A more powerful subvector Anderson and Rubin test in linear instrumental variables regression

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North American Winter Meeting

January 2018

Overview

- Consider **subvector inference in the linear IV model**, allowing for **weak instruments** but assuming **conditional homoskedasticity**

- **Background:**
  - Projection of Anderson and Rubin (AR) test (Dufour and Taamouti, Ecta 2005).

  - Guggenberger, Kleibergen, Mavroeidis, and Chen (Ecta 2012, GKMC) provide power improvement:

  - Using $\chi^2_{k-m_W,1-\alpha}$ as critical value, rather than $\chi^2_{k,1-\alpha}$ still controls asymptotic size.

  - "Worst case" occurs under strong identification.

- **HERE:** consider a **data-dependent critical value** that adapts to strength of identification.

- One main objective: computational ease.

- Show: conditional subvector AR test controls finite sample/asymptotic size & has higher power than method in GKMC.

- Test in GKMC is inadmissible.

- Proposed test has a near optimality property when $m_W = 1$.

Outline

1. Finite sample analysis

   (a) Motivation for conditional subvector AR test

   (b) Size of test when $m_W = 1$

   (c) Power analysis when $m_W = 1$

   (d) Size of test when $m_W > 1$

2. Asymptotics

Model and Objective (finite sample case)

$$
\begin{aligned}
y &= Y\beta + W\gamma + \varepsilon, \\
Y &= Z\Pi_Y + V_Y, \\
W &= Z\Pi_W + V_W,
\end{aligned}
$$

$$y \in \Re^n, Y \in \Re^{n\times m_Y}, W \in \Re^{n\times m_W}, \text{ and } Z \in \Re^{n\times k}.$$

- **Reduced form:**

$$(y \,\vdots\, Y \,\vdots\, W) = Z\,(\Pi_Y \,\vdots\, \Pi_W)\begin{pmatrix} \beta & I_{m_Y} & 0 \\ \gamma & 0 & I_{m_W} \end{pmatrix} + \underbrace{(v_y \,\vdots\, V_Y \,\vdots\, V_W)}_{V}.$$

- **Objective:** test

$$H_0 : \beta = \beta_0 \;\text{ versus }\; H_1 : \beta \neq \beta_0$$

s.t. size bounded by nominal size & "good" power.

Parameter space:

1. The error term is distributed as

$$V_i \sim \text{i.i.d. } N\left(0, \Omega\right), \; i = 1, ..., n,$$

   where $\Omega \in R^{(m+1)\times(m+1)}$ is assumed to be known and positive definite.

2. $Z \in R^{n\times k}$ fixed, and $Z'Z > 0 \; k \times k$ matrix.

- **Note:** no restrictions on reduced form parameters $\longrightarrow$ allow for weak IV.

- Many tests available for **full vector inference**

$$H_0 : \beta = \beta_0, \gamma = \gamma_0 \text{ vs } H_1 : \text{not } H_0$$

  including AR (Anderson and Rubin, 1949), LM, and CLR tests, see Kleibergen (2002), Moreira (2003, 2009).

- **Optimality properties:** Andrews, Moreira, and Stock (2006) and Chernozhukov, Hansen, and Jansson (2009).

Derived subvector procedures

- **Projection:** "inf" over parameter not under test, same critical value $\longrightarrow$ "computationally hard" and "uninformative".

- **Bonferroni:** Staiger and Stock (1997), Chaudhuri and Zivot (2011), McCloskey (2012), Wang and Doko Tchatoka (2017)...; often computationally hard, power ranking with projection unclear.

- **Plug-in approach:** Kleibergen (2004), Guggenberger and Smith (2005)...Requires strong ID of parameters not under test.

- Kleibergen (2015): subvector CLR test with correct size under weak IV and asymptotically efficient under strong IV.

- Power ranking under weak IV is unclear:
  - In just-identified case $k = m_Y + m_W$, subvector LR statistic is equal to the subvector AR statistic, and CLR cv is $\chi^2_{m_Y,1-\alpha}$.
  - Hence, less powerful than the test proposed here.

The Anderson and Rubin (1949) test

- **AR test stat** for **full vector hypothesis**

$$H_0 : \beta = \beta_0, \gamma = \gamma_0 \; vs \; H_1 : \text{not } H_0$$

- AR statistic exploits $EZ_i\varepsilon_i = 0$.

