THE BAYES CORRELATED EQUILIBRIA OF LARGE ELECTIONS

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This paper studies the Bayes correlated equilibria of large majority elections in a general environment with heterogeneous, private preferences. Voters have exogeneous private signals and a version of the Condorcet Jury Theorem holds when voters do not receive additional information. We show that any state-contingent outcome can be implemented in Bayes-Nash equilibrium by some expansion of the given private signal structure. We interpret the result in terms of the possibility of persuasion of privately informed voters by a biased sender. We show that persuasion does not require detailed knowledge of the distribution of voters' preferences. An implication of our result is that an outside analyst who only knows that voters receive at least the information assumed in the Condorcet Jury Theorem cannot make a robust prediction on the election outcome.

Elections are ubiquitous instruments of collective choice. This paper studies the Bayes correlated equilibria of standard majority elections. Correlated information might arise for example through communication of voters or through persuasion by a manipulator.[1]We treat the persuasion application most prominently: An interested party has information that is valuable for voters and tries to affect voters' choices by the strategic release of this information. Examples of interested parties holding and strategically releasing relevant information for voters are numerous. Consider the vote on a reform. Certain advantages of the reform are unknown to the public, and an informed politician can decide how to release information. Or consider the election of a CEO at an annual shareholder meeting. The Board of directors provides information on the candidates with the shareholder meeting brochure, through conversations, and presentations.

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This study revisits the general voting setting by Feddersen and Pesendorfer [1997]. There are two possible policies (outcomes), A and B. Voters' preferences over policies are heterogenous and depend on an unknown state, α or β, in a general way: Some voters may prefer A in state α, some prefer A in state β, and some may prefer A independently of the state (while others always prefer B). Preferences are drawn independently and identically across voters. Their preferences are each voters' private information. The election determines the outcome by a simple majority rule.

We explore the possibility and limits of persuasion, (Kamenica and Gentzkow [2011]): Prior to the election, a manipulator commits to an information structure, which is a joint distribution over states and signal realizations that are privately observed by the voters. We ask: can the manipulator ensure that a majority supports his favorite policy in a large election by choosing an appropriate signal?

In this setting, Feddersen and Pesendorfer [1997] have shown that, within a broad class of monotone preferences and conditionally i.i.d. signals, equilibrium outcomes of large elections are equivalent to the outcome with publicly known states ("information aggregation"). This may suggest that elections are robust. Our main result (Theorem 2) shows that, nevertheless, within the same class of monotone preferences for any possible state-contingent policy, there exists an expansion of the conditionally i.i.d. signal structure and a natural equilibrium that ensures that the manipulator's preferred policy is supported by a majority with probability close to one. In particular, the supported policy can be the opposite of the outcome with publicly known states, for every state.

At first, we consider the case in which all information of voters comes from a manipulator. Specifically, the main result for this baseline model shows that the manipulator can persuade a large electorate to elect any state-contingent policy with probability close to 1 if there is one belief about the likelihood that the state is α such that a voter with randomly drawn preferences prefers A with probability larger than 1/2 given this belief and another belief such that the probability of preferring A given this belief is smaller than 1/2 (Theorem 1). Denote these beliefs by p_A and p_B, respectively. In particular this condition guarantees that there is a belief $\bar{r}$ at which a random voter prefers A with probability of exactly 50% since we assume prefereces to be continuous in beliefs. Clearly, some such condition is necessary for persuasion to

[1]See Bergemann and Morris [2017] for a detailed analysis of how the study of Bayes correlated equilibria relates to existing work, including that on communication in games (Myerson [1991]), and Bayesian persuasion (Kamenica and Gentzkow [2011])

be effective: For example, if for all beliefs, each voter prefers A with probability less than 1/2, then, whatever the induced beliefs, in a large election the expected share of voters supporting A will be less than 1/2.
We show that the condition is sufficient. For example, when the manipulator's goal is to get A elected in both states we construct a signal structure as follows. Roughly speaking, with high probability, $1-\varepsilon$, the voters receive conditionally independent draws of a binary signal, a or b, with a being relatively more likely in state α and b relatively more likely in state β. With monotone preferences and $\varepsilon = 0$, this would generally ensure information aggregation in equilibrium as in Feddersen and Pesendorfer [1997]. However, with probability $\varepsilon > 0$, the manipulator induces an additional *state-of-confusion*: In this additional state, almost all voters will receive a common signal z while only few voters receive signals a or b. Thus, conditional on observing z, a voter knows that most other voters have also observed z. The consequence is that, in contrast to the usual calculus of strategic voting, there is essentially no further information about others' signals contained in the event of being pivotal. This is the critical observation, and it implies that voters behave essentially sincerely conditional on z. By choosing the relative probability of z in the two states appropriately, the posterior conditional on z will be $\bar{r}$, meaning, each voter prefers A with probability 1/2 and, hence, the election is close to being tied in the state-of-confusion. We show that, even from the viewpoint of the few voters observing signals a or b, conditional on the election being tied, it is likely that the other voters received the common signal z. By appropriately choosing the probabilities of a and b in the state-of-confusion, the posterior conditional on the state-of-confusion and conditional on a or b is is the belief p_A for which more than 1/2 of the voters support A. Hence, in the standard state, when there are only signals a and b, a large majority supports A. The main idea of the construction is that one can first characterize equilibrium for voters receiving a z signal and then use that characterization to extend the construction to voters receiving other signals.

We argue that persuasion is robust in various dimensions. First, the played equilibrium is simple and insures voters against errors. Specifically, the equilibrium profile is almost identical to voting sincerely given one's signal, conditional on the state-of-confusion. One may argue that this behavior is simple. In particular, voters just need interpret their own signal conditional on that state; they do not need to make any further inference about other voters' signals using the equilibrium strategy profile or have to know the preference distribution of the electorate. Furthermore, as will be explained in detail later, sincere behavior is 'safe' in the sense of being an ε best response conditional on being pivotal, for a neighborhood around the actual environment. Thus, even if a voter's belief about the environment and the equilibrium is slightly wrong, the cost of this error is small (even conditional on being pivotal). Second, the equilibrium is 'attracting'. In particular, its "basin of attraction" for the best response dynamic is essentially the full set of strategy profiles, except for the one (essentially unique) strategy profile that corresponds to the one type of other equilibrium:[2] If we start with any strategy profile that is close to but not exactly equal to that type of equilibrium and if we consider the voters best response to it and the voters best response to this best response, then the resulting strategy profile is arbitrarily close to the manipulated equilibrium when the number of voters is large (Proposition 1). Third, the sender does not need to know the exact parameters of the game (meaning, the distribution of the private preferences and the prior) when choosing the signal structure. One may interpret this by saying that the signal structure satisfies a version of the 'Wilson Doctorine' of not requiring excessive knowledge by the principal: Fix the signal structure and some parameter of the game. There will be an open set of parameters containing the fixed one such that for every parameter from this set there is a manipulated equilibrium that implements the senders preferred outcome. By way of contrast, as discussed momentarily, existing work assumes that the manipulator knows the exact preference of each individual voter and this knowledge is indeed used.

In the second part of the paper, we consider the setting in which voters already have access to exogeneous information of the form studied in Feddersen and Pesendorfer [1997]. Thus, if the manipulator adds no further information, the outcome would be as with publicly known states. We show that, by adding additional information, the manipulator can still persuade the voters effectively to elect any state-contingent policy (Theorem 2). In this setting, the manipulator does not have the ability to 'block' information in a small added state. However, the main idea of the construction of the baseline model works here, too. Again, the manipulator releases conditionally independent draws of a binary signal a or b, with a being relatively more likely in state

[2]For common values, it follows from a result by McLennan [1998] that the symmetric strategy that maximizes the voters' welfare is an equilibrium. In this equilibrium, information aggregates except in the added state. By continuity, it is impossible that the manipulator implements a prefered outcome in all equilibria with probability 1 when voters have almost common values. Our model nests almost common values. Hence, the robustness result presented here is the strongest possible.

α and b relatively more likely in state β. In an additional state, almost all voters receive a common signal z while only few voters receive signals a or b. We can first characterise equilibrium behaviour for voters receiving z. In the added state the game converges to a game with only the exogeneous signal. It is known that the equilibrium limit of such a game is uniquely determined. In particular, this pins down the behavior in the added state. We extend the construction to the other signals.
The main result has an important implication: If an outside observer only knows that voters have the preferences as in Feddersen and Pesendorfer [1997] and access to information that is at least as fine as theirs, then it is not certain that information is aggregated in equilibrium. Moreover, no robust prediction is possible if the observer cannot exclude that voters might receive additional information (Corollary 1).

For contrast, we discuss the possibility of persuasion with public signals. Suppose that the preferences are monotone, voters have no private signals about the state, and hold a prior at which a majority votes B. The manipulator's goal is to get A elected in both states. Revealing the state clearly increases the probability of the outcome A from zero to the prior probability of α. However, it is not possible to induce a posterior distribution such that a majority supports A for all posteriors. For public signals, Bayes consistency implies that the expected posterior is equal to the initial prior. So persuasion is only partial (Proposition 3).[3] When voters receive the exogeneous private signals as in Feddersen and Pesendorfer [1997] and the preferences are monotone, no persuasion is possible with public signals: When adding a public signal to the setting, this is equivalent to a shift in the common prior. However, we know that information is aggregated for all non-degenerate priors (Theorem 0). A degenerate posterior can only be induced by revealing a state, but this only helps information aggregation.

The study is related to work on information design in general (see Bergemann and Morris [2017] for a survey) and especially to persuasion with multiple receivers (e.g., Mathevet et al. [2016]) and to persuasion of a receiver with private information about its preferences (Kolotilin et al. [2015]) and with private signals about the state (Guo and Shmaya [2017]). To the best of our knowledge, our paper is one of few papers on persuasion that allow for exogeneous private signals, and the first to study persua-

[3] However, when the preferences are non-monotone, complete persuasion might be possible, for $n \to \infty$.

sion of multiple receivers with exogeneous private signals or with private preferences: In particular, we allow for general preference heterogeneity and voters' preferences are their private information. Persuasion in the context of elections has been studied in a number of studies under various restrictions. Alonso and Câmara [2015] study persuasion through a public signal that is observed by all voters simultaneously. Consequently, voters do not condition on being pivotal. Bardhi and Guo [2016a] study persuasion in elections with the unanimity rule. With unanimity, every voter needs to be persuaded, and hence the problem is similar to a single-receiver persuasion problem: in particular Bayes consistency is a central limitation.

In an extension, the authors discuss non-unanimous voting rules. There, our work shares with theirs the observation that the voters' conditioning on being pivotal allows relaxing the Bayesian consistency requirement. Wang [2013] studies private persuasion by conditionally independent signals. This rules out the type of persuasion through a state-of-confusion that we consider. We believe that correlation of signals is feasible in many natural applications. Chan et al. [2016] study persuasion with publicly known and monotone preferences through private signals when voting is costly. Since the preferences are private in our setting, the type of 'targeted' persuasion that is studied in the related work is not feasible here. When the preferences of individual voters are known, signals can be adjusted to them. Methodologically, with known preferences, a revelation principle argument implies that individual signals are binary without loss of generality. [4] In our work, persuasion is achieved differently, namely, through a state-of-confusion.

A more detailed discussion of the related literature is in Section 6 and in the conclusion. In Section 6 we also discuss in depth the existing work on failures of information aggregation, especially Mandler [2012], Feddersen and Pesendorfer [1997] (their extension to aggregate uncertainty about the preferences), and Bhattacharya [2013].

The rest of the study is organized as follows: In Section 1 we present the model. In Section 2, we discuss a binary-state version of Feddersen and Pesendorfer [1997] as in Bhattacharya [2013]. We relate the Condorcet Jury Theorem observed there (Theorem 0) to persuasion with public signals. We discuss public persuasion further

[4]Furthermore, given that there is a deterministic relation between signals and induced votes, the signal structure can be chosen such that the signal 'vote A' is pivotal in different profiles from the signal 'vote B'. For example, with 11 voters, if the induced signal profile is 6 'vote A' and 5 'vote B' signals, then only voters with a 'vote A' signal are pivotal. In our setting, the interpretation of being pivotal is independent of one's signal.

in the Online Supplement (Proposition 3). In Section 3, we show that persuasion is essentially limitless when the information designer is monopolistic (Theorem 1) and illustrate the robustness of the 'manipulated equilibrium'. In Section 4, we prove the main result of this paper by showing that persuasion is essentially limitless even when a manipulator can only add information to arbitrarily precise exogeneous private signals (Theorem 2). In particular, any state-contingent policy can be an equilibrium outcome. Section 5 discusses other equilibria (Proposition 2) and their instability and gives another interpretation of the main result: the equilibria of the game with a manipulator are the Bayes correlated equilibria a voting game as in Feddersen and Pesendorfer [1997]. Section 5 also discusses feasibility and evidence for the strategic voter paradigm. In Section 6, we discuss the paper's contribution to the existing literature and compare our results especially to other results on voter persuasion and other reported failures of information aggregation. The conclusion discusses the relation to the literature on auctions with general information structures.

1 Model

There are $2n+1$ voters, two possible election outcomes A and B, and two states of the world $\omega \in \{\alpha, \beta\} = \Omega$. Voters hold a common prior. The prior probability of α is $p_0 \in (0,1)$, and the probability of β is $1-p_0$.
Voters have heterogeneous preferences. The preferences are private information. A preference type is a pair $t = (t_\alpha, t_\beta) \in [-1,1]^2$, with t_ω the utility of A in ω. We normalise the utility of B to zero, so that t_ω is the difference of the utilities of A and of B in ω. Preference types are independently and identically distributed according to a commonly known distribution G that has a strictly positive, continuous density g.

An information structure π is a finite set of signals S and a joint distribution of signal profiles and states. We also denote by π the joint distribution. We assume that $\pi_{|\omega}$ is exchangeable with respect to the voters for all $\omega \in \Omega$.[5]

A symmetric strategy of the voters is a function of the signal s and the type t, and denoted by $\sigma : S \times [-1,1]^2 \to [0,1]$ where $\sigma(s,t)$ is the probability of type t to vote

[5]The joint distribution F of discrete random variables $Y_1, \ldots, Y_{2n+1}$ is called exchangeable if $\Pr_F(y_1 = z_1, \ldots, y_{2n+1} = z_{2n+1}) = \Pr_F(y_1 = z_{h(1)}, \ldots, y_{2n+1} = z_{h(2n+1)})$ for all realisations $(z_1, \ldots, z_{2n+1})$ and all permutation h of $\{1, \ldots, 2n+1\}$.

A after s. For any $\omega \in \Omega$, we denote by

$$\Pr(\sigma(s,t)=1|\omega) := \int_{s\in S}\int_{t\in[-1,1]^2} \sigma(s,t)dG(t)d\pi(s|\omega)$$

the probability that a citizen votes A in ω. Similarly, for any $s \in S$, we denote by

$$\Pr(\sigma(s,t)=1|s) := \int_{t\in[-1,1]^2} \sigma(s,t)dG(t)$$

the probability that a citizen who received s votes A.

Aggregate Preferences. For a given strategy σ, we use piv to denote the event in which, from the viewpoint of a given voter, n of the other $2n$ voters vote for A and n for B. In this event, if she votes A, the outcome is A, if she votes B, the outcome is B. In any other event, the outcome is independent of her vote. Thus, a strategy is optimal if and only if it is optimal conditional on piv. Given σ, a voter of type t who received s weakly prefers to vote A if and only if

$$\Pr(\alpha|s,\text{piv};\sigma,\pi)\cdot t_\alpha + (1-\Pr(\alpha|s,\text{piv};\sigma,\pi))\cdot t_\beta \quad \geq \quad 0. \tag{1}$$

A central object of our analysis is the *aggregate preference function*

$$\phi(p) := \Pr_G(\{t : p\cdot t_\alpha + (1-p)\cdot t_\beta > 0\}), \tag{2}$$

which maps a common belief p to the probability that a random type t prefers A under p.[6] Note that ϕ is continuously differentiable, since G has a continuous density.

REMARK 1 *The collection of posteriors conditional on* piv *and* s*, namely* $(\Pr(\alpha|s,\text{piv};\sigma,\pi))_{s\in S}$*, is a sufficient statistic for the unique best response (recall the inequality (1)). The possibility of writing equilibria in terms of posteriors is what makes our model easily amenable to the Bayesian persuasion literature.*

Equilibrium. Any information structure π induces a Bayesian game of voters, denoted by $\Gamma(\pi)$. We analyse the symmetric Bayes-Nash-equilibria of $\Gamma(\pi)$ in weakly undominated, pure strategies and call them (voting) equilibria.[7] Voter types $t \gg 0$ (*A-partisans*) have the weakly dominant strategy to vote for A. Voter types $t \ll 0$ (*B-partisans*) have the weakly dominant strategy to vote for B. The restriction to undominated equilibria rules out trivial equilibria: the distribution G puts strictly positive probability on voter types $t >> 0$ and $t << 0$ by the assumption that it has

[6]Occasionally, it is convenient to work with belief ratios instead of beliefs. We define $\bar{\phi}(y) := \phi(\frac{y}{1+y})$ for any $y \in \mathbb{R}$, which is equivalent to $\bar{\phi}(\frac{p}{1-p}) = \phi(p)$ for any $p \in (0,1)$. The function $\bar{\phi}$ maps belief ratios y to the probability that a random type t prefers A under y.

[7]Since we assumed that π is exchangeable with respect to the voters, a symmetric equilibrium exists.

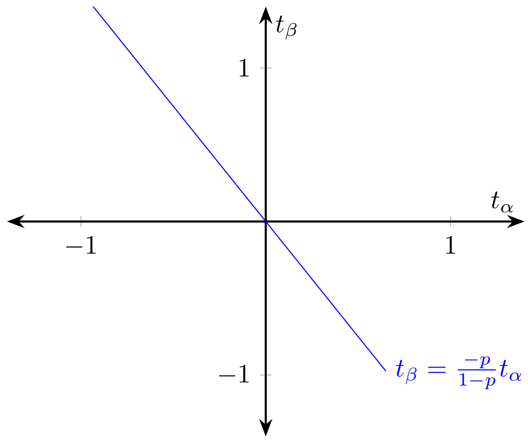

Figure 1: The curve of indifferent types is $t_\beta = \frac{-p}{1-p} t_\alpha$ for any given belief $p = \Pr(\alpha) \in (0,1)$.

a strictly positive density. Hence, there exists $\epsilon > 0$ such that for all $s \in S$, and any undominated strategy σ,

$$\epsilon < \Pr(\sigma(s,t) = 1|s) < 1 - \epsilon. \tag{3}$$

This ensures that for any $\omega \in \Omega$ and $s \in S$, and any undominated strategy σ, we have $\Pr(\text{piv}|s, \omega; \sigma) > 0$, so that the posterior $\Pr(\omega|s, \text{piv}; \sigma; \pi)$ is well-defined by Bayes' rule. The restriction to equilibria in pure strategies is without loss, because, by the inequality (1) and the continuity of G, a voter has a unique strict best response with probability 1. A strategy σ is a *cutoff strategy*, if for all $s \in S$ there exists $p_s \in [0,1]$ such that $\sigma(s,t) = 1 \Leftrightarrow t_\alpha \cdot p_s + t_\beta \cdot (1 - p_s) \geq 0$. Any best reponse is a cutoff strategy with cutoffs $p_s = \Pr(\alpha|s, \text{piv}; \sigma, \pi)$ by the inequality (1).

Information Aggregation. The full information outcome in $\omega \in \Omega$ is the outcome which is prefered by a random voter with probability weakly larger than $\frac{1}{2}$ conditional on ω. The literature on information aggregation in elections is concerned with the question of whether strategy sequences σ_n imply the full information outcome when n grows to infinity.

REMARK 2 *Given the general preference distribution G, the model nests almost common values. Moreover, it does not only include the case in which the full information outcome is A in α and B in β, but also all cases in which the full information outcome does not match the state.*

Convergence. Convergence of strategies means pointwise convergence (up to measure 0). A sequence of cutoff-strategies σ_n with cutoffs $(p_{s,n})_{s\in S}$ converges to a cutoff strategy σ with cutoffs $(p_s)_{s\in S}$ if and only if $\lim_{n\to\infty} p_{s,n} = p_s$ for all $s \in S$. When we speak of distances between two cutoff strategies, we mean the Euclidean distance. When we discuss limits of statistics of sequences of strategies, we implicitly refer to a converging subsequence such that the limit exists.[8]

2 Benchmark: Condorcet Jury theorem

In this section we analyse the situation when voters receive private signals from an exogeneous information structure π_1. The information structure π_1 sends binary signals $S_1 = \{u, d\}$ that are independently, and identically distributed across voters conditional on the state of the world $\omega \in \Omega$. We make the following assumption on the informativeness of signals,

$$1 > \Pr(u|\alpha) > \Pr(u|\beta) > 0. \tag{4}$$

Hence, signal u is indicative of α, and signal d is indicative of β. We assume that

$$\begin{aligned} &\phi(p) \text{ is strictly increasing in } p, \\ &\phi(0) < \frac{1}{2} < \phi(1). \end{aligned} \tag{5}$$

The second part $\phi(0) < \frac{1}{2} < \phi(1)$ is a mild richness condition. Given that ϕ is strictly increasing, the assumption $\phi(0) < \frac{1}{2} < \phi(1)$ excludes two trivial cases: If $\phi(p) < \frac{1}{2}$ for all $p \in [0, 1]$, in any equilibrium sequence σ_n, the probability that B is elected converges to 1. If $\phi(p) > \frac{1}{2}$ for all $p \in [0, 1]$, in any equilibrium sequence σ_n, the probability that A is elected, converges to 1. Therefore, with a slight abuse of language, we say that the preferences are 'monotone' when the preferences satisfy both conditions in (5). Note that the full information outcome is A in α and B in β when $\phi(0) < \frac{1}{2} < \phi(1)$.

Note that the model in this section describes a binary-state version of Feddersen and Pesendorfer [1997] as studied in Bhattacharya [2013].[9]

Sincere Voting. The *sincere strategy* $\hat{\sigma}$ is the strategy that acts upon the posteriors conditional on the signal s only; it is the pure strategy given by

$$\hat{\sigma}_n(s, t) = 1 \qquad \Leftrightarrow t_\alpha \cdot \Pr(\alpha|s) + t_\beta \cdot (1 - \Pr(\alpha|s)) \geq 0. \tag{6}$$

[8] All sequences that we analyse are sequences of real numbers such that there always exists a subsequence that converges in the extended reals. Typically, we show that such a limit is unique, so the sequence itself converges to the unique limit of subsequences.

[9] The notation for the function ϕ is h in Bhattacharya [2013].

When voters vote sincerely and the prior is sufficiently extreme, sincere voting does not necessarily aggregate information: For example, if p_0 is sufficiently low such that $\phi(\Pr(\alpha|u)) < \frac{1}{2}$, a random voter votes A with probability smaller than $\frac{1}{2}$ after any signal. The law of large numbers implies that B is elected with probability converging to 1. However, if priors are not too extreme, then, if voters vote sincerely and signals are relatively precise, the full information outcome is elected with probability converging to 1: For example, suppose that the prior is sufficiently close to $\phi^{-1}(\frac{1}{2})$ such that $\phi(\Pr(\alpha|u)) > \frac{1}{2}$ and $\phi(\Pr(\alpha|d)) < \frac{1}{2}$. Note that under sincere voting, the vote probabilities in each state ω are a convex combination of the vote probabilities conditional on the signals u and d: $\Pr(\hat{\sigma}(s,t) = 1|\omega) = \sum_{s\in\{u,d\}} \Pr(s|\omega)\phi(\Pr(\alpha|s))$. Hence, if in addition, signals are sufficiently precise, we have $\Pr(\hat{\sigma}(s,t) = 1|\alpha) > \frac{1}{2} > \Pr(\hat{\sigma}(s,t) = 1|\beta)$. The law of large numbers implies that the full information outcome is elected with probability converging to 1. This instance of the Condorcet Jury Theorem (Condorcet [1793]) is illustrated in Figure 3.

