Constrained principal components estimation of large approximate factor models

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Abstract

This paper studies the efficient estimation of large approximate factor models with cross-sectional dependence in panels where the number of cross-sections (N) may be larger than the number of observations (T). The traditional method of principal components (PC) achieves consistency for any path of the panel dimensions but it is inefficient as the errors are treated to be homoskedastic and uncorrelated. This paper considers a constrained principal components (Cn-PC) method to efficienctly estimate the factors and their loadings when the errors are cross-correlated. The Cn-PC solves a principal components problem subject to an explicit constraint of bounded cross-sectional dependence as stated in Chamberlain and Rothschild [1983]. The Cn-PC estimators are computationally very tractable. They are equivalent to the PCEs of a regularized form of the data covariance matrix. Unlike maximum likelihood type methods, the Cn-PC estimators do not require inverting a large covariance matrix and are valid for panels with $N \geq T$. The paper derives a convergence rate for the Cn-PC estimators of the common factors and establishes their asymptotic normality. In a Monte Carlo study, the Cn-PC estimators of the factors have good small sample properties in terms of estimation and forecasting performances relative to the existing PC estimators. The Cn-PCEs performs better than the generalized PC type estimators (Choi [2012]) as the panel dimension N approaches T.

Keywords: High dimensionality, unknown factors, principal components, cross-sectional correlation, shrinkage regression, out-of-sample forecasting
JEL Classification: C11, C13, C33, C53, C55

1 Introduction

Factor models constitute the dominant framework across many disciplines for realistic parsimonious representation of the dynamic behavior of large panels of time series.

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Principal components estimators (PCEs) of the common factors can be easily computed in panels where the cross-sectional dimension N is large and possibly larger than the sample size T. PCEs are feasible for any path of the panel dimensions and are consistent for both N and T going to infinity [Forni et al., 2009, 2005, 2004, Bai, 2003, Bai and Ng, 2003, Stock and Watson, 2002a,b]. However, principal components are not efficient in the presence of heteroscedasticity or dependence in the error term. Methods based on maximum likelihood (ML) and generalized principal components (GLS) type methods depend on estimating a high-dimensional covariance matrix, which is a challenging problem in large systems ($N > T$) when errors are dependent and heteroscedastic. The sample covariance matrix behaves optimally if N is fixed and converges to the population covariance at a rate $T^{-1/2}$. However, when $N \to \infty$, the sample covariance matrix can behave very badly and for $N > T$ cannot be inverted. One common solution in the literature is to regularize the covariance matrix. See Fan et al. [2016] for an overview.

This article is related to a large literature on factor models and a much smaller literature on estimation when N is large and the errors are cross-sectionally dependent. Common factors can be consistently estimated using principal components or maximum likelihood (ML). The fundamental result in the literature is that common factors can be consistently estimated for both N and T going to infinity, with no restrictions on the relative rates of convergence and under fairly general conditions on the time and cross-sectional dependence of the errors [Stock and Watson, 1998, 2002a,b, 2006, Bai and Ng, 2002, Bai, 2003, Kapetanios, 2010, Onatski, 2010]. These studies treat the idiosyncratic error components to be homoscedastic and uncorrelated cross-sectionally and over time. In general, although there exist well-established estimation procedures for static factor models, efficiency considerations have only received selective attention in the literature. Boivin and Ng [2006] documented through extensive simulation analysis the potential effects of the presence of dependence on the PC estimators. They find that "Weighting the data by their properties when constructing the factors also lead to improved forecasts" and that with cross-correlated errors the estimated factors may be less useful for forecasting when more series are available. The ML estimation provides a natural framework to account for heteroscedasticity and temporal dependence [Forni et al., 2004, 2009]. Doz et al. [2012] establish the properties of maximum likelihood estimators for factor models in large panels of time series under heteroscedasticity. Breitung and Tenhofen [2011] propose a two-step generalized least squares estimation that generalizes principal components to account for heteroscedasticity and serial correlation in a dynamic factor model with possibly large N. Choi [2012] considers efficient estimation using generalized least squares type PCEs to account for heteroscedasticity and dependence, but the framework is not applicable to panels with $N > T$.

The literature is even more sparse in regards to efficiency considerations in large panels with large N (possibly larger than T) and cross-correlated errors. To our knowledge, Bai and Liao [2016] is the most relevant study to this article. Bai and Liao [2016] propose ML estimation with penalization of a large covariance sparse matrix. The method produces joint estimates of the factors, the loadings and the covariance matrix and is shown to be more efficient than PC estimators (PCEs) or GLS type PCEs. Bai and Liao [2016] paper is related to a growing literature on estimating large covariance matrices [Ledoit and Wolf, 2004, 2012, Lam and Fan, 2009a]. Advances in matrix theory have opened a new line of research into consistent estimation of large matrices. Once a consistent estimate of the covariance matrix is achieved, a GLS type estimation or ML can be implemented in a two-step plug-in estimation approach. Bai and Liao [2016] presented results for both a two-step and a joint estimator. The Sparsity assumption of the errors covariance matrix requires many off-diagonal elements to be zero or nearly zero. This assumption is stronger than the weak cross-correlation of Chamberlain and Rothschild [1983].

The objective of this article is methodological and practical. This article proposes a novel PC-based estimation of factors in systems with large N (possibly larger than T) and where the errors are cross-sectionally dependent. The suggested estimator solves the PCEs problem under a constraint derived from the assumption of bounded dependence in the sense of Chamberlain and Rothschild [1983]. This constrained system can be solved using the method of principal components. The Cn-PCEs are obtained by performing eigenvalue decomposition to a regularized data covariance matrix. The constrained estimation has a dual problem that can be cast as shrinkage estimation, where the regularization is applied to the cross-sectional correlations in the data. The asymptotic properties of the Cn-PCEs of the common factors are derived using the existing techniques of Bai and Ng [2002], Bai [2003] and Choi [2012]. We derive a convergence rate for the Cn-PCEs to the population common factors and show asymptotic normality. The Cn-PCEs are computationally more attractive (than ML-based estimators) because the estimation does not require (i) explicit assumption about the structure of sparsity of the covariance matrix, or (ii) estimating and inverting large covariance matrices.

In small samples, Monte Carlo simulations suggest that the Cn-PCEs have improved accuracy compared to the PCEs and to GLS-type estimators. Applied to the problem of forecasting U.S. inflation and industrial production using the *diffusion indexes* framework of Stock and Watson [2002a], we find relative improvement in accuracy. However, the gains are not substantial and depend on the target series.

The rest of the paper is organized as follows. Section 2 reviews some results of the dynamic factor models and the method of principal components. Section 3 introduces the C-PC estimator and Section 4 establishes asymptotic convergence result and its relative efficiency to PC estimator. The small sample properties of the estimators are compared in Section 5 by means of Monte Carlo simulations. Finally, Section 5 concludes the article. Proofs are deferred to the Appendix.

Notation

The following notation is used throughout the paper: $E(.|Z_t)$ and $E_t(.)$ denote conditional expectation given variables in Z_t and given information at time t respectively, A' denotes the transpose of A, when $A = [a_{i,j}]$ is $q \times p$ matrix, $A' = [a_{j,i}]$ is of dimensions $p \times q$, $A \otimes B$ denotes the Kronecker product of matrices A and B, for $A = [a_{ij}]$ and $B = [b_{ij}]$, $A \otimes B = [a_{ij}B]$, A^{-1} denotes the inverse of a matrix A, ι_m is a m-vector of ones, I_m is an $m \times m$ identity matrix, $\text{diag}(A) = (a_{1,1}, a_{2,2}, ..., a_{n,n})$ when $A = [a_{i,j}]$, by "vector" we mean column vector, for any positive number a, $[a]$ is the largest integer smaller than or equal to a.

2 Econometric Model: Notation and Preliminaries

Let X_{it} be the observed data for the i^{th} cross-section unit at time t $(i = 1, \cdots, N, t = 1, \cdots, T)$. Consider the static factor model representation of the data:

$$X_{it} = \lambda_i' F_t + e_{it}, \tag{2.1}$$

where $F_t = \{F_{kt}\}_{1 \leq k \leq r}$, is an $r \times 1$ vector of common factors, $\lambda_i = \{\lambda_{ik}\}_{1 \leq k \leq r}$ is the corresponding vector of factor loading for cross-section unit i, and e_{it} is an idiosyncratic component.

Let $\underline{X}_1, \cdots, \underline{X}_T$ be observations from the $N-$variate response variable, the factor structure in vector form:

$$\underline{X}_t = \Lambda F_t + \underline{e}_t, t = 1, \cdots, T, \tag{1.1}$$

where F_t is the $r \times 1$ vector of common factors, Λ is an $N \times r$ matrix of factor loadings, $\Lambda = \{\lambda_1', \cdots, \lambda_N'\}$; $\underline{e}_t$ is the $N \times 1$ vector of idiosyncratic component of the model. Let Ψ_N be the covariance matrix for the $N - variate$ response variable, $\Psi_N = \mathbf{E}(\underline{X}_t \underline{X}_t')$. Then the factor structure implies a variance decomposition in the form

$$\Psi_N = \Lambda_N \Omega_F \Lambda_N' + \Omega_N, \tag{1.2}$$

where the subscript N is explicit to show that the factor structure depends on the number of cross-sections. The existence and uniqueness of the approximate factor structure requires that the largest r eigenvalues of Ψ_N are unbounded with respect to N, the remaining eigenvalues are constant (Chamberlain and Rothschild [1983],Brown [1989] and Connor and Korajczyk [1993]). The approximate factor structure of Chamberlain and Rothschild [1983] generalizes the strict factor model which assumes diagonal error covariance Ω_N to allow for a more general covariance structure of the error term allowing for both time and cross-sectional dependence amongst the errors. The correlation between the idiosyncratic components is assumed to be weak both serially and cross-sectionally to allow for identification and estimation of the factor structure. The dimension of the panel in Chamberlain and Rothschild [1983] approximate factor model can be large in both N and T. In fact the high-dimensional property is needed to derive the desirable statistical properties of the estimate of both the factors and the loadings in an approximate factor model. The model assumes weak cross-sectional correlation, which is literally defined at the limit: as the number of variables grows larger, the correlation between these variables becomes smaller. At the limit, when N goes to infinity, the correlation dies out which ensures consistent estimation of the number of factors and the space spanned by the common factors (Stock and Watson [2002a] and Bai and Ng [2002]), and inferential theory (Bai [2003], Bai and Ng [2003]). The consistency result is achieved even if the estimation method doesn't exploit features of the data, such as heterogeneity in the signal to noise ratio, and non-spherical error component.

In matrix notation, the model is written as

$$\mathbf{X} = \mathbf{F}^0\mathbf{\Lambda}^{\mathbf{0}'} + \mathbf{e}, \tag{2.2}$$

where $\mathbf{X} = [\underline{X}_1, \cdots, \underline{X}_T]'$ is the $T \times N$ matrix of observations, $\mathbf{e} = [\underline{e}_1, \cdots, \underline{e}_T]'$ is a $T \times N$ matrix of idiosyncratic errors, $\mathbf{F}^0 = [F_1^0, \cdots, F_T^0]'$ is the $T \times r$ matrix of common factors and $\mathbf{\Lambda}^0 = [\lambda_1^0, \cdots, \lambda_N^0]'$ is $N \times r$ matrix of factor loadings.

The underlying assumptions of the approximate factor structure are standard in the literature. In particular the following assumptions are made (Bai and Ng [2002] and Bai [2003]):

Assumption A1 (Factors). *$E\|F_t^0\|^4 < \infty$ and $T^{-1}\sum_{t=1}^T F_t^0 F_t^{0'} \to \Sigma_F$ as $T \to \infty$ for some positive definite matrix Σ_F.*

Assumption A2 (Factor Loadings). *$\|\lambda_i^0\| < \bar{\lambda} < \infty$, and $\|N^{-1}\sum_{i=1}^T \lambda_i^0 \lambda_i^{0'} - \Sigma_\Lambda\| \to 0$ as $N \to \infty$ for some $r \times r$ positive definite matrix Σ_Λ.*

Assumption A3 (Error term). *There exists a positive constant $M < \infty$, such that for all N and T,*

1. *$E(e_{it}) = 0, E|e_{it}|^8 \leq M$;*

2. *$E(\underline{e}_s'\underline{e}_t/N) = \gamma_N(s,t)$, $|\gamma_N(s,s)| \leq M$ for all s and $T^{-1}\sum_{s=1}^T\sum_{t=1}^T |\gamma_N(s,t)| \leq M$;*

3. *$E(e_{it}e_{jt}) = \tau_{ij,t}$ with $|\tau_{ij,t}| \leq |\tau_{ij}|$ for some τ_{ij} and for all t; in addition,*
$$N^{-1}\sum_{i=1}^N\sum_{j=1}^N |\tau_{ij}| \leq M;$$

4. *$E(e_{it}e_{js}) = \tau_{ij,ts}$ and $(NT)^{-1}\sum_{t=1}^T\sum_{s=1}^T\sum_{i=1}^N\sum_{j=1}^N |\tau_{ij,ts}| \leq M$;*

5. *for every (t,s), $E\left|N^{-1/2}\sum_{i=1}^N [e_{is}e_{it} - E(e_{is}e_{it})]\right|^4 \leq M$*

Assumption A4. *Weak Dependence between Factors and Idiosyncratic Errors:*

$$E\left(\frac{1}{N}\sum_{i=1}^N \left\|\frac{1}{\sqrt{T}}\sum_{t=1}^T F_t^0 e_{it}\right\|\right) \leq M$$

In Assumption A2, the factor loadings are nonrandom. As noted in Bai and Ng [2002], the results can be extended to the case of random factor loadings provided they are independent of the factors and the idiosyncratic errors, and $E\|\lambda_i\|^4 < M$.

This paper uses the usual normalization trick in the PC literature to enable identification of the factor structure of either $\frac{1}{T}\sum_{t=1}^T F_t F_t' = I_r$ or $\frac{1}{N}\sum_{i=1}^N \lambda_i \lambda_i' = I_r$. The model being analyzed in this paper is a static factor model in the sense that $\underline{X}_t$ has a contemporaneous relationship with the factors. Identification of the factor structure requires that the N dimensional population covariance matrix of X, Φ_N has r eigenvalues that diverge as $N \to \infty$.

The only observable quantities are the $\underline{X}_t, t = 1, \cdots, T$. The true factors, their loadings and the idiosyncratic errorsand the loadings λ_i are unknown population parameters and are the subject of this paper estimation approach. The number of factors r is generally unknown and is an important question to be addressed in the framework of factor analysis. Many estimation methods have been developped for the number of common factors in panel data with both large number of cross-section units and time series observations. For static approximate factor models like the one studied in this paper, Bai and Ng [2002](BN hereandafter) proposed a consistent estimator of r by minimizing two model information criteria which depended on both N and T. The BN estimators are consistent and are linked to the eigenvalues of Ψ_N. The finite sample properties of the BN estimators may be sensitive to the choice of the maximum possible number of factors and a prespecified threshhold function, especially in the presence of moderate to strong serial and cross-section correlation. Alessi et al. [2010] revisited the penalty in BN to add a multiplicative tuning constant based on Hallin and Liška's (2007) diverging eigenvalue method for generalized factor models. Examples of other estimators with improved finite sample properties are the "Edge Distribution" estimator of Onatski [2010], the "Eigenvalue Ratio" and "Growth Ratio" of Ahn and Horenstein [2013]. See also Onatski [2009] for inference about r, Forni et al. [2000], Amengual and Watson [2007], and Bai and Ng [2007] for dynamic factor models.

In this paper, the number of factors r is known. An interesting question though is to address the sampling behaviour of the aforementioned methods for estimating r. This is beyond the scope of this article and is left for fututre research.