- **AR test stat:**

$$AR_n(\beta_0, \gamma_0) = \frac{(y - Y\beta_0 - W\gamma_0)' P_Z (y - Y\beta_0 - W\gamma_0)}{\left(1 \,\vdots\, -\beta_0' \,\vdots\, -\gamma_0'\right) \Omega \left(1 \,\vdots\, -\beta_0' \,\vdots\, -\gamma_0'\right)'}$$

- AR stat is χ^2_k under null hypothesis; critical value $\chi^2_{k,1-\alpha}$.

- **Subvector AR statistic** for testing H_0 is given by

$$AR_n(\beta_0) = \min_{\gamma \in R^{m_W}} \frac{(\overline{Y}_0 - W\gamma)' P_Z (\overline{Y}_0 - W\gamma)}{(1 \vdots -\beta_0' \vdots -\gamma')\, \Omega\, (1 \vdots -\beta_0' \vdots -\gamma')'},$$

where $\overline{Y}_0 = y - Y\beta_0$.

- Alternative representation: Let $\hat{\kappa}_i$ for $i = 1, ..., p = 1 + m_W$ be roots of characteristic polynomial in κ

$$\left| \kappa \Omega(\beta_0) - \left(\overline{Y}_0 \vdots W\right)' P_Z \left(\overline{Y}_0 \vdots W\right) \right| = 0,$$

ordered non-increasingly, where we define

$$\Omega(\beta_0) = \begin{pmatrix} 1 & 0 \\ -\beta_0 & 0 \\ 0 & I_{m_W} \end{pmatrix}' \Omega \begin{pmatrix} 1 & 0 \\ -\beta_0 & 0 \\ 0 & I_{m_W} \end{pmatrix}.$$

Then

$$AR_n(\beta_0) = \hat{\kappa}_p.$$

- As discussed: When using $\chi^2_{k,1-\alpha}$ critical values, trivially, test has correct size;

- GKMC show that this is also true for $\chi^2_{k-m_W,1-\alpha}$ critical values.

- **Next:** AR statistic is the minimum eigenvalue of a non-central Wishart matrix.

- The roots $\hat{\kappa}_i$ solve

$$0 = \left|\hat{\kappa}_i I_{1+m_W} - \Xi'\Xi\right|, \quad i = 1, ..., p = 1 + m_W,$$

where $\Xi \sim N\left(\mathcal{M}, I_k \otimes I_p\right)$, and $\mathcal{M}$ is a $k \times p$.

- Under H_0, the noncentrality matrix becomes $\mathcal{M} = \left(0^k, \Theta_W\right)$, where

$$\Theta_W = \left(Z'Z\right)^{1/2} \Pi_W \Sigma_{V_W V_W \cdot \varepsilon}^{-1/2},$$
$$\Sigma_{V_W V_W \cdot \varepsilon} = \Sigma_{V_W V_W} - \Sigma'_{\varepsilon V_W} \sigma_{\varepsilon\varepsilon}^{-1} \Sigma_{\varepsilon V_W}$$

and

$$\begin{pmatrix} \sigma_{\varepsilon\varepsilon} & \Sigma_{\varepsilon V_W} \\ \Sigma'_{\varepsilon V_W} & \Sigma_{V_W V_W} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\beta_0 & 0 \\ -\gamma & I_{m_W} \end{pmatrix}' \Omega \begin{pmatrix} 1 & 0 \\ -\beta_0 & 0 \\ -\gamma & I_{m_W} \end{pmatrix}$$

- **Summarizing**, under H_0

$$\Xi'\Xi \sim \mathcal{W}_p\left(k, I_p, \mathcal{M}'\mathcal{M}\right),$$

non-central Wishart, with noncentrality matrix

$$\mathcal{M}'\mathcal{M} = \begin{pmatrix} 0 & 0 \\ 0 & \Theta'_W \Theta_W \end{pmatrix}$$

and

$$AR_n(\beta_0) = \kappa_{\min}(\Xi'\Xi)$$

- The distribution of the eigenvalues of a noncentral Wishart matrix only depends on the eigenvalues of the noncentrality matrix $\mathcal{M}'\mathcal{M}$.

- Hence, distribution of $\hat{\kappa}_i$ only depends on the eigenvalues of $\Theta_W'\Theta_W$, κ_i say, $i = 1, \ldots, m_W$ and $\kappa = (\kappa_1, ..., \kappa_{m_W})'$

- When $m_W = 1$, $\kappa_1 = \Theta_W'\Theta_W$ is scalar (concentration parameter for γ under Null).

Theorem: Suppose $m_W = 1$. Then, under the null hypothesis $H_0 : \beta = \beta_0$, the distribution function of the subvector AR statistic, $AR_n(\beta_0)$, is monotonically decreasing in the parameter κ_1.

