Spying in Contests

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Abstract

Two players compete for a prize and their valuations are private information. Before the contest, each player acquires a costly, noisy and private signal regarding the opponent's valuation. In equilibrium, each player's effort is non-decreasing in the posterior probability that the opponent has the same valuation. Accounting for the cost of spying, players are better off spying when the spying technology is partially but not perfectly informative. Suppose instead that each player can, at no cost, ex ante commit to disclose a signal about her valuation to the opponent, but cannot observe realizations of the signal. Then every equilibrium involves non-disclosure by at least one player, even though some disclosure by each player would benefit both.

Keywords: Spying, Espionage, All-pay auction, Information acquisition, Rotation order

JEL Classification: C72 D44 D74 D82

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1 Introduction

Winner-take-all contests, like rent-seeking contests for monopoly rights, patent races, lobbying, political campaigns and competitions for promotion, burden participants with the prospect that their investments may yield no reward. The efforts, time and resources invested in competing for the prizes are unrecoverable, and typically, only the participant with the highest investment reaps the rewards of the contest. Thus, anticipating the rivals' intentions becomes particularly valuable; learning that rivals will invest little can save on the investment to win the prize, and, conversely, learning of an excessive investment outlay by rivals would lead a firm to avoid investing in a lost cause. This paper studies players' incentives of acquiring information about the opponents prior to winner-take-all contests, and shows relevant welfare analysis.

In competing for a procurement contract, for example, suppliers spend enormous time, resources and efforts to prepare proposals for a buyer to evaluate.[1] This process is even costlier when it also involves bribing the procurement agent (Celentani and Ganuza, 2002; Burguet and Che, 2004). Since each supplier may value the contract differently, their willingness to commit resources to win the contract or to bribe the procurement agent may differ. Gathering intelligence on the opponent's valuation can prove particularly valuable. This also applies to R&D contests or patent races, where intelligences regarding the rival's new research progresses, scientists' backgrounds, or prototypes of new products/samples of new drugs, are beneficial to the firms. To obtain the intelligences in the examples above, players may hire hackers to steal information from rival's computer, investigators to search through office trash or detectives to steal files from office safe, etc.[2]

The existing literature on contests implies that players will overall not benefit from such spying in contests. Kovenock et al. (2015) show that the payoffs to players are the same when valuations of the prize are commonly known and when they are private information. However, perfect information about the opponent is extremely difficult — if not impossible — to acquire in reality. Furthermore, common knowledge assumption is not plausible in this situation as spying activities in real life often induce uncertainty about the rival's belief. For example, a firm hacked by its competitor does usually not know what intelligence the competitor has obtained and hence, remains ignorant about the competitor's belief.

[1]Airbus and Boeing spent 10 years in competing for a contract to build U.S. Air Force aerial fueling tankers. According to the chairman of European Aeronautic Defence and Space Company (now known as Airbus Group), EADS spent over $200 million in the competition.

[2]Detectives hired by Larry Ellison, the head of Oracle, bribed the cleaning staff at Microsoft's office to gather sensitive information from the office trash. Staffs of Procter&Gamble were found searching the garbage of Unilever — its competitor in the hair-care market — for "the Organics and Sunsilk brands of shampoo" which contains commercially sensible information. Large companies like General Motors, Kodak, and BP even set up their own separate competitive intelligence units to study their competitors (Billand et al., 2016).

In this paper, we present a model of contests in which players can acquire partial and private information regarding the opponent's valuation for the prize before they compete. We consider a first price all-pay auction with one indivisible prize and two players who have independent private valuations (IPV).[3] For tractability, each player's valuation is assumed to be either *high* or *low*. Before participating the contest, each player acquires a costly, noisy and private *spying* signal about her opponent's valuation. In acquiring the signal, she chooses a level of accuracy which is a continuous decision variable whose lower and upper bound corresponds to completely uninformative and perfectly informative signals, respectively.[4] She *then* observes both her valuation and signal realization, and exerts effort in the contest, i.e., bids in the auction.

We mainly show three results. First, in a simplified setting where the accuracy of the spying signal is exogenously fixed and the signal costless, we show that each player's effort in the unique symmetric equilibrium is non-decreasing in the probability she perceives that the two of them are evenly matched. If the signals are relatively noisy (hence the contest is opaque), such an equilibrium is allocative efficient. In this case, when the player's valuation is *high*, her equilibrium effort is non-decreasing in the posterior likelihood that the opponent's valuation is also *high* ("motivation effect" of spying); instead, when her valuation is *low* then her effort is non-increasing in such a likelihood ("demotivation effect" of spying). If, however, the signals are sufficiently accurate (hence the contest is relatively transparent), then the unique symmetric equilibrium is allocative inefficient. With very accurate signals, the low valuation type of the opponent is aware that he must be exposed and thus, needs to signal-jam the player by mixing his effort. This, in turn, induces the player with either the high or the low valuation to mix in the same support, hence the inefficiency. We emphasize that the unique symmetric equilibrium of the contest replicates the unique equilibrium in the all-pay auction with IPV and complete information when the exogenous accuracy reaches its lower and upper bound, respectively.

Second, a comparative statics analysis shows that players earn higher expected payoff than what they earn in the IPV setting when their signals are partially but not perfectly informative. The intuition behind this result is best explained in two parts. Firstly, a player benefits from anticipating the opponent's move based on the intelligence that she obtains from spying. In particular, spying is most profitable when the high and the low valuation players' strategies are completely different, i.e., when the contest is opaque and thus, the equilibrium is efficient. When the contest, however, becomes sufficiently transparent (so we have the inefficient equilibrium)

[3]All-pay auctions have been applied to study winner-take-all contests with sunk investments include procurement contests (Kaplan, 2012), research and development (Che and Gale, 2003; Dasgupta, 1986), rent-seeking and lobbying (Baye et al., 1993; Che and Gale, 1998, 2006; Ellingsen, 1991), and competition for promotion (Clark and Riis, 1998).

[4]Accuracy is defined by rotation order (Johnson and Myatt, 2006) for tractability.

and the two strategies are similar, spying is less profitable. In fact, the marginal return to spying is zero when the opponent believes the contest is perfectly transparent. Secondly, the player being spied on can also benefit from the espionage. For example, a firm is aware that the rival may have acquired some of the commercially-sensitive information, such as classified files, financial records, or recordings of internal meetings. Since the rival would act upon this intelligence, the firm is able to anticipate the rival's move (with some noise) as it has a sense of what conclusion the opponent might reach based on the intelligence. Therefore, both spying on and being spied on by the opponent benefit the player. We then characterize the equilibrium choice of accuracy by imposing mild conditions on the spying cost function, and show that players overall better off spying even accounting for the cost of spying.

The information structure in the model we present lies between incomplete information (Lu and Parreiras, 2014; Konrad, 2004; Amann and Leininger, 1996) and complete information setting (Baye et al., 1996; Ellingsen, 1991; Hillman and Riley, 1989). By varying the accuracy of the spying signal from completely uninformative to perfectly informative, we characterize the equilibrium in an arbitrarily large set of information structure in our first result. There has been some progresses in examining behavior in the all-pay auctions with information structures that are different from IPV and complete information, include common valuation and affiliated signals (Rentschler and Turocy, 2016; Chi et al., 2015), and interdependent valuations (Siegel, 2013; Krishna and Morgan, 1997). In these settings players can infer information about the opponent from their own valuations. The current paper is different from this line of research for the independence between each player's valuation and information about the opponent.

In the pioneer study of spying activities in contests by Baik and Shogren (1995), the authors propose a model where players can acquire information regarding their relative abilities. The model also allows partial information acquisition by restricting the distribution of signals to mean-preserving spreads. However, the contest success function in their model is neither as in Tullock contests nor as in all-pay auctions, but instead subjective: both players could believe (in "equilibrium") that they will win with, e.g., 60% probability.[5] In the current paper, we instead take a standard game theoretic approach by analyzing behavior in the Bayesian Nash Equilibrium. In addition, Zhang (2015) consider one-sided private information setting with a perfect spying technology in both all-pay auction and Tullock contest (Tullock, 1967), whereas the current paper focuses on the all-pay auction with two-sided private information and imperfect spying technology.[6]

[5]See Bolle (1996) for a criticism of the approach and Baik and Shogren (1996) for the authors' respond.

[6]In the IO literature, spying/espionage are also considered in entrant deterrence (Barrachina et al., 2014; Solan and Yariv, 2004), in price and quantity competition in duopoly (Kozlovskaya, 2016; Wang, 2016; Whitney and Gaisford, 1999), and in multi-market competitions (Billand et al., 2016).

Spying may be prohibitively costly (or illegal). The previous results imply, however, that players would benefit if they were to disclose to each other a noisy signal of their valuation. Would players disclose such information voluntarily? Section 4 considers a twist of the main model in which each player commits to disclose a signal about her own valuation to the opponent before the contest. In doing so, she chooses an accuracy for the signal which the opponent will receive. The accuracy is observable to both players, but the signal realization is only observable to the opponent. Neither disclosing nor receiving the signal incurs any cost to any player. In the patent race example, this corresponds to a firm provides, for instances, a prototype of a new product or samples of a new drug to the rival firm. With such pieces of hard evidence (which determines accuracy), the rival can conduct experiments on the product/drug to obtain relevant parameters (which corresponds to signal realizations) which are unavailable to the firm who discloses the information.[7]

Based on the second result, we show that if players disclose partially informative signals because they have set up an agreement or are required by the regulator to do so, then the total expected effort/expenditure is strictly lower than that in the IPV setting. The result implies, for example, in procurement contests money spent on bribery and efforts exerted in the process are lower on average if suppliers disclose partial information regarding their valuation for the contract. This implication is particularly important because public procurement is a hotbed for bribery, e.g., among OECD countries (Ehlermann-Cache and Others, 2007), which creates social inefficiencies. In addition, such a corrupted behavior is hard to detect. When suppliers disclose to each other, the social costs are reduced without actually detecting the bribing behavior.[8] This result is qualitatively consistent with Kovenock et al. (2015) who also focus on decentralized information disclosure but restrict attention to full disclosure, and show that full disclosure lowers total expected expenditure but leads to allocative inefficiency.[9] The current paper, however, implies that partial information disclosure reduces total expected expenditure with potentially less efficiency losses.

Nevertheless, we show, as our last result, that disclosing a partially informative signal is weakly dominated by disclosing an uninformative signal for each player. Even though partial disclosure by both players lowers ex ante expected efforts and increases payoffs, a player can

[7]For another example, consider each firm "discloses" information by choosing its security level of office buildings and firewalls. Thus, the accuracy is the security level, and the intelligences that the rival has acquired are not observable to the firm.

[8]There is a similar implication to rent-seeking and lobbying contests: bribes to politicians are social costs, and disclosure improves welfare by reducing expenditures. In the patent race example, information disclosure between firms reduces duplicated investments in R&D.

[9]Most of the studies on information disclosure in contest literature take a centralized view and analyze how a contest organizer should disclose information to players in order to maximize total effort (Lu et al., 2016; Zhang and Zhou, 2016; Chen, 2016; Serena, 2015; Denter et al., 2014).

do better by unilaterally adding noise to the signal disclosed to the opponent, as then the evenly matched opponent is more likely to be demotivated. Therefore, there does not exist any equilibrium in which both players disclose any partially informative signals.

In summary, we provide an interpretation of why we often observe spying/espionage activities but seldom information disclosure/sharing in real life winner-take-all contests, even if the latter results in a better outcome for players. We also show that spying activities overall benefit players but a better social outcome can be achieved through mandatory, albeit imperfect, disclosure between players. The findings are particularly important for wasteful contests, e.g., rent-seeking contests (Congleton et al., 2008), where resources invested by players only serve as a means to determine the winner but do not contribute to value creation (Tullock, 1967; Posner, 1974). For instance, the estimated social cost of rent-seeking for the US is 22.6 percent of GNP in 1985 (Laband and Sophocleus, 1988); and it has been long argued (since (Wright, 1983)) that patent races generate wasteful duplication of effort.

The rest of the paper proceeds as follows. Section 2 presents the main model of spying in contests. Section 3 characterizes the equilibrium effort when the accuracy is exogenously fixed, presents the comparative statics analysis, and characterizes the equilibrium choice of accuracy. Section 4 presents the model of information disclosure, the analysis on disclosure agreement, and the equilibrium choice of disclosure. Section 5 concludes.

2 The model

There are two risk neutral players, indexed by $i \in \{1,2\}$, who compete in a contest with one indivisible prize for which they have independent private valuations (IPV). Player i ($i = 1,2$) may value the prize at θ_h with probability $p_h \in (0,1)$ or at θ_l with probability $p_l = 1 - p_h$, where $\theta_h > \theta_l > 0$. Players know only their own valuations, and the distribution of the opponent's valuation. We refer to a representative player $i = 1,2$ as "she" and her opponent, player $j = 3 - i$, as "he". We also refer to a player with θ_h (θ_l) as a "high (low) valuation player".

Information acquisition (spying): Player i can acquire additional information regarding the opponent by receiving a private spying signal (hereafter "signal") about the opponent's valuation. The possible signal realization π_i is drawn from a compact set $[\underline{\pi}, \overline{\pi}]$. Player i acquires information about θ_j by choosing from a family of joint distributions over $[\underline{\pi}, \overline{\pi}] \times \{\theta_h, \theta_l\}$

$$\{F(\pi_i, \theta_j | \alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$$

indexed by $\alpha_i \in [\underline{\alpha}, \overline{\alpha}]$. We refer to $F(\pi_i, \theta_j|\alpha_i)$ the signal, α_i the accuracy (to be defined shortly), and π_i the realization of the signal. Since the conditional distribution of π_i depends only on θ_j which is independent of θ_i, π_i is thus independent of π_j.

Let $F(\cdot, \alpha_i)$ denote the marginal distribution of π_i with corresponding density $f(\cdot, \alpha_i)$, given any α_i. Furthermore, denote by $F_h(\cdot, \alpha_i)$ ($F_l(\cdot, \alpha_i)$) the conditional cumulative distribution of π_i given $\theta_j = \theta_h$ ($\theta_j = \theta_l$). Let $f_h(\cdot, \alpha_i)$ and $f_l(\cdot, \alpha_i)$ be the corresponding probability density functions, and assume both are differentiable on both arguments. We assume that w.l.o.g. $F(\cdot, \alpha_i)$ is uniform on $[0,1]$ for every given $\alpha_i \in [\underline{\alpha}, \overline{\alpha}]$,[10] i.e., $\underline{\pi} = 0, \overline{\pi} = 1$ and

$$p_h F_h(\pi_i, \alpha_i) + p_l F_l(\pi_i, \alpha_i) = \pi_i, \quad (1)$$

$$p_h f_h(\pi_i, \alpha_i) + p_l f_l(\pi_i, \alpha_i) = 1. \quad (2)$$

Posterior belief: Observing π_i leads player i to update her belief on θ_j according to Bayes' rule. Denote player i's posterior belief that player j has valuation θ_h upon receiving π_i by $\mu(\pi_i, \alpha_i)$, thus

$$\mu(\pi_i, \alpha_i) = \frac{p_h f_h(\pi_i, \alpha_i)}{p_h f_h(\pi_i, \alpha_i) + p_l f_l(\pi_i, \alpha_i)}$$

Note that, according to (2), $\mu(\pi_i, \alpha_i) = p_h f_h(\pi_i, \alpha_i)$ and $1 - \mu(\pi_i, \alpha_i) = p_l f_l(\pi_i, \alpha_i)$.

[10] Since a random variable's percentile function as another random variable is uniformly distributed, we can always transform an alternatively distributed signal to a uniformly distributed signal which contains exactly the same information.

Information order: To rank signals by accuracy, we adopt the "rotation order" which was first introduced by Johnson and Myatt (2006), and was applied to auction settings by Shi (2012).[11]

Definition 1 (Rotation order)**.** *A local change in* α *leads to a rotation of* $f_h(\pi, \alpha)$ *if, for some* π^+_α *and each* $\pi \in [0, 1]$,

$$\frac{\partial f_h(\pi, \alpha)}{\partial \alpha} \gtreqless 0 \Longleftrightarrow \pi \gtreqless \pi^+_\alpha.$$

If this holds for $\forall \alpha \in [\underline{\alpha}, \overline{\alpha}]$*, then* $\{F(\pi, \theta|\alpha)\}$ *is ordered by a sequence of rotations.*

When α increases, $f_h(\pi, \alpha)$ rotates *counter clockwise* around π^+_α, which implies $f_l(\pi, \alpha)$ rotates *clockwise* according to (2). For example, when player i increases her accuracy of signal marginally from α to α', it must be true, by definition, that $f_h(\pi, \alpha) \gtreqless f_h(\pi, \alpha')$ for $\pi \lesseqgtr \pi^+_\alpha$, and that $f_l(\pi, \alpha) \lesseqgtr f_l(\pi, \alpha')$ for $\pi \lesseqgtr \pi^+_\alpha$. See Figure 1 and 2.

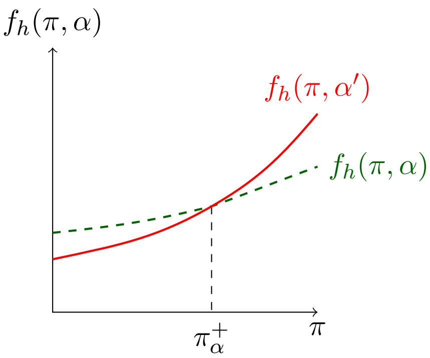

Figure 1: Increasing α to α' means counter clockwise rotation of $f_h(\pi, \alpha)$

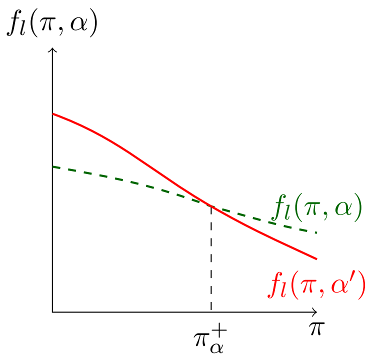

Figure 2: Increasing α to α' means clockwise rotation of $f_l(\pi, \alpha)$

When player i chooses $\alpha_i = \underline{\alpha}$, we have $f_h(\pi_i, \underline{\alpha}) = f_l(\pi_i, \underline{\alpha}) = 1$ for all $\pi_i \in [0, 1]$, see Figure 3. This is the case when any realization of the signal does not convey any information about the opponent. When player i chooses $\alpha_i = \overline{\alpha}$, we have $f_h(\pi_i, \overline{\alpha}) = 0$ if $\pi_i \le p_l$ and $f_h(\pi_i, \overline{\alpha}) = \frac{1}{p_h}$ if $\pi_i > p_l$, see Figure 4. This is the case when each realization conveys perfect information about the opponent.

Spying cost: Each player's cost of acquiring the signal is captured by a convex increasing function $C(\alpha)$ with $C(\underline{\alpha}) = 0$. Denote by $MC(\alpha) = \frac{\partial C(\alpha)}{\partial \alpha}$ the marginal cost, let $MC(\underline{\alpha}) = 0$ and $MC(\alpha) > 0$ for $\forall \alpha > \underline{\alpha}$.

Effort and payoff in the contest: Player i decides her effort b_i after observing θ_i and π_i. Thus, the contest stage of the game is a Bayesian game with two-dimensional types, and the effort of player i is a two-to-one mapping:[12] $b : \{\theta_h, \theta_l\} \times [0, 1] \to \mathbb{R}_+$.

[11] See Ganuza and Penalva (2010) for thorough discussions on signal ordering.

[12] The two-to-one mapping strategy in the contest creates complications in characterizing the equilibrium. In most of the previous studies involving multi-dimensional type space in auctions, all dimensions are payoff relevant to the player. Hence, either the equilibrium bidding strategy is monotonically increasing in both dimensions (Tan,

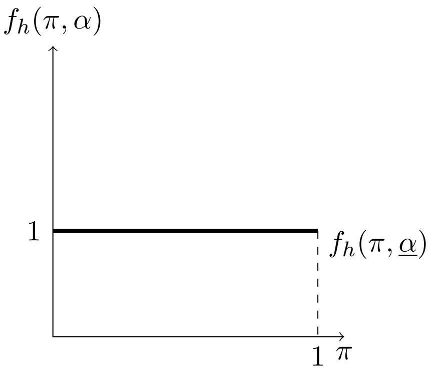

Figure 3: Completely uninformative signal $\underline{\alpha}$

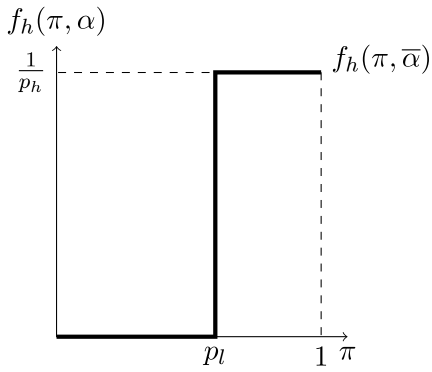

Figure 4: Perfectly informative signal $\overline{\alpha}$

Players choose their efforts in the contest simultaneously. The player who exerts higher effort wins the prize, whereas the losing player's effort is unrecoverable. Ties are broken with equal probabilities. Thus, player i with valuation θ_i exerting effort b_i earns a payoff:

$$u(b_i, b_j, \theta_i) = \begin{cases} -b_i, \text{ if } b_i < b_j \\ \theta_i - b_i, \text{ if } b_i > b_j \\ \frac{1}{2}\theta_i - b_i, \text{ if } b_i = b_j \end{cases}$$

A contest with the above payoff, $u(b_i, b_j, \theta_i)$, is also known as a first price all-pay auction.