Under the regularity conditions in Assumptions **A1-A4** (Bai and Ng [2002], Stock and Watson [2002b]), the factors and factor loadings can be consistently estimated as N and T are both large using the method of *asymptotic principal components*, Connor and Korajczyk [1989]. Technically, the principal component estimator minimizes the total sum of squares

$$V(\mathbf{\Lambda}, \mathbf{F}) \quad = \quad \mathrm{tr}\left[(\mathbf{X} - \mathbf{F}\mathbf{\Lambda}')'(\mathbf{X} - \mathbf{F}\mathbf{\Lambda}')\right], \tag{2.3}$$

subject to the normalization $\mathbf{F}'\mathbf{F}/T = I_r$. The estimator has a simple interpretation in terms of the singular value decomposition of the sample covariance of the data. Consider the spectral decomposition of the sample covariance matrix of $\mathbf{X}$, $\Psi_N = \frac{1}{T}\mathbf{X}'\mathbf{X}$:

$$\Psi_N \mathit{\Gamma} = \mathit{\Gamma} \mathit{\Delta},$$

where $\mathit{\Delta} = \mathrm{diag}(d_1, \cdots, d_N)$ is a diagonal matrix with d_l corresponding to the l^{th} highest eigenvalue of Ψ_N, and $\mathit{\Gamma} = (\varphi_1, \cdots, \varphi_N)$ is the matrix whose columns corresponds to the normalized eigenvectors of Ψ_N. The normalized PC estimator of $\mathbf{F}$ are $\widehat{F}_{k,t} = \frac{1}{\sqrt{d_k}} \varphi_k' \underline{X}_t$, for $k = 1, \cdots, r$; De Mol et al. [2008]. The PC estimator for $\mathbf{\Lambda}$ can be computed as OLS projection of $\mathbf{X}$ on the estimated $\hat{\mathbf{F}}$, $\mathbf{\Lambda} = \frac{1}{T}\mathbf{X}'\hat{\mathbf{F}} = \mathit{\Gamma}_{1:r}$. Let us assume that the processes are stationary, abstract from serial correlation and focus only on the role of the cross-sectional dependence assumption. Assumption (**A3.3**) is sufficient and necessary for the asymptotic properties of the PC estimators derived in the literature. Bai and Ng [2002] derive consistency results for the estimated number of factors, and the space spanned by the estimated factors under approximate factor model. Referring to mathematical appendix of Bai and Ng [2002], the result that PC estimated factors $\hat{F}_t$ span (up to an orthogonal rotation H^k) the space of the true factors F_t, ie, $C^2_{NT}\left(\frac{1}{T}\sum_{t=1}^{T}\|\hat{F}^k_t - H^{k'}F_t\|^2\right) = O_p(1)$ at a rate $C_{NT} = min\{\sqrt{N}, \sqrt{T}\}$, requires that $N^{-2}\|e_t'\mathbf{\Lambda}\|^2 = O_p(N^{-1})$. The latter follows from $E\left(T^{-1}\sum_{t=1}^{T}\|N^{-1/2}e_t'\mathbf{\Lambda}\|^2\right) \leq \bar{\lambda}M$, where $\|\lambda_i\| \leq \bar{\lambda} < \infty$, which is a direct implication of Assumption (**A3.3**) as shown in Bai and Ng [2002]' Lemma 1. Further more, this average convergence rate is sufficient for consistency of the estimated number of factors using Bai and Ng [2002] criteria.

Assumption (**A3.3**) doesn't explicitly play a role in the estimation of the factors and the loadings. The PC estimators in the approximate factor model are the same as those estimated in a strict factor model where the error covariance matrix is diagonal and homoscedastic.

A number of studies have shown the importance of deviation from the assumption of spherical e_{it} on the small sample properties of the PC estimators. Boivin and Ng [2006] provide an empirical assessment of the extent of which likely features of the data affect the properties of the PC factors estimates $\hat{\mathbf{F}}$. Their study finds that forecast based on weighted had smaller errors that forecasts based on OLS-PC estimation. Their result points to a need to develop more efficient estimators that fully exploit information in the data. Let us assume for the moment that the errors are independent across time and that the time and cross-sectional dynamics are separable, $E(\underline{e}_t\underline{e}_t') = \Omega$. If Ω is known, a generalized least squares type principal component (GLS-PC) estimator can be constructed by minimizing,

$$V_\Omega(\mathbf{\Lambda}, \mathbf{F}) \;=\; \operatorname{tr}\left[\Omega^{-1}(\mathbf{X} - \mathbf{F}\mathbf{\Lambda}')'(\mathbf{X} - \mathbf{F}\mathbf{\Lambda}')\right]. \tag{2.4}$$

This GLS-PC estimator is studied by Choi [2012] for the case $N < T$ with heteroscedastic errors, where $\Omega = \text{diag}[E(e^2_{1t}), \cdots, E(e^2_{Nt})]$, and with block diagonal cross section dependence with n blocks, where $\Omega = \Omega_1 \bigoplus \Omega_2 \cdots \bigoplus \Omega_n$. Breitung and Tenhofen [2011] considers similar type estimation for dynamic factor models with heteroscedasticity and serial correlation. Estimation requires an estimate for the covariance matrix Ω. Feasible estimators include, the sample covariance matrix $\hat{\Omega}_N = T^{-1}(\mathbf{X} - \hat{\mathbf{F}}\hat{\mathbf{\Lambda}}')'(\mathbf{X} - \hat{\mathbf{F}}\hat{\mathbf{\Lambda}}')$. However, for high-dimensional systems with $N > T$, $\hat{\Omega}_N$ is singular. A candidate estimate for Ω is the sample covariance matrix. However, when $N > T$, $\widehat{\Omega}$ is singular and minimizing $V_{\hat{\Omega}_N}(\mathbf{\Lambda}, \mathbf{F})$ is unfeasible. To overcome inverting a singular matrix, Boivin and Ng [2006] propose a weighting scheme that accounts for heteroscedasticity and cross correlation. The weighting scheme is then applied to the PC estimator minimize

$$V(\lambda_i, F_t, w_{it}) = \sum_{i=1}^{N} w_{iT}\sum_{t=1}^{T}(X_{it} - \lambda_i'F_t)^2, \tag{2.5}$$

where choices of the weights include (i) w_{it} is the inverse of the diagonal element of $\widehat{\Omega}_T$ estimated using data up to time T and, (ii) w_{it} is the inverse of $N^{-1}\sum_{i=1}^{N}|\widehat{\Omega}_T(i,j)|$.

In principle, if an estimator of $\hat{\Omega}^{-1}$ is available, a GLS type PC estimation can be carried out. The random matrix literature has a rich body of work on estimating large dimensional covariance matrices. Some of the results in this literature have been used in the factor model literature. The approach is to estimate a sparse covariance matrix using threshholding or penalized maximum likelihood.

Bai and Liao [2016] apply the estimator principal orthogonal component threshholding estimator of Fan et al. [2013] to derive a two-step estimator, and use a penalized likelihood (Lam and Fan [2009b]) for their proposed joint estimation procedure. In their study, two estimators are proposed. The first is a two-step estimator that minimizes the negative log-likelihood function,

$$-\mathcal{L}_1(\mathbf{\Lambda}, \Omega) = \frac{1}{N}\text{log}|\text{det}(\mathbf{\Lambda}\mathbf{\Lambda}' + \Omega_N)| + \frac{1}{N}\text{tr}\left(S_{\mathbf{X}}(\mathbf{\Lambda}\mathbf{\Lambda}' + \Omega_N)^{-1}\right), \tag{2.6}$$

where $S_{\mathbf{X}}$ is the sample covariance matrix of the data. An estimator of Ω_N is obtained in a first step estimation using threshholding. The second joint estimator they propose is an l_1-penalized maximum likelihood estimator that minimizes,

$$L_2\mathbf{\Lambda}, \Omega) = -\mathcal{L}_1(\mathbf{\Lambda}, \Omega) + \frac{1}{N}\sum_{i\neq j}\mu_T w_{ij}|\Omega_{ij}| \tag{2.7}$$

The second estimator penalizes the off-diagonal elements of the covariance matrix.

3 The Cn-PC Estimator

Let us consider Assumption (**A3.3**) of bounded cross-sectional correlation in an approximate factor structure. The eigenvalues of the error covariance matrix $\Omega = E(\underline{e}_t\underline{e}_t')$ in Chamberlain and Rothschild's (1983) factor model must be bounded. Under the assumption of (covariance) stationarity, $E(e_{it}e_{jt}) = \tau_{ij}$, all the eigenvalues of Ω are bounded by $\max_i \sum_{i=1}^N |\tau_{ij}|$. Thus Assumption (**A3.3**) is implied by the assumption of $\sum_{i=1}^N |\tau_{ij}| \leq M$ for all i and all N, Bai and Ng [2002].

Let sgn(a) denote the spatial sign function with sgn(a) $= |a|/a$ for $a \neq 0$ and sgn(0) $= 1$. Under the assumption of stationarity, the inequality in Assumption (**A3.3**) can be written as:

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\text{sgn}(\tau_{ij})\tau_{ij} \leq M, \tag{3.1}$$

where $\tau_{ij} = E(e_{it}e_{jt})$ and $e_{st} = X_{st} - \lambda_s' F_t$, for $s = i, j$.

In this article, the estimated factors solve an optimization problem that combines the PC objective function and the assumption of bounded cross-sectional correlation. The proposed Cn-PC estimation solves:

$$\underset{\lambda_i, F_t}{\text{minimize}}\ (NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}e_{it}^2 \tag{3.2}$$

$$\text{s.t}\ \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\text{sgn}(\tau_{ij})\tau_{ij} \leq M \tag{3.3}$$

Let $\mathcal{S}$ be $N \times N$ matrix with elements $[\mathcal{S}_{ij}]$ defined as,

$$\begin{aligned} \mathcal{S}_{i,i} &= 0 & (3.4) \\ \mathcal{S}_{i,j} &= \operatorname{sgn}(\tau_{ij}) \text{ for } i \neq j. & (3.5) \end{aligned}$$

The population $\mathcal{S}_{ij}$ for $i \neq j$ are generally not observed. In the following, an unfeasible estimator is derived under the assumption that $\mathcal{S}$ is known. We show below in Lemma1 that a consistent estimator of each element in $\mathcal{S}$ can be obtained. We define a feasible estimator based on an estimated $\mathcal{S}$.

3.1 The unfeasible CnPC estimator

In the following, let us assume that the population $\mathcal{S}_{ij}$ are known. This is unfeasible since in practice the sign of dependence between cross-sections is generally unknown. We argue that in many applications, institutional knowledge and theory may provide information about the direction of co-variation between variables without the knowledge of the strength of the relationship.

Let $\mathcal{L}_1(F, \Lambda) = \frac{1}{T}\sum_{t=1}^{T} \underline{e}_t' \underline{e}_t$ and $\mathcal{L}_2(F, \Lambda) = \frac{1}{NT}\sum_{t=1}^{T} \underline{e}_t' \mathcal{S} \underline{e}_t - M$. The optimization in (3.2)-(3.3) can be written as:

$$\underset{\Lambda, F}{\text{minimize}} \ \{\mathcal{L}_1(\Lambda, F, r) | \mathcal{L}_2(F, \Lambda, r) \leq 0\}, \tag{3.6}$$

under the normalization of either $T^{-1}\sum_{t=1}^{T} F_t F_t' = I_r$, or $N^{-1}\sum_{i=1}^{N} \lambda_i \lambda_i' = I_r$. This optimization problem can be solved using the theorem of Kuhn-Tucker.

Note that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are of magnitude of order N. Let us assume that the number of factors r is known and concentrate on the estimation of F and Λ for a given r. Treating the system in (3.6) as a convex programming problem, the Lagrangian is

$$\mathcal{L}(\Lambda, F, \mu) = \frac{1}{N}\mathcal{L}_1(\Lambda, F) + \mu_{NT}\mathcal{L}_2(F, \Lambda). \tag{3.7}$$

The matrix $\mathcal{S}$ has diagonal elements equal to zero and off diagonal elements that are either 1 or -1. The Lagrangian is similar to that of a shrinkage regression where the cross correlations are shrunk towards zero. The parameter μ_{NT} represents the cost/penalty for deviation of the solution from (3.1) and thus plays the role of a shrinkage factor.

Proposition 1. *The constrained principal component estimator (Cn-PC) for F^0, denoted $\hat{F}$, which solves (3.7) is $\sqrt{T}$ times the matrix consisting of the eigenvectors corresponding to the r largest eigenvalues of the matrix $\mathbf{X}\mathcal{A}_N\mathbf{X}'$, where $\mathcal{A}_N = I_N + \mu_{NT}\mathcal{S}$, where μ_{NT} is the Lagrange multiplier parameter. The Cn-PC estimator for Λ^0, denoted $\hat{\Lambda}$ is given by $\hat{\Lambda} = \frac{1}{T}\mathbf{X}\hat{F}$.*

See Appendix A.

Assumption A5 **(Error term).** *There exists a positive constant $M < \infty$, such that for all N and T,*

1. *let* $E(\underline{e}_s'\mathcal{S}\underline{e}_t/N) = \varrho_N(s,t)$*, then* $\sum_{s=1}^{T} |\varrho_N(s,t)| \leq M$ *for all* t*.*

2. *for every* $t, s,$ *and* N*, assume that* $E\left|N^{-1/2}\left[\underline{e}_s'\mathcal{S}\underline{e}_t - E\left(\underline{e}_s'\mathcal{S}\underline{e}_t\right)\right]\right|^4 \leq M$*;*

3. *for any* t *and* N*, there exists a positive constant* $M < \infty$ *such that* $E\left\|\frac{1}{\sqrt{N}}\Lambda^{0'}\mathcal{S}\underline{e}_t\right\|^2 \leq M$*.*

Theorem 3.1. *For any fixed (known)* $r \geq 1$*, there exists a suitable* $(r \times r)$ *full rank rotation matrix* $\mathcal{H}$ *such that under Assumption A1-A5*

$$\frac{1}{T}\sum_{t=1}^{T}\left\|\hat{F}_t - \mathcal{H}'F_t^0\right\|^2 = O_p(\delta_{NT}^{-2}) + O_p(\mu_{NT}^{-2}\delta_{NT}^{-2}),$$

where $\mathcal{H} = \left(\frac{\Lambda'\mathcal{A}_N\Lambda}{N}\right)\left(\frac{F'\hat{F}}{T}\right)V_{NT}^{-1}$*. Or equivalently,*

$$\omega_{NT}^2\left(\frac{1}{T}\sum_{t=1}^{T}\left\|\hat{F}_t - \mathcal{H}'F_t^0\right\|^2\right) = O_p(1),$$

where $\delta_{NT} = \min\left\{\sqrt{N}, \sqrt{T}\right\}$ *and* $\omega_{NT} = \min\left\{\delta_{NT}, \delta_{NT}\mu_{NT}\right\}$*.*

See Proof in Appendix B.

As in the standard principal components estimation, the true factors F_t^0 are identified only up to a scale. What is considered is the space spanned by the true factors identified by a rotation $\mathcal{H}F_t^0$ of F_t^0. In Theorem 3.1, the time average of squared deviations between the Cn-PC estimator and those that lie in the true factor space goes to zero as $N, T \to \infty$. The rate of convergence depends on the panel structure but also on the regularization factor μ_{NT}.

Note that the Cn-PC estimator are implicit functions of μ_{NT}. When $\mu_{NT} = O(1)$ (equivalent to $h = 0$ in Proposition below) the Cn-PC estimator of $\hat{F}$ is the principal component estimator of the factor space consisting of the eigenvectors corresponding to the r largest eigenvalues of $\mathbf{XX}'/T$ (see for example, Stock and Watson [2002a], Bai and Ng [2002], Bai [2003]). In this case, Theorem 3.1 implies the same rate of convergence as in Bai and Ng [2002] which is equal to δ_{NT} and is determined by the smaller of N or T.

Theorem 3.1 establishes conditions under which the convergence of the Cn-PC estimator is faster/slower than that of the ordinary PCEs.