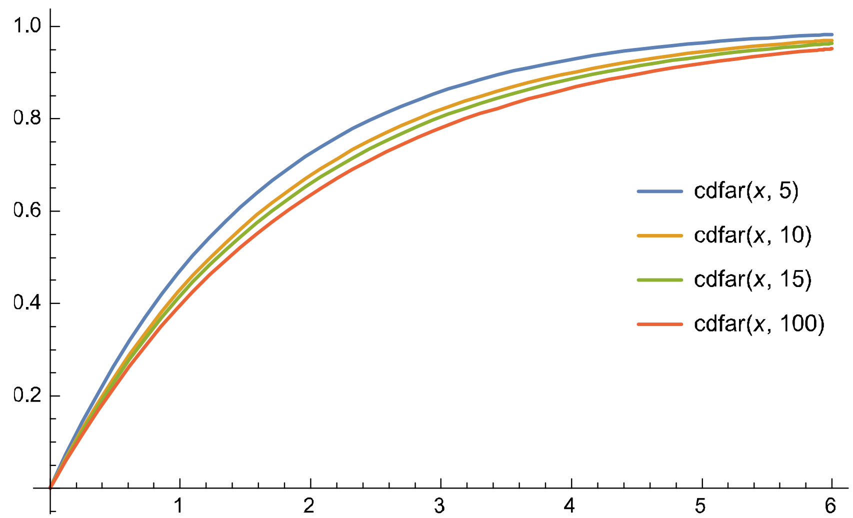


Figure 1: The cdf of the subset AR statistic with $k = 3$ instruments, for different values of $\kappa_1 = 5, 10, 15, 100$, shown in the legend on the right.

New critical value for subvector Anderson and Rubin test

- **Relevance:** If we knew κ_1 we could implement the subvector AR test with a smaller critical value than $\chi^2_{k-m_W,1-\alpha}$ which is the critical value in the case when κ_1 is "large".

- **Intuition for new critical value.** Let's assume $m_W = 1$ for simplicity.

- Under null, when κ_1 "is large", the larger root $\widehat{\kappa}_1$ is a sufficient statistic for κ_1, see Muirhead (1978).

- Muirhead provides approximate, nuisance parameter free, density of $AR_n(\beta_0) = \widehat{\kappa}_2$ given $\widehat{\kappa}_1$ (which measures strength of identification).

- The **new critical value** for the subvector AR-test at significance level $1-\alpha$ is given by

$$1-\alpha \text{ quantile of (approximation of } AR_n \text{ given } \widehat{\kappa}_1)$$

- Denote cv by

$$c_{1-\alpha}(\hat{\kappa}_1, k-m_W)$$

Depends only on α, $k-m_W$, and $\hat{\kappa}_1$.

- We find, by simulations over fine grid of values of κ_1, that test controls size.

- It improves on the GKMC procedure in terms of power.

- **Theorem:** Suppose $m_W = 1$. The subvector Anderson Rubin test that uses the new conditional critical value $c_{1-\alpha}(\hat{\kappa}_1, k - m_W)$ has correct size under the assumptions above.

Details

- Again: $\kappa_1 \geq 0$ is nonzero latent root of $\mathcal{M}'\mathcal{M}$ (nuisance parameter).

- When the root is "large", the conditional density of $AR_n(\beta_0) = \hat{\kappa}_2$ given $\hat{\kappa}_1$ can be approximated by

$$f_{\hat{\kappa}_2|\hat{\kappa}_1}(x) \sim f_{\chi^2_{k-1}}(x)\,(\hat{\kappa}_1 - x)^{1/2}\,g(\hat{\kappa}_1),$$

