A Rational Rush Theory of Financing Innovations *

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Abstract

We propose a theory of rational "*Rush*", emphasizing the quantity of rational over-investment in contrast to the theory of irrational price "Bubble". We illustrate an important friction when financing breakthrough innovations: non-excludability and spillover of uncertain knowledge due to imperfect IPR (Intellectual Property Rights, e.g. patent) protection. Facing a limited supply of new projects with uncertain return, investors make decisions about when and how many projects to invest. Investors' preemption motive will distort their incentives for patient learning about project return, thus inducing them to "rush in" to finance uncertain projects massively at a premature stage. A small positive news shock regarding the project return can greatly amplify over-investment and result in large social inefficiency. On the other hand, information externality creates free-rider motive, which can also make under-investment possible. Our empirical finding based on sectoral Venture Capital investment shows that weak IPR protection lead to excessively high investment level and more procyclicality. Broader patent rights should be granted when the uncertainty of innovation is high, although the "Rush" prevention can induce more patent race at the early R&D stage, i.e. Rush-Race shifting. We also discuss the optimal and robust patent design problems for macro and financial stability.

Keywords: rush, patent design, innovation, venture capital, bubble, preemption
JEL Codes: C73, D83, G21, G24, L26, O32, O34.

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1 Introduction

This research makes a linkage between innovation-induced economic booms with the friction of imperfect protection of Intellectual Property Rights (IPR). We also provide some empirical evidence on the relationship between the degree of IPR protection and procyclicality and volatility of Venture Capital investment. Then we use Optimal Patent Design method to correct this inefficiency, and propose a new principle for Patent design under uncertainty.

New financial products have been widely blamed as an important cause of the 2008 financial crisis. Report issued by the FCIC (Financial Crisis Inquiry Commission) claims "there was an explosion in risky subprime lending and securitization ... the GSEs (Government-sponsored enterprise) participated in the expansion of subprime and other risky mortgages, but they followed rather than led Wall Street and other lenders in the rush for fool's gold." As early as 2007, the subprime sector started to experience serious delinquencies, triggering fire sales and a credit crunch eventually. Therefore, a central question is whether, why, and how can innovation lead to over-investment and social welfare loss?

In this paper, we develop a Theory of "Rush" in financing new technologies. *By "Rush", we mean the premature massive investments in uncertain new technologies, which may be eventually proved to be futile, wrong or even harmful.* "Rush" emphasizes a sharp increase in the *quantity* of investments, in contrast with the widely used term "Bubble" which focuses on high asset price. We also illustrate an amplification mechanism through which a small shock can induce a big "Rush". Market structure matters for the scale of amplification and inefficiency.

We argue for a view of "Rational Rush", rather than "Irrational Bubble". In fact, *the market equilibrium price for innovations might be too low, not too high.* Property rights for innovations especially the breakthrough ones are oftentimes not perfectly assigned, and consequently the innovators are usually underpaid by investors. In the market equilibrium, over-investment ofttimes happens in the sense that uncertainty of innovations is undervalued due to investors' tradeoff of seizing a larger share of these underpriced innovations.

Then we point to an important friction: imperfect IPR protection and missing knowledge market especially for breakthrough innovations and General Purpose Technologies (GPT). Innovation is generally a favorable public good, which can broaden the human knowledge base, provide new products to consumers, and introduce new profitable opportunities for entrepreneurs and investors. On the other hand, innovations still bear uncertainties. Therefore the new technology needs to be examined and refined patiently. Some of them will eventually prove to be fool's gold, and should be discarded efficiently. Premature massive adoption might create big social hazards. From the social planner's view, it is optimal to firstly invest in the new products on a small scale, wait and examine the outcomes carefully, and decide whether to continue to finance them on a large scale at a later stage.

Excludability due to perfect patent protection is assumed by Romer-type innovation literature, but here we adopt an opposite assumption: *non-excludability*. Despite well-established IPR laws, there are many innovations that IPR laws cannot be applied to. Abstract ideas and business method are generally not patentable. For example, patent is rarely granted to financial innovations, though first movers of financial innovations usually can catch a larger market share, according to Tufano (1989). Due to *non-excludability*, investors face a tradeoff between the precision of learning and market share grabbing. This is an important reason why they will be inclined to deviate from the socialli efficient investment in learning of a new technology.

In contrast to the assumption of *nonrival* knowledge usage in the endogenous growth literature, there is actually *rivalry* in financing these new projects. New investment opportunities are often supplied in a limited quantity. Therefore, investors have strong incentive to capture a lion's share and preempt others. This will distort the incentives for patient learning and generate inefficiency and unnecessary hazards. There are strong complementarity between investors because they compete for limited resources in a common pool. A small news shock to the expected return of new technology can be greatly amplified. A larger number of investors will exacerbate this amplification mechanism. This mechanism for "Rush-in" bears a comparative analogy to "Bank run" in that depositors withdraw funds from a common pool of deposits. Similar to coordination problem in bank runs, investors rush in to invest in new projects, giving up the benefits of waiting and learning. This reminds us of the "Tragedy of the Commons" in an uncertain and competitive environment. Fish in the "Uncertain open-access pond" might be poisonous, and poisoning can spread too quickly due to coordination failure. We name this type of inefficiency *"The Tragedy of the Uncertain Commons"*.

This research explores the relationship between the uncertainty of innovation and patent design. We use the Optimal Patent Design method to correct the inefficiency of rush. Traditional Patent Design has not take into consideration the uncertainty and premature adoption of innovations. To discourage over-investment and too much entry when uncertainty is still high, broader patent rights should be granted to the inventor. This can mitigate social welfare loss due to over-entry into an uncertain technology. We also present and analyze the Robust Patent Design problem when the prior of the expected return is unknown. Moreover, we embed the rush mechanism in a simple endogenous growth framework, which allows endogenous choices of R&D investment, as well as endogenous choice of technology adoption.

Rush prevention can induce more patent race at the early R&D stage but this $rush - race$ $shifting$ can still improve social welfare in total, because in general rush inefficiency dominates race inefficiency.

This research makes a linkage between innovation and social risk of innovation, which the endogenous growth literature, as well as the current experimentation and preemption game literature have missed. Besides the well-known public good property of innovation, we emphasize innovations' "public bad" potential due to its innate uncertainty. Therefore, this research reveals another benefit that IPR protection have contributed implicitly: by granting Patent, it can give innovators' enough incentives to patiently learn and reduce the potential hazards of innovations to the society.

Information externality from the learning of the innovation also creates free-rider motive for investors. This makes under-investment possible especially when the expected return of an innovation is low ex ante. We derive a sharp threshold of expected return below which under-investment occurs and above which there will be over-investment. Larger number of investors will amplify over-investment and under-investment on both sides.

We provide empirical evidence on several historical "Rush" episodes. Using cross-industry Venture capital investment data, we illustrate a linkage between the degree of sectoral IPR protection and investment volatility. In addition, we point out that a new wave of "*Unicorn Rush*" has just emerged since 2010.

The friction identified and emphasized in this research is an important one: missing knowledge market and the spillover of uncertain knowledge. This has broad policy implications: anti-trust, patent design, monetary policy and social hazards regulation (e.g. FDA, EPA, CFPB). Particular attention should be paid to policies that can alleviate the coordination problem and mitigate the "Tragedy of the Uncertain Commons".

Related Literature

Endogenous Growth and Innovation-induced Social Hazards Romer (1986,1990) starts the literature on endogenous growth. This line of research emphasizes the *Nonrivalry and Excludability* feature of innovation and knowledge. This paper makes opposite assumptions: "*Rivalry*" and "*Non-excludability*" when financing new technologies.

Moreover, the potential hazards of innovation is largely omitted by the literature. Knowledge spillover is generally treated as a spread of good thing. However, the spillover of "innovation" can also have widespread adverse effects. Jones (2014) takes a first step forward to also consider potential risk of innovations. In the other paper by the author, Xie (2015b) endogenizes hazards generation and regulation in a growth framework. This paper emphasizes the effect of market structure on amplifying uncertainty and potential hazards of innovations.

Information Acquisition, Bubbles and Crisis Chari and Kehoe (2003) shows that information is important for financial crisis, and illustrates the herding effect on crisis. Angeletos, Hellwig and Pavan (2006) highlights that endogenous information generated by policy intervention can move global game from unique equilibrium to multiple equilibria. In this paper, the information structure is endogenously determined by the market structure and actions of agents.

Entry, Competition and Inefficiency Mankiw and Winston (1986) discuss the social inefficiency due to over-entry of firms. Hsieh and Moretti (2003) provides empirical evidence on this channel.

Preemption game and Patent race Reinganum (1981), and Fudenberg and Tirole (1985) apply preemption game to market entry and technology adoption. Firm will make a tradeoff between entering earlier with the possibility of acquiring a patent or a significant share of the new market and waiting for a reduced uncertainty and entry cost. Hopenhayn and Squintani (2011) extends this line of research by adding heterogeneous information to each firm.

Experimentation, learning, and Learning-by-doing (LBD) Jovanovic and Lach (1989) shows that the Learning-by-doing mechanism can result in the S-shaped diffusion across firms. Bolton and Harris (1999) extend the two-armed bandit problem to a multi-agents dynamic game, and show the coexistence of a discouraging free-rider effect and a counteracting encourage effect for experimentation.

Pastor and Veronesi (2003, 2006, 2009) argue technology bubble may be efficient, and their analysis is under the assumption of social efficient learning. An essential difference from our research is that they assume the *nonrivalry* of technological innovation and unlimited number of new projects available for investing. With this assumption, there will not be distorted incentives to induce early entry, and competitive learning in equilibrium delivers the same efficient outcome as the social optimum. However in reality, *there is significant rivalry for the limited new investment opportunities.* As revealed by the 2008 Subprime Crisis, investors had competed intensively to enter a new market too early and massively than the socially optimal level before the uncertainty of new technology is sufficiently reduced.

The paper proceeds as follows. Section 2 describes the environment and model setup. Section 3 derives the optimal allocation. Section 4 discusses ownership and market structure for new projects, and how they are related to inefficiency. Section 5 analyzes the equilibria in a decentralized economy. Section 6 analyzes the amplification mechanism of rush, and tries to derive a shadow price corresponding to the observed investment quantity. Section 7 discusses inefficiencies in both cases of over- and under-investment. Section 8 analyzes the optimal and robust patent design problem to correct abovementioned inefficiencies. Section 9 extends the Optimal Patent design problem with endogenous R&D. Section 10 provides some empirical evidence on Rush and patent protection. Section 11 discusses some alternative policies. Finally Section 12 concludes the paper and points to some future extensions.

2 The Model

The model has three periods $t = 0, 1, 2$, as shown by Figure (1).

2.1 Types of Projects

There are two types of projects: (i) the new and illiquid projects and (ii) liquidity.

(i) New and illiquid projects Each project needs one unit of investment. These new projects can be started at either $t = 0$ or $t = 1$. No matter whether a project is financed at $t = 0$ or 1, it will mature at $t = 2$. After maturity, each project will produce R units of output. Return R will be the same for all projects. R is unknown, follows a normal distribution, with a prior $R \sim N(R_0, \alpha_0^{-1})$, shared by all players.

Assumption 1 *Project invested at* $t = 0$ *cannot be liquidated at date 1.*

There will be a total number N of new projects. N is deterministic and known to all players.

(ii) Liquidity liquidity asset, which can return 1 for 1 unit of investment at any time when needed. It is equal to holding cash.

Ownership of New Projects

Because of knowledge spillover, there is no perfect ownership defined for the new projects. They look like common-pool resource to the economic agents. In Section 4, we will discuss in details the nonexcludability problem of innovation and the "Tragedy of the Uncertain Commons" as a result. Moreover, under imperfect property rights for innovations, we need to propose an allocation mechanism for these new projects (discussion in Section 2.3).

2.2 Investors

There are M symmetric investors in total, indexed by $i = 1, 2, ...M$. Each investor i is indifferent to consuming at $t = 1$ or 2, and has a linear preference as follows,

$$u(c_{i,1} + c_{i,2}) = c_{i,1} + c_{i,2} \tag{1}$$

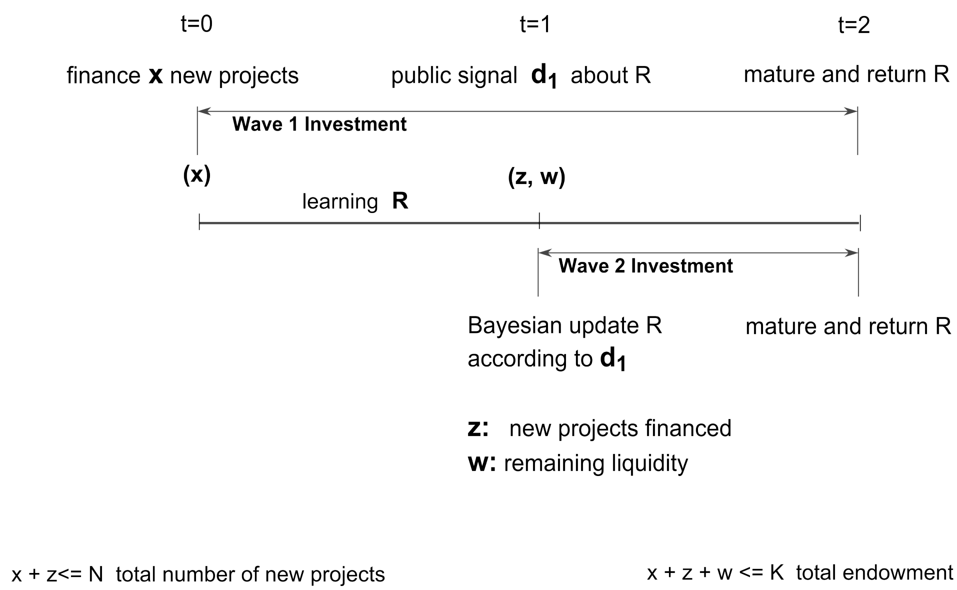


Figure 1: Timing

Each investor has an endowment of k units of capital at date 0. Investors can choose to invest in new projects at two dates: either $t = 0$ or $t = 1$,or both dates. Investors essentially select a portfolio composed of illiquid projects and liquidity.

2.3 Allocation mechanism

At $t = 0$, all investors make a simultaneous move. There is no pre-assigned property rights for all new projects at $t = 0$ (but later we will discuss Patent as a special allocation mechanism which grants certain degree of property rights to the innovator). Each investor i can pose a request for x_i new projects. Denote the aggregate requests of all Investors as X, so $X = \sum_{i=1}^{M} x_i$.The total requests from all investors can be larger than the total number of new projects N. This is also true for $t = 1$. So we need a rationing and allocation mechanism, defined as the following,

Definition 1 *An allocation mechanism allocates new projects to each investor i at $t = 0$ and 1, according to each investor's individual requests as well as all investors' aggregate requests of projects. The allocation follows two related allocation functions $h^0(\cdot)$ for $t = 0$ and $h^1(\cdot)$ for $t = 1$ respectively.*

2.3.1 Allocation at $t = 0$

We use x_i^a to denote the number of projects that investor i will actually receive from an allocation mechanism at $t = 0$. Then we have the following definition for the vector-valued function $h^0(\cdot)$ at $t = 0$,

Definition 2 *An allocation function $h^0(\cdot)$, takes the investors' request vector $\langle x_1, x_2, ..., x_M \rangle$ as input and return a unique allocation vector $(x_i^a, x_2^a, ..., x_M^a)$ as the function value:*

$$\langle x_1^a, x_2^a, ..., x_M^a \rangle = h^0 (\langle x_1, x_2, ..., x_M \rangle) \tag{2}$$

in addition, we use $h_i^0(\cdot)$ to denote the i^{th} element of $\langle x_1^a, x_2^a, ..., x_M^a \rangle$, i.e. number of projects allocated to the i^{th} investor. And $h^0(\cdot)$ must satisfy the following constraint (3),

$$\textstyle\sum_{i=1}^{M} h_i^0 (\langle x_1, x_2, ..., x_M \rangle) \leq N \tag{3}$$

We also impose the following assumption for all allocation mechanisms,

Assumption 2 *As a commitment to participating in the allocation mechanism, investor is required to finance all its allocated projects at $t = 0$ and 1.*

Then there will be $N_1 \geq 0$ new projects left for date $t = 1$,

$$N_1 = N - \textstyle\sum_{i=1}^{M} h_i^0 (\langle x_1, x_2, ..., x_M \rangle) \tag{4}$$

2.3.2 Allocation at $t = 1$

At $t = 1$,each investor i can pose a request for z_i new projects. Denote the aggregate requests of all investors as Z, so $Z = \sum_{i=1}^{M} z_i$.The remaining N_1 projects will be allocated according to a vector-valued function $h^1(\cdot)$ at $t = 0$,

Definition 3 *An allocation function $h^1(\cdot)$, takes the investors' request vector $\langle z_1, z_2, ..., z_M \rangle$ as input and return a unique allocation vector $\langle z_i^a, z_2^a, ..., z_M^a \rangle$ as the function value:*

$$\langle z_1^a, z_2^a, ..., z_M^a \rangle = h^1 (\langle z_1, z_2, ..., z_M \rangle) \tag{5}$$

subject to the following constraint (4) and (6),

$$\textstyle\sum_{i=1}^{M} h_i^1 (\langle z_1, z_2, ..., z_M \rangle) \leq N_1 \tag{6}$$

With $h^0(\cdot)$ and $h^1(\cdot)$, we can formally describe various allocation mechanisms.

2.4 Investors' Strategy Space and Strategy Profile

Each investor i choose a $3-tuple$ (x_i, z_i, w_i). x_i denotes her requested new projects at $t=0$; z_i is additional requests of new projects at $t=1$; w_j is the investments in liquidity. The strategy profile of M investors is thus $\langle (x_1, z_1, w_1), (x_2, z_2, w_2), ..., (x_M, z_M, w_M) \rangle$.

2.5 Information Structure

At $t=0$, all players have the same prior for the return of new projects: $R \sim N(R_0, \alpha_0^{-1})$.

Assume at $t=0$, there is a total number of investments x. At $t=1$, consumers and Investors receive an aggregate public signal d_1 about R,

$$d_1 = R + \epsilon_1 \tag{7}$$

d_1 is a realization of the return of new projects financed at $t=0$.The signal d_1 is not perfect because there is a component noise ϵ_1.

The endogenous noise ϵ_1 follows,

$$\epsilon_1 \sim N\left(0, \frac{1}{x^\theta}\right) \tag{8}$$

where x^θ is the precision of the noice ϵ_1. Investing in more projects can reduce the variance of ϵ_1.

Due to (7), we have $d_1 | R \sim N\left(R, \left(x^\theta\right)^{-1}\right)$. R and ϵ_1 are two independent random variables with normal distributions. The sum of them, i.e. d_1 also follows a normal distribution as follows,

$$d_1 \sim N\left(R_0, \left(x^\theta\right)^{-1} + (\alpha_0)^{-1}\right) \tag{9}$$

3 Optimal Allocation

In this section, we start from the description of the social planner's problem and then derive the optimal allocation.