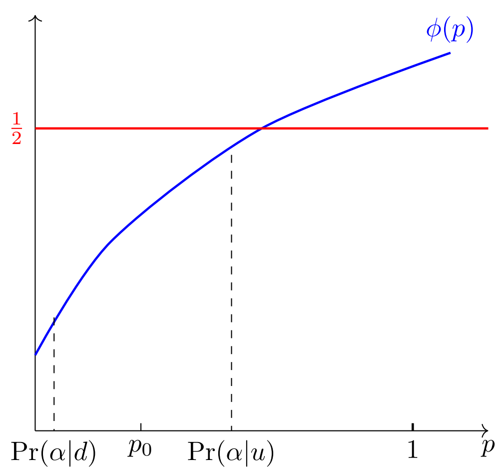

Figure 2: Under $\hat{\sigma}$, information aggregation fails with sufficiently extreme priors.

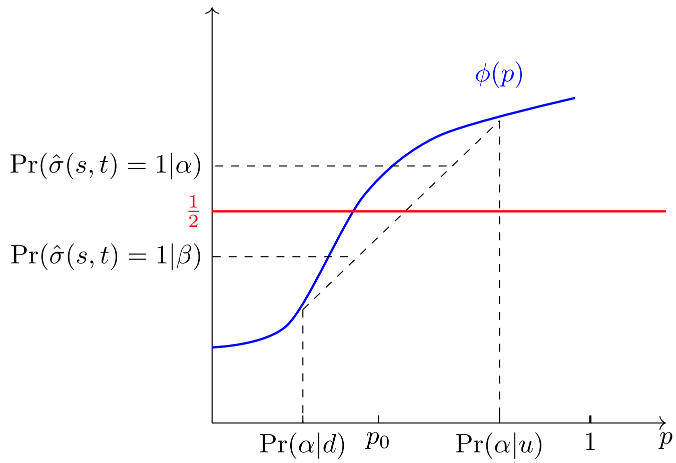

Figure 3: Under $\hat{\sigma}$, information can be aggregated with intermediate priors and sufficiently precise signals.

Strategic Voting. Bhattacharya [2013] has replicated a result by Feddersen and Pesendorfer [1997], namely that the Condorcet Jury Theorem extends to strategic voting.[10]

THEOREM 0 *(Bhattacharya [2013]).*[11] *Let voters receive private signals from an information structure* π_1 *that sends independently, and identically distributed binary signals from* $S_1 = \{u, g\}$ *with* $1 > \Pr(u|\alpha) > \Pr(u|\beta) > 0$. *Let the preferences be monotone (that is, the conditions in (5) hold). Then, for any sequence of equilibria* σ_n,

$$\begin{aligned}\lim_{n\to\infty} \Pr(A \textit{ is elected}|\alpha;\sigma_n) &= 1,\\ \lim_{n\to\infty} \Pr(B \textit{ is elected}|\beta;\sigma_n) &= 1.\end{aligned}$$

Proof. In the Appendix. ■

[10] Moreover, the result under strategic voting is stronger than under sincere voting, since information is aggregated for all priors on the state.

[11] This theorem is a special case of Theorem 1 in Bhattacharya [2013]. We provide the proof for the convenience of the reader.

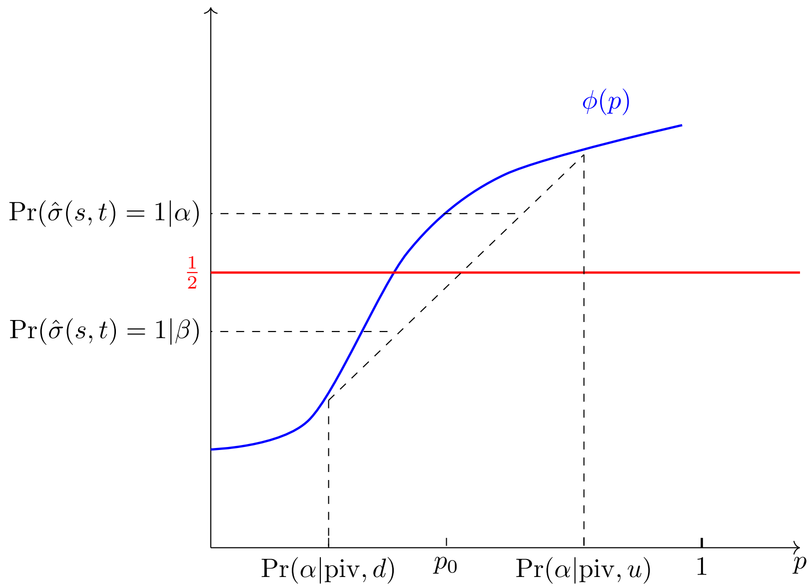

Figure 4: Condorcet Jury Theorem in Bhattacharya [2013]: In any equilibrium sequence, $\lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1|\alpha) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - \Pr(\sigma_n(s,t) = 1|\beta)$ holds.

On the Impossibility of Persuasion with Public Signals: Theorem 0 implies that persuasion is not possible with public signals when voters receive exogeneous private signals and the preferences are monotone:[12] When adding a public signal to a setting as in Feddersen and Pesendorfer [1997], this is equivalent to a shift in the common prior. However, it follows from Theorem 0 that information is aggregated for all possible non-degenerate priors. On the other hand, a degenerate prior can only be induced by revealing a state, but this only helps information aggregation.

3 No Exogeneous Private Signals

In this section, we consider the situation when all the information of voters comes from a manipulator. We consider a class of information structures with $S = \{a, b, z\}$ illustrated in Figures 5 and 6.

[12]In the Online Supplement, we analyse persuasion with public signals when voters do not receive exogeneous private signals and when the preferences are non-monotone: The optimal public signal can be found by concavification, similar to Alonso and Câmara [2015].

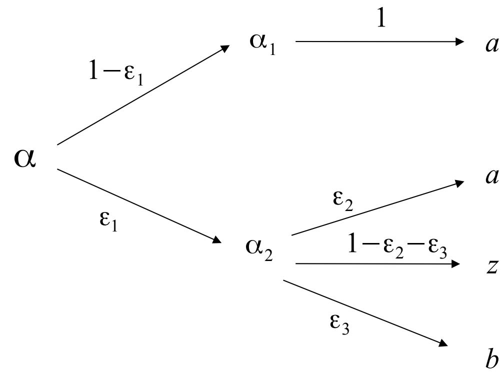

Figure 5: Distribution of signals in α

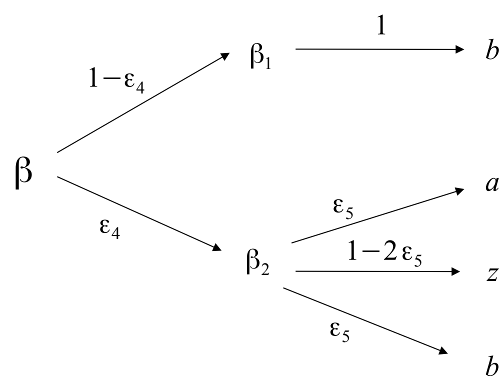

Figure 6: Distribution of signals in β

First, nature draws the state $\omega \in \{\alpha, \beta\}$ according to p_0. Then, a substate ω_j is drawn with $j \in \{1, 2\}$. Conditional on ω_j, voters receive independently and identically distributed signals $s \in \{a, b, z\}$. The probabilities by which the substates ω_j are drawn and the probabilities by which the signals are sent to voters conditional on ω_j are indicated along the arrows.

The choice of information structures serves two purposes: Note, that, when $\epsilon_i = 0$ for all $i \in \{1, \ldots, 5\}$, the information structure perfectly reveals the state. Firstly, by letting ϵ_i converge to 0 for $n \to \infty$ for any $i \in \{1, \ldots, 5\}$, we deviate marginally from the benchmark $\epsilon_i = 0$ and in this way illustrate most clearly why persuasion is possible when the receivers are a large electorate. We choose

$$\epsilon_4 = \frac{1}{2n}, \quad \epsilon_5 = \frac{1}{n^2}.^{13} \tag{7}$$

We implicitly define parameters q, l and r by

$$\epsilon_1 = \frac{1-p_0}{p_0}\frac{r}{1-r}\epsilon_4, \quad \epsilon_2 = \frac{1-r}{r}\frac{q}{1-q}\epsilon_5, \quad \epsilon_3 = \frac{1-r}{r}\frac{l}{1-l}\epsilon_5. \tag{8}$$

This way, the information structures in Figure 5 and 6 constitute a family $\pi_n(q,r,l)$. Secondly, we will be able to implement any state-contingent policy in equilibrium simply by varying the parameters q, l and r. The parameters q, l and r have an easy interpretation: A voter who received z knows that the true substate is in $\Omega_2 := \{\alpha_2, \beta_2\}$.[14] However, a voter who received $s \in \{a, b\}$ is unsure if a substate in Ω_1 or in Ω_2 holds. Supppose that he considers which state $\omega \in \Omega$ must hold if he would have the additional information that the substate is in Ω_2. The posteriors $\Pr(\alpha|s, \Omega_2; q, r, l)$ describe the result of this thought experiment. By definition of $\pi_n(q,r,l)$, they satisfy

$$\begin{aligned} \Pr(\alpha|a, \Omega_2; q, r, l) &= q \quad \text{for all } n \in \mathbb{N}, & (9)\\ \lim_{n\to\infty} \Pr(\alpha|z, \Omega_2; q, r, l) &= r, & (10)\\ \Pr(\alpha|b, \Omega_2; q, r, l) &= l \quad \text{for all } n \in \mathbb{N}. & (11) \end{aligned}$$

For any $(q,r,l) \in [0,1]^3$, the game $\Gamma_n(q,r,l)$ is the game of n voters induced by $\pi_n(q,r,l)$.

3.1 Result Without Exogeneous Private Signals

We will show that any state-contingent outcome can be implemented in some equilibrium sequence. To do so, we choose the information structure parameters q, l and r appropriately, and construct equilibrium sequences that converge to the following strategy: The Ω_2-sincere (or conditional sincere) strategy $\hat{\sigma}_{\Omega_2}(q,r,l)$ is the pure strategy under which a voter who received $s \in \{a, b, z\}$ votes A if and only if [15]

$$t_\alpha \cdot \Pr(\alpha|s, \Omega_2; q, r, l) + t_\beta \cdot (1 - \Pr(\alpha|s, \Omega_2; q, r, l)) \geq 0. \tag{12}$$

To make the analysis interesting, we assume that there exist beliefs $p_A, p_B \in [0,1]$,

[14] Similarly, we define $\Omega_1 := \{\alpha_1, \beta_1\}$ and we denote the generic element of Ω_i by ω_i.

[15] Recall the interpretation of the parameters q, l and r in terms of the posteriors $\Pr(\alpha|s, \Omega_2; q, r, l)$ in equations (9), (10) and (11).

such that[16]

$$\phi(p_A) > \frac{1}{2} > \phi(p_B). \tag{13}$$

Since ϕ is continuous, the intermediate value theorem implies that there exists $\bar{r}$ for which

$$\phi(\bar{r}) = \frac{1}{2}. \tag{14}$$

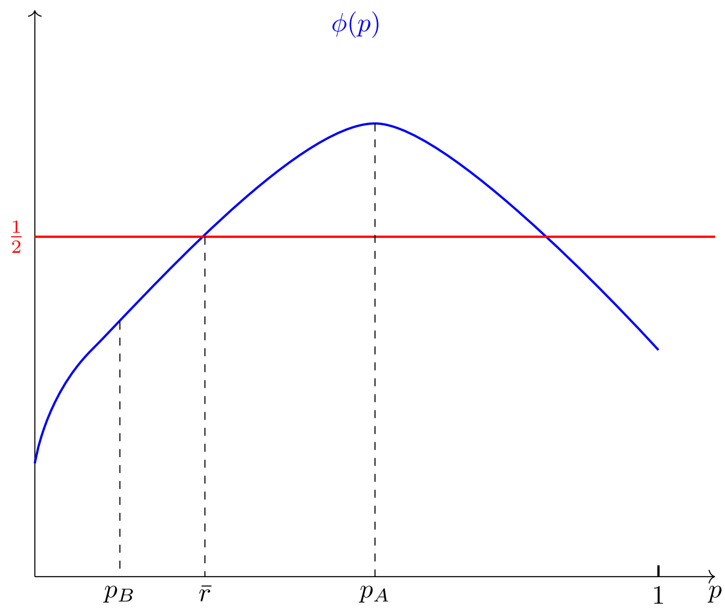

Figure 7: Under p_A, a random voter prefers A with probability larger than $\frac{1}{2}$. Under p_B, a random voter prefers B with probability larger than $\frac{1}{2}$. Under $\bar{r}$, a random voters prefers A and B with probability $\frac{1}{2}$.

Intuitively, condition (13) describes two aspects of G:

No majority of A- or B-partisans. The voters that prefer to vote B regardless of their belief do not represent a majority, for $n \to \infty$. These are the types with $t << 0$. The same holds for A-partisans.

Asymmetry of Information-Sensitive Types. There must be an asymmetry

[16]If $\phi(p) < \frac{1}{2}$ for all $p \in [0, 1]$, the election outcome is B with probability converging to 1, for $n \to \infty$, for any information structure, and for any equilibrium sequence. Conversely, if $\phi(p) > \frac{1}{2}$ for all $p \in [0, 1]$,the election outcome is A with probability converging to 1 for $n \to \infty$, for any information structure, and for any equilibrium sequence.

between the voter types who prefer A only in state α, that is, those for which $t_\alpha > 0$ and $t_\beta < 0$, and the voter types who prefer A only in state β, that is, those for which $t_\alpha < 0$ and $t_\beta > 0$. If both groups of voter types are equally likely, and the density of G is symmetric, meaning that it takes the same values at (t_α, t_β) and at $(-t_\alpha, -t_\beta)$ for all (t_α, t_β) with $t_\alpha > 0$ and $t_\beta < 0$, then the function $\phi(p) = \Pr_G(p \cdot t_\alpha + (1-p) \cdot t_\beta > 0)$ is constant in p. Then, the condition cannot be fulfilled.

Theorem 1 *For any preference distribution G that satisfies condition (13), the following holds: For any state-contingent outcome $x_\alpha \in \{A, B\}$ and $x_\beta \in \{A, B\}$, the following holds in the games with signals $\pi_n(p_{x_\alpha}, \bar{r}, p_{x_\beta})$ from the manipulator:*

- *there exists an equilibrium sequence σ_n, such that for all $\omega \in \Omega$,*
  $$\lim_{n\to\infty} \Pr(x_\omega \textit{ is elected}|\omega; \sigma_n, \pi_n(p_{x_\alpha}, \bar{r}, p_{x_\beta})) = 1,$$
- *the equilibrium sequence σ_n converges to conditional sincere voting, that is, $\lim_{n\to\infty} \sigma_n = \hat{\sigma}_{\Omega_2}(p_{x_\alpha}, \bar{r}, p_{x_\beta})$.*

Weakened Bayes Consistency Constraints: Note that, for example, under $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$, for $n \to \infty$, agents act upon the posterior p_A with probability converging to 1, independently of the prior p_0. The Bayes consistency constraints for persuasion of multiple voters vanish completely for $n \to \infty$. This is in stark contrast to persuasion of a single receiver where posteriors have the martingale property (Kamenica and Gentzkow [2011]).

3.2 Proof of Theorem 1

The proof is provided for the case in which $x_\omega = A$ for $\omega \in \{\alpha, \beta\}$. This is done for the ease of exposition. The proof for the other cases is completely analogous.

Outlook. We show that, in $\Gamma_n(p_A, \bar{r}, p_A)$, under $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ being pivotal asymptotically means that Ω_2 holds, that is,

$$\begin{aligned} & \lim_{n\to\infty} (\Pr(\alpha|\text{piv}, s; \hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)))_{s\in\{a,b,z\}} \\ = & \lim_{n\to\infty} (\Pr(\alpha|\Omega_2, s; \hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)))_{s\in\{a,b,z\}} \\ = & (p_A, \bar{r}, p_A). \end{aligned} \tag{15}$$

where we used the equations (9), (10) and (11) for the last equality and suppressed the dependence of the posteriors on the information structure $\pi_n(p_A, \bar{r}, p_A)$. Recall

Remark (1) and the equation (1): The posteriors $(\Pr(\alpha|s,\text{piv};\sigma_n,p_A,\bar{r},p_A))_{s\in\{a,b,z\}}$ are a sufficient statistic for the best response. Thus, the equation (15) means that the best response to $\hat{\sigma}_{\Omega_2}(p_A,\bar{r},p_A)$ converges to $\hat{\sigma}_{\Omega_2}(p_A,\bar{r},p_A)$.

In the states Ω_2, almost all voters receive the common signal z, for any q,r,l. Conditional on observing z, a voter knows that either α_2 or β_2 holds and that most other voters have also observed z. In fact, the probability that all voters received z in α_2 and β_2 converges to 1 for $n\to\infty$: for example $\lim_{n\to\infty}\Pr(\text{all voters received } z|\beta_2) = \lim_{n\to\infty}(1-\frac{1}{n^2})^{2n+1} = 1$.[17] Intuitively, in contrast to the usual calculus of strategic voting, there is no further information about others' signals contained in the event of being pivotal. We record

Lemma 2 *For any sequence $(\sigma_n)_{n\in\mathbb{N}}$ of strategies, it holds that*

$$\lim_{n\to\infty}\Pr(\alpha|\text{piv},z;\sigma_n,q,r,l) = \lim_{n\to\infty}\Pr(\alpha|\Omega_2,z;\sigma_n,q,r,l) = r.$$

Proof. In the Appendix. ■

Lemma 2 implies that after signal z, agents behave sincerely for $n\to\infty$. This implies that we can control the behavior of agents getting z perfectly. In particular, we can make the election arbitrarily close to being tied in ω_2 for any $\omega_2\in\{\alpha_2,\beta_2\}$ by choosing r appropriately. By definition of $\bar{r}$ in equation (14), under the belief $\bar{r}$ a random voter prefers A with probability $\frac{1}{2}$. Thus, given $\pi_n(q,\bar{r},l)$ and any equilibrium sequence σ_n,

$$\lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|z,\sigma_n;q,\bar{r},l) = \phi(\bar{r}) = \frac{1}{2}. \tag{16}$$

Lemma 3 *If*

$$\max_{\omega_2\in\Omega_2}|\lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|\omega_2;q,r,l)-\frac{1}{2}| \tag{17}$$
$$< \min_{\omega_1\in\Omega_1}|\lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|\omega_1;q,r,l)-\frac{1}{2}|$$

holds, then

$$\lim_{n\to\infty}\Pr(\alpha|a,\text{piv};\sigma_n,q,r,l) = \Pr(\alpha|s,\Omega_2;q,r) = q,$$
$$\lim_{n\to\infty}\Pr(\alpha|b,\text{piv};\sigma_n,q,r,l) = \Pr(\alpha|s,\Omega_2;q,r) = l.$$

[17] Here we used the limit description $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$ for the exponential function. Note that $(1-\frac{1}{n^2}) = (1+\frac{1}{n})(1-\frac{1}{n})$. Therefore, $\lim_{n\to\infty}(1-\frac{1}{n^2})^{2n} = (1-\frac{1}{n})^{2n}(1+\frac{1}{n})^{2n} = \lim_{n\to\infty} e^2e^{-2} = e^0 = 1$.

Hence, the unique best response to σ_n in the games $\Gamma_n(q,r,l)$ converges to $\hat{\sigma}_{\Omega_2}(q,r,l)$ for $n \to \infty$.

Proof. In the Appendix. ■

Lemma 3 shows that if the limit of the expected margin of victory in the states Ω_1 is strictly larger than the limit of the expected margin of victory in the states Ω_2 (call this the *'margin of victory condition'*), the unique best response converges to Ω_2-sincere voting $\hat{\sigma}_{\Omega_2}(q,r)$ for $n \to \infty$. Intuitively, when the margin of victory-condition holds, conditional on being tied, the states Ω_2 are infinitely more likely than the states Ω_1 for $n \to \infty$. Hence, being pivotal contains the information that the states ω_1 do not hold, but no information beyond that, by Lemma 2. This is precisely the information that Ω_2-sincere voters condition on. Hence, the best reply converges to $\hat{\sigma}_{\Omega_2}$.

More precisely, the margin of victory condition implies $\lim_{n\to\infty} \frac{\Pr(\omega_1|s,\text{piv};\sigma_n,q,r,l)}{\Pr(\omega_2'|s,\text{piv};\sigma_n,q,r,l)} = \lim_{n\to\infty} \frac{\Pr(\omega_1|s;q,r,l)}{\Pr(\omega_2'|s;q,r,l)} \frac{\Pr(\text{piv}|\omega_1;\sigma_n,q,r,l)}{\Pr(\text{piv}|\omega_2;\sigma_n,q,r,l)} = 0$ for $s \in \{a,b\}$ and any $\omega_1 \in \Omega_1$, and any $\omega_2' \in \Omega_2$. This can be seen in the following manner: The probability of the election being tied is decreasing exponentially faster in states ω_1 than in states ω_2. Conditional on the signal $s \in \{a,b\}$, states Ω_2 are less likely than states Ω_1. However, note that the ratios $\frac{\Pr(\omega_1|s;q,r,l)}{\Pr(\omega_2|s;q,r,l)}$ are only increasing at a rate proportional to n^3 for $s \in \{a,b\}$. Therefore, the exponentially decreasing term $\frac{\Pr(\text{piv}|\omega_1;\sigma_n,q,r,l)}{\Pr(\text{piv}|\omega_2;\sigma_n,q,r,l)}$ dominate and the posteriors conditional on being pivotal and conditional on $s \in \{a,b\}$ vanish on ω_1. Being pivotal contains the information that the states Ω_1 do not hold for $n \to \infty$.

Equilibrium Construction. By Lemma 3, we can control the limit behaviour of agents getting $s \in \{a,b\}$ by choosing $q = \Pr(\alpha|a,\Omega_2;q,r)$ and $l = \Pr(\alpha|b,\Omega_2;q,r)$ appropriately. We choose $q = l = p_A$ for some $p_A \in (0,1)$ that satisfies the inequality (13). In the games $\Gamma_n(p_A,\bar{r},p_A)$, under the Ω_2-sincere strategy $\hat{\sigma}_{\Omega_2}(p_A,\bar{r},p_A)$ and for $n \to \infty$, a strict majority of agents votes A after getting $s \in \{a,b\}$:

$$\lim_{n\to\infty} \Pr(\hat{\sigma}_{\Omega_2}(s,t)=1|s;p_A,\bar{r},p_A) = \phi(p_A) > \frac{1}{2} \tag{18}$$

for $s \in \{a,b\}$, where we used the inequality (13) for the last inequality. Under $\hat{\sigma}_{\Omega_2}(p_A,\bar{r},p_A)$, the limit of the expected margin of victory in the states Ω_2 is zero; see equation (16). Thus, $\hat{\sigma}_{\Omega_2}(p_A,\bar{r},p_A)$ satisfies the margin of victory condition (17) of Lemma 3. In fact, there exists $\epsilon > 0$ such that the margin of victory condition (17) holds for all $\sigma \in \mathrm{B}_\epsilon(\hat{\sigma}_{\Omega_2}(p_A,\bar{r},p_A))$.[18] So, for any $\sigma \in \mathrm{B}_\epsilon(\hat{\sigma}_{\Omega_2}(p_A,\bar{r},p_A))$, the best

[18]For any σ, we denote by $\mathrm{B}_\epsilon(\sigma)$ the set of all cutoff strategies σ' for which the Euclidean distance

reply converges to $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ for $n \to \infty$. Hence, there exists $n(\epsilon)$ such that for all $n \geq n(\epsilon)$, and for all $\sigma \in \mathrm{B}_\epsilon(\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A))$,

$$\mathrm{BR}(\sigma) \in \mathrm{B}_\epsilon(\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)). \tag{19}$$

We apply Brouwers fixed point theorem. Hence, there exists a sequence of equilibria that converges to $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$. This shows the claim of Theorem 1 for the case $x_\omega = A$ for all $\omega \in \{\alpha, \beta\}$, because under any strategy close-by to $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$, A gets elected with certainty, for $n \to \infty$.

