Optimal Public Debt with Life Cycle Motives *

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Abstract

In a seminal paper, Aiyagari and McGrattan (1998) use an infinitely lived agent model with incomplete markets to show that the U.S. government should hold a large amount of public debt. This paper revisits their result using a life cycle model and finds that the government should optimally *save*, on the order of magnitude of 60% of output. In the infinitely lived agent model, government debt increases the interest rate, encourages agents to save and relaxes liquidity constraints. In the life cycle model, we find this mechanism is quantitatively smaller since agents begin life with no wealth but quickly accumulate assets not only to buffer against income shocks but also to finance post-retirement consumption. Therefore, optimal policy switches from public debt to public savings.

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1 Introduction

In the decades preceding the Great Recession, debt to GDP ratios in advanced economies averaged over 40%. Moreover, only three advanced economies held a net level of public savings. Motivated by these basic facts, this paper asks: What is the optimal quantity of public debt that a government should hold?

In their seminal work, Aiyagari and McGrattan (1998) provide a quantitative answer to this question. Their framework is the standard incomplete markets model, in which infinitely lived households can only partially insure against the realization of idiosyncratic labor productivity shocks. Imperfect insurance against ex post labor market outcomes admits a role for government policy to improve upon the competitive equilibrium allocation ex ante. Aiyagari and McGrattan (1998) show that a government can enhance liquidity in asset markets by holding more public debt and crowding out productive capital. Asset markets endogenously respond to the relative scarcity of capital by requiring higher rates of return, which encourages households to accumulate a larger stock of precautionary savings. Indeed, Aiyagari and McGrattan (1998) find that the U.S. quantity of public debt (two-thirds the size of GDP) is optimal.

This paper revisits the question of optimal public debt by departing from the standard incomplete markets model along one key dimension: we abandon the assumption of an infinitely lived agent in favor of an explicit life cycle model. Importantly, we find that introducing a life cycle changes equilibrium savings patterns and therefore leaves less of a role for a government to improve equilibrium allocations through the insurance channel highlighted by Aiyagari and McGrattan (1998). In fact, we find that the particular nature of life cycle wealth accumulation strongly drives government policy in the other direction, toward public savings instead of public debt. Accordingly, we determine that incorporating life cycle features has a quantitatively important effect on optimal debt policy.

We obtain our results by solving for optimal policy in two model economies that are calibrated to match macroeconomic aggregates during the post-war U.S. economy. The first model is similar to that in Aiyagari and McGrattan (1998) and includes infinitely lived agents. The second model includes life cycle features such as a realistic age-wage profile, mortality risk, retirement and a Social Security program. We find that optimal policy is starkly different in the two models. In the infinitely lived agent model, it is optimal for the government to hold debt equal to approximately 20 percent of output, which is consistent with Aiyagari and McGrattan's (1998) result. In contrast, in the life cycle model, we find that it is optimal for the government to hold public savings equal to almost 60 percent of output. Not only does the optimal debt policy look quite different when one ignores life cycle features but the welfare consequences of ignoring them are non-trivial. We find that if a government implemented the infinitely lived agent model optimal 20% debt-to-output policy in the life cycle model, then life cycle agents would be worse off by nearly 0.5 percent of expected lifetime consumption. Accordingly, we find that incorporating life cycle features changes the answer to the question of whether it is optimal for the government to hold public savings or public debt.

Why do life cycle features generate a drastically different optimal policy than assuming an infinitely lived agent? We find that including life cycle features implies that an agent transitions through distinct lifetime phases that are not prevalent in the infinitely lived agent model. In particular, life cycle model agents begin their life with no savings and enter an *accumulation phase* in which they build a precautionary stock of savings to insurance against income shocks and finance their post-retirement consumption. In middle life, agents may enter a *stationary phase* in which they have accumulated a target level of assets, around which savings fluctuates.[1] Finally, older agents enter a *deaccumulation phase* in which they spend down their savings in anticipation of death. In the infinitely lived agent model, agents only experience the stationary phase. Through a series of counterfactual experiments, we demonstrate that the *accumulation phase* is the key to why life cycle features have such a large effect on optimal policy.

The reason for the large affect from the *accumulation phase* can be seen by examining one of the main mechanisms leading debt to be optimal in the infinitely lived agent model. As Aiyagari and McGrattan (1998) show, higher government debt (or lower government savings) tends to crowd out the stock of productive capital, which leads to a higher interest rate and lower wage. The relatively higher interest rate encourages agents to hold more savings, which in turn helps agents to better insure against labor earnings risk and avoid binding liquidity constraints. This channel is significantly less efficacious in the life cycle model compared to the infinitely lived agent model. In particular, in the infinitely lived agent model, agents do not experience an accumulation phase but instead experience a perpetual stationary phase. If the government holds more public debt, then the *steady state* level of aggregate savings is larger and the average agent has more wealth *ex ante*. In contrast, life cycle agents begin working life with zero wealth and immediately begin saving in the accumulation phase. Thus, although changes in the interest rate may increase the level of savings in the stationary phase for life cycle agents, these agents will still need to forgo consumption in order to invest in savings while build to the stationary phase. Ultimately, this significantly reduces the benefit of government debt from improved liquidity in the life cycle model.

Our paper is related to a well established literature that has examined the optimal level of government debt and savings in quantitative models. Following Aiyagari and McGrattan (1998), a number of studies examine the optimal level of debt in the steady state of an infinitely lived agent model. Floden (2001) finds that increasing government debt can provide welfare benefits if transfers are below optimal levels. Similarly, Dyrda and Pedroni (2016) find that it is optimal for the government to hold debt. However they find that when optimizing both taxes and debt at the same time leads to a smaller level of optimal debt than previous studies. In contrast, Röhrs and Winter (2016) find that when they include a skewed wealth distribution that more closely matches the upper tail of the U.S. wealth distribution, it is optimal for the government to save as opposed to holding debt. In contrast to these papers, we study optimal public debt and savings in a life cycle model in which individuals begin life as liquidity constrained savers and grow to become retirees that run down their savings. We find that this age-dependent savings pattern can lead to different welfare effects from government savings and debt.

[1] In life cycle models where agents live for a short enough span, agents sometimes transition directly from the accumulation phase to the deaccumulation phase skipping this stationary phase. We generally find this to be the case in our baseline life cycle model.

Our paper is also related to recent work by Dyrda and Pedroni (2016), Röhrs and Winter (2016), and Desbonnet and Weitzenblum (2012), that finds quantitatively large welfare costs of transitioning between steady states after a change in public debt. However, these studies focus on models inhabited by infinitely lived agent and do not incorporate the mechanisms prevalent in a life cycle setting. The present study does not consider steady state transitions, and instead focuses on steady state comparisons to more sharply highlight the contribution of the life cycle to the question of optimal debt and welfare.

Lastly, our paper is related to Dávila, Hong, Krusell, and Ríos-Rull (2012), whose work defines constrained inefficiency in a standard incomplete markets model and shows what forces lead to the optimality of a larger aggregate capital stock. Our paper examines a different model and achieves improved welfare through the restriction of government policy to debt policy. Despite these differences, our paper arrives at a similar conclusion that the current U.S. capital stock is too low.

The remainder of this paper is organized as follows. Section 2 illustrates the underlying mechanisms by which optimal government policy interacts with life cycle and infinitely lived agent model features. Section 3 describes the life cycle and infinitely lived agent model environments and defines equilibrium. Section 4 presents the calibration strategy and Section 5 presents quantitative results. Section 6 concludes.

2 Illustration of the Mechanisms

In this section, we discuss the mechanisms that might lead the government to hold debt or savings. We begin by discussing the core inefficiency that provides a welfare improving role for public savings or public debt. Then we turn to why the optimal policy may differ in the two models. Specifically, we discuss how life cycle features generate a pattern of savings over the life cycle that is distinct from average savings in the infinitely lived agent model. Finally, we discuss the main channels by which government debt or savings impacts individual behavior and how the strength of these channels may vary between the life cycle and infinitely lived agent models.

2.1 Pecuniary Externality

Individual agents, who are constrained by incomplete asset markets and borrowing constraints, do not internalize how their decisions impact prices. In such an environment, the price mechanism does not fully work and competitive equilibria are generically inefficient. If agents were to systematically deviate from individual optimization, then equilibrium prices could be attained that improve social welfare.

The government's public savings policy can attain higher welfare than the competitive equilibrium allocation by partially correcting this pecuniary externality. By choosing a public savings (debt) policy, the government *crowds in (out)* the supply of loanable funds that can be directed to firms for investment in productive capital. The government's public savings (debt) policy directly changes the supply (demand) for assets and therefore manipulates the equilibrium interest rate through market clearing conditions.

Because it understands the relationship between public savings and factor prices, the government can implement a welfare improving allocation that individual agents could not attain alone.

To illustrate the nature of this externality, we derive the government's optimality condition for public savings. For ease of explication, consider a simplified model in which agents live for J periods with certainty, value consumption and hours according to standard utility functions $u(c)$ and $v(h)$, respectively, and discount the future with $\beta < 1$. Each period, agents consume, save and work while collecting labor and asset income at given prices w and r, respectively. Labor productivity, denoted e, is subject to random shocks, denoted by ε. Lastly, suppose that the government does not spend $(G = 0)$ and can choose any level of public savings B'.