Social planner will allocate the total endowment of capital K to a $3-tuple$ (x, z, w). x is the capital invested in risky new projects at $t=0$. Because each illiquid project requires one unit of investment, this means that number x risky projects are invested at $t=0$. A the same time, the choice x also determines the signal d_1 for $t=1$, whose distribution follows $d_1 \sim N\left(R_0, \left(x^\theta\right)^{-1} + (\alpha_0)^{-1}\right)$.

At $t = 1$, an additional decision is to choose z : the number of additional risky projects to invest. The total number of risky projects available for investment is N, so we have,

$$x + z \leq N \tag{10}$$

This choice of z depends on the new information received at $t = 1$, i.e. signal d_1 of projects' return invested last period. There will be Bayesian Updating according to signal d_1, and the posterior belief follows (11) and (12),

$$E_1\left[R|d_1\right]=\frac{\alpha_0 R_0 + d_1 x^{\theta}}{\alpha_0 + x^{\theta}} \tag{11}$$

$$V_1\left[R|d_1\right]=\left(\alpha_0 + x^{\theta}\right)^{-1} \tag{12}$$

If the signal of project return is high enough, more risky projects will be invested. $z = 0$ if the signal is below some threshold. The remaining endowment will be the leftover liquidity w. Therefore, we have a binding resource constraint (13) ,

$$x + z + w = K \tag{13}$$

x plays the role of learning about risky project's return. However, if the return of risky project turns out to be very low, it will impose a cost due to the investment on bad projects at the beginning. x is chosen to make an optimal tradeoff between information acquisition and potential welfare loss due to investments in uncertain new projects.

3.1 Social Planner's Problem

The social planner's problem is recursively described by (14) and (15),

$$V_0 = \max_x \mathbb{E}_0\left[Rx + V_1\right] \tag{14}$$

where V_1 is the continuation value at $t = 1$, described by (15)

$$V_1 = \max_{z,w} \mathbb{E}_1\left[Rz + w|d_1\right] \tag{15}$$

$$s.t. \quad (9)\,,(10)\,,(11),(12),(13)$$

At $t = 0$, the expected return of risky projects is $E_0[R] = R_0$, which is just the prior of the return at the beginning. Whereas *at* $t = 1$, the information set will include a new signal d_1. Thus the choice of z at $t = 1$ depends on this newly generated signal d_1, which is affected by the choice x last period, according to (9).

Additional new projects will be financed if and only if $E_1 \left[R | \, d_1 \right] > 1$. Because the total endowment is significantly larger than the number of all risky projects, there can be some leftover liquidity w, even if all the remaining risky projects have been financed at $t = 1$.

The grand optimizatoin problem (14) need to be solved by backward induction.

3.2 Allocation Problem at $t = 1$

At $t = 1$, the optimization problem is (15).The public signal d_1 has been generated. The belief about return R is updated according to signal d_1.

The optimal choice for (z^*, w^*) is given by (16) ,

$$\begin{cases} z^* = N - x & w^* = K - N & if & E_1 \left[R | d_1 \right] \geq 1 \\ z^* = 0 & w^* = K - x & if & E_1 \left[R | d_1 \right] < 1 \end{cases} \tag{16}$$

The threshold value $\bar{d}_1$ of choice is when $E_1 \left[R | d_1 \right] = \frac{\alpha R_0 + d_1 x^\theta}{\alpha_0 + x^\theta} = 1$,and is given by (17),

$$\bar{d}_1 = \frac{\alpha_0 + x^\theta - \alpha_0 R_0}{x^\theta} \tag{17}$$

Given the optimal choice (16) at $t = 1$, maximized total utility derived from $t = 1$ investment now becomes,

$$\begin{cases} \frac{\alpha R_0 + d_1 x^\theta}{\alpha_0 + x^\theta} N + (K - N) & if d_1 \geq \bar{d}_1 \\ \frac{\alpha R_0 + d_1 x^\theta}{\alpha_0 + x^\theta} x + (K - x) & if d_1 < \bar{d}_1 \end{cases} \tag{18}$$

Then the planner can take the result (18) as given, and make decisions at $t = 0$.

3.3 Allocation Problem at $t = 0$

Back to the very beginning, the decision problem is to choose the optimal signal d_1 for the next period.

At $t = 0$, we only know $d_1 = R + \epsilon_1$.Because R and ϵ_1 are two independent, we have (9) .Therefore, we know signal d_1's probability distribution function $f(d_1)$ follows (19) ,

$$f(s) = \frac{1}{\sqrt{(x^\theta)^{-1} + (\alpha_0)^{-1}}} \phi \left(\frac{s - R_0}{\sqrt{(x^\theta)^{-1} + (\alpha_0)^{-1}}} \right) \tag{19}$$

Notice here the signal's probability distribution function $f(d_1)$ is a function of x. Then we can rewrite (14) as the optimization problem (20),

$$\max_{\{x\}} \left\{ \begin{array}{c} \int_{-\infty}^{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}} \left(\frac{\alpha_0 R_0+s\cdot x^\theta}{\alpha_0+x^\theta}x+K-x\right) f(s)ds \\ +\int_{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}}^{\infty} \left(\frac{\alpha_0 R_0+s\cdot x^\theta}{\alpha_0+x^\theta}N+K-N\right) f(s)ds \end{array} \right\} \tag{20}$$

$$s.t.\ (19)$$

Proposition 1 *There is a unique solution to the optimal allocation problem* (20) *.The optimal solution x to* (20) *is given by* (21)*,*

$$\frac{\Phi\left((1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+x^\theta\right)}{x^\theta}}\right)}{\phi\left((1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+x^\theta\right)}{x^\theta}}\right)} = \frac{1}{(1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+x^\theta\right)}{x^\theta}}}\left(\frac{\theta\alpha_0\left(N-x\right)}{2x\left(\alpha_0+x^\theta\right)}-1\right) \tag{21}$$

Proof. The detailed derivation and proof is provided in the appendix. ■

Corollary 1 *x^* is larger than 0.*

Experimentation at the beginning is efficient because the option value for learning is greater than 0. The new projects are essentially a kind of real option and provide an opportunity of financing new projects with potentially higher return. Therefore, investments at this stage can generate more accurate information by trying a small number of new projects. Information acquisition at the first stage can help to make better decisions later.

Corollary 2 *x^* is less than N in the optimal allocation.*

In general, financing all risky projects at $t = 0$ are not socially optimal. This is due to precaution regarding an uncertain new technology.

Corollary 3 *When $R_0 = 1$, $\theta = 1$, we have a unique closed-form solution for x^*,*

$$x^* = \frac{1}{4}\sqrt{\alpha_0\left(8N+9\alpha_0\right)} - \frac{3}{4}\alpha_0$$

and in the limit $\frac{dx^*}{d\alpha_0}$ converges to 0 *when α_0 grows to be large,*

$$\lim_{\alpha_0\to\infty}\frac{dx^*}{d\alpha_0} = 0$$

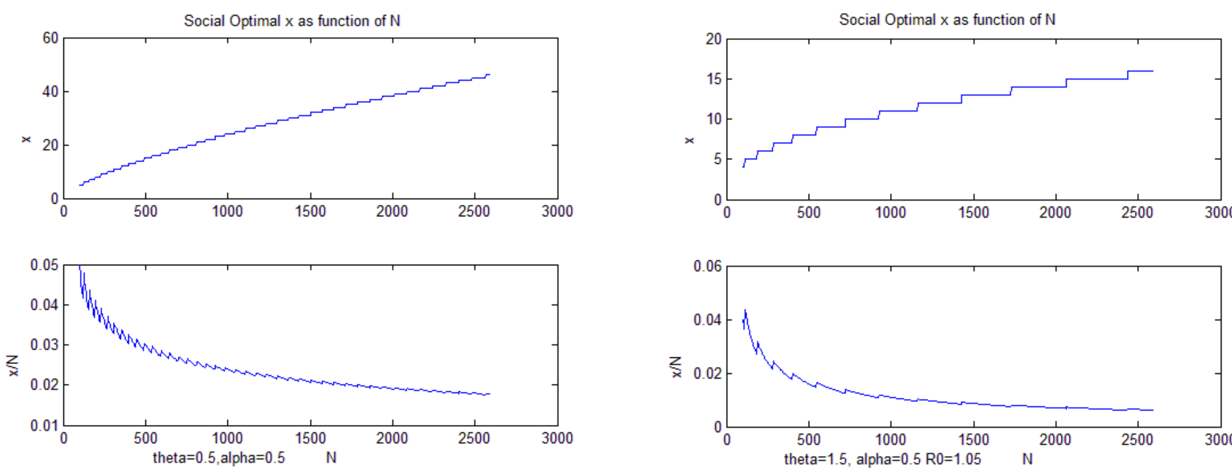
3.4 Properties and discussions

3.4.1 Number of New Projects N

As shown in the Figure 2, the optimal x^* is an increasing function of N, but the fraction x/N is a decreasing function of N. This means that the relative learning cost is declining with respect to the total number of new projects.

Proposition 2 *The social optimal investments x^* at $t = 0$ is an increasing function of N, but the ratio x/N is a decreasing function of N.*

With a larger number of new projects, the social planner tends to be more cautious because the total social value and therefore the stake of learning will increase with N. However, the relative number of projects x/N used for learning purpose decreases with N. On the other hand, this can imply that market equilibrium can impose relatively higher social cost when N grows larger.



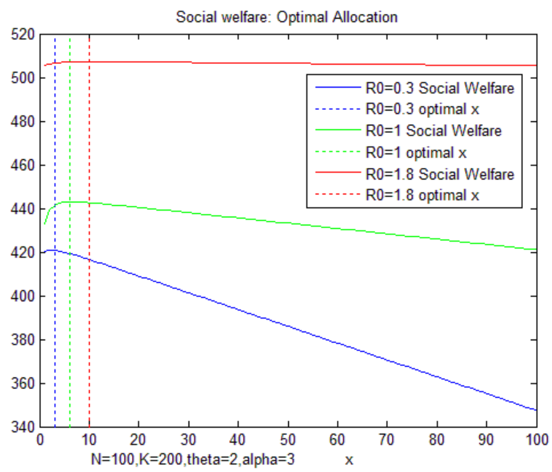
Panel A

Panel B

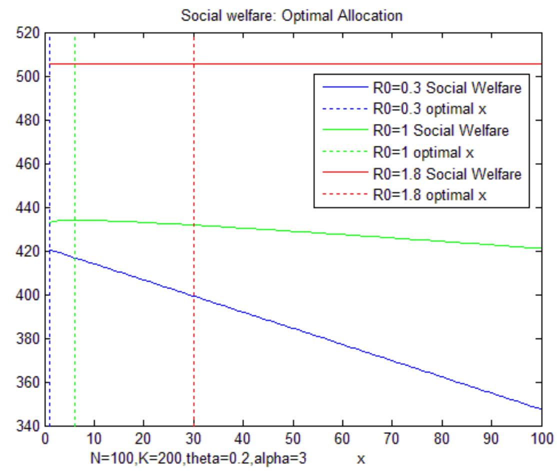
Figure 2. The number of new projects N

3.4.2 Learning efficiency θ

We want to see the effect of learning efficiency on the optimal x^*.We can see from Figure 3, that less efficient learning, as in Panel B will demands more learning when the prior of return is high, but less learning when the prior of return is low. Although the cost of learning rises with a lower θ, it is worth more investment because of a better outlook of return.



Panel A

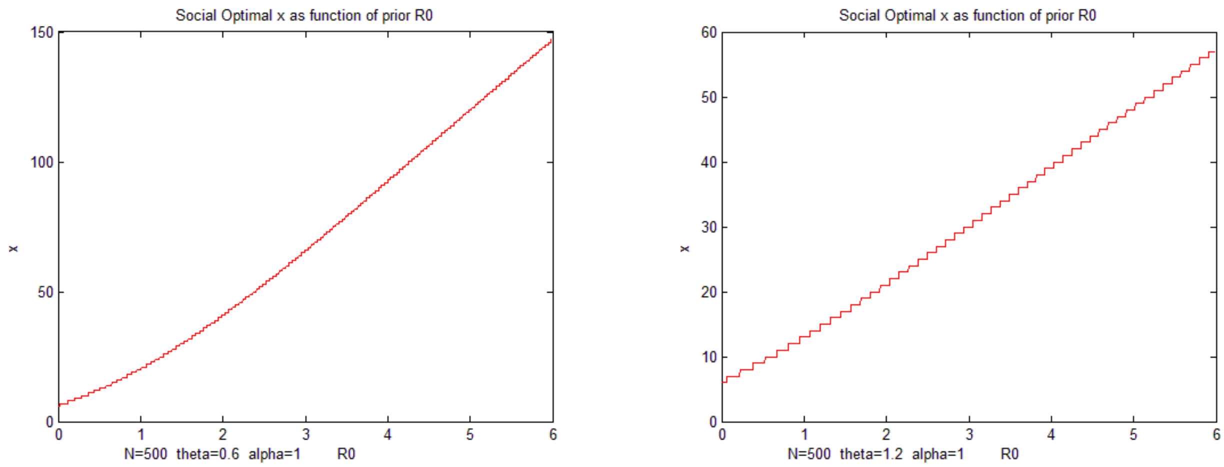


Panel B

Figure 3. Learning efficiency

3.4.3 Prior return R_0

Figure 4 shows how the socially optimal x responds to different priors of R_0. With disparate learning efficiency ($\theta = 0.6$ for the left panel, $\theta = 1.2$ for the right panel), the optimal x is smooth increasing function of R_0. Even the prior R_0 rises to as high as 6 (600% $return$ for investment), the first period investment only increases smoothly, without occurrence of any "Rush". You may now imagine market equilibrium can generate very different result than the social optimum.



Panel A Panel B

Figure 4. Optimal $x$as a function of prior R_0

4 Nonexcludability, Rivalry, and Uncertainty of Innovation: The Tragedy of the Uncertain Commons

In this section, we discuss the ownership and market structure for new technologies. Exclusive ownership of new technology encourages patient learning of new technology which can reduce potential hazards to the society. Conversely, imperfect IPR (Intellectual Property Rights) protection can aggravate the coordination failure of massive premature investments in those new but uncertain technologies.

4.1 Nonexcludability

The classical “Tragedy of the Commons” denotes a situation that individuals tend to “*overgraze*” some common pool of resource because the social cost will be shared by the group whereas an individual can keep the benefit for herself. *Nonexcludability* and the lack of property rights is one important cause of the Tragedy of the Commons.

IPR, e.g. patent, assigns exclusive property rights to the inventors, for the purpose of providing incentive to invest on *R&D*. The endogenous growth literature assumes the full patent rights to innovations. However, in reality, due to the public good nature of innovation, excludability of IPR can be easily violated in various ways. Therefore property rights are more often ill-defined for these “*New Commons*”, e.g. new technologies and new business opportunities. We will discuss several situations of nonexcludability.

4.1.1 Nonpatentability by the Patent Law

There are many types of new ideas that cannot be protected by the current Patent law. For example, "*abstract idea*" and "obvious idea" are not patentable. Most recently, the business method "surge-pricing" of *Uber* is thought to be nonpatentable: "This application is really seeking to claim the basic idea of pricing and service, which is a concept Adam Smith discussed 200 years ago." It is very difficult for *Uber* to discourage other competitors to copy its business method.

In general, financial innovations are also not eligible for patent application. In a recent case, Alice Corp. v. CLS Bank (2014), the U.S. Supreme Court made a final decision that "a computer-implemented, electronic escrow service for facilitating financial transactions covered abstract ideas ineligible for patent protection. The patents were held to be invalid because the claims were drawn to an *abstract idea*, and implementing those claims on a computer was not enough to transform that idea into patentable subject matter." Moreover, after this Supreme Court decision, the U.S. Patent and Trademark office has indeed *stopped* granting business method patents.

4.1.2 Design-arounds

It is also possible to bypass an exisiting patent. By borrowing the idea of a major patent, competitors can patent slightly modified ideas and bypass the existing one. This enroaches the monopoly of existing patent and "steals" the IPR in an implicit way. Kremer (2001) points out the relative easiness to design around vaccines patents.

4.1.3 GPT and Technological Revolutions

The "Commons" problem is especially prominant for the GPT (General Purpose Technology) and technological revolutions. A breakthrough in GPT often injects huge knowledge spillover to the whole economy. A GPT, like the Internet, can spur the invention of a lot of new products. The origional inventor of the GPT can only capture a very small share of all the profitable opportunities built on the general technology. "*New Commons*" will naturally emerge from such technological revolutions. This implies the supply of a large quantity of "free lunches".

4.2 Nonrivalry of ideas v.s. Rivalry of Investment Opportunities

On the other hand, limited profit opportunities for a new technology results in *Rivalry* for financing new technologies. This is contrary to the "*Nonrivalry of knowledge use*" assumption in the endeogenous growth literature. Knowledge or ideas indeed has the nonrival nature, especially for long run growth. Future generations can freely reuse the same idea for infinite times without depreciation. Investment opportunities for new idea or new technology go in the other way: it will disappear very quickly, and its profitability can be completely lost when the idea becomes a pure public knowledge. The opportunities of investing in profitable innovations is in a very limited supply.

4.3 Deep Uncertainty of Innovations

Some innovations have potential hazards, even very lethal ones. Many problems of innovations can be revealed only in the market. Lab experiment and *FDA-like* pre-testing can only detect a limited amount of potential hazards. Therefore, investment in a moderate scale is necessary to generate better and more subtle information for innovations.

In general, idea and knowledge spillover is believed to be beneficial to the society. However, the hasty proliferation of bad ideas or wrong models can be extremely detrimental, because the full spectrum of an innovation's benefits and risks can not be completely understood during a very short period.

Example 1 *The drug "Thalidomide" is a thrilling case. "Shortly after Thalidomide was sold in 1957, in Germany, between 5,000 and 7,000 infants were born with phocomelia (malformation of the limbs). Only 40% of these children survived." Globally, there were more than 10,000 reported malformations due to usage of Thalidomide.*

4.4 The Tragedy of the Uncertain Commons

A property with the nature of nonexcludability and rivalry is called open-access common property. Without perfect IPR protection, the tragedy of the Commons will come up. There does not exist a complete price system to discourage the massive entry into financing certain new technologies. The limited supply of profitable opportunities creates strategic complementarity between investors which incentivizes them to rush into the new market. The fear of losing market share to competitors will dilute the investor's concern for the uncertainty of innovations. Investors will finance new projects too quickly even when there are still a lot of uncertainties remaining. "Overgrazing" these *new* and "*Uncertain Commons*" can result in high social costs ex post. This leads to the *Tragedy of the Uncertain Commons*.

In general, "rush" phenomenon is often a consequence of ill-defined property rights and missing market. With uncertainty, rush will generate larger inefficiencies than under the traditional Tragedy of the Commons.