Remark 3 *We call the states Ω_2 the 'states-of-confusion', because information aggregation is not possible in Ω_2 since voters receive an almost public signal z. Moreover, by the equality (16), the election outcome is purposefully highly uncertain in Ω_2.*

Remark 4 *(Belief Trap Ω_2.)*[19] *The states Ω_2 function as a belief trap. When voters believe that being pivotal contains (only) the information that Ω_2 holds, and best respond to this belief by voting Ω_2-sincerely, behaviour can be arbitrarily manipulated by choice of r and q (compare to the equations (9)-(11)). As long as q and r are chosen such that Ω_2-sincere voting satisfies the margin of victory condition (3), voters are 'trapped' into believing that Ω_2 holds conditional on being pivotal.*

Supermajority Rules. Note that the proof of Theorem 1 does not rely on voting by simple majority rule. The result extends to any non-unanimous majority rule if we replace the condition in (13) by the assumption that there exist beliefs $p_A, p_B \in [0,1]$ with $\phi(p_A) > \frac{1}{2} > \phi(p_B)$. When the preferences are monotone, this assumption is satisfied for all $\tau \in (0,1)$. This is similar to the Condorcet Jury Theorem as in Feddersen and Pesendorfer [1997] (cf. Theorem 0) which holds for all non-unanimous majority rules.

Computational Example. We specify the preferences by the assumptions that $\Pr(\{t : t_\alpha > 0, t_\beta < 0\}) = 1$[20], and that $\phi(p) = p$ for all $p \in [0,1]$[21] which implicitly defines G. Further, we set $p_0 = \frac{1}{4}$. When $\phi(p) = p$ for all $p \in (0,1)$, then $\frac{1}{2}$ satisfies the equality (16) and $\frac{3}{4}$ satisfies the inequality (13). So, we have $\bar{r} = \frac{1}{2}$, and $p_A = \frac{3}{4}$.

of their cutoffs to the cutoffs of σ is smaller or equal to ϵ.

[19] We thank Sourav Bhattacharya for making us aware of the notion of belief traps.

[20] Note that this is slightly inconsistent with the assumption that G has a strictly positive density on $[-1,1]^2$, but is done for the simplicity of presentation.

[21] One distribution G on $[0,1] \times [-1,0]$ that induces such a uniform distribution of 'thresholds of

In the Appendix, we show that under these primitives, an equilibrium σ_n close to conditional sincere voting exists for $n \geq 200$ in the games $\Gamma_n(p_A, \bar{r}, p_A)$. In addition, in this equilibrium, A is elected with a probability of over 99%. To do so, we show that under the specified primitives, the best reponse is a self-map on the set of strategies σ satisfying $\Pr(\sigma(s,t) = 1|s') \geq 0.7$ for $s' \in \{a, b\}$, and $\Pr(\sigma(s,t) = 1|z) \in [0.45, 0.54]$ for $n \geq 200$. This yields an equilibrium in which voters with an a-or b-signal vote A with a probability of at least 70%.

3.3 Robustness

This section further analyses the games $\Gamma_n(p_A, \bar{r}, p_A)$ of Theorem 1.

Conditional Sincere Voting is Simple. The voting strategy is simple to operationalise: If we want to tell a voter to behave conditionally sincere, then this will only require the voter to calculate his personal beliefs. It would not require knowledge of G or the strategies of others. It is simple to rationalise: Conditional sincere voting is an equilibrium (limit) by the simple logic that it is optimal to condition on the states Ω_2 if the expected margin of victory is smaller in Ω_2 than in Ω_1 (cf. Lemma 3). If all voters actually condition on Ω_2 and vote Ω_2-sincerely, the underlying assumption on the order of the margin of victories is indeed true, intuitively, because voters receive an almost public signal in Ω_2, which induces a close election outcome by construction.

Basin of Attraction.[22] Recall from the proof of Theorem 1 that the best response is a self-map on an ϵ-neighbourhood of $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$, for sufficiently large n; see formula (19). This implies that, in the games $\Gamma_n(p_A, \bar{r}, p_A)$, conditional sincere voting has a non-trivial basin of attraction with respect to the best response dynamics, for sufficiently large n. In fact, something stronger is true: Denote by $BR(\sigma)$ the best response to a strategy σ, and by BR^2 the twice iterated best response, $BR^2(\sigma) =$

doubt' is given by the density

$$g(t_\alpha, t_\beta) = \begin{cases} \sqrt{1 + (\frac{t_\beta}{t_\alpha})^2} \cdot (2 \cdot \int_{|t_\alpha| > |t_\beta|} \sqrt{1 + (\frac{t_\beta}{t_\alpha})^2} dt)^{-1} & \text{if} \quad \frac{-t_\beta}{t_\alpha - t_\beta} \leq \frac{1}{2}, \\ \sqrt{1 + (\frac{t_\alpha}{t_\beta})^2} \cdot (2 \cdot \int_{|t_\alpha| > |t_\beta|} \sqrt{1 + (\frac{t_\beta}{t_\alpha})^2} dt)^{-1} & \text{if} \quad \frac{-t_\beta}{t_\alpha - t_\beta} \geq \frac{1}{2}. \end{cases}$$

[22] In this paragraph, we analyse *approximate* limit behaviour of the best response dynamics. Note that we do not prove that the best reponse correspondence converges to conditional sincere voting or or that it converges at all. In this sense, we slightly deviate from the typical use of the notion of a basin of attraction.

$BR(BR(\sigma))$. Further, for any $\epsilon > 0$, and any $n \in \mathbb{N}$ define

$$\Sigma^2(\epsilon, n) := \{\sigma : \sigma \text{ cutoff strategy for which } |BR^2(\sigma) - \hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)| < \epsilon\}.$$

Proposition 1 *(Global Basin of Attraction)*[23]
When $\phi(0) < \frac{1}{2} < \phi(1)$: For any $\epsilon > 0$, the measure of $\Sigma^2(\epsilon, n)$ in the space of cutoff-strategies $[0,1]^3$ converges to 1, for $n \to \infty$.

Thus, for an arbitrarily large set of strategy profiles, the best response dynamics will be arbitrarily close to $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ already after two iterations.
Proof. In the Appendix. ■

Sketch of Proof. We show that, for any $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$, such that for $n \geq n(\epsilon)$ all cutoff strategies σ that satisfy

$$\Big||\Pr(\sigma(s,t) = 1|\alpha_1) - \frac{1}{2}| - |\Pr(\sigma(s,t) = 1|\beta_1) - \frac{1}{2}|\Big| > n^{-\frac{1}{4}} \quad (20)$$

$$\Big|\min_{\omega_1 \in \{\alpha_1, \beta_1\}} |\Pr(\sigma(s,t) = 1|\omega_1) - \frac{1}{2}| - \min_{\omega_2 \in \{\alpha_2, \beta_2\}} |\Pr(\sigma(s,t) = 1|\omega_2) - \frac{1}{2}|\Big| > n^{-\frac{1}{4}}, \quad (21)$$

are elements of $\Sigma^2(\epsilon, n)$.

Consider any cut-off strategy that satisfies the inequalities (20) and (21). Whenever the margins of victory are larger in Ω_1 than in Ω_2, than the best response converges to conditional sincere voting $\hat{\sigma}_{\Omega_2}(q, r, l)$ for $n \to \infty$ by Lemma 3. In the Appendix, we show that a difference of $n^{-\frac{1}{4}}$ between the probability that a random voter votes A in α and the probability that a random voter votes A in β, as in the formula (21), is sufficient for this result.
Conversely, whenever the margins of victory are sufficiently smaller in Ω_1 than in Ω_2, being pivotal contains the information that states Ω_2 do not hold, for $n \to \infty$. By the same reasoning, if the difference of the margins of victory in α_1 and β_1 is sufficiently large, as in the formula (20), after signals a and b being pivotal contains the information that either α_1 does not hold or β_1 does not hold, for $n \to \infty$. In any case, under the best response, voter behaviour in Ω_1 is almost as if it is known that a specific state holds. When $\phi(0) < \frac{1}{2} < \phi(1)$, the expected margin of victory is strictly larger zero when it is known that α holds, or when it is known that β holds. Since in any equilibrium the limit of the expected margin of victory is zero in Ω_2,

[23] The result holds more generally. If we consider any random (not necessarily cutoff) strategy as the starting point, for any $\epsilon > 0$ the probability that the twice-iterated best response lies in an ϵ-neighbourhood of conditional sincere voting converges to 1, for $n \to \infty$.

the best response satisfies the margin of victory condition (17) for sufficiently large n. Consequently, the twice-iterated best response converges to $\hat{\sigma}_{\Omega_2}(q, r, l)$ by Lemma 3.

Stability and Conditional ϵ-equilibria. Recall formula (19) which says, that for any $\epsilon > 0$, any $n \geq n(\epsilon)$, and any $\sigma \in \mathrm{B}_\epsilon(\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A))$, the cutoffs of σ and $\mathrm{BR}(\sigma)$ are ϵ-close. Thus, after any signal s, any type that makes different choices under σ and $BR(\sigma)$ must be ϵ-close to the indifferent type (the cutoff of $BR(\sigma)$); consequently, the type's loss is smaller than ϵ conditional on being pivotal. We say that σ is a *conditional ϵ-equilibrium.*[24] The equilibrium limit $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ is *stable* or 'safe' in the sense that all strategies in the ϵ-neighbourhood of $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ are conditional ϵ-equilibria for $n \geq n(\epsilon)$. Proposition 1 implies that for a set of cutoff strategies σ of measure $1-\epsilon$, the twice-iterated best response $BR^2(\sigma)$ is a conditional ϵ-equilibrium when n is sufficiently large.

Level k-Implementability: Note that Proposition 1 loosely relates to the concept of level k-implementability (de Clippel et al. [2016]). For approximately any strategy (a 'behavioral anchor'), the level-2-consistent strategies are conditional ϵ-equilibria and ϵ-close to conditional sincere voting for sufficiently large n. In this sense, alternative A is level-2-implementable.

Perturbation Robustness. Consider a voter with a misspecified belief $G' \neq G$ (or alternatively a misspecified prior $p_0' \neq p_0$). Consequently, he has a wrong belief on the margin of victory in states Ω_1 and Ω_2 under conditional sincere voting $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A; p_0)$.[25] If the misspecification is small, he believes that $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A; p_0)$ satisfies the margin of victory condition (17), so his best response under the misspecification converges to conditional sincere voting $\hat{\sigma}\Omega_2(p_A, \bar{r}, p_A; p_0')$. Since conditional sincere voting is continuous in the prior, the limit of the voter's best response is close to the limit $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A; p_0)$ of the best response without misspecification, for $n \to \infty$.

Wilson Doctrine. Recall that $\pi_n(q, r)$ are functions of the prior. Consider a sender who has a misspecified prior p_0'. Suppose that the sender commits to $\pi_n(q', r', q')$ such

[24] The classical notion of ϵ-equilibrium (see e.g. Radner [1980]) is void for the voting games analysed, since the probability of being pivotal converges to 0 for $n \to \infty$. Therefore, any strategy is an ϵ-equilibrium for sufficiently large n.

[25] We use the notation $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A; p_0)$ instead of $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ to highlight the dependence of the prior.

that under p_0' the induced posteriors conditional on the signal and conditional on Ω_2 satisfy $\lim_{n\to\infty} \Pr(\alpha|z,\Omega_2;(q',r',q'),p_0') = r'$, and $\Pr(\alpha|s,\Omega_2;(q',r',l'),p_0') = q'$ for $s \in \{a,b\}$.[26] If the true prior is p_0, the actual posteriors satisfy e.g. $\lim_{n\to\infty} \frac{\Pr(\alpha|z,\Omega_2;q',r',p_0)}{\Pr(\beta|z,\Omega_2;q',r',p_0)} = r' \cdot \frac{p_0}{p_0'} \cdot \frac{1-p_0'}{1-p_0}$. Suppose the sender chooses r' such that $\phi(r') = \frac{1}{2}$ (compare to the equation (14)), and q' with $\phi(q') > \frac{1}{2}$. If his misspecification on the prior is small, also the actual posteriors q and r satisfy $|\phi(q) - \frac{1}{2}| - |\phi(r) - \frac{1}{2}| > 0$. Then, the argument of the proof of Theorem 1 goes through and there exists an equilibrium sequence that converges to conditional sincere voting and implements A for $n \to \infty$. A similar argument can be applied if the sender has a slightly misspecified belief on G. Thus, the sender does not need to know the exact parameters of the games when choosing the signal structure to persuade. One may interpret this by saying that the signal structure satisfies a version of the 'Wilson Doctrine', because it does not rely on the principal having detailed knowledge.

4 Exogeneous Private Signals

The results by Feddersen and Pesendorfer [1997] and Bhattacharya [2013] (see our Theorem 0) have shown that, within the class of monotone preferences and conditionally i.i.d. signals, equilibrium outcomes of large elections are equivalent to the outcome with publicly known states, and a version of the Condorcet Jury Theorem holds. In Section 3, we showed that a sender can manipulate elections by an almost public signal z. (Theorem 1). The possibility of almost public signals relies on the assumption that the sender is a monopolistic information provider. This is restrictive. For example, the independent media are a major source of information for voters. This section investigates the following natural question:

Is manipulation possible just by releasing *additional* information when voters already have private signals and otherwise a version of the Condorcet Jury Theorem would hold in a large election?

Formally, in this section, we study the following scenario: Voters receive both exogeneous private signals π_1 (as in Section 2), and additionally private signals $\pi_n(q,r,l)$

[26] We use the notation $\Pr(\alpha|s,\Omega_2;(q',r',q'),p_0')$ instead of $\Pr(\alpha|s,\Omega_2;q',r',q')$ to highlight the dependence of the prior.

from a manipulator. The exogeneous signals and the signals from the manipulator are assumed to be independent. As in the benchmark in Section 2, we assume that preferences are monotone, that is, we assume that the conditions in (5) holds. We adopt the definition of the games $\pi_n(q,r,l)$ and the definition of the Ω_2-sincere strategy $\hat{\sigma}_{\Omega_2}(q,r,l)$ to this setting by making the necessary modifications; see Section 3.

4.1 Result with Exogeneous Private Signals

We prove the result corresponding to Theorem 1 for the setting with exogeneous private signals. For this, we study the case when the preferences are monotone; that is the conditions in 5 hold. For our setup with exogeneous private signals it is known that information aggregation fails when the preferences are non-monotone, even without additional signals from a manipulator (see Bhattacharya [2013]).

THEOREM 2 *For any exogeneous private signals π_1 that satisfy assumption (4) and any preferences G that satisfy the conditions in (5), there exist $p_A > \bar{r} > p_B \in (0,1)$ such that for any state-contingent outcome $x_\alpha \in \{A,B\}$ and $x_\beta \in \{A,B\}$ and for the games with additional signals $\pi_n(p_{x_\alpha},\bar{r},p_{x_\beta})$ from the manipulator, the following holds:*

- *there exists an equilibrium sequence σ_n, such that for $\omega \in \{\alpha,\beta\}$,*
$$\lim_{n\to\infty} \Pr(x_\omega \textit{ is elected}|\omega;\sigma_n,\pi_n(p_{x_\alpha},\bar{r},p_{x_\beta}),\pi_1) = 1,$$
- *the equilibrium sequence σ_n converges to conditional sincere voting, that is, $\lim_{n\to\infty}\sigma_n = \hat{\sigma}_{\Omega_2}(p_{x_\alpha},\bar{r},p_{x_\beta})$.*

Unpredictability by an Outside Observer. An important branch in the information design literature characterises *robust predictions* that hold under various information structures, potentially for all Bayes correlated equilibria (Bergemann and Morris [2017]). If an outside observer only knows that voters have access to information that is at least as fine as π_1, then it is not certain that information is aggregated in equilibrium. When it cannot be excluded that voters receive additional information, Theorem 2 implies that no robust prediction can be made.

Recall that the proof of Theorem 1 relied on two main insights: Lemma 2 and Lemma 3. The fundamental insight of Lemma 3 is still true in the setting with exogeneous private signals: Whenever the limit of the expected margin of victory is strictly larger in Ω_1 than in Ω_2, then, for $n \to \infty$, being pivotal contains the information that

Ω_2 holds.
The analogue of Lemma 2 does not hold in the setting with exogeneous private signals: Unlike in the setting without exogeneous private signals, when conditioning on being pivotal in Ω_2, voters can learn something about the signal distribution of the other voters, and thereby on the state. In the setting without exogeneous private signals, the signal distribution is almost the same in α_2, and β_2: in both states, the probability that all $2n+1$ voters receive the same signal z, converges to 1, for $n \to \infty$.[27] As a consequence, a voter who receives z already learns the complete signal profile almost perfectly, and there is no further information contained in the event of being pivotal. In contrast, in the setting with exogeneous private signals, the asymptotic signal distributions in α_2 and β_2 differ: voters either receive z and u or z and d, and the likelihood of z and u is strictly higher in α_2. Therefore, the margin of victory in α_2 can be very different from the the margin of victory in β_2, and the event of being pivotal can contain information about the state, for $n \to \infty$; compare to Lemma 2.

Lemma 2 allows us to steer voter behaviour after z: Lemma 2 implies that, in equilibrium, voters behave sincerely after z . Hence, equilibrium behaviour after z is a function of the information structure, independently of behaviour after a and b, and we can steer it perfectly by choosing the parameters of the information structure. In the setting with exogeneous private signals, we cannot steer voter behaviour after z similarly. However, we show, that equilibrium behaviour after z is asymptotically independent of behaviour after a and b (see Lemma 4 below). For this, note, that conditional on Ω_2 the voting game approximately describes a game with binary signals u, d that are conditionally independent given the state α_2 or β_2. From Bhattacharya [2013], we know, that such a game has a unique equilibrium limit for $n \to \infty$, which is pinned down by equating the margin of victories in α_2 and β_2; compare to Figure 4.
It turns out, that the asymptotic independence of behaviour after z is a sufficient analogue of Lemma 2 when the preferences are monotone: We provide a proof of the possibility of persuasion in three steps, which mirror the proof in Section 3. In addition, we rationalise the constructed equilibrium behaviour as Ω_2-sincere voting, which facilitates notation, since this way, equilibrium behaviour is simply captured by the parameters of the information structure $\pi_n(q, r, l)$; compare to the equations (22) - (24). By rationalising the equilibrium as Ω_2-sincere voting, we strenghten the

[27] Recall the argument in Footnote 17.

result, since one may argue that sincere voting and Ω_2-sincere voting are simple or focal strategies; compare to the dicussion in Section 3.3.

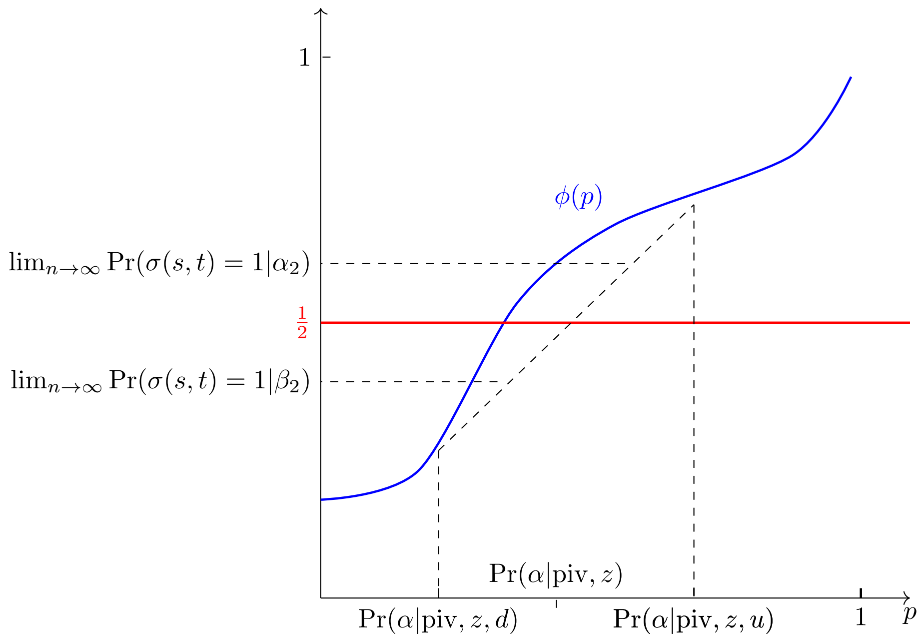

Figure 8: In Ω_2, for $n \to \infty$, voters play the unique Bhattacharya [2013]-type equilibrium σ_n with $\lim_{n\to\infty} \Pr(\sigma_n(s,t)=1|\alpha_2) - \frac{1}{2} = \lim_{n\to\infty} \frac{1}{2} - \Pr(\sigma_n(s,t)=1|\beta_2)$ (cf. Figure 4).

4.2 Proof of Theorem 2

The proof is provided for the case in which $x_\omega = A$ for all $\omega \in \Omega$. Again, this is done for the ease of exposition. The proof for the other cases is completely analogous. Analogously to the equations (9) - (11), we derive now the posteriors conditional on Ω_2 and conditional on having received signals $s \in \{a, b, z\}$ and $v \in \{u, d\}$. It holds that

$$\lim_{n\to\infty} \frac{\Pr(\alpha|v, z; \Omega_2)}{\Pr(\beta|v, z, \Omega_2)} = \frac{r}{1-r} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)}, \tag{22}$$

$$\frac{\Pr(\alpha|v, a, \Omega_2)}{\Pr(\beta|v, a, \Omega_2)} = \frac{q}{1-q} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)}, \tag{23}$$

$$\frac{\Pr(\alpha|v, b, \Omega_2;)}{\Pr(\beta|v, b, \Omega_2)} = \frac{l}{1-l} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)} \tag{24}$$

where we used the equations (9), (10) and (11) and the independence of π_1 and $\pi_n(q,r,l)$. Note that we suppress the dependence of the posteriors on the information $\pi_n(q,r,l)$ and π_1. In comparison with (9), (10) and (11), the additional terms $\frac{\Pr(v|\alpha)}{\Pr(v|\beta)}$ come from the learning through π_1.

Fix voter behaviour after a and b. Consider any $r \in (0,1)$. For any $v \in \{u,d\}$, let $p(v|r)$ denote the belief given by $\frac{p(v|r)}{1-p(v|r)} = \frac{r}{1-r} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)}$. Denote by σ^r the strategy such that, for any $v \in \{u,d\}$, it holds $\sigma^r((z,v),t) = 1$ if and only if

$$t_\alpha \cdot p(v|r) + t_\beta \cdot (1 - p(v|r)) \geq 0.$$

Let $\bar{r} \in (0,1)$ be the unique number such that under $\sigma^{\bar{r}}$ the margin of victory in α_2 is equal to the margin of victory in β_2 for $n \to \infty$,[28]

$$\begin{aligned} & \lim_{n\to\infty} |\Pr(\sigma(s,t) = 1|\alpha_2; \sigma^{\bar{r}}, \pi_n(q,r,l), \pi_1) - \frac{1}{2}| \\ = & \lim_{n\to\infty} |\Pr(\sigma(s,t) = 1|\beta_2; \sigma^{\bar{r}}, \pi_n(q,r,l), \pi_1) - \frac{1}{2}|. \end{aligned} \tag{25}$$

Note that $\bar{r}$ does not depend on the voter behaviour after a and b. Note that, for any $0 < r < \bar{r}$, under σ^r, the limit of the expected margin of victory is smaller in α_2 than in β_2. For any $1 > r > \bar{r}$, under σ^r, the limit of the expected margin of victory is larger in α_2 than in β_2.[29]

Lemma 4 *For any equilibrium sequence σ_n, it holds that*

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, z; \sigma_n, \pi_n(q,r,l), \pi_1) = \bar{r}.$$

Proof. In the Appendix.

■

We can rationalise equilibrium behaviour after z as sincere voting. We choose the information structure parameter $r = \bar{r}$. Then, Lemma 4 implies that any equilibrium sequence σ_n satisfies

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, z; \sigma_n) = \bar{r} = r = \lim_{n\to\infty} \Pr(\alpha|z, \Omega_2) = \lim_{n\to\infty} \Pr(\alpha|z),$$

where we used the equation (10) for the last second last equality and the definition of $\pi_n(q,r,l)$ for the last equality and we suppressed the dependence of the posteriors on the information structure in the notation. Thus, for $r = \bar{r}$, after z being pivotal

[28] We give a formal proof of the uniqueness of $\bar{r}$ in the Appendix. Intuitively, uniqueness follows from the monotonicity of the aggregate preference function ϕ.