The government's problem is to maximize the present value of expected utility of an agent prior to entering the economy, subject to allocations being a competitive equilibrium. The government understands how individual allocations and prices vary with public savings.

$$S(B) \equiv \max_{B' \in \mathbb{R}} \sum_{\varepsilon} \int_A \mathbb{E}_1 \sum_{j=1}^{J} \beta^{j-1} \left[u\left(c_j(a, \varepsilon; B')\right) - v\left(h_j(a, \varepsilon; B')\right) \right] d\lambda_1(a, \varepsilon)$$

$$\begin{aligned} \text{s.t.} \quad c_j(a, \varepsilon; B) + a'_j(a, \varepsilon; B') &= we(\varepsilon)h_j(a, \varepsilon; B) + (1 + r)a \\ w &= F_L(K, L) \\ r &= F_K(K, L) - \delta \\ K &= \textstyle\sum_{j=1}^{J} \sum_{\varepsilon} \int_A a d\lambda_j(a, \varepsilon; B) + B \\ L &= \textstyle\sum_{j=1}^{J} \sum_{\varepsilon} \int_A e(\varepsilon)h_j(a, \varepsilon; B) d\lambda_j(a, \varepsilon; B) \end{aligned}$$

where $\lambda_1(a, \varepsilon)$ is the joint distribution over wealth and labor productivity shocks, and (K, L) are aggregate capital and labor. Therefore, the government's optimality condition is:[2]

$$\sum_{j=1}^{J} \mu_j \sum_{\varepsilon} \int_{\mathcal{A}} \omega_j \beta \sum_{\varepsilon'} \pi_j(\varepsilon'|\varepsilon) u'(c_{j+1}) \left(\frac{a'_j}{K'} - \frac{e(\varepsilon')h_{j+1}}{L'} \right) d\lambda_j(a, \varepsilon) = 0$$

where ω_j is given by:

$$\omega_j \equiv \frac{dr'}{dK'} \cdot \frac{dK'_j}{dB'} \cdot K' - \frac{dw'}{dL'} \cdot \frac{dL'_j}{dB'} \cdot L'$$

This ω_j term captures the effect of public savings on prices by way of changes in aggregate factor inputs. A change in public savings dB' induces a change in factor prices given by $dr/dK' = F_{KK}(K', L')$ and $dw/dL' = F_{LL}(K', L')$, and a direct change in aggre-

[2]Derivations are provided in Appendix A.2.

gate capital and aggregate labor given by:

$$\frac{dK'_j}{dB'} = \frac{d}{dB'}(A'_j + B') = \sum_{\varepsilon'}\sum_{\varepsilon} \pi_j(\varepsilon'|\varepsilon) \int \frac{da'_j(a,\varepsilon)}{dB'} d\lambda_j(a,\varepsilon) + 1$$

$$\frac{dL'_j}{dB'} = \sum_{\varepsilon'} \int e(\varepsilon') \frac{dh_{j+1}(a',\varepsilon')}{dB'} d\lambda_{j+1}(a',\varepsilon').$$

If the competitive equilibrium were efficient, then $\omega_j = 0$ and a change in the government's public savings policy would be socially suboptimal. However, $\omega_j = 0$ does not generically hold in competitive equilibrium.

Instead, the government chooses policy according to a marginal utility weighted average of the individual effects of an increase in public savings. This means that the government places higher weight on low income agents who have high marginal utility of consumption. For example, low wealth agents may choose to work more hours despite low labor productivity, in which case the government assigns a high marginal utility weight to a $a'_j/K' < e_{j+1}h_{j+1}/L'$. The aggregate effect depends on the distribution $\lambda_j(a,\varepsilon)$.

Next we turn to examining the distinct features in the life cycle model that can lead to different optimal policy in the life cycle model compared to the infinitely lived agent model.

2.2 Life Cycle Phases

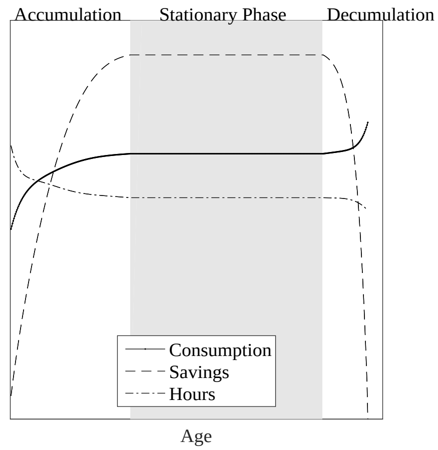
In order to highlight how the life cycle may impact optimal debt policy, it will be useful to consider the following illustrative example. Suppose that agents are born with zero wealth, work throughout their lifetimes and die with certainty within a finite number of periods. Agents face idiosyncratic labor productivity shocks and use assets to partially insure against the resulting earnings risk. Since agents do not retire, savings is accumulated only to insure against idiosyncratic shocks.

For this hypothetical economy, Figure 1 depicts cross-sectional averages for savings, hours and consumption decisions at each age. Figure 1 shows that agents experience three different phases. Agents enter the economy without any wealth and begin the *accumulation phase*, which is characterized by the accumulation of wealth for precautionary motives. While accumulating a stock of savings, agents tend to work more and consume less.

Once a cohort's average wealth provides sufficient insurance against labor productivity shocks, these agents have entered the *stationary phase*.[3] This phase is characterized by savings, hours and consumption that are remain constant in the aggregate. However, underlying constant aggregates are agents who respond to shocks by choosing differ-

[3]The stationary level of average savings is related to the "target savings level" in Carroll (1992, 1997). Given the primitives of the economy, an agent faces a tradeoff between consumption levels and consumption smoothing. The agent targets a level of savings that provides sufficient insurance while maximizing expected consumption.

Figure 1: Illustrative example of life cycle phases.



ent allocations and moving about various states within a non-degenerate distribution of savings, hours and consumption.

Finally, agents enter the *deaccumulation phase* as they approach the end of their lives. In order to smooth consumption in the final period of their lives, agents attempt to deaccumulate assets so that they are not forced to consume a large quantity immediately preceding death. Furthermore, with few periods of life remaining, agents no longer want to hold as much wealth for precautionary reasons. Thus, the average level of savings and labor supply decreases, while consumption increases slightly.

2.3 Channels By Which Public Debt Affects Agents

In this section we identify four main channels through which public debt policy affects welfare: the *level channel*, the *insurance channel*, the *income composition channel* and the *inequality channel*.[4] The level channel is a direct effect of policy on aggregate resources. It measures the change in utility across steady states of the average agent. The remaining channels measure the change in utility across steady states due to a change in the allocation of resources across heterogeneous agents. These channels operate indirectly through general equilibrium effects such as market clearing prices and changes in the wealth distribution. In the remainder of this section, we heuristically characterize how these channels differ across life cycle and infinitely lived agent economies and lead to different optimal policies.

Level Channel: Generally, higher public savings corresponds to more productive capital and, through a higher wage, encourages agents to work more hours. Higher aggregate

[4]These channels were proposed in Floden (2001) and more recently reinterpreted by Röhrs and Winter (2016) to highlight slightly different model mechanisms. While building on both previous papers, we also find it useful to reinterpret these channels in order to highlight life cycle specific mechanisms. In particular, we separate Röhrs and Winter's (2016) insurance channel into two separate channels: a self-insurance and inequality channel. In the life cycle model, the inequality channel has a distinct and important role.

capital and labor increases output and, all else equal, aggregate consumption.[5] While the average agent derives greater utility from an increase in aggregate consumption, an increase in aggregate hours corresponds to greater disutility from labor. As a result, the net welfare effect depends on the relative magnitude of these competing utilities. There is no a priori argument for why the level channel should operate more strongly in either the life cycle or infinitely lived agent economies.

Insurance Channel: Agents' welfare tends to improve as a result of better insurance against labor earnings risk. Public debt mechanically crowds out productive capital, which leads to a higher interest rate. In turn, the higher interest rate encourages agents to save more, which increases the aggregate stock of precautionary savings and decreases the likelihood that agents face binding liquidity constraints.[6]

The efficacy of the insurance channel is fundamentally different in the life cycle and infinitely lived agent models. In the infinitely lived agent model, agents do not experience an accumulation phase but instead experience a perpetual stationary phase. If the government holds more public debt, then the *steady state* level of aggregate savings is larger and the average agent has more wealth *ex ante*. Accordingly, infinitely lived agents are well insured against labor earnings risk ex ante. In contrast, life cycle agents begin working life with zero wealth and immediately begin saving in the accumulation phase. Although changes in the interest rate may increase the level of savings in the stationary phase, agents will still need to forgo consumption in order to accumulate wealth while building to the stationary phase. Therefore, this channel will be less effective in the life cycle model compared to the infinitely lived agent model.

Income Composition Channel: Agents receive labor and asset interest income. Given a wealth distribution, an increase in the interest rate will make existing assets more valuable. This price effect is prevalent in the infinitely lived agent model, as higher government debt increases the ex ante value of assets for a given wealth distribution. However, this effect essentially disappears in the life cycle model since agents enter the model with zero assets.

Inequality Channel: As cohorts age, cross-sectional variance in labor and asset income may increase. From an ex ante perspective, ex post income inequality lowers utility since agents are risk averse and prefer smooth consumption paths. Because government policy has opposing effects on factor prices, policy can lower lifetime income inequality. For example, if labor income contributes more to lifetime income inequality than asset

[5]While public savings can crowd out private savings, our quantitative findings show that the elasticity of private savings to public savings is less than one. Therefore we find that public savings increases productive capital across models. Additionally, the production technology is increasing in aggregate capital. Because we assume that the production technology exhibits decreasing marginal returns to capital, aggregate consumption and aggregate private savings may decrease when the aggregate capital stock is sufficiently large. In our quantitative experiments, however, we find that aggregate consumption is increasing in public savings when public savings varies between the calibrated debt level and the optimal policy.

[6]This channel corresponds to Aiyagari and McGrattan's (1998) liquidity provision channel, which largely accounts for the optimality of public debt in their infinitely lived agent model.

income, then increasing public debt will lower the wage and compress labor income across agents.

As demonstrated in Dávila, Hong, Krusell, and Ríos-Rull (2012), the contribution to overall inequality from labor versus interest income depends on an agent's lifespan. A shorter lifespan in the life cycle model implies that labor earnings constitute a larger fraction of total income. In the extreme of a one-period model, for example, all income is necessarily labor income. Accordingly, the government could enact a debt policy that lowers the wage rate in order to reduce total income inequality. As lifespan extends, interest income contributes more to total income inequality. This is because labor earnings disperse as agents receive a long string of positive or negative labor productivity shocks. Asset income inequality then develops as an endogenous response to labor earnings, because agents reduce their wealth in response to negative labor productivity shocks and increase it in response to positive shocks. As asset income becomes a larger source of income inequality, the government can reduce lifetime income inequality by holding less public debt (more public savings) and decreasing the interest rate. Accordingly, the inequality channel may lead the government to hold public debt in life cycle model and public savings in an infinitely lived agent model.