5 The $M-investor$ Equilibria

In this section, we discuss the market equillibrium of M symmetric investors ($M-investor$ Equilibria). We denote X_{-i} as the total number of new projects requested by other investors at $t=0: X_{-i}=\sum_{j\neq i}^{M}x_j$. The aggregate requests is $X=\sum_{j=1}^{M}x_j$. Denote Z_{-i} as the total number of new projects financed by other investors at $t=1$, i.e. $Z_{-i}=\sum_{j\neq i}^{M}z_j$.

5.1 Investor's Problem with General-form Allocation Mechanism

The optimization problem of investor i is described by (22),

$$V_{i,0}=\max_{x_i}\mathbb{E}_{i,0}\left[R\times x_i+V_{i,1}\right] \tag{22}$$

where $V_{i,1}$ is investor i's continuation value at $t=1$, defined by (23)

$$V_{i,1}=\max_{z_i,w_i}\mathbb{E}_{i,1}\left[R\times z_i+w_i|d_1\right] \tag{23}$$

$$s.t.\quad h_i^0\left(\langle x_1,x_2,...,x_M\rangle\right)+h_i^1\left(\langle z_1,z_2,...,z_M\rangle\right)+w_i\leq k \tag{24}$$

$$x_i,z_i,w_i\geq 0 \tag{25}$$

$$E_1\left[R|d_1\right]=\frac{\alpha_0R_0+d_1\left(\sum_{i=1}^{M}h_i^0\left(\langle x_1,x_2,...,x_M\rangle\right)\right)^{\theta}}{\alpha_0+\left(\sum_{i=1}^{M}h_i^0\left(\langle x_1,x_2,...,x_M\rangle\right)\right)^{\theta}} \tag{26}$$

$$V_1\left[R|d_1\right]=\left(\alpha_0+\left(\sum_{i=1}^{M}h_i^0\left(\langle x_1,x_2,...,x_M\rangle\right)\right)^{\theta}\right)^{-1} \tag{27}$$

$$d_1\sim N(R_0,\left(\sum_{i=1}^{M}h_i^0\left(\langle x_1,x_2,...,x_M\rangle\right)\right)^{-\theta}+(\alpha_0)^{-1}) \tag{28}$$

Investor's problem has been described above with the most general-form allocation mechanism. In the next section, we will introduce the baseline allocation mechanism.

5.2 Baseline Allocation Mechanism: ECPR

We firstly define an allocation mechanism which embodies rivalry and nonexcludability as discussed in the previous section. Under this mechanism, each investor receives her requested projects up to an equal share limit at both dates. This is formally defined by the following **ECPR** Allocation Mechanism (Equal Opportunity for the Common-pool Resource),

Allocation Mechanism 1 (Baseline: ECPR) *Allocation functions $h^0(\cdot)$ and $h^1(\cdot)$ are defined by (4), (29), and (30),*

$$h_i^0\left(\langle x_1, x_2, ..., x_M\rangle\right) = \begin{cases} x_i & if x_i \leq \frac{N}{M} \\ \frac{N}{M} & if x_i > \frac{N}{M} \end{cases} \tag{29}$$

$$h_i^1\left(\langle z_1, z_2, ..., z_M\rangle\right) = \begin{cases} z_i & if z_i \leq \frac{N_1}{M} \\ \frac{N_1}{M} & if z_i > \frac{N_1}{M} \end{cases} \tag{30}$$

Notice that (4), (29), and (30) have automatically satisfied constraints (3) and (6).

Common-pool Resource problem

In the ECPR mechanism, the Common-pool Resource problem is embodied by the inter-period constraint (4). The remaining projects will be reallocated equally at $t = 1$. Other investors' increase in investments at $t = 0$ will reduce the leftover quantity N_1 at $t = 1$. This raises the cost of delaying investments. Therefore, this creates complementarity between investors' x choices at $t = 0$.

5.3 A Model of Venture Capital (VC)

We argue the current model with the ECPR mechanism is consistent with the Venture Capital's (VC) investment pattern. Think each investor as a Venture Capitalist. At $t = 0$, each VC builds their specific product based on a technological breakthrough, and stays in their own niche for the first period. This is consistent with (29). Take the Shared-economy as a recent example of technological breakthrough, Uber and AirBnB have applied the same idea to different areas. But at $t = 1$, investors are allowed to enter each other's niches, as illustrated by (30).

With the ECPR mechanism, constraints in the general-form (24),(25),(26),(27),and (28) can be reduced to the following,

$$x_i + z_i + w_i \leq k \tag{31}$$

$$x_i, z_i, w_i \geq 0 \tag{32}$$

$$x_i \leq \frac{N}{M} \tag{33}$$

$$z_i \leq \frac{N - x_i - X_{-i}}{M} \tag{34}$$

$$E_1\left[R|d_1\right]=\frac{\alpha_0 R_0+d_1\left(x_i+X_{-i}\right)^{\theta}}{\alpha_0+\left(x_i+X_{-i}\right)^{\theta}} \tag{35}$$

$$V_1\left[R|d_1\right]=\left(\alpha_0+\left(x_i+X_{-i}\right)^{\theta}\right)^{-1} \tag{36}$$

$$d_1\sim N(R_0,\left(x_i+X_{-i}\right)^{-\theta}+\left(\alpha_0\right)^{-1}) \tag{37}$$

In comparison with social planner's problem (14), the decentralized problem (22) for each investor i have additional constraints (33), (34), (35), (36) and (37). These constraints will move the optimal allocation to opposite directions.

As a mapping to the allocation functions (4), (29), and (30) in the last subsection, (33) and (34) create complementarity between investors' x choices.

The other three constraints (35), (36) and (37) reflect *Information externality*, i.e. the public good nature of the signal generated by all investments at $t=0$, including other Investors'. Here we have assumed all Investors' investments contribute equally to the generation of the public signal d_1.Only the aggregate number of new projects financed *at* $t=0$, the X_a, matters for the precision of d_1. Thus other Investors' investment at $t=0$ can directly benefit investor i at $t=1$. This information externality encourages Investors to delay investment and free ride on others' effort.

Other Investors' investments can reveal more information, but this will also reduce the stock of new projects available for financing. These two effects work in contradictory directions.

5.4 Subgame Perfect Equilibrium of M-investor game

The Equilibrium concept for the M-investor game is Subgame Perfect Equilibrium *(SPE)*.

Definition 4 *(Subgame Perfect Equilibrium) In an equilibrium of M-investor game, each investor i chooses its optimal vector* (x_i^*, z_i^*, w_i^*) *and will not deviate from it, given all other Investors' optimal strategies* $\left\{\left(x_j^*, z_j^*, w_j^*\right)_{j=1\&j\neq i}^{M}\right\}$*;and this also applies to every proper subgame.*

Similar to the solution method for the social planner's problem, we use backward induction to derive the decentralized solution.

5.5 Investor i's problem at $t=1$

At $t=1$,the decision problem is whether to finance additional new risky projects, after perceiving the public signal d_1 about project return. All Investors see the same signal d_1.

There will exist a signal threshold $\bar{d}_1$ above which more investments will be made. No more investments will be made if $d_1 < \bar{d}_1$. After Bayesian updating, the expected return of new projects is given by $\frac{\alpha_0 R_0 + d_1 (x_i + X_{-i})^\theta}{\alpha_0 + (x_i + X_{-i})^\theta}$.

So the threshold $\bar{d}_1$ will make $\frac{\alpha_0 R_0 + \bar{d}_1 \times (x_i + X_{-i})^\theta}{\alpha_0 + (x_i + X_{-i})^\theta} = 1$, and is thus determined by (38),

$$\bar{d}_1 = \frac{\alpha_0 + (x_i + X_{-i})^\theta - \alpha_0 R_0}{(x_i + X_{-i})^\theta} \tag{38}$$

The optimal choice (z_i^*, w_i^*) of investor i is thus given by the following equations,

$$\begin{cases} z_i^* = \frac{N - x_i - X_{-i}}{M} & w_i^* = k - \left(x_i + \frac{N - x_i - X_{-i}}{M}\right) & if & d_1 \geq \bar{d}_1 \\ z_i^* = 0 & w_i^* = k - x_i & if & d_1 < \bar{d}_1 \end{cases} \tag{39}$$

5.6 Investor i's problem at $t = 0$

Given the $t = 1$ solution (39), Investor i's optimization problem at $t = 0$ becomes,

$$\max_{\{x_i\}} \left\{ \begin{array}{c} \int_{-\infty}^{\bar{d}_1} \left(\frac{\alpha_0 R_0 + s \cdot (x_i + X_{-i})^\theta}{\alpha_0 + (x_i + X_{-i})^\theta} x_i + k_i - x_i \right) f(s) ds \\ + \int_{\bar{d}_1}^{\infty} \left(\frac{\alpha_0 R_0 + s \cdot (x_i + X_{-i})^\theta}{\alpha_0 + (x_i + X_{-i})^\theta} \left(x_i + \frac{N - x_i - X_{-i}}{M} \right) + k_i - \left(x_i + \frac{N - x_i - X_{-i}}{M} \right) \right) f(s) ds \end{array} \right\} \tag{40}$$

$$s.t. \ (38), (41)$$

where d_1's probability distribution function $f(\cdot)$ follows (41),

$$f(s) = \frac{1}{\sqrt{(x_i + X_{-i})^{-\theta} + (\alpha_0)^{-1}}} \phi \left(\frac{s - R_0}{\sqrt{(x_i + X_{-i})^{-\theta} + (\alpha_0)^{-1}}} \right) \tag{41}$$

After careful integration, the optimization problem finally becomes,

$$\max_{x_i} \left\{ \begin{array}{c} k_i + (R_0 - 1) \left(\frac{M-1}{M} x_i + \frac{N - X_{-i}}{M} \right) \\ + \frac{N - x_i - X_{-i}}{M} (1 - R_0) \Phi \left(\frac{\frac{(1 - R_0)\left[\alpha_0 + (x_i + X_{-i})^\theta\right]}{(x_i + X_{-i})^\theta}}{\sqrt{(x_i + X_{-i})^{-\theta} + (\alpha_0)^{-1}}} \right) \\ + \frac{N - x_i - X_{-i}}{M} \frac{(x_i + X_{-i})^{\frac{\theta}{2}}}{\alpha_0^{\frac{1}{2}} \left(\alpha_0 + (x_i + X_{-i})^\theta \right)^{\frac{1}{2}}} \phi \left(\frac{\frac{(1 - R_0)\left[\alpha_0 + (x_i + X_{-i})^\theta\right]}{(x_i + X_{-i})^\theta}}{\sqrt{(x_i + X_{-i})^{-\theta} + (\alpha_0)^{-1}}} \right) \end{array} \right\} \tag{42}$$

5.7 Best response correspondence

Proposition 3 *Equation* (43) *gives the best response correspondence regarding the optimization problem* (22).

$$M - 1 + \Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right) =$$

$$\frac{(x_i+X_{-i})^{\frac{1}{2}\theta-1}}{2\left(R_0-1\right)\left(\alpha_0+(x_i+X_{-i})^\theta\right)^{\frac{3}{2}}\alpha_0^{\frac{1}{2}}}\left(\begin{array}{c} 2\left(x_i+X_{-i}\right)^{\theta+1}+2\alpha_0(x_i+X_{-i}) \\ -N\theta\alpha_0+\theta\alpha_0(x_i+X_{-i}) \end{array}\right)\phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right) \tag{43}$$

Equation (43) characterizes the best response of Investor i to the aggregate choice of all other Investors.

We can see (43) has an additional item $M-1$ in comparison with social planner's solution (21). This can move the equilibrium solution to either direction, over-invest or under-invest, conditional on the sign of the middle item (43). This implies market equilibrium level of early investment x_i can be above or below the social optimal level.

Solution (43) is compatible with either pure-strategy symmetric equilibrium or mixed-strategy $(leader, follower)$ equilibrium.

Figure 5 shows a best response function with 2 Investors. This will give a unique equilibrium.

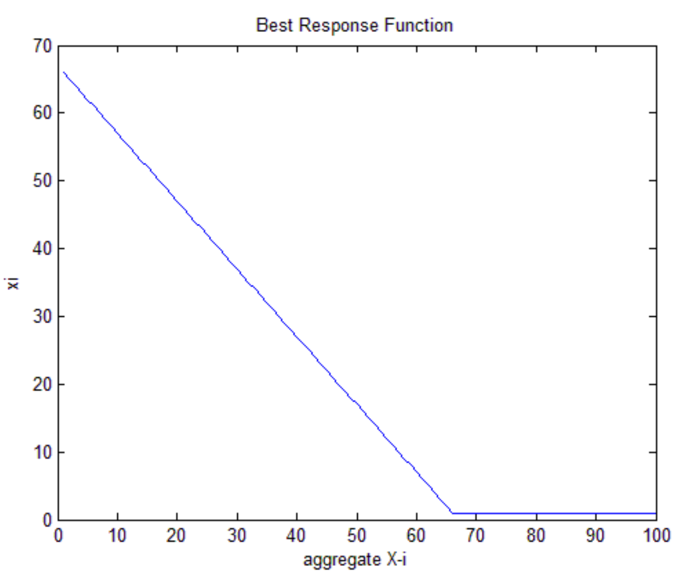


Figure 5. Best Response Correspondence

5.8 Symmetric Equilibrium

We have assumed all Investors are symmetric at the beginning, and then can get the following proposition.

Proposition 4 *There exists a unique symmetric equilibrium, and the closed-form solution of x is given by* (44)*,*

$$(M-1)+\Phi\left(\frac{(1-R_0)\sqrt{\alpha}\left((Mx)^{\theta}+\alpha\right)^{\frac{1}{2}}}{(Mx)^{\frac{\theta}{2}}}\right)=$$

$$\frac{1}{2(R_0-1)}\frac{(Mx)^{\frac{1}{2}\theta-1}}{\alpha^{\frac{1}{2}}\left(\alpha+(Mx)^{\theta}\right)^{\frac{3}{2}}}\left(\begin{array}{c}2Mx\alpha+2(Mx)^{\theta+1}\\-N\theta\alpha+\theta\alpha Mx\end{array}\right)\phi\left(\frac{(1-R_0)\sqrt{\alpha}\left((Mx)^{\theta}+\alpha\right)^{\frac{1}{2}}}{(Mx)^{\frac{\theta}{2}}}\right) \tag{44}$$

In the symmetric equilibrium, it is possible that all $x_i=0$, i.e. all Investors wait and intend to free-ride on others' information acquisition.

At first glance, it seems there can exist multiple symmetric equilibria within some parameter ranges. Assume there exist two symmetric equilibria, $\left(x_i^H\right),\left(x_i^L\right)$, where $x_i^H>x_i^L$. A $high-x$ equilibrium $\left(x_i^H\right)$ can be justified due to preemption motive. A $low-x$ equilibrium $\left(x_i^L\right)$ can be also justified due to the value of learning and free-rider motive. Multiple equilibria can exist when these two forces are close to a balance. However, uncertainty removes the multiplicity.

Corollary 4 *When $R_0=1$, $\theta=1$, we have a unique closed-form solution for x^*,*

$$x^*=\frac{\sqrt{\alpha_0(8N+9\alpha_0)}-3\alpha_0}{4M}$$

and in the limit $\frac{dx^*}{d\alpha_0}$ converges to 0 when α_0 grows to be large,

$$\lim_{\alpha_0\to\infty}\frac{dx^*}{d\alpha_0}=0$$

We can see that at $R_0=1$, the decentralized solution coincides with the social optimum.

6 Amplification and "Rush"

In this section we will discuss how a **"Rush"** can happen and the amplification mechanism in market equilibrium.

We firstly use the following numerical example to demonstrate the degree of sensitivity around some critial value of R_0.

Example 2 $\theta = 1.25$; *prior precision* $\alpha_0 = 1$; $R_0 = 1.275$*; total number of new projects N=500; two Investors M=2*

when $\Delta R_0 = 0.001$;the change of x_i is from 75 to 325,$\frac{\Delta x_i}{x_i} = 3.333$, $\frac{\Delta R_0}{R_0} = 0.0008$. Then we have a sensitivity$\frac{\left(\frac{\Delta X}{X}\right)}{\left(\frac{\Delta R_0}{R_0}\right)} = 4250$.

This means some tiny belief change $\frac{\Delta R_0}{R_0} = 0.0008$ can trigger a huge move in the market. Investors will respond with more than four thousand times amplification in early investment. A "Rush" occurs.

We will formally define **"Rush"** as the following,

Definition 5 *A Rush: The equilibrium $\hat{X}$, the first period investment, is dramatically increased by some small shock.*

A typical small shock is News shocks. For example, $\Delta R_0 > 0$, for some good news; $\Delta R_0 < 0$ for some bad news. A mathematical measure of "Rush" is defined by the following Amplication function,

Definition 6 *Amplification Function* $S(R_0)$:

$$S(R_0) = \frac{\frac{d\hat{X}}{dR_0}}{\frac{dX^*}{dR_0}}(R_0) \tag{45}$$

6.1 In Comparison with the Social Optimum

With Figure 4 in section 3, we have shown the social optimal X^* is a smooth function of prior R_0.There will never be a "rush". Social planner always prefer to conduct some small-scale experiment to learn the new projects patiently, before any massive investments. We call the social optimum choice of X^* "smooth learning" for investment. But market equilibrium will behavior in very different fashion.

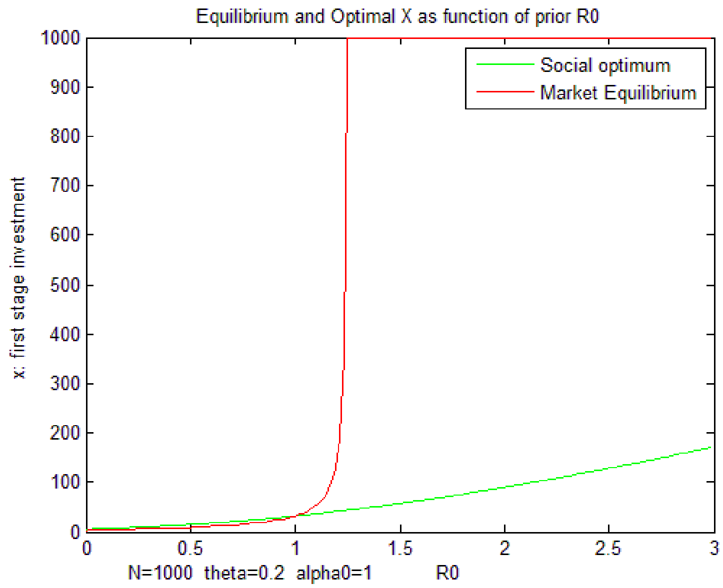


Figure 6. The generation of a **"Rush"**

Figure 6 compares the social optimum with a $2-Investor$ market equilibrium outcome. We can see a rush indeed occurs for the $2-Investor$ equilibrium around some threshold value 1.2. The amplification function $S(R_0)$ is an increasing function of R_0 until all the new projects are exhausted.

It is also worth mentioning that in this market equilibrium it is not a direct switching between small-scale experiment and full-scale investment. It is still a continuous transition though the amplification effect grows very fast. The remaining uncertainty prevents Investors from taking a vertical jump to finance all new projects.