[29] We give a formal proof of these claims about the order of the limit of the margins of victory in α_2 and β_2 under σ^r for $r \neq \bar{r}$ in the Appendix.

contains no information, for $n \to \infty$. Hence, voters vote sincerely after z or $n \to \infty$. This is analogous to Lemma 2 in the situation without exogeneous private signals π_1.

As Lemma 3 followed from Lemma 2, its analogue (with the necessary notational changes) follows from Lemma 4 when $r = \bar{r}$.

LEMMA 5 *If*

$$\begin{aligned} & \max_{\omega_2 \in \Omega_2} |\lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1 | \omega_2; \pi_n(q,\bar{r},l), \pi_1) - \frac{1}{2}| \\ < & \min_{\omega_1 \in \Omega_1} |\lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1 | \omega_1; \pi_n(q,r,l), \pi_1) - \frac{1}{2}| \end{aligned} \tag{26}$$

holds, then

$$\begin{aligned} \lim_{n\to\infty} \Pr(\alpha | a, \mathrm{piv}; \sigma_n, \pi_n(q,\bar{r},l), \pi_1) = \Pr(\alpha | s, \Omega_2; \pi_n(q,\bar{r},l), \pi_1) &= q, \\ \lim_{n\to\infty} \Pr(\alpha | b, \mathrm{piv}; \sigma_n, \pi_n(q,\bar{r},l), \pi_1) = \Pr(\alpha | s, \Omega_2; \pi_n(q,\bar{r},l), \pi_1) &= l. \end{aligned}$$

Hence, the unique best response to σ_n in the games $\Gamma_n(q,\bar{r},l)$ converges to $\hat{\sigma}_{\Omega_2}(q,\bar{r},l)$ for $n \to \infty$.

Note that Lemma 4 shows that the equilibrium play after z converges to $\sigma^{\bar{r}}$. By the definition of $\bar{r}$ through the equation (25), the limit of the expected margin of victory is the same in α_2 and β_2 under $\sigma^{\bar{r}}$. In Lemma 7 of the Appendix, we show that for any $r \in (0,1)$, under σ^r, a random voter votes A with a strictly higher probability in α_2 than in β_2, for $n \to \infty$. Consequently, in any equilibrium sequence σ_n, the limit of the expected vote share of A is strictly larger than $\frac{1}{2}$ in α_2 and strictly smaller than $\frac{1}{2}$ in β_2,

$$\lim_{n\to\infty} \Pr(\sigma(s,t) = 1 | \beta_2; \sigma_n) < \frac{1}{2} < \lim_{n\to\infty} \Pr(\sigma(s,t) = 1 | \alpha_2; \sigma_n). \tag{27}$$

Equilibrium Construction. We show that we can control the behaviour of agents getting a or b by choosing q and l appropriately. We choose $q = l = p_A$ sufficiently large such that for all $v, w \in \{u, d\}$ and $s \in \{a, b\}$,

$$\frac{\bar{r}}{1-\bar{r}} \cdot \frac{\Pr(v|\alpha)}{\Pr(v|\beta)} < \frac{p_A}{1-p_A} \cdot \frac{\Pr(w|\alpha)}{\Pr(w|\beta)}. \tag{28}$$

Recall the formulas (22)-(24) for the limits of the posteriors conditional on Ω_2 and conditional on any combination of an exogeneous private signal $v \in \{u, d\}$ and a signal $s \in \{a, b, z\}$ from the manipulator. Recall that for any $p \in (0, 1)$, we denote $\bar{\phi}(\frac{p}{1-p}) = \phi(p)$. By definition of conditional sincere voting, under $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$, a random voter who received $s \in \{a, b\}$ and $w \in \{u, d\}$ votes A with probability $\bar{\phi}(\frac{p_A}{1-p_A}\frac{\Pr(w|\alpha)}{\Pr(w|\beta)})$. A random voter who received z and $v \in \{u, d\}$ votes A with probability $\bar{\phi}(\frac{\bar{r}}{1-\bar{r}}\frac{\Pr(v|\alpha)}{\Pr(v|\beta)})$. In this section we assumed that ϕ is strictly increasing, as in the benchmark in Section 2. Consequently, the inequality (28) implies that $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ satisfies the margin of victory condition (17) of Lemma 5. Moreover, for any $\epsilon > 0$ sufficiently small, any $\sigma \in \mathrm{B}_\epsilon(\hat{\sigma}(p_A, \bar{r}, p_A))$ satisfies the margin of victory condition (17) of Lemma 5.

The ϵ-truncated Best Reponse. After receiving $s \in \{a, b, z\}$, and $v \in \{u, d\}$, a voter weakly prefers to vote A if and only if

$t_\alpha \cdot \Pr(\alpha|s, v, \text{piv}; \sigma_n, \pi_n(q, r, l), \pi_1) + t_\beta \cdot (1 - \Pr(\alpha|s, v, \text{piv}; \sigma_n, \pi_n(q, r, l), \pi_1) \geq 0.$

with

$$\frac{\Pr(\alpha|\text{piv}, s, v; \sigma_n, \pi_n(q, r, l), \pi_1)}{1 - \Pr(\alpha|\text{piv}, s, v; \sigma_n, \pi_n(q, r, l), \pi_1)} = \frac{\Pr(\alpha|\text{piv}, s; \sigma_n, \pi_n(q, r, l))}{1 - \Pr(\alpha|\text{piv}, s; \sigma_n, \pi_n(q, r, l))} \cdot \frac{\Pr(v|\alpha; \pi_1)}{\Pr(v|\beta; \pi_1)}$$

by independence of $\pi_n(q, r, l)$ and π_1. So the triple $(\Pr(\alpha|\text{piv}, s; \sigma_n, \pi_n(q, r, l), \pi_1))_{s\in\{a,b,z\}}$ is a sufficient statistic for the best response. Therefore, let us consider the best reponse as a function of belief triples. Then, for any $\epsilon > 0$, and any $(p(s))_{s\in\{a,b,z\}}$, we truncate the best response as follows: For any $s \in \{a, b, z\}$, the *ϵ-truncated best response* function sets the s-component $\Pr(\alpha|\text{piv}, s)$ of the best reponse to $p(s) - \epsilon$ if it is weakly lower than $p(s) - \epsilon$, and to $p(s) + \epsilon$ if it is weakly larger than $p(s) + \epsilon$. Otherwise, the ϵ-truncated best response equals the best response. Note that the ϵ-truncated best reponse function is a continuous function on a compact set, and therefore has a fixed point.

By Lemma 4, the limit equilibrium play after z is unique, and hence uniquely described by some posterior $\bar{r} = \lim_{n\to\infty} \Pr(\alpha|\text{piv}, z; \sigma_n, \pi_n(q, r, l), \pi_1)$.

Lemma 6 *Let p_A satisfy the equation (28). For any $\epsilon > 0$ sufficiently small, there exists $n(\epsilon) > 0$ such that for any $n \geq n(\epsilon)$, in the games $\Gamma_n(p_A, \bar{r}, p_A)$, any fixed point of the ϵ-truncated best response around $(p_A, \bar{r}, p_A)$ is interior.*

Proof. In the Appendix.

■

Note that the beliefs $(p_A, \bar{r}, p_A)$ correspond to the limit of the Ω_2-sincere voting strategy $\hat{\sigma}_\Omega(p_A, \bar{r}, p_A)$. This shows that Lemma 6 is an analogue of formula (19): Lemma 6 says that the ϵ-truncated best response has an interor fixed point in the

ϵ-neighbourhood of Ω_2-sincere voting.

Note that any interior fixed point of the ϵ-truncated best reponse is an equilibrium. Hence, Lemma 6 implies that there exists an equilibrium sequence that converges to $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$. Under any strategy σ close-by to $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$, we have $\frac{1}{2} < \lim_{n\to\infty} \Pr(\sigma(s,t) = 1|\omega_1)$ for any $\omega \in \{\alpha, \beta\}$ by the inequalities (27) and (28). Hence, the law of large numbers implies that alternative A gets elected with certainty in Ω_1 for $n \to \infty$ under the equilibrium sequence that converges to $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$. Recall that the probability of the states Ω_1 converges to 1 under $\pi_n(p_A, \bar{r}, p_A)$, that is$\lim_{n\to\infty} \Pr(\Omega_1|\pi_n(p_A, \bar{r}, p_A)) = 1$. This finishes the proof of Theorem 2 for the case with $x_\omega = A$ for all $\omega \in \{A, B\}$.

Similar to the situation without exogeneous private signals, the equilibrium construction of Theorem 2 is robust:

Perturbation Robustness. The remark in Section 3.3 about perturbation robustness applies in the same way. Hence, the equilibrium is 'safe' in the following sense: even if a voter's belief about the environment, that is about the prior or the preference distribution G, is slightly wrong, the cost of this error is small (even conditional on being pivotal).

Detail-Freeness. The sender does not need to know the exact parameters of the game when choosing the signal structure. For any $\epsilon > 0$, the sender can choose the parameters q and l of the information structure such that the inequality (28) universally holds for any prior $p_0 \in [\epsilon, 1-\epsilon]$, for any preference distribution such that $\phi' > \epsilon$, and for any exogeneous signals with $\frac{1}{2} < \Pr(u|\alpha) < 1-\epsilon$ and $\frac{1}{2} < \Pr(d|\beta) < 1-\epsilon$. The inequality (28) implies that $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ satisfies the margin of victory condition (17) of Lemma 5. Then, the remaining argument of the proof of Theorem 2 goes through and there exists an equilibrium sequence that implements A for $n \to \infty$.

5 Discussion and Remarks

5.1 Other Equilibria

We further analyze equilibrium sequences of the games $\Gamma_n(p_A, \bar{r}, p_A)$ in Section 3 with $\bar{r}$ satisfying (16), and p_A satisfying (13).

LEMMA 8 *For any preference distribution G that satisfies the conditions in 5: If $\sigma \neq \hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ is the limit of an equilibrium sequence σ_n in the games $\Gamma_n(p_A, \bar{r}, p_A)$, it satisfies*

1. *the limit of the minimum of the margins of victory in the states Ω_1 equals the limit of the minimum of the margin of victory in the states Ω_2, namely*
$$\begin{aligned} &\lim_{n\to\infty} \min_{\omega_1 \in \{\alpha_1, \beta_1\}} |\Pr(\sigma(s,t) = 1 | \omega_1; p_A, \bar{r}, p_A) - \frac{1}{2}| \\ = &\lim_{n\to\infty} \min_{\omega_2 \in \{\alpha_2, \beta_2\}} |\Pr(\sigma(s,t) = 1 | \omega_2; p_A, \bar{r}, p_A) - \frac{1}{2}|. \end{aligned} \tag{29}$$

2. *σ is a cutoff strategy with cutoffs $(p_s)_{s\in S}$ that satisfy one of the following conditions: Either $p_s = \bar{r}$ for all $s \in S$, or $p_z = \bar{r}$ and $0 < p_b < \bar{r} < p_a < 1$.*

Proof. In the Appendix. ■

Sketch of Proof. Consider σ as in the statement. Hence, Lemma 3 implies that under σ_n, the limit of the margin of victory in α_1 is weakly smaller than in α_2 or β_2, or the limit of the margin of victory in β_1 is weakly smaller than in α_2 or β_2; otherwise, the best response to σ_n converges to $\hat{\sigma}_{\Omega_2}$. Now, Lemma 2 and (16) imply that the margin of victory under σ_n converges to zero in α_2 and β_2; hence, the same must hold in either α_1 or β_1. In the Appendix, we show that property (2.) is implied by property (1.) when the preferences are monotone, that is, when they satisfy the conditions in 5.

PROPOSITION 2 *Let $\bar{r}$ and p_A satisfy (16) and (13), respectively: When the preferences are monotone, there exists an equilibrium sequence σ_n in the games $\Gamma_n(p_A, \bar{r}, p_A)$ which, for $n \to \infty$ implies the full information outcome,*
$$\lim_{n\to\infty} \Pr(A \textit{ is elected} | \alpha; \sigma_n, p_A, \bar{r}, p_A) = 1,$$
$$\lim_{n\to\infty} \Pr(B \textit{ is elected} | \beta; \sigma_n, p_A, \bar{r}, p_A) = 1.$$

The Instability of Other Equilibria. Consider any equilibrium sequence σ_n in

$\Gamma_n(p_A, \bar{r}, p_A)$ with $\lim_{n\to\infty} \sigma_n = \sigma \neq \hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$. Lemma 8 has shown that this equilibrium sequence necessarily satisfies condition (29). Proposition 1 now implies that for arbitrarily large n, arbitrarily small changes to σ_n, such that the inequalities (20) and (21) hold, suffice to enter the basin of attraction of $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ under the best response dynamics. Hence, all equilibrium sequences that do not converge to conditional sincere voting $\hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ are unstable.

5.2 Bayes Correlated Equilibria

We use the terminology of Bergemann and Morris [2016] who define *Bayes correlated equilibrium.* A *decision rule* of $\Gamma(\pi)$ is a mapping $\theta : (S \times [-1,1]^2)^{2n+1} \times \Omega \to \{A, B\}^{2n+1}$. W.l.o.g. we consider decision rules θ for which $\sigma_\theta(s_i, t_i) = \Pr(\theta_i = A | s_i, t_i)$ does not depend on $i \in \{1, \ldots, 2n+1\}$ such that σ_θ defines a symmetric strategy.

Definition 1 *A decision rule θ of $\Gamma(\pi)$ is called obedient if for any voter $i \in \{1, \ldots, 2n+1\}$, for all $s_i \in S$, for all $t_i \in [-1,1]^2$ and all $x_i \in \{A, B\}$, we have*

$$t_\alpha \Pr(\alpha|\text{piv}, s_i, t_i, \theta_i = x; \sigma_\theta) + t_\beta(1 - \Pr(\alpha|\text{piv}, s_i, t_i, \theta_i = x; \sigma_\theta)) \geq 0 \quad \textit{if} \quad x_i = A,$$
$$t_\alpha \Pr(\alpha|\text{piv}, s_i, t_i, \theta_i = x; \sigma_\theta) + t_\beta(1 - \Pr(\alpha|\text{piv}, s_i, t_i, \theta_i = x; \sigma_\theta)) \leq 0 \quad \textit{if} \quad x_i = B.$$

An obedient decision rule of $\Gamma(\pi)$ is called a Bayes correlated equilibrium of $\Gamma(\pi_1)$. If we have two information structures (π_1, S_1) and (π_2, S_2), we say that information structure (π, S) is a combination of information structures (π_1, S_1) and (π_2, S_2) if the combined information structure (π, S) is obtained by forming a product space of the signals, $S = S_1 \times S_2$, and a joint distribution of signals and states π that preserves the marginal distribution of its constituent information structures.

Definition 2 *(Combination). The information structure (π, S) is a combination of information structures (π_1, S_1) and (π_2, S_2) if*

$$\begin{aligned} T &= T_1 \times T_2, \\ \sum_{s_2 \in S_2} \pi(s_1, s_2|\omega) &= \pi_1(s_1|\omega) \quad \textit{for all } s_1 \in S_1,\ \omega \in \Omega, \\ \sum_{s_1 \in S_1} \pi(s_1, s_2|\omega) &= \pi_2(s_2|\omega) \quad \textit{for all } s_2 \in S_2,\ \omega \in \Omega. \end{aligned}$$

Note that the above definition places no restrictions on whether signals $s_1 \in S_1$ and $s_2 \in S_2$ are independent or correlated, conditional on ω, under π.

Definition 3 *(Expansion). An information structure* (π, S) *is an expansion of* (π_1, S_1) *if* (π, S) *is a combination of* (π_1, S_1) *and another information structure* (π_2, S_2).

Theorem 3 *(Bergemann and Morris [2016]). For any information structure* π_1*: A decision rule* $\theta : (S_1 \times [-1,1]^2)^{2n+1} \times \Omega \to \{A, B\}^{2n+1}$ *is a Bayes correlated equilibrium of* $\Gamma(\pi_1)$ *if and only if for some expansion* π *of* π_1*, there is a Bayes Nash equilibrium* σ *of* $\Gamma(\pi)$ *that induces* θ.

In Section 4 we studied the Bayes-Nash equilibria of games where voters receive both signals from an exogeneous information structure π_1, and independently of π_1, additional signals from the information structure $\pi_n(q, r, l)$ of a manipulator. Note that the joint information structure π of the voters is an expansion of π_1. Theorem 2 states that any state-contingent outcome can arise as the limit of Bayes-Nash equilibria of such games $\Gamma(\pi)$. In view of Theorem 3, our analysis can be understood as a characterization of the Bayes correlated equilibria of the voting games with signals π_1.

Corollary 1 *For any exgeneous information structure* π_1 *of voters: for any state-contingent outcome* $x_\alpha \in \{A, B\}$ *and* $x_\beta \in \{A, B\}$*, there exists a sequence of Bayes correlated equilibria* θ_n *of* $\Gamma(\pi_1)$ *such that under* θ_n *the probability that* x_ω *gets elected in* ω *converges to* 1 *for all* $\omega \in \Omega$.

Recall that the games $\Gamma(\pi_1)$ represent a binary-state version of Feddersen and Pesendorfer [1997] as studied in Bhattacharya [2013]. Corollary 1 shows that the Condorcet Jury Theorem (see Theorem 0) does not survive the solution concept of Bayes correlated equilibrium.

Information Design without Elicitation. Bergemann and Morris [2017] distinguish three cases of information design: when the designer is omniscient, when receivers have private information and an information designer may be able to elicit and condition on the private information (information design with elicitation) or he may be unable to do so (information design without elicitation). Information design without elicitation has the strongest constraints and hence relates to the smallest set of Bayes correlated equilibria. Note that the information structures $\pi_n(q, r, l)$ of the manipulator do not condition on the private information of voters. Also, note, that we do not allow the manipulator to correlate his signals with the exogeneous signals from π_1. Theorem 2 implies that all state-contingent outcomes can be implemented in a Bayes correlated equilibrium of a large election just by information design without elicitation, and just by using independent expansions.

5.3 Feasibility

This section explains that information aggregation is feasible in the situation of Theorem 2, but fails only because of incentives. This is in contrast to several reported failures of information aggregation in the literature: Feddersen and Pesendorfer [1997] (Section 6) show that an invertibility problem arises and information aggregation can fail when there is aggregate uncertainty with respect to the preference distribution. Chan et al. [2016] (Proposition 1) have provided an example of voter persuasion by using signals which are close to the null information structure that always sends the same signal.

When the preferences are monotone, the expected median voter prefers A in state α, and B in state β: $\phi(1) > \frac{1}{2}$, and $\phi(0) > \frac{1}{2}$. Since under $\pi_n(q, r, l)$ we have $\Pr(a|\alpha_1) = 1$ and $\Pr(b|\beta_1) = 1$, any strategy σ which prescribes to vote A after a and to vote B after b elects the full information outcome with certainty for $n \to \infty$.

5.4 Pivotal Voter Paradigm

Empirical literature has tested the pivotal voter paradigm and provided correlational and causal evidence for the effect of beliefs about other people's behaviour on political decisions: Cantoni et al. [2017] conducted a field experiment in the context of Hong Kong's pro-democracy movement. They identify a causal effect of beliefs about total turnout of protesters on individual turnout decisions. In a laboratory experiment, Guarnaschelli et al. [2000] show that actual behaviour is consistent with the hypothesis that each voter acts optimally against the strategies employed by other voters plus a random error. In another experiment, Duffy and Tavits [2008] observe a positive correlation between the propensity of voting and the beliefs of being pivotal, but subjects systematically overestimate the probability of being pivotal. Coate et al. [2008] provide descriptive evidence and show that field data from small scale-elections on Texas liquor referenda is consistent with strategic voter models in terms of predicted turnout but not in terms of margin of victory. Further evidence in favor of strategic voter models has been provided by Ladha et al. [1996].

In this paper, the assumption of strategic voting is particularly justified: The probabilities of being pivotal are exceptionally high in the states-of-confusion Ω_2 for the information structures $\pi_n(p_{x_\alpha}, \bar{r}, p_{x_\beta})$ that establish the possibility result of Theorem 2. This is because $\bar{r}$ has been chosen to make the election close to being tied in Ω_2, such that (16) holds. For sufficiently large n, when a random agent votes for A with probability $\frac{1}{2}$, the probability of being pivotal, $\binom{2n}{n}(\frac{1}{2})^{2n}$, is of order $n^{-\frac{1}{2}}$ by Stirling's formula (see Feller [1968], chapter II, formula 9.1).[30]

6 Literature review

6.1 Voter Persuasion Literature

Alonso and Câmara [2015] study persuasion of voters with public signals by using the methodology of Kamenica and Gentzkow [2011] (KG) in an environment with perfect information on the preferences. In our model, when voters receive exogeneous private signals as in Feddersen and Pesendorfer [1997], public persuasion of a large electorate is not possible, see the discussion at the end of Section 2. On the other hand, when a manipulator is a monopolistic information provider we obtain results similar to Alonso and Câmara [2015]; these results are provided in the Online Supplement. Schnakenberg [2015] considers a cheap talk model where a sender publicly persuades voters. Kolotilin et al. [2015] study persuasion of a single, privately informed receiver and show that efficient information structures do not need to screen types. Correspondingly, we showed that for large electorates with private preferences, the set of equilibrium outcomes that can be obtained by information design is the

[30] More precisely, Stirling's formula yields

$$\begin{aligned}(2n)! &\overset{\text{Stirling}}{\approx} (2\pi)^{\frac{1}{2}} 2^{2n+\frac{1}{2}} n^{2n+\frac{1}{2}} e^{-2n},\\ (n!)^2 &\overset{\text{Stirling}}{\approx} (2\pi) n^{2n+1} e^{-2n}.\end{aligned}$$

Consequently,

$$\binom{2n}{n} \approx (2\pi)^{-\frac{1}{2}} 2^{2n+\frac{1}{2}} n^{-\frac{1}{2}},$$

which yields

$$\begin{aligned}\binom{2n}{n}(\frac{1}{2})^{2n} &\approx (2\pi)^{-\frac{1}{2}} n^{-\frac{1}{2}} 2^{\frac{1}{2}}\\ &= (n\pi)^{-\frac{1}{2}}.\end{aligned}$$

same with and without the option to screen types. This results holds when the information designer is monopolistic (Theorem 1) and when voters have private signals from an independent, exogeneous source and the preferences of voters are monotone in the sense of (5) (Theorem 2).

Chan et al. [2016] (CHX) study voter persuasion with private signals and when voting for A is costly. As a consequence of the voting cost an interesting effect appears, namely that a manipulator, who tries to implement A in both states, induces minimal winning coalitions for A and maximal winning coalitions for B. Intuitively, a voter has to pay a cost when voting A, hence can only be convinced to do so when the probability of being pivotal is sufficiently large. Bardhi and Guo [2016a] (BG) study voter persuasion with private signals and focus on the unanimity rule. They allow for heterogeneous, correlated preferences that are only known to the sender. For non-unanimous rules, they show that persuasion is possible. Here, our work shares with theirs the observation that the voters' conditioning on being pivotal allows relaxing the Bayesian consistency requirement. In contrast to (CHX) and (BG), we analyse an environment that allows for non-monotone preferences. Some citizens vote for A when believing sufficiently strongly in α, and some voters prefer A when believing sufficiently strongly in β. When the sender cannot screen types, it is *a priori* unclear which beliefs he should induce with a random receiver. In this paper, the informational requirements for persuasion are considerably weak (see discussion at the end of Section 2.2); in particular, we allow for private information on the preferences and exogeneous private signals. In contrast, both (CHX) and (BG) assume perfect knowledge on preference realisations by the sender and do not allow for exogeneous private signals. Further, in contrast to (CHX) and (BG), our focus lies on persuasion of large electorates, which makes the results easily amenable to the literature on information aggregation. (CHX) and (BG) adopt an information design approach and study sender-preferred equilibria. In this paper, we study other equilibria also, and show that persuasion is robust in manifold ways (Sections 3.3 and 4).
Several other studies study persuasion of groups, but are less closely related: Liu [2016] provides results on public persuasion of privately informed voters. Bardhi and Guo [2016b] study sequential persuasion of a group of receivers.