Proposition 2. *Let* $\mu_{NT} = \delta_{NT}^{-h}$*, then the rate of convergence in Theorem 3.1 is:*

(i) $\omega_{NT}^2 = \delta_{NT}^{2(1-h)}$ *for* $h > 0$*,*

(ii) $\omega_{NT}^2 = \delta_{NT}^2$ *for* $h \leq 0$*.*

In the case of $h > 0$, $\omega_{NT}^2 < \delta_{NT}^2$ and thus the Cn-PC estimator converge (in the sense of Theorem 3.1) to factors that lie in the true factors space at a rate slower than Bai and Ng [2002] ordinary CPEs. The two methods are implying a different rotation matrix $\mathcal{H}$ which means the convergence is towards different rotation of the space spanned by the true factors. Thus the estimated factor spaces are not directly comparable.

Lemma 1. *Assume in addition that $max_{1\leq t\leq T}\sum_{s=1}^{T}\gamma_N(s,t)^2 \leq M$ for some $M < \infty$ uniformly in t, then*

$$\omega_{NT}^2 \left\| \hat{F}_t - \mathcal{H}' F_t^0 \right\|^2 = O_p(1).$$

The proof is similar to that of Theorem 3.1.

3.2 The Feasible Cn-PC estimator

The population $\mathcal{S}_{ij}, i \neq j, i, j = 1, \cdots, N$ are generally unknown. In principle, the information required is an estimate of the direction of association between two cross sections. This doesn't necessarily require estimating the covariance/correlation matrix. Any statistic that measures the ordinal association between $\mathbf{X}_i$ and $\mathbf{X}_j$ functions of the model parameters, $\mathcal{S}_{ij} \equiv \mathcal{S}(\lambda_i, \lambda_j, F_t)$. Each $\mathcal{S}_{ij}$ is a measure of ordinal association between e_{it} and e_{jt}.

An estimate of $\mathcal{S}_{ij}$ can be defined using the estimated parameters of the model,

$$\begin{aligned} \hat{\mathcal{S}}_{ij} &= \mathcal{S}\left(\hat{\lambda}_i, \hat{\lambda}_j, \hat{F}_t\right) && (3.8) \\ &= \operatorname{sgn}\left[\widehat{E}\left(\hat{e}_{it}\hat{e}_{jt}\right)\right] && (3.9) \\ &= \operatorname{sgn}\left[\frac{1}{T}\sum_{t=1}^{T}\left(X_{it} - \hat{\lambda}_i'\hat{F}_t\right)\left(X_{jt} - \hat{\lambda}_j'\hat{F}_t\right)\right] && (3.10) \end{aligned}$$

A pairwise covariance/correlation estimator for τ_{ij} can be computed only from the i^{th} and j^{th} cross sections. This is a fast and better strategy in high-dimensions with possibly sparse systems ($N > T$), Dürre et al. [2015]. Consider the sample moment estimator, $\hat{\tau}_{ij}$, for the population τ_{ij}:

$$\hat{\tau}_{ij} = \frac{1}{T}\sum_{t=1}^{T}\hat{e}_{it}\hat{e}_{jt},$$

where $\hat{e}_{kt} = X_{kt} - \hat{C}_{kt}$, where the common component estimator, $\hat{C}_{kt} = \hat{\lambda}_k'\hat{F}_t$, for $k = 1, \cdots, N$. In order to make Assumption (**A3.3**) operational, the population moments τ_{ij} are replaced by the sample moments $\hat{\tau}_{ij}$, and $\operatorname{sgn}(\tau_{ij})$ by $\operatorname{sgn}(\hat{\tau}_{ij})$.

Lemma 2 (Consistency of τ_{ij} and $\operatorname{sgn}(\hat{\tau}_{ij})$)**.** *Under assumptions A1-A4, as $T, N \to \infty$ we have*

i. $\hat{\tau}_{ij}$ converges to τ_{ij} at a rate $O_p\left(\frac{1}{T^{1/4}}\right) + O_p\left(\frac{1}{\delta_{NT}}\right)$

ii. For $\hat{\tau}_{ij} \neq 0$, plim sgn($\hat{\tau}_{ij}$) $= sgn(\tau_{ij})$

Proof in Appendix 7

3.3 Choosing M

The 'shrinkage' parameter μ_{NT} can be estimated from the objective function $\mathcal{L}(\hat{\mu})$. The system is a function of M, the amount of cross-sectional correlation allowed in the approximate factor structure. M is a tuning parameter that controls the amount of shrinkage that is applied to the estimated τ_{ij}. Clearly, μ_{NT} increases as M decreases. The relationship between μ_{NT} and M is a correspondence and not a function. Although positive values of μ_{NT} correspond to a single value of M, the value $\mu_{NT} = 0$ relates to all M in $\left[\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|E(\hat{e}(0)_{it}\hat{e}(0)_{jt})|, \infty\right)$, $\hat{e}(0)_{it}$ are the residuals from the unconstrained PCA. If the factor structure is strict, then there is no need for shrinkage.

Let $M_0 = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|T^{-1}\sum_{t=1}^{T}\hat{e}(0)_{it}\hat{e}(0)_{jt}\right|$, then values of $M < M_0$ will increase shrinkage and induce more sparsity of the error covariance matrix. The complimentary slackness conditions are used to deduce an estimate $\hat{\mu}_{NT}$ of μ_{NT}. If the constraints are not binding and $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|E(e_{it}e_{jt})| \leq M$, then the constrained maxima are the PC solution $\left(\hat{F}, \hat{\Lambda}, 0\right)$. On the other hand, if $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|E(e_{it}e_{jt})| > M$, then by the complimentary slackness we must have $\hat{\mu} > 0$ and $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|E(e_{it}e_{jt})| = M$

Chamberlain and Rothschild (1983) showed that asset prices have an approximate factor structure if the largest eigenvalue of $\Omega = E(e_te_t')$ is bounded. The largest eigenvalue of Ω is bounded by $\max_i \sum_{i=1}^{N}\|\tau_{ij}\|$, where $\tau_{ij} = E(e_{it}e_{jt})$, (Boivin and Ng [2006]).Under the assumptions of an approximate factor model, there should exist a M such that $\sum_{j=1}^{N}\|\tau_{ij}\| \leq M < \infty$ for all i and N. This assumption is vital in the development of the approximate factor structure theory. However, there is no indication as to how much cross-correlation is permitted in practice. Boivin and Ng [2006] use $\hat{\tau}^* = \max_i\hat{\tau}_i^*/N$, where $\hat{\tau}_i^* = \sum_{j=1}^{N}|T^{-1}\sum_{t=1}^{T}\hat{e}_{it}\hat{e}_{jt}|$ as indicator for M/N, which should be small and decreasing with N. That is, the bounding quantity M is of order $O_p(N)$.

There a correspondence between the tuning parameter μ_{NT} that controls the amount of regularization and the threshold M. If M is greater or equal than the $L_{1,1}-$norm of the PC regression sample covariance matrix, $M_0 = \sum_{j=1}^{N}\sum_{i=1}^{N}|\hat{\tau}_{ij}|$, $\hat{\tau}_{ij} = \sum_{t=1}^{T}\hat{e}_{it}\hat{e}_{jt}/T$, then the PCA estimator is, of course unchanged by the proposed regularization. For smaller values of M, the constrained problem shrinks the estimated cross-sectional correlations towards the origin in the $L_{1,1}$ sense. One-way to calibrate and estimate M is cross-validation. Using a normalized parameter $m = M/M_0$ to index the constrained estimates of F and Λ over a grid of values of s between 0 and 1 inclusive. The value $\hat{m}$ yielding the lowest estimated value for some risk function is selected. The risk can be measured in terms of fit of the factors estimates $\hat{F}$ and/or in terms of prediction error for factor based $h-$steps ahead forecasts. In our analysis, we present the path of solutions indexed by a fraction m of shrinkage factor of M_0.

Penalized principal components regression

A closely related optimization problem to constrained PC regression problem in (3.7) is the constrained regression

$$\underset{\lambda_i,F_t}{\text{minimize}}\ (NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}e_{it}^2+\kappa_{nt}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|E(e_{it}e_{jt})| \tag{3.11}$$

Problems (3.7) and (3.11) are equivalent (Osborne et al. [2000]). For a given $\kappa_{nt}, 0 \leq \kappa_{nt} < \infty$, there exists a $M \geq 0$ such that the two problems share the same solution, and vice versa. In (3.11), the parameter κ_{nt} is easily interpreted as shrinkage/regularization parameter applied to large cross-section correlation parameters. The Lagrange multiplier μ_{NT} is the price of deviation from the bounded cross correlation constraint imposed by the approximate factor structure. The two parameters are exchangeable for all practical purposes.

4 Limiting distributions of constrained principal component estimators

In this section, we study the asymptotic distributions of the proposed constrained PC estimators. In particular, these estimators are compared to the properties of the ordinary PC estimators of Bai [2003] and the generalized PCEs of Choi [2012].

Assumption A6. *Moments and Central Limit Theorem*

1. *for any t, N and T, there exists an $M < \infty$ such that*

$$E\left\|\frac{1}{\sqrt{NT}}\sum_{s=1}^{T}F_s^0\left[\underline{e}_s'\mathcal{S}\underline{e}_t-E\left(\underline{e}_s'\mathcal{S}\underline{e}_t\right)\right]\right\|^2 \leq M;$$

2. *for any N and T, there exists an $M < \infty$ such that*

$$E\left\|\frac{1}{\sqrt{NT}}\sum_{s=1}^{T}\Lambda^{0\prime}\mathcal{S}\underline{e}_s F_s^{0\prime}\right\|^2 \leq M;$$

3. *for each t, as $N \to \infty$,*

$$\frac{1}{\sqrt{N}}\Lambda^{0\prime}\underline{e}_t \xrightarrow{d} N(0,\varPsi_t)$$

where $\varPsi_t = \lim_{N\to\infty}\frac{1}{N}\Lambda^{0\prime}E(\underline{e}_t\underline{e}_t')\Lambda^0$;

4. *for each i, as $T \to \infty$,*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t^0 e_{it} \xrightarrow{d} N(0,\Phi_i),$$

where $\Phi_i = plim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\ E(F_t^0F_t^{0\prime}e_{it}e_{is})$.

Assumption A7 (**Factor Loadings***). *$\|N^{-1}\Lambda^{0\prime}\mathcal{A}_N\Lambda^0 - \Sigma_{\Lambda*}\| \to 0$ as $N \to \infty$ for some $r \times r$ positive definite matrix $\Sigma_{\Lambda*}$.*

Assumption A8. *The eigenvalues of the $r \times r$ matrix $(\Sigma_{\Lambda*} \cdot \Sigma_F)$ are distinct.*

Assumption A9. *The tuning parameter μ_{NT} satisfies:*

1. *$\frac{1}{\delta_{NT}} = o(\mu_{NT})$, $\frac{\sqrt{N}}{\sqrt{T}\delta_{NT}} = o(\mu_{NT})$ and $\mu_{NT} = o(1)$;*
2. *$\mu_{NT}\sum_{i\neq j}|\tau_{ij}| \to 0$.*

Theorem 4.1. *Suppose that Assumptions A1-A7 hold.*

1. *If $\frac{\sqrt{N}}{T\mu_{NT}} \to 0$,*

$$\sqrt{N}\left(\hat{F}_t - \mathcal{H}'F_t^0\right) \xrightarrow{d} N(0, V^{-1/2}\mathcal{Q}\Psi_t\mathcal{Q}'V^{-1/2}). \tag{4.1}$$

Proof in Appendix B.2

Efficiency of Cn-PC estimator

The main motivation of this paper is to improve on the existing estimators in terms of efficiency. The ordinary PCEs have asymptotic distribution (Theorem 1 of Bai and Ng [2003]):

$$\sqrt{N}\left(\hat{F}_t - H'F_t^0\right) \xrightarrow{d} N(0, V_{opc}^{-1/2}Q_{opc}\Psi_t Q_{opc}' V_{opc}^{-1/2}), \tag{4.2}$$

where $Q_{opc} = \Sigma_\Lambda^{-1/2}\Upsilon_{opc}V_{opc}^{1/2}$, Υ_{opc} is eigenvector of $\Sigma_\Lambda^{1/2}\Sigma_F\Sigma_\Lambda^{1/2}$, and $V_{opc} = Q_{opc}\Sigma_\Lambda Q_{opc}'$.

It is not obvious to compare the asymptotic variance covariance matrices in (4.1) and (4.2) because the Cn-PC estimator and PCEs are estimating different objects. These estimators are estimating different rotations of the true factors because H in the PCEs and $\mathcal{H}$ in the Cn-PC estimator are generally different.

Consider the case of a factor structure with one common factor. This is an interesting case where H and $\mathcal{H}$ are identical and equal to the scalar $\Sigma_F^{-1/2}$. In this case, PCEs and Cn-PC estimator are estimating the same object $F_t/\sqrt{\Sigma_F}$. Since in this case (of $r = 1$), $\Upsilon = \Upsilon_{opc} =\equiv 1$, $\mathcal{Q} = Q_{opc} = \Sigma_F^{-1/2}$, then the Cn-PC estimator of F_t^0

$$\hat{F}_t \backsimeq \frac{F_t^0}{\sqrt{\Sigma_F}} + \frac{1}{\sqrt{N}}N\left(0, \frac{1}{\Sigma_F}\Sigma_{\Lambda*}^{-1}\Psi_t\Sigma_{\Lambda*}^{-1}\right) \tag{4.3}$$

and the PCEs have

$$\hat{F}_{t,opc} \backsimeq \frac{F_t^0}{\sqrt{\Sigma_F}} + \frac{1}{\sqrt{N}}N\left(0, \frac{1}{\Sigma_F}\Sigma_\Lambda^{-1}\Psi_t\Sigma_\Lambda^{-1}\right), \tag{4.4}$$

where

$$\Sigma_{\Lambda *} = \Sigma_\Lambda + \mu_{NT} \text{ plim } \frac{\Lambda' \mathcal{S} \Lambda}{N} \geq \Sigma_\Lambda, \tag{4.5}$$

because $\mathcal{S}$ is positive definite and $\mu_{NT} \geq 0$. In this case, the Cn-PC estimator are more efficient than the PCEs with ratio of (asymptotic) variances equal to:

$$\frac{V(\hat{F}_{t,opc})}{V(\hat{F}_t)} = \left(1 + \mu_{NT} \text{ plim } \frac{\Lambda' \mathcal{S} \Lambda}{N}\right)^2.$$

5 Monte Carlo Simulations

5.1 Simulations designs

This section presents the Monte Carlo experiments designed to study the small sample properties of the proposed Cn-PC estimator and their performance relative to the ordinary PCEs in the presence of cross-correlated errors. The experimental design for the Monte Carlo simulation adopts the same covariance structure as in Boivin and Ng [2006]. Let the total number of cross-sections N be divided into three groups of sizes N_1, N_2 and N_3 such as, $N = N_1 + N_2 + N_3$. Let the errors u_{it} be the building blocks for the errors dynamics with $u_{it} \sim N(0,1), i = 1, \cdots, N$, and construct the errors e_{it} where

$$N_1 : e_{it} = \sigma_1 u_{it},$$

$$N_2 : e_{it} = \sigma_2 u_{it},$$

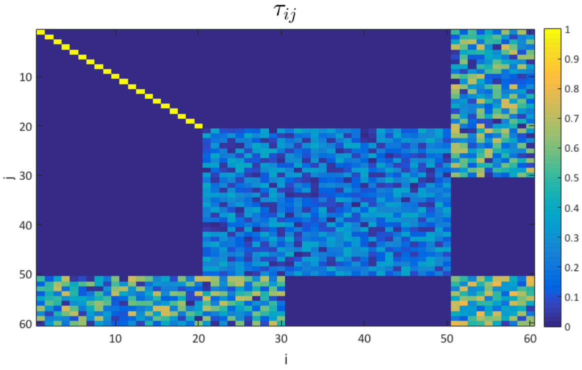
$$N_3 : e_{it} = \sigma_3 \tilde{e}_{it}, \tilde{e}_{it} = u_{it} + \textstyle\sum_{j=1}^{C} \rho_{ij} u_{jt}$$

In this experiment, the errors in the first N_1 series are mutually uncorrelated, the errors in the next N_2 are also mutually uncorrelated but their variance differ from the first series, $\sigma_2^2 > \sigma_1^2$. Cross correlation is introduced in the last N_3 series. The latter are correlated with a proportion C from the N_1 group. The cross correlation matrix Ω_{13} therefore has $C \cdot N_3$ non-zero elements. The correlation coefficients ρ_{ij} denote cross-correlation of series $i \in \{1, N_1\}$ and $j \in \{N_1 + N_2 + 1, N\}$ and is drawn from a uniform distribution $U[0.05, 0.7]$. The error variance in the third group is $\sigma_3^2 = \sigma_1^2$.The error covariance matrix takes the form:

$$\begin{aligned}
\Omega_{ii} &= \sigma_1^2, & 1 &\leq i \leq N_1 \\
\Omega_{ii} &= \sigma_2^2, & N_1 + 1 &\leq i \leq N_1 + N_2 \\
\Omega_{ii} &= \sigma_3^2, & N_1 + N_2 + 1 &\leq i \leq N \\
\Omega_{ij} &= \ 0, & 1 &\leq i, j \leq N_1 + N_2 \\
\Omega_{ij} &= \sigma_1 \sigma_3 \rho_{ij}, & i \leq C, N_1 + N_2 + 1 &\leq j \leq N.
\end{aligned}$$

Figure 1 displays an example of the pattern of dependence structure in the simulation design. The variances are equal to one, and thus the graph represents a sparsity

Figure 1: Sparsity of the errors covariance matrix, $N = 50$



plot for both the covariance and the correlation matrix. In this example, there is clustering of correlations between group 1 and group 3 as well as within group 1 and group 3 series.