Timing: Firstly, player i chooses the accuracy α_i for the signal to be acquired on the opponent. Secondly, Nature determines the valuation profile according to the prior distribution and player i $(i = 1, 2)$ observes θ_i. Thirdly, according to θ_j and α_i, Nature determines a signal realization π_i observed by player i. Finally, player i chooses her effort b_i according to her private information (θ_i, π_i). The timing of the game is also shown in Figure 5.

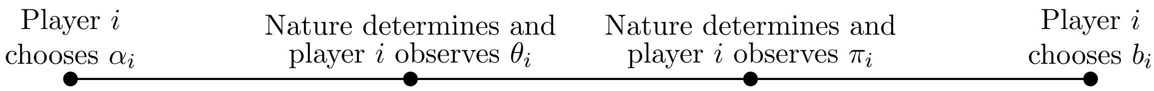

Figure 5: Timing of the game $(i = 1, 2)$

For the rest of the paper, we make the following assumptions:

Assumption 1. *Monotonic likelihood ratio property (MLRP): Given any* $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $\frac{f_h(\pi,\alpha)}{f_l(\pi,\alpha)}$ *is non-decreasing in* $\pi \in [0, 1]$.

Assumption 2. *The family of signals,* $\{F(\pi_i, \theta_j|\alpha_i)\}_{\alpha_i \in [\underline{\alpha}, \overline{\alpha}]}$*, is rotation ordered for* $\forall \alpha_i \in [\underline{\alpha}, \overline{\alpha}]$ *around a sequence of rotation points* $\pi^+_{\alpha_i}$*, where* $i = 1, 2$*, and* $j = 3 - i$*.*

2016), or the two-dimensional signal can be translated into a summary statistic which is monotonically increasing in bids (Goeree and Offerman, 2003). In general, it is difficult to characterize the equilibrium due to "monotonicity is not naturally defined" (Tan, 2016) or to non-existence of equilibrium (Jackson, 2009).

Assumption 1 implies the posterior belief $\mu(\pi_i, \alpha_i)$ is non-decreasing in π_i fixing α_i. Assumption 2 guarantees that players' family of signals are rotation ordered, and the sequence of rotation points are the same across players.

3 Analysis of spying in contests

3.1 Exogenous accuracy

In this section we study a simplified model where each player exogenously receives a free, noisy and private spying signal about the opponent's valuation. The accuracy of each player's signal is common knowledge.

To characterize the equilibrium effort in the contest, we start by showing some useful properties of the symmetric, pure strategy, and allocative efficient equilibrium. An equilibrium of the contest is allocative efficient if type (θ_h, s) of each player's effort is higher than type (θ_l, t)'s effort with probability one for any $s, t \in [0, 1]$.

Lemma 1. *Given any $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, then in any symmetric, allocative efficient, pure strategy equilibrium of the contest, the following must be true:*

1. *Monotonicity: type (θ_h, π) of player i's equilibrium effort, denote by $b_h(\pi, \alpha)$, is non-decreasing in π and type (θ_l, π) of player i's equilibrium effort, denote by $b_l(\pi, \alpha)$, is non-increasing in π;*

2. *Continuity: both players' strategies are continuous without any atoms;*

3. *Initial conditions: $b_l(1, \alpha) = 0$ and $b_l(0, \alpha) = b_h(0, \alpha)$.*

See all the proofs in the Appendix. Lemma 1 implies that the effort in any symmetric, pure strategy, allocative efficient equilibrium is non-decreasing in the posterior that the opponent has the same valuation as the player. Lemma 2 below provides the necessary condition for the existence of such an equilibrium.

Lemma 2 (Efficiency)**.** *There exists a symmetric, pure strategy, allocative efficient equilibrium in the contest only if $\frac{f_h(\pi,\alpha)}{f_l(\pi,\alpha)} \geq \frac{\theta_l}{\theta_h}$ for all $\pi \in [0, 1]$.*

Definition 2. *Denote by $\widehat{\alpha}$ the highest possible accuracy of signals that satisfy $\frac{f_h(\pi,\alpha)}{f_l(\pi,\alpha)} \geq \frac{\theta_l}{\theta_h}$ for all $\pi \in [0, 1]$.*

Lemma 2 implies that in any symmetric, pure strategy, allocative efficient equilibria of the contest, it must be true that $\alpha \in [\underline{\alpha}, \widehat{\alpha}]$.[13] The necessary condition given in the lemma imposes a lower bound on the likelihood ratio which in turn, imposes an upper bound on the accuracy of the signal.

In light of Lemma 1 and 2, we now characterize the symmetric, pure strategy, and allocative efficient equilibrium. The expected payoffs of types (θ_l, π) and (θ_h, π) of player i when choosing b, given that player j plays the equilibrium efforts $b_h(\pi, \alpha)$ and $b_l(\pi, \alpha)$, are given by:

$$\widetilde{U}_l(b|\pi,\alpha) = \theta_l \left[1-\mu(\pi,\alpha)\right] \int_{b_l^{-1}(b,\alpha)}^{1} f_l(\Pi,\alpha) d\Pi - b \tag{3}$$

$$\widetilde{U}_h(b|\pi,\alpha) = \theta_h \left[\left[1-\mu(\pi,\alpha)\right] + \mu(\pi,\alpha) \int_0^{b_h^{-1}(b,\alpha)} f_h(\Pi,\alpha) d\Pi \right] - b \tag{4}$$

where $b_l^{-1}(b, \alpha)$ and $b_h^{-1}(b, \alpha)$ are the inverse of effort functions by player j, and they in fact represent the information type of the low and the high valuation player j who chooses b, respectively. By the first order conditions w.r.t. b from (3) and (4):

$$\begin{aligned} \frac{\partial b_l(\pi,\alpha)}{\partial \pi} &= -\left[1-\mu(\pi,\alpha)\right] f_l(\pi,\alpha)\theta_l \\ \frac{\partial b_h(\pi,\alpha)}{\partial \pi} &= \mu(\pi,\alpha) f_h(\pi,\alpha)\theta_h \end{aligned}$$

These can be solved based on the initial conditions in Lemma 1, and the solutions are given in Proposition 1. Furthermore, Proposition 1 also characterizes the allocative inefficient equilibrium where the necessary condition given in Lemma 2 is not satisfied.

Proposition 1. *Suppose Assumption 1 is satisfied.*

- *If $\alpha \in [\underline{\alpha}, \widehat{\alpha}]$, then there exists a unique symmetric equilibrium in which type (θ_l, π) and (θ_h, π) play the following pure strategies, $b_l(\pi, \alpha)$ and $b_h(\pi, \alpha)$, respectively:*

$$\begin{aligned} b_l(\pi,\alpha) &= \theta_l \int_\pi^1 \left[1-\mu\left(\Pi,\alpha\right)\right] dF_l(\Pi,\alpha), \\ b_h(\pi,\alpha) &= \theta_h \int_0^\pi \mu\left(\Pi,\alpha\right) dF_h(\Pi,\alpha) + \theta_l \int_0^1 \left[1-\mu\left(\Pi,\alpha\right)\right] dF_l(\Pi,\alpha). \end{aligned}$$

- *If $\alpha \in (\widehat{\alpha}, \overline{\alpha}]$, then there exists a unique symmetric equilibrium in which type (θ_l, π) and*

[13]By MLRP and rotation order, for all $\alpha \in [\underline{\alpha}, \widehat{\alpha}]$ the condition in Lemma 2 is satisfied for all $\pi \in [0, 1]$, and for all $\alpha \in (\widehat{\alpha}, \overline{\alpha}]$ the condition is not satisfied for at least $\pi = 0$.

(θ_h, π) with $\pi > \pi^*$ play the following pure strategies, $b_l(\pi,\alpha)$ and $b_h(\pi,\alpha)$, respectively:

$$\begin{aligned} b_l(\pi,\alpha) &= \theta_l \int_{\pi}^{1} [1-\mu(\Pi,\alpha)]\, dF_l(\Pi,\alpha), \\ b_h(\pi,\alpha) &= \theta_h \int_{\pi^*}^{\pi} \mu(\Pi,\alpha)\, dF_h(\Pi,\alpha) \\ &+ \theta_l \int_{\pi^*}^{1} [1-\mu(\Pi,\alpha)]\, dF_l(\Pi,\alpha) + \frac{\theta_h\theta_l}{p_h\theta_l + p_l\theta_h}\pi^*; \end{aligned}$$

and type (θ_l, π) and (θ_h, π) with $\pi \le \pi^*$ mix over $[b_l(\pi^*,\alpha), b_h(\pi^*,\alpha)]$ according to CDF $\sigma_l(b|\pi,\alpha)$ and $\sigma_h(b|\pi,\alpha)$, respectively:

$$\begin{aligned} \sigma_l(b|\pi,\alpha) &= \sigma_h(b|\pi,\alpha) \\ &= \frac{p_h\theta_l + p_l\theta_h}{\theta_h\theta_l\pi^*}\left(b - \theta_l \int_{\pi^*}^{1} [1-\mu(\Pi,\alpha)]\, dF_l(\Pi,\alpha)\right), \end{aligned}$$

where π^* is given by

$$\theta_l \int_0^{\pi^*} f_l(\Pi,\alpha) d\Pi = \theta_h \int_0^{\pi^*} f_h(\Pi,\alpha) d\Pi. \tag{5}$$

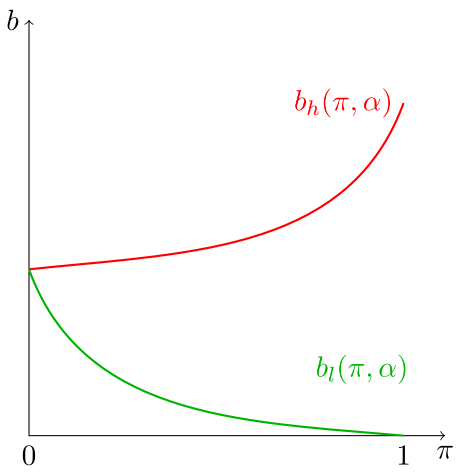

Figure 6: Allocative efficient equilibrium

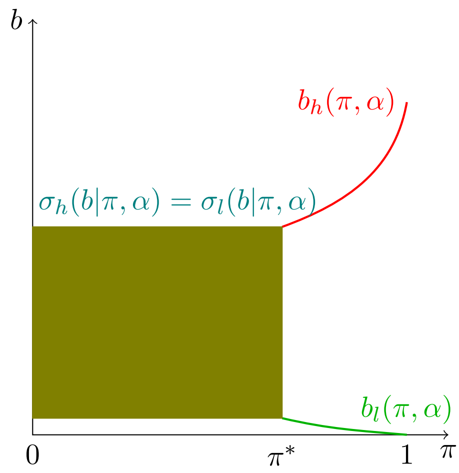

Figure 7: Allocative inefficient equilibrium

See Figure 6 for the equilibrium with $\alpha \in [\underline{\alpha}, \widehat{\alpha}]$ and Figure 7 for the equilibrium with $\alpha \in (\widehat{\alpha}, \overline{\alpha}]$. On the one hand, when the signals are not informative enough, i.e., $\alpha \in [\underline{\alpha}, \widehat{\alpha}]$, the equilibrium allocation is efficient, because players can still hide behind their private information. In such an equilibrium, the belief that the opponent has the high valuation — induced by higher π — encourages the high valuation player to compete more aggressively to increase the odds of winning ("motivation effect"), and discourages the low valuation player to compete more conservatively to save the cost of effort ("demotivation effect").

On the other hand, when players' signals are sufficiently informative, i.e., $\alpha \in (\widehat{\alpha}, \overline{\alpha}]$, we have inefficient allocation in equilibrium. In this case, player i is very likely to receive, for example, relatively low signal realizations ($\pi_i < \pi^*$) when the opponent has the low valuation. Thus, she can deviate her efforts to the level just above the support of the low valuation opponent's effort to win (almost) for sure with minimum effort. The opponent is aware of this fact and thus, mixes the effort in order to signal-jam player i. Hence, both the high and the low valuation player i who receives $\pi_i \leq \pi^*$ randomize in the interval where player j mixes.

Corollary 1. *In the equilibrium as given by Proposition 1, the following must be true:*

(i) Denote by $M_l(\pi, \alpha)$ and $M_h(\pi, \alpha)$ the equilibrium expected payoff for types (θ_l, π) and (θ_h, π), respectively. Then, for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, $M_l(\pi, \alpha)$ and $M_h(\pi, \alpha)$ are non-increasing in π:

$$\begin{aligned} \frac{\partial M_l(\pi,\alpha)}{\partial \pi} &= p_l \theta_l \frac{\partial f_l(\pi,\alpha)}{\partial \pi} \int_\pi^1 f_l(\Pi,\alpha) d\Pi \leq 0 \\ \frac{\partial M_h(\pi,\alpha)}{\partial \pi} &= -p_h \theta_h \frac{\partial f_h(\pi,\alpha)}{\partial \pi} \int_\pi^1 f_h(\Pi,\alpha) d\Pi \leq 0 \end{aligned}$$

where the equalities are only satisfied when $\pi = 1$.

(ii) The pure strategies are weakly convex in π:

$$\begin{aligned} \frac{\partial^2 b_l(\pi,\alpha)}{\partial \pi^2} &= -2 p_l \theta_l f_l(\pi,\alpha) \frac{\partial f_l(\pi,\alpha)}{\partial \pi} \geq 0 \\ \frac{\partial^2 b_h(\pi,\alpha)}{\partial \pi^2} &= 2 p_h \theta_h f_h(\pi,\alpha) \frac{\partial f_h(\pi,\alpha)}{\partial \pi} \geq 0 \end{aligned}$$

and the mixed strategies are independent of π:

$$\frac{\partial \sigma_l(b|\pi,\alpha)}{\partial \pi} = \frac{\partial \sigma_h(b|\pi,\alpha)}{\partial \pi} = 0.$$

The proofs of all the corollaries in the paper except Corollary 2 are straightforward and thus, are omitted. Part (i) of Corollary 1 implies that it is never a good news that the opponent is more likely to have the high valuation. Part (ii) of the corollary suggests that the competition becomes fiercer when player i is more confident that the two players are evenly matched.

When there is a marginal increase in the accuracy, the equilibrium efforts are more sensitive to a marginal change of π, see Corollary 2.

Corollary 2 (Sensitivity)**.** *The slope of $b_h(\pi, \alpha)$ and $b_l(\pi, \alpha)$ are increasing (decreasing) in α for $\pi > (<) \pi_\alpha^+$, i.e.,*

$$\frac{\partial^2 b_h(\pi,\alpha)}{\partial \pi \partial \alpha}, \frac{\partial^2 b_l(\pi,\alpha)}{\partial \pi \partial \alpha} \lesseqgtr 0, \text{ for } \pi \lesseqgtr \pi_\alpha^+.$$

Furthermore, $\frac{\partial \pi^*}{\partial \alpha} > 0$.

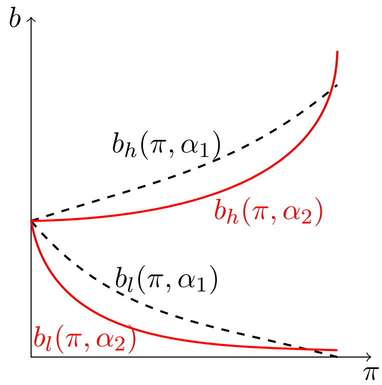

Figure 8: Rotation and sensitivity: rotation from α_1 to α_2 decreases the slope of efforts for $\pi < \pi_\alpha^+$ and increases the slope for $\pi > \pi_\alpha^+$

See Figure 8 for this result and the Appendix for the proof of this corollary. With a marginally more informative signal, the slopes of $b_h(\pi, \alpha)$ and $b_l(\pi, \alpha)$ are decreased for $\pi < \pi_\alpha^+$ and are increased for $\pi > \pi_\alpha^+$. Intuitively, when the signal becomes more informative, the high valuation player would not increase her effort as much as before in response to a marginal increase of π in the interval $[0, \pi_\alpha^+)$, but she would increase her effort more than before if π is in $(\pi_\alpha^+, 1]$. The reason is that the former interval indicates that the opponent is likely to have the low valuation, whereas the latter indicates that the opponent is likely to have the high valuation. Similar intuition applies to $b_l(\pi, \alpha)$.

The second half of Corollary 2 suggests that the probability that both the high and the low valuation player randomize, i.e., π^*, increases in the accuracy.[14]

3.2 Endogenous choice of accuracy

According to the timing of the spying game given in Section 2, each player chooses an accuracy for the spying signal before they learn their valuations. Therefore, the incentive to spy depends on the expected payoff in the contest stage.

We define the following expected payoffs, $V_i(\alpha_i, \alpha_j)$ and $U_i(\alpha_i, \alpha_j)$, for the contest stage.

Definition 3. $V_i(\alpha_i, \alpha_j)$: *player i's ex ante equilibrium expected payoff in the contest stage when player i (j) has acquired a signal with accuracy* α_i (α_j) *and* (α_i, α_j) *is common knowledge.*

Definition 4. $U_i(\alpha_i, \alpha_j)$: *player i's maximum expected payoff in the contest stage when player i (j) has acquired a signal with accuracy* α_i (α_j) *but player j (wrongly) believes that player i also has* α_j.

[14] In fact, when $\alpha = \overline{\alpha}$, the inefficient equilibrium in Proposition 1 replicates the non-monotonic mixed strategy equilibrium of all-pay auction with complete information. It is also worthwhile to point out that when $\alpha = \underline{\alpha}$, the efficient equilibrium as given in Proposition 1 replicates the monotonic mixed strategy equilibrium of all-pay auction with independent private value.

The expected payoff $V_i(\alpha_i, \alpha_j)$ is the equilibrium payoff when the profile of accuracies are (α_i, α_j). Therefore, when $\alpha_i = \alpha_j = \alpha \in [\underline{\alpha}, \overline{\alpha}]$, $V_i(\alpha, \alpha)$ represents player i's expected payoff in the symmetric equilibrium shown in Proposition 1.[15] However, $U_i(\alpha_i, \alpha_j)$ is not defined for equilibrium and may only equal to the equilibrium payoff $V_i(\alpha_i, \alpha_j)$ when $\alpha_i = \alpha_j = \alpha$, i.e., $U_i(\alpha, \alpha) = V_i(\alpha, \alpha)$. Denote by $AMU_i(\alpha_i, \alpha_j) = \frac{\partial U_i(\alpha_i, \alpha_j)}{\partial \alpha_i}$ the corresponding marginal expected payoff.

Lemma 3. *$AMU_i(\alpha_i, \alpha) > 0$ for $\alpha_i \leq \alpha \in [\underline{\alpha}, \widehat{\alpha}]$, and $AMU_i(\alpha_i, \alpha) \geq 0$ for $\alpha_i \leq \alpha \in (\widehat{\alpha}, \overline{\alpha})$. Finally, $AMU_i(\alpha_i, \overline{\alpha}) = 0$ for any $\alpha_i \leq \overline{\alpha}$.*

Lemma 3 states that player i can strictly increase her payoff by increasing α_i when the opponent believes the contest is relatively opaque, i.e., both players have $\alpha \in (\underline{\alpha}, \widehat{\alpha}]$, and can weakly increase her payoff when he believes the contest is relatively transparent, i.e., both players have $\alpha \in (\widehat{\alpha}, \overline{\alpha})$. In the former we have an efficient equilibrium in which the high and the low valuation players' efforts differ from each other completely. In this case, spying is strictly profitable: $AMU_i > 0$. This benefit of spying, however, is not as strong in the latter: $AMU_i \geq 0$, because the high and the low valuation players' distributions of effort are identical with positive probability. After all, spying is not really helpful if players with different valuations behave similarly. Hence, the main intuition behind Lemma 3 is that the return to spying depends crucially on how much the equilibrium effort varies with valuation, which further depends on the level of transparency of the contest (in the opponent's belief). In fact, the marginal return to spying is driven down to zero when the opponent believes that the contest is perfectly transparent, as indicated by the last part of the lemma.