3 Economic Environment

In this section, we present both the Life Cycle model and the Infinitely Lived Agent model. Given that there are many common features across models, we will first focus on the Life Cycle model in detail before providing an overview of the Infinitely Lived Agent model.

3.1 Life Cycle Model

3.1.1 Production

Assume there exist a large number of firms that sells goods in perfectly competitive product markets, purchase inputs from perfectly competitive factor markets and each operate an identical constant returns to scale production technology, $Y = ZF(K, L)$. These assumptions on primitives admit a representative firm. The representative firm chooses capital (K) and labor (L) inputs in order to maximize profits, given an interest rate r, a wage rate w, a level of total factor productivity Z and capital depreciation rate $\delta \in (0, 1)$.

3.1.2 Consumers

Demographics: Let time be discrete and let each model period represent a year. Each period, the economy is inhabited by J overlapping generations of individuals. Age J is each agent's exogenous terminal period of life. Before period J all living agents face mortality risk. Conditional on living to age j, agents have a probability s_j of living to age

$j+1$, with a terminal age probability given by $s_J = 0$. Each period a new cohort is born and the size of each successive newly born cohort grows at a constant rate $g_n > 0$.

Agents who die before age J may hold savings since mortality is uncertain. These savings are treated as accidental bequests and are equally divided across each living agent in the form of a lump-sum transfer, denoted Tr.

Preferences: Agents rank lifetime paths of consumption and labor, denoted $\{c_j, h_j\}_{j=1}^{J}$, according to the following preferences:

$$\mathbb{E}_1 \sum_{j=1}^{J} \beta^{j-1} s_j \left[u(c_j) - \zeta_j' v(h_j) \right]$$

where β is the time discount factor. Expectations are taken with respect to the stochastic processes governing labor productivity. Furthermore, $u(c)$ and $v(h)$ are instantaneous utility functions over consumption and labor hours, respectively, satisfying standard conditions. Lastly, ζ_j' is a retirement decision that is described immediately below.

Retirement: Agents choose their retirement age, which is denoted by J_{ret}. A retired agent may not sell labor hours and the decision is irreversible. Agents endogenously determine retirement age in the interval $j \in [\underline{J}_{ret}, \bar{J}_{ret}]$ and are forced to retire after age $\bar{J}_{ret}$. Let $\zeta_j' \equiv \mathbb{1}(j < J_{ret})$ denote an indicator variable that equals one when an agent chooses to continue working and zero upon retirement.

Labor Earnings: Agents are endowed with one unit of time per period, which they split between leisure and market labor. During each period of working life, an agent's labor earnings are $we_j h_j$, where w is the wage rate per efficiency unit of labor, e_j is the agent's idiosyncratic labor productivity drawn at age j and h_j is the time the agent chooses to work at age j.

Following Kaplan (2012), we assume that labor productivity shocks can be decomposed into four sources:

$$\log(e_j) = \kappa + \theta_j + \nu_j + \epsilon_j$$

where (i) $\kappa \overset{iid}{\sim} \mathcal{N}(0, \sigma_\kappa^2)$ is an individual-specific fixed effect that is drawn at birth, (ii) $\{\theta_j\}_{j=1}^{J}$ is an age-specific fixed effect, (iii) ν_j is a persistent shock that follows an autoregressive process given by $\nu_{j+1} = \rho \nu_j + \eta_{j+1}$ with $\eta \overset{iid}{\sim} \mathcal{N}(0, \sigma_\nu^2)$ and $\eta_1 = 0$, and (iv) $\epsilon_j \overset{iid}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$ is a per-period transitory shock.

For notational compactness, we denote the relevant state as a vector $\varepsilon_j = (\kappa, \theta_j, \nu_j, \epsilon_j)$ that contains each element necessary for computing contemporaneous labor earnings and forming expectations about future labor earnings. Denote the Markov process governing the process for ε by $\pi_j(\varepsilon_{j+1}|\varepsilon_j)$ for each $j = 1, \ldots, \bar{J}_{ret}$ and for each ε_j, ε_{j+1}. Lastly, define the function $e(\varepsilon_j) \equiv e_j$.

Insurance: Agents have access to a single asset, a non-contingent one-period bond denoted a_j with a market determined rate of return of r. Agents may take on a net debt position, in which case they are subject to a borrowing constraint that requires their debt position be bounded below by $\underline{a} \in \mathbb{R}$. Agents are endowed with zero initial wealth, such that $a_1 = 0$ for each agent.

3.1.3 Government Policy

The government (i) consumes an exogenous amount G, (ii) collects linear Social Security taxes τ_{ss} on all pre-tax labor income below an amount $\bar{x}$, (iii) distributes lump-sum Social Security payments b_{ss} to retired agents, (iv) distributes accidental bequests as lump-sum transfers Tr, and (v) taxes each individual's taxable income according to an increasing and concave function $\Upsilon(\cdot)$.

Social Security: The model's Social Security system consists of taxes and payments. The social security payroll tax is given by τ_{ss} with a per-period cap denoted by $\bar{x}$. We assume that half of the social security contributions are paid by the employee and half by the employer. Therefore, the consumer pays a payroll tax given by: $(1/2)\ \tau_{ss} \min\{weh, \bar{x}\}$. Social security payments are computed using an averaged indexed monthly earnings (AIME) that summarizes an agents lifetime labor earnings. The AIME is denoted by $\{x_j\}_{j=1}^{J}$ and is given by:

$$x_{j+1} = \left\{ \begin{array}{ll} \frac{1}{j}\left(\min\{we_j h_j, \bar{x}\} + (j-1)x_j\right) & \text{for } j \leq 35 \\ \max\left\{x_j, \frac{1}{j}\left(\min\{we_j h_j, \bar{x}\} + (j-1)x_j\right)\right\} & \text{for } j \in (35, J_{ret}) \\ x_j & \text{for } j \geq J_{ret} \end{array} \right\}$$

The AIME is a state variable for determining future benefits. Benefits consists of a base payment and an adjusted final payment. The base payment, denoted by $b^{ss}_{base}(x_{J_{ret}})$, is computed as a piecewise-linear function over the individual's average labor earnings at retirement $x_{J_{ret}}$:

$$b^{ss}_{base}(x_{J_{ret}}) = \left\{ \begin{array}{lll} \tau_{r1} & \text{for} & x_{J_{ret}} \in [0, b^{ss}_1) \\ \tau_{r2} & \text{for} & x_{J_{ret}} \in [b^{ss}_1, b^{ss}_2) \\ \tau_{r3} & \text{for} & x_{J_{ret}} \in [b^{ss}_2, b^{ss}_3) \end{array} \right\}$$

Lastly, the final payment requires an adjustment that penalizes early retirement and credits delayed retirement. The adjustment is given by:

$$b_{ss}(x_{J_{ret}}) = \left\{ \begin{array}{lll} (1 - D_1(J_{nra} - J_{ret}))b^{ss}_{base}(x_{J_{ret}}) & \text{for} & \underline{J}_{ret} \leq J_{ret} < J_{nra} - 1 \\ (1 + D_2(J_{ret} - J_{nra}))b^{ss}_{base}(x_{J_{ret}}) & \text{for} & J_{nra} \leq J_{ret} \leq \bar{J}_{ret} \end{array} \right\}$$

where $D_i(\cdot)$ are functions governing the benefits penalty or credit, $\underline{J}_{ret}$ is the earliest age agents can retire, J_{nra} is the "normal retirement age" and $\bar{J}_{ret}$ is the latest age an agent can retire.

Net Government Transfers: Taxable income is defined as labor income and capital income net of social security contributions from an employer. Denote taxable income by:

$$y(h,a,\varepsilon) \equiv we(\varepsilon)h + r(a+Tr) - \frac{\tau_{ss}}{2}\min\{we(\varepsilon)h, \bar{x}\}$$

The government taxes each individual's taxable income according to the function $\Upsilon(y(h,a,e))$.

Define the functions $T_j(\cdot)$ and $T_j^{ret}(\cdot)$ as the government's total transfers to agents in social security payments and unemployment benefits net of income taxation and social security payroll taxation. Recall that $\zeta = 1$ if an agent continues to work and $\zeta = 0$ if an agent has retired. Define net transfers during an agent's working ages (if $j < J_{ret}$) and retirement (if $j \geq J_{ret}$) as:

$$T(h,a,\varepsilon,x,\zeta) = \left\{ \begin{array}{cc} -\frac{\tau_{ss}}{2}\min\{we(\varepsilon)h,\bar{x}\} - \Upsilon(y(h,a,\varepsilon)) & \text{if } \zeta = 1 \\ b_{ss}(x) - \Upsilon(r(a+Tr)) & \text{if } \zeta = 0 \end{array} \right\}$$

Public Savings and Budget Balance: Each period, the government accumulates savings, denoted B', and collects asset income rB. The resulting government budget constraint is:

$$G + B' - B = rB + \Upsilon_y$$

where Υ_y is aggregate revenues from income taxation and G is an unproductive level of government expenditures.[7] The model's Social Security system is self-financing and therefore does not appear in the governmental budget constraint.