The amplification function $S(R_0)$ and its derivative $S'(R_0)$ increase in R_0. And the fact amplification function $S(R_0)$ is increasing in R_0 also brings about the concern for social welfare loss.

6.2 Amplification as a Function of N

At any given R_0, if the prior is good enough, a higher N will add more impetus for an investor to preempt the market.

Figure 7 displays the case of a huge $N = 10000$. A huge number of new projects can be exhausted instantaneously because the amplification function $S(R_0)$ keeps growing very quickly.

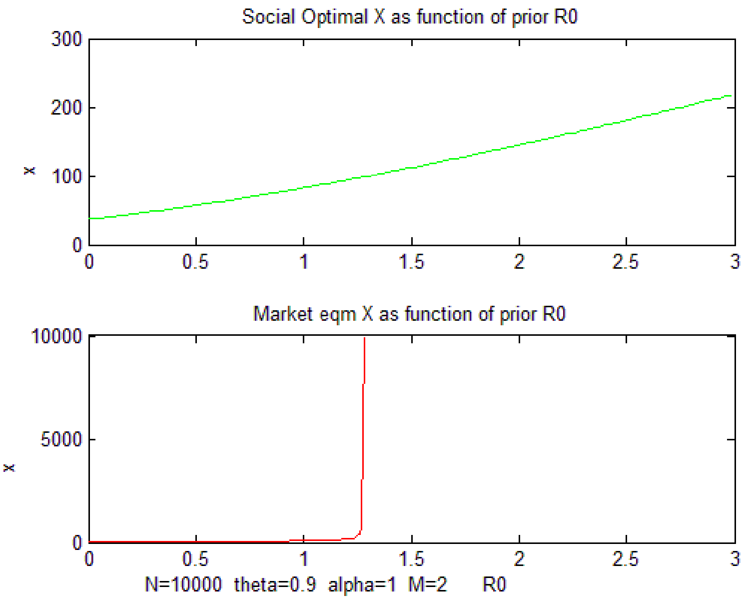


Figure 7. "Rush" for huge N

When there is over-investment, we can prove $\underset{N\to\infty}{lim} S(R_0)|_{\hat{X}=N} = \infty$; but $S(R_0) < \infty$,for any R_0,when $N < \infty$.

This implies that larger quantity of new projects, can be drained in an accelerated manner. Without constraint in available deposit, higher N will lead to expeditious investments. The total social welfare loss is an increasing function of N because the upper limit N can be reached very fast no matter how large is N. The uncertainty of innovation will be magnified by the number of new projects.

6.3 Amplification as a function of M

Previous discussions focus on the 2-Investor equilibrium. We can easily generalize the amplification function to $M - Investor$ setup: $S(R_0, M)$. $S(R_0, M)$ is non-decreasing in M, and welfare loss is also non-decreasing in M.

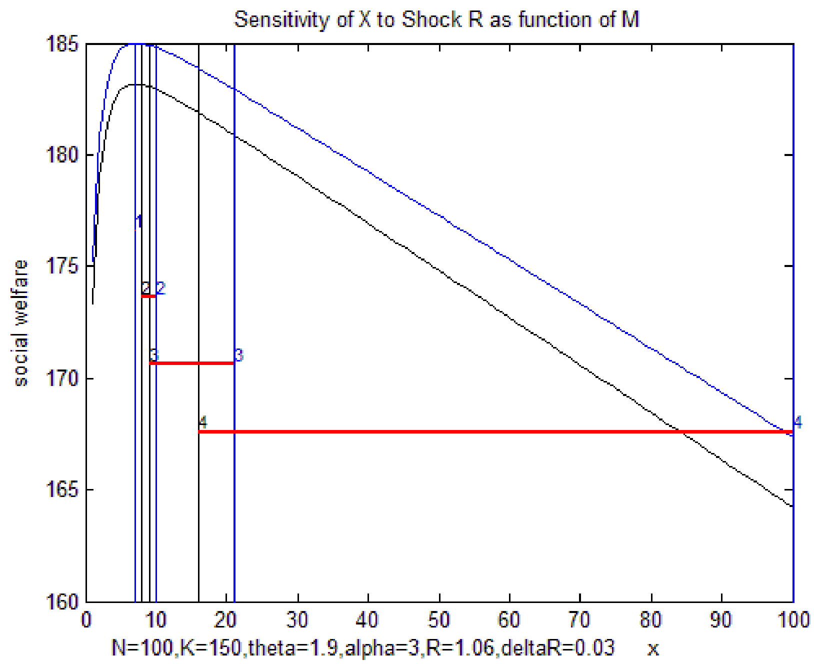


Figure 8. Amplification and M

Figure 8 shows the different x for $M - Investor$ equiliria ($M = 1, 2, 3, 4$). After a small shock to R_0,the new equilibrium x is illustrated with blue lines. The horizontal lines indicate the increase in x for different M after the shock. The increase in equilibrium x is an increasing function of Investor number M.

6.4 "Pricing" a Rush: Shadow Asset Price

Since the property rights for innovations are not perfectly defined, the investment market for innovations is somewhat missing. Therefore the market price for financing innovations might be just misleading. Investors usually underpay for innovations.

In this section, we try to derive the "shadow price" (i.e. the shadow return) for new technologies. We cannot observe the real price but can see the equilibrium quantity $\hat{x}$. A thought experiment is to back out the shadow return of new projects by referring to the optimal allocation. For example, in Figure 7, the top panel shows the relationship between optimal x^* and prior mean return R_0, while the bottom panel shows the relationship between equilibrium $\hat{x}$ and prior mean return R_0.

Denote the socially optimal solution $x^* = X(R_0)$ as a function of R_0,and the its inverse function $R = X^{-1}(x)$ gives the corresponding R value. Then we substitute the market equilibrium $\hat{x}$ into the inverse function $X^{-1}(\cdot)$ and get the shadow return in (46),

$$\hat{R} = X^{-1}(\hat{x}) \tag{46}$$

The solution to (46) can be finally derived by combining (44) and (21).

Corresponding to the super high level of equilibrium $\hat{x}$, the prior mean return R_0 must reach an excessively high level $X^{-1}(\hat{x})$.

"Bubble" usually refers to the excessively high asset price. But we argue for the view of "**Rational Rush**", rather than "**Irrational Bubble**". In fact, *the market equilibrium price for innovations is too low, not too high.* The property rights for innovations are often not perfectly defined, and consequently the innovators are usually underpaid. In the market equilibrium, over-investment ofttimes happens in the sense that uncertainty of innovations is underpriced due to the tradeoff of seizing a larger share of the underpriced innovations.

7 Inefficiencies: Over-investment or Under-investment

In this section, we will discuss the causes and conditions for inefficiencies in the market equilibria.

There are two kind of externalities embedded in the model: **(i)** imperfect property rights and common pool resource problem, and **(ii)** information externalities.

(i) Common-pool resource and coordination cost

The limited supply of new investment opportunities creates complementarities between Investors. The ownership of these new investment opportunities is not clearly defined and results in the "*Commons*" problem. Without well-defined property rights, price cannot work properly to impede a rush into the new and uncertain market. This will lead to over-investment and "the Tragedy of the Uncertain Commons". The Option value of learning and discarding bad innovations is reduced due to over-investment at an early stage.

(ii) Information externality

Information externality is generated because the public signal d_1 of project return at $t = 1$ is equally contributed by all Investor's investments at $t = 0$. Given that d_1 can be perfectly seen by all Investors, each of them wants to delay investment and wait for the signal created by other Investors' efforts. This free-rider motive always exist but will play a more evident role when the prior of return R_0 is low.

Interestingly, there is a watershed between over-investment and under-investment, and we have the following proposition,

Proposition 5 *There exists a threshold* $\bar{R}_0 = 1$,

(i) above which decentralized investors over-invest relative to the social optimum;

(ii) below which decentralized Investors under-invest in equilibrium.

See the appendix for the details of proof.

7.1 Welfare loss due to over-investment in early stage

Competition between Investors will incentivize them to concern more about the market share of new projects than their uncertain return. Figure 9 compares the social optimal investment with a $2 - Investor$ market equilibrium. The social optimal x^* is 8, while each investor will invest in 49 new projects in equilibrium. The total market equilibrium investments are 98, close to exhausting all the new investment opportunities at the very beginning. The top panel of Figure 9 shows that social welfare (the y axis) is achieved at the highest level when $x = x^*$ $(= 8)$. It contrasts with a $sub - optimal$ social welfare level (the dotted line) corresponding to the equilibrium x $(= 98)$. The coordination cost dominates the information externality. The welfare loss is mainly due to the abandoned option value of learning.

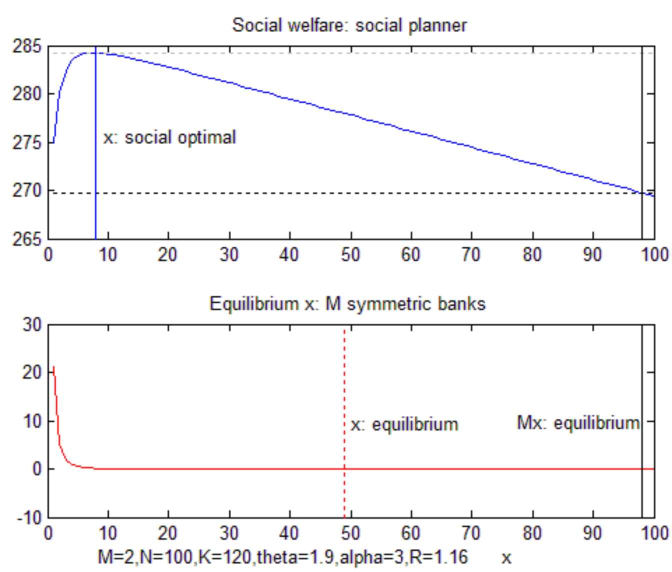


Figure 9. Over-investment in 2-Investor Equilibrium

7.2 Welfare loss due to under-investment in early stage

There are also circumstances that information externality plays a major role so that under-investment happens. In particular, when the prior of mean return R_0 is less than 1, Investors have much less incentive to invest early in the new market, because the benefit of preemption is relatively small. Investors tend to delay investments and wait to watch the public signal. Figure 10 shows that the 2-Investor equilibrium has lower x than the social optimal x^*. Admittedly, we can see from the top panel that social welfare loss is not very significant in this under-investment situation.

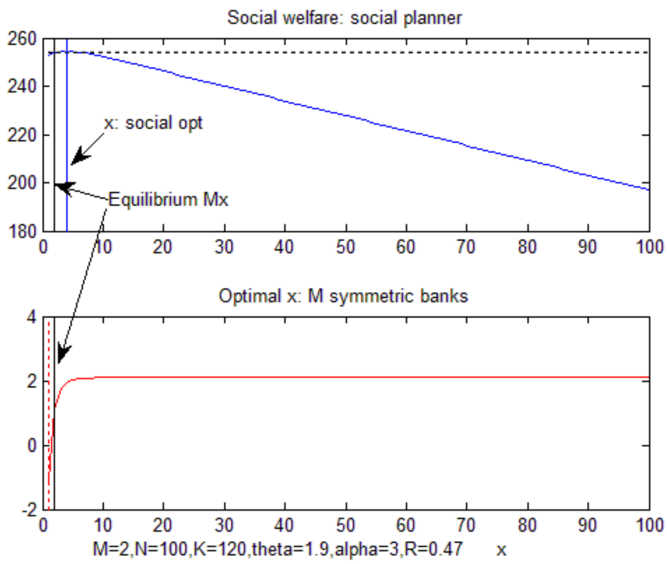


Figure 10. Under-investment in 2-Investor Equilibrium

7.3 Amplification at the threshold value $\bar{R}_0$

The switch between under-investment and over-investment will increase the curvature of amplification function $S(R_0)$ at the threshold value $\bar{R}_0$.The free-rider motive will amplify a news shock around the $\bar{R}_0$.

7.4 Welfare loss and M

Increase in the number of Investors will exacerbate both the coordination problem and information externality. For the over-investment circustance, coordination cost is aggravated more severely than information free-riding problem. Therefore, the net over-investment incentive is magnified by a larger M. We have the following proposition.

Proposition 6 (Inefficiency and M) *For the over-investment circumstance:*

(i) The aggregate quantity of illiquid projects X financed at $t = 0$ is a non-decreasing function of the number of banks M;

(ii) Social welfare is a non-increasing function of the number of banks M;

(iii) The limiting case: there exists a threshold $\bar{M}$, above which all N illiquid projects will be financed at $t = 0$.

This can be easily proved as an extension to the proof of Proposition 5.

A mirror proposition can also be proved for the case of under-investment.

Figure 11 shows how the equilibrium x increases with M, and correspondingly how the social welfare decreases with M.

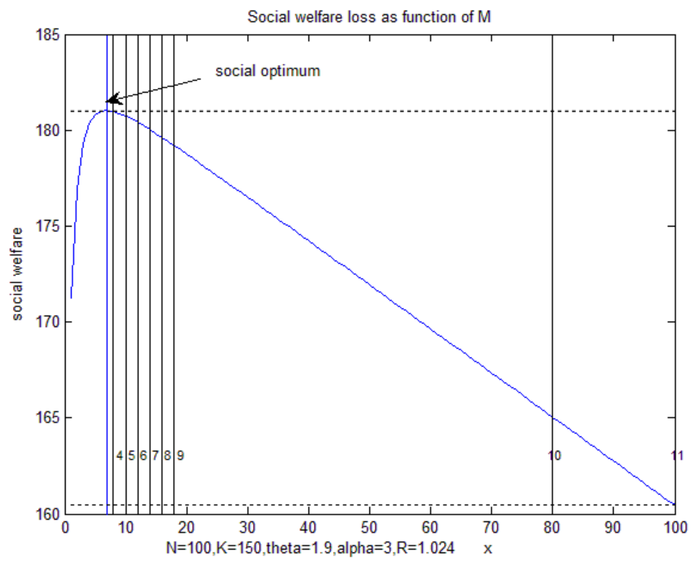


Figure 11. Welfare loss as a function of M

7.5 The Number of New Projects N

The number of new projects also matter for the total welfare loss. From the previous discussion of social planner's problem and Figure 2 we know the relative cost of learning $\frac{x}{N}$ is a decreasing function of N. This implies the benefit of learning increases with M. On the other hand, from the Proposition of Amplification Function and Figure 7, we know around the threshold level R_0,any large number of projects can be exhausted very quickly due to the property of the amplification function. The welfare loss due to "**Rush**" will be multiplied by the number of projects N.

8 Optimal and Robust Macro-Patent Design

In this section, we will discuss the *Patent Design problem*, as a correction to the inefficiency of over-investment. At the end of the section, we will also prescribe the *Robust Patent Design problem* for a generalized setup with ambigious priors.

8.1 Flexible Patent Mechanism

We assume the investor indexed by s is the innovator who creates the breakthrough technology, and is guaranteed the ownership of a fixed shares (μ_0, μ_1) of new projects based on that breakthrough technology. The vector (μ_0, μ_1) embodies the Patent policy. μ_0 and μ_1 represent the monopolistic share for the innovator at $t = 0$ and $t = 1$ respectively, as illustrated by the following allocation mechanism. Under full patent protection, $\mu_0 = \mu_1 = 1$. However, full patent protection is not socially optimal with the possibility of sequential innovation (or with deadweight loss of monopoly).

Allocation Mechanism 2 (Flexible Patent) *Allocation functions $h^0(\cdot)$ and $h^1(\cdot)$ are defined by (4), (47), (48), with the patent policy (μ_0, μ_1),*

$$h_i^0\left((x_1, x_2, ..., x_M)\right) = \begin{cases} x_i & if x_{i=s} \leq \mu_0 N \\ \mu_0 N & if x_{i=s} > \mu_0 N \\ x_i & if x_{i \neq s} \leq \frac{(1-\mu_0)N}{M-1} \\ \frac{(1-\mu_0)N}{M-1} & if x_{i \neq s} > \frac{(1-\mu_0)N}{M-1} \end{cases} \tag{47}$$

$$h_i^1\left((z_1, z_2, ..., z_M)\right) = \begin{cases} z_i & if z_{i=s} \leq \mu_1 N_1 \\ \mu_1 N_1 & if z_{i=s} > \mu_1 N_1 \\ z_i & if z_{i \neq s} \leq \frac{(1-\mu_1)N_1}{M-1} \\ \frac{(1-\mu_1)N_1}{M-1} & if z_{i \neq s} > \frac{(1-\mu_1)N_1}{M-1} \end{cases} \tag{48}$$

This Patent Mechanism is flexible in that disparate degrees of monopolistic power are allowed for different stages.

8.2 Correcting Rush Inefficiency with Patent

We want to use patent to discourage over-investment and reduce the inefficiency due to rush. Patent as a monopolistic power also imposes cost, which we will model as the barrier to sequential innovation.

For the following discussion, we focus on the $2 - investor$ case for simplicity.

8.2.1 At $t = 0$: First-wave experimental investment in the market

We assume at $t = 0$, one investor succeeds in making a breakthrough innovation and gets the patent. We call her the leader. The other investor is then named a copycat. This breakthrough will directly generate N new projects available for financing. Each of the project has return R.

Patent Policy

According to the Flexible Patent Allocation Mechanism, there is a total share μ_0 of N new projects reserved for the leader for $t = 0$. The other investor can only stay in her own niche of $(1 - \mu_0) N$ new projects at $t = 0$.

Sequential Innovation

The copycat (but in this sense, not a pure copycat) can combine the breakthrough technology with her own innovation. The sequential innovation will generate a multiplier effect by expanding the number of projects he financed by a factor χ, and $\chi > 1$. This can be thought as a "buy one get $\chi - 1$ free" bonus due to sequential innovation. This captures the social welfare gain of sequential innovation which expands the applications of the technological breakthrough. Too strong patent protection can discourage sequential innovation and reduce social welfare.

8.2.2 At $t = 1$: Second-wave investment and systemic adoption decision

In this stage, the leader can finance up to $\mu_1 \cdot N_1$ new projects. The copycat can only invest in his own niche of $(1 - \mu_1) N_1$ projects.

8.3 The Optimal Patent Design Problem

The patent policy is defined by the vector (μ_0, μ_1).