6.2 Information Aggregation Literature

The literature has identified several circumstances in which information may fail to aggregate. We discuss the studies that are most closely related:
Feddersen and Pesendorfer [1997] (FP, Section 6) show that an invertibility problem causes a failure when there is aggregate uncertainty with respect to the preference distribution conditional on the state. Bhattacharya [2013] (BH) shows that failure can happen when preference monotonicity is violated in a model otherwise akin to (FP). However, when the preferences are monotone information is aggregated perfectly. In this paper, we studied the Bayes corrrelated equilibria of the model in (BH): We showed that a monopolistic information provider can implement any state-contigent outcome in a robust equilibrium (Theorem 1, Proposition 1). In Section 4, we showed that we can add signals to the model with monotone preferences as in (BH) and thereby cause failure. In particular, we showed that any state-contingent outcome is implementable even when voters receive exogeneous private information from an independent source (Theorem 2).
In a pure common-values setting, Mandler [2012] (MA) shows that failure can happen when there is aggregate signal uncertainty conditional on the state. The study does not discuss persuasion, but the results can be understood in terms of it: Signals are sent independently and are identically distributed conditional on a binary state and conditional on a substate, as in this paper. The substate captures the signal precision $q = \Pr(a|\omega)$ and is continuous with density $h_\omega(q)$. (MA) shows that, for $n \to \infty$, any limit of an equilibrium sequence can be described by an intersection point q^* of the scaled densities $\Pr(\omega) \cdot h_\omega(q)$, and vice versa (Proposition 1, the discussion before, and Proposition 2). In such a q^*-equilibrium sequence, alternative A is elected with certainty, for $n \to \infty$, if $q > q^*$ realises, and B is elected with certainty if $q < q^*$ realises, or vice versa (depending on whether $\Pr(\alpha) \cdot h_\alpha(q)$ crosses from below or above). Hence, A can be implemented with arbitrary high probability in an equilibrium sequence, by design of the scaled densities. However, all equilibrium sequences are coequal in terms of robustness, unlike in this paper. In particular, the continuity of the densities prevents implementation in a uniquely robust equilibrium, as is illustrated in the following example: Let $\Pr(\alpha) = \frac{1}{10}$. Consider scaled densities that are single-crossing at $\epsilon > 0$, with $\Pr(\alpha) \cdot h_\alpha(q)$ crossing from below. Thus, any equilibrium sequence implements A, for $q > \epsilon$ and B for $q < \epsilon$. For all $q > \epsilon$, we must

have

$$(1 - \Pr(\alpha)) \cdot h_\beta(q) \quad < \quad \Pr(\alpha) \cdot h_\alpha(q)$$

This implies that

$$\frac{9}{10}(1 - \Pr(q < \epsilon|\beta)) \quad < \quad \frac{1}{10} \cdot (1 - \Pr(q < \epsilon|\alpha))$$

Consequently,

$$(1 - \Pr(q < \epsilon|\beta)) \quad < \quad \frac{1}{9}.$$

This implies that alternative B is elected at least with probability $\frac{8}{9}$ in β, the more likely state, for sufficiently large n.

Gerardi et al. [2009] study aggregation of expert information by an uninformed decision maker. By giving each expert a small change of being a dictator, information can be perfectly extracted at a marginal loss, while implementing any intended outcome otherwise. The states-of-confusion Ω_2 serve a similar role in our analysis.

7 Conclusion

In this paper, we studied the Bayes correlated equilibria of non-unanimous majority elections and showed that no bounds on equilibrium outcomes exist that hold across all information structures and all Bayesian equilibria when the electorate is large (Corollary 1). Bergemann et al. [2016] and Du [2017] studied Bayes correlated equilibria of common value auctions, and in particular calculated the minimum revenue across all models of information and all Bayesian equilibria for the mechanisms that maximize minimum revenue. In comparison to the auctions literature, we innovated methodologically and characterised the Bayes correlated equilibrium outcomes also when a minimum level of information is imposed, and this minimum level of information can be arbitrarily precise. By correlating the signals of voters, the information designer can implement any state-contingent outcome. In the future, we hope to analyse differences and common features of auctions and voting through the lens of information design. We hope that our work on Bayes corrrelated equilibria of large elections inspires insights on revenue bounds in common value auctions with minimal information or on Bayes correlated equilibria of large auctions more generally. The intuition of our main result that a manipulator can implement any state-contingent outcome by information design without elicitation may carry over to many different and more general voting settings: for example in settings with more than two possible outcomes (like in Bouton and Castanheira [2012]) or with ethical voters (like in Evren [2012]). A detailed analysis of these settings is left for future work.

8 Appendix

8.1 Appendix A: Preliminaries and Proof of Theorem 0

LEMMA 1 *Suppose* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are such that* $\lim_{n\to\infty} x_n \in (0,1)$, $\lim_{n\to\infty} y_n \in (0,1)$.

(i) If $\lim_{n\to\infty} \left| |x_n - \frac{1}{2}| - |y_n - \frac{1}{2}| \right| \cdot n^2 < c$ *for some* $c > 0$, *then* $\lim_{n\to\infty} \left[\frac{x_n(1-x_n)}{y_n(1-y_n)}\right]^n = 1$.

(ii) If $\lim_{n\to\infty} |x_n - \frac{1}{2}| \cdot n^{\frac{1}{2}} = c$ *for some* $c \in \mathbb{R}$, *and* $y_n = \frac{1}{2}$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty} \left[\frac{x_n(1-x_n)}{y_n(1-y_n)})\right]^n = \mu$ *for some* $\mu > 0$.

(iii) If $\lim_{n\to\infty}(|x_n - \frac{1}{2}| - |y_n - \frac{1}{2}|) \cdot n^{\frac{1}{2}} = \infty$, *then* $\lim_{n\to\infty} \left[\frac{x_n(1-x_n)}{y_n(1-y_n)}\right]^n = 0$.

Proof. Let $\tilde{x}_n = x_n - \frac{1}{2}$, and $\tilde{y}_n = y_n - \frac{1}{2}$, and use this to rewrite as follows

$$
\begin{aligned}
\frac{x_n(1-x_n)}{y_n(1-y_n)} &= \frac{(\tilde{x}_n + \frac{1}{2})(1 - \frac{1}{2} - \tilde{x}_n)}{(\tilde{y}_n + \frac{1}{2})(1 - \frac{1}{2} - \tilde{y}_n)} \\
&= \frac{(\frac{1}{2}^2 - (\tilde{x}_n)^2)}{(\frac{1}{2}^2 - (\tilde{y}_n)^2)} \\
&= [\frac{\frac{1}{4} - (\tilde{y}_n)^2 + (\tilde{y}_n)^2 - (\tilde{x}_n)^2}{\frac{1}{4} - (\tilde{y}_n)^2}] \\
&= \left[1 - \frac{(\tilde{x}_n)^2 - (\tilde{y}_n)^2}{\frac{1}{4} - (\tilde{y}_n)^2}\right]. \qquad (30)
\end{aligned}
$$

Proof of (i): Split the sequence $|\tilde{x_n}| - |\tilde{y_n}|$ into maximally two subsequences, one sequence consisting of the weakly positive elements, and the other of the negative elements. First, we consider the subsequence of the weakly positive elements, and suppose for convenience, that this is the sequence itself. Denote $a_n := (\tilde{x}_n)^2 - (\tilde{y}_n)^2$ and $b_n := |\tilde{x_n}| - |\tilde{y_n}|$. It holds that

$$
\begin{aligned}
& a_n \\
= \; & 2b_n|\tilde{y}_n| + (b_n)^2 \\
< \; & b_n + (b_n)^2. \qquad (31)
\end{aligned}
$$

For all n sufficiently large, we have

$$\begin{aligned} & a_n \\ < \; & \frac{c}{n^2} + (\frac{c}{n^2})^2 \\ < \; & \frac{2c}{n^2}. \end{aligned} \tag{32}$$

where we used the formula (31) and the assumption $\lim_{n\to\infty} |\tilde{x}_n| - |\tilde{y}_n| \cdot n^2 < c$ for the first inequality. Now, on the one hand

$$\lim_{n\to\infty} (1 - \frac{\tilde{x_n}^2 - \tilde{y_n}^2}{\frac{1}{4} - \tilde{y_n}^2})^n \leq 1. \tag{33}$$

On the other hand,

$$\begin{aligned} \lim_{n\to\infty} (1 - \frac{\tilde{x_n}^2 - \tilde{y_n}^2}{\frac{1}{4} - \tilde{y_n}^2})^n \geq & \lim_{n\to\infty} (1 - \frac{1}{n^2} \frac{2c}{\frac{1}{4} - \tilde{y_n}^2})^n \\ \geq & \lim_{n\to\infty} (1 - \frac{1}{n} m)^n \quad \text{for all} \quad m \in \mathbb{R}^{>0} \\ = & e^{-m} \quad \text{for all} \quad m \in \mathbb{R}^{>0}. \end{aligned} \tag{34}$$

The inequality on the first line follows from the formula (32). For the second inequality, recall that $\lim_{n\to\infty} y_n \in (0,1)$. Hence, for n sufficiently large, $\frac{1}{4} - \tilde{y}_n^2$ is bounded above by $\frac{1}{4}$ and below by a constant strictly larger than 0. So $\frac{2c}{\frac{1}{4} - \tilde{y}_n^2}$ is bounded. From the inequality (34), we obtain that

$$\lim_{n\to\infty} \left[\frac{x_n(1-x_n)}{y_n(1-y_n)}\right]^n \geq 1. \tag{35}$$

We can perform the same calculation as above for the subsequence of all negative elements $|\tilde{x_n}| - |\tilde{y_n}| < 0$, but we have to replace all greater equal signs with smaller equal signs and vice versa. In any case, the inequalities (30), (33) and (35) together yield that $\lim_{n\to\infty} \left[\frac{x_n(1-x_n)}{y_n(1-y_n)}\right]^n = 1$. This finishes the proof of (i).

We have

$$\begin{aligned} & \lim_{n\to\infty} \left[\frac{x_n(1-x_n)}{y_n(1-y_n)}\right]^n \\ = \; & \lim_{n\to\infty} \left[1 - \frac{(\tilde{x}_n)^2 - (\tilde{y}_n)^2}{\frac{1}{4} - (\tilde{y}_n)^2}\right] \\ = \; & e^{-\lim_{n\to\infty} \frac{n \cdot \left[(\tilde{x}_n)^2 - (\tilde{y}_n)^2\right]}{\frac{1}{4} - (\tilde{y}_n)^2}} \end{aligned} \tag{36}$$

where the equality on the first line follows from the equality (30).

Proof of (ii): By assumption, we have, that $\lim_{n\to\infty} |\tilde{x}_n| \cdot n^{\frac{1}{2}} = c$ for some $c \in \mathbb{R}$. So,

$$\begin{aligned}\lim_{n\to\infty} (\tilde{x}_n)^2 \cdot n &= \lim_{n\to\infty} (|\tilde{x}_n| \cdot n^{\frac{1}{2}})^2 \\ &= c^2. \end{aligned} \tag{37}$$

We obtain

$$\begin{aligned}&\lim_{n\to\infty} \left[\frac{x_n(1-x_n)}{y_n(1-y_n)}\right]^n \\ = \;& e^{-\lim_{n\to\infty} \frac{n\cdot(\tilde{x}_n)^2}{\frac{1}{4}}} > 0,\end{aligned}$$

where the first equality follows from the assumption that $y_n = \frac{1}{2}$ and the equality (36). The inequality follows from the equality (37).

Proof of (iii): By assumption, we have that $\lim_{n\to\infty}(|\tilde{x}_n| - |\tilde{y_n}|) \cdot n^{\frac{1}{2}} = \infty$. So,

$$\begin{aligned}\lim_{n\to\infty} [(\tilde{x}_n)^2 - (\tilde{y}_n)^2] \cdot n &= \lim_{n\to\infty} |(\tilde{x}_n - \tilde{y}_n)(\tilde{x}_n + \tilde{y}_n) \cdot n| \\ &= \lim_{n\to\infty} |(\tilde{x}_n - \tilde{y}_n) \cdot n^{\frac{1}{2}}| \cdot |(\tilde{x}_n + \tilde{y}_n) \cdot n^{\frac{1}{2}}| \\ &\geq \lim_{n\to\infty} \left[(|\tilde{x}_n| - |\tilde{y}_n|) \cdot n^{\frac{1}{2}}\right]^2 \\ &= \infty. \end{aligned} \tag{38}$$

The inequality on the third line follows from the reverse triangle inequality. We obtain

$$\begin{aligned}&\lim_{n\to\infty} \left[\frac{x_n(1-x_n)}{y_n(1-y_n)}\right]^n \\ = \;& e^{-\lim_{n\to\infty} \frac{n\cdot\left[(\tilde{x}_n)^2 - (\tilde{y}_n)^2\right]}{\frac{1}{4} - (\tilde{y}_n)^2}} \\ = \;& 0,\end{aligned}$$

where the first equality follows from the equality (30) and the second equality follows from the inequality (38).

■

THEOREM 0 *(Bhattacharya [2013]).*[31] *Let voters receive private signals from an information structure π_1 that sends independently, and identically distributed binary signals from $S_1 = \{u, g\}$ with $1 > \Pr(u|\alpha) > \Pr(u|\beta) > 0$. Let the preferences be monotone (that is, the conditions in (5) hold). Then, for any sequence of equilibria*

[31]This theorem is a special case of Theorem 1 in Bhattacharya [2013]. We provide the proof for the convenience of the reader.

σ_n,

$$\begin{aligned}\lim_{n\to\infty} \Pr(A \textit{ is elected}|\alpha;\sigma_n) &= 1,\\ \lim_{n\to\infty} \Pr(B \textit{ is elected}|\beta;\sigma_n) &= 1.\end{aligned}$$

Proof. The assumption that $\Pr(u|\alpha) > \Pr(u|\beta)$ (see (4)) implies that

$$\frac{\Pr(\alpha|u)}{\Pr(\beta|u)} = \frac{p_0}{1-p_0}\cdot\frac{\Pr(u|\alpha)}{\Pr(u|\beta)} > \frac{p_0}{1-p_0}\cdot\frac{\Pr(d|\alpha)}{\Pr(d|\beta)} = \frac{\Pr(\alpha|d)}{\Pr(\beta|d)}. \tag{39}$$

For any sequence of strategies σ_n and all $n \in \mathbb{N}$,

$$\begin{aligned}\Pr(\sigma_n(s,t)=1|u) &= \phi(\Pr(\alpha|u,\mathrm{piv};\sigma_n))\\ &> \phi(\Pr(\alpha|d,\mathrm{piv};\sigma_n))\\ &= \Pr(\sigma_n(s,t)=1|d).\end{aligned} \tag{40}$$

where the equality on the first line follows from the characterisation of the best response in (1) and the definition of the aggregate preference function ϕ in (2). The inequality on the second line follows from the inequality (39) and since we assumed that ϕ is strictly increasing; see formula (5).

We want to show that the inequality (40) carries over when taking limits $n \to \infty$. For this, we show by contradiction that for all equilibrium sequences σ_n, we have

$$\lim_{n\to\infty} \frac{\Pr(\mathrm{piv}|\alpha;\sigma_n)}{\Pr(\mathrm{piv}|\beta;\sigma_n)} \in \mathbb{R}_{++}. \tag{41}$$

Suppose that $\lim_{n\to\infty} \frac{\Pr(\mathrm{piv}|\alpha;\sigma_n)}{\Pr(\mathrm{piv}|\beta;\sigma_n)} = 0$. This implies that $\lim_{n\to\infty}\Pr(\alpha|\mathrm{piv},s;\sigma_n) = 0$ for all $s \in S$. For all $\omega \in \Omega$, we have

$$\begin{aligned}\lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|\omega) &= \lim_{n\to\infty}\sum_{s\in S}\Pr(s|\omega)\cdot\phi(\Pr(\alpha|\mathrm{piv},s;\sigma_n))\\ &= \sum_{s\in S}\Pr(s|\omega)\cdot\phi(\lim_{n\to\infty}\Pr(\alpha|\mathrm{piv},s;\sigma_n))\\ &= \phi(0)\\ &< \frac{1}{2}.\end{aligned} \tag{42}$$

The equality on the first line follows from the characterisation of the best response in (1) and the definition of the aggregate preference function ϕ in (2). The equality on the second line follows from the continuity of ϕ. The inequality on the last line holds by the assumption that B is the full information outcome in β; recall the conditions in (5). Denote $x_n = \Pr(\sigma_n(s,t)=1|\alpha)$, and $y_n = \Pr(\sigma_n(s,t)=1|\beta)$. Note that the inequalities (42), (40) and (4) together imply that, for n sufficiently large, we have $\frac{1}{2} > x_n > y_n$. Recall (3), hence $y_n > 0$ for all $n \in \mathbb{N}$ which implies $(y_n - \frac{1}{2})^2 < \frac{1}{4}$. For all n sufficiently large, we have

$$\begin{aligned}
\frac{\Pr(\alpha|\text{piv};\sigma_n)}{\Pr(\beta|\text{piv};\sigma_n)} &= \frac{\Pr(\text{piv}|\alpha;\sigma_n)}{\Pr(\text{piv}|\beta;\sigma_n)}\frac{p_0}{1-p_0} \\
&= \Big[\frac{(x_n(1-x_n)}{(y_n(1-y_n)}\Big]^n\frac{p_0}{1-p_0} \\
&= [1-\frac{(|x_n-\frac{1}{2}|)^2-(|y_n-\frac{1}{2}|)^2}{\frac{1}{4}-(y_n-\frac{1}{2})^2}\Big]^n\cdot\frac{p_0}{1-p_0} \\
&\geq \frac{p_0}{1-p_0}. \qquad (43)
\end{aligned}$$

where the inequality on the third line follows from (30). The inequality on the last line follows because $\frac{1}{2}>x_n>y_n$ implies $|x-\frac{1}{2}|>|y-\frac{1}{2}|$, and because $(y_n-\frac{1}{2})^2<\frac{1}{4}$. Note that (43) is a contradiction to our starting hypothesis $\lim_{n\to\infty}\frac{\Pr(\text{piv}|\alpha;\sigma_n)}{\Pr(\text{piv}|\beta;\sigma_n)}=0$. By an analogous argument we arrive at a contradiction if we assume that $\lim_{n\to\infty}\frac{\Pr(\text{piv}|\alpha;\sigma_n)}{\Pr(\text{piv}|\beta;\sigma_n)}=\infty$. This proves the claim (41).

Consider now any sequence of equilibria σ_n. The claim (41) implies that $\lim_{n\to\infty}\frac{\Pr(\alpha|u,\text{piv};\sigma_n)}{\Pr(\beta|u,\text{piv};\sigma_n)}\in\mathbb{R}_{++}$, and we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{\Pr(\alpha|u,\text{piv};\sigma_n)}{\Pr(\beta|u,\text{piv};\sigma_n)} &= \frac{\Pr(\alpha|u)}{\Pr(\beta|u)}\cdot\lim_{n\to\infty}\frac{\Pr(\text{piv}|\alpha;\sigma_n)}{\Pr(\text{piv}|\beta;\sigma_n)} \\
&> \frac{\Pr(\alpha|d)}{\Pr(\beta|d)}\cdot\lim_{n\to\infty}\frac{\Pr(\text{piv}|\alpha;\sigma_n)}{\Pr(\text{piv}|\beta;\sigma_n)} \\
&= \lim_{n\to\infty}\frac{\Pr(\alpha|d,\text{piv};\sigma_n)}{\Pr(\beta|d,\text{piv};\sigma_n)}, \qquad (44)
\end{aligned}$$

where the inequality on the second line follows from (39). We have

$$\begin{aligned}
\lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|u) &= \lim_{n\to\infty}\phi(\Pr(\alpha|u,\text{piv};\sigma_n)) \\
&= \phi(\lim_{n\to\infty}\Pr(\alpha|u,\text{piv};\sigma_n)) \\
&> \phi(\lim_{n\to\infty}\Pr(\alpha|d,\text{piv};\sigma_n)) \\
&= \lim_{n\to\infty}\phi(\Pr(\alpha|d,\text{piv};\sigma_n)) \\
&= \lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|d), \qquad (45)
\end{aligned}$$

where the inequality on the third line follows from the inequality (44). The inequality (45) finishes the proof that the inequality (40) carries over when taking limits $n\to\infty$.

We obtain

$$\begin{aligned}\lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|\alpha) &= \sum_{s'\in S}\Pr(s'|\alpha)\cdot\lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|s') \\ &> \sum_{s'\in S}\Pr(s'|\beta)\cdot\lim_{n\to\infty}\Pr(\sigma_n(s',t)=1|s') \\ &= \lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|\beta). \qquad (46)\end{aligned}$$

where the inequality on second line follows from the assumption that signal u is more likely in α than in β (see 4) and from the inequality (45).

Now, we prove by contradiction that for any sequence of equilibria σ_n,

$$\lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|\alpha)-\frac{1}{2}=\lim_{n\to\infty}\frac{1}{2}-\Pr(\sigma_n(s,t)=1|\beta). \qquad (47)$$

Denote again $x_n=\Pr(\sigma_n(s,t)=1|\alpha)$, and $y_n=\Pr(\sigma_n(s,t)=1|\beta)$. Suppose that the equality (47) does not hold. Note that the inequality (46) translates as $\lim_{n\to\infty}x_n > \lim_{n\to\infty}y_n$. Consequently, we must have $\lim_{n\to\infty}|x_n-\frac{1}{2}|-|y_n-\frac{1}{2}|\neq 0$. Recall that

$$\begin{aligned}\frac{\Pr(\alpha|\text{piv};\sigma_n)}{\Pr(\beta|\text{piv};\sigma_n)} &= \frac{p_0}{1-p_0}\frac{\Pr(\text{piv}|\alpha;\sigma_n)}{\Pr(\text{piv}|\beta;\sigma_n)} \\ &= \Big[\frac{(x_n(1-x_n)}{(y_n(1-y_n)}\Big]^n.\end{aligned}$$

If $\lim_{n\to\infty}|x_n-\frac{1}{2}|-|y_n-\frac{1}{2}|>0$, Lemma 1 (iii) implies that $\lim_{n\to\infty}\frac{\Pr(\alpha|\text{piv};\sigma_n)}{\Pr(\beta|\text{piv};\sigma_n)}=0$. If $\lim_{n\to\infty}|y_n-\frac{1}{2}|-|x_n-\frac{1}{2}|>0$, Lemma (iii) implies that $\lim_{n\to\infty}\frac{\Pr(\beta|\text{piv};\sigma_n)}{\Pr(\alpha|\text{piv};\sigma_n)}=0$, and hence $\lim_{n\to\infty}\frac{\Pr(\alpha|\text{piv};\sigma_n)}{\Pr(\beta|\text{piv};\sigma_n)}=\infty$. This implies that $\lim_{n\to\infty}\frac{\Pr(\text{piv}|\alpha;\sigma_n)}{\Pr(\text{piv}|\beta;\sigma_n)}\notin\mathbb{R}_{++}$ which contradicts (41). Hence, (47) must hold for any equilibrium sequence. The inequality (47) together with the inequality (46) implies that

$$\lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|\alpha)>\frac{1}{2}>\lim_{n\to\infty}\Pr(\sigma_n(s,t)=1|\beta). \qquad (48)$$

Therefore, it follows from the law of large numbers that

$$\begin{aligned}\lim_{n\to\infty}\Pr(\text{A is elected}|\alpha;\sigma_n) &= 1, \\ \lim_{n\to\infty}\Pr(\text{B is elected}|\beta;\sigma_n) &= 1.\end{aligned}$$

This finishes the proof of Theorem 0.