The common factors and the loadings are fixed throughout the simulation, which corresponds to analysis conditional on F^0 and Λ^0. The number of factors r is known and fixed. The study considers two cases, r equal to one and two. The panel dimension takes combinations of $T = 50, 100$, and $N = 50, 100, 150$. Data are generated through $X_{it} = \sum_{m=1}^{r} \lambda_{im} F_{mt} + e_{it}$. Our Monte Carlo results are based on $L = 2,000$ repetitions.

For each repetition $l = 1, \cdots, L$, the Monte Carlo experiment is carried out as follows.

(i) Compute the ordinary principal components estimators of $\hat{F}^{(l)}_{OLS-PC}$, $\hat{\Lambda}^{(l)'}_{OLS-PC}$ and the estimated errors $\hat{e}^{l}_{OLS-PC} = \mathbf{X}^{(l)} - \hat{\Lambda}^{(l)'}_{OLS-PC} \hat{F}^{(l)}_{OLS-PC}$. Using the sample covariance matrix $\hat{\Omega}_{OLS-PC} = \hat{e}'\hat{e}/NT$, construct an estimate for sign matrix, $\hat{\mathcal{S}}^{(l)}$.

(ii) Given a value of $M = m \cdot M_0$, where $m \in [0, 1]$, compute $\left(\hat{F}^{(l)}, \hat{\mu}^{l}_{NT}\right)$:

(a) Begin with a starting value $\mu_{NT} = \mu_0$, here we take $\mu_0 = 0.5\sqrt{tr(\hat{e}'\hat{e})/tr(\hat{e}'\mathcal{A}_N \hat{e})}$, and $\mathcal{A}_\mu = I_N - \mu\mathcal{S}$, find the optimal solution to the dual objective function $\mathcal{L}(\mu)$:

$$\hat{\mu}_{NT} = \arg\max_{\mu} (NT)^{-1} \left[\text{tr } \mathbf{X}\mathcal{A}_\mu \mathbf{X}' - \text{tr } \hat{F}'_\mu \mathbf{X}\mathcal{A}_\mu \mathbf{X}' \hat{F}_\mu \right] - M, \tag{5.1}$$

where $\hat{F}_\mu$ is $\sqrt{T}$ times eigenvectors corresponding to the largest r eigenvalues of $\Psi_{N,\mu} = \frac{1}{T}\mathbf{X}'\mathcal{A}_\mu\mathbf{X}$. This is iterated to convergence and to optimal values $\hat{F}^{(l)}, \hat{\mu}^{(l)}_{NT}$.

(b) Compute the Cn-PC estimator for the loadings as a linear projection of $\mathbf{X}$ on $\hat{F}^{(l)}$: $\hat{\Lambda}^{(l)} = \frac{1}{T}\mathbf{X}'\hat{F}^{(l)}$.

(iii) Compute the following measures of performance.

- *Percentage explained variation.* Boivin and Ng [2006] use the percentage of variation in the true factors captured by the estimated structure,

$$S^{(l)}_{\hat{F},F^0} \quad = \quad \frac{\mathrm{tr}\left(F^{0\prime}\hat{F}^{(l)}\left(\hat{F}^{(l)\prime}\hat{F}^{(l)}\right)^{-1}\hat{F}^{(l)\prime}F^0\right)}{\mathrm{tr}(F^{0\prime}F^0)}.$$

- *Small sample bias.* The estimated factors and the true factors are not directly comparable. The estimated factors span a transformation of the true factors. In comparing the small sample bias of the Cn-PC estimator and the benchmark PCEs, one has to account for the differences in the rotation matrices H and $\mathcal{H}$. We compute the small sample bias of the (rotated) factors $\tilde{F}_t \equiv \mathcal{H}^{-1}\hat{F}_t$:

  $$\mathrm{bias}^{(l)} \quad = \quad \frac{1}{L}\sum_{l=1}^{L}\tilde{F}^{(l)}_{tk} - F^0_{tk}, \tag{5.2}$$

  for $k = 1$ and $t = 1, [T/2], T$.

- *Empirical mean squared errors (MSEs).* For each $\hat{F}^{(l)}_t$, we compute

  $$MSEs^{(l)} = r^{-1}\left\|\hat{F}^{(l)}_t - F^{0(l)}_t\right\|^2. \tag{5.3}$$

5.2 The 'Diffusion Index' framework

Consider the forecasting model whereby we are interested in the one h-ahead forecast of a series y_t. The series to be forecasted in both Monte Carlos are generated by

$$y_{t+h} = \beta_0 + \sum_{j=1}^{r}\beta_j F^0_{jt} + \epsilon_{t+h} \equiv y_{F^0,t+h|t} + \epsilon_{t+h},$$

where $\epsilon_t \sim N(0,\sigma^2_\epsilon)$, and σ^2_ϵ is chosen such that the R^2 of the forecasting equation is κ_y. The infeasible diffusion index forecast is $\hat{y}_{F^0,t+h|t}$, which only requires estimation of β. The feasible diffusion index forecast is denoted $\hat{y}_{\hat{F},t+h|t}$, which requires estimation of both the factors and β. A forecast using the observed N series is not feasible if N is large. However, one can use the factor structure of X_{it} in equation (2.1) and use $F^0_t \equiv \{F^0_{jt}\}^r_{j=1}$ to account for the important drivers of common variation in $\mathbf{X}$:

$$\hat{y}_{F^0,t+h|\mathcal{I}_t} = \hat{\beta}_0 + F^{0\prime}_t\hat{\beta}. \tag{5.4}$$

This forecast is unfeasible since the true factors F^0_t are unobserved. Given estimates $\hat{F}_{t,N} \equiv \{\hat{F}_{jt,N}\}^{\hat{r}}_{j=1}$, using the data from the N series and conditional on information at time $\mathcal{I}_t$, a feasible factor augmented forecast, also known as a 'diffusion index' forecast (Stock and Watson [2002a]), is

$$\hat{y}_{\hat{F}_{t,N},t+1|\mathcal{I}_t} = \hat{\beta}_0 + \hat{F}'_{t,N}\hat{\beta}. \tag{5.5}$$

Table 1: Small sample bias and standard errors for the estimated factors $\tilde{F}_{t,1}$, for $t = [T/2], T$ and $r = 1$

		Cn-PC				PCE			
		$\tilde{F}_{[T/2],1}$		$\tilde{F}_{T,1}$		$\tilde{F}_{[T/2],1}$		$\tilde{F}_{T,1}$	
T	N	bias	std	bias	std	bias	std	bias	std
50	50	-0.018	0.190	-0.092	0.163	-0.024	0.037	-0.139	0.097
	100	0.001	0.179	-0.207	0.092	0.033	0.031	-0.025	0.008
	150	-0.155	0.136	0.293	0.137	-0.211	0.103	0.353	0.166
100	50	-0.004	0.118	0.008	0.092	-0.046	0.025	0.121	0.015
	100	-0.128	0.102	-0.109	0.106	-0.120	0.079	-0.114	0.054
	150	-0.168	0.105	-0.049	0.115	-0.126	0.063	-0.013	0.053
150	50	-0.007	0.089	0.026	0.078	-0.032	0.053	0.070	0.081
	100	0.018	0.097	-0.113	0.086	0.054	0.022	-0.180	0.032
	150	0.093	0.062	0.031	0.065	-0.000	0.020	-0.065	0.021

The results are for the sampling distribution of $\tilde{F}_t = \mathcal{J}^{-1}\hat{F}_t$, $\mathcal{J} = \mathcal{H}$ for Cn-PC and $\mathcal{J} = H$ for PCE. The shrinkage factor M is chosen by a 10−fold cross-validation.

The feasible 'diffusion index' forecast requires the estimation of both F_t and β and thus depends on the properties of the 'generated' regressors $\hat{F}_{t,N}$.

We compute the empirical mean-squared-forecast errors (MSFE) and, Boivin and Ng [2006]

$$MSFE_{\hat{y}^{\hat{F}},\hat{y}^{F^0}} = \frac{1}{J}\sum_{t=T}^{T+J-1}\left(\hat{y}_{F^0,t+1|t} - \hat{y}_{\hat{F},t+1|t}\right)^2 \tag{5.6}$$

$$S_{\hat{\beta},\beta} = \frac{1}{J}\sum_{t=T}^{T+J-1}\left(y_{\hat{F},t+1|t} - \hat{y}_{\hat{F},t+1|t}\right)^2 \tag{5.7}$$

The statistic(5.6) measures the loss in forecast accuracy due to F_t being unobserved and estimated. If the estimated factors are consistent and span the same space as the true factors, the difference in forecasting performance of the two predictors $\hat{F}_{t,N}$ and F_t^0 should be negligible and $S_{\hat{y}^{\hat{F}},\hat{y}^{F^0}}$ close to one. The larger is $S_{\hat{y}^{\hat{F}},\hat{y}^{F^0}}$, the closer are the 'diffusion index' forecasts to those generated by the (infeasible) forecasts based on observed factors. The statistic in (5.7) assesses the accuracy of the 'diffusion index' forecasts relative to the conditional mean forecasts which requires only estimation of F_t. Smaller values of $S_{\hat{\beta},\beta}$ are desirable.

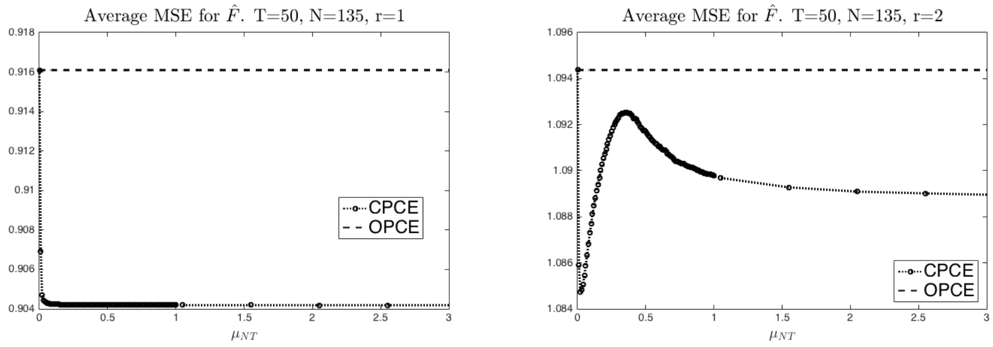
A pseudo-out-of-sample forecasting experiment with 10 years rolling window corresponds to $T = 120$ and is carried out for 10 years into the future, $J = 120$. The panel size in this experiment are $T = 120$ and $N = 131$ to reflect the panel dimensions commonly used in macroeconomic forecasting.

5.3 Simulation results

Fixed M

Table 1 reports the small sample bias and sample standard deviation of the estimated factors $\tilde{F}_{tj}$ for $j = 1$. For the sake of brevity, the results are computed for t set to equal

Figure 2: Accuracy of the Cn-PC estimators of common factors: Empirical MSEs



Note: The MSEs are computed for the rotated estimated factor matrix $\tilde{F}_t = \mathcal{J}\hat{F}_t$. The shrinkage factor M is equal to M_0 for which the constrained problem has the same solution as its dual penalized PC regression.

$[T/2]$ and T. The number of factors in this design experiment is $r = 1$. Note that in all of Monte Carlo results, the number of factors r is assumed to be known. The threshold $M = \overline{M}$ is selected with 10-fold cross validation. Results show that, overall, the proposed constrained estimators (Cn-PC estimator) have smaller bias compared to the ordinary principal components estimators (PCEs). The sample standard deviation of the Cn-PC estimator is larger than those of the PCEs for panel dimensions considered in the experiment.

Figure 2 displays the sample MSEs (5.3) for the rotated factors matrix $\widetilde{F}_t$, estimated using the Cn-PC over a grid of values for the regularization parameter μ_{NT} and for a given M. The results shown are equivalent to the penalized PC estimator that solves (3.11). Results for the PCEs are displayed in dashed line. The left panel is for the case with one true factor and the right panel is for the case of two factors in the population model. As expected, the proposed technique with $\mu_{NT} = 0$ gives the same factors' accuracy in terms of MSEs as the standard principal components method. As the penalization increases, the MSEs for model with $r = 1$ decrease sharply. For $r = 2$ DGP, the MSEs of $\tilde{F}_t$ also reaches a stable value after some dynamics for small μ_{NT}. The relationship between MSE and μ_{NT} is not monotonic.

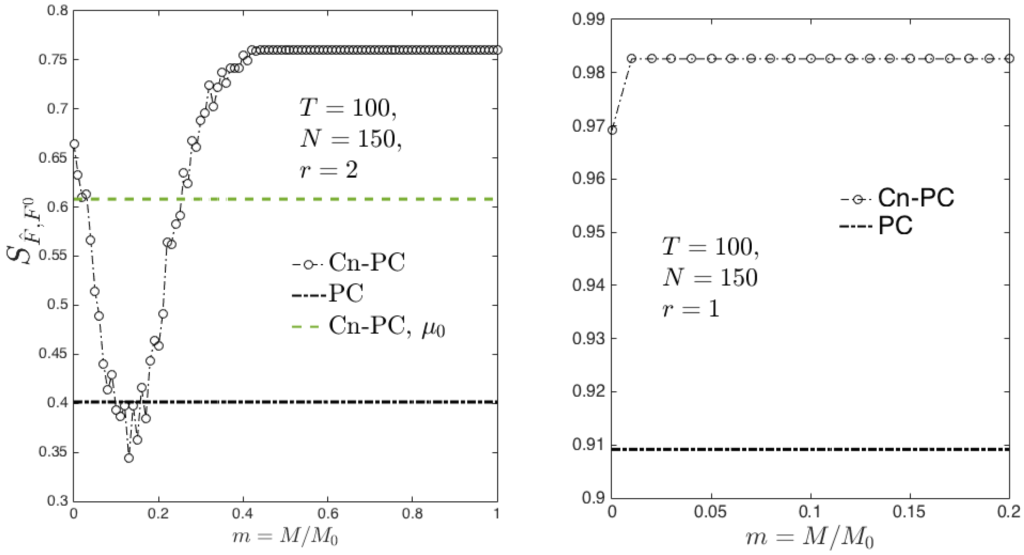
Figure (4) displays the MSEs for the Cn-PC estimates of the common factors F (right panel) and the common components $\hat{C}$ (left panel). It is observed that the GLS-PC delivers a 22% decrease in the MSE of the estimated factors. The Cn-PC estimator proves more accurate than the GLS-PC, with potential gains in MSE ranging from 5% to 35%, depending on M.

Consider the case of the common factors in the left panel. Accuracy of the Cn-PC, as measured by the MSE, increases as N becomes large. This is not the case for the PC estimators where the gains in forecasting accuracy are very small.