8.3.1 Copycat's Problem at $t = 0$

$$V_{c,0} = \max_{x_c} \mathbb{E}_{c,0} \left[R \times \chi x_c + V_{c,1} \right] \tag{49}$$

$$V_{c,1} = \max_{z_c, w_c} \mathbb{E}_{c,1} \left[R \times \chi z_c + w_c | d_1 \right] \tag{50}$$

subject to

$$x_c + z_c + w_c \leq k - y_c$$
$$x_c, z_c, w_c \geq 0$$

$$x_c \leq (1 - \mu_0)N \tag{51}$$

$$z_c \leq (1 - \mu_1)(N - x_s - x_c) \tag{52}$$

$$E_1[R|d_1] = \frac{\alpha_0 R_0 + d_1 (x_c + x_s)^{\theta}}{\alpha_0 + (x_c + x_s)^{\theta}} \tag{53}$$

$$V_1[R|d_1] = \left(\alpha_0 + (x_c + x_s)^{\theta}\right)^{-1} \tag{54}$$

$$d_1 \sim N(R_0, (x_c + x_s)^{-\theta} + (\alpha_0)^{-1}) \tag{55}$$

8.3.2 Leader (patent holder)'s Problem at $t = 0$

$$V_{s,0} = \max_{x_s} \mathbb{E}_{s,0}[R \times x_s + V_{s,1}] \tag{56}$$

$$V_{s,1} = \max_{z_s, w_s} \mathbb{E}_{s,1}[R \times z_s + w_s | d_1] \tag{57}$$

subject to

$$x_s + z_s + w_s \leq k - y_s$$
$$x_s, z_s, w_s \geq 0$$

$$x_s \leq \mu_0 N \tag{58}$$

$$z_s \leq \mu_1 (N - x_s - x_c) \tag{59}$$

and (53),(54),(55)

8.3.3 The Choice of Optimal Patent Policy

Finally, designing optimal patent is just to choose the vector (μ_0, μ_1) to maximize the value function V_0 at $t = 0$,

$$\max_{\{(\mu_0, \mu_1)\}} \{V_0\} \tag{60}$$

$$V_0 = \omega V_{s,0} + (1-\omega) V_{c,0} \tag{61}$$

where V_0 is the weighted sum of utilities of the two types of players. ω is the patent holder's population share or the successful probability of becoming a patent holder. Here we simply assume equal probability of becoming a leader and a copycat, $\omega = 0.5$. In a later subsection where we will explicitly model the R&D investment, these probabilities will be endogenously determined by the choice of early R&D investments.

8.4 Solution Method

8.4.1 At t=1

The copycat's threshold $\bar{d}_{c,1}$ will make $\chi\frac{\alpha_0 R_0 + \bar{d}_1 (x_c+x_s)^\theta}{\alpha_0 + (x_c+x_s)^\theta} = 1$, and is thus determined by (62),

$$\bar{d}_{c,1} = \frac{\alpha_0 + (x_c+x_s)^\theta - \chi\alpha_0 R_0}{\chi (x_c+x_s)^\theta} \tag{62}$$

The optimal choice (z_c^*, w_c^*) of the *copycat* is thus given by the following equations,

$$\begin{cases} z_c^* = (1-\mu_1)(N - x_s - x_c) & w_i^* = k - x_c - z_c^* & if & d_1 \geq \bar{d}_{c,1} \\ z_c^* = 0 & w_i^* = k - y_c - x_c & if & d_1 < \bar{d}_{c,1} \end{cases} \tag{63}$$

The leader's threshold $\bar{d}_{s,1}$ will make $\frac{\alpha_0 R_0 + \bar{d}_1 (x_c+x_s)^\theta}{\alpha_0 + (x_c+x_s)^\theta} = 1$, and is thus determined by (64),

$$\bar{d}_{s,1} = \frac{\alpha_0 + (x_c+x_s)^\theta - \alpha_0 R_0}{(x_c+x_s)^\theta} \tag{64}$$

The optimal choice (z_s^*, w_s^*) of the *leader* is thus given by the following equations,

$$\begin{cases} z_s^* = \mu_1 (N - x_s - x_c) & w_s^* = k - x_s - z_s^* & if & d_1 \geq \bar{d}_{s,1} \\ z_s^* = 0 & w_s^* = k - y_s - x_s & if & d_1 < \bar{d}_{s,1} \end{cases} \tag{65}$$

8.4.2 At t=0

Given the $t=1$ solution (63), *copycat*'s optimization problem at $t=0$ becomes,

$$V_{c,0} = \max_{\{x_c\}} \left\{ \begin{array}{c} \int_{-\infty}^{\bar{d}_{c,1}} \left(\chi\frac{\alpha_0 R_0 + s\cdot(x_c+x_s)^\theta}{\alpha_0 + (x_c+x_s)^\theta} x_c + k - y_c - x_c \right) f(s) ds \\ + \int_{\bar{d}_{c,1}}^{\infty} \left(\begin{array}{c} \chi\frac{\alpha_0 R_0 + s\cdot(x_c+x_s)^\theta}{\alpha_0 + (x_c+x_s)^\theta} (x_c + (1-\mu_1)(N - x_s - x_c)) \\ +k - (y_c + x_c + (1-\mu_1)(N - x_s - x_c)) \end{array} \right) f(s) ds \end{array} \right\} \tag{66}$$

$$s.t.\ (62), (67), (51)$$

where d_1's probability distribution function $f(\cdot)$ follows (67),

$$f(s) = \frac{1}{\sqrt{(x_c + x_s)^{-\theta} + (\alpha_0)^{-1}}}\phi\left(\frac{s - R_0}{\sqrt{(x_c + x_s)^{-\theta} + (\alpha_0)^{-1}}}\right) \tag{67}$$

Similarly, *leader's* problem at $t = 0$ becomes,

$$V_{s,0} = \max_{\{x_s\}} \left\{ \begin{array}{c} \int_{-\infty}^{\bar{d}_{s,1}} \left(\frac{\alpha_0 R_0 + s \cdot (x_c + x_s)^{\theta}}{\alpha_0 + (x_c + x_s)^{\theta}} x_s + k - y_s - x_s \right) f(s) ds \\ + \int_{\bar{d}_{s,1}}^{\infty} \left(\begin{array}{c} \frac{\alpha_0 R_0 + s \cdot (x_c + x_s)^{\theta}}{\alpha_0 + (x_c + x_s)^{\theta}} (x_s + \mu_1 (N - x_s - x_c)) \\ + k - (y_s + x_s + \mu_1 (N - x_s - x_c)) \end{array} \right) f(s) ds \end{array} \right\} \tag{68}$$

$$s.t.\ (64), (67), (58)$$

8.5 Numerical Result

In this subsection, we will numerically solve the optimal patent problem (60).

Figure 12 illustrates a numerical solution to (60). The x-axis corresponds to μ_1, and the y-axis corresponds to μ_0. Function values denote the social welfare. With a setup $R_0 = 1.1, \alpha_0 = 3$, $\chi = 1.01, \theta = 1$, we get the optimal patent $(\mu_0^* = 0.76, \mu_1^* = 0.65)$. That is, $\mu_0^* \geq \mu_1^*$.

In fact, this result is robust to various paramter combinations. In general, we present the following result (69),

Result 1 (Uncertainty of Innovation and Patent Design)

$$\mu_0^* \geq \mu_1^* \tag{69}$$

This implies that higher uncertainty demands stricter patent protection. At an early stage of technology adoption, larger monopolistic power should be granted to the innovator or first-mover, and discourage the rush of other investors.

At first glance, you may think μ_1^* should be equal to $1/2$, which means no monopolistic power for the leader. But if so, the leader will over-invest at $t = 0$ because she expects decreased investment opportunities at $t = 1$.

Granting monopolistic power is a way to prevent the premature spread of uncertain knowledge which can potentially cause damages. After uncertainty dwindles with learning, the monopolistic power of the patent holder should be reduced to facilitate sequential innovation.

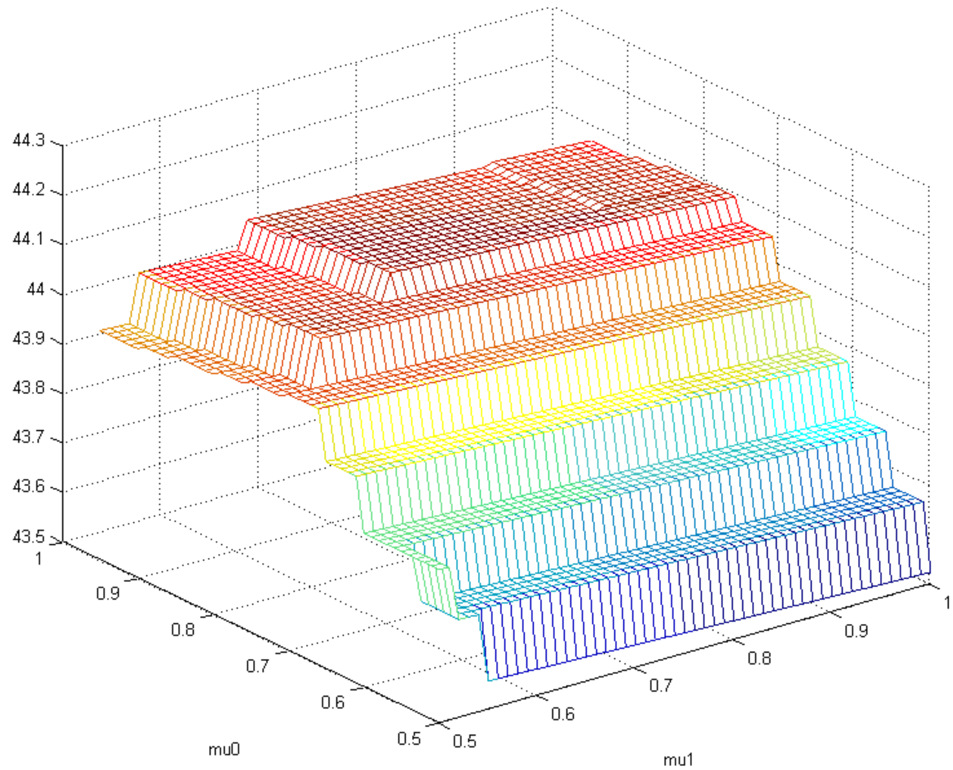


Figure 12. 2-Stage Optimal Patent

8.6 Tradeoffs in a sum

Better Patent protection leads to,

1. (+) lower cost of over-investment under uncertainty;

2. (−) lower sequential innovation;

Number (1) is a new force that traditional patent design has not taking into account. This will imply a stricter patent protection than the traditional patent literature.

8.7 Robust Macro-Patent Design

Another challenge facing patent designer is the existence of uncertainty even for the prior return of a new technology. The prior return R_0 can also be a random variable, following certain distribution $R_0 \sim F$. Here we can assume $R_0 \sim Uniform(\underline{R}, \bar{R})$ for simplicity.

Assume the social planner (patent designer) needs to consider the worst-case senario. This results in the following Robust Patent Design problem,

$$\max_{\{(\mu_0,\mu_1)\}} \min_{\{R_0\}} \{V_0\} \tag{70}$$

where V_0 is defined by (61) as before.

The problem (70) takes into consideration extreme cases of both low and high R_0. The possibility of a very high R_0 requires more stringent patent protection to prevent excessive investment. This implies determining a high upper bound of monopolistic power to deter big "rush".

On the other hand, a low R_0 implies weak private incentive to explore the new technology relative to the social optimum. This also demands better patent protection, and even with necessary government subsidies like the Orphan drug case.

9 Extension: Optimal Patent Design with Endogenous R&D

In this section, we prescribe and discuss the patent design problem in a more general form. We add an additional period $t = -1$ when initial $R\&D$ investment happens, as shown in Figure 13.

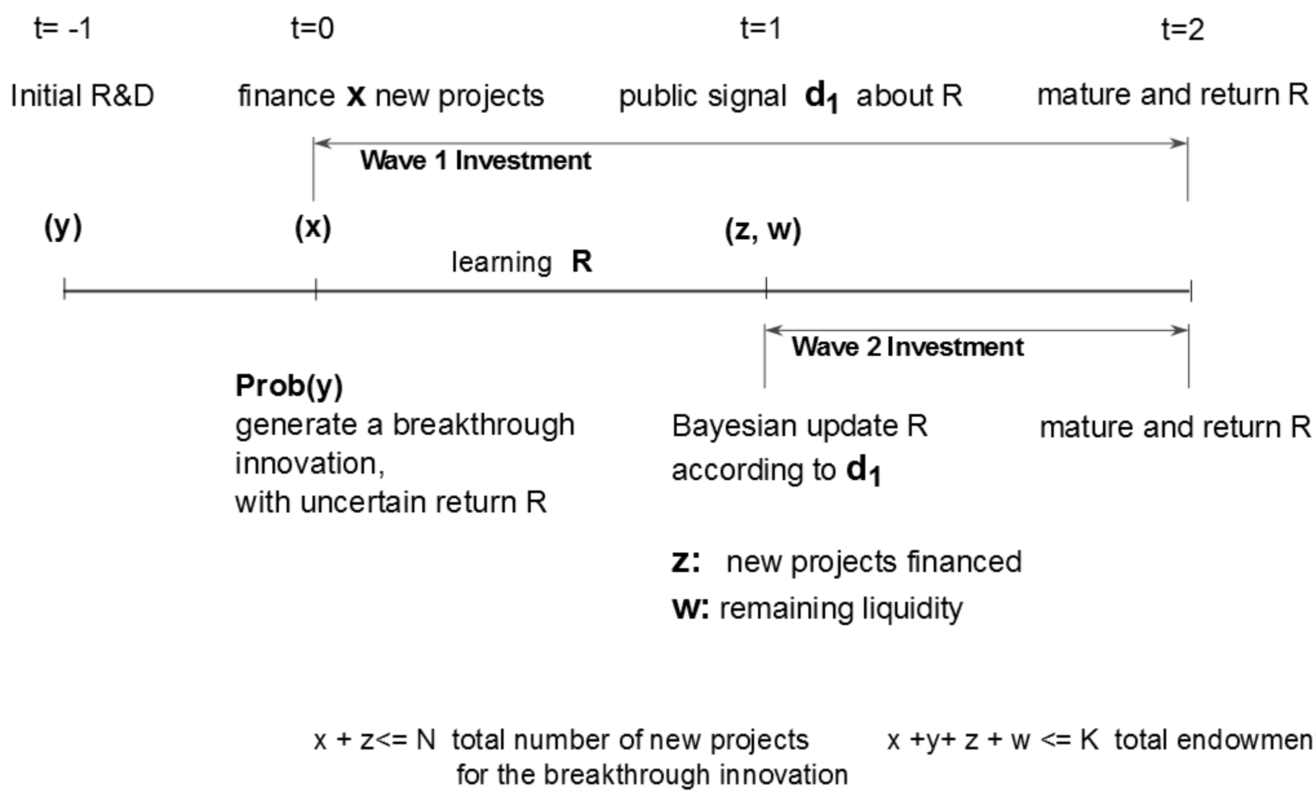


Figure 13. Timing with R&D

9.1 At $t = -1$: Lab R&D stage

Each investor $i \in \{1, 2\}$ will invest an amount y_i as $R\&D$ expenditure at $t = -1$. Each of them will generate an innovation with return r_i. But there can only be one breakthrough innovation with return R. We assume only one investor will succeed in discovering and winning the breakthrough, with an independent probability $p(y_i)$. $p(y_i)$ is an increasing, concave function. We call the breakthrough innovator the *leader*, denoted with subscript s. The other investor is named *copycat*, denoted with subscript c.

The probabilities for all four possible contingencies is,

$$\begin{cases} (1-p(y_i))(1-p(y_j)) & \text{no investor makes a breakthrough} \\ p(y_i)(1-p(y_j)) & \text{only investor } i \text{ makes a breakthrough} \\ p(y_j)(1-p(y_i)) & \text{only investor } j \text{ makes a breakthrough} \\ p(y_i)p(y_j) & \text{both } i \text{ and } j \text{ make a breakthrough} \end{cases} \tag{71}$$

When both investors make the breakthrough, the planner randomly assigns patent to one of them.

We assume the return of all innovations are,

$$\begin{cases} r_i = 1 & i \neq s \\ r_i = R & i = s \end{cases} \tag{72}$$

.

9.2 Investor's problem at $t = -1$

The two investors are identical at $t = -1$.

The probability for investor i to successfully get the patent of a breakthrough innovation is $p(y_i)(1-p(y_j)) + \frac{p(y_i)p(y_j)}{2}$;

the probability for the other investor j get the patent is $p(y_j)(1-p(y_i)) + \frac{p(y_i)p(y_j)}{2}$

the probability of no breakthrough innovation is $(1-p(y_i))(1-p(y_j))$.

The aggregate successful probability of a breakthrough for the society is,

$$p(y_i) + p(y_j) - p(y_i)p(y_j) \tag{73}$$

So we can write investor $i\prime s$ optimization problem as (74),

$$V_{-1} = \max_{y_i} \mathbb{E}_{-1} \begin{bmatrix} \left(p(y_j)(1-p(y_i)) + \frac{p(y_i)p(y_j)}{2}\right) V_{c,0} \\ + \left(p(y_i)(1-p(y_j)) + \frac{p(y_i)p(y_j)}{2}\right) V_{s,0} \\ + (1-p(y_i))(1-p(y_j)) V_{n,0} \\ -y_i \end{bmatrix} \tag{74}$$

Value of the case of no-breakthrough $V_{n,0}$ is,

$$V_{n,0} = N \times 1 \tag{75}$$

where the project return is just equal to 1.

9.3 Rush and Patent Race: the Rush-Race shifting

The Generlized Optimal Patent design problem is a tradeoff between the following forces. Better Patent protection leads to,

1. (+) higher aggregate R&D expenditures, and higher probability of a breakthrough;
2. (−) lower sequential innovation;
3. (+) lower cost of over-investment under uncertainty;
4. (−) patent race at the very beginning

More stringent patent protection can discourage rush and over-investment at a later stage ($t = 0$), but it will aggravate the traditional "patent race" at the earlier stage. Despite this $Rush - Race$ shifting, the welfare loss due to patent race in $R\&D$ will be at a smaller scale than the loss due to rush in investments at a later stage. There is very few empirical evidence on $Patent\ Race$ although there have been many theoretical research on it. In contrast, the magnitude of over-investment at later stage is usually much larger, as shown by the empirical evidence of the next section.

10 Empirical Evidence

Economic historian Kindleberger (1978) cites Minsky's argument that any speculative bubble and crisis starts with a "displacement" or innovation or some exogenous macroeconomic shock. This will grow to be a speculative bubble, over-investment, and eventual crash. The mechanism modeled in this research is consistent with the empirical descriptions of Kindleberger and Minsky. In contrast to their emphasis on irrational factors and mania, here we attribute these innovation-induced economic booms to rational rushes in financing innovations and inefficiencies due to coordination failure.

In this section, we study three cases and provide some suggestive evidence on rush in terms of quantity of investment as well as the linkage between the degree of patent protection and over-investment.

10.1 Three Historical Episodes of Technology "Rush"

10.1.1 The 1990's "Internet Rush"

Figure 14 compares several scaled time series during the 1990's Internet "Rush", with year 1995 set as 100.