■

8.2 Appendix B: No Exogeneous Private Signals

Lemma 2 *For any sequence $(\sigma_n)_{n\in\mathbb{N}}$ of strategies, it holds that*

$$\lim_{n\to\infty}\Pr(\alpha|\text{piv},z;\sigma_n,q,r,l)=\lim_{n\to\infty}\Pr(\alpha|\Omega_2,z;\sigma_n,q,r,l)=r.$$

Proof. Let $x_n:=\Pr(\sigma_n(s,t)=1|\alpha_2;\sigma_n,q,r,l)$, and $y_n:=\Pr(\sigma_n(s,t)=a|\beta_2;\sigma_n,q,r,l)$.

Then,

$$\begin{aligned}\frac{\Pr(\text{piv}|\alpha_2;\sigma_n,q,r)}{\Pr(\text{piv}|\beta_2;\sigma_n,q,r)} &= \frac{\binom{2n}{n}(x_n)^n(1-x_n)^n}{\binom{2n}{n}(y_n)^n(1-y_n)^n} \\ &= \left[\frac{x_n(1-x_n)}{y_n(1-y_n)}\right]^n. \end{aligned} \tag{49}$$

We show that the requirements of Lemma 1, (i) are fulfilled: Firstly, recall (3) and that there exists $\epsilon > 0$ such that $\lim_{n\to\infty} x_n \in (\epsilon, 1-\epsilon)$, and $\lim_{n\to\infty} y_n \in (\epsilon, 1-\epsilon)$. We rewrite

$$x_n = \lim_{n\to\infty} \sum_{s'\in\{a,b,z\}} \Pr(s'|\alpha_2)\Pr(\sigma_n(s,t)=1|s'),$$

and

$$y_n = \lim_{n\to\infty} \sum_{s'\in\{a,b,z\}} \Pr(s'|\beta_2)\Pr(\sigma_n(s,t)=1|s').$$

In the states Ω_2, almost all voters receive the same signal z: The probability that a voter receives a signal $s \neq z$ is smaller than cn^{-2} for some $c > 0$. Consequently, we have for all $n \in \mathbb{N}$,

$$|x_n - y_n| < cn^{-2}.$$

Therefore,

$$|x_n - y_n|n^2 < c,$$

and

$$\begin{aligned}\left||x_n - \frac{1}{2}| - |y_n - \frac{1}{2}|\right|n^2 &\leq |x_n - y_n|n^2 \\ &< c,\end{aligned}$$

where the inequality on the first line follows by application of the reverse triangle inequality. We conclude that the requirements of Lemma 1, (i) are fulfilled. It follows from Lemma 1, (i), and (49) that

$$\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2;\sigma_n,q,r,l)}{\Pr(\text{piv}|\beta_2;\sigma_n,q,r,l)} = 1. \tag{50}$$

It follows from (50) that

$$\begin{aligned} & \lim_{n\to\infty} \Pr(\alpha|\Omega_2, z, \text{piv}; \sigma_n, q, r, l) \\ = & \lim_{n\to\infty} \Pr(\alpha|\Omega_2, z; \sigma_n, q, r, l) \\ = & \; r. \end{aligned}$$

For the equality on the last line recall that the parameter r captures the limit of the posterior conditional on Ω_2 and z; see (9). ■

LEMMA 3 *If*

$$\begin{aligned} & \max_{\omega_2\in\Omega_2} |\lim_{n\to\infty} \Pr(\sigma_n(s,t)=1|\omega_2;q,r,l)-\frac{1}{2}| \\ < & \min_{\omega_1\in\Omega_1} |\lim_{n\to\infty} \Pr(\sigma_n(s,t)=1|\omega_1;q,r,l)-\frac{1}{2}| \end{aligned} \tag{17}$$

holds, then

$$\begin{aligned} \lim_{n\to\infty} \Pr(\alpha|a,\text{piv};\sigma_n,q,r,l) = \Pr(\alpha|s,\Omega_2;q,r) &= q, \\ \lim_{n\to\infty} \Pr(\alpha|b,\text{piv};\sigma_n,q,r,l) = \Pr(\alpha|s,\Omega_2;q,r) &= l. \end{aligned}$$

Hence, the unique best response to σ_n in the games $\Gamma_n(q,r,l)$ converges to $\hat{\sigma}_{\Omega_2}(q,r,l)$ for $n\to\infty$.

Proof. Set $x_n := \Pr(\sigma_n(s,t)=1|\omega_1;\sigma_n,q,r,l)$, and $y_n := \Pr(\sigma_n(s,t)=1|\omega_2;\sigma_n,q,r,l)$. For $s\in\{a,b\}$, we have

$$\begin{aligned} \frac{\Pr(\omega_1|s,\text{piv};\sigma_n,q,r,l)}{\Pr(\omega_2|s,\text{piv};\sigma_n,q,r,l)} &= \frac{\Pr(\omega_1|q,r,l)}{\Pr(\omega_2|q,r,l)}\cdot\frac{\Pr(s|\omega_1;q,r,l)}{\Pr(s|\omega_2;q,r,l)}\cdot\frac{\Pr(\text{piv}|\omega_1;\sigma_n,q,r,l)}{\Pr(\text{piv}|\omega_2;\sigma_n,q,r,l)} \\ &\le c\cdot n^3\cdot\frac{\Pr(\text{piv}|\omega_1;\sigma_n,q,r,l)}{\Pr(\text{piv}|\omega_2;\sigma_n,q,r,l)} \end{aligned}$$

where the inequality on the second line holds for all n sufficiently large, and some constant $c>0$ that only depends on q and r. We rewrite

$$\begin{aligned} \frac{\Pr(\text{piv}|\omega_1;\sigma_n,q,r)}{\Pr(\text{piv}|\omega_2;\sigma_n,q,r)} &= \Big[\frac{x_n(1-x_n)}{y_n(1-y_n)}\Big]^n \\ &= \Big[1-\frac{(x_n-\frac{1}{2})^2-(y_n-\frac{1}{2})^2}{\frac{1}{4}-(y_n-\frac{1}{2})^2}\Big]^n. \end{aligned} \tag{51}$$

where the inequality on the last line follows from the equation (30). By the assumption of Lemma 3, we have $\lim_{n\to\infty}(x_n-\frac{1}{2})^2-(y_n-\frac{1}{2})^2>0$. Note that by (3) there exists $\epsilon>0$ such that $(y_n-\frac{1}{2})^2\in[\epsilon,\frac{1}{4}-\epsilon]$. Consequently, $\frac{(x_n-\frac{1}{2})^2-(y_n-\frac{1}{2})^2}{\frac{1}{4}-(y_n-\frac{1}{2})^2}$ converges to a strictly positive number. But then if follows from the equation (51) that for any $c\in\mathbb{R}$ and any $\omega\in\{\alpha,\beta\}$,

$$\lim_{n\to\infty} c\cdot n^3\cdot\frac{\Pr(\text{piv}|\omega_1;\sigma_n,q,r)}{\Pr(\text{piv}|\omega_2;\sigma_n,q,r)} = 0. \tag{52}$$

The following shows that the posterior after some signal $s\in\{a,b\}$ and conditional on being pivotal, converges to the posterior conditional on $s \in \{a, b\}$ and Ω_2 for $n \to \infty$:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\Pr(\alpha|s, \text{piv}; \sigma_n, q, r, l)}{\Pr(\beta|s, \text{piv}; \sigma_n, q, r, l)} \\
= &\lim_{n\to\infty} \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\sum_{j=1,2} \Pr(\alpha_j|\alpha; q, r, l) \cdot \Pr(s|\alpha_j; q, r, l) \cdot \Pr(\text{piv}|\alpha_j, s; \sigma_n, q, r, l)}{\sum_{j=1,2} \Pr(\beta_j|\beta; q, r, l) \cdot \Pr(s|\beta_j, q, r, l) \cdot \Pr(\text{piv}|\beta_j, s; \sigma_n, q, r, l)} \\
= &\lim_{n\to\infty} \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\alpha_2|\alpha; q, r, l) \cdot \Pr(s|\alpha_2; q, r, l) \cdot \Pr(\text{piv}|\alpha_2, s; \sigma_n, q, r, l)}{\Pr(\beta_2|\beta; q, r, l) \cdot \Pr(s|\beta_2; q, r, l) \cdot \Pr(\text{piv}|\beta_2, s; \sigma_n, q, r, l)} \\
= &\lim_{n\to\infty} \frac{\Pr(\alpha)}{\Pr(\beta)} \frac{\Pr(\alpha_2|\alpha; q, r, l) \cdot \Pr(s|\alpha_2; q, r, l)}{\Pr(\beta_2|\beta; q, r, l) \cdot \Pr(s|\beta_2; q, r, l)} \\
= &\lim_{n\to\infty} \frac{\Pr(\alpha|s, \Omega_2; \sigma_n, q, r, l)}{\Pr(\beta|s, \Omega_2; \sigma_n, q, r, l)} \qquad (53) \\
= &\begin{cases} \frac{q}{1-q} & \text{for} \quad s = a, \\ \frac{l}{1-l} & \text{for} \quad s = b. \end{cases} \qquad (54)
\end{aligned}$$

The equality on the third line follows from the equality (52). The equality on the fourth line follows from Lemma 2. For the application of Lemma 2 note that the probability that a random voter $j \in -i$ votes for A is independent of voter i's signal, since signals are independent conditional on ω_2 for $\omega_2 \in \{\alpha_2, \beta_2\}$. The equality on the last line follows from the eqaution (9) and (11) which say that the limit of the posterior conditional on Ω_2 and conditional on $s \in \{a, b\}$ is described by the information structure parameters q and l. This finishes the first part of Lemma 3.

Recall that inequality (1) shows that the best response is fully described by the posteriors conditional on being pivotal and conditional on s. A consequence of Lemma 2 is that after z the best response converges to acting optimally upon the posterior belief conditional on Ω_2 and conditional on z, as the following shows:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\Pr(\alpha|z, \text{piv}; q, r, l)}{\Pr(\beta|z, \text{piv}; q, r, l)} \\
= &\lim_{n\to\infty} \frac{p_0}{1-p_0} \cdot \frac{\Pr(z|\alpha; q, r, l)}{\Pr(z|\beta; q, r)} \cdot \frac{\Pr(\text{piv}|z, \alpha; q, r, l)}{\Pr(\text{piv}|z, \beta; q, r, l)} \\
= &\lim_{n\to\infty} \frac{p_0}{1-p_0} \cdot \frac{\Pr(z, \Omega_2|\alpha; q, r, l)}{\Pr(z, \Omega_2|\beta; q, r, l)} \cdot \frac{\Pr(\text{piv}|z, \alpha_2; q, r, l)}{\Pr(\text{piv}|z, \beta_2; q, r, l)} \\
= &\lim_{n\to\infty} \frac{p_0}{1-p_0} \cdot \frac{\Pr(z, \Omega_2|\alpha; q, r, l)}{\Pr(z, \Omega_2|\beta; q, r, l)} \\
= &\frac{\Pr(\alpha|z, \Omega_2; q, r, l)}{\Pr(\beta|z, \Omega_2; q, r, l)},
\end{aligned}$$

where the equality on the third line follows from Lemma 2. It follows from the eqaution (53) that the best response after s for $s \in \{a, b\}$ converges also to acting optimally upon the posterior belief conditional on Ω_2 and conditional on s, when $n \to \infty$. This is by definition the strategy $\hat{\sigma}_{\Omega_2}(q, r, l)$. This finishes the second part of Lemma 3 ■

Computational Example. Consider any strategy σ which satisfies $\Pr(\sigma(s,t) = 1) \geq 0.7$ for $s \in \{a, b\}$, and $\Pr(\sigma(s,t) = 1|z) \in [0.45, 0.54]$. For $n \geq 200$, we have the following bounds on voting probabilities conditional on the substates ω_j for $\omega \in \{\alpha, \beta\}$ and $j \in \{1, 2\}$,

$$\begin{aligned} \Pr(\sigma(s,t) = 1|\omega_1) &\geq 0.7 \text{ for } \omega \in \{\alpha, \beta\}, \\ \Pr(\sigma(s,t) = 1|\omega_2) &\geq 0.45 \text{ for } \omega \in \{\alpha, \beta\}, \\ \Pr(\sigma(s,t) = 1|\omega_2) &\leq 0.54 + \max(1, \frac{q}{1-q})\frac{2}{n^2} \\ &\leq 0.55, \end{aligned} \tag{55}$$

where the inequality on the last line holds for all $n \geq 200$.

Step 1: Posterior Ratios Conditional on piv**.**
For any $\omega_1 \in \{\alpha_1, \beta_1\}, \omega_2' \in \{\alpha_2, \beta_2\}$, we have

$$\begin{aligned} &\frac{\Pr(\text{piv}|\omega_2')}{\Pr(\text{piv}|\omega_1)} \\ \geq\ & \left[1 + \min_{\omega_1\{\alpha_1,\beta_2\},\omega_2'\in\{\alpha_2,\beta_2\}} \frac{(\Pr(\sigma(s,t)=1|\omega_1) - \frac{1}{2})^2 - (\Pr(\sigma(s,t)=1|\omega_2') - \frac{1}{2})^2}{\frac{1}{4} - (\Pr(\sigma(s,t)=1|\omega_1) - \frac{1}{2})^2}\right]^n, \\ \geq\ & (1 + \frac{\frac{9}{100} - \frac{1}{400}}{\frac{1}{4} - \frac{9}{100}})^n \\ \geq\ & (1 + \frac{35}{64})^n \\ \geq\ & (\frac{3}{2})^n. \end{aligned} \tag{56}$$

where the inequality on the second line follows from the ineqaulity (30) in the proof of Lemma 1, (i). The inequality on the second line follows from the bounds for the voting probabilities (55). Note that for any $x, y \in [0, \frac{1}{2}]$ it holds that

$$\begin{aligned} |x^2 - y^2| &= |(x+y)(x-y)| \\ &\leq |x - y|. \end{aligned} \tag{57}$$

For all $n \geq 200$, the ratio of pivotal probabilities in α_2 and β_2 is bounded by

$$\begin{aligned}
&\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \\
\geq\ & (1 - \frac{|(\Pr(\sigma(s,t)=1|\alpha_2) - \frac{1}{2})^2 - (\Pr(\sigma(s,t)=1|\beta_2) - \frac{1}{2})^2|}{\frac{1}{4} - (\Pr(\sigma(s,t)=1|\beta_2) - \frac{1}{2})^2})^n \\
\geq\ & (1 - \frac{|\Pr(\sigma(s,t)=1|\alpha_2) - \Pr(\sigma(s,t)=1|\beta_2)|}{\frac{1}{4} - (\Pr(\sigma(s,t)=1|\beta_2) - \frac{1}{2})^2})^n \\
\geq\ & (1 - \frac{\frac{q}{1-q} \cdot \frac{2}{n^2}}{\frac{1}{4}})^n \\
=\ & (1 - \frac{\frac{6}{n^2}}{\frac{1}{4}})^n \\
\geq\ & 0.885. \qquad (58)
\end{aligned}$$

Here, the inequality on the second line follows from the equation (30) in the proof of Lemma 1, (i) and the expression (49) for the ratio of pivotal probabilities in terms of voting probabilities. The inequality on the third line follows from (57). The inequality on the fourth line follows since a random voter receives a signal $s \neq z$ with probability of at most $\frac{p_A}{1-p_A}\frac{2}{n^2}$ in both α_2 and β_2 such that this represents an upper bound for the difference in the voting probabilities. The equality on the fifth line follows from plugging in $p_A = \frac{3}{4}$, and the inequality on the last line holds for all $n \geq 200$.

Step 2: Posterior Ratios Conditional on piv **and** $s \in \{a, b\}$**.**

The posterior belief ratios conditional on being pivotal and conditional on $s \in \{a, b\}$ satisfy

$$\begin{aligned}
&\frac{\Pr(\alpha|\text{piv}, s)}{\Pr(\beta|\text{piv}, s)} \\
=\ & \frac{p_0}{1-p_0} \cdot \frac{\sum_{j=1,2} \Pr(\alpha_j|\alpha; p_A, \bar{r}, p_A) \cdot \Pr(s|\alpha_j; p_A, \bar{r}, p_A) \cdot \Pr(\text{piv}|\alpha_j, s; \sigma_n, p_A, \bar{r}, p_A)}{\sum_{j=1,2} \Pr(\beta_j|\beta; p_A, \bar{r}, p_A) \cdot \Pr(s|\beta_j, p_A, \bar{r}, p_A) \cdot \Pr(\text{piv}|\beta_j, s; \sigma_n, p_A, \bar{r}, p_A)}, \\
\geq\ & \frac{1}{3} \cdot \frac{\frac{3}{2n} \cdot \frac{3}{n^2} \cdot \Pr(\text{piv}|\alpha_2)}{\frac{1}{2n^3} \cdot \Pr(\text{piv}|\beta_2) + (1 - \frac{1}{2n}) \cdot \frac{2}{3} \cdot \Pr(\text{piv}|\beta_1)} \\
\geq\ & 3 \cdot \frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2) + 2n^3(1 - \frac{1}{2n}) \cdot \frac{2}{3} \cdot \Pr(\text{piv}|\beta_1)} \qquad (59)
\end{aligned}$$

The inequality on the third line follows from plugging in the value $\frac{1}{3}$ for $\frac{p_0}{1-p_0}$, the value $\frac{3}{2n}$ for $\Pr(\alpha_2|\alpha; p_A, \bar{r}, p_A)$, the value $\frac{3}{n^2}$ for $\Pr(s|\alpha_2; p_A, \bar{r}, p_A)$, the value $\frac{1}{n}$ for $\Pr(\beta_2|\beta; p_A, \bar{r}, p_A)$, the value $\frac{1}{n^2}$ for $\Pr(s|\beta_2; p_A, \bar{r}, p_A)$, the value $(1 - \frac{1}{2n})$ for $\Pr(\beta_1|\beta; p_A, \bar{r}, p_A)$, the upper bound $\frac{2}{3}$ for $\Pr(s|\beta_1; p_A, \bar{r}, p_A)$. For $n \geq 200$, we have that $2n^3(1-\frac{1}{2n})\frac{2}{3}\frac{3}{2}^{-n} \leq 10^{-27}$. It follows from the inequality (56) that $\Pr(\text{piv}|\beta_1) \leq (\frac{3}{2})^{-n}\Pr(\text{piv}|\beta_2)$. Together with the inequality (59), it follows that for $s \in \{a,b\}$ and $n \geq 200$,

$$\frac{\Pr(\alpha|\text{piv},s)}{\Pr(\beta|\text{piv},s)} \geq 3 \cdot \frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \cdot \frac{1}{1+10^{-27}}. \tag{60}$$

It follows from the inequalities (60) and (58) that for $s \in \{a,b\}$ and $n \geq 200$,

$$\frac{\Pr(\alpha|\text{piv},s)}{\Pr(\beta|\text{piv},s)} \geq 3 \cdot 0.884 = 2.652$$

Therefore, for $s \in \{a,b\}$,

$$\Pr(BR(\sigma)(s,t)=1) = \Pr(\alpha|\text{piv},s) \geq \frac{2.652}{1+2.652} > 0.7,$$

where $BR(\sigma)$ denotes the best response to σ. The first equality follows from the assumption that the thresholds of doubt are uniformly distributed on $[0,1]$.

Step 3: Posterior Ratios Conditional on piv and z.
We have

$$\begin{aligned} \frac{\Pr(\alpha|\text{piv},z)}{\Pr(\beta|\text{piv},z)} &= \frac{p_0}{1-p_0}\frac{\Pr(z|\alpha)}{\Pr(z|\beta)}\frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \\ \geq & \quad \frac{p_0}{1-p_0}\frac{\Pr(\alpha_2|\alpha)\Pr(z|\alpha_2)}{\Pr(\beta_2|\beta)\Pr(z|\beta_2)} \cdot 0.885 \\ \geq & \quad \frac{1-\frac{6}{n^2}}{1-\frac{2}{n^2}} \cdot 0.885 \\ \overset{n\geq 200}{\geq} & \quad 0.88. \end{aligned} \tag{61}$$

where the inequality on the second line follows from the inequality (58) and since z is only received in the states α_2 and β_2. The inequality on the third line follows from the assumptions of the example which imply that $\frac{p_0}{1-p_0} = \frac{\Pr(\beta_2|\beta)}{\Pr(\alpha_2|\alpha)}$ and from plugging in the value $1-\frac{6}{n^2}$ for $\Pr(z|\alpha_2)$ and the value $1-\frac{2}{n^2}$ for $\Pr(z|\beta_2)$.

Therefore

$$\Pr(BR(\sigma)(s,t)=1|z) = \Pr(\alpha|\text{piv},z) \geq \frac{0.88}{1+0.88} \geq 0.46. \tag{62}$$

where first equality follows from the assumption that the thresholds of doubt are uniformly distributed on $[0,1]$.

On the other hand, we have

$$\begin{aligned}
& \frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \\
\leq\ & (1+\frac{|(\Pr(\sigma(s,t)=1|\alpha_2)-\frac{1}{2})^2-(\Pr(\sigma(s,t)=1|\beta_2)-\frac{1}{2})^2|}{\frac{1}{4}-(\Pr(\sigma(s,t)=1|\beta_2)-\frac{1}{2})^2})^n \\
\leq\ & (1+\frac{|\Pr(\sigma(s,t)=1|\alpha_2)-\Pr(\sigma(s,t)=1|\beta_2)|}{\frac{1}{4}-(\Pr(\sigma(s,t)=1|\beta_2)-\frac{1}{2})^2})^n \\
\leq\ & (1+\frac{\frac{q}{1-q}\cdot\frac{2}{n^2}}{\frac{1}{4}-\frac{1}{400}})^n \\
=\ & (1+\frac{\frac{6}{n^2}}{\frac{1}{4}-\frac{1}{400}})^n \\
\leq\ & 1.13
\end{aligned} \tag{63}$$

Here, the inequality on the second line follows from the equation (30) in the proof of Lemma 1, (i) and the expression (49) for the ratio of pivotal probabilities in terms of voting probabilities. The inequality on the third line follows from the ineqaulity (57). The inequality on the fourth line follows since a random voter receives a signal $s \neq z$ with probability of at most $\frac{p_A}{1-p_A}\frac{2}{n^2}$ in both α_2 and β_2 such that this represents an upper bound for the difference in the voting probabilities. The equality on the fifth line follows from plugging in $p_A = \frac{3}{4}$, and the inequality on the last line holds for all $n \geq 200$.

We obtain

$$\frac{\Pr(\alpha|\text{piv},z)}{\Pr(\beta|\text{piv},z)} \leq \frac{1-\frac{6}{n^2}}{1-\frac{2}{n^2}}\cdot 1.13 \leq 1.13.$$

where the first inequality follows from the analogous argument as used for the inequality (61). Therefore

$$\Pr(BR(\sigma)(s,t)=1|z) = \Pr(\alpha|\text{piv},z) \leq \frac{1.13}{1+1.13} < 0.54. \tag{64}$$

where first equality follows from the assumption that the thresholds of doubt are uniformly distributed on $[0,1]$.

Step 4: Fixed Point Argument.

It follows from the inequalities (61)-(64) that the best response is a self-map on the set of strategies which satisfy $\Pr(\sigma(s,t)=1|s') \geq 0.7$ for any $s' \in \{a,b\}$, and $\Pr(\sigma(s,t)=1|z) \in [0.45, 0.54]$. Evaluation of the binomial distribution shows that $\Pr(\mathcal{B}(2n+1,x)) > n) \geq 0.999999$ if $n \geq 200$ and $x \geq 0.7$. Therefore, for any $n \geq 200$, the Brouwer fixed point theorem yields an equilibrium[32] which satisfies

$$\begin{aligned} & \Pr(A \text{ is elected}) \\ \geq\ & \Pr(A \text{ is elected}|\Omega_1) \cdot \min_{\omega} \Pr(\Omega_1|\omega) \\ \geq\ & 0.999999 \cdot (1 - \frac{3}{2n}) \\ \geq\ & 99\%, \end{aligned}$$

where the last inequality holds for all $n \geq 200$.

8.3 Appendix C: Robustness

Note that we suppress the dependence of the information structure in the notation in Appendix C-Appendix E.

Proposition 1 *(Global Basin of Attraction)*[33]
When $\phi(0) < \frac{1}{2} < \phi(1)$: For any $\epsilon > 0$, the measure of $\Sigma^2(\epsilon, n)$ in the space of cutoff-strategies $[0,1]^3$ converges to 1, for $n \to \infty$.