5.3.1 M indexed path

Figure 3 displays the path of the statistic $S_{\hat{F},F^0}$ indexed by $m = M/M_0$. Note that in this experiment, the Cn-PC estimator $\hat{F}_t$ and $\hat{\mu}_{NT}$ are jointly estimated. The Cn-PC

Figure 3: Accuracy of Cn-PC estimators of common factors $\hat{F}$: $S_{\hat{F},F^0}$



estimator results are shown in circle-dot dashed line and the PCEs in bold dashed line. The left panel plots the results for $r = 2$ and the right panel for one factor model, $r = 1$. The right panel shows that the PCEs is doing a pretty good job with statistic values in the 90% range. However, there is also a clear advantage of the Cn-PC estimator with values that range from 0.97 to 0.98. For the two-factors model on the left side of the figure, the estimated factors span less perfectly the true space of the true factors. The explained variation in the true factors for the PCEs is low in the 40% range. The Cn-PC estimator improves the ability of the factors estimates to span the factor space with values reaching 0.75. The plot suggests that the relation between M and $S_{\hat{F},F}$ is not monotonic. There are some values of M for which the Cn-PC estimators do slightly worse than the standard PC.

These plots display how the estimated statistics are affected by the model threshold selection method. These can be used as graphical tools for the selection of the parameter M in a similar way as the *ridge-trace* plot is used in the context of ridge regression [Hoerl et al., 1975]. Such plots provide a visual assessment of the effect on coefficient of the choice of the ridge regularization parameter, thus allowing the analyst to make a more informed decision. The selected M would correspond to the threshold value at which the value of the statistic of interest stabilizes.

In Table 2, the estimator GLS-PC refers to Choi [2012] estimator that uses PCE sample covariance estimator to compute a feasible generalized PC efficient estimator. The OLS-PC are very inaccurate in terms of $S_{\hat{F},F^0}$. The GLS-PC performs better in case of T large and N small. However, as N becomes larger, PC-GLS becomes less accurate. When N is large, GLS-PC performs poorly with S_{F,F^0} considerably lower than the ones for the PC and Cn-PC estimators. The low accuracy of PC-GLS can be explained by the poor accuracy and unstable estimator of the covariance matrix when N is large and close to T.

Figure 4: Empirical mean squared errors (MSEs) of Cn-PC estimators of common factors $\hat{F}$ and common components $\hat{C}$

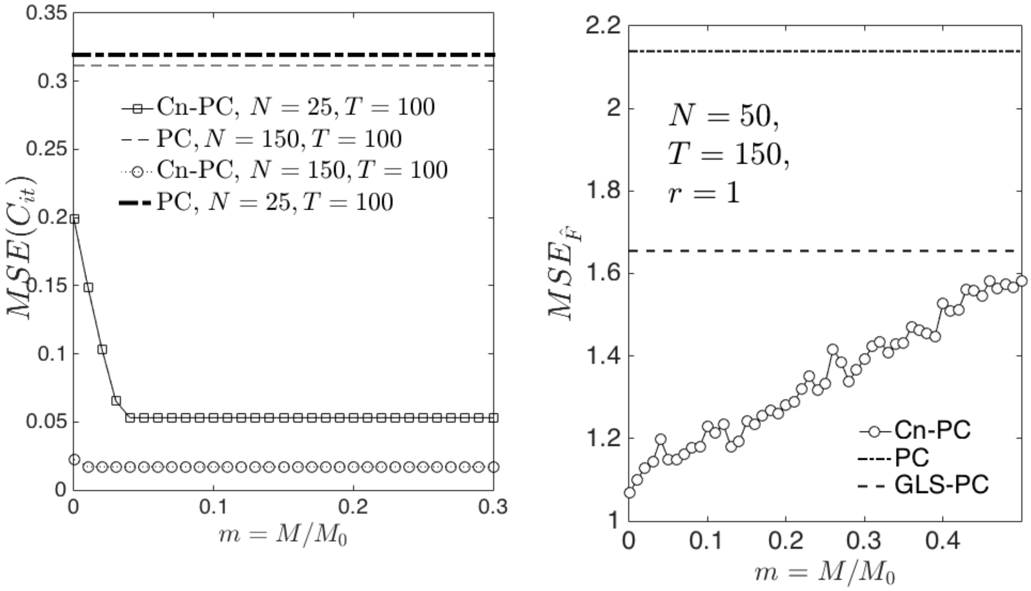


Table 2: Efficiency of estimated common factors: $S_{\hat{F},F^0}$ and $MSE_{\hat{F}}$

		$S_{\hat{F},F^0}$			$MSE_{\hat{F}}$		
T	N	PC	Cn-PC	PC-GLS	PC	Cn-PC	PC-GLS
100	25	0.13	0.319	0.434	2.17	2.16	2.13
	50	0.12	0.382	0.157	1.84	1.76	1.77
150	50	0.10	0.337	0.185	1.78	1.95	1.97
	100	0.10	0.580	0.078	1.83	1.89	1.94
55	50	0.24	0.505	0.072	1.94	1.95	1.75
50	25	0.26	0.341	0.252	1.96	1.36	2.02

Sample correlations

Similar to the use ridge-trace plot which shows graphically the effect of the shrinkage parameter on the coefficients in the linear regression model, one can look at the effect of the threshold M on the elements $|\hat{\tau}_{ij}|$ and the sample cross-section correlations. We use this strategy to select the threshold M for the results in this section.

Figure 5 shows histograms of the sampling distribution of $\hat{\omega}_{ij}$ for a selection of values for i and j. We select cases where $\omega^0_{ij} = 0$ and $\omega^0_{ij} \neq 0$ in the population model. The dotted vertical line marks the true population value. The Cn-PC estimators are shown in the black colored histogram.

In the top two panels, the results show that for the PCEs estimates of the sample correlations, the distribution is almost symmetric around zero and fat tailed. The Cn-PC estimator estimates are much smaller and concentrated around a small average value. This observation is independent of the true population value. This is due to the fact that the Cn-PC estimator is shrinking the average absolute value of these correlations. The shrinkage is not applied to each correlation coefficient.

The Cn-PC estimator's correlations are shrunk relative to the PCEs. This reduction in the size of the correlations is less significant for the case of $N = T = 150$, although the spread is still smaller.

Figure 6 shows the sampling distribution of maximum average cross-correlation $\hat{\tau}^*$. The results show that overall, the estimated $\hat{\tau}^*$ based on Cn-PC estimator are lower than those based on PCEs. In the first panel with $N = 50$ and $T = 100$, $\hat{\tau}^*$ for Cn-PC estimator support ranges from 0.02 to 0.46, while for PCEs the range starts from 0.47 to 0.66.

These results depend on the panel dimension. For $T = N = 150$, the results are less promising, although the distribution of $\hat{\tau}^*$ is skewed to the left, favoring lower values.

Simulated forecasts

Figure 7 displays the statistics $S_{\hat{y},y}$ and $S_{\hat{\beta},\beta_0}$ indexed by $m = M/M_0$. The dotted-dashed circle line plots the results for Cn-PC estimator, while the benchmark PCEs are shown in the straight dashed line. The plot also shows the results for the weighted-PC estimator [Boivin and Ng, 2006], which uses as weights w_{iT} equal to the inverse of $N^{-1}\sum_{j=1}^{N}|\hat{\Omega}_{ij}|$ for each error e_{it} in the PC objective function. The results correspond to a panel with $T = 120$ and $N = 130$, to reflect the panel dimensions that are encountered in macroeconomic forecasting and arbitrage pricing applications. The plots correspond to averages over 1000 replications.

As expected, the weighted-PC estimator outperforms the PCEs with smaller values of $S_{\hat{y},y}$ and $S_{\hat{\beta},\beta_0}$. The shrinkage factor $M = m \cdot M_0$ matters for the performance of the Cn-PC. Unlike the results we have documented earlier with respect to the accuracy of the factors, there is no pattern to the relationship between M and the diffusion index forecasts. But the results show that, for small values of m, the Cn-PC can outperform the weighted-PC by sizable margins.

6 Empirical Example

This section applies the Cn-PC estimator to a forecasting experiment for the U.S. Index of Industrial Production (IPS10) and Consumer Price Index (PUNEW) using the dataset provided by Stock and Watson [2002a]. The data include real variables such as sectoral industrial production, employment and hours worked; nominal variables such as consumer and price indexes, wages, money aggregates; stock prices and exchange rates. The data series are transformed to achieve stationarity: monthly growth rates for real variables (e.g. industrial production, sales) and first differences for variables already expressed in rates (e.g. unemployment rate, capacity utilization). The dataset comprises of monthly observations from 1959:01 to 2003:12 and 131 time series. The sample is divided into an in-sample portion of size $T = 120$ (1959:01 to 1969:12) and an out-of-sample evaluation portion with first date December 1970 and last date December

Table 3: Pseudo-out-of-sample mean squared forecasts errors for US inflation and industrial production

		IPS10				PUNEW			
		$r = 10$		$r = 5$		$r = 10$		$r = 5$	
h=12		PC	Cn-PC	PC	Cn-PC	PC	Cn-PC	PC	Cn-PC
1970-2002	$MSFE$	0.51	0.51	0.52	0.50	0.64	0.62	0.57	0.57
	Var	0.85	0.85	0.66	0.66	0.53	0.53	0.60	0.60
1970-1985	$MSFE$	0.32	0.31	0.31	0.31	0.43	0.40	0.38	0.38
	Var	0.95	0.94	0.75	0.75	0.45	0.45	0.56	0.56
1985-2002	$MSFE$	1.09	1.08	1.13	1.11	1.65	1.63	1.46	1.40
	Var	0.53	0.50	0.39	0.43	0.87	0.85	0.77	0.75

		IPS10				PUNEW			
		$h = 1$		$h = 4$		$h = 1$		$h = 4$	
r=7		PC	Cn-PC	PC	Cn-PC	PC	Cn-PC	PC	Cn-PC
1970-2002	$MSFE$	0.72	0.70	0.57	0.57	0.78	0.75	0.67	0.67
	Var	0.42	0.38	0.56	0.56	0.27	0.27	0.37	0.37
1970-1985	$MSFE$	0.66	0.61	0.49	0.49	0.75	0.71	0.56	0.55
	Var	0.46	0.43	0.56	0.56	0.26	0.25	0.42	0.41
1985-2002	$MSFE$	0.86	0.86	0.86	0.86	0.82	0.82	0.97	0.97
	Var	0.28	0.28	0.54	0.54	0.28	0.28	0.25	0.25

2003. Therefore, there are a total of $M = 397$ out-of-sample evaluation points split into pre- and post-1985 periods with cat-off date December 1984. The models and parameters are re-estimated and the 12-step-ahead forecasts are computed for each month $t = T + 12, \cdots T + 12 + M - 1$ using a rolling window scheme that uses the most recent 10 years of monthly data, that is data indexed $t - 12 - T + 1, \cdots, t - 12$.

In this empirical example, the Cn-PC estimator is computed using a threshold parameter M that is chosen using a ten-fold cross-validation.

Table 3 reports the mean squared forecasts error (MSFE) relative to the random walk and the variance (var) of the forecasts relative to the variance of the series to be forecast. We consider three sample periods and consider different values for the forecast horizon h. The number of factors r is selected using Bai and Ng [2002] information criterion IC_{p_1}, which returns and estimate of $\hat{r} = 7$. We also show results for arbitrary values of $r = 5, 10$.

It is observed that the gains in forecast accuracy depend on the sample period and on the target series. Generally, the gains are not significant and range from 0% to 6% decrease in the pseudo-out-of-sample mean-squared forecast errors.

Consumer price Index forecasts appear to benefit the most from incorporating dependence features using the Cn-PC estimators of the predictors $\hat{F}_t$. These benefits are more appreciable during the period of post moderation of 1985-2002. This is supported by the findings in the literature. During this period, predictability of the price and output series is problematic partly because of the instabilities in the data and of the inflation targeting policy of the Federal Reserve Bank.

7 Conclusion

This paper proposes a novel PC-based method for incorporating the features of cross-correlation in the data in large factor models. The method allows for approximate factor structure in the sense of Chamberlain and Rothschild [1983] and embeds the assumption of bounded cross-sectional dependence as external information in the PC method. This constrained estimation is easily implemented within the classical principal components analysis. The method does not require inverting a large covariance matrix and works through a shrinkage mechanism applied to the sample cross correlations. The Monte Carlo results show that the Cn-PC estimator is generally more efficient than the PC and GLS-PC for large systems. Applied to real data, the results suggest that improvements in the accuracy of the estimated factors do not always lead to improvements into the forecasts accuracy, but that the results depend on the target series and on the forecast horizon.

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Appendix A1: Cn-PC Estimators

The critical points of the function (3.7) are found by solving the first order conditions on the feasible set:

$$(I): \frac{\partial \mathcal{L}(\Lambda, F)}{\partial \Lambda}|_{\hat{\Lambda},\hat{F}} = 0 \tag{7.1}$$

$$(II): \frac{\partial \mathcal{L}(\Lambda, F)}{\partial F}|_{\hat{\Lambda},\hat{F}} = 0 \tag{7.2}$$

$$M \geq (NT)^{-1}\sum_{t=1}^{N} \underline{\hat{e}}_t' \mathcal{S} \underline{\hat{e}}_t, \quad \hat{\mu}_{NT} \geq 0, \quad \hat{\mu}_{NT}\left(M - (NT)^{-1}\sum_{t=1}^{N} \underline{\hat{e}}_t' \mathcal{S} \underline{\hat{e}}_t\right) = 0 \tag{7.3}$$

The conditions in (7.3) are known as the complementary slackness. The first two sets of conditions in 7.1 and 7.2, lead to the following:

$$(I) : \sum_{t=1}^{T} (I_N + \hat{\mu}_{NT}\mathcal{S})\, \underline{\hat{e}}_t \hat{F}_t' = 0 \tag{7.4}$$

$$\hat{\Lambda} = \left(\sum_{t=1}^{T} \underline{X}_t F_t'\right)\left(\sum_{t=1}^{T} F_t F_t'\right)^{-1} \tag{7.5}$$

$$(II) : \sum_{t=1}^{T} \hat{\Lambda}' (I_N + \hat{\mu}_{NT}\mathcal{S})\, \underline{\hat{e}}_t = 0 \tag{7.6}$$

$$\hat{F}_t = \left(\hat{\Lambda}' (I_N + \hat{\mu}_{NT}\mathcal{S})\, \hat{\Lambda}\right)^{-1} \hat{\Lambda}' (I_N + \hat{\mu}_{NT}\mathcal{S})\, \underline{X}_t \tag{7.7}$$

Substituting (7.5) into the Lagrangian and imposing the identification restriction $F'F/T = I_r$, this concentrates out Λ to obtain a reduced Lagrangian that is a function of F and μ:

$$\begin{aligned}
\mathcal{L}(\hat{F}, \hat{\mu}_{NT}, r) &= (NT)^{-1}\sum_{t=1}^{T} \underline{\hat{e}}_t' \underline{\hat{e}}_t - \mu\left[M/N - (N^2T)^{-1}\sum_{t=1}^{N} \underline{\hat{e}}_t' \mathcal{S} \underline{\hat{e}}_t\right] \\
&= (NT)^{-1}\text{trace}\left[\hat{e}\,(I_N + \hat{\mu}_{NT}\mathcal{S})\,\hat{e}\right] - M \\
&= \frac{\text{trace } X\,(I_N + \hat{\mu}_{NT}\mathcal{S})\,X'}{NT} - \frac{\text{trace } \hat{F}'\,(\mathbf{X}\,(I_N + \hat{\mu}_{NT}\mathcal{S})\,\mathbf{X}')\,\hat{F}}{NT} - \hat{\mu}_{NT}M
\end{aligned}$$

For a given $\hat{\mu}_{NT}$, the optimization problem is identical to maximizing trace $F'\left(\frac{\mathbf{X}(I_N+\hat{\mu}_{NT}\mathcal{S})\mathbf{X}'}{T}\right)F$ with respect to F. The estimated factor matrix, denoted by $\hat{F}_{\hat{\mu}_{NT}}$ to the latter problem is the matrix with columns consisting of the principal components of, $\mathbf{X}\,(I_N + \hat{\mu}_{NT}\mathcal{S})\,\mathbf{X}'$. Technically, consider the spectral decomposition of the matrix of,

$$\Psi_{N,\hat{\mu}}' = \mathbf{X}\,(I_N + \hat{\mu}_{NT}\mathcal{S})\,\mathbf{X}',$$

$$\Psi_{N,\hat{\mu}}\mathit{\Gamma}_{\hat{\mu}} = \mathit{\Gamma}_{\hat{\mu}}\mathit{\Delta}_{\hat{\mu}},$$

where $\mathit{\Delta}_{\hat{\mu}} = \text{diag}(d_{1,\hat{\mu}}, \cdots, d_{N,\hat{\mu}})$ is a diagonal matrix with $d_{a,\hat{\mu}}$ corresponding to the a^{th} highest eigenvalue of $\Psi_{N,\hat{\mu}}$, and $\mathit{\Gamma}_{\hat{\mu}} = (\varphi_{1,\hat{\mu}}, \cdots, \varphi_{N,\hat{\mu}})$ is the matrix whose columns corresponds to the normalized eigenvectors of $\Psi_{N,\hat{\mu}}$. The 'normalized' constrained PC estimators (Cn-PC estimator) of $\mathbf{F}(\hat{\mu})$ are $\widehat{F}_{k,t} = \frac{1}{\sqrt{d_{k,\hat{\mu}}}}\varphi_{k,\hat{\mu}}'\underline{X}_t$, for $k = 1, \cdots, r$.