Lemma 4. *$U_i(\underline{\alpha}, \alpha) > V_i(\underline{\alpha}, \underline{\alpha})$ for $\alpha \in [\underline{\alpha}, \widehat{\alpha}]$, and $U_i(\underline{\alpha}, \alpha) \geq V_i(\underline{\alpha}, \underline{\alpha})$ for $\alpha \in (\widehat{\alpha}, \overline{\alpha}]$. Finally, $U_i(\underline{\alpha}, \overline{\alpha}) = V_i(\underline{\alpha}, \underline{\alpha})$.*

As long as player j believes that player i has the same accuracy as he does, player i is better off being spied on even if she does not spy on player j. When player j holds that belief, he responds to his spying signal realization π_j according to Proposition 1, which leaves player i an advantage: Player i is well aware of θ_i and thus, is aware of the exact distribution of π_j. Even though she has no information regarding θ_j in addition to the prior, she is able to anticipate j's move (with some noise) based on her knowledge of θ_i. As this advantage is absent when none of the players spies on each other, Lemma 4 implies that player i benefits from being spied on.

In summary, Lemma 3 implies that player i benefits from spying on the opponent and Lemma 4 implies that she also benefits from being spied on. We can then derive the ex ante

[15] In Proposition 4 in Section 4.1, we consider asymmetric accuracy, i.e., $\alpha_i \neq \alpha_j$. Thus, $V_i(\alpha_i, \alpha_j)$ represents player i's expected payoff in the asymmetric equilibrium shown in Proposition 4.

equilibrium expected payoff of player i in the contest stage:

$$V_i(\alpha,\alpha) = U_i(\underline{\alpha},\alpha) + \int_{\underline{\alpha}}^{\alpha} AMU_i(t,\alpha)dt,$$

and show, by combining the two lemmas, that this payoff is higher when $\alpha \in (\underline{\alpha}, \overline{\alpha})$ than when players do not spy at all, as is stated in Proposition 2.

Proposition 2. *When $\alpha_1 = \alpha_2 = \alpha \in (\underline{\alpha}, \widehat{\alpha}]$, player i's ($i = 1,2$) expected payoff satisfies $V_i(\alpha,\alpha) > V_i(\underline{\alpha},\underline{\alpha})$. Alternatively, when $\alpha_1 = \alpha_2 = \alpha \in (\widehat{\alpha}, \overline{\alpha})$, player i's ($i = 1,2$) expected payoff satisfies $V_i(\alpha,\alpha) \geq V_i(\underline{\alpha},\underline{\alpha})$. Finally, when $\alpha_1 = \alpha_2 = \overline{\alpha}$, player i's ($i = 1,2$) expected payoff satisfies $V_i(\overline{\alpha},\overline{\alpha}) = V_i(\underline{\alpha},\underline{\alpha})$.*

Note that $V_i(\underline{\alpha},\underline{\alpha}) = V_i(\overline{\alpha},\overline{\alpha})$, i.e., all-pay auction is payoff equivalent across private value and complete information, is first shown by Morath and Münster (2008) and then confirmed by Kovenock et al. (2015).

Corollary 3 (Expected Effort)**.** *Players exert strictly less total expected effort when $\alpha \in (\underline{\alpha}, \overline{\alpha}]$ than when they do not spy at all:*

$$2\left[p_h \int_0^1 b_h(\Pi,\alpha)d\Pi + p_l \int_0^1 b_l(\Pi,\alpha)d\Pi\right] < (1-p_h^2)\theta_l + p_h^2\theta_h,$$

where $(1-p_h^2)\theta_l + p_h^2\theta_h$ is the revenue in the complete information all-pay auction.

Proposition 2 implies Corollary 3. When $\alpha \in (\underline{\alpha}, \widehat{\alpha}]$, i.e., the unique equilibrium is efficient, the social surplus is a constant: $p_l^2\theta_l + (1-p_l^2)\theta_h$. When $\alpha \in (\widehat{\alpha}, \overline{\alpha}]$, i.e., the unique equilibrium is inefficient, the social surplus is even less than $p_l^2\theta_l + (1-p_l^2)\theta_h$. Thus, in both cases the fact that players are better off implies the total expected effort is lower. See below for a numerical example.

Example 1. *Suppose $\theta_h = 2, \theta_l = 1, p_h = p_l = \frac{1}{2}$. For $\alpha \in [0,+\infty]$, $f_h(\pi,\alpha)$ and $f_l(\pi,\alpha)$ are given by:*

$$f_h(\pi,\alpha) = \begin{cases} 0 & \text{if } \pi < \frac{1}{2} - \frac{1}{\alpha}; \\ 1 + \alpha(\pi - \frac{1}{2}) & \text{if } \pi \in [\frac{1}{2} - \frac{1}{\alpha}, \frac{1}{2} + \frac{1}{\alpha}]; \\ 2 & \text{if } \pi > \frac{1}{2} + \frac{1}{\alpha}. \end{cases}$$

and

$$f_l(\pi,\alpha) = \begin{cases} 2 & \text{if } \pi < \frac{1}{2} - \frac{1}{\alpha}; \\ 1 - \alpha(\pi - \frac{1}{2}) & \text{if } \pi \in [\frac{1}{2} - \frac{1}{\alpha}, \frac{1}{2} + \frac{1}{\alpha}]; \\ 0 & \text{if } \pi > \frac{1}{2} + \frac{1}{\alpha}. \end{cases}$$

Thus, the signal is rotation ordered around a single point $\pi_\alpha^+ = \frac{1}{2}$ as α changes in $[0, +\infty]$, and a simple calculation shows $\widehat{\alpha} = \frac{2}{3}$. Based on Proposition 1, when $\alpha \in [0, \frac{2}{3}]$ the unique symmetric equilibrium is efficient, and when $\alpha \in (\frac{2}{3}, +\infty]$ the unique symmetric equilibrium is inefficient.

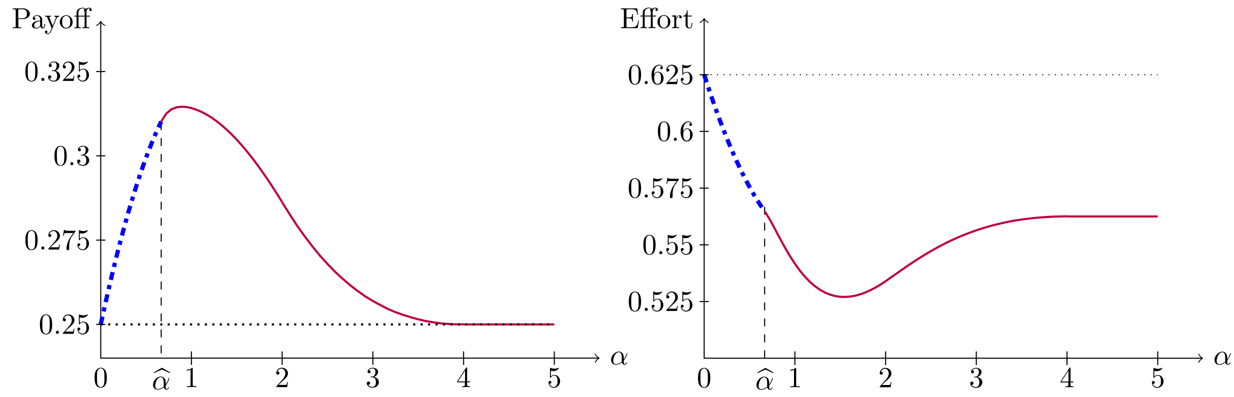

Figure 9: Expected payoff (left panel) and effort (right panel) of one player
The parts with $\alpha \leq \widehat{\alpha}$ ($\alpha > \widehat{\alpha}$) correspond to efficient (inefficient) equilibrium.

The expected payoff of player i is plotted in the left panel of Figure 9 and the expected effort the right panel. The payoff is equal to 0.25 in both IPV ($\alpha = 0$) and complete information setting ($\alpha = +\infty$), and is strictly higher for $\alpha \in (0, \widehat{\alpha}]$ and weakly higher for $\alpha \in (\widehat{\alpha}, +\infty]$. The expected effort in IPV setting ($\alpha = 0$) is 0.625 which is always higher than when $\alpha > 0$.

Now, we characterize the equilibrium choice of accuracy and show that spying improves players' welfare even accounting for the cost of spying. Since we assume that the spying cost function is symmetric, we focus on symmetric equilibria. Let $MU_i(\alpha) = AMU_i(\alpha, \alpha)$, and note that Lemma 3 implies $MU_i(\underline{\alpha}) > 0$, $MU_i(\overline{\alpha}) = 0$, and $MU(\alpha) \geq 0$ for $\alpha \in (\underline{\alpha}, \overline{\alpha})$.

Proposition 3. *There always exists a spying cost function $C(\alpha)$ of which the marginal cost $MC(\alpha)$ crosses with $MU_i(\alpha)$ from below only once. Given such $C(\alpha)$:*

1. *There exists a unique symmetric equilibrium of the spying game (α^*, α^*) where $\alpha^* \in (\underline{\alpha}, \overline{\alpha})$ satisfies $MU_i(\alpha^*) = MC(\alpha^*)$;*

2. *Each player is better off than either no spying at all or all players receive a perfectly informative signal for free, i.e., $V_i(\alpha^*, \alpha^*) - C(\alpha^*) \geq V_i(\underline{\alpha}, \underline{\alpha}) = V_i(\overline{\alpha}, \overline{\alpha})$,*

where $i = 1, 2$.

Proposition 3 shows that when the spying cost function satisfies some mild conditions, there exists a symmetric equilibrium in which players acquire a partially informative signal and earn higher expected payoff net of the spying cost. According to Corollary 3, Proposition 3 also suggests that spying activities reduce the total expected effort in the contest.

According to Lemma 3, any increasing $MC(\cdot)$ that satisfies $MC(\underline{\alpha}) = 0$ crosses with $MU_i(\cdot)$ from below for at least once. Yet, we cannot conclude that the two *only* cross once without making additional assumptions on the information order to guarantee decreasing $MU_i(\cdot)$. This is the reason that Proposition 3 is not established for arbitrary convex spying cost functions. In fact, the conditions on $MC(\cdot)$ are easily satisfied as long as the cost function is convex enough, for example, when $C(\alpha) = (\alpha - \underline{\alpha})^\eta$ and η is large.

4 Information disclosure in contests

Given the previous results that players are better off when they both acquire a partially informative signal about the opponent, a question naturally arises: Do players have incentives to disclose their private information to each other? With information disclosure, their payoffs should be even higher since they do not have to pay to spy.

To study information disclosure, we mildly modify the model in Section 2 by letting each player chooses the accuracy of the opponent's signal. In other words, contrary to the spying situation where player i chooses α_i, she now chooses α_j, i.e., the accuracy of the signal about θ_i, and player j chooses α_i, i.e., the accuracy of the signal about θ_j. Disclosure is free and players can observe the accuracies of both players' signals before exerting efforts in the contest. However, despite that she discloses the signal/chooses the accuracy, player i does not observe any realizations of the signal she discloses — only her opponent does. Companies in a patent race may disclose information by allowing the opponent to, e.g., run some experiments or tests on prototypes of products or samples of drugs. Company A can decide what samples to provide to company B, yet the results of experiments or tests is not available to company A.[16]

The timing of the information disclosure game is given by Figure 10. First, player j chooses the accuracy α_i for the signal to be received by his opponent, player i. Second, Nature determines valuation profile according to the prior distribution and each player observes her own valuation. Third, Nature determines signal realization π_i according to θ_j and α_i and player i observes it. Finally, player i chooses effort b_i based on her private information (θ_i, π_i).

[16] Lobbying firms can disclose either essential information (high accuracy), e.g., backgrounds of lobbyists, or inessential information (low accuracy), e.g., administrative expenses, to each other. Yet, they don't know how the opponent will interpret these information. In particular, the information regarding the connection between a lobbyist and a government official, disclosed by the opponent, might be perceived as either valuable or insignificant. However, such interpretation of the information is unknown to the opponent.

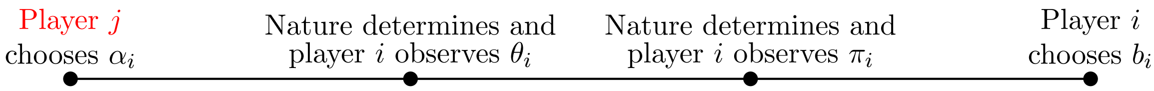

Figure 10: Timing of information disclosure in the contest ($i = 1, 2$ and $j = 3 - i$)

4.1 Exogenous asymmetric accuracy

Since players can observe each other's accuracy and their choices of accuracy may differ, we first characterize equilibrium efforts with asymmetric exogenous accuracy.

Denote by $b_{il}(\pi, \alpha_i, \alpha_j)$ and $b_{ih}(\pi, \alpha_i, \alpha_j)$ the efforts of type (θ_l, π) and (θ_h, π) of player i given the profile of accuracy (α_i, α_j). Similar to the symmetric case, player i's equilibrium effort is monotonic condition on her valuation, see Lemma 5.

Lemma 5. *Suppose $\alpha_1 \neq \alpha_2$, then in any allocative efficient, pure strategy equilibrium of the contest, the following must be true for player $i = 1, 2$:*

1. *Monotonicity: type (θ_h, π) of player i's effort is non-decreasing in π and type (θ_l, π) of player i's effort is non-increasing in π;*
2. *Continuity: both players' strategies are continuous without any atom;*
3. *Common support: $b_{1l}(1, \alpha_1, \alpha_2) = b_{2l}(1, \alpha_2, \alpha_1)$ and $b_{1h}(1, \alpha_1, \alpha_2) = b_{2h}(1, \alpha_2, \alpha_1)$;*
4. *Initial conditions: $b_{1l}(1, \alpha_1, \alpha_2) = b_{2l}(1, \alpha_2, \alpha_1) = 0$ and $b_{1l}(0, \alpha_1, \alpha_2) = b_{1h}(0, \alpha_1, \alpha_2) = b_{2l}(0, \alpha_2, \alpha_1) = b_{2h}(0, \alpha_2, \alpha_1)$.*

Part 1 of Lemma 5 implies that type $(\theta_l, 1)$ of both players chooses the lowest effort, 0, and type $(\theta_h, 1)$ of both players choose the highest effort, in the allocative efficient, pure strategy equilibrium. Part 3 indicates that the upper bound of each valuation of player's effort must be the same. Part 4 is useful later in solving the equilibrium effort and in proving the uniqueness of such an equilibrium.

Lemma 6 below provides the necessary conditions for existence of any symmetric, pure strategy and allocative efficient equilibria.

Lemma 6 (Efficiency)**.** *Suppose $\alpha_1 \neq \alpha_2$, then there exists a symmetric, pure strategy, allocative efficient equilibrium in the contest only if $\frac{f_h(\pi,\alpha_i)}{f_l(\pi,\alpha_i)} \geq \frac{\theta_l}{\theta_h}$ for all $\pi \in [0, 1]$ and $i = 1, 2$.*

For simplicity, we restrict attention to such equilibria. This boils down to assuming $\alpha_1, \alpha_2 \in [\underline{\alpha}, \widehat{\alpha}]$, as stated in the Assumption 3.

Assumption 3.

$$\frac{f_h(\pi,\alpha_i)}{f_l(\pi,\alpha_i)} \geq \frac{\theta_l}{\theta_h}, \text{ for all } \pi \in [0,1] \text{ and } i = 1, 2.$$

This contest is asymmetric in the sense that players' accuracy of signals about the opponent are different, yet there may still exist symmetric equilibria in which players' strategies conditioning on their private information are the same.

We now derive the symmetric equilibrium of the contest given $\alpha_1 \neq \alpha_2$ and Assumption 3. Denote by $b_{il}^{-1}(b,\alpha_i,\alpha_j)$ and $b_{ih}^{-1}(b,\alpha_i,\alpha_j)$ the inverse effort of the low and the high valuation types of player i whose equilibrium effort is b. According to Lemma 5, the expected payoff for type (θ_l,π) and (θ_h,π) of player j when choosing an effort b can be written as $\widetilde{U}_{jl}(b|\pi,\alpha_j,\alpha_i)$ and $\widetilde{U}_{jh}(b|\pi,\alpha_j,\alpha_i)$, respectively:

$$\begin{aligned}
\widetilde{U}_{jl}(b|\pi,\alpha_j,\alpha_i) &= \theta_l[1-\mu(\pi,\alpha_j)]\int_{b_{il}^{-1}(b,\alpha_i,\alpha_j)}^{1} f_l(\Pi,\alpha_i)d\Pi - b \\
\widetilde{U}_{jh}(b|\pi,\alpha_j,\alpha_i) &= \theta_h\left[(1-\mu(\pi,\alpha_j)) + \mu(\pi,\alpha_j)\int_0^{b_{ih}^{-1}(b,\alpha_i,\alpha_j)} f_h(\Pi,\alpha_i)d\Pi\right] - b
\end{aligned}$$

By the first order conditions and the initial conditions provided in part 4 of Lemma 5, the equilibrium strategy in the contest is characterized in Proposition 4 below.

Proposition 4. *If Assumption 3 is satisfied, then the unique symmetric equilibrium of the contest is given by:*

$$\begin{aligned}
b_{il}(\pi,\alpha_i,\alpha_j) &= \theta_l \int_\pi^1 \left[1-\mu\left(\Pi,\alpha_i\right)\right] dF_l(\Pi,\alpha_j) \\
b_{ih}(\pi,\alpha_i,\alpha_j) &= \theta_h \int_0^\pi \mu\left(\Pi,\alpha_i\right) dF_h(\Pi,\alpha_j) + \theta_l \int_0^1 \left[1-\mu\left(\Pi,\alpha_i\right)\right] dF_l(\Pi,\alpha_j)
\end{aligned}$$

where $i = 1, 2$ and $j = 3 - i$.

Recall that $1-\mu(\cdot,\alpha_i) = p_l f_l(\cdot,\alpha_i)$ and $\mu(\cdot,\alpha_i) = p_h f_h(\cdot,\alpha_i)$, it then becomes clear that α_i and α_j are interchangeable in the effort functions given in Proposition 4: $b_{il}(\pi,\alpha_i,\alpha_j) = b_{il}(\pi,\alpha_j,\alpha_i)$ and $b_{ih}(\pi,\alpha_i,\alpha_j) = b_{ih}(\pi,\alpha_j,\alpha_i)$. Therefore, even if the two players have different accuracies, their equilibrium strategies are symmetric.

Based on Proposition 4, the sensitivity of player i's effort to π depends on both her own and the opponent's accuracies, see Corollary 4. Similar to the symmetric accuracy setting, both players' accuracies increase the sensitivity of each player's effort.

Corollary 4 (Sensitivity). *The slope of $b_{ih}(\pi,\alpha_i,\alpha_j)$ and $b_{il}(\pi,\alpha_i,\alpha_j)$ are increasing (decreas-*

ing) in both α_i and α_j for $\pi > (<)\pi^+_{\alpha_k}$, i.e.,

$$\frac{\partial^2 b_{ih}(\pi,\alpha_i,\alpha_j)}{\partial\pi\partial\alpha_k}, \frac{\partial^2 b_{il}(\pi,\alpha_i,\alpha_j)}{\partial\pi\partial\alpha_k} \lesseqgtr 0, \text{ for } \pi \lesseqgtr \pi^+_{\alpha_k},$$

where $k = i, j$, and $i = 1, 2, j = 3 - i$.

We now turn to the comparative statics on accuracies correspond to Proposition 4.

Proposition 5. *For any $\alpha_i, \alpha_j \in [\underline{\alpha}, \widehat{\alpha}]$, the expected payoff of player i in the contest stage satisfies $V_i(\alpha_i, \alpha_j) > V_i(\underline{\alpha}, \underline{\alpha})$, where $i = 1, 2, j = 3 - i$.*

Corollary 5 (Expected Effort)**.** *Players exert strictly less total expected effort when $\alpha_i \neq \alpha_j$ and $\alpha_i, \alpha_j \in (\underline{\alpha}, \widehat{\alpha}]$ than when they do not receive any signal:*

$$\Sigma^2_{i=1}\left[p_h \int_0^1 b_{ih}(\Pi,\alpha_i,\alpha_j)d\Pi + p_l \int_0^1 b_{il}(\Pi,\alpha_i,\alpha_j)d\Pi\right] < (1-p_h^2)\theta_l + p_h^2\theta_h,$$

where $i = 1, 2, j = 3 - i$.

