continuously differentiable. Let $u_c(\cdot)$ denote the marginal utility of consumption. Associating the multiplier θ with the wealth constraint, optimal consumption is characterized by the first-order condition:

$$e^{(r-\rho)t} u_c(c(t), q(t)) = \theta$$

To analyze the value of life, let $\delta(t)$ be a perturbation on the mortality intensity with $\int_0^T \delta(t) dt = 1$, and consider

$$S^{\varepsilon}(t) = \exp\left[-\int_0^t (\mu(s) - \varepsilon\delta(s)) ds\right], \text{where } \varepsilon > 0$$

Let $c^{\varepsilon}(t)$ represent the equilibrium variation in $c(t)$ caused by this perturbation. As shown in Rosen (1988), the marginal utility of this life-extension is given by

$$\left.\frac{\partial \mathbb{E}U}{\partial \varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial}{\partial \varepsilon} \int_0^T e^{-\rho t} S^{\varepsilon}(t) u\big(c^{\varepsilon}(t), q(t)\big) dt\right|_{\varepsilon=0}$$

$$= \int_0^T \left[e^{-\rho t} u(c(t), q(t)) + e^{-rt}\theta\big(m(t) - c(t)\big)\right]\left[\int_0^t \delta(s) ds\right] S(t) dt$$

A canonical choice for $\delta(\cdot)$ is the Delta-Dirac function, so that the mortality rate is perturbed at $t = 0$ and remains unaffected otherwise. Dividing the result by the marginal utility of wealth, θ, then yields the marginal value of life-extension that is commonly called the value of a statistical life (VSL):

$$VSL \equiv \int_0^T e^{-rt} S(t) \left(\frac{u(c(t), q(t))}{u_c(c(t), q(t))} + m(t) - c(t)\right) dt \tag{1}$$

VSL corresponds to the value that the individual places on a marginal reduction in risk of death in the current period. For example, it is the amount that 1,000 people would be collectively willing to pay to eliminate a current risk that is expected to kill one of them. It is equal to the present discounted value of lifetime consumption, plus the change in net savings.

Alternatively, one might be interested in valuing a treatment that reduces mortality evenly across all ages. In this case $\delta(\cdot)$ is more appropriately modeled with a uniform distribution function, yielding

$$\int_0^T e^{-rt} \left(\frac{u(c(t), q(t))}{u_c(c(t), q(t))} + m(t) - c(t)\right) \frac{t}{T} S(t) dt$$

This puts greater weight on later years of life.

It is also useful to characterize the value of a statistical life-year, which is the value of a one-period change in survival from the perspective of current time:

$$v(t) \equiv \frac{u\big(c(t), q(t)\big)}{u_c\big(c(t), q(t)\big)} + m(t) - c(t)$$

The value of statistical life depends on consumption and the quality of life. Define the elasticity of intertemporal substitution as:

$$\frac{1}{\sigma} \equiv -\frac{u_{cc}c}{u_c}$$

In addition, define the elasticity of quality of life with respect to the marginal utility of consumption as:

$$\eta \equiv \frac{u_{cq}q}{u_c}$$

When this term is positive, the marginal utility of consumption is higher in healthier states, and vice-versa.

Taking logarithms of the first-order condition for consumption and differentiating with respect to time yields the rate of change for consumption over the life cycle:

$$\frac{\dot{c}}{c} = \sigma(r - \rho) + \sigma\eta\frac{\dot{q}}{q} \qquad (2)$$

If one assumes that $r > \rho$, and that the marginal utility of consumption is higher when health status is better, then life-cycle consumption will have the inverted U-shape observed in real-world data.[4]

Note the crucial feature of the conventional model that consumption growth over the life-cycle is independent of mortality risk, because the individual is fully insured against that risk. This feature in turn implies that the rate of change in the value of a life-year is also not a function of mortality risk:

$$\frac{\dot{v}}{v} = \left(\frac{u}{u_c}\frac{1}{\sigma v} + \frac{c}{v}\right)\frac{\dot{c}}{c} + \left(\frac{-\eta}{v} + \frac{qu_q}{uv}\right)\frac{\dot{q}}{q} + \frac{\dot{m}}{v}$$

Although changes in mortality do not affect the rate of change in the value of a statistical life-year, inspection of equation (1) reveals that an increase in mortality lowers the value of a statistical life by reducing the expected net present value of lifetime utility. The only effect of mortality risk on VSL operates by changing the probability that the individual will live to enjoy consumption in a particular year.

[4] Consumption climbs early in life as the benefits to savings diminish. It declines later in life when quality of life deteriorates. This inverted U-shape for the age profile of consumption has been widely documented across different countries and goods (Carroll and Summers 1991; Banks, Blundell et al. 1998; Fernandez-Villaverde and Krueger 2007).

II.B. The uninsured value of life

To illustrate the effects of annuitization, we consider a model without any annuitization possibilities. In our calibration exercises later, we will consider various partial annuitization schemes. To characterize the model without annuitization, we employ the Yaari (1965) model of consumption behavior under mortality risk. The consumer's maximization problem is now:

$$\max_{c(t)} \int_0^T e^{-\rho t} S(t) u(c(t), q(t)) dt$$

$$s.t. W(0) = W_0,$$

$$W(t) \geq 0, W(T) = 0,$$

$$\dot{W} = rW(t) + m(t) - c(t)$$

If the non-negative wealth constraint binds, then the solution to the consumer's problem is simply to set $c(t) = m(t)$. Otherwise, the solution is to maximize subject to the constraint on the law of motion for wealth. We focus here on the latter, nontrivial case.

The consumer's first-order condition for consumption is:

$$e^{(r-\rho)t} S(t) u_c(c(t), q(t)) = \theta$$

Unlike in the case of perfect markets, the survival function enters the consumer's first-order condition for optimal consumption. Instead of setting the discounted marginal utility of consumption equal to the marginal utility of wealth, the consumer sets the *expected* discounted marginal utility of consumption at time t equal to the marginal utility of wealth. This effectively shifts consumption to earlier ages in the life-cycle. This is rational because consumption allocated to later time periods will not be enjoyed in the event of an early death.

The expression for the marginal utility of life extension is the same as in the case of perfect markets:

$$\begin{aligned}
\left.\frac{\partial \mathbb{E}U}{\partial \varepsilon}\right|_{\varepsilon=0} &= \left.\frac{\partial}{\partial \varepsilon} \int_0^T e^{-\rho t} S^{\varepsilon}(t) u\big(c^{\varepsilon}(t), q(t)\big) dt\right|_{\varepsilon=0} \\
&= \int_0^T e^{-\rho t} \left[\int_0^t \delta(s) ds\right] S(t) u(c(t), q(t)) dt + \int_0^T e^{-\rho t} S(t) u(c(t), q(t)) \left.\frac{\partial c^{\varepsilon}(t)}{\partial \varepsilon}\right|_{\varepsilon=0} dt \\
&= \int_0^T e^{-\rho t} \left[\int_0^t \delta(s) ds\right] S(t) u(c(t), q(t)) dt + \theta \frac{\partial}{\partial \varepsilon} \int_0^T e^{-rt} c^{\varepsilon}(t) dt \\
&= \int_0^T e^{-\rho t} \left[\int_0^t \delta(s) ds\right] S(t) u(c(t), q(t)) dt
\end{aligned}$$

where the last equality follows from application of the budget constraint.[5] Choosing again the Delta-Dirac function for $\delta(\cdot)$ and dividing the result by the marginal utility of wealth, θ, yields an expression for VSL that differs from the perfect markets case:

[5] The budget constraint $W(T) = 0$ implies $\int_0^T e^{-rt} c^{\varepsilon}(t) dt = W_0 + \int_0^T e^{-rt} m(t) dt$, which is equal to a constant.

$$VSL = \int_0^T e^{-\rho t} S(t) \frac{u\big(c(t), q(t)\big)}{u_c\big(c(0), q(0)\big)} dt = \int_0^T e^{-rt} \frac{u\big(c(t), q(t)\big)}{u_c\big(c(t), q(t)\big)} dt \tag{3}$$

As before, the value of statistical life is proportional to the expected discounted (lifetime) utility of consumption, and inversely proportional to the marginal utility of consumption. There is no effect of net savings. It is well known that removing annuity markets lowers lifetime utility (Yaari 1965). We also argued earlier, and will show more formally below, that removing these markets shifts consumption to earlier ages, thereby lowering the marginal utility of consumption, at least at those ages. Thus, the effect of annuity markets on VSL is in general ambiguous.

The expression for the value of a statistical life year, v, is the same under both perfect and imperfect markets because it is calculated from the perspective of current time and thus is unaffected by differential discounting. However, the actual values will in general differ because annuitization alters the time profile of consumption.