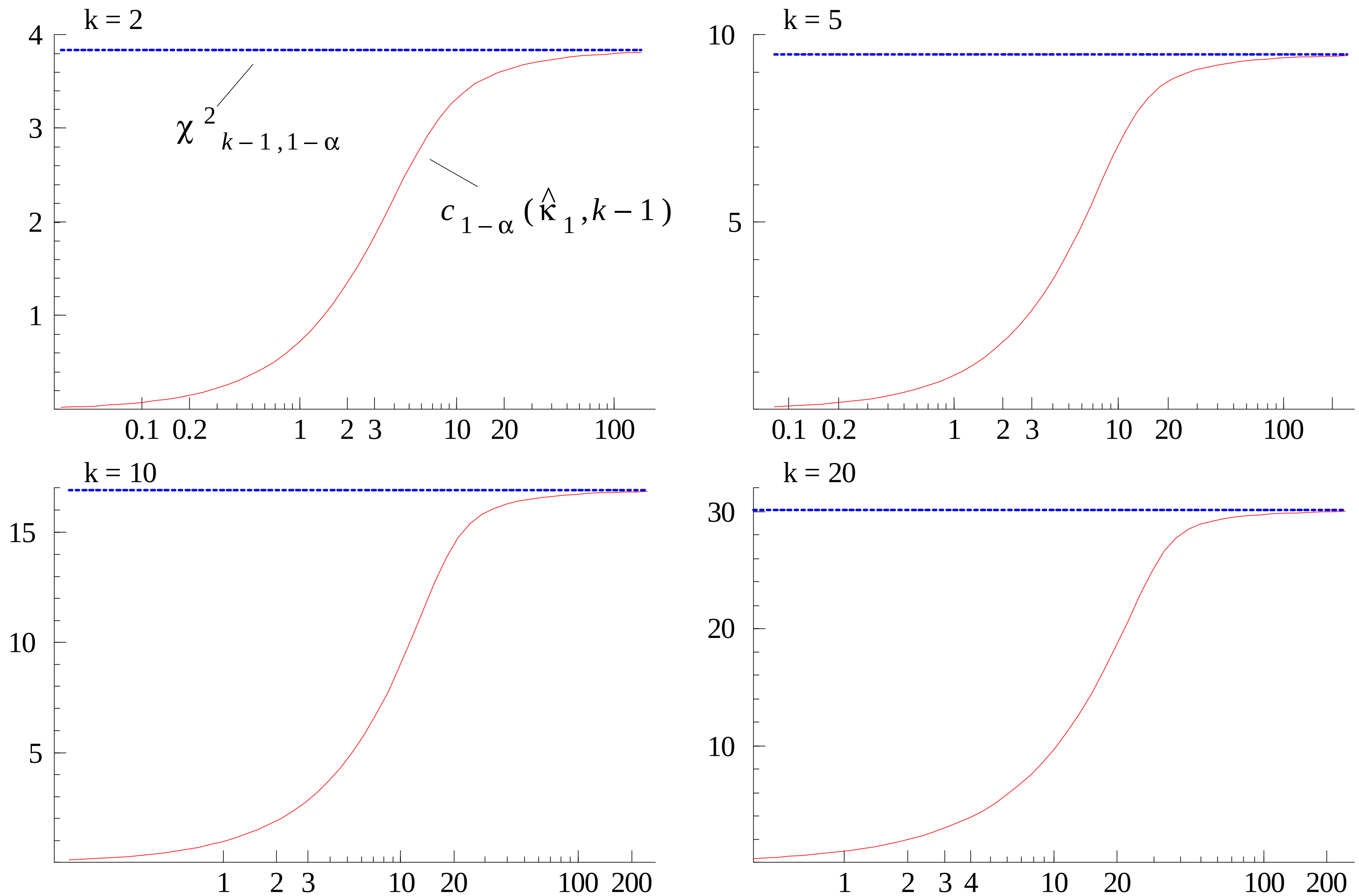
  where $f_{\chi^2_{k-1}}$ is the density of a χ^2_{k-1} and g is a function that does not depend on κ_1. (Muirhead, 1978 due to Leach, 1969).

- Analytical formula for g.

- Conditional quantiles can be computed by numerical integration.

- Conditional critical values can be tabulated $\longrightarrow$ implementation of new test is trival and fast.