Proof. Since the measure of the cutoff strategies σ that satisfy (20) and (21) converges to 1, it is sufficient to show that all such strategies are elements of $\Sigma^2(\epsilon, n)$.

Case 1: $\min_{\omega_1 \in \{\alpha_1, \beta_1\}} |\Pr(\sigma(s,t) = 1|\omega_1) - \frac{1}{2}| - \min_{\omega_2 \in \{\alpha_2, \beta_2\}} |\Pr(\sigma(s,t) = 1|\omega_2) - \frac{1}{2}| > n^{-\frac{1}{4}}$
Note that $\lim_{n\to\infty} \frac{n^{-\frac{1}{4}}}{n^{-\frac{1}{2}}} = \lim_{n\to\infty} n^{\frac{1}{4}} = \infty$. By application of Lemma 1, (iii) to $x_n = \Pr(\sigma(s,t) = 1|\Omega_1)$ and $y_n = \Pr(\sigma(s,t) = 1|\Omega_2)$, we obtain that $\lim_{n\to\infty} \Pr(\Omega_2|s, \text{piv}; \sigma) = 1$ for $s \in \{a, b\}$. Being pivotal contains the information that Ω_2 holds, for $n \to \infty$, and no information beyond that by Lemma 2. Consequently $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, s; \sigma) = \Pr(\alpha|\Omega_2, s; \sigma)$, so the cutoffs of the best response to σ converge to the cutoffs of Ω_2-sincere voting.[34] Hence, for any $\epsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that $|\text{BR}(\sigma) - \hat{\sigma}_{\Omega_2}(\bar{q}, \bar{r})| < \epsilon$ for all $n \geq \bar{n}$. Therefore $\text{BR}(\sigma)$ satisfies the assumption of Case 1 for n

[32] Note that this equilibrium is not degenerate, that is, $\Pr(\sigma(s,t) = 1) \in (0,1)$, since $\Pr(\sigma(z,t) = 1) \in (0.45, 0.54)$.

[33] The result holds more generally. If we consider any random (not necessarily cutoff) strategy as the starting point, for any $\epsilon > 0$ the probability that the twice-iterated best response lies in an ϵ-neighbourhood of conditional sincere voting converges to 1, for $n \to \infty$.

[34] A more detailed version of the proof until here can be done in complete analogy to the proof of Lemma 3.

sufficiently large. Iteration of the argument shows that $\sigma \in \Sigma(\epsilon, n)$ for n sufficiently large.

Case 2: $\min_{\omega_2 \in \{\alpha_2, \beta_2\}} |\Pr(\sigma(s,t) = 1|\omega_2) - \frac{1}{2}| - \min_{\omega_1 \in \{\alpha_1, \beta_1\}} |\Pr(\sigma(s,t) = 1|\omega_1) - \frac{1}{2}| > n^{-\frac{1}{4}}$

By application of Lemma 1, (iii) to $x_n = \Pr(\sigma(s,t) = 1|\Omega_2)$ and $y_n = \Pr(\sigma(s,t) = 1|\Omega_1)$, we obtain that $\lim_{n\to\infty} \Pr(\Omega_1|s, \text{piv}; \sigma) = 1$ for $s \in \{a, b\}$. If $|\Pr(\sigma(s,t) = 1|\alpha_1) - \frac{1}{2}| - |\Pr(\sigma(s,t) = 1|\beta_1) - \frac{1}{2}| > n^{-\frac{1}{4}}$, we apply Lemma 1, (iii) to $x_n = \Pr(\sigma(s,t) = 1|\alpha_1)$ and $y_n = \Pr(\sigma(s,t) = 1|\beta_1)$, and obtain that

$$\lim_{n\to\infty} \Pr(\alpha_1|s, \text{piv}; \sigma) = 0 \tag{65}$$

for any $s \in \{a, b\}$. If $|\Pr(\sigma(s,t) = 1|\beta_1) - \frac{1}{2}| - |\Pr(\sigma(s,t) = 1|\alpha_1) - \frac{1}{2}| > n^{-\frac{1}{4}}$, we apply Lemma 1, (iii) to $x_n = \Pr(\sigma(s,t) = 1|\beta_1)$ and $y_n = \Pr(\sigma(s,t) = 1|\alpha_1)$, and obtain that

$$\lim_{n\to\infty} \Pr(\beta_1|s, \text{piv}; \sigma, q, r) = 0 \tag{66}$$

for any $s \in \{a, b\}$. Being pivotal contains the information that either α_1 or β_1 holds, for $n \to \infty$. We have

$$\begin{aligned} & \lim_{n\to\infty} \Pr(\sigma(s,t) = 1|\alpha_1) \\ = & \lim_{n\to\infty} \phi(\Pr(\alpha|a, \text{piv}; \sigma)) \\ = & \phi(\lim_{n\to\infty} \Pr(\alpha|a, \text{piv}; \sigma)) \in \{\phi(0), \phi(1)\}. \end{aligned} \tag{67}$$

The equality on the second line follows from the definition of the aggregate preference function ϕ in (1). The equality on the second line follows from the continuity of ϕ, and the inclusion on the second line follows from (65) and (66). Analogously,

$$\begin{aligned} & \lim_{n\to\infty} \Pr(\sigma(s,t) = 1|\beta_1) \\ = & \lim_{n\to\infty} \phi(\Pr(\alpha|b, \text{piv}; \sigma)) \\ = & \phi(\lim_{n\to\infty} \Pr(\alpha|b, \text{piv}; \sigma)) \in \{\phi(0), \phi(1)\}. \end{aligned} \tag{68}$$

Recall that we assume $\phi(0) < \frac{1}{2}$ and $\phi(1) > \frac{1}{2}$; see (5). So, the margin of victory under the best response is non-zero in α_1 and β_1 for n sufficiently large. However, by equation (16), the margin of victory in Ω_2 is zero, for $n \to \infty$. So, the best response satisfies the margin of victory condition (17) of Lemma 3, and consequently the twice iterated best response converges to Ω_2-sincere voting. ■

8.4 Appendix D: Exogeneous Private Signals

Note that we suppress the dependence of the information structure in the notation in Appendix C-Appendix E. The following Lemma is needed in the proof of Lemma 4, and in the proof of uniqueness of $\bar{r}$ satisfying (25).

Lemma 7 *For any voter behaviour after a and b and any $r \in (0,1)$,*

$$\lim_{n\to\infty} \Pr(\sigma(s,t)=1|\alpha_2;\sigma^r) > \lim_{n\to\infty} \Pr(\sigma(s,t)=1|\beta_2;\sigma^r).$$

Proof. Fix voter behaviour after a and b. Consider any $r \in (0,1)$. The assumption that $\Pr(u|\alpha) > \Pr(u|\beta)$ (see (4)) implies that

$$\frac{p(u|r)}{1-p(u|r)} = \frac{r_z}{1-r_z} \cdot \frac{\Pr(u|\alpha)}{\Pr(u|\beta)} > \frac{r_z}{1-r_z} \cdot \frac{\Pr(d|\alpha)}{\Pr(d|\beta)} = \frac{p(d|r)}{1-p(d|r)}, \tag{69}$$

where $p(u|r)$ and $p(d|r)$ are defined through the equalities; compare to the main text. Recall that we defined the strategy σ^r as the pure strategy that votes A after z and $v \in \{u,d\}$ if $p(v)t_\alpha + (1-p(v))t_\beta \geq 0$.

$$\begin{aligned}
& \lim_{n\to\infty} \Pr(\sigma^r(s,t)=1|\alpha_2) \\
= & \sum_{s' \in \{a,b,z\}\times\{u,d\}} \lim_{n\to\infty} \Pr(s'|\alpha_2)\cdot \Pr(\sigma^r(s,t)=1|s') \\
= & \sum_{s' \in \{z\}\times\{u,d\}} \lim_{n\to\infty} \Pr(s'|\alpha_2)\cdot \Pr(\sigma^r(s,t)=1|s') \\
= & \sum_{v\in\{u,d\}} \Pr(v|\alpha)\cdot\phi(p(v)) \qquad (70) \\
> & \sum_{v\times\{u,d\}} \Pr(v|\beta)\cdot\phi(p(v)) \\
= & \lim_{n\to\infty} \Pr(\sigma^r(s,t)=1|\beta_2;\sigma^r). \qquad (71)
\end{aligned}$$

The equality on the third line holds, because the probability of receiving z in α_2 converges to 1 by definition of the information structures $\pi_n(q,r,l)$; see (7) - (8). The equation on the fourth line follows from the definition of σ^r and the definition of the aggregate preference function ϕ in (2). The inequality on the fifth line follows from the assumption that $\Pr(u|\alpha) > \Pr(u|\beta)$, the inequality (69) and since we assumed that ϕ is strictly increasing; see formula (5). ■

Proof of the uniqueness and the existence of $\bar{r}$ satisfying (25)

Existence. Let $x(r) = \lim_{n\to\infty} \Pr(\sigma^r(s,t)=1|\alpha_2)$, and $y(r) = \lim_{n\to\infty} \Pr(\sigma^r(s,t) =$

$1|\beta_2)$. Note that

$$\begin{aligned}\lim_{r\to 0} x(r) &= \lim_{r\to 0} \sum_{v\in\{u,d\}} \Pr(v|\alpha)\cdot\phi(p(v|r)) \\ &= \phi(0) \qquad (72)\\ &< \frac{1}{2}, \qquad (73)\end{aligned}$$

where the equality on the first line restates the formula (70) for the limit of the voting probability $x(r)$ for $n \to \infty$. The equality on the second line follows from the definition of $p(v|r)$; see (69). The inequality on the third line follows since we assumed that the full information outcome in β is B; compare to (5). Analogously,

$$\begin{aligned}\lim_{r\to 1} x(r_z) &= \lim_{r\to 1} \sum_{v\in\{u,d\}} \Pr(v|\alpha)\cdot\phi(p(v|r)) \\ &= \phi(1) \qquad (74)\\ &> \frac{1}{2}. \qquad (75)\end{aligned}$$

For any $r \in (0,1)$ sufficiently small, we have

$$\frac{1}{2} \;>\; x(r) > y(r),$$

where the first inequality follows from the inequality (73) and the continuity of ϕ, and the second inequality from Lemma 7. This implies that, for r sufficiently small, the right hand side of (25) is larger than the left hand side of (25). Conversely, for any $r \in (0,1)$ sufficiently large, we have

$$\frac{1}{2} \;<\; y(r) < x(r).$$

where the first inequality follows from the inequality (75) and the continuity of ϕ, and the second inequality from Lemma 7. This implies that, for r sufficiently large, the right hand side of (25) is smaller than the left hand side of (25). We obtain

$$\begin{aligned}&|x(r) - \frac{1}{2}| - |y(r) - \frac{1}{2}| \\ =\;& |(\sum_{v=u,d} \Pr(v|\alpha)\cdot\bar{\phi}(\frac{r}{1-r}\cdot\frac{\Pr(v|\alpha)}{\Pr(v|\beta)})) - \frac{1}{2}| \qquad (76)\\ &-|(\sum_{v=u,d} \Pr(v|\beta)\cdot\bar{\phi}(\frac{r}{1-r}\cdot\frac{\Pr(v|\alpha)}{\Pr(v|\beta)})) - \frac{1}{2}|.\end{aligned}$$

from plugging in the formula (70) for the limits of the voting probability in α_2 and β_2, $x(r)$ and $y(r)$, for $n \to \infty$. Hence, the function $|x(r) - \frac{1}{2}| - |y(r) - \frac{1}{2}|$ is continuous in r, since $\bar{\phi}$ is continuous in r[35]. The intermediate value theorem implies that the function $|x(r) - \frac{1}{2}| - |y(r) - \frac{1}{2}|$ has at least one zero $\bar{r} \in (0,1)$. In other words, there exists $\bar{r}$ that satisfies the condition (25), which means that under $\sigma^{\bar{r}}$ the limit of the

[35] Recall that for any $p \in (0,1)$, we have defined $\bar{\phi}(\frac{p}{1-p}) = \phi(p)$ and that ϕ is continuous.

margin of victory in α_2 is the same as in β_2.

Uniqueness. Consider any $\bar{r}$ such that $\sigma^{\bar{r}}$ satisfies the condition (25). Let $\bar{x} = \lim_{n\to\infty} \Pr(\sigma^{\bar{r}}(s,t) = 1|\alpha_2)$, $\bar{y} = \lim_{n\to\infty} \Pr(\sigma^{\bar{r}}(s,t) = 1|\beta_2)$. It follows from Lemma 7 that $\bar{x} > \bar{y}$. It follows from the definition of $\bar{r}$ that $|\bar{x} - \frac{1}{2}| = |\bar{y} - \frac{1}{2}|$. Consequently, $\bar{x} > \frac{1}{2} > \bar{y}$. Recall the formula (70) (or the formula (76)) which expresses the limits of the voting probabilities in α_2 and β_2, $x(r)$ and $y(r)$, as functions of values of ϕ. If $0 < r < \bar{r}$, we have $y(r) < \bar{y} < \frac{1}{2}$ since ϕ is strictly increasing in r. Also, either $\bar{x} > x(r) \geq \frac{1}{2}$, or $y(r) < x(r) < \frac{1}{2}$. In any case, we have $|x(r) - \frac{1}{2}| - |y(r) - \frac{1}{2}| < |\bar{x} - \frac{1}{2}| - |\bar{y} - \frac{1}{2}| = 0$. If $1 > r > \bar{r}$, then $x > \bar{x} > \frac{1}{2}$ and either $\frac{1}{2} \geq y > \bar{y}$ or $x > y > \frac{1}{2}$. In any case, we have $|x(r) - \frac{1}{2}| - |y(r) - \frac{1}{2}| > |\bar{x} - \frac{1}{2}| - |\bar{y} - \frac{1}{2}| = 0$. This shows that $\bar{r}$ is the unique zero of the function $|x(r) - \frac{1}{2}| - |y(r) - \frac{1}{2}|$, hence the unique $r \in (0,1)$ such that σ^r satisfies the equality (25).

Lemma 4 *For any equilibrium sequence σ_n, it holds that*

$$\lim_{n\to\infty} \Pr(\alpha|\text{piv}, z; \sigma_n, \pi_n(q,r,l), \pi_1) = \bar{r}.$$

Proof. Consider any equilibrium sequence σ_n. Let $r_{z,n} = \Pr(\alpha|\text{piv}, z; \sigma_n)$, and

$$x_n = \Pr(\sigma(s,t) = 1|\alpha_2; \sigma_n), \quad (77)$$

$$y_n = \Pr(\sigma(s,t) = 1|\beta_2; \sigma_n). \quad (78)$$

We lead the assumption $\lim_{n\to\infty} r_{z,n} \neq \bar{r}_z$ to a contradiction.
Case 1: If $\lim_{n\to\infty} r_{z,n} = 0$, we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \phi(0) < \frac{1}{2}. \quad (79)$$

where the second equality restates the expression (72) for the limit of the voting probability in α_2. The first equality holds, because we can derive the same expression for the limit of the voting probability in β_2. The inequality follows from the inequality (72). Note that for n sufficiently large, this implies that $\frac{1}{2} > x_n > y_n$, where the first inequality follows from (79) and the second inequality from Lemma 7. Recall that the presence of partisans implies that the voting probability in β_2 is strictly interior, that is, $y_n \in (\epsilon, 1-\epsilon)$ for all $n \in \mathbb{N}$ and some $\epsilon > 0$; compare to (3). Consequently, for all

n sufficiently large,

$$\begin{aligned} \frac{\Pr(\alpha_2|\text{piv})}{\Pr(\beta_2|\text{piv})} &= \frac{\Pr(\text{piv}|\alpha_2)}{\Pr(\text{piv}|\beta_2)} \cdot \frac{p_0}{1-p_0} \\ &= \Big[\frac{(x_n(1-x_n)}{(y_n(1-y_n)}\Big]^n \cdot \frac{p_0}{1-p_0} \\ &= [1 - \frac{(x_n-\frac{1}{2})^2-(y_n-\frac{1}{2})^2}{\frac{1}{4}-(y_n-\frac{1}{2})^2}\Big]^n \cdot \frac{p_0}{1-p_0} \\ &\geq \frac{p_0}{1-p_0}. \end{aligned} \tag{80}$$

where the equality on the third line follows from (30). The inequality on the fourth line follows since the factor on the left of the fourth line is weakly larger than 1. This follows from the observations that $\frac{1}{2} > x_n > y_n$ and that $y_n \in (\epsilon, 1-\epsilon)$. The bound 80 implies a contradiction to the assumption of Case 1 that $\lim_{n\to\infty} r_{z,n} = 0$, because $p_0 \in (0,1)$ and $\lim_{n\to\infty} r_{z,n} = \lim_{n\to\infty} \Pr(\alpha|z, \text{piv}; \sigma_n) = \lim_{n\to\infty} \Pr(\alpha|\Omega_2, \text{piv}; \sigma_n)$.

Case 2: Suppose that $\lim_{n\to\infty} r_{z,n} = 1$. We arrive at a contradiction by an analogous argument as for Case 1.

Case 3: Suppose that $0 < \lim_{n\to\infty} r_{z,n} < \bar{r}$. Recall that $\bar{r}$ is the unique number such that under $\sigma^{\bar{r}}$ the limit of the expected margin of victory in α_2 is the same as is β_2. From the proof of this uniqueness result it follows that the assumption $0 < \lim_{n\to\infty} r_{z,n} < \bar{r}$ implies that the limit of the expected margin of victory in α_2, that is $\lim_{n\to\infty} |x_n - \frac{1}{2}|$, is strictly smaller than the limit of the expected margin of victory in in β_2, that is, $\lim_{n\to\infty} |y_n - \frac{1}{2}|$. Apply Lemma 1, (iii) to see that $\lim_{n\to\infty} \frac{(y_n(1-y_n)}{(x_n(1-x_n)})^n = \frac{\Pr(\text{piv}|\beta_2;\sigma_n)}{\Pr(\text{piv}|\alpha_2;\sigma_n)} = 0$. Conditional on being tied, α_2 is infinitely more likely than β_2, for $n \to \infty$. We obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{r_{z,n}}{1-r_{z,n}} &= \lim_{n\to\infty} \frac{\Pr(\alpha|z, \text{piv}; \sigma_n)}{\Pr(\beta|z, \text{piv}; \sigma_n)} \\ &= \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2; \sigma_n)}{\Pr(\text{piv}|\beta_2; \sigma_n)} \frac{p_0}{1-p_0} \frac{\Pr(\alpha_2|\alpha)\Pr(z|\alpha_2)}{\Pr(\beta_2|\beta)\Pr(z|\beta_2)} \\ &= \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2; \sigma_n)}{\Pr(\text{piv}|\beta_2; \sigma_n)} \frac{r}{1-r} \\ &= \infty. \end{aligned} \tag{81}$$

The equality on the first line follows from the definition of $r_{z,n}$ as the posterior conditional on being pivotal and z. The equality on the third line follows from the formula (10) for the limit of the posterior conditional on being pivotal and z. The equality on the fourth line follows from the observation that $\lim_{n\to\infty} \frac{\Pr(\text{piv}|\beta_2;\sigma_n)}{\Pr(\text{piv}|\alpha_2;\sigma_n)} = 0$. The equality (81) implies a contradiction to the assumption $\lim_{n\to\infty} r_{z,n} < \bar{r_z} \in (0,1)$ of Case 3.

Case 4: Suppose that $1 > \lim_{n\to\infty} r_{z,n} > \bar{r_z}$.[36] Recall that $\bar{r}$ is the unique number such that under $\sigma^{\bar{r}}$ the limit of the expected margin of victory in α_2 is the same as is β_2. From the proof of this uniqueness result it follows that the assumption $1 > \lim_{n\to\infty} r_{z,n} > \bar{r_z}$ implies that the limit of the expected margin of victory in α_2, that is $\lim_{n\to\infty} |x_n - \frac{1}{2}|$, is strictly larger than the limit of the expected margin of victory in in β_2, that is, $\lim_{n\to\infty} |y_n - \frac{1}{2}|$. Apply Lemma 1, (iii) to see that $\lim_{n\to\infty} \frac{(x_n(1-x_n)}{(y_n(1-y_n)})^n = \frac{\Pr(\text{piv}|\alpha_2;\sigma_n)}{\Pr(\text{piv}|\beta_2;\sigma_n)} = 0$. Conditional on being tied, β_2 is infinitely more likely than α_2, for $n \to \infty$. We obtain

$$\begin{aligned}
\lim_{n\to\infty} \frac{r_{z,n}}{1-r_{z,n}} &= \lim_{n\to\infty} \frac{\Pr(\alpha|z,\text{piv};\sigma_n)}{\Pr(\beta|z,\text{piv};\sigma_n)} \\
&= \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2;\sigma_n)}{\Pr(\text{piv}|\beta_2;\sigma_n)} \frac{p_0}{1-p_0} \frac{\Pr(\alpha_2|\alpha)\Pr(z|\alpha_2)}{\Pr(\beta_2|\beta)\Pr(z|\beta_2)} \\
&= \lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2;\sigma_n)}{\Pr(\text{piv}|\beta_2;\sigma_n)} \frac{r}{1-r} \\
&= 0. \qquad (82)
\end{aligned}$$

The equality on the first line follows from the definition of $r_{z,n}$ as the posterior conditional on being pivotal and conditional on z. The equality on the third line follows from the formula (10) for the limit of the posterior conditional on being pivotal and z. The equality on the fourth line follows from the observation that $\lim_{n\to\infty} \frac{\Pr(\text{piv}|\alpha_2;\sigma_n)}{\Pr(\text{piv}|\beta_2;\sigma_n)} = 0$. The equality (81) implies a contradiction to the assumption $\lim_{n\to\infty} r_{z,n} > \bar{r_z} \in (0,1)$ of Case 4.

■

Lemma 6 *Let p_A satisfy the equation (28). For any $\epsilon > 0$ sufficiently small, there exists $n(\epsilon) > 0$ such that for any $n \geq n(\epsilon)$, in the games $\Gamma_n(p_A, \bar{r}, p_A)$, any fixed point of the ϵ-truncated best response around $(p_A, \bar{r}, p_A)$ is interior.*

Proof. Consider any fixed point $(p(s))_{s\in\{a,b,z\}}$ of the ϵ-truncated best response function around $(p_A, p_A, \bar{r})$ with p_A satisfying the inequality (28) and $\epsilon > 0$ small enough such that any $\sigma \in \mathrm{B}_\epsilon(\hat{\sigma}(p_A, \bar{r}, p_A))$ satisfies the margin of victory condition (26) of Lemma 5. In particular, this holds for the fixed point by construction. Consequently, for $s \in \{a, b\}$, the s-component of the best response to the fixed point

[36] The proof of Case 4 is analogous to the proof of Case 3. We provide it for the sake of completeness.

converges to p_A for $n \to \infty$. Hence, the s-component of the fixed point is interior for $s \in \{a, b\}$ and n sufficiently large. Suppose that the z-component of the fixed point is at the lower bound $\bar{r} - \epsilon$. Then the remark after Equation (25) implies that the margin of victory in α_2 is strictly smaller than in β_2, for any n sufficiently large. Let $y_n = \Pr(\sigma(s,t) = 1|\alpha_2; (p(s))_{s\in\{a,b,z\}})$ and $x_n = \Pr(\sigma(s,t) = 1|\beta_2; (p(s))_{s\in\{a,b,z\}})$. Then, we have $|y_n - \frac{1}{2}| < |x_n - \frac{1}{2}|$. We obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left[\frac{(x_n(1-x_n)}{y_n(1-y_n)}\right]^n \\ &= \lim_{n\to\infty} \frac{\Pr(\alpha_2|\text{piv}; (p(s))_{s\in\{a,b,z\}})}{\Pr(\beta_2|\text{piv}; (p(s))_{s\in\{a,b,z\}})} \\ &= \lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv}, z; (p(s))_{s\in\{a,b,z\}})}{\Pr(\beta|\text{piv}, z; (p(s))_{s\in\{a,b,z\}})}. \qquad (83) \end{aligned}$$

The equality on the first line follows from Lemma 1, (iii). Intuitively, conditional on being tied, the substate α_2 is infinitely more likely than than β_2 for $n \to \infty$. Hence, being pivotal contains the information that α_2 holds conditional on Ω_2. The equality on the third line follows, because z is received with probability converging to 1 in states Ω_2, and with probability 0 in states Ω_1; hence, for any $\omega \in \{\alpha, \beta\}$, the substate ω_2 holds almost surely if ω holds and z is received and vice versa. Equality (83) implies that the z-component $\lim_{n\to\infty} \Pr(\alpha|\text{piv}, z; (p(s))_{s\in\{a,b,z\}})$ of the best response converges to 1. In particular it is weakly larger than $\bar{r} + \epsilon$, for n sufficiently large. Hence the truncation of it, that is, the z-component of the fixed point must be at the upper bound $\bar{r} + \epsilon$ when n is sufficiently large. This contradicts the assumption that the z-component of the fixed point is at the lower bound $\bar{r} - \epsilon$.