In particular, the life-cycle consumption profile of the non-annuitized individual depends explicitly on mortality risk. Taking logarithms of the first-order condition for consumption and differentiating with respect to time yields:

$$\frac{\dot{c}}{c} = \sigma(r - \rho) + \sigma\eta\frac{\dot{q}}{q} - \sigma\mu(t) \tag{4}$$

Comparing this result to the standard case, given by equation (2), reveals both similarities and differences. As in the standard, fully annuitized model, the non-annuitized consumption profile described by equation (4) changes shape when the rate of time preference is above or below the rate of interest and when the quality of life changes. Unlike in the standard model, however, the consumption profile described by equation (4) depends explicitly on the mortality rate, $\mu(t)$. Higher rates of mortality depress the rate of consumption growth over the life-cycle. This rate of growth is always higher in the fully annuitized case, in which the last term drops out of the consumption growth equation (4). Put another way, removing the annuity market "pulls consumption earlier" in the life-cycle.

An appealing feature of the uninsured model is that it generates an inverted U-shape for the profile of consumption under quite natural assumptions. In particular, low income early in life and high mortality risk later in life are sufficient conditions for the inverted U-shape consumption profile. One need not impose the ad hoc assumptions on the signs of $r - \rho$ or η that are necessary in the fully annuitized model (Murphy and Topel 2006).

Because it depends on consumption, the life-cycle profile of the value of a statistical life-year also depends on mortality:

$$\frac{\dot{v}}{v} = \left(\frac{1}{\sigma} + \frac{c}{v}\right)\frac{\dot{c}}{c} + \left(\frac{qu_q}{u} - \eta\right)\frac{\dot{q}}{q} \tag{5}$$

An important implication of this model is that willingness to pay for longevity depends critically on the life-cycle mortality profile (see equation 4). Holding quality of life constant, increases in the mortality rate will raise v, the value of a statistical life-year. This is evident from the definition of $v \equiv \frac{u}{u_c}$. That is, mortality will shift forward the value of life. As societies become richer and live longer, the fraction of wealth spent on health will depend not just on the income elasticity of health, but also on the degree of survival uncertainty they face. We return to this point in our empirical exercise. Furthermore, our results imply that public programs such as Social Security that increase annuitization levels will affect society's willingness to pay for longevity, thereby creating a feedback loop that could dampen or increase program expenditures.[6] As a general matter, the model demonstrates that the degree of annuitization influences how people value gains in longevity.

In the next section, we allow mortality to be stochastic so that we can investigate the effect of health shocks on the value of life. Before turning to that analysis, we pause to note that suffering a health shock is similar to removing access to annuity markets, which exposes an individual to mortality risk. As we have just demonstrated, this shifts the value of a life-year forward, thereby increasing the value of early life-years and decreasing the value of later life-years. The net effect on VSL is ambiguous. As we shall see, health shocks have a similar effect.

III. THE VALUE OF LIFE WHEN MORTALITY IS STOCHASTIC

The previous analysis demonstrates that mortality risk affects the value of a statistical life when annuity markets are incomplete. Earlier analyses of the value of life have overlooked this relationship by assuming complete annuitization. However, the conventional framework is ill-equipped to study the influence of mortality risk for another reason as well. Prior analysis, just like our model above, treats the mortality rate as a nonrandom parameter (cf, Murphy and Topel, 2006). Thus, shifts in mortality risk reflect preordained and anticipated changes in mortality. In the real world, however, neither the timing nor the size of shifts in mortality risk is known.

We now extend our analysis to include random mortality risk and show that our prior results continue to hold. Specifically, we assume that the mortality rate now depends on the individual's health state. Let Y_t be a continuous-time Markov chain with finite state space $Y = \{1,2, \dots n\}$, where states are strictly ordered from most healthy (1) to least healthy (n).[7] Denote the transition intensities by:

$$\lambda_{ij}(t) = \lim_{h \to 0} \frac{1}{h} \mathbb{P}[Y_{t+h} = j | Y_t = i]$$

$$\lambda_{ii}(t) = -\sum_{j \neq i} \lambda_{ij}(t)$$

The mortality rate at time t is defined as

$$\mu(t) = \sum_{j=1}^{n} \overline{\mu}_j(t) \mathbf{1}\{Y_t = j\}$$

where $\left\{\overline{\mu}_j(t)\right\}$ are exogenous and $\mathbf{1}\{Y_t = j\}$ is an indicator variable equal to 1 if the individual is in state j at time t and 0 otherwise. The ordering of health states implies

[6] Philipson and Becker (1998) make the important, but distinct, point that the moral hazard effects of public annuity programs also increase an individual's willingness to pay for longevity gains.

[7] The finite state assumption is only a mild restriction because of the approximation property of Markov chains.

$$\mathbb{E}\left[\exp\left\{-\int_0^t \mu(s)ds\right\}\middle| Y_t = i\right] = \mathbb{E}[S(t)|Y_t = i] > \mathbb{E}[S(t)|Y_t = j]\ \forall i, j > i$$

III.A. The fully annuitized value of life

Even when mortality is stochastic, annuitization breaks the link between consumption growth and mortality risk. We assume a full menu of life annuities is available where consumers can choose consumption streams, $c_{Y_t}(t)$, that depend on the current health state, Y_t. As before, we assume that annuities are actuarially fair.

The consumer's maximization problem is:

$$\max_{c_{Y_t}(t)} \mathbb{E}\left[\int_0^T e^{-\rho t} S(t) u\big(c_{Y_t}(t), q_{Y_t}(t)\big)dt \middle| Y_0\right] \tag{6}$$

$$\text{s.t.}\ \mathbb{E}\left[\int_0^T e^{-rt} S(t) c_{Y_t}(t)dt \middle| Y_0\right] \le \mathbb{E}\left[W_0 + \int_0^T e^{-rt} S(t) m_{Y_t}(t)dt \middle| Y_0\right] \equiv \overline{W}(0, Y_0)$$

where current wealth at time t in state i is $\overline{W}(t, i)$. Define the objective function as

$$J(u, i) = \mathbb{E}\left[\int_0^{T-u} e^{-\rho t} \exp\left\{-\int_0^t \mu(u+s)ds\right\} u\big(c_{Y_{u+t}}(u+t), q_{Y_{u+t}}(u+t)\big)dt \middle| Y_u = i\right]$$

Define the optimal value function as

$$V(t, Y_t) = V\big(t, \overline{W}_t, Y_t\big) = \max_{c_{Y_t}(t)}\{J(t, Y_t)\}$$

From standard arguments, we know that if V and its partial derivatives are continuous, then V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}
&\big(\rho + \overline{\mu}_i(t)\big)V(t, i) \\
&\qquad = \max_{c_i(t)}\Bigg\{u\big(c_i(t), q_i(t)\big) \\
&\qquad + \sum_{k=1}^{n} \frac{\partial V(t, i)}{\partial W(t, k)}\Bigg[\big(r + \overline{\mu}_i(t)\big)\overline{W}(t, k) - c_{Y_k}(t) \\
&\qquad + \sum_{l \ne k} \lambda_{kl}(t)\big[\overline{W}(t, k) - \overline{W}(t, l)\big]\Bigg] + \frac{\partial V(t, i)}{\partial t} + \sum_{j \ne i} \lambda_{ij}(t)[V(t, j) - V(t, i)]\Bigg\}
\end{aligned} \tag{7}$$

We are interested in understanding how optimal consumption, and thus the value of life, changes over the life-cycle in this problem. In order to derive analytic expressions, we follow Parpas and Webster (2013), who demonstrate that it is possible to formulate a stochastic optimization problem as a deterministic problem that takes $V(t, j), j \ne i$, along with the corresponding optimal policies, as exogenous.

Claim 1:

Define $\tilde{S}(i,t) \equiv \exp\left\{-\int_0^t \overline{\mu}_i(s) + \sum_{j\neq i} \lambda_{ij}(s)\, ds\right\}$. The optimal value function, $\tilde{V}_t(\overline{W}_u, i) = e^{-\rho t}\tilde{S}(i,t)V_t(\overline{W}_t, i)$, for the following auxiliary deterministic optimization problem also satisfies the HJB given by equation (7):

$$V_0(\overline{W}_0, i) = \max_{c_i(t)}\left[\int_0^T e^{-\rho t}\tilde{S}(i,t)\left(u(c_i(t), q_i(t)) + \sum_{j\neq i}\lambda_{ij}(t)V\left(t, \overline{W}(t,j)\right)\right)dt\right] \tag{8}$$

$$\text{s.t.} \frac{\partial \overline{W}(t,i)}{\partial t} = \big(r + \mu(t)\big)\overline{W}(t,i) - c_i(t) + \sum_{j\neq i}\lambda_{ij}(t)\big[\overline{W}(t,i) - \overline{W}(t,j)\big]$$

Proof: see appendix

As shown in Bertsekas (2005), the Hamiltonian for the maximization problem (8) is:

$$\begin{aligned} H(\overline{W}_t, c_i(t), p_t) = e^{-\rho t}\tilde{S}(i,t)&\left(u(c_i(t), q_i(t)) + \sum_{j\neq i}\lambda_{ij}(t)V\left(t, \overline{W}(t,j)\right)\right) \\ &+ \sum_{k=1}^{n} p_t^{(k)}\left[\big(r + \overline{\mu}_k(t)\big)\overline{W}(t,k) - c_k(t) + \sum_{l\neq k}\lambda_{kl}(t)\big[\overline{W}(t,k) - \overline{W}(t,l)\big]\right]\end{aligned} \tag{9}$$

Claim 2:

The consumer's first-order condition for the Hamiltonian (9) is

$$e^{(r-\rho)t}u_c\big(c_{Y_t}(t), q_{Y_t}(t)\big) = \theta$$

Proof of claim 2:

The adjoint equations for the Hamiltonian (9) are:

$$\dot{p}_t^{(i)} = -p_t^{(i)}\left(r + \bar{\mu}_i(t) + \sum_{j\neq i}\lambda_{ij}(t)\right) + \sum_{l\neq i}\lambda_{li}(t)p_t^{(l)} \text{ and}$$

$$\dot{p}_t^{(k)} = e^{-\rho t}\tilde{S}(i,t)\lambda_{ik}(t)\frac{\partial V\left(t, \overline{W}(t,j)\right)}{\partial \overline{W}(t,j)} - p_t^{(k)}\left(r + \bar{\mu}_k + \sum_{l\neq k}\lambda_{kl}(t)\right) + \sum_{l\neq k}\lambda_{lk}(t)p_t^{(l)}$$

for $k \neq i$. Assume that $p_t^{(k)} = 0$, $k \neq i$. (We will verify this at the end of the proof.) Then this implies:

$$p_t^{(i)} = \theta e^{-rt}\tilde{S}(i,t)$$

where θ is a constant. Note also that the first-order condition of the Hamiltonian with respect to $c_i(t)$ is

$$e^{-rt}\tilde{S}(i,t)u_c(c_i(t), q_i(t)) = p_t^{(i)}$$

Setting these last two equations equal to each other then yields the desired result.

To verify that $p_t^{(k)} = 0$, $k \neq i$, note that the first-order condition implies $\partial V(t,i)/\partial W(t,i) = \theta\, e^{(r-\rho)t}$, so that the adjoint equation for $k \neq i$ is

$$\dot{p}_t^{(k)} = \overbrace{-\theta e^{-rt}\tilde{S}(i,t)}^{p_t^{(i)}}\lambda_{ik}(t) + p_t^{(i)}\lambda_{ik}(t) - p_t^{(k)}\left(r + \bar{\mu}_k + \sum_{l\neq k}\lambda_{kl}(t)\right) + \sum_{l\neq\{k,i\}}\lambda_{lk}(t)p_t^{(l)}$$

QED

If we assume that q is independent of the health state, e.g., if full income insurance is available, then we can obtain the life-cycle profile of consumption by differentiating the first-order condition with respect to t. Doing so shows that the dynamics are the same as in the deterministic case:

$$\frac{\dot{c}_{Y_t}}{c_{Y_t}} = \frac{\dot{c}}{c} = \sigma(r-\rho) + \sigma\eta\frac{\dot{q}}{q}$$

This result demonstrates that annuitization insulates the consumer from mortality risk even when mortality is stochastic.

To analyze the value of life, we again let $\delta(t)$ be a perturbation on the mortality intensity with $\int_0^T \delta(t)dt = 1$.

Claim 3:

The marginal utility of life extension takes the same form as in the deterministic case:

$$\frac{\partial \mathbb{E}U}{\partial\varepsilon}\bigg|_{\varepsilon=0} = \mathbb{E}\left[\int_0^T \left[e^{-\rho t}u\big(c_{Y_t}(t), q_{Y_t}(t)\big) + e^{-rt}\theta(m_{Y_t}(t) - c_{Y_t}(t))\right]\left(\int_0^t \delta(s)ds\right)S(t)dt\,\middle|\, Y_0\right]$$

Proof of Claim 3:

The marginal utility of life extension is defined as

$$\frac{\partial \mathbb{E}U}{\partial\varepsilon}\bigg|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\mathbb{E}\left[\int_0^T e^{-\rho t}\exp\left\{-\int_0^t \mu(s) - \varepsilon\delta(s)ds\right\}\left(u\left(c_{Y_t}^{\varepsilon}(t), q_{Y_t}(t)\right)\right)dt\,\middle|\, Y_0\right]\Bigg|_{\varepsilon=0}$$

where $c^{\varepsilon}(t)$ represent the equilibrium variation in $c(t)$ caused by this perturbation. Then

$$\begin{aligned}
\frac{\partial \mathbb{E}U}{\partial\varepsilon}\bigg|_{\varepsilon=0} &= \mathbb{E}\left[\int_0^T e^{-\rho t}\left(\int_0^t \delta(s)ds\right)\exp\left\{-\int_0^t \mu(s)ds\right\}u\big(c_{Y_t}(t), q_{Y_t}(t)\big)dt\,\middle|\, Y_0\right] \\
&\quad + \mathbb{E}\left[\int_0^T e^{-\rho t}\exp\left\{-\int_0^t \mu(s)ds\right\}u_c\left(c_{Y_t}^{\varepsilon}(t), q_{Y_t}(t)\right)\frac{\partial c_{Y_t}^{\varepsilon}(t)}{\partial\varepsilon}\bigg|_{\varepsilon=0}dt\,\middle|\, Y_0\right] \\
&= \mathbb{E}\left[\int_0^T e^{-\rho t}\left(\int_0^t \delta(s)ds\right)\exp\left\{-\int_0^t \mu(s)ds\right\}u\big(c_{Y_t}(t), q_{Y_t}(t)\big)dt\,\middle|\, Y_0\right] \\
&\quad + \theta\mathbb{E}\left[\int_0^T e^{-rt}\exp\left\{-\int_0^t \mu(s)ds\right\}\frac{\partial c_{Y_t}^{\varepsilon}(t)}{\partial\varepsilon}\bigg|_{\varepsilon=0}dt\,\middle|\, Y_0\right]
\end{aligned}$$

Finally, the budget constraint implies

$$0 = \frac{\partial W_0}{\partial\varepsilon}\bigg|_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\mathbb{E}\left[\int_0^T e^{-rt}\exp\left\{-\int_0^t \mu(s) - \varepsilon\delta(s)ds\right\}\left(c_{Y_t}^{\varepsilon}(t) - m_{Y_t}(t)\right)dt\,\middle|\, Y_0\right]\Bigg|_{\varepsilon=0}$$

$$= \mathbb{E}\left[\int_0^T e^{-rt}\left(\int_0^t \delta(s)ds\right)\exp\left\{-\int_0^t \mu(s)ds\right\}\left(c_{Y_t}(t) - m_{Y_t}(t)\right)dt\middle| Y_0\right]$$
$$+ \mathbb{E}\left[\int_0^T e^{-rt}\exp\left\{-\int_0^t \mu(s)ds\right\}\frac{\partial c^{\varepsilon}_{Y_t}(t)}{\partial \varepsilon}\bigg|_{\varepsilon=0} dt\middle| Y_0\right]$$

Plugging this last result into the expression for $\frac{\partial \mathbb{E}U}{\partial \varepsilon}\Big|_{\varepsilon=0}$ then yields the desired result.

QED

Choosing again the Delta-Dirac function for $\delta(\cdot)$ and dividing the result by the marginal utility of wealth, θ, shows that the value of statistical life also takes the same form as in the deterministic case:

$$VSL = \mathbb{E}\left[\int_0^T e^{-rt}\left[\frac{u\left(c_{Y_t}(t), q_{Y_t}(t)\right)}{u_c\left(c_{Y_t}(t), q_{Y_t}(t)\right)} + m_{Y_t}(t) - c_{Y_t}(t)\right]S(t)dt\middle| Y_0\right] \tag{10}$$

These results demonstrate that stochastic mortality, by itself, does not alter the basic insights offered by (Murphy and Topel 2006; Rosen 1988) as long as the full annuitization assumption maintained in their models is met. As we shall see in the next section, however, stochastic mortality matters in an environment where annuities are absent.

This stochastic setting also allows us to derive an expression for value of prevention, i.e., the value of a reduction in the probability of transitioning to a different health state. This is not possible in a deterministic environment, where there is effectively only one health state. For the purposes of this exercise, we will assume that it is only possible to transition to worse health states, i.e., $\lambda_{ij} = 0$ for $j < i$.

The model with deterministic mortality considers only the value of preventing death. The stochastic mortality model expands the set of states to include varying levels of illness, not just death. Thus, we can analyze the value of preventing the transition to a worse health state. For concreteness, suppose that transition to a new health state corresponds to the onset of a new chronic illness like diabetes, hypertension, lung disease, cancer, or the like. We can use this concept to study the value not just of a life saved, but of an illness prevented. We call this the value of a statistical illness (VSI).

To analyze the VSI, let $\delta_{ij}(t)$ be a perturbation on $\lambda_{ij}(t)$, where $\sum_{j\neq i}\int_0^T \delta_{ij}(t)dt = 1$.