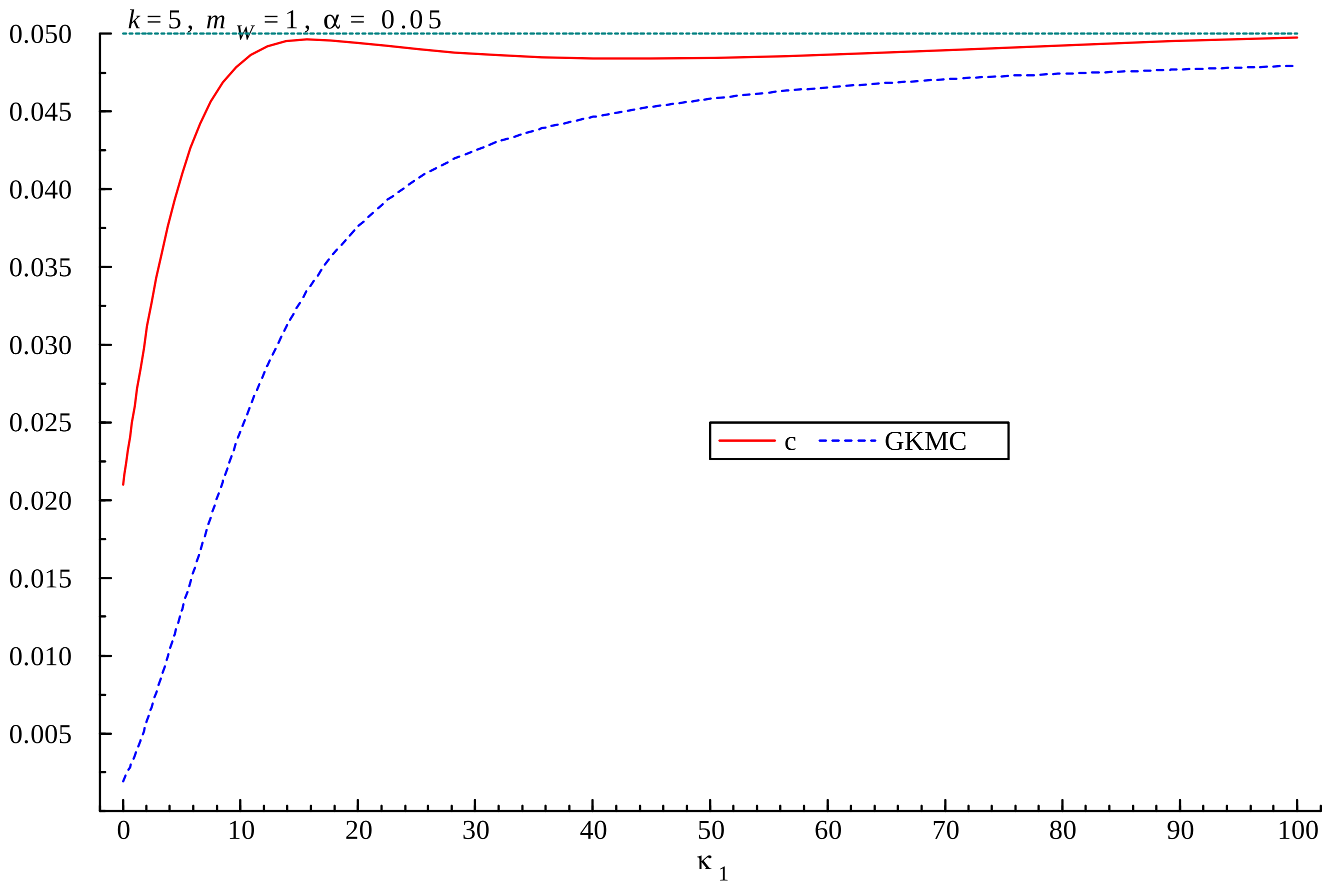
- They are increasing in $\hat{\kappa}_1$ and converging to quantiles of χ^2_{k-1}.



Critical value function $c_{1-\alpha}(\widehat{\kappa}_1, k-1)$ for $\alpha = 0.05$.

Table of conditional critical values

$\alpha = 5\%, \quad k - m_W = 4$											
$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv	$\hat{\kappa}_1$	cv
0.22	0.2	2.00	1.8	3.92	3.4	6.10	5.0	8.95	6.6	14.46	8.2
0.44	0.4	2.23	2.0	4.17	3.6	6.41	5.2	9.40	6.8	15.88	8.4
0.65	0.6	2.46	2.2	4.43	3.8	6.73	5.4	9.89	7.0	17.85	8.6
0.87	0.8	2.70	2.4	4.69	4.0	7.05	5.6	10.42	7.2	20.89	8.8
1.10	1.0	2.94	2.6	4.96	4.2	7.39	5.8	11.01	7.4	26.42	9.0
1.32	1.2	3.18	2.8	5.24	4.4	7.75	6.0	11.68	7.6	39.82	9.2
1.54	1.4	3.42	3.0	5.52	4.6	8.13	6.2	12.44	7.8	114.76	9.4
1.77	1.6	3.67	3.2	5.81	4.8	8.52	6.4	13.35	8.0	+.Inf	9.5



Null rejection frequency of subset AR test based on conditional (red) and χ^2_{k-1} (blue) critical values, as function of κ_1. 10000 MC simulations with importance sampling over a grid of 42 points.

Power

- The subvector AR statistic is the LR statistic for testing $H_0' : \rho(A) \leq m_W$ against $H_1' : \rho(A) = m_W + 1$ for $A = E\left[Z'\left(y - Y\beta_0 \vdots W\right)\right]$, where the data is $Z'\left(y - Y\beta_0 \vdots W\right)$.