3.1.4 Consumer's Problem

The agent's state variables consist of asset holdings a, labor productivity shocks $\varepsilon \equiv (\kappa, \theta, \nu, \epsilon)$, Social Security contribution (AIME) variable x and retirement status ζ. The agent's recursive problem is:

$$V_j(a,\varepsilon,x,\zeta) = \max_{c,a',h,\zeta'} \left[u(c) - \zeta' v(h)\right] + \beta s_j \sum_{\varepsilon'} \pi_j(\varepsilon'|\varepsilon) V_{j+1}(a',\varepsilon',x',\zeta')$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq \zeta' we(\varepsilon)h + (1+r)(a+Tr) + T(h,a,\varepsilon,x,\zeta') \qquad (1) \\ a' &\geq \underline{a} \\ \zeta' &\in \{\underline{\zeta}_j,\ \bar{\zeta}_j \cdot \zeta\} \end{aligned}$$

We define $\underline{\zeta}_j \equiv \mathbb{1}(j < \underline{J}_{ret})$, which equals one when an agent is too young to retire and equals zero thereafter. Additionally we define $\bar{\zeta}_j \equiv \mathbb{1}(j \leq \bar{J}_{ret})$, which equals zero for all

[7] Two recent papers, Röhrs and Winter (2016) and Chaterjee, Gibson, and Rioja (2016) have relaxed the standard Ramsey assumption that government expenditures are unproductive. Both papers show that public savings is optimal with productive government expenditures, intuitively because there is an additional benefit to aggregate output.

ages after an agent must retire and equals one beforehand. Therefore the agent's recursive problem nests all three stages of life: working life, near-retirement and retirement.[8]

3.1.5 Recursive Competitive Equilibrium

Agents are heterogeneous with respect to their age $j \in \mathbf{J} \equiv \{1, \ldots, J\}$, wealth $a \in \mathbf{A}$, labor productivity $\varepsilon \in \mathbf{E}$, average lifetime earnings $x \in \mathbf{X}$ and retirement status $\zeta \in \mathbf{R} \equiv \{0, 1\}$. Let $\mathbf{S} \equiv \mathbf{A} \times \mathbf{E} \times \mathbf{X} \times \mathbf{R}$ be the state space and $\mathcal{B}(\mathbf{S})$ be the Borel σ-algebra on $\mathbf{S}$. Let $\mathbf{M}$ be the set of probability measures on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$. Then $(\mathbf{S}, \mathcal{B}(\mathbf{S}), \lambda_j)$ is a probability space in which $\lambda_j(S) \in \mathbf{M}$ is a probability measure defined on subsets of the state space, $S \in \mathcal{B}(\mathbf{S})$, that describes the distribution of individual states across age-j agents. Denote the fraction of the population that is age $j \in \mathbf{J}$ by μ_j. For each set $S \in \mathcal{B}(\mathbf{S})$, $\mu_j \lambda_j(S)$ is the fraction of age $j \in \mathbf{J}$ and type $S \in \mathbf{S}$ agents in the economy. We can now define a recursive competitive equilibrium of the economy.

Definition (Equilibrium): Given a government policy $(G, B, B', \Upsilon, \tau_{ss}, b_{ss})$, a *stationary recursive competitive equilibrium* is (i) an allocation for consumers described by policy functions $\{c_j, a'_j, h_j, \zeta'_j\}_{j=1}^{J}$ and consumer value function $\{V_j\}_{j=1}^{J}$, (ii) an allocation for the representative firm (K, L), (iii) prices (w, r), (iv) accidental bequests Tr, and (v) distributions over agents' state vector at each age $\{\lambda_j\}_{j=1}^{J}$ that satisfy:

(1) Given prices, policies and accidental bequests, $V_j(a, \varepsilon, x)$ solves the Bellman equation (1) with associated policy functions $c_j(a, \varepsilon, x, \zeta)$, $a'_j(a, \varepsilon, x, \zeta)$, $h_j(a, \varepsilon, x, \zeta)$ and $\zeta'_j(a, \varepsilon, x, \zeta)$.

(2) Given prices (w, r), the representative firm's allocation minimizes cost: $r = ZF_K(K, L) - \delta$ and $w = ZF_L(K, L)$

(3) Accidental bequests, Tr, from agents who die at the end of this period are distributed equally across next period's living agents:

$$(1 + g_n)Tr = \sum_{j=1}^{J} (1 - s_j)\mu_j \int a'_j(a, \varepsilon, x, \zeta) d\lambda_j(a, \varepsilon, x, \zeta)$$

(4) Government policies satisfy budget balance:

$$G + (B' - B) = rB + \Upsilon_y$$

[8] During an agent's working life (ages $j < \underline{J}_{ret}$) the agent's choice set for retirement is $\{\underline{\zeta}_j, \bar{\zeta}_j\} = \{1, 1\}$ and therefore the agent must continue working. Near retirement (ages $\underline{J}_{ret} \leq j \leq \bar{J}_{ret}$), the agent's choice set is $\{\underline{\zeta}_j, \bar{\zeta}_j\} = \{0, 1\}$ and the agent may retire by choosing $\zeta' = 0$. Lastly, if an agent has retired either because he chose retirement at a previous date ($\zeta = 0$) or because of mandatory retirement ($j > \bar{J}_{ret}$), then the choice set is $\{0, 0\}$ and $\zeta' = \zeta = 0$.

aggregate income tax revenue is given by:

$$\Upsilon_y \equiv \sum_{j=1}^{J} \mu_j \int \Upsilon\big(y(h(a,\varepsilon,x,\zeta),a,\varepsilon,\zeta)\big)\ d\lambda_j(a,\varepsilon,x,\zeta)$$

(5) Social security is self-financing:

$$\sum_{j=1}^{J} \mu_j \int \zeta_j'(a,\varepsilon,x,\zeta)\tau_{ss} \min\{we(\varepsilon)h_j(a,\varepsilon,x,\zeta),\bar{x}\} d\lambda_j(a,\varepsilon,x,\zeta)$$
$$= \sum_{j=1}^{J} \mu_j \int (1-\zeta'(a,\varepsilon,x,\zeta))b_{ss}(x) d\lambda_j(a,\varepsilon,x,\zeta) \tag{2}$$

(6) Given policies and allocations, prices clear asset and labor markets:

$$K - B = \sum_{j=1}^{J} \mu_j \int a\ d\lambda_j(a,\varepsilon,x,\zeta)$$

$$L = \sum_{j=1}^{J} \mu_j \int \zeta'(a,\varepsilon,x,\zeta)e(\varepsilon)h_j(a,\varepsilon,x,\zeta)\ d\lambda_j(a,\varepsilon,x,\zeta)$$

and the allocation satisfies the resource constraint (guaranteed by Walras' Law):

$$\sum_{j=1}^{J} \mu_j \int c_j(a,\varepsilon,x,\zeta) d\lambda_j(a,\varepsilon,x,\zeta) + G + K' = ZF(K,L) + (1-\delta)K$$

(7) Given consumer policy functions, distributions across age j agents $\{\lambda_j\}_{j=1}^{J}$ are given recursively from the law of motion $T_j^* : \mathbf{M} \to \mathbf{M}$ for all $j \in \mathbf{J}$ such that T_j^* is given by:

$$\lambda_{j+1}(\mathcal{A}\times\mathcal{E}\times\mathcal{X}\times\mathcal{R}) = \sum_{\zeta\in\{0,1\}} \int_{A\times E\times X} Q_j\left((a,\varepsilon,x,\zeta),\mathcal{A}\times\mathcal{E}\times\mathcal{X}\times\mathcal{R}\right) d\lambda_j$$

where $\mathcal{S} \equiv \mathcal{A}\times\mathcal{E}\times\mathcal{X}\times\mathcal{R} \subset \mathbf{S}$, and $Q_j : \mathbf{S}\times\mathcal{B}(\mathbf{S}) \to [0,1]$ is a transition function on $(\mathbf{S},\mathcal{B}(\mathbf{S}))$ that gives the probability that an age-j agent with current state $\mathbf{s} \equiv (a,\varepsilon,x,\zeta)$ transits to the set $\mathcal{S} \subset \mathbf{S}$ at age $j+1$. The transition function is given by:

$$Q_j\left((a,\varepsilon,x,\zeta),\mathcal{S}\right) = \left\{ \begin{array}{cl} s_j \cdot \pi_j(\mathcal{E}|\varepsilon)^{\zeta} & \text{if } a_j'(\mathbf{s}) \in \mathcal{A},\ x_j'(\mathbf{s}) \in \mathcal{X},\ \zeta'(\mathbf{s}) \in \mathcal{R} \\ 0 & \text{otherwise} \end{array} \right\}$$

where agents that continue working and transition to set $\mathcal{E}$ choose $\zeta'(\mathbf{s}) = 1$, while agents that transition from working life to retirement choose $\zeta'(\mathbf{s}) = 0$. For $j = 1$, the distribution λ_j reflects the invariant distribution $\pi_{ss}(\varepsilon)$ of initial labor productivity over $\varepsilon = (\kappa, \theta_1, 0, \epsilon_1)$.

(8) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that $K' = K$, $B' = B$, $w' = w$, $r' = r$, and $\lambda'_j = \lambda_j$ for all $j \in \mathbf{J}$.

3.2 Infinitely Lived Agent Model

The infinitely lived agent model differs from the life cycle model in three ways. First, agents in the infinitely lived agent model have no mortality risk ($s_j = 1$ for all $j \geq 1$) and lifetimes are infinite ($J \to \infty$). Second, labor productivity no longer has an age-dependent component ($\theta_j = 0$ for all $j \geq 1$). Lastly, there is no retirement ($J_{ret} \to \infty$ such that $\zeta_j = 1$ for all $j \geq 1$) and there is no Social Security program ($\tau_{ss} = 0$ and $b_{ss}(x) = 0$ for all x).

Accordingly, we study a stationary recursive competitive equilibrium in which the initial endowment of wealth and labor productivity shocks no longer affects individual decisions and the distribution over wealth and labor productivity is time invariant.

Definition (Equilibrium): Given a government policy (G, B, B', Υ), a *stationary recursive competitive equilibrium* is (i) an allocation for consumers described by policy functions (c, a', h) and consumer value function V, (ii) an allocation for the representative firm (K, L), (iii) prices (w, r), and (v) a distribution over agents' state vector λ that satisfy:

(1) Given prices and policies, $V(a, \varepsilon)$ solves the following Bellman equation:

$$V(a,\varepsilon) = \max_{c,a',h} \; \big[u(c) - v(h)\big] + \beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) V(a', \varepsilon')$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq we(\varepsilon)h + (1+r)a + \Upsilon(y(h,a,\varepsilon)) \\ a' &\geq \underline{a} \end{aligned} \tag{3}$$

with associated policy functions $c(a, \varepsilon)$, $a'(a, \varepsilon)$ and $h(a, \varepsilon)$.

(2) Given prices (w, r), the representative firm's allocation minimizes cost.