Claim 4:

The marginal utility of preventing an illness is given by

$$\left.\frac{\partial \mathbb{E}U}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_0^T \left(e^{-\rho t} u\left(c_{Y_t}(t), q_{Y_t}(t)\right) + \sum_{j \neq i} \lambda_{ij}(t)\, V\left(t, \overline{W}(t,j)\right) \right.$$

$$\left. + \theta e^{-rt} \left(m_{Y_t}(t) - c_{Y_t}(t) - \sum_{j \neq i} \lambda_{ij}(t)\, \overline{W}(t,j) \right) \right) \left(\int_0^t \sum_{j \neq i} \delta_{ij}(s)\, ds \right) \tilde{S}(i,t)$$

$$- \left(e^{-\rho t} \sum_{j \neq i} \lambda_{ij}(t)\, V\left(t, \overline{W}(t,j)\right) - \theta \sum_{j \neq i} \delta_{ij}(t)\, \overline{W}(t,j) \right) \tilde{S}(i,t) dt$$

PROOF: see appendix

The first term in the expression represents the marginal utility of a reduction in the probability of exiting state $Y_t = i$, and is analogous to the expression for $\left.\frac{\partial \mathbb{E}U}{\partial \varepsilon}\right|_{\varepsilon=0}$ for life-extension. The second term represents the loss in marginal utility from the reduction in probability of transitioning to other possible states.

As in the life-extension case, it is helpful to choose the Delta-Dirac function for $\delta(\cdot)$, so that the probability is perturbed at $t = 0$ and remains unaffected otherwise. It is also helpful to consider a reduction in the probability for only one state, j_0, so that $\delta_{ij}(t) = 0\ \forall j \neq j_0$. Dividing the resulting expression by the marginal utility of wealth, θ, then yields what we term the value of statistical illness (VSI):

$$VSI = \mathbb{E}\left[\int_0^T e^{-rt} \left[\frac{u\left(c_{Y_t}(t), q_{Y_t}(t)\right)}{u_c\left(c_{Y_t}(t), q_{Y_t}(t)\right)} + m(t) - c(t) \right] S(t) dt \middle| Y_0 = i \right] - \mathbb{E}\left[\int_0^T e^{-rt} \left[\frac{u\left(c_{Y_t}(t), q_{Y_t}(t)\right)}{u_c\left(c_{Y_t}(t), q_{Y_t}(t)\right)} + m(t) - c(t) \right] S(t) dt \middle| Y_0 = j_0 \right] \tag{11}$$

Thus VSI is the difference in VSL between the two different health states. If state j_0 is death, then VSI simplifies to VSL (see equation 10). It is straightforward to generalize equation (11) to include a treatment that prevents multiple diseases: in that case, VSI is equal to VSL in the healthy state minus the weighted average of the VSL's across the multiple disease states.

This result is consistent with the notion that, all things equal, it is more valuable to prevent serious diseases than mild diseases, as measured by VSL. It also implies that, all things equal, it is more valuable to prevent diseases that occur among young, healthy individuals (i.e., those with a high VSL) than among old, sick individuals.

Interestingly, VSI is not a function of $\lambda_{ij}(t)$. This implies that the (marginal) prevention of two distinct, but equally lethal, diseases is equally valuable, even if one of the diseases is more prevalent than the other. This is not true for diseases that affect quality of life (Lakdawalla, Malani, and Reif 2017).

III.B. The uninsured value of life

The consumer's maximization problem is:

$$\max_{c_{Y_t}(t)} \mathbb{E}\left[\int_0^T e^{-\rho t} S(t) u\big(c_{Y_t}(t), q_{Y_t}(t)\big) dt \middle| Y_0\right]$$

$$\text{s.t.}\, W(0) = W_0,$$

$$W(t) \geq 0, W(T) = 0,$$

$$\dot{W} = rW(t) + m_{Y_t}(t) - c_{Y_t}(t)$$

As before, we will focus on the case where the non-negative wealth constraint does not bind. The optimal value function V satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}\big(\rho + \overline{\mu}_i(t)\big) V(t,i) &\\ = \max_{c_i(t)} \Bigg\{ & u\big(c_i(t), q_i(t)\big) + \frac{\partial V(t,i)}{\partial W(t)} [rW(t) + m(t) - c_i(t)] + \frac{\partial V(t,i)}{\partial t} \\ & + \sum_{j \neq i} \lambda_{ij}(t) [V(t,j) - V(t,i)] \Bigg\}\end{aligned} \tag{12}$$

Claim 5:

As in the deterministic case, one can find an auxiliary deterministic optimization problem whose value function satisfies the HJB (12). The Hamiltonian for this deterministic problem is

$$H(W_t, c_i(t), p_t) = e^{-\rho t} \tilde{S}(i,t) \left(u\big(c_i(t), q_i(t)\big) + \sum_{j \neq i} \lambda_{ij}(t) V(t, j, W(t)\right) + p_t[rW(t) - c_i(t) + m_i(t)]$$

PROOF: see appendix (to be done)

The first-order condition for the Hamiltonian is

$$u_c\big(c_i(t), q_i(t)\big) = \frac{p_t}{e^{-\rho t} \tilde{S}(i,t)} \tag{13}$$

Taking logs, differentiating with respect to t, and rearranging then shows that the life-cycle profile of consumption, conditional on being in state $Y_t = i$, is given by

$$\frac{\dot{c}_i}{c_i} = \sigma(r - \rho) + \sigma\eta \frac{\dot{q}}{q} - \sigma \overline{\mu}_i(t) - \sigma \sum_{j \neq i} \lambda_{ij}(t) \left[1 - \frac{u_c\big(c_j(t), q_j(t)\big)}{u_c\big(c_i(t), q_i(t)\big)} \right] \tag{14}$$

As in the deterministic case, the rate of change is a declining function of the individual's current mortality rate, $\overline{\mu}_i(t)$: removing the annuity market "pulls consumption earlier" in the life-cycle. Unlike in the deterministic case, there is now an additional source of mortality risk, captured by the fourth term in equation (14).

What happens to consumption when the consumer transitions to a different health state? This question cannot be posed in a deterministic setting, and it is here that the stochastic mortality model departs significantly from the traditional, deterministic mortality model. Under stochastic mortality, consumption can exhibit discrete jumps when transitioning to different states. Intuitively, transitioning to a state where the current mortality and future expected mortality are high will shift consumption forward (see Figure 2), and vice versa.

Next, we turn to deriving expressions for the value of life.

Claim 6:

The marginal utility of life extension takes the same form as in the deterministic case:

$$\left.\frac{\partial \mathbb{E}U}{\partial \varepsilon}\right|_{\varepsilon=0} = \mathbb{E}\left[\int_0^T \left[e^{-\rho t}\left(\int_0^t \delta(s)ds\right)S(t)u\big(c_{Y_t}(t), q_{Y_t}(t)\big)\right]dt\,\middle|\,Y_0\right]$$

Proof:

The marginal utility of life extension is defined as

$$\begin{aligned}\left.\frac{\partial \mathbb{E}U}{\partial \varepsilon}\right|_{\varepsilon=0} &= \frac{\partial}{\partial \varepsilon}\mathbb{E}\left[\int_0^T e^{-\rho t}\exp\left\{-\int_0^t \mu(s)-\varepsilon\delta(s)ds\right\}\left(u\left(c^{\varepsilon}_{Y_t}(t), q_{Y_t}(t)\right)\right)dt\,\middle|\,Y_0\right]\Bigg|_{\varepsilon=0}\\ &= \mathbb{E}\left[\int_0^T e^{-\rho t}\left(\int_0^t \delta(s)ds\right)S(t)\left(u\left(c_{Y_t}(t), q_{Y_t}(t)\right)\right)dt\,\middle|\,Y_0\right]\\ &\qquad + \mathbb{E}\left[\int_0^T e^{-\rho t}S(t)\left(u_c\left(c^{\varepsilon}_{Y_t}(t), q_{Y_t}(t)\right)\right)\left.\frac{\partial c^{\varepsilon}_{Y_t}(t)}{\partial \varepsilon}\right|_{\varepsilon=0}dt\,\middle|\,Y_0\right]\end{aligned}$$

where $c^{\varepsilon}(t)$ represent the equilibrium variation in $c(t)$ caused by this perturbation. We conclude the proof by showing that the latter term is equal to 0:

$$\begin{aligned}&\mathbb{E}\left[\int_0^T e^{-\rho t}S(t)\left(u_c\left(c^{\varepsilon}_{Y_t}(t), q_{Y_t}(t)\right)\right)\left.\frac{\partial c^{\varepsilon}_{Y_t}(t)}{\partial \varepsilon}\right|_{\varepsilon=0}dt\,\middle|\,Y_0\right]\\ &\qquad = \mathbb{E}\left[\int_0^T \mathbb{E}\left[e^{-\rho t}\exp\left\{-\int_0^t \mu(s)ds\right\}\middle|\,Y_t\right]\left(u_c\left(c^{\varepsilon}_{Y_t}(t), q_{Y_t}(t)\right)\right)\left.\frac{\partial c^{\varepsilon}_{Y_t}(t)}{\partial \varepsilon}\right|_{\varepsilon=0}dt\,\middle|\,Y_0\right]\\ &\qquad = \theta\mathbb{E}\left[\int_0^T e^{-rt}\left.\frac{\partial c^{\varepsilon}_{Y_t}(t)}{\partial \varepsilon}\right|_{\varepsilon=0}dt\,\middle|\,Y_0\right] = \theta\frac{\partial}{\partial \varepsilon}\mathbb{E}\left[\int_0^T e^{-rt}c^{\varepsilon}_{Y_t}(t)dt\,\middle|\,Y_0\right]\Bigg|_{\varepsilon=0}\\ &\qquad = \theta\frac{\partial}{\partial \varepsilon}\left[W_0 + \int_0^T e^{-rt}m(t)dt\right]\Bigg|_{\varepsilon=0} = 0\end{aligned}$$