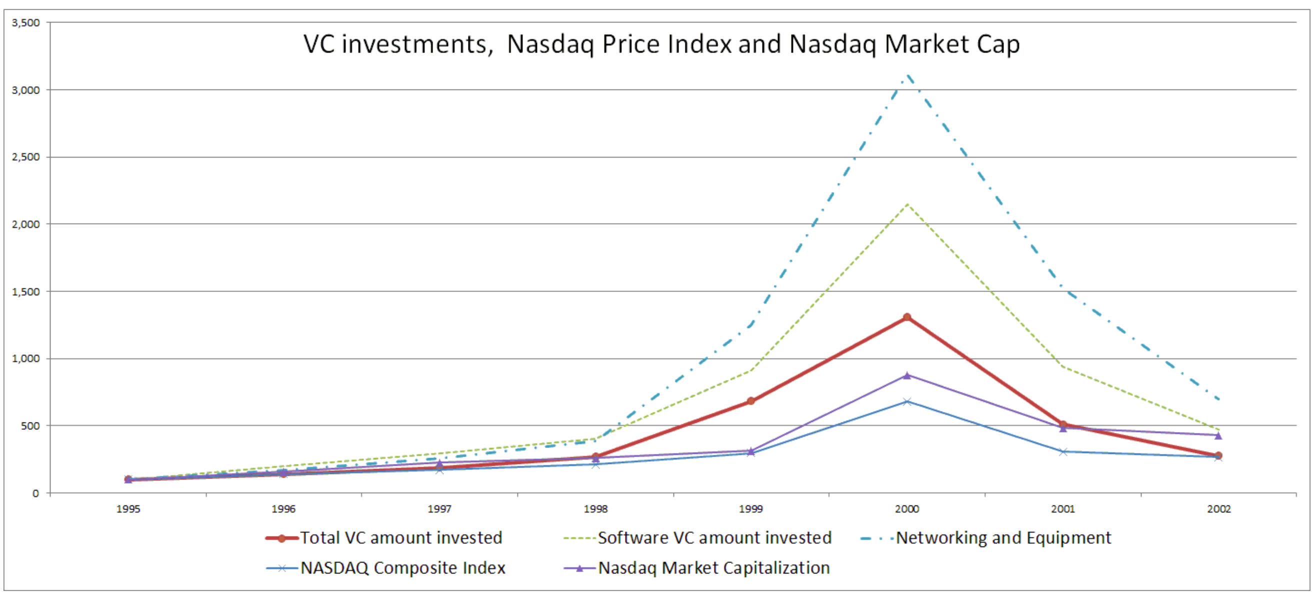


Figure 14: VC investments v.s. Nasdaq Price and Market Cap

Price, as illustrated by the Nasdaq index, cannot capture all the abrupt increase in investment. If we look at the quantity of Venture Capital investments, we see VC investments have grown much faster than Nasdaq price index and the Nasdaq Market Cap. From 1995 to 2000, the total amount of venture capital investment increased by 12 times, while the NASDAQ price index only quadrupled.

The new technologies had actually been undervalued due to knowledge spillovers and non-excludability in applying the General Purpose Technology, the Internet.

10.1.2 The 2000's "Subprime Rush"

The cause of the 2008 Financial Crisis was largely attributed to excessive subprime mortgage lending and securitization. Subprime mortgage and securitization are financial innovations, which are not patentable in general. We have seen a large number of subprime mortgage lenders enter the market and compete heavily to lend to new subprime borrowers. There were limited number of potential subprime borrowers. Nevertheless, both securitization and subprime mortgage contract have revealed to have innate defects in the design. This "Subprime Rush" engendered excessive lending and eventually ended up with the Great Recession.

The uncertainty of financial innovations can easily raise systemic risk in the economy if a rush happens. According to the principle illustrated in the Optimal Patent design section, broader patent rights should be granted to the financial innovators.

10.1.3 The 2010's "Unicorn Rush"

Most recently, we are witnessing a new (potential) rush, the phenomenon of the rise of "Unicorns". "Unicorns" refer to technology startups with at least $1 billion valuation, based on fundraising. Figure 15 displays the trend of Unicorns from 2009 to Nov 2015.

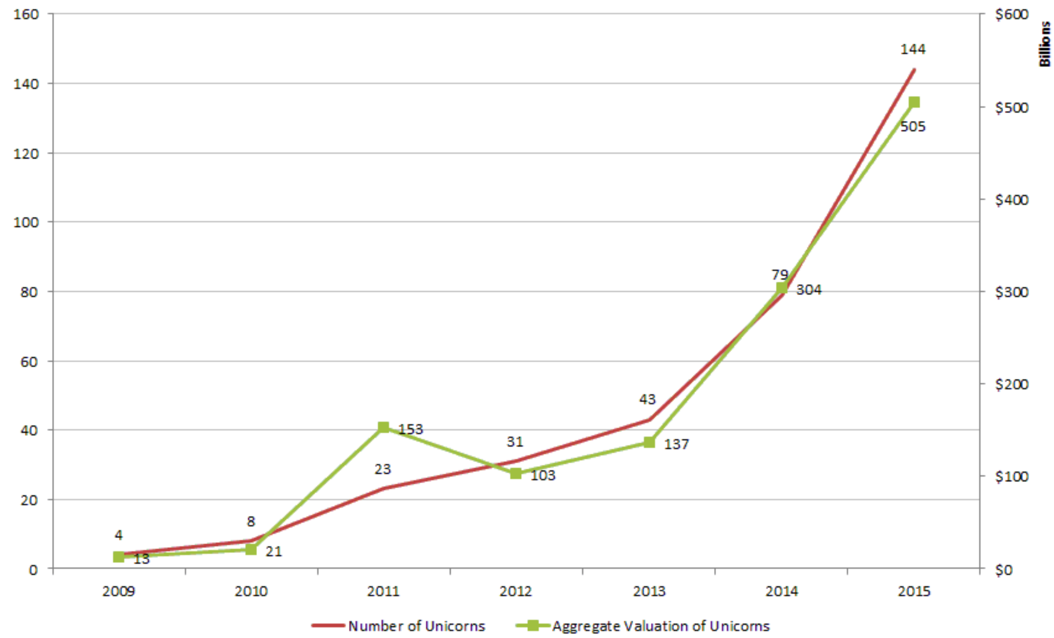


Figure 15: The Trend of Unicorns[1]

Among the list of Unicorns[2], dominating ones are based on social media, Mobile technology, shared economy, and new drugs. The top Unicorn is Uber, with 51 $Billion till November 2015, and the second one is Xiaomi, a new Chinese mobile phone company. Airbnb takes the third position, Airbnb shares the same fundamental technology with Uber, but has the application in a different niche. This is consistent with our theoretical model above.

Unicorns have blurred the borderline between early VC investment and IPO, and they might have grown too big relative to the social optimal size. From Figure 15, we see an accelerated growth of unicorns from 2013 on: both the number and aggregate value of the unicorns almost quadrupled in only 2 years. This reminds us the Internet Rush and a jump in VC investments from 1998, as shown in Figure 14.

10.2 Cross-industry VC investments and Patent protection

According to the theoretical part of this research, we will test the hypothesis of whether sectors with less de facto IPR protection will be more procyclical and display more severe “Rush”.

[1] Data Source: CB Insights and Thomson Reuters.

[2] see the latest updates from http://fortune.com/unicorns/ and http://graphics.wsj.com/billion-dollar-club/?co=Square

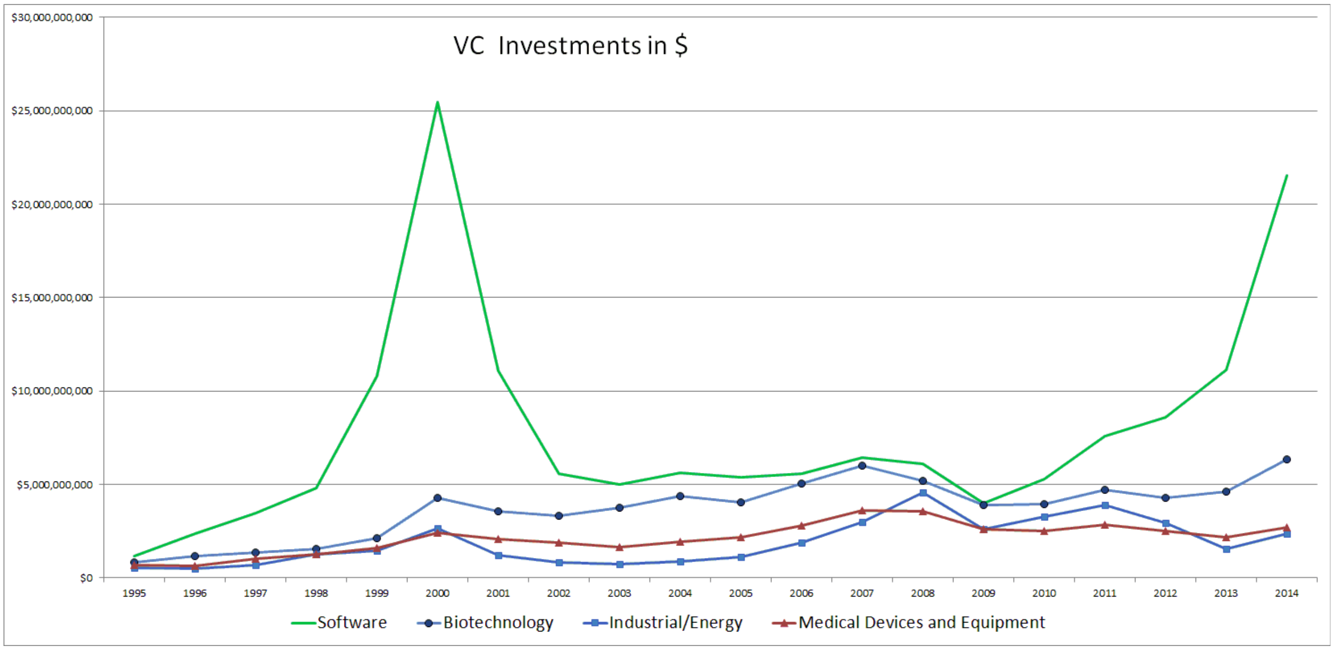


Figure 16: VC investments by sectors[3]

Figure 16 compares the Venture capital investments across several industries. We can see from Figure 16 that biotech and medical devices were rushed on a much smaller scale than the software industry. The volatility of VC investments in software industry is huge relative to other sectors. This is because the idea of software is very easy to copy and duplicate, even though it can be patented de jure. Therefore, the concept of "De facto IPR protection" is important. The essential idea of a new software can be easily designed around and so it is not protected as well as pharmacy although both of them have de jure patents.

Within the biotech industry, different types of drugs also have variations in de facto patent protection. Chemical drugs (based on Chemical molecule) are relatively easy to design around or copy with small changes, while biomolecules drugs are difficult to copy because the production process has subtleties, which can be thought of as business secret in some sense.

Industries with weaker de facto IPR protection display more volatile and procyclical pattern of VC investments.

[3]Data Source: CB Insights and Thomson Reuters.

11 Alternative Policy Instruments

In general, it should be socially optimal to subsidize the earliest adoption of new technology, but put some restrictions on followers' "rushing" into it.

11.1 Patent policy contingent on expected return

An straightforward policy is to set "tax rates" contingent on innovations' expected returns. In general, the optimal policy is to tax innovations with high expected returns and use the proceeds to subsidize innovations with low expected returns, such as the Orphan drugs.

11.2 Anti-trust and Competition Policy

For some new industries and new products, if without appropriate patent protection, anti-trust regulation should be relaxed. Allowing certain monopolistic power at the early stage of technology adoption can help to mitigate some unexpected hazards.

11.3 Product Liability, Patent and Social Regulation

Patent not only provides incentives for inventing new products, but also helps to ameliorate hazards. Because the innovator has exclusive IPR of the new product, the liability of the new product is also exclusively assigned to the innovator. It implies that the innovator will be fully responsible for the negative side of her innovation. This can mitigate moral hazards and other inefficiencies of insufficient learning.

Thus there is a linkage or substitute effect between patent protection and product liability.

For products that are generally nonpatentable, to mitigate potential hazards, it demands stricter liability system or screening. Because financial products are generally not available for patent, the establishment of the CFPB (Consumer Financial Protection Bureau) can play an important role in alleviating the side effects of financial innovations.

FDA sets a standard or require the least information to reduce uncertainty of an innovation. This can mitigate the *uncertainty spillover.*, and reduce the proportions of bad models on the market.

11.4 Monetary v.s. IPR Policy: cheap credit or cheap knowledge

Monetary policy plays a big role in controlling credit to firms and investors. It has been pointed out that loose monetary policy and thus cheap credit was culprit for the major financial crises. However, technological breakthroughs often inject too much “liquidity” of underpriced knowledge, due to its nature of public good. Relative to the monetary policy which broadly affects the whole economy, will a counter-cyclical IPR policy be more accurate in targeting the sectors which market failure of IPR is most significant?

12 Conclusions

Innovations not only create better investment opportunities but also bear uncertainties and potential hazards. Patient learning are necessary to reduce the uncertainties to a socially optimal level. However, this can be distorted by Investor’s incentive to capture a lion’s share of the new market. Facing a limited supply of new projects, investors prefer to enter the market prematurely. An inefficient "**rush**" often happens after the launch of some breakthrough technology.

We assume the knowledge of learning can be publicly seen. Interestingly, the incentive for free-riding the knowledge generated by other investors’ early investments can mitigate the coordination problem to some extent. On the other hand, this information externality can lead to under-investment at early stage especially when the prior of mean return is low.

Our findings imply that granting some monopolistic power can help to remove inefficiencies in decentralized learning and investments in innovations. Financial innovations are generally not protected by patent laws. The traditional literature of innovation and patent emphasizes the benefit and public good nature of innovations. This research calls attention to the other side of innovation: uncertainty and potential social hazards. Premature diffusion and rush in financing an innovation can bring about significant and unexpected "public bad". Appropriate mechanism needs to be designed to make Investors internalize the negative externalities.

The core mechanism illustrated by this research has broad policy implications. We suggest the following applications and extensions for future research,

Effort level for learning

We can allow the Investors to make a tradeoff between the quantity of projects to finance and the effort level put on each project. This tradeoff is important especially when Investors face resource constraints.

Bank Runs and sytemic risk

A rush in financing innovations can increase the probability of runs. Consumers also have coordination problems regarding their withdrawal decisions. Suboptimal learning of decentralized Investors can amplify the coordination failure of consumers. Therefore, "rushing-in" at the asset side leads to "running out" at the liability side. The author illustrates this mechanism in a separate paper, Xie (2015c).

Technological improvement and refinement

The innovator can keep on improving the product and removing hazards after learning. Therefore, at a later stage, the expected return can be higher due to learning and improvement.

Endogenous Growth Model with Rush

Embed this mechanism of Rush into an endogenous growth model. We can calibrate an endogenous growth model with this amplification mechanism to match the "rush" phenomena in real data.

A New Quantity Theory of Money

This research points out an important friction, knowledge spillover and non-excludability for using uncertain knowledge. This implies the dysfunction of the price mechanism, when pricing major innovations and technological breakthroughs. Therefore, the mainstream monetary regime for targeting price level will be insufficient. During a technology rush, an inflation-targeting monetary framework will miss the "target": there seems to be no need to intervene when there is high growth and low inflation. Therefore, this research implies a monetary framework to target both price and quantity. In my coauthored paper Hufbauer and Xie (2009), we show the correlation of our definition of a broader money aggregate, the De Facto Money (DFM), and financial instability. An optimal monetary framework with special attention to quantity can be derived based on the theoretical mechanism of this research and the empirical work in Hufbauer and Xie (2009).

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13 Appendix

13.1 Proof of Proposition 1

Given the optimization problem,

$$\max_{\{x\}}\left\{\begin{array}{c}\int_{-\infty}^{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}\left(\frac{\alpha_0 R_0+s\cdot x^{\theta}}{\alpha_0+x^{\theta}}x+K-x\right)f(s)ds\\+\int_{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}^{\infty}\left(\frac{\alpha_0 R_0+s\cdot x^{\theta}}{\alpha_0+x^{\theta}}N+K-N\right)f(s)ds\end{array}\right\} \tag{76}$$

where the distribution $f(s)$ follows,

$$f(s)=\frac{1}{\sqrt{\left(x^{\theta}\right)^{-1}+\left(\alpha_0\right)^{-1}}}\phi\left(\frac{s-R_0}{\sqrt{\left(x^{\theta}\right)^{-1}+\left(\alpha_0\right)^{-1}}}\right) \tag{77}$$

There are three major steps. Firstly integrate (76), and secondly take first-order conditions to get a closed-form solution of x. Thirdly, we prove the uniqueness of social planner's solution.

13.1.1 Step 1: Integration

(i) Integrate the lower part of (76)

$\int_{-\infty}^{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}\left(\frac{\alpha R_0+s\cdot x^{\theta}}{\alpha+x^{\theta}}x+K-x\right)f(s)ds$

$=\int_{-\infty}^{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}\left(\frac{\alpha R_0+s\cdot x^{\theta}}{\alpha+x^{\theta}}x-x+K\right)f(s)ds$

$=\int_{-\infty}^{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}\left(\frac{s\cdot x^{\theta}}{\alpha+x^{\theta}}x\right)f(s)ds+\left(K-x+\frac{x\alpha R_0}{\alpha+x^{\theta}}\right)\int_{-\infty}^{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}f(s)ds$

(ii) Integrate the lower part of (76)

$\int_{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}^{\infty}\left(\frac{\alpha R_0+s\cdot x^{\theta}}{\alpha+x^{\theta}}N+K-N\right)f(s)ds$

$=\int_{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}^{\infty}\left(\frac{s\cdot x^{\theta}}{\alpha+x^{\theta}}N\right)f(s)ds+\left(K-N+\frac{\alpha R_0 N}{\alpha+x^{\theta}}\right)\int_{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}^{\infty}f(s)ds$

rearrange the sum of the two outcomes above, we have

$\left(\left(K-x+\frac{x\alpha R_0}{\alpha+x^{\theta}}\right)\int_{-\infty}^{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}f(s)ds+\left(K-N+\frac{\alpha R_0 N}{\alpha+x^{\theta}}\right)\int_{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}^{\infty}f(s)ds\right)+$
$\frac{x^{\theta}}{\alpha+x^{\theta}}\frac{1}{1-\lambda}\left(x\int_{-\infty}^{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}sf(s)ds+N\int_{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}^{\infty}sf(s)ds\right)$

and we integrate the two parts respectively,

(a) The first part

$\left(K-x+\frac{x\alpha R_0}{\alpha+x^{\theta}}\right)\int_{-\infty}^{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}f(s)ds+\left(K-N+\frac{\alpha R_0 N}{\alpha+x^{\theta}}\right)\int_{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}^{\infty}f(s)ds$

because $f(s)=\frac{1}{\sqrt{(x^{\theta})^{-1}+(\alpha_0)^{-1}}}\phi\left(\frac{s-R_0}{\sqrt{(x^{\theta})^{-1}+(\alpha_0)^{-1}}}\right)$,

we have $\int_{-\infty}^{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}f(s)ds=\Phi\left(\frac{s-R_0}{\sqrt{(x^{\theta})^{-1}+(\alpha_0)^{-1}}}\right)\Big|_{-\infty}^{\frac{\alpha+x^{\theta}-\alpha R_0}{x^{\theta}}}$

therefore $\left(K-x+\frac{x\alpha R_0}{\alpha+x^\theta}\right)\int_{-\infty}^{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}} f(s)ds+\left(K-N+\frac{\alpha R_0 N}{\alpha+x^\theta}\right)\int_{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}}^{\infty} f(s)ds$