- $H_0 : \beta = \beta_0$ implies H_0' but the converse is not true:

  – H_0' holds iff $\rho\left(\Pi_Y\left(\beta - \beta_0\right) \vdots \Pi_W\right) \leq m_W$, which includes $H_1 : \beta \neq \beta_0$ when $H_0' \backslash H_0$ holds, i.e., if Π_W is rank deficient or $\Pi_Y(\beta - \beta_0) \in span(\Pi_W)$.

- Under H_0', $(\hat{\kappa}_1, ..., \hat{\kappa}_p)$ are distributed as eigenvalues of $\mathcal{W}_p\left(k, I_p, \mathcal{M}'\mathcal{M}\right)$ with rank deficient noncentrality.

- Thus, every test $\varphi(\hat{\kappa}_1, ..., \hat{\kappa}_p) \in [0, 1]$ that has size α under H_0 must also have size α under H_0', so cannot have power exceeding size under alternatives $H_0' \backslash H_0$.

- In other words, size α tests $\varphi(\hat{\kappa}_1, ..., \hat{\kappa}_p)$ can only have nontrivial power under alternatives $\rho(A) = m_W + 1$.

- We use this insight to derive a power envelope for tests of the form $\varphi(\hat{\kappa}_1, ..., \hat{\kappa}_p)$.

- Consider only the case $m_W = 1$.

- Testing $\rho(\mathcal{M}) \leq 1$ against $\rho(\mathcal{M}) = 2$, where $\Xi \sim N(\mathcal{M}, I)$.

- Equivalently, $H_0' : \kappa_2 = 0,\ \kappa_1 \geq \kappa_2$ against $H_1' : \kappa_2 > 0, \kappa_1 \geq \kappa_2$.

- Maximal invariant is $\hat{\kappa}_1, \hat{\kappa}_2$ (Muirhead, 2009, Section 10.2).

- Likelihood (James, 1964)

$$lik(\kappa|\hat{\kappa}) = \exp\left(-\frac{\kappa_1 + \kappa_2}{2}\right)\ {}_0F_1^{(2)}\left(\frac{k}{2}; \frac{1}{4}\begin{pmatrix}\kappa_1 & 0 \\ 0 & \kappa_2\end{pmatrix}, \begin{pmatrix}\hat{\kappa}_1 & 0 \\ 0 & \hat{\kappa}_2\end{pmatrix}\right)$$

- Computed using the algorithms developed by Koev and Edelman (2008), available in C and Matlab.