QED

Choosing once again the Delta-Dirac function for $\delta(\cdot)$ and dividing the result by the marginal utility of wealth at time $t = 0$ shows that the value of statistical life also takes the same form as in the deterministic case:

$$VSL = \frac{\mathbb{E}\left[\int_0^T\left[e^{-\rho t}S(t)u\big(c_{Y_t}(t), q_{Y_t}(t)\big)\right]dt\,\middle|\,Y_0\right]}{\mathbb{E}\left[e^{(r-\rho)t}S(t)u_c\big(c_{Y_t}(t), q_{Y_t}(t)\big)\middle|Y_0\right]} = \mathbb{E}\left[\int_0^T e^{-rt}\frac{u\big(c_{Y_t}(t), q_{Y_t}(t)\big)}{u_c\left(c_{Y_t}(t), q_{Y_t}(t)\right)}dt\,\middle|\,Y_0\right]$$

Claim 7:

The marginal utility of preventing an illness is given by:

$$\frac{\partial \mathbb{E}U}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \int_0^T e^{-\rho t}\tilde{S}(i,t)\left[\int_0^t \sum_{j\neq i} \delta_{ij}(s)\, ds \left(u\big(c_{Y_t}(t), q_{Y_t}(t)\big) + \sum_{j\neq i} \lambda_{ij}(t)\, V\left(t, \overline{W}(t,j)\right)\right) - \sum_{j\neq i} \lambda_{ij}(t)\, V\left(t, \overline{W}(t,j)\right)\right] dt$$

Proof: See appendix

Choosing the Delta-Dirac function for disease j as done previously then yields the expression for the value of statistical illness:

$$VSI = \frac{V(0, W(0), i) - V(0, W(0), j)}{u_c\left(c_{Y_i}(0), q_{Y_i}(0)\right)} = VSL(i) - VSL(j)\frac{u_c\left(c_{Y_j}(0), q_{Y_j}(0)\right)}{u_c\left(c_{Y_i}(0), q_{Y_i}(0)\right)}$$

If we assume that state $j > i$, and that consumption jumps when entering a worse health state, then $u_c\left(c_{Y_j}(0), q_{Y_j}(0)\right) < u_c\left(c_{Y_i}(0), q_{Y_i}(0)\right)$.

IV. ESTIMATES OF THE VALUE OF LIFE

IV.A. Calibration framework

We will work with the discrete time analogue of our model and abstract from the role of quality of life, since aggregate, nationally representative data on quality-of-life trends are not generally available. Health states are ordered from most healthy (Y_1) to least healthy (Y_n). Denote the transition probabilities by:

$$p_{ij}(t) = \mathbb{P}[Y_t = j | Y_{t-1} = i]$$

As in the continuous time model, the mortality rate at time t depends on the individual's health state:

$$q_t = \sum_{j=1}^{n} \overline{q}_t^j \mathbf{1}\{Y_t = j\}$$

where $\left\{\overline{q}_t^j\right\}$ are given and $\mathbf{1}\{Y_t = j\}$ is an indicator variable equal to 1 if the individual is in state j at time t and 0 otherwise. The probability of surviving from period t to period s is denoted as $S_t(s)$, where

$$S_t(t) = 1,$$

$$S_t(s) = S_t(s-1)(1 - q_{s-1}), s > t$$

Let c_t be consumption in period t, w_t (non-annuitized) wealth, ρ the utility discount rate, and r the interest rate. Assume that in each period the consumer receives an exogenously determined income,

m_t, and that the maximum lifespan of a consumer is T (i.e., $q_T = 1$).[8] Our baseline model assumes there is no bequest motive, although we relax this assumption in a later exercise.

The consumer's maximization problem is

$$\max_{\{c_t\}} \mathbb{E}_0 \left[\sum_{t=0}^{T} e^{-\rho t} S_t(s) u(c_t) \right]$$

subject to

$$w_0 \text{ given}$$
$$w_t \geq 0$$
$$w_{t+1} = (w_t + m_t - c_t) e^r$$

We assume that utility takes a CRRA form:

$$u(c) = \frac{c^{1-\gamma} - \underline{c}^{1-\gamma}}{1-\gamma}$$

We have normalized the utility of death at zero. The consumer receives positive utility if she consumes an amount greater than $\underline{c}$, which represents a subsistence level of consumption. Consuming an amount less than $\underline{c}$ generates utility that is worse than death.

The parameter γ is the inverse of the elasticity of intertemporal substitution, an important determinant of the value of life and the value of annuitization. We set $\gamma = 0.95$ in our analyses. As points of reference, Murphy and Topel (2006) argue that γ is approximately equal to 1, but Brown (2001) uses survey data to estimate a mean value of $\gamma = 3.95$.

We employ dynamic programming techniques to solve for the optimal consumption path. The value function is defined as:

$$V_t(w_t, i) = \max_{\{c_t\}} \mathbb{E} \left[\sum_{s=t}^{T} e^{-\rho(s-t)} S_t(s) u(c_s) \middle| Y_t = i \right]$$

We can use the value function to rewrite the optimization problem as a recursive Bellman equation:

$$V_t(w_t, i) = \max_{\{c_t\}} \left[u(c_t) + \frac{1-q_t}{e^{\rho}} \sum_{j=1}^{N} p_{ij}(t) V_{t+1}((w_t + m_t - c_t)e^r, j) \right]$$

Once we have solved for the optimal consumption path, we can use the analytical formulas derived in the previous section to calculate the value of life. The value of a statistical life-year from the perspective of current time is equal to

[8] Hubbard, Skinner, and Zeldes (1995) show that failing to include a "welfare floor" in the budget constraint causes life-cycle models to overestimate savings for low-income households. Our calibration exercises model median-income individuals, however, for whom this issue is less important.

$$v_t = {c_t}^{\gamma}\left(\frac{{c_t}^{1-\gamma} - \underline{c}^{1-\gamma}}{1-\gamma}\right) = \frac{c_t}{1-\gamma} - \frac{{c_t}^{\gamma}\underline{c}^{1-\gamma}}{1-\gamma}$$

Under complete annuitization, the value of a statistical life at time t is equal to

$$VSL_t^* = \frac{1}{S_t}\sum_{\tau=t}^{T} e^{-r\tau}\mathbb{E}[S_\tau v_\tau]$$

Under incomplete annuitization, this expression becomes

$$VSL_t = \frac{1}{S_t}\sum_{\tau=t}^{T} e^{-r\tau}\mathbb{E}[v_\tau]$$

We assume throughout that $r = \rho = 0.03$ (Siegel 1992).

We also make the following simplifying assumptions, which allow us to compute an exact solution to this problem: (1) Income, m_t, is not survival contingent; (2) negative wealth is allowed; and (3) $\underline{c} = 0$. Assumptions (1) and (2) imply an equivalence between income and wealth, so we ignore income in our analyses and set initial wealth equal to $500,000 (at age 50). Assumption (3) allows for an exact solution to the recursive problem, and also simplifies the calculations of VSL:

$$VSL_t^* = \frac{\overline{c}}{(1-\gamma)S_t}\sum_{\tau=t}^{T} e^{-r\tau}\mathbb{E}[S_\tau]$$

$$VSL_t = \frac{1}{(1-\gamma)S_t}\sum_{\tau=t}^{T} e^{-r\tau}\mathbb{E}[c_\tau]$$

We plan to relax these three assumptions, and allow for bequest motives, in the next revision of this paper.

We obtain mortality data from the Future Elderly Model (FEM), a widely published microsimulation model that employs nationally representative data from the Health and Retirement Study (Michaud et al. 2011; Goldman et al. 2005; Lakdawalla, Goldman, and Shang 2005; Goldman et al. 2009; Lakdawalla et al. 2009; Goldman et al. 2013; Michaud et al. 2012; Goldman et al. 2010). The FEM uses real-world risks of disease incidence, and mortality rates by disease state, in order to estimate longevity for people over the age of 50 with different comorbid conditions.[9] This is quite useful for our current purposes, because it provides us with an empirically relevant set of estimates for what mortality risk looks like under different disease states.

Each health state in the FEM corresponds to the number (0-3) of impaired activities of daily living (ADL) and the number of chronic conditions (0-4), for a total of $4 \times 5 = 20$ health states. For each health state and age, the FEM estimates the probability of dying, and the probability of transitioning to each of the other health states in the next year. The FEM model is estimated separately by sex (male or female) and smoking status (smoker or nonsmoker).

[9] A description of its methodology is available at healthpolicy.app.box.com/FEMTechdoc.

IV.B. Results

We are interested in understanding how the value of life differs along two different dimensions: (1) fully annuitized vs uninsured; and (2) deterministic vs stochastic mortality. As shown in the theory section, full annuitization causes the rate of change in consumption and the (average) value of life to be the same for both the deterministic and stochastic cases. Thus, we will show results for three cases: fully annuitized; uninsured, deterministic mortality; and uninsured, stochastic mortality.