(3) Government policies satisfy budget balance:

$$G + (B' - B) = rB + \Upsilon_y$$

aggregate income tax revenue is given by:

$$\Upsilon_y \equiv \int \Upsilon\big(y(h(a,\varepsilon), a, \varepsilon)\big) \; d\lambda(a,\varepsilon)$$

(4) Given policies and allocations, prices clear asset and labor markets:

$$K - B = \int a \; d\lambda(a,\varepsilon)$$

$$L = \int e(\varepsilon) h(a, \varepsilon) \, d\lambda(a, \varepsilon)$$

and the allocation satisfies the resource constraint (guaranteed by Walras' Law):

$$\int c(a, \varepsilon) d\lambda(a, \varepsilon) + G + K' = ZF(K, L) + (1 - \delta)K$$

(5) Given consumer policy functions, the distribution over wealth and productivity shocks is given recursively from the law of motion $T^* : \mathbf{M} \to \mathbf{M}$ such that T^* is given by:

$$\lambda'(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} Q_j \left((a, \varepsilon), \mathcal{A} \times \mathcal{E} \right) d\lambda$$

where $\mathcal{S} \equiv \mathcal{A} \times \mathcal{E} \subset \mathbf{S}$, and $Q : \mathbf{S} \times \mathcal{B}(\mathbf{S}) \to [0, 1]$ is a transition function on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$ that gives the probability that an agent with current state $\mathbf{s} \equiv (a, \varepsilon)$ transits to the set $\mathcal{S} \subset \mathbf{S}$ in the next period. The transition function is given by:

$$Q\left((a, \varepsilon), \mathcal{S}\right) = \left\{ \begin{array}{ll} \pi(\mathcal{E}|\varepsilon) & \text{if } a'(\mathbf{s}) \in \mathcal{A}, \\ 0 & \text{otherwise} \end{array} \right\}$$

(6) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that $K' = K$, $B' = B$, $w' = w$, $r' = r$, and $\lambda' = \lambda$.

3.3 Balanced Growth Path

Following Aiyagari and McGrattan (1998), we will further assume that total factor productivity, Z, grows over time at rate $g_z > 0$. In both the life cycle model and infinitely lived agent model, we will study a balanced growth path equilibrium in which all aggregate variables grow at the same rate as output. Denote the growth rate of output as g_y. Refer to Appendix A.1 for a formal construction of the balanced growth path for this set of economies.

4 Calibration

One subset of parameters are assigned values without needing to solve the model. The other subset of parameters are estimated using a simulated method of moments procedure that minimizes the distance between model generated moments and empirical ones. Table 1 summarizes the target and value for each parameter.

Demographics: We set the conditional survival probabilities $\{s_j\}_{j=1}^{J}$ according to Bell and Miller (2002) and impose $s_J = 0$. We set the populational growth rate to $g_n = 0.011$ to match annual population growth in the US.

Production: The production function is assumed to be Cobb-Douglas of the form $F(K, L) = K^\alpha L^{1-\alpha}$ where $\alpha = 0.36$ is the income share accruing to capital. The depreciation rate is to $\delta = 0.0833$ which allows the model to match the empirically observed investment-to-output ratio.

Preferences: The utility function is is separable in the utility over consumption and disutility over labor:

$$u(c) - \zeta v(h) = \frac{c^{1-\sigma}}{1-\sigma} - \zeta \left(\chi_1 \frac{h^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} + \chi_2 \right) .$$

This functional form implies that the utility is constant relative risk aversion where σ controls the risk aversion. We set the coefficient of relative risk aversion $\sigma = 2$ consistent with Conesa et al. (2009) and Aiyagari and McGrattan (1998). The disutility over labor exhibits a constant intensive margin Frisch elasticity. We choose $\gamma = 0.5$ such that the Frisch elasticity consistent with the majority of the related literature as well as the estimates in Kaplan (2012).

We calibrate the labor disutility parameter χ_1 so that the cross sectional average of hours is one third of the time endowment. Finally, χ_2 is a fixed utility cost of earning labor income before retirement. The fixed cost generates an extensive margin decision through a non-convexity in the utility function. We choose χ_2 to match the empirical observation that seventy percent of the population has retired by the normal retirement age.

Labor Productivity Process: We take the labor productivity process from the estimates in Kaplan (2012) based on the estimates from the PSID data.[9] The deterministic labor productivity profile, $\{\theta_j\}_{j=1}^{\bar{J}_{ret}}$, is (i) smoothed by fitting a quadratic function in age, (ii) normalized such that the value equals unity when an agent enters the economy, and (iii) extended to cover ages 20 through 69 which we define as the last period in which agents are assumed to be able to participate in the labor activities ($\bar{J}_{ret}$).[10] The permanent, persistent, and transitory idiosyncratic shocks to individual's productivity are distributed normal with a mean of zero. The remaining parameters are also set in accordance with the estimates in Kaplan (2012): $\rho = 0.958$, $\sigma_\kappa^2 = 0.065$, $\sigma_\nu^2 = 0.017$ and $\sigma_\epsilon^2 = 0.081$. We discretize all three of the shocks in order to solve the model, representing the transitory shock with two states, the permanent shock with two states, and the persistent shock with five states. For expositional convenience, we refer to the two different states of the permanent shock as high and low ability types.

Government: Consistent with Aiyagari and McGrattan (1998) we set government debt equal to two-thirds of output. We set government consumption equal to 15.5 percent of output consistent. This ratio corresponds to the average of government expenditures to GDP from 1998 through 2007.[11]

[9] For details on estimation of this process, see Appendix E in Kaplan (2012).

[10] The estimates in Kaplan (2012) are available for ages 25-65.

[11] We exclude government expenditures on Social Security since they are explicitly included in our model.

Income Taxation: The income tax function and parameter values are from Gouveia and Strauss (1994). The functional form is:

$$\Upsilon(y) = \tau_0 \left(y - \left(y^{-\tau_1} + \tau_2 \right)^{-\frac{1}{\tau_1}} \right)$$

The authors find that $\tau_0 = 0.258$ and $\tau_1 = 0.768$ closely match the U.S. tax data. When calibrating the model we set τ_2 such that the government budget constraint is satisfied.

Social Security: We set the normal retirement age to 66. Consistent with the minimum and maximum retirement ages in the U.S. Social Security system, we set the interval in which agents can retire to the ages 62 and 70. The early retirement penalty and later retirement credits are set in accordance with the Social Security program. In particular, if agents retire up to three years before the normal retirement age agents benefits are reduced by 6.7 percent for each year they retire early. If they choose to retire four or five years before the normal retirement age benefits are reduced by an additional 5 percent for these years. If agents choose to delay retirement past normal retirement age then their benefits are increased by 8 percent for each year they delay. The marginal replacement rates in the progressive Social Security payment schedule $(\tau_{r1}, \tau_{r2}, \tau_{r3})$ are also set at their actual respective values of 0.9, 0.32 and 0.15. The bend points where the marginal replacement rates change (b_1^{ss}, b_2^{ss}, b_3^{ss}) and the maximum earnings ($\bar{x}$) are set equal to the actual multiples of mean earnings used in the U.S. Social Security system so that b_1^{ss}, b_2^{ss} and $b_3^{ss} = \bar{x}$ occur at 0.21, 1.29 and 2.42 times average earnings in the economy. We set the payroll tax rate, τ_{ss} such that the program's budget is balanced. In our baseline model the payroll tax rate is 10.3 percent, roughly equivalent with the statutory rate.[12]

Infinitely Lived Agent Model: The infinitely lived agent model does not have a age-dependent wage profile. For comparability across models, we replace the age-dependent wage profile with the population-weighted average θ_j. Lastly, we recalibrate the parameters (β, χ) to the same targets as in the life cycle model.

5 Quantitative Analysis

Having described how we use external data to discipline the models' structural parameters, we use calibrated model to measure optimal policy across the life cycle and infinitely lived agent models. Then we perform a series of counterfactual experiments to highlight the model mechanisms that generate differences in optimal policy across the models.

[12] Although the payroll tax rate in the U.S. economy is slightly higher than our calibrated value, the OASDI program includes additional features outside of the retirement benefits.

Table 1: Calibration Targets and Parameters for Baseline Economy.

Description	Parameter	Value	Target or Source
Demographics			
Maximum Age	J	100	By Assumption
Min/Max Retirement Age	$\underline{J}_{ret}, \bar{J}_{ret}$	62, 70	Social Security Program
Population Growth	g_n	1.1%	Conesa et al (2009)
Survival Rate	$\{s_j\}_{j=1}^{J}$	—	Bell and Miller (2002)
Preferences and Borrowing			
Coefficient of RRA	σ	2.0	Kaplan (2012)
Frisch Elasticity	γ	0.5	Kaplan (2012)
Coefficient of Labor Disutility	χ_1	55.3	Avg. Hours Worked = 1/3
Fixed Utility Cost of Labor	χ_2	1.038	70% retire by NRA
Discount Factor	β	1.012	Capital/Output = 2.7
Borrowing Limit	$\underline{a}$	0	By Assumption
Technology			
Capital Share	α	0.36	NIPA
Capital Depreciation Rate	δ	0.0833	Investment/Output = 0.255
Productivity Level	Z	1	Normalization
Output Growth	g_y	1.85%	NIPA
Labor Productivity			
Persistent Shock, autocorrelation	ρ	0.958	Kaplan (2012)
Persistent Shock, variance	σ_{ν}^2	0.017	Kaplan (2012)
Permanent Shock, variance	σ_{κ}^2	0.065	Kaplan (2012)
Transitory Shock, variance	σ_{ϵ}^2	0.081	Kaplan (2012)
Mean Earnings, Age Profile	$\{\theta\}_{j=1}^{\bar{J}_{ret}}$	—	Kaplan (2012)
Government Budget			
Government Consumption	G/Y	0.155	NIPA Average 1998-2007
Government Savings	B/Y	-0.667	NIPA Average 1998-2007
Marginal Income Tax	τ_0	0.258	Gouveia and Strauss (1994)
Income Tax Progressivity	τ_1	0.786	Gouveia and Strauss (1994)
Income Tax Progressivity	τ_2	4.541	Balanced Budget
Social Security			
Payroll Tax	τ_{ss}	0.103	Social Security Program
SS Replacement Rates	$\{\tau_{ri}\}_{i=1}^{3}$	See Text	Social Security Program
SS Replacement Bend Points	$\{b_i^{ss}\}_{i=1}^{3}$	See Text	Social Security Program
SS Early Retirement Penalty	$\{\kappa_i\}_{i=1}^{3}$	See Text	Social Security Program

5.1 Policy Experiments

Government's Planning Problem: The government places value on agents according to an ex-ante Utilitarian social welfare function. Therefore, the government chooses public savings B to maximize the expected lifetime utility of a newborn agent subject to a governmental budget constraint. The following defines the government's welfare maximization problem in the Life Cycle model:

$$\begin{aligned} S(V_1, \lambda_1) \equiv \max_B & \int V_1(a, \varepsilon, x; B)\, d\lambda_1(a, \varepsilon, x; B) \\ \text{s.t.} \quad & G = rB + \Upsilon_y(\tau_0, B) \\ & \text{Social Security [Equation (2)]} \end{aligned}$$

where the value function V_1, distribution function λ_1 and policy functions embedded in equation (2) are determined in competitive equilibrium and depend on the government's choice of public savings.