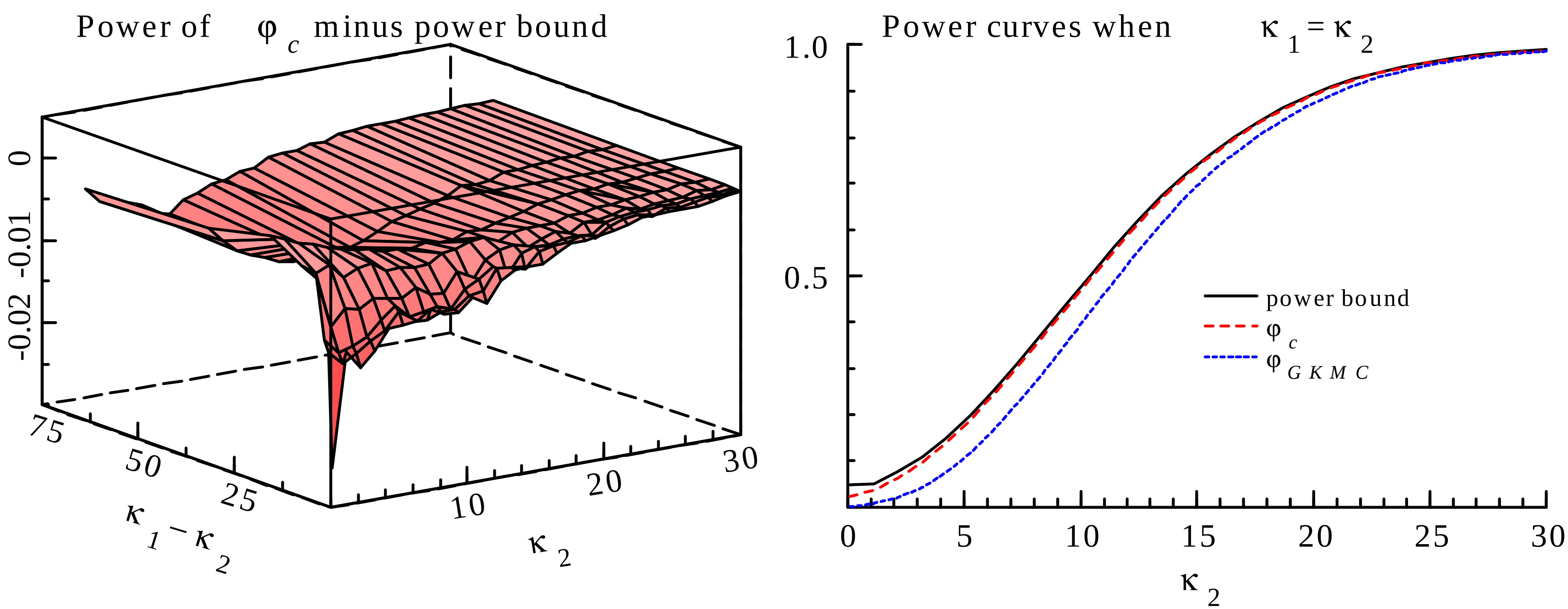
Power bounds

- Point-optimal power bounds for reduced rank testing problem using least favourable distribution Λ^{LF} over nuisance parameter κ_1.

- Two methods: Andrews Moreira and Stock (JoE, 2008, Sec 4.2) – AMS.
  - assumes one-point Λ^{LF}, gives lower and upper bounds on envelope.

- Elliott Mueller and Watson (Ecma 2015, Lemma 1) – ALFD (Approximate LF distn).

- Implementation: 42 points evenly spaced in log-scale between 0 and 99.



Power of conditional subvector AR test $\varphi_c(\hat{\kappa}) = 1_{\{\hat{\kappa}_2 > c_{1-\alpha}(\hat{\kappa}_1, k-1)\}}$ relative to power bound (left) and power of φ_c, $\varphi_{GKMC}(\hat{\kappa}) = 1_{\left\{\hat{\kappa}_2 > \chi^2_{k-1,1-\alpha}\right\}} = 1_{\{\hat{\kappa}_2 > c_{1-\alpha}(\infty, k-1)\}}$ and bound at $\kappa_1 = \kappa_2$ (right) for $k = 5$. Computed using 10000 MC replications.

- Little scope for power improvement over proposed test.

Size for $m_W > 1$

When $m_W = 1$ the new subvector AR test has correct size and uniformly improves the power of the test in GKMC.

$\longrightarrow$ Generalize this result to any m_W.

We define a new subvector AR test that rejects when

$$AR_n(\beta_0) > c_{1-\alpha}(\kappa_{\max}\left(\Xi'\Xi\right), k - m_W).$$

Note: We condition on the LARGEST eigenvalue of the Wishart matrix.

Show now that this test has correct size and has uniformly larger power than the test in GKMC.

Theorem: Under the null $H_0 : \beta = \beta_0$, there exists a random orthogonal matrix O, such that for

$$\tilde{\tilde{\Xi}} = \Xi O \in R^{k\times p}, \text{ and its upper left submatrix } \tilde{\tilde{\Xi}}_{11} \in R^{k-m_W+1\times 2}$$

$\tilde{\tilde{\Xi}}'_{11}\tilde{\tilde{\Xi}}_{11}$ is a non-central Wishart 2×2 matrix of order $k - m_W + 1$ (cond'l on O), whose noncentrality matrix, $\tilde{\mathcal{M}}'_1\tilde{\mathcal{M}}_1$ say, is of reduced rank.

It then follows that

$$\begin{aligned}
\textbf{(i)} \quad AR_n(\beta_0) &= \kappa_{\min}\left(\Xi'\Xi\right) = \kappa_{\min}\left(\tilde{\tilde{\Xi}}'\tilde{\tilde{\Xi}}\right) \\
&\leq \kappa_{\min}\left(\tilde{\tilde{\Xi}}'_{11}\tilde{\tilde{\Xi}}_{11}\right) \leq \kappa_{\max}\left(\tilde{\tilde{\Xi}}'_{11}\tilde{\tilde{\Xi}}_{11}\right) \\
&\leq \kappa_{\max}\left(\tilde{\tilde{\Xi}}'\tilde{\tilde{\Xi}}\right) = \kappa_{\max}\left(\Xi'\Xi\right)
\end{aligned}$$

and thus

$$
\begin{aligned}
& P(AR_n(\beta_0) > c_{1-\alpha}(\kappa_{\max}\left(\Xi'\Xi\right), k - m_W)) \\
& \leq P(\kappa_{\min}\left(\tilde{\Xi}'_{11}\tilde{\Xi}_{11}\right) > c_{1-\alpha}(\kappa_{\max}\left(\tilde{\Xi}'_{11}\tilde{\Xi}_{11}\right), k - m_W)) \\
& \leq \alpha,
\end{aligned}
$$

where the last inequality follows from the case $m_W = 1$ (by conditionning on O).