To summarize,

$$\begin{aligned}
\hat{F}_{\hat{\mu}_{NT}} &: \sqrt{T} \times \text{first } r \text{ principal components of } \mathbf{X}\,(I_N + \hat{\mu}_{NT}\mathcal{S}))\mathbf{X}', && (7.8)\\
\hat{\Lambda}_{\hat{\mu}_{NT}} &: \hat{\Lambda}_{\hat{\mu}_{NT}} = \mathbf{X}'\hat{F}_{\hat{\mu}_{NT}}/T, && (7.9)\\
\mu &: M = (NT)^{-1}\sum_{t=1}^{N} \underline{\hat{e}}_t' \mathcal{S} \underline{\hat{e}}_t && (7.10)
\end{aligned}$$

I solve for $(\hat{F}_{\hat{\mu}}, \hat{\mu})$ which minimizes the reduced Lagrangian $\mathcal{L}(F, \mu)$ in (7.8) subject to the constraint $F'F/T = I_r$. The problem can be solved as in the standard primal-dual procedure, whereby the Lagrangian is further concentrated to a reduced function of μ, after replacing F by $\hat{F}(\mu)$. The dual problem solves the maximum of the concentrated objective function, $\mathcal{L}(\mu)$, which is equal to:

$$(NT)^{-1}\left[\text{tr } X\left(I_N + \hat{\mu}_{NT}\mathcal{S}\right)X' - \text{tr } \hat{F}'_{\hat{\mu}_{NT}}\left(\mathbf{X}\left(I_N + \hat{\mu}_{NT}\mathcal{S}\right)\mathbf{X}'\right)\hat{F}_{\hat{\mu}_{NT}}\right] - \hat{\mu}_{NT}M \quad (7.11)$$

APPENDIX B: Proofs of Main results

B.1. Proof of Theorem 1

The main results in this paper can be proven using some of the Lemma's of Bai and Ng [2002] and Bai [2003]. I will therefore omit many of the details that are not worth reporting. In all of the proofs, I assume that the true number of factors (in the population) r is known.

Proof of Theorem 1 Let V_{NT} be an $r \times r$ matrix consisting of the largest eigenvalues of the matrix $\frac{1}{NT}\mathbf{X}\left(I_N + \mu_{NT}\mathcal{S}\right)\mathbf{X}'$ in descending order. Denote $\mathcal{A}_N \equiv I_N + \mu_{NT}\mathcal{S}$. The Cn-PC estimator estimates for the common factors $\hat{F}$ are defined as the eigenvectors corresponding to the largest eigenvalues of the matrix $\mathbf{X}\mathcal{A}_N\mathbf{X}'$ and thus satisfy the relation $\hat{F} = \frac{1}{NT}\mathbf{X}\mathcal{A}_N\mathbf{X}'\hat{F}V_{NT}^{-1}$ by the definition of eigenvalues and eigenvectors. Let the rotation matrix $\mathcal{H} = \left(\frac{\Lambda'\mathcal{A}_N\Lambda}{N}\right)\left(\frac{F'\hat{F}}{T}\right)V_{NT}^{-1}$. Then the following relation originates from Choi [2012] (if $\mathcal{A}_N$ is replaced with Ω^{-1})who generalized the original expressions in Bai [2003] and Bai and Ng [2002] (corresponding to $\mu_{NT} = 0$),

$$\begin{aligned}
\hat{F} - F\mathcal{H} &= \frac{1}{NT}\left(\mathbf{X}\mathcal{A}_N\mathbf{X}'\right)\hat{F}V_{NT}^{-1} - \frac{1}{NT}F\left(\Lambda'\mathcal{A}_N\Lambda\right)F'\hat{F}V_{NT}^{-1} \\
&= \frac{1}{NT}\left(\mathbf{X}\mathcal{A}_N\mathbf{X}' - F\left(\Lambda'\mathcal{A}_N\Lambda\right)F'\right)\hat{F}V_{NT}^{-1} \\
&= \frac{1}{NT}\left(\mathbf{e}\mathcal{A}_N\mathbf{e}' + \mathbf{e}\mathcal{A}_N\Lambda F' + F\Lambda'\mathcal{A}_N\mathbf{e}\right)\hat{F}V_{NT}^{-1}.
\end{aligned}$$

In vector form, the relation becomes,

$$\hat{F}_t - \mathcal{H}'F_t = \frac{1}{NT}V_{NT}^{-1}\hat{F}'\left(\mathbf{e}\mathcal{A}_N\underline{e}_t + F^0\Lambda'\mathcal{A}_N\underline{e}_t + \mathbf{e}\mathcal{A}_N\Lambda F_t^0\right) \quad (7.12a)$$

$$= V_{NT}^{-1}\left(\frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\underline{e}_s'\mathcal{A}_N\underline{e}_t + \frac{1}{NT}\sum_{s=1}^{T}\hat{F}_sF_s^{0'}\Lambda'\mathcal{A}_N\underline{e}_t + \frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\underline{e}_s'\mathcal{A}_N\Lambda F_t^0\right) \quad (7.12b)$$

$$= V_{NT}^{-1}\left(a_{NT,t} + b_{NT,t} + c_{NT,t}\right) \quad (7.12c)$$

Let $\mathcal{A}_{Nj}$ be the j^th column of the matrix $\mathcal{A}_N$ with elements $\mathcal{A}_{N,ij}$, then we can write:

$$a_{NT,t} = \frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\underline{e}_s'\mathcal{A}_N\underline{e}_t \quad = \quad \frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\sum_{j=1}^{N}\sum_{i=1}^{N}e_{is}\mathcal{A}_{N,ij}e_{jt}$$
$$= \quad \frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\left[\sum_{l=1}^{N}\left(e_{ls}e_{lt} + \mu_{NT}\sum_{k\neq l}^{N}\mathcal{S}_{il}e_{is}e_{lt}\right)\right]$$

Note that the latter comes from the fact that the elements of $\mathcal{A}_N = I_N + \mu_{NT}\mathcal{S}$ are equal to $\mathcal{A}_{N,ii} = 1$ for $i = 1,\cdots,N$ and $\mathcal{A}_{N,ij} = \mu_{NT}\mathcal{S}_{ij}$ for $1 \leq i \neq j \leq N$.

$$\begin{aligned}
a_{NT,t} &= \frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\sum_{l=1}^{N}e_{ls}e_{lt} + \mu_{NT}\frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\sum_{k\neq l}^{N}\mathcal{S}_{il}e_{is}e_{lt}\\
&= \frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\sum_{l=1}^{N}e_{ls}e_{lt} + \mu_{NT}\frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\left(\sum_{l=1}^{N}\sum_{i=1}^{N}\mathcal{S}_{il}e_{is}e_{lt} - \sum_{l=1}^{N}e_{ls}e_{lt}\right)\\
&= \frac{1}{NT}\sum_{s=1}^{T}\sum_{l=1}^{N}\hat{F}_se_{ls}e_{lt}(1-\mu_{NT}) + \mu_{NT}\frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\sum_{l=1}^{N}\sum_{i=1}^{N}\mathcal{S}_{il}e_{is}e_{lt}\\
&= \left[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t) + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\varsigma_{st}\right](1-\mu_{NT}) + \mu_{NT}\left[\frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\sum_{l=1}^{N}\sum_{i=1}^{N}\mathcal{S}_{il}e_{is}e_{lt}\right],
\end{aligned}$$

where $\gamma_N(s,t) = E\left(N^{-1}\sum_{i=1}^{N}e_{it}e_{is}\right)$ and $\varsigma_{st} = \frac{\underline{e}_s'\underline{e}_t}{N} - \gamma_N(s,t)$ are defined as in Bai and Ng [2002]. Similarly, we can write:

$$b_{NT,t} \quad = \quad \left[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st}\right](1-\mu_{NT}) + \mu_{NT}\left[\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\left(\frac{\Lambda^{0\prime}(I_N+\mathcal{S})\underline{e}_t}{N}\right),\right]$$

where $\eta_{st} = F_s^{0\prime}\Lambda^{0\prime}\underline{e}_t/N$. The last term $c_{NT,t}$ is equal to $b_{NT,t}$ since, $\underline{e}_s'\Lambda F_t^0/N = \eta_{st}$.

Using the Cauchy-Schwarz inequality, that states $(\sum_{s=1}^{m}z_s)^2 \leq m\sum_{s=1}^{m}z_s^2$, we have

$$\|\hat{F}_t - \mathcal{H}'F_t^0\|^2 \leq 3\left(\|a_{NT,t}\|^2 + \|b_{NT,t}\|^2 + \|c_{NT,t}\|^2\right),$$

and

$$\frac{1}{T}\sum_{t=1}^{T}\|\hat{F}_t - \mathcal{H}'F_t^0\|^2 \leq \frac{3}{T}\sum_{t=1}^{T}\left(\|a_{NT,t}\|^2 + \|b_{NT,t}\|^2 + \|c_{NT,t}\|^2\right).$$

Now

$$\begin{aligned}
\|a_{NT,t}\|^2 \quad &\leq \quad 3(1-\mu_{NT})^2T^{-2}\|\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t)\|^2 + 3(1-\mu_{NT})^2T^{-2}\|\sum_{s=1}^{T}\hat{F}_s\varsigma_{st}\|^2\\
&+ \quad 3\mu_{NT}^2T^{-2}\|\sum_{s=1}^{T}\hat{F}_s\sum_{l=1}^{N}\sum_{i=1}^{N}\mathcal{S}_{il}e_{is}e_{lt}/N\|^2.
\end{aligned}$$

Bai and Ng [2002] in the proof of their Theorem 1 in page 213, show that

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T}\|T^{-1}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t)\|^2 &= O_p\left(T^{-1}\right), \\
T^{-1}\|\sum_{s=1}^{T}\hat{F}_s\varsigma_{st}/T\|^2 &= O_p\left(N^{-1}\right.
\end{aligned}$$

For the last term in $\sum_{t=1}^{T} a_{NT,t}/T$:

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T}\|\sum_{s=1}^{T}\hat{F}_s\sum_{l=1}^{N}\sum_{i=1}^{N}\mathcal{S}_{il}e_{is}e_{lt}/NT\|^2 &= \\
\sum_{s=1}^{T}\hat{F}_s\sum_{l=1}^{N}\sum_{i=1}^{N}\mathcal{S}_{il}e_{is}e_{lt}/NT &= \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_N(s,t)+\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\varrho_N(s,t)
\end{aligned}$$

where

$$\xi_N(s,t)=N^{-1}\underline{e}_s'\mathcal{S}\underline{e}_t-\varrho_N(s,t)$$

Now,

$$\begin{aligned}
\|\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\varrho_N(s,t)\| &\leq \left(\frac{1}{T}\sum_{s=1}^{T}\|\hat{F}_s\|^2\right)^{1/2}\left(\frac{1}{T}\sum_{s=1}^{T}|\varrho_N(s,t)|^2\right)^{1/2} \\
&= O_p(1)\cdot O\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

because of the normalization $\hat{F}'\hat{F}/T=I_r$ and Assumption A5(1). Thus, $\frac{1}{T}\sum_{t=1}^{T}\|\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\varrho_N(s,t)\|^2 = O_p\left(\frac{1}{T}\right)$. Now,

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\|\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_N(s,t)\|^2 &\leq \frac{1}{T}\left(\frac{1}{T}\sum_{s=1}^{T}\|\hat{F}_s\|\right)^{1/2}\left[\frac{1}{T^2}\sum_{s=1}^{T}\sum_{s'=1}^{T}\left(\sum_{t=1}^{T}\xi_N(s,t)\xi_N(s',t)\right)^2\right]^{1/2} \\
&\leq \frac{1}{T}O_p(1)\cdot\frac{T}{N}=O_p\left(\frac{1}{N}\right)
\end{aligned}$$

since $E\left(\sum_{t=1}^{T}\xi_N(s,t)\xi_N(s',t)\right)^2 \leq T^2\ \max_{s,t}\ E\left|\xi_N(s,t)\right|^4$, and from Assumption A5(2),

$$E\left|\xi_N(s,t)\right|^4=\frac{1}{N^2}E\left|N^{-1/2}\left[\underline{e}_s'\mathcal{S}\underline{e}_t-E(\underline{e}_s'\mathcal{S}\underline{e}_t)\right]\right|^4\leq N^{-2}M.$$

To summarize,

$$\frac{1}{T}\sum_{t=1}^{T}\|a_{NT,t}\|^2=\left[O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{N}\right)\right](2\mu_{NT}^2-2\mu_{NT}+1)$$

For $b_{NT,t}$,

$$\frac{1}{T}\sum_{t=1}^{T}\|b_{NT,t}\|^2 \leq \frac{1}{T}\sum_{t=1}^{T}\|\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st}\|^2(1-\mu_{NT})^2+\mu_{NT}^2\frac{1}{T}\sum_{t=1}^{T}\|\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\left(\frac{F_s^{0'}\Lambda^{0'}\mathcal{S}\underline{e}_t}{N}\right)\|^2.$$

The proof of Theorem 1 in Bai and Ng [2002] show that: $\frac{1}{T}\sum_{t=1}^{T}\|\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st}\|^2 = O_p\left(N^{-1}\right)$. Consider the second term,

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\left(\frac{F_s^{0'}\Lambda^{0'}\mathcal{S}\underline{e}_t}{N}\right)\right\|^2 &\leq \frac{1}{NT}\sum_{t=1}^{T}\left[\left(\frac{1}{T}\sum_{s=1}^{T}\|\hat{F}_s\|^2\right)\left(\frac{1}{T}\sum_{s=1}^{T}\|F_s^0\|^2\right)\left\|\frac{\Lambda^{0'}\mathcal{S}\underline{e}_t}{\sqrt{N}}\right\|^2\right]\\ &= O_p\left(\frac{1}{N}\right),\end{aligned}$$

because of Assumption A5(3), and Assumption A1(1). Thus,

$$\frac{1}{T}\sum_{t=1}^{T}\|b_{NT,t}\|^2 = O_p\left(\frac{1}{N}\right)\left[\mu_{NT}^2+(\mu_{NT}-1)^2\right]$$

Combining all the results, we have

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\left\|\hat{F}_t-\mathcal{H}'F_t\right\|^2 &= \left[O_p\left(\frac{1}{T}\right)+O_p\left(\frac{1}{N}\right)\right]\left(2\mu_{NT}^2-2\mu_{NT}+1\right)\\ &= V_{NT}^{-1}\left[O_p\left(\delta_{NT}^{-2}\right)+O_p\left(\mu_{NT}^2\delta_{NT}^{-2}\right)\right].\end{aligned}$$