When debt changes, prices and the distribution of taxable income change, thereby changing the revenues from the tax policies. We adjust the Social Security payroll tax rate τ_{ss} to ensure that Social Security is self-financing. Furthermore we adjust the income tax parameter τ_0 to ensure that the government budget is balanced. We choose to use τ_0 to balance the government budget instead of the other income taxation parameters (τ_1, τ_2) so that the average income tax rate is used to clear the budget, as opposed to changing in the progressivity of the income tax policy. The average tax rate is the closest analogue to the flat tax that Aiyagari and McGrattan (1998) use to balance the government's budget in their model.

The social welfare function in the infinitely lived agent model is identical except that the value function and distribution over wealth and idiosyncratic income shocks are age-independent. Furthermore, there is no retirement or Social Security program in the infinitely lived agent model.

Welfare Decomposition: Underlying the response of aggregates to the level of debt are heterogeneous responses by individual agents. In order to better understand the distributional effects of a policy change on welfare, we will decompose the *consumption equivalent variation (CEV)*.

Denote the change in welfare due to a change in policy by Δ_{CEV}. We decompose Δ_{CEV} into a level effect (Δ_l) and a distributional effect (Δ_d). We further decompose the level and distributional effects into a consumption effect $(\Delta_{C_l}, \Delta_{C_d})$ and a labor hours effect $(\Delta_{H_l}, \Delta_{H_d})$ such that:[13]

$$(1 + \Delta_{CEV}) = (1 + \Delta_l) \cdot (1 + \Delta_d) = [(1 + \Delta_{C_l})(1 + \Delta_{H_l})] \cdot [(1 + \Delta_{C_d})(1 + \Delta_{H_d})]$$

[13] More generally, we follow Floden (2001) in characterizing four components of the CEV: a level effect (Δ_L), an insurance effect (Δ_I), a redistribution effect (Δ_R) and a labor hours effect (Δ_H). We combine the insurance and redistribution effects to form the "distributional effect". A derivation of the decomposition is contained in Appendix A.3.

The consumption effects measure the percent of lifetime consumption agents would be willing to forgo in order to attain the expected present value of utility from the optimal government policy. The labor hours effects similarly measure the percent of lifetime consumption agents would be willing to forgo in order to attain the expected present value of labor disutility from the optimal government policy.

The consumption and hours level effects simply measure the change in welfare due to a change in aggregate consumption or hours, respectively. The distribution effects measure the change in welfare due to changes in the allocation of resources across agents, perhaps due to changes in uncertainty over ex post realizations of idiosyncratic shocks and in the ex ante distribution of endowments.[14]

5.2 Optimal Public Policy

Optimal Policy: We find that the life cycle model prescribes a starkly different optimal policy than does the infinitely lived agent model. In the infinitely lived agent model, the government optimally holds debt. Consistent with Aiyagari and McGrattan (1998), we find that the optimal public debt to output ratio is 20%. In contrast, in the life cycle model, the optimal government policy is to accumulate savings. Optimal public savings to output ratio is 60%, as reported in Table 2.

Table 2: Prices and Aggregates Across Models

	Life Cycle			**Infinitely Lived**		
	Base.	**Opt.**	**% Δ**	**Base.**	**Opt.**	**% Δ**
Output	0.93	1.01	8.6	1.16	1.17	1.5
Capital/Output	2.70	3.01	11.5	2.70	2.74	1.9
Priv. Sav./Output	3.37	2.42	-28.2	3.35	2.96	-11.7
Pub. Sav./Output	-0.67	0.59	189.0	-0.67	-0.22	67.5
Labor	0.53	0.54	2.2	0.66	0.67	0.4
Interest Rate	5.0%	3.6%	-1.4	5.0%	4.8%	-0.2
Wage	1.12	1.19	6.3	1.12	1.13	1.0

Consumption Equivalence: While the infinitely lived agent model prescribes that the government hold public debt, the life cycle model's optimal policy prescribes accumulating public savings. What is the welfare loss from incorrectly implementing a public debt policy?

We propose a calculation that determines the welfare consequences of ignoring the life cycle when deriving optimal policy. Suppose that the government implements the

[14] Note that the Life Cycle model only assumes ex ante heterogeneity with respect to the permanent component of labor productivity. While allowing for heterogeneity in the initial wealth distribution could allow this channel to play a larger role in welfare changes, the PSID and SCF document small dispersion in individuals' wealth upon entering the labor market.

optimal debt policy from an infinitely lived agent economy when the true economy is a life cycle economy. We then quantify the welfare loss from implementing a suboptimal debt policy using consumption equivalent variation (CEV), which is measured as the percent of lifetime consumption that an agent would be willing to pay ex ante in order to live in a world with an optimal public savings policy instead of a suboptimal public debt policy.

Table 3 reports the consumption equivalent variation. We find that an 80 percentage point difference in fiscal policy corresponds to a welfare loss of approximately 0.5% of expected lifetime consumption. The welfare loss is economically significant, demonstrating that ignoring life cycle features when determining optimal debt policy will have nontrivial welfare effects.

Table 3: Welfare Decompositions

(% Change)	**Life Cycle**
Overall CEV	0.44
Level (Δ_l)	0.90
Consumption (Δ_{C_l})	1.30
Hours (Δ_{H_l})	-0.40
Distribution (Δ_d)	-0.46
Consumption (Δ_{C_d})	0.07
Hours (Δ_{H_d})	-0.53

Life Cycle model welfare decomposition, comparing -20% debt with the optimal 60% savings to output.

Welfare Decomposition: We decompose the CEV as described in Section 5.1. The results are reported in Table 3. Overall, the CEV represents a tradeoff between a positive level effect and negative distribution effect. The net effect is positive as the CEV is generated primarily by a large, positive levels effect in consumption, which reflects a 1.3% increase in aggregate consumption available to agents. As the government increases the level of public savings, productive capital increases. A larger capital stock increases aggregate output and aggregate consumption. However, the welfare gain from adopting optimal policy is partially offset by a negative hours level effect, which corresponds to an increase in average labor disutility. Figure 2 graphs average hours, consumption and savings decisions by age. The graphs compare life cycle profiles in the life cycle model's optimal policy and the suboptimal debt policy. As the government decreases debt and accumulates savings, aggregate capital increases thereby generating a higher equilibrium wage. In the graphs, we see that with a higher wage agents suffer greater labor disutility due to working more and delaying retirement.

Lastly, the consumption distribution effect is relatively small but positive. Apparently, relative to the suboptimal policy, the optimal policy does not generate a large increase in welfare due to a better allocation of consumption across agents, states and dates. On the other hand, the hours distribution effect is relatively high (and negative) because a higher wage especially encourages agents with high labor productivity to work more at the margin.

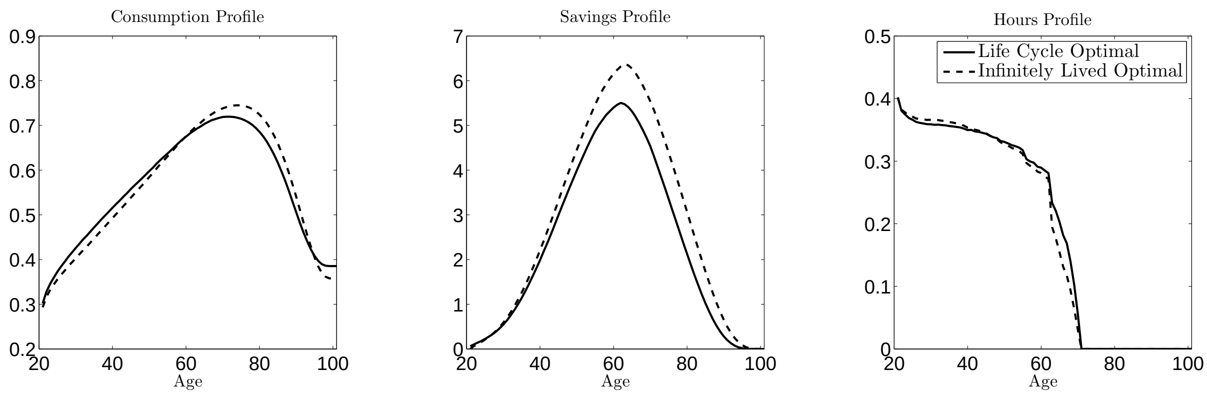


Figure 2: Solid lines are cross-sectional averages for consumption, savings, and hours by age in the life cycle economy under its optimal public savings policy. The dashed lines are cross-sectional averages for the suboptimal debt policy from the infinintely lived agent economy.

The Accumulation Phase: The welfare decomposition shows that the optimality of public savings in the life cycle model corresponds to a stronger level effect than distribution effect. Intriguingly, if we decompose welfare in the infinitely lived agent model from a public savings policy of 60% of output to the optimal debt to output ratio of 21%, then we find that the distribution effect dominates. Table **??** shows that the tradeoff between level and distribution effects reverses in the infinitely lived agent model.

