Equilibrium Selection in Auctions and High-Stakes Games

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Introduction

- The ineffectiveness of traditional Nash equilibrium refinements in some interesting auction games had led some researchers to use *ad hoc* refinements:
  - Menu auction: “truthful” or “coalition-proof” equilibrium
  - GSP auction: “locally envy-free” equilibrium
  - 2^{nd}-price auction: “tremble-robust” equilibrium

- Can a *general* Nash refinement lead to similar predictions?
  - Can it further illuminate the previous analyses?

Contributions of this paper

1. Introduces *quasi*-perfect equilibrium* for general *extensive* games.
2. Introduces a new model of *high stakes games*, in which each strategy is reviewed and approved before it is played. This leads to a normal-form refinement that we call “test-set equilibrium.”
3. *Applies test-set equilibrium* to the three cited models, leading to new, deeper analyses.

HIGH-STAKES GAMES

Notation

A game in normal form $\Gamma = (N, S, u)$

- Players $N = \{1, \dots, N\}$
- Pure strategy sets $(S_n)_{n=1}^{N}$
- Payoff functions $(u_n)_{n=1}^{N}$

Mixed strategy profiles $\sigma \in \Pi_{n=1}^{N} \Delta(S_n)$

Player n's pure best responses $BR_n(\sigma_{-n}) \subseteq S_n$

High-Stakes Versions

- Given any finite game in normal form $\Gamma = (N, S, u)$, a high-stakes version is an extensive game $\Gamma(c)$ indexed by $c > 0$, as follows.

- In $\Gamma(c)$, each player $n \in N$ acts independently, making three moves.
  1. Player n's first agent chooses a pure strategy $s_n \in S_n$
  2. Player n's second agent then reviews the choice and either
     - *Approves*, in which case s_n is played in Γ, or
     - *Disapproves*, in which case we go to step 3.
  3. Player n chooses a pure strategy $s_n' \in S_n$, which is played in Γ.

- The *outcome* of behavioral strategies b for $\Gamma(c)$ is a profile σ for Γ.

$$\bar{\pi}_n(b) = \begin{cases} \pi_n(\sigma) & \text{if } n \text{ approves on the path} \\ \pi_n(\sigma) - c & \text{otherwise} \end{cases}$$

Quasi*-perfect Equilibrium

A Nash equilibrium refinement for extensive forms in which:

- Each agent trembles expects that its own future agents will *not* tremble (van Damme, 1984).
- Players may
  - have different beliefs, and
  - believe that other agent's trembles are correlated,
  - have only beliefs that are not "too extreme":
    - each player assigns probability of order ε to any single tremble and of lower order than ε to any multiple trembles.
    - See also Bagwell and Ramey (1991), for a similar restriction in multi-player signaling games.

Definition: Quasi*-perfect equilibrium

- A *behavior strategy profile* b is a *quasi*-perfect equilibrium* of an extensive game $\bar{\Gamma}$ if there is
  - a profile $(\tau_n^m)_{n,m=1}^N$ of completely mixed behavior strategies;
  - a sequence of distributions $(\{d^{t,m}\}_{t=1}^{\infty})_{m=1}^{N}$ on the possible paths of play z; and
  - sequences of positive real numbers $\{\varepsilon_t\} \to 0$ and $\{\delta_t\} \to 0$ such that

1. Every player n, information set, $u \in U_n$, and index t, n's choice is maximizing:
   $$\bar{\pi}_{nu}(d^{t,n} \backslash_u b_n) = \max_{b_n' \in B_n} \bar{\pi}_{nu}(d^{t,n} \backslash_u b_n')$$
2. For every player m, terminal node z, and index t, beliefs satisfy:
   $$d^{t,m}(z) \geq (1 - \varepsilon_t \delta_t) \prod_{n=1}^{N} \prod_{u \in U_n} \big((1 - \varepsilon_t) b_{nu}(z_u) + \varepsilon_t \tau_{nu}^m(z_u)\big),$$

Test-Set Condition

Given a normal form Γ, the "test set" is:

$$T(\sigma) = \bigcup_{n=1}^{N} \{(\sigma_{-n}, s_n) : s_n \in BR_n(\sigma_{-n})\}$$

Informally, $T(\sigma)$ is the set of "most likely trembles."

Definition

A strategy profile σ satisfies the *test-set condition* if, for all $n \in N$, there is no $\hat{\sigma}_n \in \Delta(S_n)$ such that

$u_n(\sigma'_{-n}, \hat{\sigma}_n) \geq u_n(\sigma'_{-n}, \sigma_n)$ for all $\sigma' \in T(\sigma)$, and

$u_n(\sigma'_{-n}, \hat{\sigma}_n) > u_n(\sigma'_{-n}, \sigma_n)$ for some $\sigma' \in T(\sigma)$.

- The test-set condition rules out strategies that are weakly dominated when others' play is in $T(\sigma)$.

Definition

A strategy profile σ of Γ is a *test-set equilibrium* if and only if it is a Nash equilibrium in undominated strategies that satisfies the test-set condition.

Main Result

Theorem

1. For all $c > 0$, σ is a Nash equilibrium of Γ if and only if it is the outcome of some Nash equilibrium b of $\Gamma(c)$.
2. For all finite Γ, σ is a *test-set equilibrium* of Γ if and only if there exists a $\bar{c} > 0$ such that for all $c \in (0, \bar{c})$, σ is the outcome of some *"quasi*-perfect equilibrium"* b of $\Gamma(c)$.

Intuition

- Necessity of Test-set Equilibrium
  - In quasi*-perfect equilibrium,
    - all strategies by opponents have positive probability, so all players choose undominated strategies, and
    - for c small (since all expect that their own future agents will not tremble), if at most one agent trembles, then each player is still playing a best response to the equilibrium profile. Since all expect zero or one trembles to be most likely, all play a strategy that is not dominated against such profiles.

- Sufficiency of Test-set Equilibrium
  - Given any test-set equilibrium σ, we can construct player n's beliefs about others' trembles in $\Gamma(c)$ that justify playing σ_n in quasi*-perfect equilibrium (by applying the separating hyperplane theorem).

This Paper

- A new refinement: *test-set equilibrium*
  - Defined for general games in normal form
  - Similar, but different, selections in the three applications

- In the ______________________________, test-set equilibrium is
  auction

  slightly ______________ than ________________________.
  stronger/weaker original refinement

This Paper

- A new refinement: *test-set equilibrium*
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- In the ______first-price menu auction______, test-set equilibrium is
  auction

  slightly ___weaker___ than ___truthful equilibrium___.
  stronger/weaker original refinement

This Paper

- A new refinement: *test-set equilibrium*
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- In the generalized second price auction ________________________________, test-set equilibrium is

  auction

  slightly stronger ______________ than locally envy-free equilibrium ___________________________.

  stronger/weaker original refinement

This Paper

- A new refinement: *test-set equilibrium*
  - Defined for general games in normal form
  - Similar, but different, selections in the three applications

- In the __second price, common value auction__, test-set equilibrium is
  auction
  slightly __weaker__ than __tremble robust equilibrium__.
  stronger/weaker original refinement

Test-set equilibrium

CONCLUSION

Conclusion

- Test-set equilibrium
  - is a general game theoretic equilibrium refinement,
  - is consistent with the same strategy choices as certain related high stakes versions of the game,
  - makes selections in three auction games similar to those made based on intuitive arguments by the original authors, but
  - makes selections that do not coincide exactly, providing insight into the detailed logic used in earlier papers.

END