(ii) new conditional test is uniformly more powerful than test in GKMC (because $c_{1-\alpha}(\cdot, k - m_W))$ is increasing and converging to $\chi^2_{k-m_W,1-\alpha}$ as argument goes to infinity).

Asymptotic case

- **Parameter space** $\mathcal{F}$ under the null hypothesis $H_0 : \beta = \beta_0$. Let $U_i = (\varepsilon_i, V'_{W,i})'$ and F distribution of (U_i, V_{Yi}, Z_i)

  $\mathcal{F}$ is set of all $(\gamma, \Pi_W, \Pi_Y, F)$ s.t.

  $\gamma \in R^{m_W}, \Pi_W \in R^{k \times m_W}, \Pi_Y \in R^{k \times m_Y}$,
  $E_F(||T_i||^{2+\delta}) \leq B$, for $T_i \in \{Z_i\varepsilon_i, vec(Z_i V'_{W,i}), V_{W,i}\varepsilon_i, \varepsilon_i, V_{W,i}, Z_i\}$,
  $E_F(Z_i(\varepsilon_i, V'_{Wi}, V'_{Yi})) = 0$,
  $E_F(vec(Z_i U'_i)(vec(Z_i U'_i))') = (E_F(U_i U'_i) \otimes E_F(Z_i Z'_i))$,
  $\kappa_{\min}(A) \geq b$ for $A \in \{E_F(Z_i Z'_i), E_F(U_i U'_i)\}$

  for some $b > 0$, $B < \infty$, where $\kappa_{\min}(\cdot)$ is smallest eigenvalue, "$\otimes$" Kronecker product, $vec(\cdot)$ column vectorization.

- Subvector AR stat equals

$$AR_n(\beta_0) = \kappa_{\mathsf{min}} \left(\left(\frac{\overline{Y}' M_Z \overline{Y}}{n-k} \right)^{-1/2} (\overline{Y}' P_Z \overline{Y}) \left(\frac{\overline{Y}' M_Z \overline{Y}}{n-k} \right)^{-1/2} \right)$$

where

$$\overline{Y} := (y - Y\beta_0 \vdots W) \in R^{n \times (1+m_W)}$$

- GKMC showed $\varphi_{GKMC} = 1_{\left\{AR_n(\beta_0) > \chi^2_{k-m_W,1-\alpha}\right\}}$ has correct asymptotic size for parameter space $\mathcal{F}$.

- **Current paper:** $\varphi_c = 1_{\{AR_n(\beta_0) > c_{1-\alpha}(\widehat{\kappa}_{\mathsf{max}}, k-m_W)\}}$ has correct asy size.

Asymptotic Size of conditional subvector AR test

- The derivation of asymptotic size follows the method of Andrews Cheng and Guggenberger (2011).

- The complication relative to GKMC is that we need joint limiting distribution of $\widehat{\kappa}_1, ..., \widehat{\kappa}_p$, not just the minimum, $\widehat{\kappa}_p$.

- Fortunately, we can use the results of Andrews and Guggenberger (2015) on limit distribution of eigenvalues of quadratic forms.

- It turns out that joint limit depends only on localization parameters corresponding to the singular values of

$$(E_F Z_i Z_i')^{1/2}(\Pi_W \gamma, \Pi_W)\Omega(\beta_0)^{-1/2},$$

which correspond to singular values of Θ_W (concentration matrix) in the finite sample case.

- Hence, replicates the finite sample, normal, fixed IV, known variance matrix setup.

- Correct asymptotic size then follows from correct finite sample size.

Takeaways

- We can obtain uniform power improvement over the subvector AR test in GKMC by using data-dependent critical values.

- We propose one such test whose conditional cv's are easy to compute and can be tabulated.

- In the case $m_W = 1$, i.e., when there is a single endogenous regressor whose coefficient is unrestricted under H_0, the proposed cv's are an increasing function of a first-stage F statistic for that regressor.

- There is little scope for further power improvement when $m_W = 1$ – our proposed test is nearly optimal.

Current work: Drop assumption of conditional homoskedasticity $\longrightarrow$ allow for **heteroskedasticity**.

- Lee (2014) found an example in which the subvector AR with $\chi^2_{k-m_W,1-\alpha}$ cv's overrejects when the covariance matrix does not have Kronecker product form.

- Importantly, this does not apply to iid data.

- So far, we have found correct size of the heteroskedasticity robust subvector AR test that uses $\chi^2_{k-m_W,1-\alpha}$ cv's when $m_W = 1$ and $k = 2$.

- We are working on generalizing this to higher dimensions.