The last step is to characterize the convergence of the matrix V_{NT},

$$\begin{aligned}\|V_{NT}\| &= \frac{1}{T}\left\|\hat{F}'\left(\mathbf{X}\mathcal{A}_N\mathbf{X}'\right)\hat{F}\right\|\\ &\leq \left\|\hat{F}'\hat{F}/T\right\|\|\mathbf{X}\mathcal{A}_N\mathbf{X}'/N\|\\ &\leq O_p(1)\cdot\mu_N^2\left(\|\mathbf{X}\mathbf{X}'/N\|\,\|\mathbf{X}\mathbf{X}'/N\|\right)\\ &= \mu_N^2 O_p(1).\end{aligned}$$

At last,

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\left\|V_{NT}^{-1}\left(\hat{F}_t-\mathcal{H}'F_t\right)\right\|^2 &\leq \|V_{NT}\|^2\left[\frac{1}{T}\sum_{t=1}^{T}\left\|\hat{F}_t-\mathcal{H}'F_t\right\|^2\right]\\ &\leq \mu_{NT}^{-2}\left[O_p\left(\delta_{NT}^{-2}\right)+O_p\left(\mu_{NT}^2\delta_{NT}^{-2}\right)\right].\end{aligned}$$

and thus,

$$\frac{1}{T}\sum_{t=1}^{T}\left\|\left(\hat{F}_t-\mathcal{H}'F_t\right)\right\|^2 \leq \left[O_p\left(\mu_{NT}^{-2}\delta_{NT}^{-2}\right)+O_p\left(\delta_{NT}^{-2}\right)\right].$$

■

B.2. Proof of Theorem 2

From (7.12),

$$\begin{aligned}\hat{F}_t - \mathcal{H}'F_t &= V_{NT}^{-1}\left(\frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\underline{e}_s'\mathcal{A}_N\underline{e}_t + \frac{1}{NT}\sum_{s=1}^{T}\hat{F}_sF^{0\prime}\Lambda'\mathcal{A}_N\underline{e}_t + \frac{1}{NT}\sum_{s=1}^{T}\hat{F}_s\underline{e}_s'\mathcal{A}_N\Lambda F_t^0\right)\\ &= V_{NT}^{-1}\left(a_{NT,t} + b_{NT,t} + c_{NT,t}\right)\\ &= V_{NT}^{-1}\left[I + \mu_N II\right],\end{aligned}$$

where

$$I = \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\gamma_N(s,t) + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\varsigma_{st} + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\eta_{st} + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\xi_{st},$$

where $\xi_{st} = F_t^{0\prime}\Lambda^0\underline{e}_s/N$, and

$$II = \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\frac{\underline{e}_t'\mathcal{S}\underline{e}_s}{N} + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\left(\frac{F_s^{0\prime}\Lambda^{0\prime}\mathcal{S}\underline{e}_t}{N}\right) + \frac{1}{T}\sum_{s=1}^{T}\hat{F}_s\left(\frac{F_t^{0\prime}\Lambda^{0\prime}\mathcal{S}\underline{e}_s}{N}\right).$$

Lemma 3. *If $\mu_{NT} = O(1)$. Then*

$$\hat{F}_t - \mathcal{H}'F_t = V_{NT}^{-1}\left[O_p\left(\frac{1}{\sqrt{T}\omega_{NT}}\right) + O_p\left(\frac{1}{\sqrt{N}\omega_{NT}}\right) + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{N}\omega_{NT}}\right)\right] \tag{7.13}$$

Lemma 3 follows from the earlier result of Theorem 3.1 and the proof can be carried out in similar way as that of [Bai, 2003, Lemma A.2 pages. 159–160].

The limiting distribution is determined by the dominant term in the expression 7.13 which depends on the panel dimensions and on the tuning parameter.

Lemma 4. *Let $\sqrt{N}/T\mu_{NT} \to 0$. Then under Assumptions A1-A7,*

$$\sqrt{N}\left(\hat{F}_t - \mathcal{H}'F_t\right) = V_{NT}^{-1}\left(\frac{\sum_{s=1}^{T}\hat{F}_sF_s^{0\prime}}{T}\right)\left[\left(\frac{\Lambda^{0\prime}\underline{e}_t}{\sqrt{N}}\right) + \mu_{NT}\left(\frac{\Lambda^{0\prime}\mathcal{S}\underline{e}_t}{\sqrt{N}}\right)\right] + o_p(1) \tag{7.14}$$

We have $\Lambda^{0\prime}\mathcal{S}\underline{e}_t/\sqrt{N} = O_p(1)$ by Assumption 5(3) and $\mu_{NT} = o_p(1)$ by Assumption 7(1) thus

$$\sqrt{N}\left(\hat{F}_t - \mathcal{H}'F_t\right) = V_{NT}^{-1}\left(\frac{\sum_{s=1}^{T}\hat{F}_sF_s^{0\prime}}{T}\right)\left(\frac{\Lambda^{0\prime}\underline{e}_t}{\sqrt{N}}\right) + o_p(1) \tag{7.15}$$

By Assumption A6(3),

$$\left(\frac{\Lambda^{0\prime}\underline{e}_t}{\sqrt{N}}\right) \xrightarrow{d} N(0,\Psi_t).$$

Lemma 5. *Under Assumptions A1-A5,*

(i)

$$V_{NT} = \frac{1}{T}\hat{F}'\left(\frac{\mathbf{X}\mathcal{A}_N\mathbf{X}'}{TN}\right)\hat{F} \xrightarrow{p} V,$$

(ii)

$$\frac{\hat{F}'F^0}{T}\left(\frac{\Lambda^{0'}\mathcal{A}_N\Lambda^0}{N}\right)\frac{\hat{F}'F^0}{T} \xrightarrow{p} V$$

Lemma 6. *Under Assumptions A1-A4 and A7,*

$$plim_{T,N\to\infty}\frac{\hat{F}'F^0}{T} = \mathcal{Q},$$

where $\mathcal{Q}$ is an invertible $r \times r$ matrix given by $\mathcal{Q} = V^{1/2}\varUpsilon\Sigma_{\Lambda*}^{-1/2}$, with V consisting of eigenvalues (in descending order) of $\Sigma_{\Lambda*}\cdot\Sigma_F$ and $\varUpsilon$ is the corresponding matrix of eigenvectors.

Proof. The result in lemma 6 can be proven using the same methods as in the proof of Proposition 1 in Bai [2003]. Key elements of the proof. By Lemma 5(ii) and $\mathbf{X} = F^0\Lambda^{0'} + e$, we have (respectively):

$$\left(\frac{\Lambda^{0'}\mathcal{A}_N\Lambda^0}{N}\right)^{1/2} T^{-1}F^{0'}\left(\frac{\mathbf{X}\mathcal{A}_N\mathbf{X}'}{N}\right)\hat{F} = \left(\frac{\Lambda^{0'}\mathcal{A}_N\Lambda^0}{N}\right)^{1/2}\left(\frac{F^{0'}\hat{F}}{T}\right)V_{NT},$$

and

$$(B_{NT} + d_{NT}R_{NT}^{-1})R_{NT} = R_{NT}V_{NT},$$

where $B_{NT} = \left(\frac{\Lambda^{0'}\mathcal{A}_N\Lambda^0}{N}\right)^{1/2}\left(\frac{F^{0'}F^0}{T}\right)\left(\frac{\Lambda^{0'}\mathcal{A}_N\Lambda^0}{N}\right)^{1/2}$, and $R_{NT} = \left(\frac{\Lambda^{0'}\mathcal{A}_N\Lambda^0}{N}\right)^{1/2}\left(\frac{F^{0'}\hat{F}}{T}\right)$. Let $\varUpsilon_{NT} = R_{NT}V_{NT}^*$ with V_{NT}^* the matrix consisting of diagonal elements of $R_{NT}'R_{NT}$. Then $(B_{NT} + d_{NT}R_{NT}^{-1})\varUpsilon_{NT} = \varUpsilon_{NT}V_{NT}$, which implies that $\varUpsilon_{NT}$ is an eigenvector of $B_{NT} + d_{NT}R_{NT}^{-1}$, and we have

$$\mathcal{Q} = \text{plim}\ \frac{F^{0'}\hat{F}}{T} = \text{plim}\ \left(\frac{\Lambda^{0'}\mathcal{A}_N\Lambda^0}{N}\right)^{-1/2}\varUpsilon_{NT}V_{NT}^* = \Sigma_{\Lambda*}^{-1/2}\varUpsilon V^{1/2},$$

where $\varUpsilon$ is the eigenvectors for the matrix $B = \text{plim}\ B_{NT} + d_{NT}R_{NT}^{-1} = \Sigma_{\Lambda*}^{1/2}\Sigma_F\Sigma_{\Lambda*}^{1/2}$.

Proof of Theorem 7.15 follows due to Lemma 3 and Lemma 5 and we have limiting distribution of $\sqrt{N}\left(\hat{F}_t - \mathcal{H}'F_t^0\right)$ is therefore a $N(0,\Xi_t)$ where

$$\Xi_t = V^{-1}\mathcal{Q}\Psi_t\mathcal{Q}'V^{-1} \tag{7.16}$$

■

B.3. Proof of consistency of $\hat{\mathcal{S}}_{ij}$, Lemma2

The paper proceeds by estimating the elements of $\mathcal{S}$ using pairwise covariance/correlation estimators. The requirements for a fast strategy to compute estimators that are defined in high-dimensions with possibily sparse systems where $N > T$, suggests using pairwise covariance/correlation estimators (Dürre et al. [2015]). Consider an estimator for τ_{ij} given by the sample covariance:

$$\hat{\tau}_{ij} = \frac{1}{T}\sum_{t=1}^{T}\left(X_{it} - \hat{C}_{it}\right)\left(X_{jt} - \hat{C}_{jt}\right),$$

where $\hat{C}_{kt} = \hat{\lambda}_k' \hat{F}_t$.■

Proof of Lemma 2(i).

The first part of proving the consistency of $\mathcal{S}_{ij,T,N}$ is to chow that:

$$\text{plim}_{N,T\to\infty}\hat{\tau}_{ij} = E(e_{it}e_{jt}) = \tau_{ij}.$$

Consider the sample covariance between cross section i and cross-section j:

$$\begin{aligned}\hat{\tau}_{ij} &= \frac{1}{T}\sum_{t=1}^{T}\left(e_{it} - (\hat{C}_{it} - C_{it}^0)\right)\left(e_{jt} - (\hat{C}_{jt} - C_{jt}^0)\right)\\ &= \frac{1}{T}\sum_{t=1}^{T} e_{it}e_{jt} - \frac{1}{T}\sum_{t=1}^{T} e_{it}(\hat{C}_{jt} - C_{jt}^0) - \frac{1}{T}\sum_{t=1}^{T} e_{jt}(\hat{C}_{it} - C_{it}^0) + \frac{1}{T}\sum_{t=1}^{T}(\hat{C}_{it} - C_{it}^0)(\hat{C}_{jt} - C_{jt}^0)\\ &= I_{ij} - II_{ij} - II_{ij} + IV_{ij}\end{aligned}$$

Consider the second and third terms above, for any $i \neq j = 1, \cdots, N$, we have

$$|II_{ij}| = \left|\frac{1}{T}\sum_{t=1}^{T}\left(\hat{C}_{it} - C_{it}^0\right)e_{jt}\right| \leq \left(\frac{1}{T}\sum_{t=1}^{T}\left(\hat{C}_{it} - C_{it}^0\right)^2\right)^{1/2} \cdot \left(\frac{1}{T}\sum_{t=1}^{T} e_{jt}^2\right)^{1/2}.$$

Now

$$\hat{C}_{it} - C_{it}^0 = (\hat{F}_t - H'F_t^0)'H^{-1}\lambda_i^0 + \hat{F}_t'(\hat{\lambda}_i - H^{-1}\lambda_i^0), \tag{7.17}$$

from Bai [2003] Appendix C. Because $(a+b)^2 \leq 2(a^2+b^2)$, we have

$$(\hat{C}_{it} - C_{it}^0)^2 \leq 2\left[\left((\hat{F}_t - H'F_t^0)'H^{-1}\lambda_i^0\right)^2 + \left(\hat{F}_t'(\hat{\lambda}_i - H^{-1}\lambda_i^0)\right)^2\right], \tag{7.18}$$

and

$$\frac{1}{T}\sum_{t=1}^{T}(\hat{C}_{it} - C_{it}^0)^2 \leq 2\left(\frac{1}{T}\sum_{t=1}^{T}\left[(\hat{F}_t - H'F_t^0)'H^{-1}\lambda_i^0\right]^2 + \frac{1}{T}\sum_{t=1}^{T}\left[\hat{F}_t'(\hat{\lambda}_i - H^{-1}\lambda_i^0)\right]^2\right) = 2(I + II).$$

The first term,

$$I = \frac{1}{T}\sum_{t=1}^{T}\left[(\hat{F}_t - H'F_t^0)'H^{-1}\lambda_i^0\right]^2 \leq \frac{1}{T}\sum_{t=1}^{T}\|\hat{F}_t - H'F_t^0\|^2 \cdot \|H^{-1}\|^2 \cdot \|\lambda_i^0\|^2.$$

Note that $\|H\| \leq \|\hat{F}'\hat{F}/T\|^{1/2}\|F^{0'}F^0/T\|^{1/2}\|\Lambda^{0'}\Lambda^0/N\|$ and therefore depends on both N and T. By Assumptions A and B, $\|H\| = O_p(1)$ because each of the matrix norms in $\|H\|$ is stochastically bounded, Bai and Ng [2002]. Therefore,

$$\frac{1}{T}\sum_{t=1}^{T}\left[(\hat{F}_t - H'F_t^0)'H^{-1}\lambda_i^0\right]^2 = O_p\left(\frac{1}{\delta_{NT}^2}\right) \cdot O_p(1) \cdot \|\lambda_i\|^2 = O_p\left(\frac{1}{\delta_{NT}^2}\right) \cdot O_p(1) \cdot \overline{\lambda}^2,$$

following from Assumption A2 and $\|\lambda_i\| \leq \overline{\lambda} < \infty$. The second term,

$$II = \frac{1}{T}\sum_{t=1}^{T}\left[\hat{F}_t'(\hat{\lambda}_i - H^{-1}\lambda_i^0)\right]^2 \quad \leq \quad \|\hat{\lambda}_i - H^{-1}\lambda_i^0\|^2\frac{1}{T}\sum_{t=1}^{T}\|\hat{F}_t\|^2$$