As one might have expected, even when accuracies are asymmetric each player's ex ante expected payoff in the contest is again higher than when players do not spy on each other. The intuition and proof are similar to that of the symmetric accuracies case. Also similar to the symmetric accuracy setting, the total expected effort is lower when $\alpha_i, \alpha_j \in (\underline{\alpha}, \widehat{\alpha}]$ than when players do not spy at all. Hence, Proposition 5 and Corollary 5 partially check the robustness of Proposition 2 and Corollary 3, when players are asymmetric in accuracies.

4.2 Disclosure agreement

The players in an information disclosure agreement commit to simultaneously disclose a signal to the opponent with pre-specified accuracies. We refer to the disclosure agreement where players commit to disclose signals with accuracy profile (α_i, α_j) as the "information disclosure agreement (α_i, α_j)". The following result suggests that an agreement to disclose partially informative signals is beneficial to both players.

Proposition 6. *In any information disclosure agreement (α_i, α_j) where $\alpha_i, \alpha_j \in (\underline{\alpha}, \widehat{\alpha}]$, player i ($i = 1, 2$) is strictly better off than no disclosure agreement or full disclosure agreement, i.e.,*

$$V_i(\alpha_i, \alpha_j) > V_i(\underline{\alpha}, \underline{\alpha}) = V_i(\overline{\alpha}, \overline{\alpha}) \quad \text{for} \quad \alpha_i, \alpha_j \in (\underline{\alpha}, \widehat{\alpha}].$$

In any symmetric information disclosure agreement (α, α) *where* $\alpha \in (\widehat{\alpha}, \overline{\alpha})$*, player* i *(*$i = 1, 2$*) is weakly better off than no disclosure agreement or full disclosure agreement, i.e.,*

$$V_i(\alpha, \alpha) \geq V_i(\underline{\alpha}, \underline{\alpha}) = V_i(\overline{\alpha}, \overline{\alpha}) \quad \text{for} \quad \alpha \in (\widehat{\alpha}, \overline{\alpha}).$$

Proposition 6 follows directly from Proposition 3 and 5 and thus, the proof of which is omitted. The results suggest that disclosure agreement strictly improves welfare when the contest is relatively opaque. This result also suggests that there is some loss of generality to only compare full disclosure with full concealment.

Corollary 6 below implies that an information disclosure agreement can reduce total effort/ expenditure in the contest.

Corollary 6. *Players exert strictly less total expected effort in any information disclosure agreement* (α_i, α_j) *with* $\alpha_i, \alpha_j \in (\underline{\alpha}, \widehat{\alpha}]$ *or with* $\alpha_i = \alpha_j = \alpha \in (\widehat{\alpha}, \overline{\alpha}]$ *than when there is no information disclosure.*

4.3 Endogenous information disclosure

Would players obey the disclosure agreement if there is no external enforcement? To solve the equilibrium disclosure decision, note that player i chooses α_j to maximize her equilibrium expected payoff in the contest, denote by $V_i(\alpha_i, \alpha_j)$, given α_i. In other words, α_j is chosen to best response to α_i. The best response function of player i is derived by the first order condition of her equilibrium expected payoff in the contest, $V_i(\alpha_i, \alpha_j)$, w.r.t. α_j:

$$\frac{\partial V_i(\alpha_i, \alpha_j)}{\partial \alpha_j} = \int_0^1 \left[\frac{\theta_h + \frac{\theta_l}{p_l}}{\theta_h - \theta_l} - f_h(\Pi, \alpha_i) \left(\frac{\theta_h + \frac{p_h \theta_l}{p_l}}{\theta_h - \theta_l} - \Pi \right) - F_h(\Pi, \alpha_i) \right] \frac{\partial f_h(\Pi, \alpha_j)}{\partial \alpha_j} d\Pi. \quad (6)$$

It then follows that no player would obey the information disclosure agreement $\alpha \in (\underline{\alpha}, \widehat{\alpha}]$ as it is strictly dominant to choose $\underline{\alpha}$ when the opponent chooses $\alpha > \underline{\alpha}$. See Lemma 7 below.

Lemma 7. *If player* j *chooses* $\alpha_i \in (\underline{\alpha}, \widehat{\alpha}]$*, then* $\frac{\partial V_i(\alpha_i, \alpha_j)}{\partial \alpha_j} < 0$*, i.e., player* i *strictly prefers to choose* $\alpha_j = \underline{\alpha}$*.*

Player i wants to avoid the motivation effect and take advantage of the demotivation effect on the opponent. Specifically, type (θ_h, π) of player i finds it profitable to lower the accuracy of the signal she discloses, as then the high valuation opponent is relatively more likely to receive low realizations which demotivates him. Similarly, type (θ_l, π) of player i also finds it profitable to lower the accuracy of the signal she discloses, as then the low valuation opponent is relatively more likely to receive high realizations which demotivates him as well. Therefore, player i earns higher expected payoff by decreasing the accuracy of the signal she discloses to the opponent. Note that this intuition relies on the fact that Assumption 3 is satisfied and hence, the equilibrium is efficient.

When the opponent discloses an uninformative signal, then player i is indifferent about the accuracy of the signal she discloses. This is shown in Lemma 8 below.

Lemma 8. *When player j chooses $\alpha_i = \underline{\alpha}$, i.e., discloses an uninformative signal, then the disclosure decision of player i is irrelevant to her own expected payoff, i.e., $V_i(\underline{\alpha}, \alpha_j) = p_h p_l (\theta_h - \theta_l)$ for all $\alpha_j \in [\underline{\alpha}, \widehat{\alpha}]$.*

When $\alpha_i = \underline{\alpha}$, the distribution of π_i which player i receives is uniform. Suppose player i has the high valuation, then in equilibrium the probability that she wins when she receives π_i is exactly π_i, condition on that the opponent has the high valuation as well. Similarly, suppose player i has the low valuation, then in equilibrium the probability that she wins is $1 - \pi_i$, condition on that the opponent also has the low valuation. Since these probabilities of winning are always uniformly distributed and are unaffected by α_j, player i's expected payoff is also unaffected by it. Lemma 7 and 8 jointly imply the following result.

Proposition 7. *There does not exist any equilibrium in which each player discloses an informative signal.*

Proposition 7 follows directly from Lemma 7 and 8, thus, the proof is omitted. Even though information disclosure can improve total welfare, it cannot rely on decentralized voluntary information disclosure by players, because at least one player would prefer to disclose nothing informative to her opponent. Nevertheless, there exists a continuum of equilibria in which one player does not disclose, i.e., chooses $\underline{\alpha}$, and the other player discloses a partially informative signal, i.e., chooses an accuracy strictly above $\underline{\alpha}$. The implication is that the regulator may be able to set up a minimum information disclosure requirement which specifies the accuracy of the signal that each player should disclose.

5 Conclusion

When players spy on each other, the additional information about the opponent allows them to coordinate, i.e., only exert higher effort when it is more likely that the opponent is evenly matched with the player. Such a coordination improves players' welfare even taking the cost of spying into account. This, however, is only true when spying is costly so that players acquire partially informative signals. An information disclosure agreement in which players commit to disclose a partially informative signal to each other can achieve an even better outcome, as the cost on spying is saved. However, players would unilaterally deviate by disclosing an uninformative signal if there is no external power to enforce the agreement. This is due to the incentive of players to avoid the motivation effect and to induce the demotivation effect on the opponent.

This paper yields differential policy implications dependent on the nature of contests. For contests with wasteful efforts, e.g., rent-seeking and lobbying, it is advisable for regulators to impose a minimum disclosure requirement which specifies the minimum accuracy of signals players disclose to each other. For contests with productive efforts, e.g., sports tournaments, promotion contests and sales competitions, banning spying and disclosure maximizes total effort.

We provide a useful analytical framework of all-pay contests with endogenous information structure. The model is applied to study endogenous information acquisition (spying) and endogenous information disclosure (sharing), and is potentially applicable to other endogenous information settings, including centralized information disclosure, discriminatory information disclosure, ex post information acquisition, etc.

6 Appendix: Proofs of lemmas and propositions

We first prove Lemma 9 which is useful in proving the main results.

Lemma 9. *Suppose a differentiable function $f(x)$, $x \in [a,b]$, satisfies $f(x) < 0$ for $x \in [a,c)$, and $f(x) > 0$ for $x \in (c,b]$, where $c \in (a,b)$. Furthermore, suppose $\int_a^c f(x)dx = -\int_c^b f(x)dx$. Then, for any continuous and non-decreasing function $g(x)$, the following is true:*

$$\int_a^b g(x)f(x)dx \geq 0.$$

where the equality satisfies only when $g(x)$ is a constant.

Proof of Lemma 9. Divide the integral into two sub-integrals on $[a,c)$ and $(c,b]$, and apply the intermediate value theorem for these sub-integrals, we have:

$$\begin{aligned}\int_a^b g(x)f(x)dx &= \int_a^c g(x)f(x)dx + \int_c^b g(x)f(x)dx \\ &= g(\nu)\int_a^c f(x)dx + g(\xi)\int_c^b f(x)dx \\ &= [g(\xi) - g(\nu)]\int_c^b f(x)dx\end{aligned}$$

where $\nu \in [a,c)$ and $\xi \in (c,b]$. If $g(x)$ is not a constant then $g(\xi) > g(\nu)$ and thus, $\int_a^b g(x)f(x)dx$ is positive. If $g(x)$ is a constant then $\int_a^b g(x)f(x)dx = 0$. □

Now, we turn to the proofs of the main results.

For the allocative efficient equilibrium, we will refer to the interval $[b_l(1,\alpha), b_l(0,\alpha)]$, i.e., the equilibrium support of low valuation types, as the "low pure support"; and refer to the interval $[b_h(0,\alpha), b_h(1,\alpha)]$, i.e., the equilibrium support of high valuation types, as the "high pure support".

For the allocative inefficient equilibrium, let's refer to the support of the high valuation type's pure strategy, $[b_h(\pi^*,\alpha), b_h(1,\alpha)]$, as the "high pure support" and correspondingly, refer to $[b_l(1,\alpha), b_l(\pi^*,\alpha)]$ as the "low pure support". Finally, the support of the mixed strategy for both type (θ_l,π) and (θ_h,π) with $\pi < \pi^*$, i.e., $[b_l(\pi^*,\alpha), b_h(\pi^*,\alpha)]$, is referred to as the "mixed support".

Denote by $\widetilde{U}_l^k(b|\pi,\alpha_i,\alpha)$ and $\widetilde{U}_h^k(b|\pi,\alpha_i,\alpha)$ where $k \in \{h,m,l\}$ the expected payoff of type (θ_l,π) and (θ_h,π) of player i with $\pi > \pi^*$, respectively, when she chooses b in the high $(k = h)$, low pure support $(k = l)$, and the mixed support $(k = m)$, given that player j chooses α and believes that player i has chosen the same, whereas player i, in fact, chooses α_i. Similarly, denote by $\widetilde{U}_{ml}^k(b|\pi,\alpha_i,\alpha)$ and $\widetilde{U}_{mh}^k(b|\pi,\alpha_i,\alpha)$ where $k \in \{h,m,l\}$ the expected payoff of type (θ_l,π) and (θ_h,π) of player i where $\pi \leq \pi^*$, respectively, when she chooses b in the high $(k = h)$, low pure support $(k = l)$, and the mixed support $(k = m)$.

Proof of Lemma 1

We start by proving part 1 of the lemma. Suppose in a symmetric, pure strategy equilibrium with efficient allocation, we have $b_h(\pi_1,\alpha) < b_h(\pi_2,\alpha)$ for $\pi_1 > \pi_2$. Then it must be true that type (θ_h,π_1) finds the cost of increasing her effort from $b_h(\pi_1,\alpha)$ to $b_h(\pi_2,\alpha)$ dominates the gain from such an increase of effort, formally:

$$b_h(\pi_2,\alpha) - b_h(\pi_1,\alpha) \geq \mu(\pi_1,\alpha)\Pr\{b_h(\pi_1,\alpha) \leq b_j < b_h(\pi_2,\alpha)\}\theta_h.$$

where b_j is player j's effort. In other words, the cost must outweigh the gain to prevent type (θ_h, π_1) from deviating to $b_h(\pi_2, \alpha)$. However, type (θ_h, π_2)'s gain must outweigh her cost of such an increase of effort:

$$b_h(\pi_2, \alpha) - b_h(\pi_1, \alpha) \leq \mu(\pi_2, \alpha) \Pr\{b_h(\pi_1, \alpha) \leq b_j < b_h(\pi_2, \alpha)\}\theta_h.$$

where b_j is player j's effort. Combining the two conditions, we have $\mu(\pi_2, \alpha) \geq \mu(\pi_1, \alpha)$ which contradicts the fact that $\pi_1 > \pi_2$, due to Assumption 1. A similar argument can prove that $b_l(\pi_1, \alpha) \leq b_l(\pi_2, \alpha)$ for any $\pi_1 > \pi_2$.

To prove continuity of the strategies, i.e., part 2 of the lemma, suppose there exists a discontinuous point on $b_h(\pi, \alpha)$, say $\hat{\pi} \in (0, 1)$, such that $b_h(\hat{\pi}, \alpha) < b_h(\hat{\pi}+\epsilon, \alpha)$ for an arbitrarily small ϵ. Then type $(\theta_h, \hat{\pi}+\epsilon)$ will find it profitable to deviate to some $\hat{b} \in (b_h(\hat{\pi}, \alpha), b_h(\hat{\pi}+\epsilon, \alpha))$. Similarly, suppose there exists a discontinuous point on $b_l(\pi, \alpha)$, $\tilde{\pi} \in (0, 1)$, such that $b_l(\tilde{\pi}, \alpha) > b_l(\tilde{\pi} + \epsilon, \alpha)$ for arbitrarily small ϵ. Then type $(\theta_l, \tilde{\pi})$ will find it profitable to deviate to some $\tilde{b} \in (b_l(\tilde{\pi} + \epsilon, \alpha), b_l(\tilde{\pi}, \alpha))$.

To prove that there is no atom on any player's effort, suppose there exists p and q such that $1 > q > p > 0$ and that $b_h(x, \alpha) = b$ where $x \in [p, q]$ and b is a constant. Then type $(\theta_h, p - \epsilon)$ will find it profitable to deviate to choose $b + \epsilon$, as the gain of such deviation will be $\mu(p - \epsilon, \alpha) \int_p^q f_h(\Pi, \alpha) d\Pi$ and the cost is negligible when ϵ is arbitrarily small. A similar argument can show that there is no atom on $b_l(\pi, \alpha)$.

Finally, for part 3, given part 1 is true, type $(\theta_h, 0)$ chooses the lowest effort among all types with valuation θ_h, whereas type $(\theta_l, 0)$ chooses the highest among all types with valuation θ_l. If $b_h(0, \alpha) > b_l(0, \alpha)$ then type $(\theta_h, 0)$ will be strictly better off by lowering her effort by a small amount ϵ satisfying $b_h(0, \alpha) - \epsilon \geq b_l(0, \alpha)$, thus $b_h(0, \alpha) = b_l(0, \alpha)$. Again, by part 1, the lowest effort is made by type $(\theta_l, 1)$ among all types, thus any positive effort is strictly dominated by choosing zero for $(\theta_l, 1)$. $\square$

Proof of Lemma 2

Note that allocative efficiency implies that $b_h(\pi, \alpha) \geq b_l(\pi, \alpha)$ for all $\pi \in [0, 1]$ fixing α, and that the probability of tie between a high valuation and a low valuation player is zero. Then to rule out the incentive for type (θ_h, π_i) to deviate to $b_l(\pi_i, \alpha)$, the following must be true:

$$\begin{aligned}
& b_h(\pi_i, \alpha) - b_l(\pi_i, \alpha) \\
\leq\ & [\mu(\pi_i, \alpha) Pr\{b_j < b_h(\pi_i, \alpha) | (\theta_h, \theta_h)\} + [1 - \mu(\pi_i, \alpha)]\, Pr\{b_j \geq b_l(\pi_i, \alpha) | (\theta_h, \theta_l)\}]\, \theta_h \\
=\ & \mu(\pi_i, \alpha) \int_0^{\pi_i} f_h(\Pi, \alpha) \theta_h d\Pi + (1 - \mu(\pi_i, \alpha)) \int_0^{\pi_i} f_h(\Pi, \alpha) \theta_h d\Pi \\
=\ & \int_0^{\pi_i} f_h(\Pi, \alpha) \theta_h d\Pi
\end{aligned} \tag{7}$$

In other words, the cost saved from choosing the lower effort (LHS of (7)) must be less than the gain forgone (RHS of (7)). This ensures that type (θ_h, π_i) does not want to deviate to $b_l(\pi, \alpha)$. However, type (θ_l, π_i) should find her cost saved by choosing the lower effort outweighs her gain forgone:

$$\begin{aligned}
& b_h(\pi_i, \alpha) - b_l(\pi_i, \alpha) \\
\geq\ & [\mu(\pi_i, \alpha) Pr\{b_j < b_h(\pi_i, \alpha) | (\theta_l, \theta_h)\} + (1 - \mu(\pi_i, \alpha)) Pr\{b_j \geq b_l(\pi_i, \alpha) | (\theta_l, \theta_l)\} \theta_l \\
=\ & \mu(\pi_i, \alpha) \int_0^{\pi_i} f_l(\Pi, \alpha) \theta_l d\Pi + (1 - \mu(\pi_i, \alpha)) \int_0^{\pi_i} f_l(\Pi, \alpha) \theta_l d\Pi \\
=\ & \int_0^{\pi_i} f_l(\Pi, \alpha) \theta_l d\Pi
\end{aligned}$$

Combining the two conditions:

$$\int_0^{\pi_i} f_h(\Pi,\alpha)\theta_h d\Pi \geq \int_0^{\pi_i} f_l(\Pi,\alpha)\theta_l d\Pi$$

we then have $\frac{f_h(\pi,\alpha)}{f_l(\pi,\alpha)} \geq \frac{\theta_l}{\theta_h}$ for all $\pi_i \in [0,1]$. □

Proof of Proposition 1

The proposition is proved by checking whether: (1) the equilibrium efforts given in the proposition are indeed optimal among all efforts in their own equilibrium support, e.g., $b_h(\pi_1,\alpha)$ instead of $b_h(\pi_2,\alpha)$ is optimal for type (θ_h,π_1) where $\pi_1 \neq \pi_2$; and (2) type $(\theta_h,\cdot)$ $((\theta_l,\cdot))$ does not find it profitable to deviate to any effort in each other's support, e.g., $(\theta_h,\cdot)$ does not find it profitable to deviate to the low support.

For the allocative efficient equilibrium, we start by showing (1) is true. Given that player j chooses his strategy according to the proposition, suppose type (θ_l,π) of player i chooses an alternative effort level $b_l(s,\alpha)$, then her expected payoff is

$$\widetilde{U}_l^l(b_l(s,\alpha)|\pi,\alpha,\alpha) \quad = \quad \theta_l \int_s^1 [\mu(\Pi,\alpha)-\mu(\pi,\alpha)]dF_l(\Pi,\alpha)$$

Thus,

$$\widetilde{U}_l^l(b_l(\pi,\alpha)|\pi,\alpha,\alpha) - \widetilde{U}_l^l(b_l(s,\alpha)|\pi,\alpha,\alpha) = \theta_l \int_\pi^s [\mu(\Pi,\alpha)-\mu(\pi,\alpha)]dF_l(\Pi,\alpha) \geq 0$$

regardless of whether $\pi \geq s$ or $\pi < s$.

Suppose type (θ_h,π) of player i chooses an alternative effort level $b_h(t,\alpha)$, then her expected payoff is

$$\begin{aligned}\widetilde{U}_h^h(b_h(t,\alpha)|\pi,\alpha,\alpha) \quad &= \quad \theta_h\left[(1-\mu(\pi,\alpha)) + \mu(\pi,\alpha)\int_0^t f_h(\Pi,\alpha)d\Pi\right] \\ &\quad -\theta_h \int_0^t \mu(\Pi,\alpha)\, dF_h(\Pi,\alpha) - \theta_l \int_0^1 [1-\mu(\Pi,\alpha)]\, dF_l(\Pi,\alpha)\end{aligned}$$

Again, compare this payoff to the equilibrium payoff:

$$\widetilde{U}_h^h(b_h(\pi,\alpha)|\pi,\alpha,\alpha) - \widetilde{U}_h^h(b_h(t,\alpha)|\pi,\alpha,\alpha) = \theta_h \int_t^\pi [\mu(\pi,\alpha)-\mu(\Pi,\alpha)]\, dF_h(\Pi,\alpha) \geq 0$$

regardless of $\pi \geq t$ or $\pi < t$. Thus, the strategy given in the proposition is indeed optimal for players if they choose efforts in the equilibrium support.