A possible reason for the dominance of the distribution effect in the infinitely lived agent model, instead of the dominance of the level effect as in the life cycle model, is that the insurance channel operates more strongly in the absence of an accumulation phase. In order to quantify the importance of asset accumulation in determining the strength of the insurance channel, we construct an approximation to the infinitely lived agent economy that features an accumulation phase. Relative to the infinitely lived agent model, the counterfactual model mainly differs from the infinitely lived agent model in that agents are endowed with zero wealth. In order to make the accumulation phase relevant, we assume agents have finite lifespans and die at the end of $J = 1000$ periods. In practice, we find that $J = 1000$ is a sufficiently large terminal age to mimic the infinitely lived agent model, such that welfare calculations do not depend on late-life utility. Therefore, by construction, the fundamental difference between the counterfactual model and the infinitely lived agent model is the building phase.[15]

In the counterfactual model, we find that optimal policy is public savings that is 2.35 times as large as output. Compared to the infinitely lived agent model's optimal debt to output ratio of -0.22, eliminating the accumulation phase generates a very large amount

[15]Neither the infinitely lived nor the counterfactual model feature any age-dependent features (e.g., no mortality risk, no age-dependent wage profile, no retirement and no Social Security).

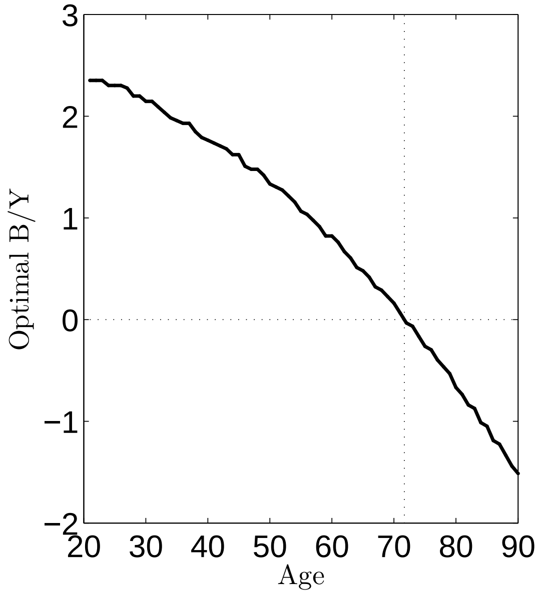


Figure 3: Optimal debt to output ratios associated with initial endowments from age x agents.

of optimal public savings.[16]

Next, using the calibrated counterfactual model, we conduct a partial equilibrium computational experiment to further isolate the impact of the accumulation phase on optimal policy. Suppose that the government chooses policy according to an alternative social welfare criterion that places less weight on utility during youth than does the ex ante Utilitarian welfare criterion. In particular, suppose that the alternative social welfare criterion places full weight on the expected present value of utility as of a given age $j^* > 1$, and zero weight on utility from ages 1 to $j^* - 1$. Government policy, therefore, maximizes agents' expected utility as of age j^*, subject to allocations being determined in competitive equilibrium:

$$\tilde{S}(V_{j^*}, \lambda_{j^*}) \equiv \max_{B} \left\{ \int V_{j^*}(a, \varepsilon; B) \, d\lambda_{j^*}(a, \varepsilon; B) \quad \text{s.t.} \quad G = rB + Y_y(\tau_0, B) \right\}.$$

Figure 3 plots the optimal policy under the alternative welfare criterion as a function of threshold age, j^*. We observe that optimal policy monotonically decreases from the public savings to output ratio of 2.35 when $j^* = 21$ to an optimal debt policy when $j^* \geq 71$. Increasing the threshold age defers the government's concern for agents to later in life. Once the threshold age is sufficiently high, the government no longer places value on agents' utility during the accumulation phase and finds it optimal to hold public debt as opposed to savings.

Across models, higher public debt (lower public savings) crowds out the productive capital stock and leads to a higher interest rate. The higher interest rate encourages agents to save, which improves self insurance. However, because agents must forgo

[16]In order to make quantitative comparisons across models, the counterfactual model's parameters are recalibrated to match all relevant the targets described in Section 4.

consumption to build wealth, the accumulation phase introduces a utility cost that reduces the potential welfare benefits from superior self insurance. Therefore, when the alternative social welfare criterion ignores the utility cost of wealth accumulation (for a sufficiently high threshold age j^*), the welfare benefit from government debt is much larger and optimal policy switches from public savings to debt.

5.3 Decomposing the Effects of Life Cycle Features

In this section we verify that the accumulation phase alone is sufficient to generate an optimal public debt policy. In what follows, we focus on comparisons between the life cycle model and two counterfactual models that isolate the effects of lifespan and age-dependent features (e.g., mortality risk, age-dependent wage profile, retirement and Social Security) on optimal policy. We find that, in fact, lifespan and age-dependent features generate a force for *higher* optimal public savings. In this way, optimal policy responds strongly to demographics as well as income composition.

In order to quantify the effects of lifespan and age-dependent model features, we consider two counterfactual economies. We refer to these two counterfactual economies as the “Short Life" and “Long Life" economies, respectively. The first economy, the Short Life economy, is a version of the life cycle model that excludes all age-dependent features (e.g., no mortality risk, no age-dependent wage profile, no retirement and no Social Security) while maintaining the same lifespan of $J = 81$ periods. The second economy, the Long Life economy, also removes age-dependent model features, but now extends the lifetime to $J = 401$ periods. The Short Life counterfactual economy allows us to isolate the age-dependent features as a source of optimal policy differences across life cycle and infinitely lived agent models, while the Long Life counterfactual economy allows us to isolate lifespan. Note that differences in optimal policy cannot be attributed to differences in initial endowments because agents in each economy are endowed with zero initial wealth and experience an accumulation phase.[17]

The first row in Table 4 reports the optimal policy across the life cycle and counterfactual models. We find that removing age-dependent model features while keeping lifespan fixed drives optimal savings-to-output from 60% to 200%. Eliminating age-dependent model features generates a very large amount of optimal public savings nearly four times as large as the optimal policy in the life cycle model. This difference in optimal policy mainly conflates an extension of expected lifetime due to the elimination of mortality risk, and an extension of working life due to the elimination of mandatory retirement. However, extending agents' lifespan alone has sizable effects on optimal public savings. We find that extending the lifespan from $J = 81$ periods to $J = 401$ periods additionally increases optimal savings-to-output from 200% to 250%.

Why does optimal public savings increase with expected lifespan? Recalling **??**, we observe that the maximum aggregate consumption corresponding to the Short and Long lifespan counterfactual economies is less than the respective optimal levels of aggregate consumption. While government policy could achieve an allocation that maximizes ag-

[17] In order to make quantitative comparisons across models, the each counterfactual model's parameters are recalibrated to match all relevant the targets described in Section 4.

Table 4: Optimal Policy and Measures of Income Composition and Inequality

	Life Cycle	Counterfactuals: Short Lifespan (81 periods)	Counterfactuals: Long Lifespan (401 periods)
Optimal Policy (B/Y)	0.59	2.00	2.48
Optimal Interest Rate	3.6%	2.6%	2.5%
Optimal Wage	1.19	1.25	1.26
Average share of lifetime labor income			
Baseline	0.88	0.86	0.92
Optimal	0.92	0.96	0.98
% Change	4.9	11.1	6.2
Coefficient of Variation: lifetime total income			
Baseline	0.36	0.32	0.31
Optimal	0.35	0.30	0.27
% Change	-1.8	-7.1	-13.8

gregate consumption, it chooses an optimal policy that overshoots the maximum level of aggregate consumption. Because public savings leads to a lower interest rate, the insurance channel does not motivate the government's policy.

Instead, optimal policy shifts the composition of lifetime income in order to decrease *income inequality*. Comparing the baseline to optimal economies in Table 4, lifetime labor income as a share lifetime total income increases while cross-sectional variation in lifetime total income decreases.[18] Furthermore, using Table 4 to compare the life cycle to counterfactual models, increasing expected lifespan leads to optimal policy that increases the share of lifetime labor income and the variation in lifetime total income. When agents live longer, they are exposed to greater ex post uncertainty and desire more insurance ex ante. As depicted in Figure 4, the wealth distribution endogenously becomes more unequal as expected lifespan increases. This is because lucky agents experience long periods of high labor productivity and accumulate large precautionary saving stocks to insure consumption against the probability of negative labor productivity growth. Increased public savings lowers the interest rate, discourages wealth accumulation and effectively mitigates spread in the wealth distribution as cohorts age. Figure 4 shows that optimal policy reverses the pattern of wealth inequality, with less wealth inequality

[18]We define lifetime asset income and lifetime labor earnings by,

$$\sum_{j=1}^{J} s_j \left(\frac{1}{1+r}\right)^{j-1} ra_j \quad \text{and} \quad \sum_{j=1}^{J} s_j \left(\frac{1}{1+r}\right)^{j-1} we_j h_j$$

respectively. Lifetime total income is the sum of lifetime labor and asset incomes. For the Short and Long Lifespan counterfactual models, there is no mortality risk and $s_j = 1$ for all $j = 1, \ldots, J$.

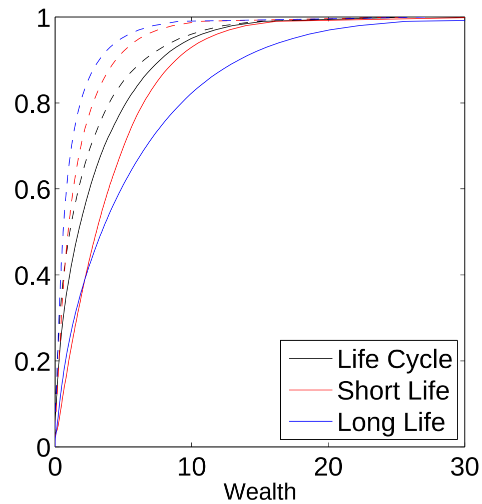


Figure 4: Cumulative Distribution Function for Wealth. Solid lines represent the baseline economy and dashed lines represent economies with optimal policy.

associated with longer expected lifespans.