Figure 3 shows VSLY, by age, for these three cases. The red line, which corresponds to a fully annuitized consumer, is flat, demonstrating that annuitization allows a consumer to perfectly smooth her consumption (and thus her value of life) over the life cycle. Compared to this baseline, the value of life is shifted forward when the consumer is exposed to mortality risk (green and blue lines). Interestingly, the shift is more pronounced in the uninsured deterministic case (green line) than in the stochastic case (blue line). Intuitively, consumption is shifted into the future in the stochastic case because this allows consumers to take advantage of the ability to increase their consumption later in life, in the event that they fall ill.

The values displayed in Figure 3 for the stochastic case have been averaged across the 20 different health states, weighted by the respective probabilities. Unlike in the deterministic cases, the stochastic case allows one to calculate VSLY by age for particular health states. Figure 4 provides an example where a healthy individual (health state 1) suffers a health shock at 70 that transitions her to health state 12. As expected, her value of life shifts forward significantly at that age, compared to an individual who never suffers a health shock (dashed line).

V. CONCLUSION

The economic theory surrounding the value of life has been put to many important uses. Yet, like most theories, it suffers from a few anomalies that appear at odds with intuition, common sense, or empirical facts. We have demonstrated that several of these anomalies are easily explained without abandoning the standard framework, simply by relaxing its strong assumptions around the completeness of annuity markets. Moreover, relaxing this assumption generates a number of new predictions with implications for health policy and behavior. In particular, we show diminishing returns to life-extension. A given gain in longevity is more valuable to a consumer that has less life remaining, and vice-versa. In addition, we demonstrate an interaction between annuity policy and health policy: Completing the annuity market may significantly increase the value of life-extension, especially for the elderly.

Diminishing returns to life-extension yield a number of subsidiary predictions. First, the onset of fatality risk creates a short-term spike in the value of statistical life, but the lucky ones that survive such risks over the long-term value life-years by less. This explains the anecdotal perception of desperation among the newly diagnosed cancer patient, along with complacency among long-term survivors who may feel they are "playing with house money." Second, the value of a statistical life-year will tend to vary across types of risk, not just across types of people. It is more valuable to add one month of life for a patient facing a highly fatal disease than for one facing a much milder ailment. Third, contrary to the old saying, treatment might be more valuable than prevention, at least when the expected gain in longevity is held fixed. Many healthcare researchers decry the unwillingness of policymakers and patients to invest in preventive activities (Dranove 1998; Finkelstein and Brown 2006). Our findings suggest there may be an economic basis for this unwillingness, other than market failures, time-discounting, or myopia. Finally, public programs that expand the market for annuities might simultaneously boost the demand for lifeextending technologies. Intuitively, annuities calm consumer fears about outliving their wealth and thus enable more aggressive investments in life-extension. Viewed differently, our results also show that market failures in annuities affect the value of statistical life, and thus the socially optimal level of health care spending.

Our analysis raises a number of important questions for further research. First, how does the value of longevity vary with endogenous demand for quality of life? Elsewhere, we have studied how incomplete health insurance enhances the value of medical technology that improves quality of life, because such technology acts as insurance by compressing the difference in utility between the sick and healthy states (Lakdawalla, Malani, and Reif 2017). Less clear is how demands for the quantity and quality of life interact with financial market incompleteness of various kinds. Second, what does the generalized value of life model mean for the value of different kinds of medical technologies? For instance, the model suggests that short-term survival gains for high-risk diseases are more valuable than previously believed, but very long-term survival gains might actually be less valuable than previously believed. Finally, what are the implications for the empirical literature on the value of statistical life? Empirical analysis has typically proceeded under the assumption that different kinds of mortality risk are all valued the same way, as long as they imply similar changes in the probability of dying (Viscusi and Aldy 2003; Hirth et al. 2000; Mrozek and Taylor 2002). Our framework casts doubt on this assumption and suggests the need for a more nuanced empirical approach. This missing insight may be one reason for the widely disparate estimates in the empirical literature on the value of a statistical life.

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VII. APPENDIX

VI.A. Mathematical appendix

Proof of Claim 1:

In a similar context, Parpas and Webster (2013) show that it is possible to formulate a stochastic problem as a deterministic optimization problem that takes the J's in other states as given. Proceeding similarly, we can write the objective function as:

$$J(u,i) = \int_0^{T-u} e^{-\rho t} \exp\left\{-\int_0^t \overline{\mu}_i(u+s) + \sum_{j\neq i} \lambda_{ij}(u+s)\, ds\right\}\Bigg(u(c_i(u+t), q_i(u+t)) + \sum_{j\neq i} \lambda_{ij}(u+t) J(u+t,j)\Bigg) dt$$

Similarly, current wealth at time u in state i, including the value of future labor income, pays for future consumption such that:

$$\overline{W}(u,i) = \mathbb{E}\left[\int_0^{T-u} e^{-rt} \exp\left\{-\int_0^t \mu(u+s)ds\right\} c_{Y_{u+t}}(u+t)dt \middle| Y_u = i\right]$$

$$= \int_0^{T-u} e^{-rt} \exp\left\{-\int_0^t \mu(u+s) + \sum_{j\neq i} \lambda_{ij}(u+s)\, ds\right\}\left(c_i(u+t) + \sum_{j\neq i} \lambda_{ij}(u+t)\, \overline{W}(u+t,j)\right) dt$$

This in turn implies

$$\frac{\partial \overline{W}(t,i)}{\partial t} = \left(r + \overline{\mu}_i(t)\right) \overline{W}(t,i) - c_i(t) + \sum_{j\neq i} \lambda_{ij}(t)\left[\overline{W}(t,i) - \overline{W}(t,j)\right]$$

Hence, for $V\left(t, \overline{W}_t, Y_t\right) = \max_{c_{Y_t}(t)} \{J(t, Y_t)\}$, we obtain the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}
&\left(\rho + \overline{\mu}_i(t)\right) V(t,i) \\
&\qquad = \max_{c_i(t)} \left\{ u\left(c_i(t), q_i(t)\right) \right. \\
&\qquad + \sum_{k=1}^{n} \frac{\partial V(t,i)}{\partial W(t,k)} \left[\left(r + \overline{\mu}_i(t)\right) \overline{W}(t,k) - c_{Y_k}(t) + \sum_{l\neq k} \lambda_{kl}(t)\left[\overline{W}(t,k) - \overline{W}(t,l)\right]\right] \\
&\qquad \left. + \frac{\partial V(t,i)}{\partial t} + \sum_{j\neq i} \lambda_{ij}(t)[V(t,j) - V(t,i)] \right\}
\end{aligned}$$

As in Parpas and Webster (2013), the HJB corresponds to the deterministic auxiliary problem that takes $V(t,j) = V\left(t, \overline{W}(t,j), j\right), j \neq i$—and corresponding optimal policies—as exogenous inputs, which themselves are solutions to a stochastic optimal control problem. The solution may depend on the state vector $\left(\overline{W}(t,1), \ldots, \overline{W}(t,n)\right)$, even though the exogenous $V(t,j) = V\left(t, \overline{W}(t,j), i\right)$ will depend solely on the starting wealth in state $\overline{W}(t,j)$.

More precisely, consider the deterministic optimization problem:

$$V_0\left(\overline{W}_0, i\right) = \max_{c_i(t)} \left\{\int_0^T e^{-\rho t} \tilde{S}(i,t)\left(u(c_i(t), q_i(t)) + \sum_{j\neq i} \lambda_{ij}(t) V\left(t, \overline{W}(t,j)\right)\right) dt\right\}$$

subject to the law of motion of wealth derived above and where $\tilde{S}(i,t) = \exp\left\{-\int_0^t \overline{\mu}_i(s) + \sum_{j\neq i} \lambda_{ij}(s)\, ds\right\}$. Denote the optimal value-to-go as

$$\tilde{V}_u\left(\overline{W}_u, i\right) = \max_{c_i(t)} \left\{\int_u^T e^{-\rho t} \tilde{S}(i,t)\left(u(c_i(t), q_i(t)) + \sum_{j\neq i} \lambda_{ij}(t) V\left(t, \overline{W}(t,j)\right)\right) dt\right\}$$

Setting $\tilde{V}_t(\overline{W}_u, i) = e^{-\rho t}\tilde{S}(i,t)V_t(\overline{W}_t, i)$ then demonstrates that V satisfies the HJB.