Now we turn to (2) by checking whether type (θ_h,π) of player i finds it profitable to deviate to any effort in the low pure support. This requires a comparison of type (θ_h,π) of player i's expected payoff in the allocative efficient equilibrium, $\widetilde{U}_h^h(b_h(\pi,\alpha)|\pi,\alpha,\alpha)$, to the maximum expected payoff from deviation. When deviating to $\beta \in [b_l(1,\alpha), b_l(0,\alpha)]$, i.e., the low pure support, the expected payoff of type (θ_h,π) of player i, given that player j plays the allocative efficient equilibrium $b_l(\pi,\alpha)$, is:

$$\widetilde{U}_h^l(\beta|\pi,\alpha,\alpha) = \theta_h [1-\mu(\pi,\alpha)] \int_{b_l^{-1}(\beta,\alpha)}^1 f_h(\Pi,\alpha)d\Pi - \beta. \tag{8}$$

Among all $\beta \in [b_l(1,\alpha), b_l(0,\alpha)]$, player i would prefer to choose the optimal effort: $\beta^* = \arg\max_\beta \widetilde{U}_h^l(\beta|\pi,\alpha,\alpha)$. The optimal deviation effort, β^*, can be found by the first order condition with respect to β. Let type (θ_l,t) be the one who chooses β^* in equilibrium, i.e.,

$b_l(t,\alpha)=\beta^*$. We can find t by the FOC of $\widetilde{U}^l_h(\beta|\pi,\alpha,\alpha)$ w.r.t β, and rearrange:

$$1-\mu(\pi,\alpha)=\frac{\theta_l}{\theta_h}\frac{f_l(t,\alpha)}{f_h(t,\alpha)}(1-\mu(t,\alpha)) \tag{9}$$

To show that $b_l(t,\alpha)$ is indeed the optimal deviation, let's compare the expected payoff of (θ,π) from bidding $b_l(t,\alpha)$ to bidding $b_l(s,\alpha)$ where $s\neq t$:

$$\begin{aligned}&\left\{\theta_h[1-\mu(\pi,\alpha)]\int_t^1 f_h(\Pi,\alpha)d\Pi-b_l(t,\alpha)\right\}-\left\{\theta_h[1-\mu(\pi,\alpha)]\int_s^1 f_h(\Pi,\alpha)d\Pi-b_l(s,\alpha)\right\}\\ = &\int_t^s\left[\frac{f_h(\Pi,\alpha)}{f_h(t,\alpha)}[1-\mu(t,\alpha)]-\frac{f_l(\Pi,\alpha)}{f_l(t,\alpha)}[1-\mu(\Pi,\alpha)]\right]d\Pi\\ \geq &\int_t^s[\mu(\Pi,\alpha)-\mu(t,\alpha)]\,d\Pi\geq 0\end{aligned}$$

Note that both the LHS and the RHS of (9) are decreasing functions of their arguments, π and t, respectively. Since $\frac{\theta_l}{\theta_h}\frac{f_l(t,\alpha)}{f_h(t,\alpha)}\leq 1$, thus $\pi\geq t$. Then there must exist $\widehat{s}\in[0,1]$ such that

$$1-\mu(\widehat{s},\alpha)\equiv\frac{\theta_l}{\theta_h}\frac{f_l(0,\alpha)}{f_h(0,\alpha)}[1-\mu(0,\alpha)] \tag{10}$$

For $\pi<\widehat{s}$, the LHS of the equation (9) is always strictly larger than the RHS, for all $t\in[0,1]$. This implies the first order derivative is positive and thus type (θ_h,π) does not want to deviate to the low pure support, whenever $\pi<\widehat{s}$. On the other hand, if $\pi\geq\widehat{s}$, there is always a unique interior solution of $t\in[0,1]$ satisfying equation (9) given π. In this case, we need to directly compare the equilibrium payoff to with the maximum deviation expected payoff which can be calculated by plugging β^* into (8). The first order derivative of the difference between the equilibrium expected payoff and the maximum deviation payoff, i.e., $\widetilde{U}^h_h(b_h(\pi,\alpha)|\pi,\alpha,\alpha)-\widetilde{U}^l_h(\beta^*|\pi,\alpha,\alpha)$, w.r.t π, is, in fact, non-negative:

$$\frac{\partial\left(\widetilde{U}^h_h(b_h(\pi,\alpha)|\pi,\alpha,\alpha)-\widetilde{U}^l_h(\beta^*|\pi,\alpha,\alpha)\right)}{\partial\pi}=\theta_h\mu'(\pi,\alpha)\left(\int_0^\pi f_h(\Pi,\alpha)d\Pi-\int_0^t f_h(\Pi,\alpha)d\Pi\right)\geq 0$$

This suggests this difference is non-decreasing in π. By (10) it can be proved that $\widetilde{U}^h_h(b_h(\widehat{s},\alpha)|\widehat{s},\alpha,\alpha)-\widetilde{U}^l_h(b_l(0,\alpha)|\pi,\alpha,\alpha)=\theta_h\mu'(\widehat{s},\alpha)\int_0^{\widehat{s}}f_h(\Pi,\alpha)d\Pi>0$. Thus, $\widetilde{U}^h_h(b_h(\pi,\alpha)|\pi,\alpha,\alpha)-\widetilde{U}^l_h(\beta^*|\pi,\alpha,\alpha)>0$ for all $\pi\in[\widehat{s},1]$, and type (θ_h,π) does not find it profitable to deviate from equilibrium strategy. Using exactly the same method, we can prove that type (θ_l,π) does not find it profitable to deviate to any bid in the high pure support. Thus, the proof is omitted. The uniqueness of the equilibrium is due to the initial conditions given in the Lemma 1.

For the inefficient equilibrium, the proof involves checking whether the type who plays pure strategy finds it profitable to deviate to the mixed strategy support and vice versa. Given MLRP, there must exists a $\hat{\pi}\in(0,1)$ such that $f_h(\hat{\pi},\alpha)/f_l(\hat{\pi},\alpha)=\theta_l/\theta_h$, fixing $\alpha\in[\widehat{\alpha},\overline{\alpha}]$. In other words, for all $\pi<\hat{\pi}$ we have $f_h(\pi,\alpha)/f_l(\pi,\alpha)\leqslant\theta_l/\theta_h$ and for all $\pi>\hat{\pi}$ we have $f_h(\pi,\alpha)/f_l(\pi,\alpha)\geqslant\theta_l/\theta_h$.

Before going into the proof, note that by the definition of π^* given in the proposition, we have

$$\theta_h\int_0^{\pi^*}f_h(\Pi,\alpha)d\Pi-\theta_l\int_0^{\pi^*}f_l(\Pi,\alpha)d\Pi\;=\;0 \tag{11}$$

By the assumption that the marginal distribution of π is uniform, we also have:

$$p_h \int_0^{\pi^*} f_h(\Pi,\alpha)d\Pi + p_l \int_0^{\pi^*} f_l(\Pi,\alpha)d\Pi \;=\; \pi^* \tag{12}$$

Consider $\int_0^{\pi^*} f_h(\Pi,\alpha)d\Pi$ and $\int_0^{\pi^*} f_l(\Pi,\alpha)d\Pi$ as unknown variables, then (11) and (12) form a linear system of equations which can be used to calculate:

$$\int_0^{\pi^*} f_h(\Pi,\alpha)d\Pi \;=\; \frac{\theta_l \pi^*}{p_h\theta_l + p_l\theta_h} \tag{13}$$

$$\int_0^{\pi^*} f_l(\Pi,\alpha)d\Pi \;=\; \frac{\theta_h \pi^*}{p_h\theta_l + p_l\theta_h} \tag{14}$$

Now, to prove the allocative inefficient equilibrium, we first show that any types with $\pi \leqslant \pi^*$ is indifferent to any effort in the mixed support, i.e. $b \in [b_l(\pi^*,\alpha), b_h(\pi^*,\alpha)]$. For the type (θ_h,π) of player i with $\pi \leqslant \pi^*$, the expected payoff is given by:

$$\begin{aligned}
\widetilde{U}^m_{mh}(b|\pi,\alpha,\alpha) &= \theta_h(1-\mu(\pi,\alpha))\left(\int_{\pi^*}^1 f_h(\Pi,\alpha)d\Pi + \int_0^{\pi^*} \sigma_l(b|\Pi,\alpha)f_h(\Pi,\alpha)d\Pi\right) \\
&\quad +\theta_h\mu(\pi,\alpha)\int_0^{\pi^*} \sigma_h(b|\Pi,\alpha)f_h(\Pi,\alpha)d\Pi - b \\
&= \theta_h(1-\mu(\pi,\alpha))\int_{\pi^*}^1 f_h(\Pi,\alpha)d\Pi - \theta_l\int_{\pi^*}^1 (1-\mu(\Pi,\alpha))\, f_l(\Pi,\alpha)d\Pi \\
&\quad + \Bigg(\underbrace{\frac{p_h\theta_l + p_l\theta_h}{\theta_l\pi^*}\int_0^{\pi^*} f_h(\Pi,\alpha)d\Pi - 1}_{=0}\Bigg) b
\end{aligned}$$

By (13) it can be checked that $\widetilde{U}^m_{mh}(b|\pi,\alpha,\alpha))$ is invariant of the effort b.

For type (θ_l,π) of player i with $\pi \leqslant \pi^*$, the expected payoff is given by:

$$\begin{aligned}
\widetilde{U}^m_{ml}(b|\pi,\alpha,\alpha) &= \theta_l\left[1-\mu(\pi,\alpha)\right]\int_{\pi^*}^1 f_l(\Pi,\alpha)d\Pi - \theta_l\int_{\pi^*}^1 \left[1-\mu\left(\Pi,\alpha\right)\right] f_l(\Pi,\alpha)d\Pi \\
&\quad + \Bigg(\underbrace{\frac{p_h\theta_l + p_l\theta_h}{\theta_h\pi^*}\int_0^{\pi^*} f_l(\Pi,\alpha)d\Pi - 1}_{=0}\Bigg) b
\end{aligned}$$

By (14) it can be checked that $\widetilde{U}^m_{ml}(b|\pi,\alpha,\alpha)$ is invariant of b. Therefore, all the types who play mixed strategy in equilibrium are indeed indifferent in their equilibrium support. Next, we check whether these types find it profitable to choose an effort outside of their equilibrium support.

If type (θ_h,π) with $\pi \leqslant \pi^*$ deviates to choose an effort in the high pure support $\beta \in (b_h(\pi^*,\alpha), b_h(1,\alpha)]$, then the expected payoff from such a deviation is:

$$\widetilde{U}^h_{mh}(\beta|\pi,\alpha,\alpha) = \theta_h\left[1-\mu(\pi,\alpha)+\mu(\pi,\alpha)\left(\int_0^{\pi^*} f_h(\Pi,\alpha)d\Pi + \int_{\pi^*}^{b_h^{-1}(\beta,\alpha)} f_h(\Pi,\alpha)d\Pi\right)\right] - \beta$$

Take the first order derivative w.r.t. β we have:

$$\frac{\theta_h \mu(\pi,\alpha) f_h(t,\alpha)}{b_h'(t,\alpha)} - 1 = \frac{\mu(\pi,\alpha)}{\mu(t,\alpha)} - 1 < 0$$

where $b_h(t,\alpha) = \beta$, and $t > \pi^*$. The above expression is strictly negative because $\pi < \pi^* < t$. This suggests type (θ_h, π) with $\pi \leqslant \pi^*$ will find it unprofitable to deviate to any effort in the interval $(b_h(\pi^*,\alpha), b_h(1,\alpha)]$.

If type (θ_h, π) with $\pi \leqslant \pi^*$ deviate to any effort $\beta \in [0, b_l(s^*,\alpha))$, then the expected payoff from such deviation is:

$$\widetilde{U}^l_{mh}(\beta|\pi,\alpha,\alpha) = \theta_h(1-\mu(\pi,\alpha)) \int_{b_l^{-1}(\beta,\alpha)}^1 f_h(\Pi,\alpha) d\Pi - \beta$$

The first order derivative w.r.t β shows

$$\frac{\theta_h(1-\mu(\pi,\alpha)) f_h(s,\alpha)}{b_l'(s,\alpha)} - 1 = \frac{\theta_h \left[1-\mu(\pi,\alpha)\right] f_h(s,\alpha)}{\theta_l \left[1-\mu(s,\alpha)\right] f_l(s,\alpha)} - 1 > 0$$

where $b_l(s,\alpha) = \beta$, i.e. suppose type (θ_l, s) chooses β in equilibrium. The above is strictly positive because $s > \pi^* > \pi$ which implies $\frac{\theta_h f_h(s,\alpha)}{\theta_l f_l(s,\alpha)} \geqslant 1$ and $1-\mu(\pi,\alpha) > 1-\mu(s,\alpha)$. This suggests type (θ_h, π) with $\pi \leqslant \pi^*$ finds it unprofitable to deviate to any effort in the interval $[0, b_l(\pi^*,\alpha))$.

Similarly, type (θ_l, π) of player i with $\pi \leqslant \pi^*$ does not find it profitable to deviate to either the high or the low pure support. The approach of proof is the same and thus, omitted.

Next, we need to check whether the types who play pure strategy in equilibrium, i.e. type (θ_h, π) and type (θ_l, π) with $\pi > \pi^*$, want to deviate to $[b_l(\pi^*,\alpha), b_h(\pi^*,\alpha)]$, i.e. the mixed support. It can be easily checked that if type (θ_h, π) with $\pi > \pi^*$ deviates to choose $\beta \in (b_l(\pi^*,\alpha), b_h(\pi^*,\alpha)]$, then their expected payoff is invariant of β. Specifically,

$$\widetilde{U}^m_h(\beta|\pi,\alpha,\alpha) = \theta_h(1-\mu(\pi,\alpha)) \int_{\pi^*}^1 f_h(\Pi,\alpha) d\Pi - \theta_l \int_{\pi^*}^1 (1-\mu(\Pi,\alpha)) f_l(\Pi,\alpha) d\Pi$$

Thus, the difference between the equilibrium expected payoff $V^m(\theta_h,\pi,\alpha)$ and the maximum expected payoff from deviating to the mixed support can be calculated:

$$\begin{aligned} V^m(\theta_h,\pi,\alpha) - \widetilde{U}^m_h(\beta|\pi,\alpha,\alpha) &= \theta_h \int_{\pi^*}^{\pi} (\mu(\pi,\alpha) - \mu(\Pi,\alpha)) f_h(\Pi,\alpha) d\Pi \\ &> 0 \end{aligned}$$

Similarly, we know that if types (θ_l, π) with $\pi > \pi^*$ deviates to choose $\beta \in (b_l(\pi^*,\alpha), b_h(\pi^*,\alpha)]$, then the expected payoff is invariant of bid β. Specifically,

$$\widetilde{U}^m_l(b|\pi,\alpha,\alpha) = \underbrace{\theta_l \int_{\pi^*}^{\pi} (\mu(\Pi,\alpha) - \mu(\pi,\alpha)) f_l(\Pi,\alpha) d\Pi}_{<0} + \theta_l \int_{\pi}^{1} (\mu(\Pi,\alpha) - \mu(\pi,\alpha)) f_l(\Pi,\alpha) d\Pi$$

The equilibrium expected payoff is given by:

$$V^m(\theta_l,\pi,\alpha) = \theta_l \int_{\pi}^{1} (\mu(\Pi,\alpha) - \mu(\pi,\alpha)) f_l(\Pi,\alpha) d\Pi$$

which is larger than $\widetilde{U}^m_l(b|\pi,\alpha,\alpha)$. Thus, all types who play pure strategy in equilibrium do not want to deviate to the support of mixed strategy. The fact that type (θ_h, π) and (θ_l, π) of player i with $\pi \geqslant \pi^*$ does not want to deviate to either the low or the high pure support follows directly from the proof of the allocative efficient equilibrium. For uniqueness, note that π^* is unique due to MLRP, thus the lower bound of $b_h(\pi, \alpha)$, i.e. $b_h(\pi^*, \alpha)$, is unique. The lower bound of $b_l(\pi, \alpha)$ must be unique, i.e. zero. Finally, there does not exist any equilibrium in which $\sigma_h(b|\pi, \alpha)$ or $\sigma_l(b|\pi, \alpha)$ has jumps or atoms in efforts, due to the same argument as the proof of the pure strategies being continuous in Lemma 1. □

Proof of Corollary 2

The proof of the first part of the corollary is obvious, thus is omitted. Here we show the calculation of $\frac{\partial \pi^*}{\partial \alpha}$. Rewrite equation (5) in Proposition 1 to

$$\int_0^{\pi^*} \left[f_l(\Pi, \alpha)\theta_l - f_h(\Pi, \alpha)\theta_h\right] d\Pi = 0$$

and take first order derivative w.r.t α, we have

$$\frac{\partial \pi^*}{\partial \alpha} = \frac{\int_0^{\pi^*} \left[\frac{\partial f_h(\Pi,\alpha)}{\partial \alpha}\theta_h - \frac{\partial f_l(\Pi,\alpha)}{\partial \alpha}\theta_l\right] d\Pi}{f_l(\pi^*, \alpha)\theta_l - f_h(\pi^*, \alpha)\theta_h}$$

Since $\int_0^{\pi^*} \frac{\partial f_h(\Pi,\alpha)}{\partial \alpha} d\Pi < 0$ and $\int_0^{\pi^*} \frac{\partial f_l(\Pi,\alpha)}{\partial \alpha} d\Pi > 0$, and since we have $f_l(\pi^*, \alpha)\theta_l - f_h(\pi^*, \alpha)\theta_h < 0$, it must be true that $\frac{\partial \pi^*}{\partial \alpha} > 0$. □

Proof of Lemma 3

First, we focus on the case when $\alpha \leq \widehat{\alpha}$, i.e., when the unique equilibrium is allocative efficient. If type (θ_h, π) of player i chooses an effort in the high pure support, then her expected payoff is:

$$\widetilde{U}_h^h(b|\pi, \alpha_i, \alpha) = \theta_h \left[1 - \mu(\pi, \alpha_i) + \mu(\pi, \alpha_i) \int_0^{b_h^{-1}(b,\alpha)} f_h(\Pi, \alpha) d\Pi\right] - b. \tag{15}$$

where $b_h^{-1}(b, \alpha)$ is the inverse of the equilibrium pure strategy that player j plays. By the first order condition w.r.t b we know that type (θ_h, π) of player i must find it optimal to choose $b_h(s, \alpha)$ where s satisfies

$$f_h(\pi, \alpha_i) = f_h(s, \alpha), \tag{16}$$

which also implies $f_l(\pi, \alpha_i) = f_l(s, \alpha)$. To show that $b_h(s, \alpha)$ is indeed the optimal effort, let's compare the expected payoff from bidding $b_h(s, \alpha)$ to an effort $b_h(\eta, \alpha)$ where $\eta \neq s$:

$$\begin{aligned}
\widetilde{U}_h^h(b_h(s, \alpha)|\pi, \alpha_i, \alpha) - \widetilde{U}_h^h(b_h(\eta, \alpha)|\pi, \alpha_i, \alpha) &= p_h \int_\eta^s \left[f_h(\pi, \alpha_i) - f_h(\Pi, \alpha)\right] f_h(\Pi, \alpha) d\Pi \\
&= p_h \int_\eta^s \left[f_h(s, \alpha) - f_h(\Pi, \alpha)\right] f_h(\Pi, \alpha) d\Pi \geq 0
\end{aligned}$$

Note that $\pi = s$ if and only if $\alpha_i = \alpha$, and that

$$\frac{\partial f_h(\pi, \alpha_i)}{\partial \pi} = \frac{\partial f_h(s, \alpha)}{\partial s}\frac{\partial s}{\partial \pi} > 0$$

which suggests $\frac{\partial s}{\partial \pi} > 0$. When $\alpha_i \leq \alpha$, by definition of rotation order, there always exists an s satisfying (16) for all $\pi \in [0, 1]$. Therefore, the maximum expected payoff for type (θ_h, π) when she chooses the optimal effort in the high pure support is indeed $\widetilde{U}_h^h(b_h(s, \alpha)|\pi, \alpha_i, \alpha)$.