Lastly, **??** shows that the life cycle model's optimal policy exhibits less aggregate consumption than the maximum. This suggests that not only is expected lifespan too short for the inequality channel to exert a strong effect, but there is a force that operates against the level channel. With retirement, agents work a shorter portion of their lives. As a result, savings serves as both an insurance instrument against idiosyncratic labor productivity shocks and a vehicle for financing post-retirement consumption. Agents have an increased incentive to accumulate savings early in life, which makes their savings decisions inelastic. In the absence of mandatory retirement, agents can work their entire lives and use savings entirely as an insurance instrument. It follows that savings in the Short Life counterfactual economy will be more elastic. Table 5 confirms that savings becomes more elastic as agents' expected lifespan (and working life) increases. The table shows the elasticity of aggregate private savings with respect to the public savings to output ratio (evaluated at baseline policy $B/Y = -0.67$) increases with expected lifespan.

Table 5: Aggregate Private Savings Elasticity

	Counterfactuals		
Life Cycle	**Short Lifespan (81 periods)**	**Long Lifespan (401 periods)**	**Infinitely Lived**
0.106	0.130	0.154	0.190

6 Conclusion

This paper measured the optimal quantity of public debt in a variant of the incomplete markets model that allows for an explicit life cycle. We find that it is optimal for the government to hold *savings* equal to 60% of output when life cycle features are included. In contrast, we find that it is optimal for the government to hold *debt* equal to 20% of output when these life cycle features are excluded. Furthermore, there are economically significant welfare consequences from not accounting for life cycle features when determining the optimal policy. We find that if a government implemented the infinitely lived agent model's optimal 20% debt-to-output policy in the life cycle model, then life cycle agents would be worse off by nearly one-half percent of expected lifetime consumption.

The substantial difference in optimal policies across the two models is primarily due to the effectiveness of debt policy to encourage agents to hold precautionary savings. Generally, higher government debt (or decreasing government savings) tends to crowd out the stock of productive capital, and leads to a higher interest rate which encourages agents to hold more savings. However, the efficacy of this channel is significantly less in the life cycle model compared to the infinitely lived agent model. In particular, in the infinitely lived agent model, agents do not experience an accumulation phase but instead experience a perpetual stationary phase. If the government holds more public debt, then the *steady state* level of aggregate savings is larger and the average agent has more wealth *ex ante*. In contrast, life cycle agents enter the model with zero wealth and immediately begin saving in the accumulation phase. Thus, although changes in the interest rate may increase the level of savings in the stationary phase for life cycle agents, these agents will still need to forgo consumption in order to invest in savings while build to the stationary phase. Ultimately, this significantly reduces the benefit of government debt from improved liquidity in the life cycle model.

When using quantitative models to answer economic questions, economists are constantly faced with a tradeoff between tractability and realism. These results demonstrate that when examining the welfare consequences of public debt or savings it is crucial to forgo the more tractable infinitely lived agent model and utilize a life cycle model.

References

AIYAGARI, S. R. AND E. R. MCGRATTAN (1998): "The optimum quantity of debt," *Journal of Monetary Economics*, 42, 447–469.

BELL, F. AND M. MILLER (2002): "Life Tables for the United States Social Security Area 1900 - 2100," .

CARROLL, C. D. (1992): "The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence," *Brookings Papers on Economic Activity*, 23, 61–156.

——— (1997): "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis," *Quarterly Journal of Economics*, 112, 1–55.

Chaterjee, S., J. Gibson, and F. Rioja (2016): "Optimal Public Debt Redux," Unpublished.

Conesa, J. C., S. Kitao, and D. Krueger (2009): "Taxing Capital? Not a Bad Idea after All!" *American Economic Review*, 99, 25–48.

Dávila, J., J. Hong, P. Krusell, and J.-V. Ríos-Rull (2012): "Constrained Efficiency in the Neoclassical Growth Model With Uninsurable Idiosyncratic Shocks," *Econometrica*, 80, 2431–2467.

Desbonnet, A. and T. Weitzenblum (2012): "Why Do Governments End Up with Debt? Short-Run Effects Matter," *Economic Inquiry*, 50, 905–919.

Dyrda, S. and M. Pedroni (2016): "Optimal fiscal policy in a model with uninsurable idiosyncratic shocks," Unpublished.

Floden, M. (2001): "The effectiveness of government debt and transfers as insurance," *Journal of Monetary Economics*, 48, 81–108.

Gouveia, M. and R. Strauss (1994): "Effective Federal Individual Income Tax Functions: An Exploratory Empirical Analysis," *National Tax Journal*, 47, 317–339.

Kaplan, G. (2012): "Inequality and the life cycle," *Quantitative Economics*, 3, 471–525.

Röhrs, S. and C. Winter (2016): "Reducing Government Debt in the Presence of Inequality," Unpublished.

A Appendix

A.1 Construction of the Balanced Growth Path

[TO BE COMPLETED]

A.2 Pecuniary Externality

[TO BE COMPLETED]

A.3 Welfare Decomposition

Proposition 1: *If preferences are additively separable in utility over consumption, $u(c)$, and disutility over hours, $v(h)$, then welfare changes can be decomposed as:*

$$(1+\Delta_{CEV}) = (1+\Delta^L)\underbrace{(1+\Delta^I)(1+\Delta^R)}_{\equiv(1+\Delta_D)}(1+\Delta^H)$$

Proof: Consider two economies, $i \in \{1,2\}$. Define ex ante welfare in economy $i \in \{1,2\}$ as:

$$S^i = S^i_c + S^i_h \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j u\left(c^i_j\right) \right] d\lambda^i_1 + \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j \zeta^i_{j+1} v\left(h^i_j\right) \right] d\lambda^i_1$$

Denote the Consumption Equivalent Variation (CEV) by Δ_{CEV}, which can be defined as the percent of lifetime consumption that an agent inhabiting economy $i = 1$ would pay in order to inhabit economy $i = 2$:

$$(1 + \Delta_{CEV})^{1-\sigma} S^1_c + S^1_h = S^2$$

Furthermore, define an individual's certainty equivalent consumption as the level $\bar{c}(a, \varepsilon, x, \zeta)$ such that the individual is indifferent between consuming $\bar{c}(a, \varepsilon, x, \zeta)$ at every age with certainty and consuming according to policy function $\{c_j(a, \varepsilon, x, \zeta)\}_{j=1}^{J}$ with uncertainty. That is, $\bar{c}(a, \varepsilon, x, \zeta)$ is defined by:

$$S^i_c \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j u\left(c^i_j\right) \right] d\lambda^i_1 = \left(\sum_{j=1}^{J} \beta^{j-1} s_j \right) \int u\left(\bar{c}^i(a_1, \varepsilon_1, x_1, \zeta_1)\right) d\lambda^i_1$$

which implies the definition of aggregate certainty equivalent consumption:

$$\bar{C}^i \equiv \int \bar{c}^i_1(a_1, \varepsilon_1, x_1, \zeta_1) d\lambda^i_1$$

Therefore, if agents only consume their certainty equivalent consumption allocation, then they only face ex ante risk in their consumption. Define the *redistribution effect* by a comparison between consuming an individual and aggregate certainty equivalent consumption allocation:

$$\int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j u\left((1-\omega^i_R)\bar{C}^i\right) \right] d\lambda^i_1 = \left(\sum_{j=1}^{J} \beta^{j-1} s_j \right) \int u\left(\bar{c}^i(a_1, \varepsilon_1, x_1, \zeta_1)\right) d\lambda^i_1$$

which implies:

$$1 - \omega^i_R = \frac{(S^i_c / \sum_{j=1}^{J} \beta^{j-1} s_j)^{\frac{1}{1-\sigma}}}{\bar{C}^i}$$

and

$$1 + \Delta^R = \frac{1 - \omega^2_R}{1 - \omega^1_R} = \frac{(S^2_c / S^1_c)^{\frac{1}{1-\sigma}}}{\bar{C}^2 / \bar{C}^1}$$

Likewise, we can define the *uncertainty effect* as a comparison between consuming at each age, the aggregate consumption allocation:

$$C^i = \sum_{j=1}^{J} \mu_j \int c_j^i(a, \varepsilon, x, \zeta) d\lambda_j^i$$

and the aggregate certainty equivalent consumption, $\bar{C}^i$. Then:

$$\int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j u \left((1 - \omega_I^i) C^i \right) \right] d\lambda_1^i = \left(\sum_{j=1}^{J} \beta^{j-1} s_j \right) \int u \left(\bar{C}^i \right) d\lambda_1^i$$

which implies:

$$1 - \omega_I^i = \frac{\bar{C}^i}{C^i} \quad \text{and} \quad 1 + \Delta_I = \frac{1 - \omega_I^2}{1 - \omega_I^1}$$

Lastly, define the labor disutility effect Δ^H as the percent of lifetime consumption that an individual would pay to change their hours allocation:

$$(1 + \Delta^H)^{1-\sigma} S_c^2 = S_c^2 + (S_h^2 - S_h^1)$$

Proceeding from the definition of the CEV, we can decompose welfare as follows:

$$\begin{array}{ccccccccc} (1 + \Delta_{CEV}) & = & (1 + \Delta^L) & \cdot & (1 + \Delta^I) & \cdot & (1 + \Delta^R) & \cdot & (1 + \Delta^H) \\ \left(\dfrac{S^2 - S_h^1}{S_c^1} \right)^{\frac{1}{1-\sigma}} & = & (C^2 / C^1) & \cdot & \dfrac{\bar{C}^2 / \bar{C}^1}{C^2 / C^1} & \cdot & \dfrac{(S_c^2 / S_c^1)^{\frac{1}{1-\sigma}}}{\bar{C}^2 / \bar{C}^1} & \cdot & \dfrac{\left((S^2 - S_h^1) / S_c^1 \right)^{\frac{1}{1-\sigma}}}{(S_c^2 / S_c^1)^{\frac{1}{1-\sigma}}} \end{array}$$

Canceling terms on the right hand side of the expression readily shows the decomposition holds as desired. Decomposing the labor hours effect follows similar reasoning. ■