From Proof of Theorem 2 in Bai [2003], the first term in (II),

$$\hat{\lambda}_i - H^{-1}\lambda_i^0 \quad = \quad H'\frac{1}{T}\sum_{s=1}^{T}F_s^0 e_{is} + \frac{1}{T}\hat{F}'\left(F^0 - \hat{F}H^{-1}\right)\lambda_i + \frac{1}{T}\left(\hat{F} - F^0 H\right)e_i$$

$$\|\hat{\lambda}_i - H^{-1}\lambda_i^0\|^2 \leq 4\left(a_i + b_i + c_i\right),$$

where:

$$\begin{aligned}
a_i \quad &= \quad \|H'\frac{1}{T}\sum_{s=1}^{T}F_s^0 e_{is}\|^2 \leq \|H\|^2\|\frac{1}{T}\sum_{s=1}^{T}F_s^0 e_{is}\|^2 \\
\|\frac{1}{T}\sum_{s=1}^{T}F_s^0 e_{is}\|^2 \quad &= \quad \frac{1}{T^2}\sum_{s=1}^{T}\sum_{l=1}^{T}F_s^{0'}F_l^0 e_{is}e_{il} \leq \left(\frac{1}{T^2}\sum_{l=1}^{T}\left(F_s^{0'}F_l^0\right)^2\right)^{1/2}\left(\frac{1}{T^2}\sum_{s=1}^{T}\sum_{l=1}^{T}(e_{is}e_{il})^2\right)^{1/2} \\
&\leq \quad \frac{1}{\sqrt{T}}\left(\frac{1}{T}\sum_{s=1}^{T}\|F_s^0\|^2\right)^{1/2} \cdot \left(\frac{1}{T^2}\sum_{s=1}^{T}\sum_{l=1}^{T}(e_{is}e_{il})^2\right)^{1/2},
\end{aligned}$$

now, $E\left[(e_{is}e_{il})^2\right] \leq \max_s E|e_{is}|^4 \leq M$ (Assumption A3 (1)), then we have

$$a_i \leq O_p(1) \cdot O_p\left(\frac{1}{\sqrt{T}}\right) \cdot O(1).$$

Now, consider the term b_i,

$$\begin{aligned} b_i &= \left\| \frac{1}{T} \hat{F}' \left(F^0 - \hat{F} H^{-1} \right) \lambda_i \right\|^2 \\ &\leq \left\| \frac{1}{T} \sum_{t=1}^{T} \hat{F}_t \left(F_t^0 - \hat{F}_t H^{-1} \right)' \right\|^2 \cdot \|\lambda_i\|^2 \\ &\leq \left(\frac{1}{T} \sum_{t=1}^{T} \|F_t^0 - \hat{F}_t H^{-1}\|^2 \right) \cdot \left(\frac{1}{T} \sum_{t=1}^{T} \|\hat{F}_t\|^2 \right) \cdot \|\lambda_i\|^2 \\ \left(\frac{1}{T} \sum_{t=1}^{T} \|F_t^0 - \hat{F}_t H^{-1}\|^2 \right) &= \|H^{-1}\|^2 \left(\frac{1}{T} \sum_{t=1}^{T} \|\hat{F}_t - F_t^0 H\|^2 \right) = O_p(1) \cdot O_p\left(\frac{1}{\delta_{NT}^2} \right) \end{aligned}$$

Thus,

$$b_i \leq O_p\left(\frac{1}{\delta_{NT}^2} \right) \cdot O(1) \cdot \overline{\lambda}^2.$$

The last term,

$$\begin{aligned} c_i &= \frac{1}{T^2} \left\| \left(\hat{F} - F^0 H \right)' e_i \right\| = \frac{1}{T^2} \left\| \sum_{s=1}^{T} (\hat{F}_t - H F_t^0) e_{it} \right\| \\ &\leq \left(\frac{1}{T} \sum_{t=1}^{T} \|\hat{F}_t - H F_t^0\|^2 \right) \cdot \left(\frac{1}{T} \sum_{t=1}^{T} e_{it}^2 \right) \\ &\leq O_p\left(\frac{1}{\delta_{NT}^2} \right) \cdot O(1) \end{aligned}$$

Collecting the terms together for the expressions (I) and (II), we have:

$$\frac{1}{T} \sum_{t=1}^{T} (\hat{C}_{it} - C_{it}^0)^2 \leq O_p\left(\frac{1}{\sqrt{T}} \right) + O_p\left(\frac{1}{\delta_{NT}^2} \right). \tag{7.19}$$

Now going back to $\hat{\tau}_{ij}$, we can derive the rate for IV_{ij} similarly to II_{ij}. We have, $IV_{ij} = O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\delta_{NT}^2}\right)$. Therefore, we can conclude,

$$\hat{\tau}_{ij} = \frac{1}{T} \sum_{t=1}^{T} e_{it} e_{jt} - O_p\left(\frac{1}{T^{1/4}} \right) - O_p\left(\frac{1}{\delta_{NT}} \right). \tag{7.20}$$

■

Proof of Lemma 2(ii).

Estimation of $\mathcal{S}$ is likewise done pairwise by estimating $\mathcal{S}_{ij}, i, j = 1, \cdots, N$. Each entry $\mathcal{S}_{ij,T}$ is compted only from the i^{th} and j^{th} cross sections, thus the computing time increases quadratically with N. Under the assumption of stationarity $\tau_{ij,t} = \tau_{ij} = E(e_{it}e_{jt})$. The population $\mathcal{S}_{ij} = \text{sgn}(\tau_{ij})$:

$$\mathcal{S}_{ij} \;=\; 1(\tau_{ij} \geq 0) - 1(\tau_{ij} < 0), \tag{7.21}$$

where $1(\mathcal{A}) = 1$ if '$\mathcal{A}$' is correct and 0 otherwise. Let $\mathcal{S}_{ij,T}$ be an estimator for $\mathcal{S}_{ij}$ defined as:

$$\mathcal{S}_{ij,T} \;=\; 1(\hat{\tau}_{ij} \geq 0) - 1(\hat{\tau}_{ij} < 0). \tag{7.22}$$

The expected value $E(\mathcal{S}_{ij,T})$:

$$E(\mathcal{S}_{ij,T}) \;=\; P\left(\hat{\tau}_{ij} \geq 0\right) - P\left(\hat{\tau}_{ij} < 0\right). \tag{7.23}$$

Given the result in equation(7.20),

$$\begin{aligned} E(\mathcal{S}_{ij,T}) &= P\left(\frac{1}{T}\sum_{t=1}^{T} e_{it}e_{jt} - O_p\left(\frac{1}{T^{1/4}}\right) - O_p\left(\frac{1}{\delta_{NT}}\right) \geq 0\right) \\ &- P\left(\frac{1}{T}\sum_{t=1}^{T} e_{it}e_{jt} - O_p\left(\frac{1}{T^{1/4}}\right) - O_p\left(\frac{1}{\delta_{NT}}\right) < 0\right). \end{aligned}$$

As $N, T \to \infty$,

$$\begin{aligned} E(\mathcal{S}_{ij,T}) &\to P\left(\lim_{N,T} \to \infty \frac{\sum_{t=1}^{T} e_{it}e_{jt}}{T} \geq 0\right) - P\left(\lim_{N,T} \to \infty \frac{\sum_{t=1}^{T} e_{it}e_{jt}}{T} < 0\right) \\ &= 1(\tau_{ij} \geq 0) - 1(\tau_{ij} < 0) = \mathcal{S}_{ij} \end{aligned}$$

The latter result because τ_{ij} is a deterministic population parameter and is either ≥ 0 or < 0, and thus the respective probabilities are identically 1 or zero. ■

7.1 Proof of efficiency of Cn-PC for $r = 1$

For the case of $r = 1$, we have

$$\Sigma_{\Lambda*} = \Sigma_\Lambda + \mu_{NT} \text{ plim } \frac{\Lambda'\mathcal{S}\Lambda}{N} \geq \Sigma_\Lambda \tag{7.24}$$

Now consider the second term,

$$\Lambda'\mathcal{S}\Lambda = \sum_{i \neq j}\sum_{j} \lambda_i'\lambda_j\mathcal{S}_{ij}$$

If there is a factor structure in the population model, we have for $i \neq j = 1, \cdots, N$:

$$\begin{aligned} X_{it} &= \lambda_i'F_t + e_{it} && (7.25) \\ X_{jt} &= \lambda_j'F_t + e_{jt} && (7.26) \\ E\left[X_{it}X_{jt}\right] &= E\left[(\lambda_i'F_t)(\lambda_j'F_t)\right] + E\left[e_{it}e_{jt}\right] && (7.27) \\ &= E\left[\lambda_i'\lambda_j F_t'F_t\right] + E\left[e_{it}e_{jt}\right]. && (7.28) \end{aligned}$$

A factor structure implies that the dynamics of the N cross sections are driven by the common factors F_t. Let's assume for the sake of argument, that we condition on F_t. In this case, we would expect that the sign of $E\left[X_{it}X_{jt}|F_t\right]$ will be driven by the sign of $E\left[\lambda_i'\lambda_j\right]$. If this holds, then the sign of $E\left[e_{it}e_{jt}\right]$ is the same as the sign of $E\left[\lambda_i'\lambda_j\right]$. We can conclude that

$$\text{sgn}\left[\text{plim}\ \frac{\Lambda'\mathcal{S}\Lambda}{N}\right] \geq 0.$$

In this case, the Cn-PC estimator are more efficient than the PCEs with ratio of (asymptotic) variances equal to:

$$\frac{V(\hat{F}_{t,opc})}{V(\hat{F}_t)} = \left(1 + \mu_{NT}\ \text{plim}\ \frac{\Lambda'\mathcal{S}\Lambda}{N}\right)^2.$$

Figure 5: Distribution of $\hat{\Omega}_{i,j}$

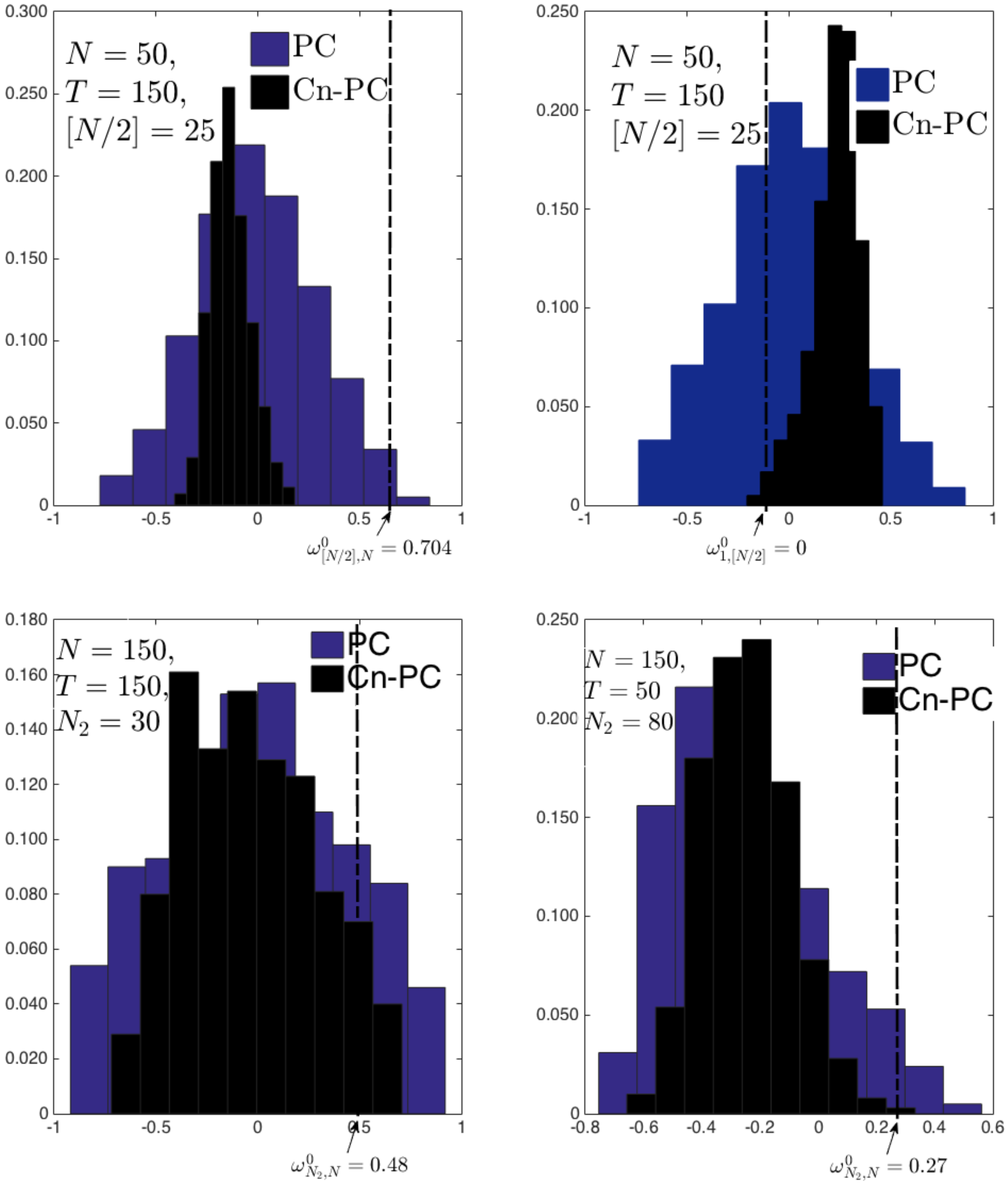


Figure 6: Sampling distribution of $\hat{\tau}^*$

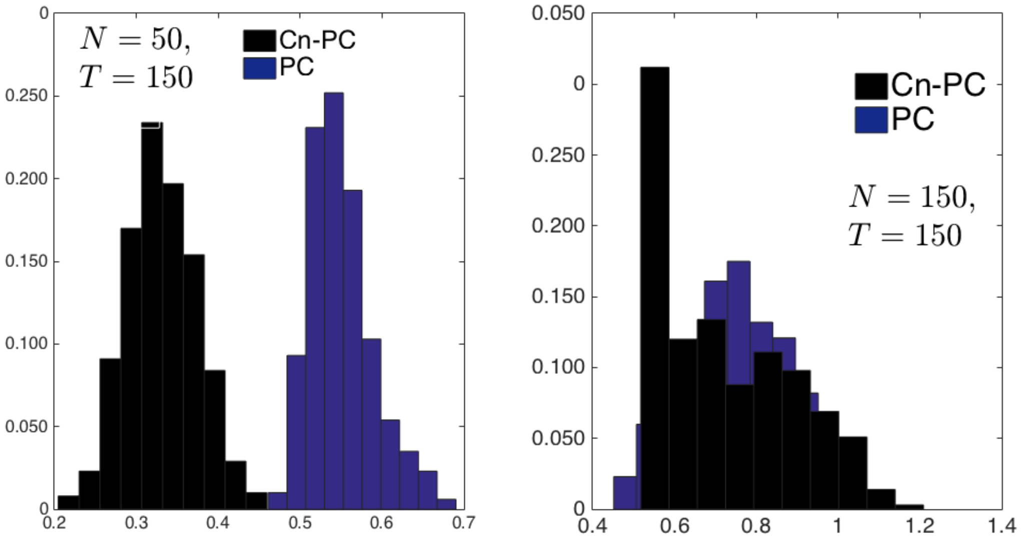
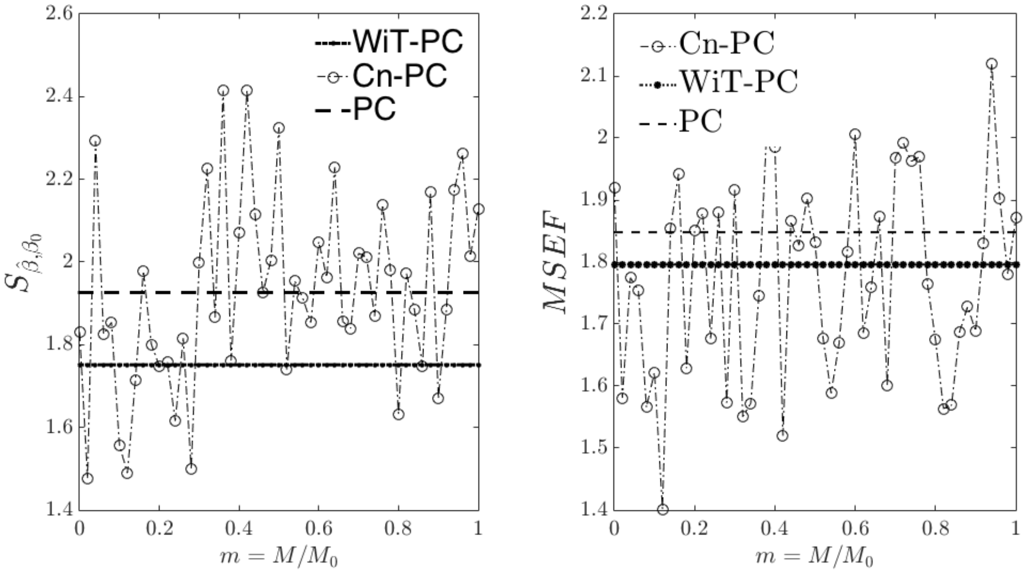


Figure 7: Accuracy of the Diffusion Index forecasts


WiT-PC
Cn-PC
-PC
$S_{\hat{\beta},\beta_0}$
$m = M/M_0$
Cn-PC
WiT-PC
PC
$MSEF$
$m = M/M_0$