Suppose instead that type (θ_h, π) of player i chooses an effort in the low pure support, then her expected payoff is:

$$\widetilde{U}_h^l(b|\pi,\alpha_i,\alpha) = \theta_h\left[1-\mu(\pi,\alpha_i)\right]\int_{b_l^{-1}(b,\alpha)}^{1} f_h(\Pi,\alpha)d\Pi - b \tag{17}$$

where $b_l^{-1}(b,\alpha)$ is the inverse of the equilibrium pure strategy that player j will play. The first order condition w.r.t b requires:

$$f_l(\pi,\alpha_i) = \frac{f_l(\widehat{s},\alpha)}{f_h(\widehat{s},\alpha)}\frac{\theta_l}{\theta_h} f_l(\widehat{s},\alpha)$$

meaning that player i would find it optimal to exert an effort $b_l(\widehat{s},\alpha)$. Note that $f_l(s,\alpha) = f_l(\pi,\alpha_i) < f_l(\widehat{s},\alpha)$, thus, $s > \widehat{s}$. To show that $b_l(\widehat{s},\alpha)$ is indeed the optimal, let's compare the expected payoff from bidding $b_l(\widehat{s},\alpha)$ to bidding $b_l(\eta,\alpha)$ where $\eta \neq \widehat{s}$:

$$\begin{aligned} & \widetilde{U}_h^l(b_l(\widehat{s},\alpha)|\pi,\alpha_i,\alpha) - \widetilde{U}_h^l(b_l(\eta,\alpha)|\pi,\alpha_i,\alpha) \\ = \; & p_l\theta_l \int_{\widehat{s}}^{\eta}\left[\frac{f_l(\widehat{s},\alpha)}{f_h(\widehat{s},\alpha)} f_l(\widehat{s},\alpha) - \frac{f_l(\Pi,\alpha)}{f_h(\Pi,\alpha)} f_l(\Pi,\alpha)\right] \geq 0 \end{aligned}$$

The maximum expected payoff for type (θ_h,π) when choosing an effort level in the low pure support is thus,

$$\widetilde{U}_h^l(b_l(\widehat{s},\alpha)|\pi,\alpha_i,\alpha) = \theta_h\left[1-\mu(\pi,\alpha_i)\right]\int_{\widehat{s}}^{1} f_h(\Pi,\alpha)d\Pi - \theta_l\int_{\widehat{s}}^{1}\left[1-\mu\left(\Pi,\alpha\right)\right] f_l(\Pi,\alpha)d\Pi$$

The difference between the two maximum expected payoffs is

$$\begin{aligned} & \widetilde{U}_h^h(b_h(s,\alpha)|\pi,\alpha_i,\alpha) - \widetilde{U}_h^l(b_l(\widehat{s},\alpha)|\pi,\alpha_i,\alpha) \\ = \; & \theta_h\int_0^s\left[\underbrace{\mu(s,\alpha)-\mu\left(\Pi,\alpha\right)}_{>0}\right] f_h(\Pi,\alpha)d\Pi \\ & +\theta_h\int_0^{\widehat{s}}\left[\left[1-\mu(\pi,\alpha_i)\right]-\left[1-\mu\left(\Pi,\alpha\right)\right]\frac{f_l(\Pi,\alpha)}{f_h(\Pi,\alpha)}\frac{\theta_l}{\theta_h}\right] f_h(\Pi,\alpha)d\Pi \\ > \; & \theta_h\int_0^s\left[\mu(\pi,\alpha_i)-\mu\left(\Pi,\alpha\right)\right] f_h(\Pi,\alpha)d\Pi + \theta_h\int_0^{\widehat{s}}\left[\mu\left(\Pi,\alpha\right)-\mu(\pi,\alpha_i)\right] f_h(\Pi,\alpha)d\Pi \\ > \; & \theta_h\int_0^{\widehat{s}}\left[\mu(\pi,\alpha_i)-\mu\left(\Pi,\alpha\right)\right] f_h(\Pi,\alpha)d\Pi + \theta_h\int_0^{\widehat{s}}\left[\mu\left(\Pi,\alpha\right)-\mu(\pi,\alpha_i)\right] f_h(\Pi,\alpha)d\Pi \\ = \; & 0 \end{aligned}$$

Therefore, type (θ_h,π) of player i's maximum expected payoff when choosing α_i is $\widetilde{U}_h^h(b_h(s,\alpha)|\pi,\alpha_i,\alpha)$, and thus, the marginal expected payoff from increasing α_i is given by

$$\frac{\partial}{\partial\alpha_i}\widetilde{U}_h^h(b_h(s,\alpha)|\pi,\alpha_i,\alpha) = -p_h\theta_h\frac{\partial f_h(\pi,\alpha_i)}{\partial\alpha_i}\int_s^1 f_h(\Pi,\alpha)d\Pi \tag{18}$$

Now we turn to the types with the low valuation. If type (θ_l,π) of player i chooses an effort in the low pure support, then her expected payoff is:

$$\widetilde{U}_l^l(b|\pi,\alpha_i,\alpha) = \theta_l\left[1-\mu(\pi,\alpha_i)\right]\int_{b_l^{-1}(b,,\alpha)}^{1} f_l(\Pi,\alpha)d\Pi - b \tag{19}$$

where $b_l^{-1}(b,,\alpha)$ is the inverse of the equilibrium pure strategy that player j will play. By the first order condition w.r.t b we know that type (θ_l,π) of player i should optimally choose $b_l(t,\alpha)$

where t satisfies

$$f_l(\pi, \alpha_i) = f_l(t, \alpha), \tag{20}$$

which also implies $f_h(\pi, \alpha_i) = f_h(t, \alpha)$. Note that

$$\frac{\partial f_l(\pi, \alpha_i)}{\partial \pi} = \frac{\partial f_l(t, \alpha)}{\partial t}\frac{\partial t}{\partial \pi} < 0$$

which implies $\frac{\partial t}{\partial \pi} > 0$. Note also that (16) and (20) suggests $t = s$. To show that $b_l(t, \alpha)$ is indeed optimal, let's compare the corresponding expected payoff to the payoff from bidding $b_l(\eta, \alpha)$ where $\eta \neq t$:

$$\widetilde{U}_l^l(b_l(t, \alpha)|\pi, \alpha_i, \alpha) - \widetilde{U}_l^l(b_l(\eta, \alpha)|\pi, \alpha_i, \alpha) = p_h \int_t^\eta \left[f_h(\eta, \alpha) - f_h(\Pi, \alpha)\right] f_h(\Pi, \alpha) d\Pi \geq 0$$

When $\alpha_i \leq \alpha$, by definition of rotation order, there always exists t satisfying (20) for all $\pi \in [0, 1]$. Therefore, the maximum expected payoff for type (θ_l, π) in the low pure support is given by:

$$\widetilde{U}_l^l(b_l(t, \alpha)|\pi, \alpha_i, \alpha) = \theta_l \int_t^1 \left[\mu\left(\Pi, \alpha\right) - \mu(\pi, \alpha_i)\right] f_l(\Pi, \alpha) d\Pi$$

Suppose instead that type (θ_l, π) of player i chooses an effort in the high pure support, then by exactly the same approach as above, we can show that the maximum expected payoff of doing so is:

$$\begin{aligned}
\widetilde{U}_l^h(b_h(\widehat{t}, \alpha)|\pi, \alpha_i, \alpha) &= p_h \theta_l \int_0^{\widehat{t}} \left[f_h(\pi, \alpha_i) - f_h(\Pi, \alpha)\frac{f_h(\Pi, \alpha)}{f_l(\Pi, \alpha)}\frac{\theta_h}{\theta_l}\right] f_l(\Pi, \alpha) d\Pi \\
&\quad + p_h \theta_l \int_0^1 \left[f_h\left(\Pi, \alpha\right) - f_h(\pi, \alpha_i)\right] f_l(\Pi, \alpha) d\Pi
\end{aligned}$$

where $\widehat{t} = \widehat{s}$. Again, we need to compare the two maximum expected payoffs to determine whether type (θ_l, π) of player i should choose an effort in the low pure support or the high pure support. It turns out that the former earns type (θ_l, π) a higher expected payoff:

$$\begin{aligned}
& \widetilde{U}_l^h(b_h(\widehat{t}, \alpha)|\pi, \alpha_i, \alpha) \\
= \; & p_h \theta_l \int_0^{\widehat{t}} \left[f_h(\pi, \alpha_i) - f_h(\Pi, \alpha)\frac{f_h(\Pi, \alpha)}{f_l(\Pi, \alpha)}\frac{\theta_h}{\theta_l}\right] f_l(\Pi, \alpha) d\Pi \\
& + p_h \theta_l \int_0^t \left[f_h\left(\Pi, \alpha\right) - f_h(\pi, \alpha_i)\right] f_l(\Pi, \alpha) d\Pi + p_h \theta_l \int_t^1 \left[f_h\left(\Pi, \alpha\right) - f_h(\pi, \alpha_i)\right] f_l(\Pi, \alpha) d\Pi \\
< \; & p_h \theta_l \int_0^t \left[1 - \frac{f_h(\Pi, \alpha)}{f_l(\Pi, \alpha)}\frac{\theta_h}{\theta_l}\right] f_h\left(\Pi, \alpha\right) f_l(\Pi, \alpha) d\Pi + p_h \theta_l \int_t^1 \left[f_h\left(\Pi, \alpha\right) - f_h(\pi, \alpha_i)\right] f_l(\Pi, \alpha) d\Pi \\
< \; & p_h \theta_l \int_t^1 \left(f_h\left(\Pi, \alpha\right) - f_h(\pi, \alpha_i)\right) f_l(\Pi, \alpha) d\Pi \\
= \; & \widetilde{U}_l^l(b_l(t, \alpha)|\pi, \alpha_i, \alpha)
\end{aligned}$$

Therefore, type (θ_l, π) of player i's maximum expected payoff is $\widetilde{U}_l^l(b_l(t, \alpha)|\pi, \alpha_i, \alpha)$, and thus, the marginal expected payoff from increasing α_i is given by

$$\frac{\partial}{\partial \alpha_i}\widetilde{U}_l^l(b_l(t, \alpha)|\pi, \alpha_i, \alpha) = -p_h \theta_l \frac{\partial f_h(\pi, \alpha_i)}{\partial \alpha_i} \int_t^1 f_l(\Pi, \alpha) d\Pi \tag{21}$$

Now, it can be shown that the ex ante marginal expected payoff from increasing α_i is positive:

$$\begin{aligned} AMU_i(\alpha_i,\alpha) &= \int_0^1 \left[p_l \frac{\partial}{\partial \alpha_i}\widetilde{U}_l^l(b_l(t,\alpha)|\pi,\alpha_i,\alpha) + p_h \frac{\partial}{\partial \alpha_i}\widetilde{U}_h^h(b_h(s,\alpha)|\pi,\alpha_i,\alpha)\right] d\pi \qquad (22)\\ &= p_h \int_0^1 \left[\underbrace{-p_l\theta_l \int_s^1 f_l(\Pi,\alpha)d\Pi - p_h\theta_h \int_s^1 f_h(\Pi,\alpha)d\Pi}_{A}\right] \frac{\partial f_h(\pi,\alpha_i)}{\partial \alpha_i} d\pi \\ &> 0 \end{aligned}$$

The second equality is due to (18), (21), and $s = t$. Recall that s is increasing in π, thus, the term A is increasing in π. Hence, according to Lemma 9, $AMU_i(\alpha_i,\alpha)$ is positive.

We have proved that $AMU_i(\alpha_i,\alpha) > 0$ given that $\alpha_i \le \alpha \le \widehat{\alpha}$. Now, we prove that $AMU_i(\alpha_i,\alpha) \ge 0$ also holds when $\alpha \ge \widehat{\alpha}$ and $\alpha_i \le \alpha$. Note that now the opponent, player j believes he is in a symmetric equilibrium in which both the high and the low valuation types of player play mixed strategy when their information type is lower than π^*.

When type (θ_l,π) of player i chooses an effort in the low pure support, if there is an interior solution, then the maximum expected payoff in this support is:

$$\widetilde{U}_l^l(b_l(s,\alpha)|\pi,\alpha_i,\alpha) = \theta_l \int_s^1 \left(\mu(\Pi,\alpha) - \mu\left(\pi,\alpha_i\right)\right) f_l(\Pi,\alpha)d\Pi \ge 0 \qquad (23)$$

where s satisfies $f_l(\pi,\alpha_i) = f_l(s,\alpha)$ and $s \in [\pi^*,1]$.

When type (θ_l,π) of player i chooses an effort in the mixed support, the expected payoff is

$$\widetilde{U}_l^m(b|\pi,\alpha_i,\alpha) = \theta_l \int_{\pi^*}^1 \left(\mu(\Pi,\alpha) - \mu\left(\pi,\alpha_i\right)\right) f_l(\Pi,\alpha)d\Pi \qquad (24)$$

which is invariant of effort b. Note that for any $s \in [\pi^*,1]$, we have $\widetilde{U}_l^m(b|\pi,\alpha_i,\alpha) > \widetilde{U}_l^l(b_l(s,\alpha)|\pi,\alpha_i,\alpha)$. If the solution of s to $f_l(\pi,\alpha_i) = f_l(s,\alpha)$ is less than π^*, then it also indicates that choosing an effort in the low pure support earns less payoff than choosing in the mixed support. Therefore, the optimal effort of type (θ_l,π) of player i is never in the low pure support.

When type (θ_l,π) of player i chooses an effort in the high pure support, the expected payoff is:

$$\widetilde{U}_l^h(b|\pi,\alpha_i,\alpha) = \theta_l \left(1 - \mu(\pi,\alpha_i) + \mu(\pi,\alpha_i)\int_0^{b_h^{-1}(b,\alpha)} f_l(\Pi,\alpha)d\Pi\right) - b$$

where $b_h^{-1}(b,\alpha)$ is the inverse of the pure strategy of player j in the high pure support. The first order condition w.r.t. b gives

$$f_h(\pi,\alpha_i) = \frac{f_h(\widehat{s},\alpha)}{f_l(\widehat{s},\alpha)}\frac{\theta_h}{\theta_l} f_h(\widehat{s},\alpha) \qquad (25)$$

where $\widehat{s} \equiv b_h^{-1}(b,\alpha)$. To show that $b_h(\widehat{s},\alpha)$ is indeed optimal, we again compare the corresponding expected payoff to the payoff from an effort $b_h(\eta,\alpha)$ where $\eta \ne \widehat{s}$:

$$\widetilde{U}_l^h(b_h(\widehat{s},\alpha)|\pi,\alpha_i,\alpha) - \widetilde{U}_l^h(b_h(\eta,\alpha)|\pi,\alpha_i,\alpha) = p_h\theta_l \int_\eta^{\widehat{s}} \left[f_h(\pi,\alpha_i) - \frac{\theta_h}{\theta_l}\frac{f_h(\Pi,\alpha)}{f_l(\Pi,\alpha)} f_h(\Pi,\alpha)\right] d\Pi \ge 0$$

Thus, when there exists $\widehat{s} \in [\pi^*,1]$ satisfying the above first order condition, then there is an optimal effort in the high pure support, $b_h(\widehat{s},\alpha)$, which player i should choose. In this case, plug the optimal effort into the expected payoff to obtain the maximum expected payoff:

$$
\begin{aligned}
&\widetilde{U}_l^h(b_h(\widehat{s},\alpha)|\pi,\alpha_i,\alpha) \\
= \quad & \theta_l \int_{\pi^*}^1 \left(\mu(\Pi,\alpha)-\mu(\pi,\alpha_i)\right) f_l(\Pi,\alpha)d\Pi + p_h\theta_l \int_{\pi^*}^{\widehat{s}} \left(\underbrace{f_h(\pi,\alpha_i) - \frac{\theta_h f_h(\Pi,\alpha)}{\theta_l f_l(\Pi,\alpha)} f_h(\Pi,\alpha)}_{>0} \right) f_l(\Pi,\alpha)d\Pi
\end{aligned}
$$

It is can be shown that $\widetilde{U}_l^h(b_h(\widehat{s},\alpha)|\pi,\alpha_i,\alpha) > \widetilde{U}_l^m(b|\pi,\alpha_i,\alpha)$ as long as there exists $\widehat{s} \in [\pi^*,1]$ satisfying (25). When the $\widehat{s}$ satisfying the first order condition is not in $[\pi^*,1]$, then the optimal effort is not in the high pure support, and must be in the mixed support, which means it is optimal to randomize in the mixed support.

In summary, since $\widehat{s}$ is increasing with π, for lower realizations of π — lower than the value which induces $\widehat{s} < \pi^*$ — the optimal effort is to randomize in the mixed support, whereas for higher realizations of π — higher than the value which induces $\widehat{s} > \pi^*$ — the optimal effort is in the high pure support.

Now we take the first order derivative of $\widetilde{U}_l^h(b_h(\widehat{s},\alpha)|\pi,\alpha_i,\alpha)$ and $\widetilde{U}_l^m(b|\pi,\alpha_i,\alpha)$ w.r.t α_i:
For $\widehat{s} < \pi^*$ (i.e., when the optimal effort is in the mixed support):

$$\frac{\partial}{\partial\alpha_i}\widetilde{U}_l^m(b|\pi,\alpha_i,\alpha) = p_h\theta_l \frac{\partial f_h(\pi,\alpha_i)}{\partial\alpha_i} \int_{\pi^*}^1 f_l(\Pi,\alpha)d\Pi$$

For $\widehat{s} > \pi^*$ (i.e., when the optimal effort is in the high pure support):

$$\frac{\partial}{\partial\alpha_i}\widetilde{U}_l^h(b_h(\widehat{s},\alpha)|\pi,\alpha_i,\alpha) \quad = \quad p_h\theta_l \frac{\partial f_h(\pi,\alpha_i)}{\partial\alpha_i} \int_{\pi^*}^1 f_l(\Pi,\alpha)d\Pi + p_h\theta_l \frac{\partial f_h(\pi,\alpha_i)}{\partial\alpha_i} \int_{\pi^*}^{\widehat{s}} f_l(\Pi,\alpha)d\Pi$$

Combining the two, the marginal expected payoff for type (θ_l,π) is given by:

$$p_h\theta_l \frac{\partial f_h(\pi,\alpha_i)}{\partial\alpha_i} \int_{\pi^*}^1 f_l(\Pi,\alpha)d\Pi + \max\left\{0, p_h\theta_l \frac{\partial f_h(\pi,\alpha_i)}{\partial\alpha_i} \int_{\pi^*}^{\widehat{s}} f_l(\Pi,\alpha)d\Pi\right\}$$

Next, we turn to type (θ_h,π) of player i's optimal effort given she chooses α_i. When she chooses an effort in the high pure support, the maximum expected payoff of doing so is:

$$
\begin{aligned}
&\widetilde{U}_h^h(b_h(t,\alpha)|\pi,\alpha_i,\alpha) \\
= \quad & \theta_h \int_{\pi^*}^t \underbrace{[\mu(\pi,\alpha_i)-\mu(\Pi,\alpha)]}_{>0} f_h(\Pi,\alpha)d\Pi + p_l\theta_h \int_{\pi^*}^1 \left(f_l(\pi,\alpha_i) - \frac{f_l(\Pi,\alpha)\theta_l}{f_h(\Pi,\alpha)\theta_h} f_l(\Pi,\alpha) \right) f_h(\Pi,\alpha)d\Pi
\end{aligned}
$$

where $t \equiv f_h^{-1}(f_h(\pi,\alpha_i),\alpha)$. By the same method as before, it can be shown that $b_h(t,\alpha)$ is indeed the optimal among all other efforts in the same support.

When type (θ_h,π) of player i chooses an effort in the low pure support, the maximum expected payoff of doing so is:

$$\widetilde{U}_h^l(b_l(\widehat{t},\alpha)|\pi,\alpha_i,\alpha) \quad = \quad p_l\theta_h \int_{\widehat{t}}^1 \left(f_l(\pi,\alpha_i) - \frac{f_l(\Pi,\alpha)\theta_l}{f_h(\Pi,\alpha)\theta_h} f_l(\Pi,\alpha) \right) f_h(\Pi,\alpha)d\Pi$$

which is less than $\widetilde{U}_h^h(b_h(t,\alpha)|\pi,\alpha_i,\alpha)$, as it must be true that $\widehat{t} \in [\pi^*,1]$ whenever there exists such an interior solution.

When type (θ_h, π) of player i chooses an effort in the mixed support, the expected payoff is:

$$\widetilde{U}_h^m(b|\pi, \alpha_i, \alpha) \quad = \quad p_l\theta_h \int_{\pi^*}^1 \left(f_l(\pi, \alpha_i) - \frac{f_l(\Pi, \alpha)\theta_l}{f_h(\Pi, \alpha)\theta_h} f_l(\Pi, \alpha) \right) f_h(\Pi, \alpha) d\Pi$$

which is, again, invariant of b. It is easy to see that $\widetilde{U}_h^m(b|\pi, \alpha_i, \alpha) \geq \widetilde{U}_h^l(b_l(\widehat{t}, \alpha)|\pi, \alpha_i, \alpha)$.