$=\frac{E-x+\frac{x\alpha R_0}{\alpha+x^\theta}}{1-\lambda}\Phi\left(\frac{\frac{\left(\alpha+x^\theta\right)(1-R_0)}{x^\theta}}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)+\frac{E-N+\frac{\alpha R_0 N}{\alpha+x^\theta}}{1-\lambda}\left[1-\Phi\left(\frac{\frac{\left(\alpha+x^\theta\right)(1-R_0)}{x^\theta}}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)\right]$

$=\frac{E-N+\frac{\alpha R_0 N}{\alpha+x^\theta}}{1-\lambda}+\frac{N-x+\frac{x\alpha R_0-N\alpha R_0}{\alpha+x^\theta}}{1-\lambda}\Phi\left(\frac{\frac{\left(\alpha+x^\theta\right)(1-R_0)}{x^\theta}}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

$=\frac{E-N+\frac{\alpha R_0 N}{\alpha+x^\theta}}{1-\lambda}+\frac{(N-x)}{1-\lambda}\left(\frac{\alpha+x^\theta-\alpha R_0}{\alpha+x^\theta}\right)\Phi\left(\frac{\frac{\left(\alpha+x^\theta\right)(1-R_0)}{x^\theta}}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

(b) The second part

$\frac{x^\theta}{\alpha+x^\theta}\frac{1}{1-\lambda}\left(x\int_{-\infty}^{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}} sf(s)ds+N\int_{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}}^{\infty} sf(s)ds\right)$

we need to firstly derive $\int sf(s)ds$

$\int sf(s)ds=\int s\phi\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)d\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

$=\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}\int\frac{s}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\phi\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)d\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

$=\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}\int\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\phi\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)d\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)+$
$\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}\int\frac{R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\phi\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)d\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

$=\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}\int\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)\phi\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)d\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)+$
$R_0\int\phi\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)d\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

$=\left(-\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}\right)\phi\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)|+R_0\Phi\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)|$

then

$x\int_{-\infty}^{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}} sf(s)ds$

$=-x\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}\phi\left(\frac{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)+xR_0\Phi\left(\frac{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

and

$N\int_{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}}^{\infty} sf(s)ds$

$=N\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}\phi\left(\frac{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)+NR_0\Phi\left(\frac{s-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)|_{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}}^{\infty}$

$=NR_0+N\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}\phi\left(\frac{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)-NR_0\Phi\left(\frac{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

Add up the two parts' integration results, we have the final integration result*:*

$\frac{E-N+\frac{\alpha R_0 N}{\alpha+x^\theta}}{1-\lambda}+\frac{(N-x)}{1-\lambda}\left(\frac{\alpha+x^\theta-\alpha R_0}{\alpha+x^\theta}\right)\Phi\left(\frac{\frac{\left(\alpha+x^\theta\right)(1-R_0)}{x^\theta}}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

$+\frac{x^\theta}{\alpha+x^\theta}\frac{NR_0}{1-\lambda}+\frac{x^\theta}{\alpha+x^\theta}\frac{N-x}{1-\lambda}\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}\phi\left(\frac{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)-\frac{x^\theta R_0}{\alpha+x^\theta}\frac{N-x}{1-\lambda}\Phi\left(\frac{\frac{\alpha+x^\theta-\alpha R_0}{x^\theta}-R_0}{\sqrt{(x^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

which can be simplified to

$\frac{1}{1-\lambda}\left(E+N\left(R_0-1\right)\right) \quad + \quad \frac{1}{1-\lambda}\left(N-x\right)\left(1-R_0\right)\Phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \quad +$
$\frac{1}{1-\lambda}\frac{x^{\frac{\theta}{2}}\left(N-x\right)}{\alpha_0^{\frac{1}{2}}\left(\alpha_0+x^{\theta}\right)^{\frac{1}{2}}}\phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)$

13.1.2 Step 2: First-order condition

now the optimization problem becomes,

$$\max_x \left\{ \begin{array}{c} K+N\left(R_0-1\right) \\ +\left(N-x\right)\left(1-R_0\right)\Phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \\ +\frac{x^{\frac{\theta}{2}}\left(N-x\right)}{\alpha_0^{\frac{1}{2}}\left(\alpha_0+x^{\theta}\right)^{\frac{1}{2}}}\phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \end{array} \right\} \tag{78}$$

the derivative of $\frac{\left(1-R\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}$
$\frac{d}{dx}\left\{\frac{\left(1-R\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right\} = \frac{1}{2x^{\frac{1}{2}\theta+1}}\alpha^{\frac{3}{2}}\frac{\theta}{\sqrt{\alpha+x^{\theta}}}\left(R-1\right)$

(i) derivative of $\left(N-x\right)\left(1-R_0\right)\Phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)$
$\frac{d}{dx}\left\{\left(N-x\right)\left(1-R_0\right)\Phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)\right\}$
$=\left(R_0-1\right)\Phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)-\left(N-x\right)\left(1-R_0\right)^2\frac{1}{2x^{\frac{1}{2}\theta+1}}\alpha^{\frac{3}{2}}\frac{\theta}{\sqrt{\alpha+x^{\theta}}}\phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)$

(ii) derivative of $\frac{x^{\frac{\theta}{2}}\left(N-x\right)}{\alpha_0^{\frac{1}{2}}\left(\alpha_0+x^{\theta}\right)^{\frac{1}{2}}}\phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)$
$\frac{d}{dx}\left\{\frac{x^{\frac{\theta}{2}}\left(N-x\right)}{\alpha_0^{\frac{1}{2}}\left(\alpha_0+x^{\theta}\right)^{\frac{1}{2}}}\phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)\right\}$
$=-\frac{1}{2}\frac{x^{\frac{1}{2}\theta-1}}{\sqrt{\alpha}\left(\alpha+x^{\theta}\right)^{\frac{3}{2}}}\left(2x\alpha+2xx^{\theta}-N\alpha\theta+x\alpha\theta\right)\phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \quad -$
$\frac{x^{\frac{\theta}{2}}\left(N-x\right)}{\alpha_0^{\frac{1}{2}}\left(\alpha_0+x^{\theta}\right)^{\frac{1}{2}}}\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\frac{1}{2x^{\frac{1}{2}\theta+1}}\alpha^{\frac{3}{2}}\frac{\theta}{\sqrt{\alpha+x^{\theta}}}\left(R-1\right)\phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)$
$=-\frac{1}{2}\frac{x^{\frac{1}{2}\theta-1}}{\sqrt{\alpha}\left(\alpha+x^{\theta}\right)^{\frac{3}{2}}}\left(2x\alpha+2xx^{\theta}-N\alpha\theta+x\alpha\theta\right)\phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \quad +$
$\frac{1}{2x^{\frac{1}{2}\theta+1}}\alpha^{\frac{3}{2}}\theta\frac{N-x}{\sqrt{\alpha+x^{\theta}}}\left(R-1\right)^2\phi\left(\frac{\left(1-R_0\right)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)$

then the sum of these two derivatives, and equalizes it with 0,

$$(R_0-1)\,\Phi\left(\frac{(1-R_0)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)-(N-x)\,(1-R_0)^2\,\frac{1}{2x^{\frac{1}{2}\theta+1}}\alpha^{\frac{3}{2}}\frac{\theta}{\sqrt{\alpha+x^{\theta}}}\phi\left(\frac{(1-R_0)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)$$
$$-\frac{1}{2}\frac{x^{\frac{1}{2}\theta-1}}{\sqrt{\alpha}(\alpha+x^{\theta})^{\frac{3}{2}}}\left(2x\alpha+2xx^{\theta}-N\alpha\theta+x\alpha\theta\right)\phi\left(\frac{(1-R_0)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)+\frac{1}{2x^{\frac{1}{2}\theta+1}}\alpha^{\frac{3}{2}}\theta\frac{N-x}{\sqrt{\alpha+x^{\theta}}}(R-1)^2\,\phi\left(\frac{(1-R_0)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)=0$$

rearrange items to get,

$$\Phi\left(\frac{(1-R_0)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)(R_0-1)$$
$$=\phi\left(\frac{(1-R_0)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)\left\{\begin{array}{c}\frac{1}{2}\frac{x^{\frac{1}{2}\theta-1}}{\sqrt{\alpha}(\alpha+x^{\theta})^{\frac{3}{2}}}\left(2x\alpha+2xx^{\theta}-N\alpha\theta+x\alpha\theta\right)\\ +(N-x)\,(1-R_0)^2\,\frac{1}{2x^{\frac{1}{2}\theta+1}}\alpha^{\frac{3}{2}}\frac{\theta}{\sqrt{\alpha+x^{\theta}}}\\ -\frac{1}{2x^{\frac{1}{2}\theta+1}}\alpha^{\frac{3}{2}}\theta\frac{N-x}{\sqrt{\alpha+x^{\theta}}}(R-1)^2\end{array}\right\}$$

which can be further simplifed to,

$$\frac{\Phi\left(\frac{(1-R_0)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)}{\phi\left(\frac{(1-R_0)(\alpha+x^{\theta})^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)}$$
$$=\frac{1}{2}\frac{1}{R_0-1}\left\{\frac{x^{\frac{1}{2}\theta-1}}{\sqrt{\alpha}(\alpha+x^{\theta})^{\frac{3}{2}}}\left(2x\alpha+2xx^{\theta}-N\alpha\theta+x\alpha\theta\right)\right\}$$
$$=\frac{1}{2}\frac{1}{R_0-1}\left\{\frac{2x^{\frac{\theta}{2}}}{\sqrt{\alpha(\alpha+x^{\theta})}}-\frac{x^{\frac{1}{2}\theta-1}\theta\sqrt{\alpha}}{(\alpha+x^{\theta})^{\frac{3}{2}}}(N-x)\right\}$$

finally we have,

$$\frac{\Phi\left(\frac{(1-R_0)\left(\alpha+x^{\theta}\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)}{\phi\left(\frac{(1-R_0)(\alpha+x^{\theta})^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)}=\frac{1}{2}\frac{1}{R_0-1}\left\{\frac{2x^{\frac{\theta}{2}}}{\sqrt{\alpha\left(\alpha+x^{\theta}\right)}}-\frac{x^{\frac{1}{2}\theta-1}\theta\sqrt{\alpha}}{(\alpha+x^{\theta})^{\frac{3}{2}}}(N-x)\right\} \tag{79}$$

13.1.3 Step 3. Uniqueness

denote $G=(1-R_0)\sqrt{\frac{\alpha_0(\alpha_0+x^{\theta})}{x^{\theta}}}$

$\sqrt{\frac{\alpha_0(\alpha_0+x^{\theta})}{x^{\theta}}}=\sqrt{\frac{\alpha_0^2}{x^{\theta}}+\alpha_0}$ is always a decreasing function of x, when $x\geq 1$

Equation (79) can be rearranged to be

$$G\frac{\Phi(G)}{\phi(G)}=\frac{\theta\alpha_0(N-x)}{2x\left(\alpha_0+x^{\theta}\right)}-1 \tag{80}$$

$\frac{\theta\alpha_0(N-x)}{2x(\alpha_0+x^{\theta})}$ is always decreasing in x for $x>0$ and $x<N$.

$\frac{d}{dG}\left(\frac{\Phi(G)}{\phi(G)}\right)=1+\Phi(G)\left(\frac{1}{\phi(G)}\right)'=1+\Phi(G)\cdot\left(-\frac{1}{\phi^2(G)}\right)\cdot(-G\phi(G))=1+G\frac{\Phi(G)}{\phi(G)}$

$\frac{d\left(\frac{G\cdot\Phi(G)}{\phi(G)}\right)}{dG}=\frac{d\left(G\cdot\frac{\Phi(G)}{\phi(G)}\right)}{dG}=\frac{\Phi(G)}{\phi(G)}+G\cdot\left(1+G\frac{\Phi(G)}{\phi(G)}\right)$

$=G+\frac{\Phi(G)}{\phi(G)}\left(G^2+1\right)$

For any $G>0$, $\frac{d\left(\frac{G\cdot\Phi(G)}{\phi(G)}\right)}{dG}>0$;

and in fact, for any $G >$G̰$= -7.48$, $\frac{d\left(\frac{G\cdot\Phi(G)}{\phi(G)}\right)}{dG} > 0$.

G̰ is tiny enough, and $G = (1-R_0)\sqrt{\frac{\alpha_0(\alpha_0+x^\theta)}{x^\theta}}$ will never be smaller than G̰ in reasonable way.

So generally $\frac{G\cdot\Phi(G)}{\phi(G)}$ is *an increasing function of G.*

When $R_0 > 1$,$(1-R_0) < 0$,

$G(x) = (1-R_0)\sqrt{\frac{\alpha_0(\alpha_0+x^\theta)}{x^\theta}}$ is increasing function of x;

so $LHS = \frac{G(x)\cdot\Phi(G(x))}{\phi(G(x))}$ *is increasing function of x.*

RHS *is decreasing function of x.*

there is a unique solution at their intersection.

When $R_0 < 1$,$(1-R_0) > 0$,

$G(x) = (1-R_0)\sqrt{\frac{\alpha_0(\alpha_0+x^\theta)}{x^\theta}}$ is decreasing function of x;

so $LHS = \frac{G(x)\cdot\Phi(G(x))}{\phi(G(x))}$ *is decreasing function of x.*

and RHS *is decreasing function of x.*

However, the slope of the RHS is higher than the LHS when x is large enough. The LHS and RHS curve will cross once and only once, and (80) has a unique solution x.

■

13.2 Proof of Corollary 1

$$\max_x \left\{ \begin{array}{c} K + N(R_0-1) \\ +(N-x)(1-R_0)\Phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \\ +(N-x)\frac{x^{\frac{\theta}{2}}}{\alpha_0^{\frac{1}{2}}(\alpha_0+x^\theta)^{\frac{1}{2}}}\phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \end{array} \right\} \tag{81}$$

$\underset{x\to 0}{lim}\left\{\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right\} = \left\{\begin{array}{lll} \infty & if & R_0 < 1 \\ -\infty & if & R_0 \geq 1 \end{array}\right.$

therefore

(i) if $R_0 < 1, x \to 0$

$\underset{x\to 0}{lim}\left\{\Phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)\right\} = 1;$

$(N-x)(1-R_0)\Phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \to N(1-R_0) = (N-1)(1-R_0) + (1-R_0)$

$\frac{x^{\frac{\theta}{2}}(N-x)}{\alpha_0^{\frac{1}{2}}(\alpha_0+x^\theta)^{\frac{1}{2}}}\phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \to 0$

if $x = 1$, then $(1-R_0)\Phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) = (1-R_0)\Phi\left((1-R_0)(\alpha_0+1)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}\right)$

$\frac{x^{\frac{\theta}{2}}}{\alpha_0^{\frac{1}{2}}(\alpha_0+x^\theta)^{\frac{1}{2}}}\phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) = \frac{1}{\alpha_0^{\frac{1}{2}}(\alpha_0+1)^{\frac{1}{2}}}\phi\left((1-R_0)(\alpha_0+1)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}\right)$

(ii) if $R_0 > 1, x \to 0$

$\underset{x\to 0}{lim}\left\{\Phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)\right\} = 0; (N-x)(1-R_0)\Phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \to 0$

$$\frac{x^{\frac{\theta}{2}}(N-x)}{\alpha_0^{\frac{1}{2}}(\alpha_0+x^\theta)^{\frac{1}{2}}}\phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)\to 0$$

but any $0 < x < N$ will make $(N-x)(1-R_0)\,\Phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) > 0$; $\frac{x^{\frac{\theta}{2}}(N-x)}{\alpha_0^{\frac{1}{2}}(\alpha_0+x^\theta)^{\frac{1}{2}}}\phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) > 0$

so $x = 0$ is not optimal.

■

13.3 Proof of Corollary 2

$$\max_x \left\{ \begin{array}{c} K + N(R_0 - 1) \\ +(N-x)(1-R_0)\,\Phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \\ +\frac{x^{\frac{\theta}{2}}(N-x)}{\alpha_0^{\frac{1}{2}}(\alpha_0+x^\theta)^{\frac{1}{2}}}\phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) \end{array} \right\} \tag{82}$$

at $x = N$,both$(N-x)(1-R_0)\,\Phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)$ and $\frac{x^{\frac{\theta}{2}}(N-x)}{\alpha_0^{\frac{1}{2}}(\alpha_0+x^\theta)^{\frac{1}{2}}}\phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right)$ are equal to 0.

because $\Phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) > 0$ and $\phi\left(\frac{(1-R_0)\left(\alpha+x^\theta\right)^{\frac{1}{2}}\alpha_0^{\frac{1}{2}}}{x^{\frac{\theta}{2}}}\right) > 0$

when $(1-R_0) < 0$,any $x < N$ will have a higher utility than when $x = N$.

■

13.4 Proof of Corollary 3

$$(1-R_0)\frac{\Phi\left((1-R_0)\sqrt{\frac{\alpha_0(\alpha_0+x^\theta)}{x^\theta}}\right)}{\phi\left((1-R_0)\sqrt{\frac{\alpha_0(\alpha_0+x^\theta)}{x^\theta}}\right)} = \frac{1}{\sqrt{\frac{\alpha_0(\alpha_0+x^\theta)}{x^\theta}}}\left(\frac{\theta\alpha_0(N-x)}{2x\left(\alpha_0+x^\theta\right)} - 1\right)$$

When $R_0 \longrightarrow 1$,

$$\lim_{R_0\longrightarrow 1}(1-R_0)\frac{\Phi\left((1-R_0)\sqrt{\frac{\alpha_0(\alpha_0+x^\theta)}{x^\theta}}\right)}{\phi\left((1-R_0)\sqrt{\frac{\alpha_0(\alpha_0+x^\theta)}{x^\theta}}\right)} = 0 = \lim_{R_0\longrightarrow 1}\frac{1}{\sqrt{\frac{\alpha_0(\alpha_0+x^\theta)}{x^\theta}}}\left(\frac{\theta\alpha_0(N-x)}{2x\left(\alpha_0+x^\theta\right)} - 1\right)$$

so we have $\frac{\theta\alpha_0(N-x)}{2x\left(\alpha_0+x^\theta\right)} - 1 = 0$

$\theta\alpha_0(N-x) = 2x\left(\alpha_0 + x^\theta\right)$

That is, x is the solution to $2x^{\theta+1} + \alpha_0(2+\theta)x - \theta\alpha_0 N = 0$

When $\theta = 0$, $2x + 2\alpha_0 x = 0$; so $x = 0$.