QED

Proof of Claim 4:

Working from equation (8) in the text, the marginal utility of prevention is given by

$$\frac{\partial \mathbb{E}U}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\int_0^T e^{-\rho t}\exp\left\{-\int_0^t \overline{\mu}_i(s) + \sum_{j\neq i}\left[\lambda_{ij}(s) - \varepsilon\delta_{ij}(s)\right]ds\right\}\left(u\left(c^{\varepsilon}_{Y_t}(t), q_{Y_t}(t)\right)\right.$$
$$\left. + \sum_{j\neq i}\left[\lambda_{ij}(t) - \varepsilon\delta_{ij}(t)\right]V\left(t, \overline{W}^{\varepsilon}(t,j)\right)\right)dt\Bigg|_{\varepsilon=0}$$

where $c^{\varepsilon}(t)$ and $\overline{W}^{\varepsilon}(t,j)$ represent the equilibrium variations in $c(t)$ and $\overline{W}(t,j)$ caused by this perturbation. This yields

$$\frac{\partial \mathbb{E}U}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \int_0^T e^{-\rho t}\left(\int_0^t \sum_{j\neq i}\delta_{ij}(s)\,ds\right)\tilde{S}(i,t)\left(u\left(c_{Y_t}(t), q_{Y_t}(t)\right) + \sum_{j\neq i}\lambda_{ij}(t)\,V\left(t, \overline{W}(t,j)\right)\right)$$
$$- e^{-\rho t}\tilde{S}(i,t)\sum_{j\neq i}\delta_{ij}(t)\,V\left(t, \overline{W}(t,j)\right)$$
$$+ e^{-\rho t}\tilde{S}(i,t)\left(\underbrace{u_c\left(c_{Y_t}(t), q_{Y_t}(t)\right)}_{\theta e^{-(r-\rho)t}}\frac{\partial c^{\varepsilon}_{Y_t}(t)}{\partial \varepsilon}\bigg|_{\varepsilon=0}\right.$$
$$\left. + \sum_{j\neq i}\lambda_{ij}(t)\underbrace{V_W\left(t, \overline{W}(t,j)\right)}_{\theta e^{-(r-\rho)t}}\frac{\partial \overline{W}^{\varepsilon}(t,j)}{\partial \varepsilon}\bigg|_{\varepsilon=0}\right)dt$$

Next, we note that the budget constraint implies

$$0 = \frac{\partial W_0}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\int_0^T e^{-rt}\exp\left\{-\int_0^t \overline{\mu}_i(s) + \sum_{j\neq i}\left[\lambda_{ij}(s) - \varepsilon\delta_{ij}(s)\right]ds\right\}\left(c^{\varepsilon}_{Y_t}(t) - m_{Y_t}(t\right.$$
$$\left. + \sum_{j\neq i}\left[\lambda_{ij}(t) - \varepsilon\delta_{ij}(t)\right]\overline{W}^{\varepsilon}(t,j)\right)dt\Bigg|_{\varepsilon=0}$$

$$
\begin{aligned}
= \int_0^T e^{-rt} \left(\int_0^t \sum_{j \neq i} \delta_{ij}(s)\, ds \right) \tilde{S}(i,t) \left(c_{Y_t}(t) - m_{Y_t}(t + \sum_{j \neq i} \lambda_{ij}(t) \overline{W}(t,j) \right) dt \\
- e^{-rt} \tilde{S}(i,t) \sum_{j \neq i} \delta_{ij}(t)\, \overline{W}(t,j) \\
+ e^{-rt} \tilde{S}(i,t) \left(\frac{\partial c_{Y_t}^{\varepsilon}(t)}{\partial \varepsilon} \bigg|_{\varepsilon=0} + \sum_{j \neq i} \delta_{ij}(t) \frac{\partial \overline{W}^{\varepsilon}(t,j)}{\partial \varepsilon} \bigg|_{\varepsilon=0} \right) dt
\end{aligned}
$$

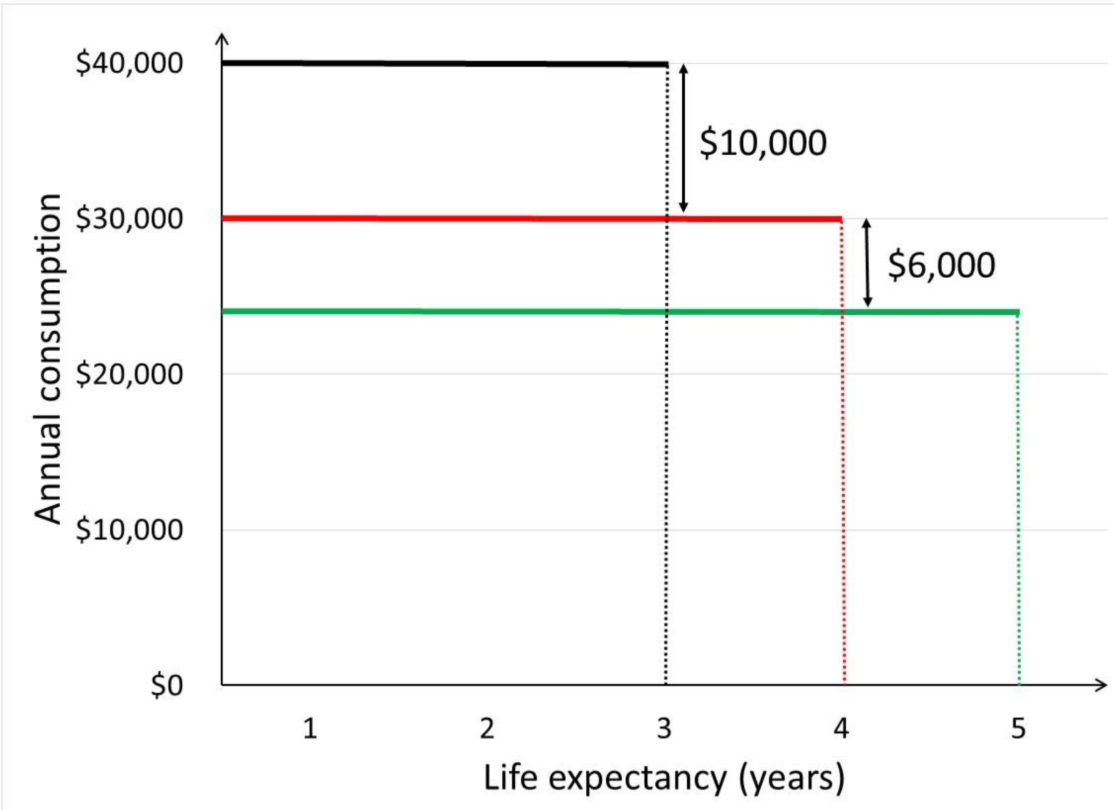
Substituting in then yields

$$
\begin{aligned}
\frac{\partial \mathbb{E} U}{\partial \varepsilon} \bigg|_{\varepsilon=0} = \int_0^T e^{-\rho t} \left(\int_0^t \sum_{j \neq i} \delta_{ij}(s)\, ds \right) \tilde{S}(i,t) \left(u\left(c_{Y_t}(t), q_{Y_t}(t) \right) + \sum_{j \neq i} \lambda_{ij}(t)\, V\left(t, \overline{W}(t,j) \right) \right) \\
- e^{-\rho t} \tilde{S}(i,t) \sum_{j \neq i} \delta_{ij}(t)\, V\left(t, \overline{W}(t,j) \right) \\
- \theta e^{-rt} \left(\int_0^t \sum_{j \neq i} \delta_{ij}(s)\, ds \right) \tilde{S}(i,t) \left(c_{Y_t}(t) - m_{Y_t}(t + \sum_{j \neq i} \lambda_{ij}(t) \overline{W}(t,j) \right) \\
+ \theta e^{-rt} \tilde{S}(i,t) \sum_{j \neq i} \delta_{ij}(t)\, \overline{W}(t,j) dt
\end{aligned}
$$

QED

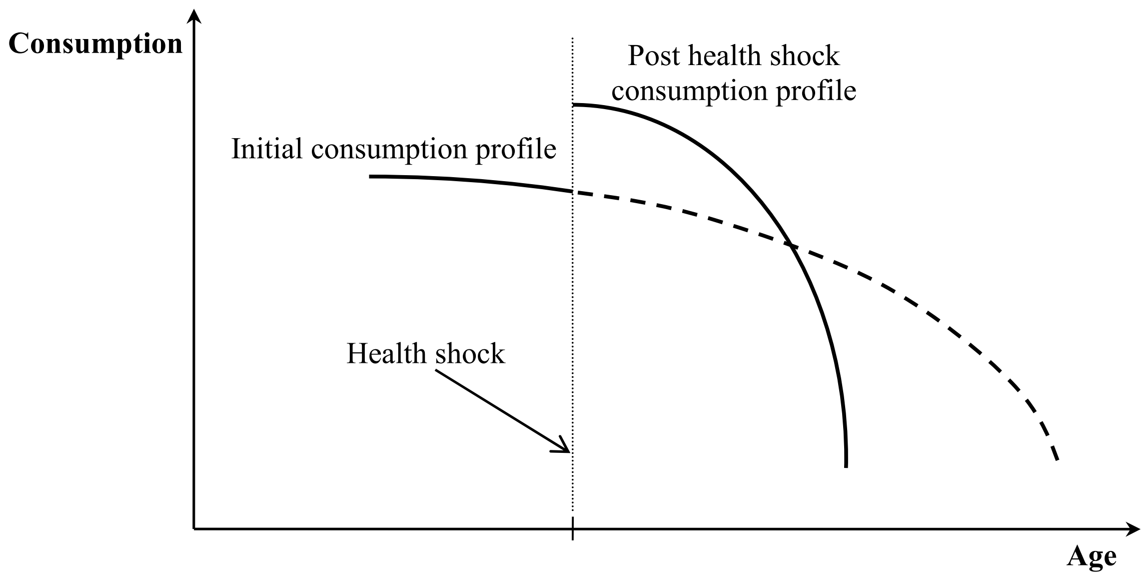
VIII. TABLES AND FIGURES

Figure 1. Annual consumption for consumer with $120,000 in wealth and a life expectancy of 3, 4, or 5 years