In summary, given that t is increasing in π, for lower values of π such that $t < \pi^*$ the optimal effort lies in the mixed support, and for higher values of π such that $t > \pi^*$ the optimal effort lies on the high pure support.

Now we take the first order derivative of $\widetilde{U}_h^m(b|\pi, \alpha_i, \alpha)$ and $\widetilde{U}_h^h(b_h(t, \alpha)|\pi, \alpha_i, \alpha)$ w.r.t α_i:

For $t < \pi^*$ (i.e., when the optimal effort is in the mixed support):

$$\frac{\partial}{\partial \alpha_i} \widetilde{U}_h^m(b|\pi, \alpha_i, \alpha) = -p_h\theta_h \frac{\partial f_h(\pi, \alpha_i)}{\partial \alpha_i} \int_{\pi^*}^1 f_h(\Pi, \alpha) d\Pi$$

For $t > \pi^*$ (i.e., when the optimal effort is in the high pure support):

$$\frac{\partial}{\partial \alpha_i} \widetilde{U}_h^h(b_h(t, \alpha)|\pi, \alpha_i, \alpha) \quad = \quad -p_h\theta_h \frac{\partial f_h(\pi, \alpha_i)}{\partial \alpha_i} \int_t^1 f_h(\Pi, \alpha) d\Pi$$

Combining the two, the marginal expected payoff for type (θ_h, π) is given by:

$$-p_h\theta_h \frac{\partial f_h(\pi, \alpha_i)}{\partial \alpha_i} \int_{\max\{t, \pi^*\}}^1 f_h(\Pi, \alpha) d\Pi$$

We are now able to calculate the ex ante marginal expected payoff of player i w.r.t α_i:

$$\begin{aligned} & AMU_i(\alpha_i, \alpha) \\ = \quad & p_h \int_0^1 \frac{\partial f_h(\pi, \alpha_i)}{\partial \alpha_i} \underbrace{\left\{ p_l\theta_l \max\left\{0, \int_{\pi*}^{\widehat{s}} f_l(\Pi, \alpha) d\Pi \right\} - p_h\theta_h \int_{\max\{t, \pi^*\}}^1 f_h(\Pi, \alpha) d\Pi \right\}}_{F} d\pi \\ > \quad & 0 \end{aligned}$$

Since both t and $\widehat{s}$ are increasing in π, thus the term F above is weakly increasing in π. According to the Lemma 9, we have $AMU_i(\alpha_i, \alpha) \geq 0$ for $\alpha > \widehat{\alpha}$.

We now turn to the last part of the lemma. Recall from Section 2, when the signals players receive are perfectly informative, we have $f_h(\pi_i, \overline{\alpha}) = 0$ if $\pi_i \leq p_l$ and $f_h(\pi_i, \overline{\alpha}) = \frac{1}{p_h}$ if $\pi_i > p_l$; and correspondingly, that $f_l(\pi_i, \overline{\alpha}) = \frac{1}{p_l}$ if $\pi_i \leq p_l$ and $f_l(\pi_i, \overline{\alpha}) = 0$ if $\pi_i > p_l$. Note that in this case $\pi^* = \frac{p_l\theta_h + p_h\theta_l}{\theta_h} > \pi_{\overline{\alpha}}^+ \equiv p_l$.

When type (θ_l, π) of player i chooses an effort in the low pure support, the expected payoff given that player j playing the symmetric equilibrium with $\overline{\alpha}$ is:

$$\widetilde{U}_l^l(b|\pi, \alpha_i, \overline{\alpha}) = \theta_l \left[1 - \mu(\pi, \alpha_i)\right] \int_{b_l^{-1}(b, \overline{\alpha})}^1 f_l(\Pi, \overline{\alpha}) d\Pi - b = 0$$

In fact, the low pure support in this case is condensed to a point at zero. When type (θ_l, π) of player i chooses an effort in the mixed support, the expected payoff is

$$\widetilde{U}_l^m(b|\pi, \alpha_i, \overline{\alpha}) = \theta_l \int_{\pi^*}^1 \left(\mu(\pi, \alpha_i) - \mu\left(\Pi, \overline{\alpha}\right)\right) f_l(\Pi, \overline{\alpha}) d\Pi = 0$$

When type (θ_l, π) of player i chooses an effort $b_h(\kappa, \overline{\alpha})$ (where $\kappa > \pi^*$) in the high pure support, the expected payoff is:

$$\begin{aligned}
\widetilde{U}_l^h(b|\pi,\alpha_i,\overline{\alpha}) &= \theta_l\left[1-\mu(\pi,\alpha_i)+\mu(\pi,\alpha_i)\int_0^{\kappa} f_l(\Pi,\overline{\alpha})d\Pi\right]-b_h(\kappa,\overline{\alpha}) \\
&= \theta_l - b_h(\kappa,\overline{\alpha}) \\
&= \frac{\theta_h}{p_h}(p_l-\kappa) \\
&\leq 0
\end{aligned}$$

The second equality is due to player j's signal realization π_i must be below p_l given that player i has θ_l, as the signal is perfectly informative. The inequality is because of $\kappa \geq \pi^* > p_l$. Thus, type (θ_l, π) of player i does not want to choose any effort in the high pure support. In summary, her maximum expected payoff is zero.

Now we turn to type (θ_h, π) of player i. When she has α_i, and suppose she chooses an effort in the high pure support, the expected payoff is:

$$\widetilde{U}_h^h(b|\pi,\alpha_i,\overline{\alpha}) = \theta_h\left((1-\mu(\pi,\alpha_i))+\mu(\pi,\alpha_i)\int_0^{b_h^{-1}(b,\overline{\alpha})} f_h(\Pi,\overline{\alpha})d\Pi\right)-b$$

and by taking the first order derivative w.r.t. b we have $\frac{f_h(\pi,\alpha_i)}{f_h(t,\overline{\alpha})}-1<0$ where $t \in (\pi^*,1)$. Thus, player i does not want to choose any effort in the high pure support and again, she finds it optimal to choose an effort in the mixed support, which earns her zero expected payoff.

Since the low valuation type of player j choose zero with positive probability, type (θ_h, π) of player i would never choose zero with positive probability. In summary, her optimal effort must always lie in the mixed support.

When type (θ_h, π) of player i chooses an effort in the mixed support, the expected payoff is:

$$\widetilde{U}_h^m(b|\pi,\alpha_i,\overline{\alpha}) = p_l\theta_h f_l(\pi,\alpha_i)\int_{\pi^*}^1 f_h(\Pi,\overline{\alpha})d\Pi$$

The first order derivative of $\widetilde{U}_h^m(b|\pi,\alpha_i,\overline{\alpha})$ w.r.t α_i is:

$$\frac{\partial}{\partial\alpha_i}\widetilde{U}_h^m(b|\pi,\alpha_i,\overline{\alpha}) = -p_h\theta_h\frac{\partial f_h(\pi,\alpha_i)}{\partial\alpha_i}\int_{\pi^*}^1 f_h(\Pi,\overline{\alpha})d\Pi$$

We are now able to calculate the marginal expected payoff of player i w.r.t α_i:

$$\begin{aligned}
AMU_i(\alpha_i,\overline{\alpha}) &= p_h\int_0^1 \frac{\partial}{\partial\alpha_i}\widetilde{U}_h^m(b|\pi,\alpha_i,\overline{\alpha})d\pi \\
&= -p_h^2\theta_h\int_0^1 \frac{\partial f_h(\pi,\alpha_i)}{\partial\alpha_i}d\pi\int_{\pi^*}^1 f_h(\Pi,\overline{\alpha})d\Pi \\
&= 0
\end{aligned}$$

□

Proof of Lemma 4

Similar to the previous lemma, we prove this lemma in two separated cases: when $\alpha \leq \widehat{\alpha}$ and when $\alpha > \widehat{\alpha}$.

When $\alpha \leq \widehat{\alpha}$, let $\alpha_i = \underline{\alpha}$, then the maximum expected payoff of player i when she has θ_h

and θ_l are given by

$$\begin{aligned}
\widetilde{U}_l^l(b_l(t,\alpha)|\pi,\underline{\alpha},\alpha) &= p_h\theta_l\int_t^1 [f_h(\Pi,\alpha)-f_h(\pi,\underline{\alpha})]\,dF_l(\Pi,\alpha)>0\\
\widetilde{U}_h^h(b_h(s,\alpha)|\pi,\underline{\alpha},\alpha) &= p_h\underbrace{\int_0^s [1-f_h(\Pi,\alpha)]\,[f_h(\Pi,\alpha)\theta_h-f_l(\Pi,\alpha)\theta_l]\,d\Pi}_{>0}\\
&\quad +p_l\theta_l\underbrace{\int_s^1 [1-f_l(\Pi,\alpha)]\,f_l(\Pi,\alpha)d\Pi}_{>0}+p_l\,(\theta_h-\theta_l)
\end{aligned}$$

where $t=s$ is given by $f_h(s,\alpha)=f_h(\pi,\underline{\alpha})=1$ and $f_l(t,\alpha)=f_l(\pi,\underline{\alpha})=1$. and thus

$$U_i(\underline{\alpha},\alpha)=p_h\widetilde{U}_h^h(b_h(s,\alpha)|\pi,\underline{\alpha},\alpha)+p_l\widetilde{U}_l^l(b_l(t,\alpha)|\pi,\underline{\alpha},\alpha)>p_hp_l(\theta_h-\theta_l)=V_i(\underline{\alpha},\underline{\alpha}).$$

When $\alpha>\widehat{\alpha}$, again let $\alpha_i=\underline{\alpha}$. We first show that the expected payoff of player i with θ_l is positive, and then show that the expected payoff when she has θ_h is larger than $p_l(\theta_h-\theta_l)$.

Suppose, on the one hand, if $\widehat{s}<\pi^*$, then the expected payoff of player i with θ_l is $\widetilde{U}_l^m(b|\pi,\underline{\alpha},\alpha)$. In the proof of Lemma 3, we have shown that $\widetilde{U}_l^m(b|\pi,\alpha_i,\alpha)\geq\widetilde{U}_l^l(b_l(s,\alpha)|\pi,\alpha_i,\alpha)\geq 0$ for $\alpha_i\leq\alpha$, hence $\widetilde{U}_l^m(b|\pi,\underline{\alpha},\alpha)\geq 0$. On the other hand, suppose $\widehat{s}\geq\pi^*$, type (θ_l,π) of player i has an expected payoff of $\widetilde{U}_l^h(b_h(\widehat{s},\alpha)|\pi,\underline{\alpha},\alpha)$, which has been shown to be positive in the proof of Lemma 3.

Let us turn to type (θ_h,π) of player i. Recall from (5) that $\int_0^{\pi^*}(f_h(\Pi,\alpha)\theta_h-f_l(\Pi,\alpha)\theta_l)\,d\Pi=0$. Since we have $\int_0^1(f_h(\Pi,\alpha)\theta_h-f_l(\Pi,\alpha)\theta_l)\,d\Pi=\theta_h-\theta_l$, hence

$$\int_{\pi^*}^1\left(1-\frac{f_l(\Pi,\alpha)}{f_h(\Pi,\alpha)}\frac{\theta_l}{\theta_h}\right)f_h(\Pi,\alpha)\theta_h d\Pi=\theta_h-\theta_l.$$

When $t>\pi^*$, her expected payoff is given by

$$\begin{aligned}
&\widetilde{U}_h^h(b_h(t,\alpha)|\pi,\underline{\alpha},\alpha)\\
= {}& p_h\theta_h\int_{\pi^*}^t (1-f_h(\Pi,\alpha))\,f_h(\Pi,\alpha)d\Pi+p_l\theta_h\int_{\pi^*}^1\left(1-\frac{f_l(\Pi,\alpha)\theta_l}{f_h(\Pi,\alpha)\theta_h}f_l(\Pi,\alpha)\right)f_h(\Pi,\alpha)d\Pi\\
> {}& \int_{\pi^*}^t p_lf_l(\Pi,\alpha)\left(\frac{f_h(\Pi,\alpha)\theta_h}{f_l(\Pi,\alpha)\theta_l}-1\right)f_l(\Pi,\alpha)\theta_l d\Pi+p_l(\theta_h-\theta_l)\\
\geq {}& p_l(\theta_h-\theta_l)
\end{aligned}$$

The equality satisfies when $\alpha=\overline{\alpha}$.

Now, we consider the case when $t\leq\pi^*$. Since $f_l(\Pi,\alpha)<f_l(t,\alpha)=f_l(\pi,\overline{\alpha})=1$ for all $\Pi>t$, we have

$$\begin{aligned}
\widetilde{U}_h^m(b|\pi,\underline{\alpha},\alpha) &= p_l\theta_h\int_{\pi^*}^1\left(1-\frac{f_l(\Pi,\alpha)\theta_l}{f_h(\Pi,\alpha)\theta_h}f_l(\Pi,\alpha)\right)f_h(\Pi,\alpha)d\Pi\\
&> p_l\theta_h\int_{\pi^*}^1\left(1-\frac{f_l(\Pi,\alpha)\theta_l}{f_h(\Pi,\alpha)\theta_h}\right)f_h(\Pi,\alpha)d\Pi\\
&= p_l(\theta_h-\theta_l).
\end{aligned}$$

for $\alpha<\overline{\alpha}$. Therefore, we have proved the second part of the lemma.

Finally, $U_i(\underline{\alpha},\overline{\alpha})=V_i(\underline{\alpha},\underline{\alpha})$ follows directly from $AMU_i(\alpha_i,\overline{\alpha})=0$ from Lemma 3, because $V_i(\underline{\alpha},\underline{\alpha})=V_i(\overline{\alpha},\overline{\alpha})=U_i(\underline{\alpha},\overline{\alpha})+\int_{\underline{\alpha}}^{\overline{\alpha}}AMU_i(t,\overline{\alpha})dt$. □

Proof of Proposition 2

The equilibrium expected payoff of player i can be written as

$$V_i(\alpha,\alpha)=U_i(\underline{\alpha},\alpha)+\int_{\underline{\alpha}}^{\alpha} AMU_i(t,\alpha)dt.$$

According to Lemma 3 and 4, when $\alpha\in(\underline{\alpha},\widehat{\alpha}]$, $U_i(\underline{\alpha},\alpha)>V_i(\underline{\alpha},\underline{\alpha})$ and $AMU_i(\alpha_i,\alpha)>0$, thus, $V_i(\alpha,\alpha)>V_i(\underline{\alpha},\underline{\alpha})$. Alternatively, when $\alpha\in(\widehat{\alpha},\overline{\alpha}]$, $U_i(\underline{\alpha},\alpha)\geq V_i(\underline{\alpha},\underline{\alpha})$ and $AMU_i(\alpha_i,\alpha)\geq 0$, thus, $V_i(\alpha,\alpha)\geq V_i(\underline{\alpha},\underline{\alpha})$. Finally, $V_i(\underline{\alpha},\underline{\alpha})=V_i(\overline{\alpha},\overline{\alpha})$ because $U_i(\underline{\alpha},\overline{\alpha})=V_i(\underline{\alpha},\underline{\alpha})$ and $AMR_i(\alpha_i,\overline{\alpha})=0$. □

Proof of Proposition 3

Given Lemma 3, it must be true that $MU_i(\underline{\alpha})>0$, $MU_i(\alpha)\geq 0$ for $\alpha\in(\underline{\alpha},\overline{\alpha})$, and $MU_i(\overline{\alpha})=0$. Hence, any increasing convex cost functions with $MC(\underline{\alpha})=0$ crosses with $MU_i(\alpha)$ from below for at least once. As long as the cost function is convex enough, the two only cross once. In that case, there must exist a α^* such that $MU_i(\alpha^*)=MC(\alpha^*)$.

The condition $MU_i(\alpha^*)=MC(\alpha^*)$ suggests the following:

$$V_i(\alpha^*,\alpha^*)-C(\alpha^*)\geq U_i(\underline{\alpha},\alpha^*)-C(\underline{\alpha})\geq V(\underline{\alpha},\underline{\alpha})=V_i(\overline{\alpha},\overline{\alpha})$$

Therefore, player i's expected payoff in the entire game is higher than when both players choosing $\underline{\alpha}$, i.e., not spying on each other. Furthermore, by $V(\underline{\alpha},\underline{\alpha})=V(\overline{\alpha},\overline{\alpha})$, player i's expected payoff in the game is also higher than when both players receive a perfect signal about the opponent for free. □

Proof of Lemma 5

We start by proving part 1. Suppose in a pure strategy equilibrium with efficient allocation, we have $b_{ih}(\pi_1,\alpha_i,\alpha_j)<b_{ih}(\pi_2,\alpha_i,\alpha_j)$ for some $\pi_1>\pi_2$. Then type (θ_h,π_1) of player i must find the cost of increasing her effort from $b_{ih}(\pi_1,\alpha_i,\alpha_j)$ to $b_{ih}(\pi_2,\alpha_i,\alpha_j)$ dominates the gain from such an increase, formally:

$$b_{ih}(\pi_2,\alpha_i,\alpha_j)-b_{ih}(\pi_1,\alpha_i,\alpha_j)\geq\mu(\pi_1,\alpha_i)\Pr\{b_{ih}(\pi_1,\alpha_i,\alpha_j)\leq b_{jh}<b_{ih}(\pi_2,\alpha_i,\alpha_j)\}\theta_h.$$

where b_{jh} is player j's effort. However, type (θ_h,π_2) would require her gain outweighs her cost of such increase of effort:

$$b_{ih}(\pi_2,\alpha_i,\alpha_j)-b_{ih}(\pi_1,\alpha_i,\alpha_j)\leq\mu(\pi_2,\alpha_i)\Pr\{b_{ih}(\pi_1,\alpha_i,\alpha_j)\leq b_{jh}<b_{ih}(\pi_2,\alpha_i,\alpha_j)\}\theta_h.$$

Combining the two condition, we have $\mu(\pi_2,\alpha_i)\geq\mu(\pi_1,\alpha_i)$ which contradicts to $\pi_1>\pi_2$, due to Assumption 1. Similar arguments can prove that $b_{il}(\pi_1,\alpha_i,\alpha_j)\leq b_{il}(\pi_2,\alpha_i,\alpha_j)$ for any $\pi_1>\pi_2$.

To prove continuity of the equilibrium effort in part 2, suppose there exists a discontinuous point on player i's effort $b_{ih}(\pi,\alpha_i,\alpha_j)$, $\hat{\pi}\in(0,1)$, such that $b_{ih}(\hat{\pi},\alpha_i,\alpha_j)<b_{ih}(\hat{\pi}+\epsilon,\alpha_i,\alpha_j)$ for an arbitrarily small ϵ. Then type $(\theta_h,b_{jh}^{-1}(b_{ih}(\hat{\pi}+\epsilon,\alpha_i,\alpha_j),\alpha_i,\alpha_j))$ of player j will find it profitable to deviate to some $\hat{b}\in(b_{ih}(\hat{\pi},\alpha_i,\alpha_j),b_{ih}(\hat{\pi}+\epsilon,\alpha_i,\alpha_j))$, where $b_{jh}^{-1}(\cdot,\alpha_j,\alpha_i)$ is the inverse of player j's effort. Similarly, suppose there exists a discontinuous point on $b_{il}(\pi,\alpha_i,\alpha_j)$, $\tilde{\pi}\in(0,1)$, such that $b_{il}(\tilde{\pi},\alpha_i,\alpha_j)>b_{il}(\tilde{\pi}+\epsilon,\alpha_i,\alpha_j)$ for arbitrarily small ϵ. Then type $(\theta_l,b_{jl}^{-1}(b_{il}(\tilde{\pi},\alpha_i,\alpha_j),\alpha_j,\alpha_i))$ of player j will find it profitable to deviate to some $\tilde{b}\in(b_{il}(\tilde{\pi}+\epsilon,\alpha_i,\alpha_j),b_{il}(\tilde{\pi},\alpha_i,\alpha_j))$, where $b_{jl}^{-1}(\cdot,\alpha_j,\alpha_i)$ is the inverse of player j's effort.