When $\theta = 1$,

we have $2x^2 + 3\alpha_0 x - \alpha_0 N = 0$, and the solution is given by,

$\left(\frac{1}{4}\sqrt{\alpha_0(8N+9\alpha_0)} - \frac{3}{4}\alpha_0, -\frac{1}{4}\sqrt{\alpha_0(8N+9\alpha_0)} - \frac{3}{4}\alpha_0\right)$

finally we have the following unique positive solution for x,

$x=\frac{1}{4}\sqrt{\alpha_0\left(8N+9\alpha_0\right)}-\frac{3}{4}\alpha_0$
In the limit
$\lim\limits_{\alpha_0\to\infty}\frac{dx}{d\alpha_0}=\lim\limits_{\alpha_0\to\infty}\frac{8N+18\alpha_0}{8\sqrt{9\alpha_0^2+8N\alpha_0}}-\frac{3}{4}=\lim\limits_{\alpha_0\to\infty}\frac{8N+18\alpha_0}{24\alpha_0}-\frac{3}{4}=0$

■

13.5 Proof of Proposition 2

13.5.1 x with respect to N

When $(1-R_0)<0$, the LHS of (80) is a decreasing function of x. The RHS of (80) is an increasing function of x.

Because for $\frac{\sqrt{\frac{x^\theta}{\alpha_0(\alpha_0+x^\theta)}}}{1-R_0}\frac{\theta\alpha_0}{2x(\alpha_0+x^\theta)}\left(N-x\right)$, $\sqrt{\frac{x^\theta}{\alpha_0(\alpha_0+x^\theta)}}>0$ and $\frac{\theta\alpha_0}{2x(\alpha_0+x^\theta)}>0, 1-R_0<0$, so an increase in N will shift down the RHS.

The intersection x will rise.

This outcome also applies when $(1-R_0)>0$.

13.5.2 x/N with respect to N

the RHS of (80) can be rearranged to be $\frac{\sqrt{\frac{x^\theta}{\alpha_0(\alpha_0+x^\theta)}}}{1-R_0}\left(\frac{\theta\alpha_0\left(\left[\frac{N}{x}\right]-1\right)}{2(\alpha_0+x^\theta)}-1\right)$.

■

13.6 Proof of Proposition 3

Derivation of the best response function

The threshold of signal:$\bar{d}_1=\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}$

$$\max_{\{x_i\}}\left\{\begin{array}{c}\int_{-\infty}^{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}\left(\frac{\alpha R_0+s\cdot(x_i+X_{-i})^\theta}{\alpha+(x_i+X_{-i})^\theta}x_i+k-x_i\right)f(s)ds\\+\int_{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}^{\infty}\left(\frac{\alpha R_0+s\cdot(x_i+X_{-i})^\theta}{\alpha+(x_i+X_{-i})^\theta}\left(x_i+\frac{N-x_i-X_{-i}}{M}\right)+\left(k-\left(x_i+\frac{N-x_i-X_{-i}}{M}\right)\right)\right)f(s)ds\end{array}\right\}\tag{83}$$

with $f(s)=\frac{1}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\phi\left(\frac{s-R_0}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right)$

Integration Part

$\int_{-\infty}^{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}\left(\frac{\alpha R_0+s\cdot(x_i+X_{-i})^\theta}{\alpha+(x_i+X_{-i})^\theta}x_i+k-x_i\right)f(s)ds$

$=\int_{-\infty}^{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}\left(\frac{\alpha R_0+s\cdot(x_i+X_{-i})^\theta}{\alpha+(x_i+X_{-i})^\theta}x_i-x_i+k\right)f(s)ds$

$=\int_{-\infty}^{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}\left(\frac{x_i(x_i+X_{-i})^\theta}{\alpha+(x_i+X_{-i})^\theta}\right)sf(s)ds+\left(k-x_i+\frac{x_i\alpha R_0}{\alpha+(x_i+X_{-i})^\theta}\right)\int_{-\infty}^{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}f(s)ds$

and

$$\int_{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}^{\infty}\left(\frac{\alpha R_0+s\cdot(x_i+X_{-i})^\theta}{\alpha+(x_i+X_{-i})^\theta}\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)+\left(k-\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)\right)\right)f(s)ds$$
$$=\int_{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}^{\infty}\left(\frac{(x_i+X_{-i})^\theta}{\alpha+(x_i+X_{-i})^\theta}\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)\right)sf(s)ds$$
$$+\left(k-\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)+\frac{\alpha R_0\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)}{\alpha+(x_i+X_{-i})^\theta}\right)\int_{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}^{\infty}f(s)ds$$

(1) The coefficient on $\int f(s)ds$

$$\int_{-\infty}^{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}f(s)ds=\Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right)$$

$$\left(k-x_i+\frac{x_i\alpha R_0}{\alpha+(x_i+X_{-i})^\theta}\right)\int_{-\infty}^{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}f(s)ds$$
$$+\left(k-\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)+\frac{\alpha R_0\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)}{\alpha+(x_i+X_{-i})^\theta}\right)\int_{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}^{\infty}f(s)ds$$
$$=\left(k-x_i+\frac{x_i\alpha R_0}{\alpha+(x_i+X_{-i})^\theta}\right)\Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right)$$
$$+\left(k-\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)+\frac{\alpha R_0\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)}{\alpha+(x_i+X_{-i})^\theta}\right)\left(1-\Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right)\right)$$
$$=k-\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)+\frac{\alpha R_0\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)}{\alpha+(x_i+X_{-i})^\theta}+\Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right)\left(\frac{N-X}{M}-\frac{N-X}{M}\frac{\alpha R_0}{\alpha+(x_i+X_{-i})^\theta}\right)$$

(2) The coefficient on $\int sf(s)ds$

$$\int_{-\infty}^{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}\left(\frac{x_i(x_i+X_{-i})^\theta}{\alpha+(x_i+X_{-i})^\theta}\right)sf(s)ds+\int_{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}^{\infty}\left(\frac{(x_i+X_{-i})^\theta}{\alpha+(x_i+X_{-i})^\theta}\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)\right)sf(s)ds$$
$$=\frac{X^\theta}{\alpha+X^\theta}\left(x_i\int_{-\infty}^{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}sf(s)ds+\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)\int_{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}}^{\infty}sf(s)ds\right)$$
$$=\frac{X^\theta}{\alpha+X^\theta}\frac{\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)R_0}{1-\lambda}+\frac{X^\theta}{\alpha+X^\theta}\frac{\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)-x_i}{1-\lambda}\sqrt{(X^\theta)^{-1}+(\alpha_0)^{-1}}\phi\left(\frac{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}-R_0}{\sqrt{(X^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$$
$$-\frac{X^\theta}{\alpha+X^\theta}\frac{\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)-x_i}{1-\lambda}R_0\Phi\left(\frac{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}-R_0}{\sqrt{(X^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$$

(3) The final integration result

$$=k-\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)+\frac{\alpha R_0\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)}{\alpha+(x_i+X_{-i})^\theta}+\frac{X^\theta}{\alpha+X^\theta}\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)R_0$$
$$+\Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right)\left(\begin{array}{c}\frac{N-X}{M}\left(\frac{\alpha+(x_i+X_{-i})^\theta-\alpha R_0}{\alpha+(x_i+X_{-i})^\theta}\right)\\ -\frac{X^\theta}{\alpha+X^\theta}\left(\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)-x_i\right)R_0\end{array}\right)$$
$$+\frac{X^\theta}{\alpha+X^\theta}\left(\left(\frac{M-1}{M}x_i+\frac{N-X_{-i}}{M}\right)-x_i\right)\sqrt{(X^\theta)^{-1}+(\alpha_0)^{-1}}\phi\left(\frac{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}-R_0}{\sqrt{(X^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$$

and it can be simplifed to the following optimization problem,

$$\max_x \left(\begin{array}{c} k + (R_0 - 1)\left(\frac{M-1}{M}x_i + \frac{N-X_{-i}}{M}\right) \\ +\frac{N-x_i-X_{-i}}{M}(1-R_0)\,\Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right) \\ +\frac{N-x_i-X_{-i}}{M}\frac{(x_i+X_{-i})^{\frac{\theta}{2}}}{\alpha^{\frac{1}{2}}\left(\alpha+(x_i+X_{-i})^\theta\right)^{\frac{1}{2}}}\phi\left(\frac{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}-R_0}{\sqrt{(X^\theta)^{-1}+(\alpha_0)^{-1}}}\right) \end{array} \right)$$

Taking First-order Conditions with respect to x

$$\frac{d}{dx}\left\{(1-R_0)\frac{\left((x_i+X_{-i})^\theta+\alpha_0\right)^{\frac{1}{2}}}{(x_i+X_{-i})^{\frac{\theta}{2}}}\sqrt{\alpha_0}\right\} = \frac{R-1}{2}\frac{\alpha^{\frac{3}{2}}\theta}{(x+X_{-i})^{\frac{1}{2}\theta+1}\sqrt{\alpha+(x+X_{-i})^\theta}}$$

(i) Part 1 derivative

derivative for $\frac{N-x_i-X_{-i}}{M}(1-R_0)\,\Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right)$

$$= \frac{R_0-1}{M}\Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right) - \frac{N-x_i-X_{-i}}{M}\frac{(R-1)^2}{2}\frac{\alpha^{\frac{3}{2}}\theta}{(x+X_{-i})^{\frac{1}{2}\theta+1}\sqrt{\alpha+(x+X_{-i})^\theta}}\phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right)$$

(ii) Part 2 derivative

derivative for $\frac{N-x_i-X_{-i}}{M}\frac{(x_i+X_{-i})^{\frac{\theta}{2}}}{\alpha^{\frac{1}{2}}\left(\alpha+(x_i+X_{-i})^\theta\right)^{\frac{1}{2}}}\phi\left(\frac{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}-R_0}{\sqrt{(X^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$

$$= -\frac{(X_{-i}+x)^{\frac{1}{2}\theta-1}}{2\left(\alpha+(X_{-i}+x)^\theta\right)^{\frac{3}{2}}M\alpha^{\frac{1}{2}}}\left(\begin{array}{c} 2X_{-i}\left(X_{-i}+x\right)^\theta + 2x\left(X_{-i}+x\right)^\theta \\ +2X_{-i}\alpha + 2x\alpha - N\theta\alpha \\ +X_{-i}\theta\alpha + x\theta\alpha \end{array}\right)\phi\left(\frac{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}-R_0}{\sqrt{(X^\theta)^{-1}+(\alpha_0)^{-1}}}\right)$$

$$-\frac{1}{2M}\theta\frac{\alpha^{\frac{3}{2}}}{\sqrt{\alpha+(X_{-}+x)^\theta}}\frac{(R-1)^2}{(X_{-}+x)^{\frac{1}{2}\theta+1}}\left(X_{-i}-N+x\right)\phi\left((1-R_0)\sqrt{\alpha_0}\frac{\left((x_i+X_{-i})^\theta+\alpha_0\right)^{\frac{1}{2}}}{(x_i+X_{-i})^{\frac{\theta}{2}}}\right)$$

(iii) The final FOC result with respect to x

$$M-1+\Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right) = \frac{(x_i+X_{-i})^{\frac{1}{2}\theta-1}}{2\left(R_0-1\right)\left(\alpha_0+(x_i+X_{-i})^\theta\right)^{\frac{3}{2}}\alpha_0^{\frac{1}{2}}}$$

$$\times\left(\begin{array}{c} 2\left(x_i+X_{-i}\right)^{\theta+1}+2\alpha_0(x_i+X_{-i}) \\ -N\theta\alpha_0+\theta\alpha_0(x_i+X_{-i}) \end{array}\right)\phi\left(\frac{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}-R_0}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right) \tag{84}$$

■

13.7 Proof of Proposition 4

from the Best-response function of x derived in Proposition 2,

$$M-1+\Phi\left(\frac{\frac{(1-R_0)\left[\alpha_0+(x_i+X_{-i})^\theta\right]}{(x_i+X_{-i})^\theta}}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right)$$

$$= \frac{(x_i+X_{-i})^{\frac{1}{2}\theta-1}}{2(R_0-1)\left(\alpha_0+(x_i+X_{-i})^\theta\right)^{\frac{3}{2}}\alpha_0^{\frac{1}{2}}}\left(\begin{array}{c} 2\left(x_i+X_{-i}\right)^{\theta+1}+2\alpha_0(x_i+X_{-i}) \\ -N\theta\alpha_0+\theta\alpha_0(x_i+X_{-i}) \end{array}\right)\phi\left(\frac{\frac{\alpha_0+(x_i+X_{-i})^\theta-\alpha R_0}{(x_i+X_{-i})^\theta}-R_0}{\sqrt{(x_i+X_{-i})^{-\theta}+(\alpha_0)^{-1}}}\right)$$

for the symmetric equilibrium $x = x_i = \frac{X_{-i}}{M-1}$
then the final M-Investor symmetric equilibrium is,

$(M-1) + \Phi\left(\frac{(1-R_0)\sqrt{\alpha_0}\left((Mx)^{\theta}+\alpha_0\right)^{\frac{1}{2}}}{(Mx)^{\frac{\theta}{2}}}\right)$
$= \frac{1}{R_0-1}\frac{1}{2}\frac{(Mx)^{\frac{1}{2}\theta-1}}{\alpha_0^{\frac{1}{2}}\left(\alpha_0+(Mx)^{\theta}\right)^{\frac{3}{2}}}\left(2Mx\alpha_0 + 2\left(Mx\right)^{\theta+1} - N\theta\alpha_0 + \theta\alpha_0 Mx\right)\phi\left(\frac{(1-R_0)\sqrt{\alpha_0}\left((Mx)^{\theta}+\alpha_0\right)^{\frac{1}{2}}}{(Mx)^{\frac{\theta}{2}}}\right)$
This can be simplifed to (85),

$$(M-1)+\Phi\left((1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}}\right) = \phi\left((1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}}\right)\times\frac{1}{(1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}}}\left[\frac{\theta\alpha\left(N-Mx\right)}{2Mx\left(\alpha+(Mx)^{\theta}\right)} - 1\right] \tag{85}$$

13.7.1 Uniqueness

denote $G\left(x\right) = (1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}}$
$\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}} = \sqrt{\frac{\alpha_0^2}{(Mx)^{\theta}}+\alpha_0}$ is always a decreasing function of x, when $x \geq 1$
Equation (85) can be rearranged to be,

$$\underbrace{\frac{(M-1)\left[(1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}}\right]}{\phi\left((1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}}\right)}}_{\text{LHS1}} + \underbrace{\frac{\left[(1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}}\right]\Phi\left((1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}}\right)}{\phi\left((1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}}\right)}}_{\text{LHS2}} = \underbrace{\left[\frac{\theta\alpha_0\left(N-Mx\right)}{2Mx\left(\alpha_0+(Mx)^{\theta}\right)} - 1\right]}_{\text{RHS}}$$

and further simplified to (86),

$$\underbrace{\frac{(M-1)\,G}{\phi\left(G\right)}}_{\text{LHS1}} + \underbrace{\frac{G\Phi\left(G\right)}{\phi\left(G\right)}}_{\text{LHS2}} = \underbrace{\left[\frac{\theta\alpha_0\left(N-Mx\right)}{2Mx\left(\alpha_0+(Mx)^{\theta}\right)} - 1\right]}_{\text{RHS}} \tag{86}$$

Monotonicity of $\frac{G}{\phi(G)}$
$\frac{d\left(\frac{G}{\phi(G)}\right)}{dx} = \frac{1}{\phi(G)} - \frac{G}{(\phi(G))^2}\left[-G\phi\left(G\right)\right] = \frac{1}{\phi(G)} + \frac{G^2}{\phi(G)} = \frac{1+G^2}{\phi(G)} > 0$
$\frac{G}{\phi(G)}$ *is always an increasing function of* G, and does not depend on the sign of G.
Monotonicity of $\frac{G\Phi(G)}{\phi(G)}$
As proved before, generally $\frac{G\cdot\Phi(G)}{\phi(G)}$ is *an increasing function of* G.
Monotonicity of RHS
RHS $=\frac{\theta\alpha_0(N-Mx)}{2Mx\left(\alpha_0+(Mx)^{\theta}\right)}$is always decreasing in x for $x > 0$ and $x < N$.
When $R_0 > 1$,$(1-R_0) < 0$,
$G\left(x\right) = (1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^{\theta}\right)}{(Mx)^{\theta}}}$ is an increasing function of x;
so $LHS = LHS1 + LHS2$ *is increasing functions of x.*
RHS *is decreasing function of x.* There is a unique solution at their intersection.
When $R_0 < 1$,$(1-R_0) > 0$,

$G(x) = (1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^\theta\right)}{(Mx)^\theta}}$ is an decreasing function of x;

so *LHS is a decreasing function of x*, and *RHS is also a decreasing function of x.*

However, the slope of the RHS is larger than the LHS when x grows large enough. The LHS and RHS curve will cross once and only once, and (80) has a unique solution x.

■

13.8 Proof of Proposition 5

The Threshold R_0

$$\underbrace{\frac{(M-1)\left[(1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^\theta\right)}{(Mx)^\theta}}\right]}{\phi\left((1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^\theta\right)}{(Mx)^\theta}}\right)}}_{\text{LHS1}}+\underbrace{\frac{\left[(1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^\theta\right)}{(Mx)^\theta}}\right]\Phi\left((1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^\theta\right)}{(Mx)^\theta}}\right)}{\phi\left((1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^\theta\right)}{(Mx)^\theta}}\right)}}_{\text{LHS2}}=\underbrace{\left[\frac{\theta\alpha_0\left(N-Mx\right)}{2Mx\left(\alpha_0+(Mx)^\theta\right)}-1\right]}_{\text{RHS}}$$

denote $G(x) = (1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^\theta\right)}{(Mx)^\theta}}$,then we have,

$$\underbrace{\frac{(M-1)\,G}{\phi(G)}}_{\text{LHS1}}+\underbrace{\frac{G\Phi(G)}{\phi(G)}}_{\text{LHS2}}=\underbrace{\left[\frac{\theta\alpha_0\left(N-Mx\right)}{2Mx\left(\alpha_0+(Mx)^\theta\right)}-1\right]}_{\text{RHS}}$$

When $R_0 > 1$,$(1-R_0) < 0$,

As have been proved before, $LHS1, LHS2$ a*re increasing functions of x, and $LHS = LHS1 + LHS2$ is also an increasing function of x.*

But $G(x) = (1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^\theta\right)}{(Mx)^\theta}} < 0$, and $LHS1 < 0$.

$LHS2$ is an increasing functions of x; adding $LHS1$ to it will shift the original $LHS2$ curve downwards.

Compare to the social planner's solution x, the equilibrium x̂ will be larger, and there will be overinvestment.

When $R_0 < 1$,$(1-R_0) > 0$,

$G(x) = (1-R_0)\sqrt{\frac{\alpha_0\left(\alpha_0+(Mx)^\theta\right)}{(Mx)^\theta}} > 0$, and it is an decreasing function of x;

so *LHS is a decreasing function of x*, and *RHS is also a decreasing function of x.*

However, the slope of the RHS is larger than the LHS when x grows large enough. The LHS and RHS curve will cross once and only once, and (80) has a unique solution x. RHS curve will cross the LHS curve from above once and only once.

$LHS1 > 0$. So adding $LHS1$ to $LHS2$ will shift the original $LHS2$ curve up.

This will make the interaction point x̂ between LHS and RHS smaller. That is, there will be underinvestment relative to the social optimal level.

■