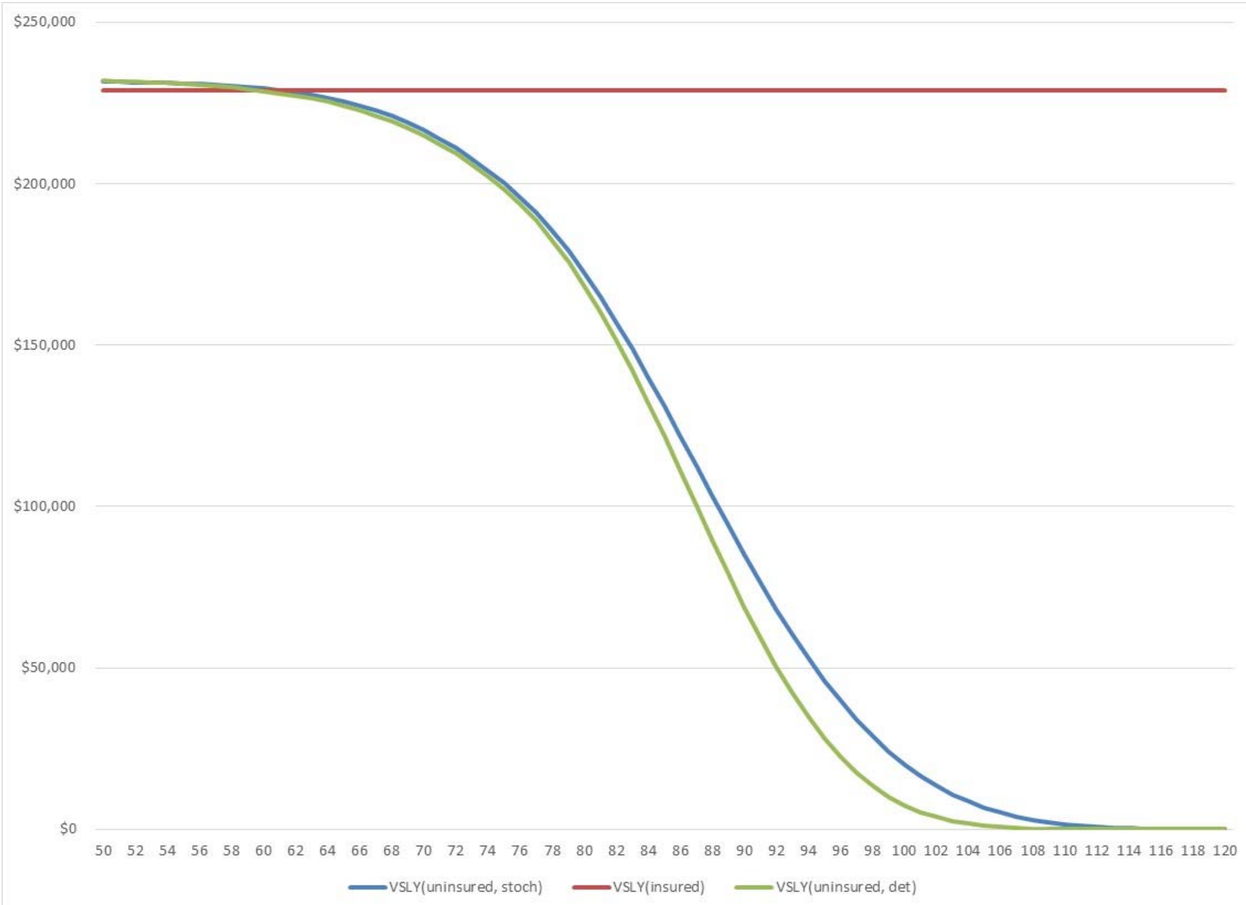
Notes: This illustrative example assumes there is no uncertainty in mortality and the consumer discount rate is equal to the interest rate, so that the optimal consumption profile is flat. Increasing life expectancy from 3 to 4 years reduces annual consumption by $10,000. Increasing it from 4 to 5 years reduces annual consumption by $6,000.

Figure 2. Effect of a negative health shock on life-cycle consumption when the consumer does not have an annuity



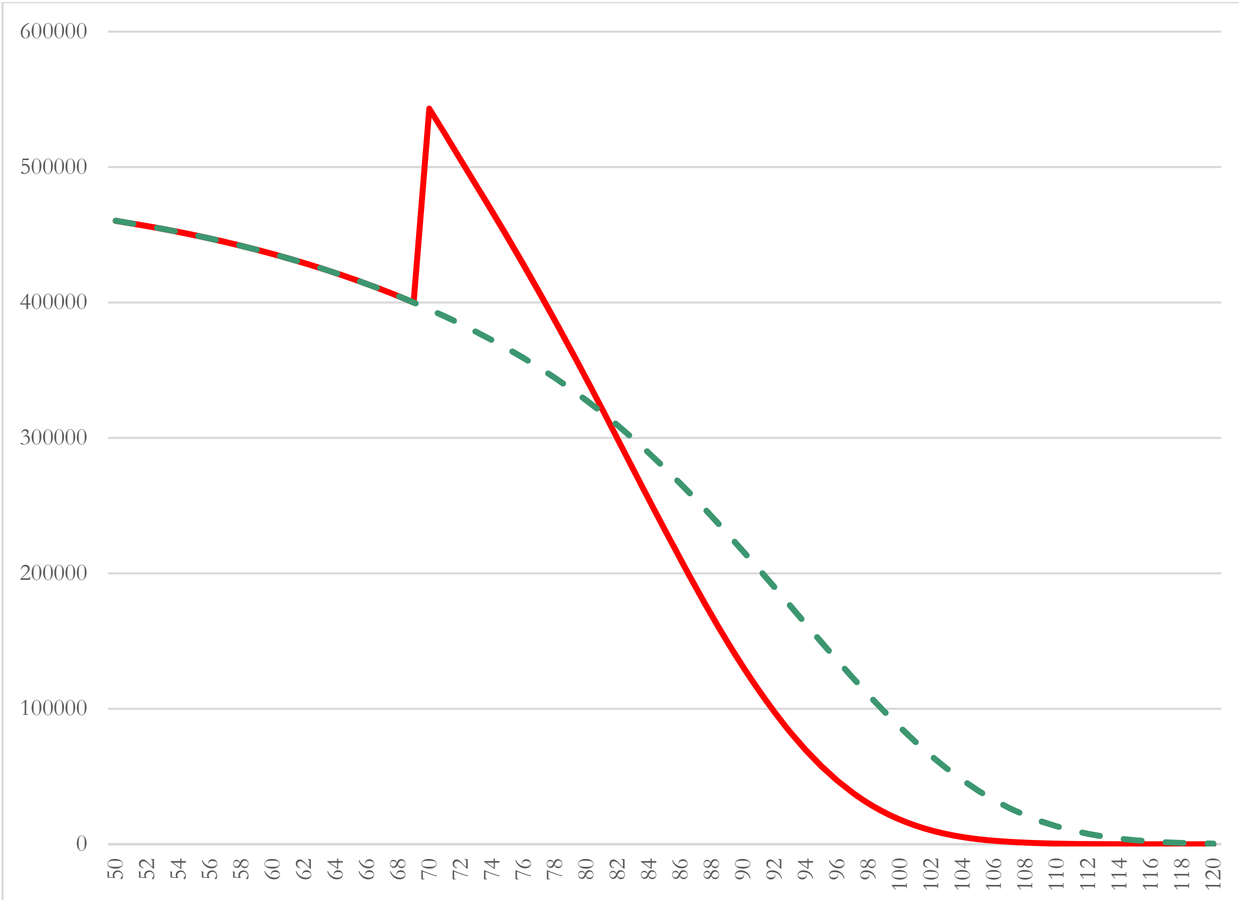
Notes: The health shock increases annual mortality or, equivalently, reduces expected survival. This causes the consumer to "spend down" her wealth.

Figure 3. Value of statistical life-year (VSLY), by age.



Note: The red line displays VSLY, by age, for a fully insured (annuitized) consumer. The green (blue) line corresponds to an uninsured consumer in a deterministic (stochastic) setting. (For the stochastic case, the figure reports VSLY that has been probability-weighted across the 20 different health states.) At age 50, VSL(insured) = VSL(uninsured, det) < VSL(uninsured, stoch). Comparing the insured case (red line) to the uninsured cases (blue and green lines) demonstrates that the value of life (and consumption) shifts forward when the consumer is exposed to mortality risk. Consumption shifts forward more in the uninsured deterministic case than in the uninsured stochastic case.

Figure 4. VSLY by age, health shock at age 70



Notes: This red line displays VSLY for a healthy uninsured consumer who suffers a health shock (moving from health state 1 to health state 12 out of 20) at age 70. The dashed line displays VSLY for a health uninsured consumer who never suffers a health shock.