To prove that there is no atom on any player's effort, suppose there exists p and q such that $1>q>p>0$ and that $b_{ih}(x)=b$ where $x\in[p,q]$ and b is a constant. Then by continuity there must be a type $(\theta_h,b_{jh}^{-1}(b-\epsilon,\alpha_j,\alpha_i),\alpha_j,\alpha_i)$ of player j who chooses $b-\epsilon$, and he will find it profitable to deviate to choose $b+\epsilon$, as the gain of such deviation will be $\mu(b_{jh}^{-1}(b-\epsilon),\alpha_j)\int_p^q f_h(\Pi,\alpha_i)d\Pi>0$ and the cost is negligible when ϵ is arbitrarily small. A similar argument can show that there is no atom on $b_{il}(\pi, \alpha_i, \alpha_j)$.

For part 3, given that part 1 is true, type $(\theta_h, 1)$ of player i chooses the highest effort among all types, whereas type $(\theta_l, 1)$ chooses the lowest effort among all types, in an allocative efficient equilibrium. Thus, it must be true that $b_{il}(1, \alpha_i, \alpha_j) = b_{jl}(1, \alpha_j, \alpha_i) = 0$ as these are the lower bound of equilibrium support. They must be the same and cannot be positive. It must also be true that $b_{ih}(1, \alpha_i, \alpha_j) = b_{jh}(1, \alpha_j, \alpha_i)$ as these are the highest effort exerted by players, and in any equilibrium the equality is satisfied.

Finally, by part 1, type $(\theta_h, 0)$ of player i chooses the lowest effort among all types with valuation θ_h, whereas type $(\theta_l, 0)$ chooses the highest among all types with valuation θ_l. Suppose $b_{ih}(0, \alpha_i, \alpha_j) > b_{il}(0, \alpha_i, \alpha_j)$, then it implies that there is a gap in the equilibrium support of player i's effort. This cannot be part of any equilibrium as then player j would not choose any effort in $[b_{il}(0, \alpha_i, \alpha_j), b_{ih}(0, \alpha_i, \alpha_j)]$, which contradicts the optimality of $b_{ih}(0, \alpha_i, \alpha_j)$, as then player i would want to deviate to a $b \in (b_{il}(0, \alpha_i, \alpha_j), b_{ih}(0, \alpha_i, \alpha_j))$. Suppose $b_{ih}(0, \alpha_i, \alpha_j) < b_{il}(0, \alpha_i, \alpha_j)$, then in any equilibrium with efficient allocation, it must be true that $b_{jl}(0, \alpha_j, \alpha_i) \leq b_{ih}(0, \alpha_i, \alpha_j) < b_{il}(0, \alpha_i, \alpha_j) \leq b_{jh}(0, \alpha_j, \alpha_i)$. But then this implies there is a gap in the equilibrium support of player j's effort, which we have shown above to be impossible in any equilibrium with efficient allocation. Therefore, in any equilibrium with efficient allocation, it must be true that $b_{il}(0, \alpha_i, \alpha_j) = b_{ih}(0, \alpha_i, \alpha_j)$ for $i = 1, 2$. Without loss of generality, suppose $b_{il}(0, \alpha_i, \alpha_j) = b_{ih}(0, \alpha_i, \alpha_j) > b_{jl}(0, \alpha_j, \alpha_i) = b_{jh}(0, \alpha_j, \alpha_i)$, then this contradicts efficient allocation as $b_{il}(0, \alpha_i, \alpha_j) > b_{jh}(0, \alpha_j, \alpha_i)$. Thus, in any pure strategy equilibrium with efficient allocation, it must be true that $b_{il}(0, \alpha_i, \alpha_j) = b_{ih}(0, \alpha_i, \alpha_j) = b_{jl}(0, \alpha_j, \alpha_i) = b_{jh}(0, \alpha_j, \alpha_i)$. □

Proof of Proposition 4

There are two steps to take to prove the proposition. First, we show that the equilibrium strategies of each valuation type given in the proposition are indeed the optimal strategy in their equilibrium support. Second, we show that each type do not want to deviate to any effort level outside of their equilibrium support. Note that in this proof we only have low pure support and high pure support, since we focus on allocative efficient equilibrium.

Given that player j chooses his strategy according to the proposition, suppose type (θ_l, π) of player i chooses an alternative effort level $b_{il}(\zeta, \alpha_i, \alpha_j)$, then her expected payoff is

$$\widetilde{U}^l_{il}(b_{il}(\zeta, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) \quad = \quad \theta_l \int_\zeta^1 [\mu(\Pi, \alpha_i) - \mu(\pi, \alpha_i)] dF_l(\Pi, \alpha_j)$$

Thus,

$$\widetilde{U}^l_{il}(b_{il}(\pi, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) - \widetilde{U}^l_{il}(b_{il}(\zeta, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) = \theta_l \int_\pi^\zeta [\mu(\Pi, \alpha_i) - \mu(\pi, \alpha_i)] dF_l(\Pi, \alpha_j) \geq 0$$

regardless of whether $\pi \geq \zeta$ or $\pi < \zeta$.

Suppose type (θ_h, π) of player i chooses an alternative effort level $b_{ih}(\xi, \alpha_i, \alpha_j)$, then her expected payoff is

$$\begin{aligned} \widetilde{U}^h_{ih}(b_{ih}(\xi, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) \quad &= \quad \theta_h \left[(1 - \mu(\pi, \alpha_i)) + \mu(\pi, \alpha_i) \int_0^\xi f_h(\Pi, \alpha_j) d\Pi\right] \\ &\quad - \theta_h \int_0^t \mu\left(\Pi, \alpha_i\right) dF_h(\Pi, \alpha_j) - \theta_l \int_0^1 \left[1 - \mu\left(\Pi, \alpha_i\right)\right] dF_l(\Pi, \alpha_j) \end{aligned}$$

Again, compare this payoff to the equilibrium payoff:

$$\begin{aligned} & \widetilde{U}^h_{ih}(b_{ih}(\pi, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) - \widetilde{U}^h_{ih}(b_{ih}(\xi, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) \\ = \quad & \theta_h \int_\xi^\pi \left[\mu(\pi, \alpha_i) - \mu\left(\Pi, \alpha_i\right)\right] dF_h(\Pi, \alpha_j) \geq 0 \end{aligned}$$

regardless of $\pi \geq \xi$ or $\pi < \xi$. Thus, the strategy given in the proposition is indeed optimal for players if they choose efforts in the equilibrium support.

Now we turn to the case when each valuation type deviates by choosing an effort in the other valuation type's support, e.g., the high valuation type chooses an effort in the support of the low valuation type's support. When type (θ_h, π) of player i deviates to an effort β in the low pure support of player j, that is $\beta \in [0, b_{jl}(0, \alpha_j, \alpha_i)]$, then the expected payoff given the opponent playing equilibrium strategy $b_{jl}(\pi, \alpha_j, \alpha_i)$ is:

$$\widetilde{U}^l_{ih}(\beta|\pi, \alpha_i, \alpha_j) = \theta_h[1 - \mu(\pi, \alpha_i)] \int_{b_{jl}^{-1}(\beta, \alpha_j, \alpha_i)}^{1} f_h(\Pi, \alpha_j) d\Pi - \beta.$$

Among all the possible deviating efforts player i would prefer to deviate to the effort that maximizes the deviation expected payoff, i.e., $\beta^* = \arg\max_\beta \widetilde{U}^l_{ih}(\beta|\pi, \alpha_i, \alpha_j)$. β^* can be found by the first order condition with respect to β:

$$f_l(\pi, \alpha_i) = \frac{\theta_l}{\theta_h} \frac{f_l(\widehat{t}, \alpha_j)}{f_h(\widehat{t}, \alpha_j)} f_l(\widehat{t}, \alpha_i) \tag{26}$$

where $\widehat{t}$ is given by $b_{jl}(\widehat{t}, \alpha_j, \alpha_i) = \beta^*$, i.e., type $(\theta_l, \widehat{t})$ of player j bids β^* in equilibrium. It is easy to check that both sides of (26) are decreasing functions of their arguments, π and $\widehat{t}$, respectively. Furthermore, Assumption 3 implies $f_l(\pi, \alpha_i) \leq f_l(\widehat{t}, \alpha_i)$ and thus, $\pi \geq \widehat{t}$. Then, there must exist some $\widehat{\pi}$ satisfying

$$f_l(\widehat{\pi}, \alpha_i) \equiv \frac{\theta_l}{\theta_h} \frac{f_l(0, \alpha_j)}{f_h(0, \alpha_j)} f_l(0, \alpha_i)$$

If the equality in Assumption 3 is satisfied at $\pi = 0$, then $\widehat{\pi} = 0$. For $\pi < \widehat{\pi}$, we always have the LHS of the equation (26) strictly larger than the RHS, for all $\widehat{t} \in [0, 1]$. This implies the first order derivative is positive and thus type (θ_h, π) of player i doesn't want to deviate.

For $\pi \geq \widehat{\pi}$, there always exists a unique solution of equation (26) given π. In this case, we need to directly compare the equilibrium payoff with the payoff of choosing β^*. The difference between the equilibrium expected payoff and the optimal deviation payoff:

$$\begin{aligned} & \widetilde{U}^h_{ih}(b_{ih}(\pi, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) - \widetilde{U}^l_{ih}(b_{il}(\widehat{t}, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) \\ = \; & p_h\theta_h \int_0^\pi \left[f_h(\pi, \alpha_i) - f_h(\Pi, \alpha_i)\right] f_h(\Pi, \alpha_j) d\Pi + p_l\theta_h \int_0^{\widehat{t}} \left(f_l(\pi, \alpha_i) - \frac{f_l(\Pi, \alpha_i)\theta_l}{f_h(\Pi, \alpha_j)\theta_h} f_l(\Pi, \alpha_j)\right) d\Pi \end{aligned} \tag{27}$$

is increasing with π, as its first order derivative w.r.t π is positive (since $\pi \geq \widehat{t}$):

$$\begin{aligned} & \frac{\partial}{\partial \pi} \left(\widetilde{U}^h_{ih}(b_{ih}(\pi, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) - \widetilde{U}^l_{ih}(b_{il}(\widehat{t}, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j)\right) \\ = \; & p_h \frac{\partial f_h(\pi, \alpha_i)}{\partial \pi} \theta_h \left(\int_0^\pi f_h(\Pi, \alpha_j) d\Pi - \int_0^{\widehat{t}} f_h(\Pi, \alpha_j) d\Pi\right) \geq 0 \end{aligned}$$

Note that the equality is due to equation (26). Since we also know that

$$\begin{aligned} & \widetilde{U}^h_{ih}(b_{ih}(\widehat{\pi}, \alpha_i, \alpha_j)|\widehat{\pi}, \alpha_i, \alpha_j) - \widetilde{U}^l_{ih}(b_{il}(0, \alpha_i, \alpha_j)|\widehat{\pi}, \alpha_i, \alpha_j) \\ = \; & p_h\theta_h \int_0^{\widehat{\pi}} \left[f_h(\widehat{\pi}, \alpha_i) - f_h(\Pi, \alpha_i)\right] f_h(\Pi, \alpha_j) d\Pi > 0 \end{aligned}$$

the difference (27) is thus, positive. Therefore, type (θ_h, π) of player i does not find it profitable to deviate to any effort in the low pure support.

When a type (θ_l, π) of player i deviates to an effort level β in the high pure support, that is $\beta \in [b_{jl}(0, \alpha_j, \alpha_i), b_{jh}(1, \alpha_j, \alpha_i)]$, the expected payoff given the opponent playing equilibrium strategy $b_{jh}(\pi, \alpha_j, \alpha_i)$ is:

$$\widetilde{U}_{il}^h(\beta|\pi, \alpha_i, \alpha_j) = \theta_l \left[\mu(\pi, \alpha_i) \int_0^{b_{jh}^{-1}(\beta, \alpha_j, \alpha_i)} f_l(\Pi, \alpha_j) d\Pi + (1 - \mu(\pi, \alpha_i)) \right] - \beta.$$

Again, we find the optimal deviation effort $\beta^* = \arg\max_\beta \widetilde{U}_{il}^h(\beta|\pi, \alpha_i, \alpha_j)$ by the first order condition with respect to β:

$$f_h(\pi, \alpha_i) = \frac{\theta_h}{\theta_l} \frac{f_h(\widehat{s}, \alpha_j)}{f_l(\widehat{s}, \alpha_j)} f_h(\widehat{s}, \alpha_i) \tag{28}$$

Thus, $\beta^* = b_{jh}(\widehat{s}, \alpha_j, \alpha_i)$. It is easy to check that both sides of (28) are increasing functions of their arguments, π and $\widehat{s}$, respectively. Furthermore, Assumption 3 implies $f_h(\pi, \alpha_i) \geq f_h(\widehat{s}, \alpha_i)$ and thus, $\pi \geq \widehat{s}$. Then, there must be some $\widehat{\widehat{\pi}}$ satisfies

$$f_h(\widehat{\widehat{\pi}}, \alpha_i) = \frac{\theta_h}{\theta_l} \frac{f_h(0, \alpha_j)}{f_l(0, \alpha_j)} f_h(0, \alpha_i)$$

If the equality in condition (3) is satisfied at $\pi = 0$, then we must have $\widehat{\widehat{\pi}} = 0$. For $\pi < \widehat{\widehat{\pi}}$, we always have the LHS of the equation (28) strictly less than the RHS, for all $\widehat{s} \in [0, 1]$. This implies the first order derivative is negative and thus type (θ_l, π) doesn't want to deviate.

For $\pi \geq \widehat{\widehat{\pi}}$, there always exists a unique $\widehat{s}$ satisfying equation (28) given π. In this case, we need to compare the equilibrium payoff with the payoff of choosing β^*. The difference between the two

$$\begin{aligned}
& \widetilde{U}_{il}^l(b_{il}(\pi, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) - \widetilde{U}_{il}^h(b_{ih}(\widehat{s}, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) \\
= \ & p_l \theta_l \int_0^\pi [f_l(\Pi, \alpha_i) - f_l(\pi, \alpha_i)] f_l(\Pi, \alpha_j) d\Pi \\
+ \ & p_h \int_0^{\widehat{s}} \left[\frac{f_h(\Pi, \alpha_j)\theta_h}{f_l(\Pi, \alpha_j)\theta_l} f_h(\Pi, \alpha_i) - f_h(\pi, \alpha_i) \right] f_l(\Pi, \alpha_j) \theta_l d\Pi
\end{aligned}$$

is positive because its first order derivative w.r.t π is positive:

$$\begin{aligned}
& \frac{\partial}{\partial \pi} \left(\widetilde{U}_{il}^l(b_{il}(\pi, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) - \widetilde{U}_{il}^h(b_{ih}(\widehat{s}, \alpha_i, \alpha_j)|\pi, \alpha_i, \alpha_j) \right) \\
= \ & p_h \theta_l f_h'(\pi, \alpha_i) \left(\int_0^\pi f_l(\Pi, \alpha_j) d\Pi - \int_0^{\widehat{s}} f_l(\Pi, \alpha_j) d\Pi \right) \geq 0
\end{aligned}$$

and

$$\begin{aligned}
& \widetilde{U}_{il}^l(b_{il}(\widehat{\widehat{\pi}}, \alpha_i, \alpha_j)|\widehat{\widehat{\pi}}, \alpha_i, \alpha_j) - \widetilde{U}_{il}^h(b_{ih}(0, \alpha_i, \alpha_j)|\widehat{\widehat{\pi}}, \alpha_i, \alpha_j) \\
= \ & p_l \theta_l \int_0^{\widehat{\widehat{\pi}}} \left[f_l(\widehat{\widehat{\pi}}, \alpha_i) - f_l(\pi, \alpha_i) \right] f_l(\Pi, \alpha_j) d\Pi > 0.
\end{aligned}$$

Thus, there is no profitable deviation for any type of player i. □

Proof of Proposition 5

We start by a new definition in analogous to $U_i(\alpha_i, \alpha_j)$ in the main text.

Definition 5. *Let $U_i(\eta, \alpha_i, \alpha_j)$ be player i's maximum expected payoff in the contest stage by choosing η when player j chooses α_j and (wrongly) believes that player i has chosen α_i.*

Furthermore, let $AMU_i(\eta, \alpha_i, \alpha_j) = \frac{\partial U_i(\eta,\alpha_i,\alpha_j)}{\partial \eta}$ *be the corresponding marginal expected payoff.*

The above definition is different from Definition 4 in player j's belief. In the current asymmetric setting, player j believes that the profile of accuracies is (α_i, α_j) but in fact, it is (η, α_j). Thus, j plays according to the asymmetric equilibrium as given in Proposition 4.

First we show the following lemmas:

Lemma 10. $AMU_i(\eta, \alpha_i, \alpha_j) > 0$ *for all* $\eta \leq \alpha_i$ *and* $\eta, \alpha_i, \alpha_j \in [\underline{\alpha}, \widehat{\alpha}]$.

Lemma 11. $U_i(\underline{\alpha}, \alpha_i, \alpha_j) > V_i(\underline{\alpha}, \underline{\alpha})$ *for* $\alpha_i, \alpha_j \in (\underline{\alpha}, \widehat{\alpha}]$.

The proofs of the above lemmas follow directly from the proofs of Lemma 3 and 4, respectively and thus, are omitted. The equilibrium expected payoff of player i can thus be rewritten to:

$$V_i(\alpha_i, \alpha_j) = U_i(\underline{\alpha}, \alpha_i, \alpha_j) + \int_{\underline{\alpha}}^{\alpha_i} AMU_i(t, \alpha_i, \alpha_i)dt > V_i(\underline{\alpha}, \underline{\alpha})$$

where $\alpha_i, \alpha_j \in (\underline{\alpha}, \widehat{\alpha}]$. This completes the proof. $\square$

Proof of Lemma 7

The terms inside the bracket of (6), $L(\Pi)$, is monotonically decreasing with Π as

$$\frac{\partial L(\Pi)}{\partial \Pi} = -\frac{\partial f_h(\Pi, \alpha_j)}{\partial \Pi}\left(\frac{\theta_h + \frac{p_h \theta_l}{p_l}}{\theta_h - \theta_l} - \Pi\right) < 0$$

Thus, applying Lemma 9 we can prove that $\frac{\partial V_i(\alpha_i,\alpha_j)}{\partial \alpha_j} < 0$. $\square$

Proof of Lemma 8

Denote by $M_{ih}(\pi, \alpha_i, \alpha_j)$ and $M_{il}(\pi, \alpha_i, \alpha_j)$ player i's equilibrium expected payoff when she is type (θ_h, π) and (θ_l, π), respectively.

$$\begin{aligned} M_{ih}(\pi, \alpha_i, \alpha_j) &= p_h \theta_h \int_0^{\pi} \left[f_h(\pi, \alpha_i) - f_h(\Pi, \alpha_i)\right] dF_h(\Pi, \alpha_j) \\ &\quad - p_l \theta_l \int_0^1 f_l(\Pi, \alpha_i) dF_l(\Pi, \alpha_j) + p_l \theta_h f_l(\pi, \alpha_i) \qquad (29) \\ M_{il}(\pi, \alpha_i, \alpha_j) &= p_l \theta_l \int_{\pi}^1 \left[f_l(\pi, \alpha_i) - f_l(\Pi, \alpha_i)\right] f_l(\Pi, \alpha_j) d\Pi \qquad (30) \end{aligned}$$

Then, the ex ante interim expected payoff for θ_h and θ_l can be found by integrating (29) and (30) over π, respectively:

$$\begin{aligned} M_{ih}(\alpha_i, \alpha_j) &= p_h \theta_h \int_0^1 \int_0^{\pi} \left[f_h(\pi, \alpha_i) - f_h(\Pi, \alpha_i)\right] f_h(\Pi, \alpha_j) d\Pi d\pi \\ &\quad + p_l \int_0^1 \int_0^1 \left[f_l(\pi, \alpha_i)\theta_h - f_l(\Pi, \alpha_i)\theta_l\right] f_l(\Pi, \alpha_j) d\Pi d\pi \\ M_{il}(\alpha_i, \alpha_j) &= p_l \theta_l \int_0^1 \int_{\pi}^1 f_l(\pi, \alpha_j) f_l(\Pi, \alpha_i) d\Pi d\pi - p_l \theta_l \int_0^1 \int_{\pi}^1 f_l(\Pi, \alpha_j) f_l(\Pi, \alpha_i) d\Pi d\pi \end{aligned}$$

And thus, the ex ante expected payoff for player i can be calculated by

$$V_i(\alpha_i, \alpha_j) = p_h M_{ih}(\alpha_i, \alpha_j) + p_l M_{il}(\alpha_i, \alpha_j)$$

Let $\alpha_i = \underline{\alpha}$, i.e., when the opponent shares no information to player i, then player i's ex ante expected payoff is a constant: $V_i(\alpha_i, \alpha_j) = p_h p_l (\theta_h - \theta_l)$. $\square$

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