Conversely, suppose that the z-component of the fixed point is at the upper bound $\bar{r} + \epsilon$. An analogous argument leads to the implication that the z-component of the fixed point must instead be at the lower bound $\bar{r} - \epsilon$, and hence to a contradiction. This finishes the proof that any fixed point of the ϵ-truncated best reponse is interior, hence, the proof of Lemma 5. ■

8.5 Appendix E: Other Equilibria

Note that we mostly suppress the dependence of the information structure in the notation in Appendix C-Appendix E.

Lemma 8 *For any preference distribution G that satisfies the conditions in 5: If $\sigma \neq \hat{\sigma}_{\Omega_2}(p_A, \bar{r}, p_A)$ is the limit of an equilibrium sequence σ_n in the games $\Gamma_n(p_A, \bar{r}, p_A)$,* *it satisfies*

1. *the limit of the minimum of the margins of victory in the states* Ω_1 *equals the limit of the minimum of the margin of victory in the states* Ω_2*, namely*
$$\begin{aligned}&\lim_{n\to\infty}\min_{\omega_1\in\{\alpha_1,\beta_1\}}|\Pr(\sigma(s,t)=1|\omega_1;p_A,\bar r,p_A)-\frac{1}{2}|\\ =\ &\lim_{n\to\infty}\min_{\omega_2\in\{\alpha_2,\beta_2\}}|\Pr(\sigma(s,t)=1|\omega_2;p_A,\bar r,p_A)-\frac{1}{2}|.\end{aligned} \tag{29}$$

2. σ *is a cutoff strategy with cutoffs* $(p_s)_{s\in S}$ *that satisfy one of the following conditions: Either* $p_s=\bar r$ *for all* $s\in S$*, or* $p_z=\bar r$ *and* $0<p_b<\bar r<p_a<1$*.*

Proof. Consider an equilibrium sequence $(\sigma_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\sigma_n=\sigma\neq\hat\sigma_{\Omega_2}(p_A,\bar r,p_A)$. Recall that in any equilibrium sequence, the margin of victory in α_2 and β_2 converges to zero by Lemma 2. As a consequence, the negation of condition (1.) is equivalent to
$$\begin{aligned}&\lim_{n\to\infty}\min_{\omega_1\in\{\alpha_1,\beta_1\}}|\Pr(\sigma(s,t)=1|\omega_1)-\frac{1}{2}|\\ >\ &\lim_{n\to\infty}\max_{\omega_2\in\{\alpha_2,\beta_2\}}|\Pr(\sigma(s,t)=1|\omega_2)-\frac{1}{2}|,\end{aligned}$$

that is, the margin of victory condition (26) of Lemma 3. Hence, if condition (1.) does not hold, it follows from Lemma 3 that the best response to σ_n converges to conditional sincere voting $\hat\sigma_{\Omega_2}(p_A,\bar r,p_A)$ and not to σ. This yields a contradiction to the assumption that $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of equilibria.[37] This shows that condition (1.) must hold for any equilibrium sequence with $\lim_{n\to\infty}\sigma_n=\sigma\neq\hat\sigma_{\Omega_2}(p_A,\bar r,p_A)$.
Condition (2.) is an implication of (1.): Suppose that condition (1.) holds. First, σ is a cutoff strategy with cutoffs $p_s=\lim_{n\to\infty}\Pr(\alpha|\text{piv},s;\sigma_n)$ for $s\in\{a,b,z\}$ since σ is the limit of the equilibrium sequence σ_n and since any equilibrium is a cut-off strategy as a consequence of formula (1). It follows from the definition of $\pi_n(p_A,\bar r,p_A)$ that $\frac{\Pr(a|\alpha)}{\Pr(a|\beta)}>\frac{\Pr(b|\alpha)}{\Pr(b|\beta)}$. Hence,
$$\begin{aligned}&\frac{\Pr(\alpha|\text{piv},a;\sigma_n)}{1-\Pr(\alpha|\text{piv},a;\sigma_n)}\\ =\ &\frac{p_0}{1-p_0}\cdot\frac{\Pr(a|\alpha)}{\Pr(a|\beta)}\cdot\frac{\Pr(\text{piv}|\alpha;\sigma_n)}{\Pr(\text{piv}|\beta;\sigma_n)}\\ >\ &\frac{p_0}{1-p_0}\cdot\frac{\Pr(b|\alpha)}{\Pr(b|\beta)}\cdot\frac{\Pr(\text{piv}|\alpha;\sigma_n)}{\Pr(\text{piv}|\beta;\sigma_n)}\\ =\ &\frac{\Pr(\alpha|\text{piv},b;\sigma_n)}{1-\Pr(\alpha|\text{piv},b;\sigma_n)}.\end{aligned} \tag{84}$$

[37] We omit an alternative proof that uses Proposition 1.

So, we must have $p(a) \geq p(b)$. Recall that we assumed that ϕ is strictly increasing (see condition (5)), hence

$$\phi(p(a)) \geq \phi(p(b)). \tag{85}$$

For at least some $\omega_1 \in \{\alpha_1, \beta_1\}$, under σ_n, the limit of the expected vote share of A in ω_1 must satisfy

$$\frac{1}{2} = \lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1|\omega_1) \tag{86}$$

This follows from the assumption that condition (1.) holds and the observation that the limit of the margin of victory in α_2 and β_2 is zero in equilibrium. We rewrite

$$\begin{aligned} \lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1|\omega_1) &= \lim_{n\to\infty} \sum_{s\in\{a,b\}} \Pr(s|\omega_1) \cdot \phi(\Pr(\alpha|s, \text{piv}; \sigma_n)) \\ &= \sum_{s\in\{a,b\}} \Pr(s|\omega_1) \cdot \phi(p_s), \end{aligned} \tag{87}$$

where the equality on the first line follows from the characterisation of the best reponse in formula (1) and the definition of the aggregate preference function ϕ in (2). The equality on the third line follows from the definition of the cutoffs p_s. Now, it follows from equation (87) that if $\phi(p_s) = 1/2$ for all $s \in \{a,b\}$, then clearly $\lim_{n\to\infty} |\Pr(\sigma(s,t) = 1|\omega_1) - 1/2| = 0$ for all $\omega_1 \in \{\alpha_1, \beta_1\}$, so in fact (1.) holds. On the other hand, if $\phi(p_s) \neq 1/2$ for some $s \in \{a,b\}$, it follows from the formulas (85) - (87) that condition (1.) can only hold if $\phi(p_a) > 1/2 > \phi(p_b)$. Since we assumed that ϕ is strictly increasing (see condition (5)), this is equivalent to $p_a > \bar{r} > p_b$.

The formula (84) shows that $p_a = 0 \Leftrightarrow p_b = 0$, and $p_a = 1 \Leftrightarrow p_b = 1$. We have already shown that $p_a \geq \bar{r}_z \in (0,1)$, and that $p_b \leq \bar{r}_z \in (0,1)$. Therefore it must be that $p_s \notin \{0,1\}$ for $s \in \{a,b\}$. ■

The following Lemma is needed in the proof of Proposition 2.

LEMMA 9 *For any sequence of strategies σ_n and any $\omega \in \{\alpha, \beta\}$ denote $c := \lim_{n\to\infty}(\Pr(\sigma_n(s,t) = 1|\omega) - \frac{1}{2}) \cdot n^{\frac{1}{2}}$ (we allow for $c = \pm\infty$). Then*

$$\lim_{n\to\infty} \Pr(A \textit{ gets elected}|\omega; \sigma_n) = \Phi(2c),$$

where $\Phi(\cdot)$ denotes the cumulative distribution of the standard normal distribution.

Proof. Denote $x_n := \Pr(\sigma_n(s,t) = 1|\omega)$. By using the normal approximation[38]

$$\mathcal{B}(2n+1, x_n) \simeq \mathcal{N}((2n+1)x_n, (2n+1)x_n(1-x_n)),$$

[38] For this normal approximation we cannot rely on the standard central limit theorem, because x_n varies with n. However, the central limit theorem for triangular sequences holds for triangular

we see that the probability that A wins the election in ω converges to

$$\Phi(\frac{\frac{1}{2}(2n+1)-(2n+1)\cdot x_n}{((2n+1)x_n(1-x_n))^{\frac{1}{2}}}).$$

Taking limits $n\to\infty$, gives us

$$\begin{aligned}
\lim_{n\to\infty}\Phi(\frac{\frac{1}{2}(2n+1)-(2n+1)\cdot x_n}{(2n+1)x_n(1-x_n))^{\frac{1}{2}}}) &= \lim_{n\to\infty}\Phi(\frac{(2n+1)\frac{1}{2}-(2n+1)(\frac{1}{2}+(x_n-\frac{1}{2}))}{(2n+1)^{\frac{1}{2}}(x_n(1-x_n))^{\frac{1}{2}}}) \\
&= \lim_{n\to\infty}\Phi((2n+1)^{\frac{1}{2}}(x_n-\frac{1}{2})(x_n(1-x_n))^{-\frac{1}{2}}) \\
&= \lim_{n\to\infty}\Phi((x_n(1-x_n))^{-\frac{1}{2}}2^{\frac{1}{2}}c)=\Phi(4^{\frac{1}{2}}c),
\end{aligned}$$

where the last equality holds, because either $c\in\{\infty,-\infty\}$, or $\lim_{n\to\infty}x_n=\frac{1}{2}$. ■

Proposition 2 *Let $\bar{r}$ and p_A satisfy (16) and (13), respectively: When the preferences are monotone, there exists an equilibrium sequence σ_n in the games $\Gamma_n(p_A,\bar{r},p_A)$ which, for $n\to\infty$ implies the full information outcome,*

$$\lim_{n\to\infty}\Pr(A \textit{ is elected}|\alpha;\sigma_n,p_A,\bar{r},p_A)=1,$$
$$\lim_{n\to\infty}\Pr(B \textit{ is elected}|\beta;\sigma_n,p_A,\bar{r},p_A)=1.$$

Proof. Step 1 (Equilibrium Construction): Any strategy σ entails probabilities by which a random citizen votes A in ω_j, for $\omega\in\Omega$ and $j\in\{1,2\}$, denoted by $\Pr(\sigma(s,t)=1|\omega_j)$. These probabilities are a sufficient statistic for the posteriors conditional on being pivotal and having received s, for any $s\in S$. Then, it follows from the characterisation of the best response in (1) that these probabilities are a sufficient statistic for the unique best response. Hence, we can write the best response as a function in these probabilities $\Pr(\sigma(s,t)=1|\omega_j)$. Consider the modified best response function that sets the probability to vote for A in β_1, $\Pr(\sigma(s,t)=1|\beta_1)$, to $\frac{1}{2}$ whenever this probability is weakly larger than $\frac{1}{2}$ under the actual best response. The modified best reponse is continuous, and an self-map on the closed and convex set of strategies that imply that B receives in expectation $\frac{1}{2}$ or more of the votes in β_1. It follows from the Brouwer Fixed Point Theorem that the modified best response function has a fixed point and we claim that any fixed point is interior for any n sufficiently large. It follows from the construction that this is sufficient to show that any fixed point corresponds to an equilibrium.

Suppose otherwise. Then, there exists a (sub)sequence of fixed points for which the

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sequences of Bernoulli distributions $\mathcal{B}(y,x)$ with x bounded away from 0 and 1, by an application of the Berry-Esseen-Theorem. For this, recall formula (3), namely that there exists $\epsilon>0$ such that $x_n,y_n\in(\epsilon,1-\epsilon)$ for all $n\in\mathbb{N}$ and any sequence of strategies σ_n.

vote share of B in β_1 is exactly one half. We lead a slightly more general case to a contradiction. Consider any sequence of fixed points σ_n for which the vote share of B in β_1 converges to $\frac{1}{2}$ relatively fast: More precisely, assume that

$$\lim_{n\to\infty}(\Pr(\sigma_n(s,t)=1|\beta_1)-\frac{1}{2})\cdot n^{\frac{1}{2}}=c \tag{88}$$

for some $c \leq 0$. For any $\omega_2 \in \{\alpha_2, \beta_2\}$, the probability of being pivotal in ω_2 is maximal when the probability to vote A in ω_2 is exactly $\frac{1}{2}$. Even then, the ratio of the probability of being pivotal in β_1 and the probability of being pivotal in ω_2 does not converge to zero by application of Lemma 1, (ii) to $x_n = \Pr(\sigma(s,t)=1|\beta_1)$, and $y_n = \frac{1}{2}$. This implies that β_1 is infinitely more likely than α_2 and β_2 conditional on being pivotal and having received signal a or b,

$$\lim_{n\to\infty}\frac{\Pr(\omega_2|s,\mathrm{piv};\sigma_n)}{\Pr(\beta_1|s,\mathrm{piv};\sigma_n)}=0$$

for $s \in \{a,b\}$ (since the signals a and b have probability less than $\frac{1}{n}$ in ω_2 and probability $\frac{1}{2}$ or $\frac{2}{3}$ in β_1). So the posteriors conditional on being pivotal and conditional on a or b vanish on Ω_2. Then,

$$\begin{aligned} & \lim_{n\to\infty}\Pr(\alpha|\mathrm{piv},b;\sigma_n) \\ = & \lim_{n\to\infty}\Pr(\alpha|\Omega_1,b) \\ = & 0, \end{aligned} \tag{89}$$

where the equality on the last line follows, since we defined the information structures $\pi_n(q,l,r)$ such that $\Pr(a|\alpha_1)=\Pr(b|\beta_1)=1$. Hence,

$$\begin{aligned} & \lim_{n\to\infty}\Pr(\mathrm{BR}(\sigma_n)(s,t)=1|\beta_1) \\ = & \lim_{n\to\infty}\Pr(\mathrm{BR}(\sigma_n)(s,t)=1|b) \\ = & \lim_{n\to\infty}\phi(\Pr(\alpha|\mathrm{piv},b;\sigma_n) \\ = & \phi(0) \\ < & \frac{1}{2}, \end{aligned} \tag{90}$$

where the equality on the third line follows from the characterisation of the best response in (1) and the definition of the aggregate preference function ϕ. The equality on the fourth line follows from the equality (89). For the last step, recall that we assumed that $\phi(0) < \frac{1}{2}$ (compare to (5)). It follows from (90) that the fixed point σ_n is strictly interior for any n sufficiently large. This contradicts with the assumption that the equality (88) holds.

Step 2 (Information Aggregation): So far we showed that there exist equilibria that correspond to (interior) fixed points of a modified best reponse, for any n sufficiently large. We claim, that any sequence of such equilibria aggregates information, meaning that, , for $n \to \infty$, A gets elected with probability converging to 1 in α, and B in with probability converging to 1 in β (recall the assumptions made in (5)). Consider any sequence of interior fixed points σ_n.

Information aggregation in β: Suppose that the probability that B gets elected in β does not converge to 1. Hence it does not converge to 1 in β_1 either, since we defined the information structures such that $\lim_{n\to\infty} \Pr(\beta_1|\beta) = 1$). It follows from Lemma 9 that then necessarily the probability that a random citizen votes for B in β_1 must converge to $\frac{1}{2}$ sufficiently fast, namely that $\lim_{n\to\infty}(\Pr(\sigma_n(s,t) = 1|\beta_1) - \frac{1}{2}) \cdot n = c$ for some $c \in \mathbb{R}$. Since σ_n is a sequence of fixed points of the modified best response, it must hold that $c \leq 0$. However, in the preceding paragraph we lead the assumption that $\lim_{n\to\infty}(\Pr(\sigma_n(s,t) = 1|\beta_1) - \frac{1}{2}) \cdot n = c$ for some $c \leq 0$ to a contradiction.

Information aggregation in α: We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\Pr(\alpha|\text{piv}, a; \sigma_n)}{\Pr(\beta|\text{piv}, a; \sigma_n)} &= \lim_{n\to\infty} \frac{\sum_{i=1,2} \Pr(\alpha_i|\text{piv}; \sigma_n) \cdot \Pr(a|\alpha_i)}{\Pr(\beta_2|\text{piv}; \sigma_n) \cdot \Pr(a|\beta_2)} \\
&\geq \lim_{n\to\infty} \frac{\Pr(\alpha_2|\text{piv}; \sigma_n) \cdot \Pr(a|\alpha_2)}{\Pr(\beta_2|\text{piv}; \sigma_n) \cdot \Pr(a|\beta_2)} \\
&= \frac{p_A}{1 - p_A}. \qquad (91)
\end{aligned}$$

where the equalitiy on the first line follow from Bayes rule. The equality on the second line follows, since we defined the information structures such that $\Pr(a|\alpha_1) = \Pr(b|\beta_1) = 1$. The equality on the last line follows, since the limit of the posterior conditional on Ω_2 and a equals p_A as shown in equation (9). Now, we obtain that

$$\begin{aligned}
\lim_{n\to\infty} \Pr(\sigma_n(s,t) = 1|\alpha_1)) &= \phi(\Pr(\alpha|\text{piv}, a; \sigma_n)) \\
&\geq \phi(p_A) \\
&> \frac{1}{2}, \qquad (92)
\end{aligned}$$

where the equality on the first line follows from the characterisation of the best response in (1) and the definition of the aggregate preference function ϕ. The inequality on the second line follows from the inequality (91) and since we assumed that ϕ is strictly increasing; see the assumptions in (5). The inequality on the last line is simply the defining property of p_A, see (13). The inequality (92) means that the probability that a random citizen votes A in α_1 is strictly larger than $\frac{1}{2}$. It follows from the law of large numbers that the probability that A gets elected in α_1 converges to 1, for $n \to \infty$. Since we defined the information structures such that $\lim_{n\to\infty} \Pr(\alpha_1|\alpha) = 1$, it follows that information aggregates in α under any sequence of equilibria that correspond to fixed points of the modified best response. ■

9 Online Supplement

9.1 Persuasion With Public Signals

Alonso and Câmara [2015] study persuasion of voters through the release of public signals (and when voters do not receive signals from another exogeneous source). Relative to Alonso and Câmara [2015] we made two departures in the model in Section 3 without exogeneous private signals: We allowed for the release of private signals, and we allowed for private preferences. This section illustrates that, for large n, the departure to private preferences only becomes substantial when also allowing for private signals.

For public signals, the unique equilibrium in undominated strategies is sincere voting. In the limit, for $n \to \infty$, the sender is therefore perfectly informed about the aggregate voter behaviour as a function of induced beliefs, no matter if we assume private preferences or not. The posteriors conditional on the public signal have the martingale property: The expected value of the posteriors is equal to the prior. More formally, it follows from Kamenica and Gentzkow [2011] (Proposition 1) that the set of feasible posteriors for an information structure with m public signals $(s_1, \dots, s_m)$ is given by

$$\{(p(s_1), \dots, p(s_m)) : \exists(x_1, \dots x_m) : \sum_{i=1}^{m} x_i = 1 \text{ and } \sum_{i=1}^{m} x_i \cdot p(s_i) = p_0\}$$

For any public information structure, the (not necessarily convex) set W of posteriors $p \in [0, 1]$ that lead to election of A with certainty in the limit $n \to \infty$ is given by all p for which $\phi(p) > \frac{1}{2}$[39]. The set of posteriors p that lead to election of B with certainty in the limit $n \to \infty$ is given by all p for which $\phi(p) < \frac{1}{2}$.

For the ease of comparison with section 3 and the possibility result of Theorem 1, we establish a possibility result for public signals. We restrict w.l.o.g. to the case in

[39] Recall the definition of ϕ from Section 1.

which $x_\omega = A$ for all $\omega \in \Omega$.[40]

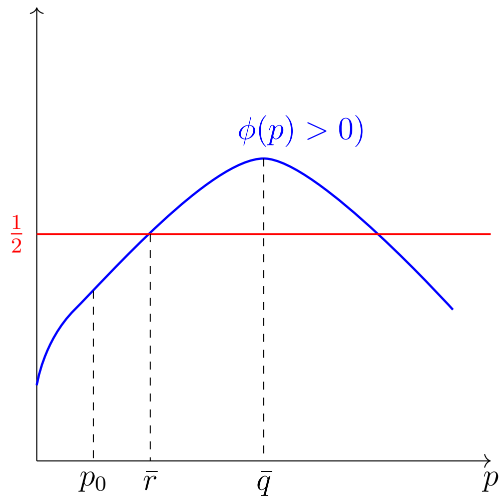
Proposition 3 *If there exist* $0 \leq p_1 \leq p_0 \leq p_2 \leq 1$ *such that for* $i = 1, 2$, *we have* $\phi(p_i) > \frac{1}{2}$, *then there exists a public information structure* π *such that, for* $n \to \infty$, *the probability that* A *gets elected converges to* 1 *in the sequence of sincere voting equilibria. The converse also holds.*

Proof. If there exists a belief $p_1 \in W$ weakly lower than the prior belief, $p_1 \leq p_0$, and a belief $p_2 \in W$ weakly larger than the prior belief, $p_2 \geq p_0$, then there exists a binary, public information structure with signals a and b which realises these beliefs as posteriors, $p(a) = p_1$, and $p(b) = p_2$.[41] In the induced sincere voting equilibrium sequence, A is elected with certainty in the limit $n \to \infty$.
If there does not exist $p_1 \in W$ with $p_1 \leq p_0$ or if there does not exist $p_2 \in W$ with $p_2 \geq p_0$, then for every public information structure there exists at least one signal s such that after s alternative A does not get elected with certainty for $n \to \infty$. This is because by the martingale property there exists at least one signal s which induces a posterior larger or equal to the prior, $p(s) \geq p_0$, and at least one signal s which induces a posterior smaller or equal to the prior, $p(s) \leq p_0$. ■

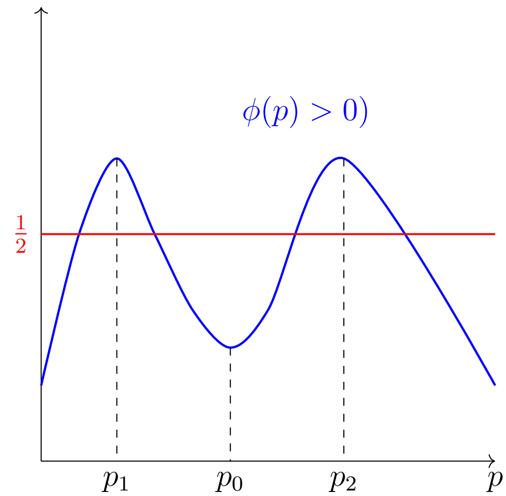
The following picture illustrates the difference between persuasion with private signals and persuasion with public signals when voters do not have exogeneous private signals:

[40] Generally, for all state-contingent outcomes $x(\alpha) \in \{A, B\}$ and $x(\beta) \in \{A, B\}$, we can use concavification techniques as in Kamenica and Gentzkow [2011] to describe sequences of public signals for which the limit of the probability that $x(\omega)$ is elected is maximal.

[41] This is because we can write $p_0 = p_1 + (p_2 - p_1)x = (1 - x)p_1 + xp_2$ for some $x \in [0, 1]$.



Persuasion with private signals.



Persuasion with public signals.

Different Winning Coalitions: Note that an optimal public information structure induces different winning coalitions, as in Alonso and Câmara [2015]. On the one hand, the winning coalition is inherently random, because the preference types of voters are random. On the other hand, when voters hold a common posterior $p_2 \geq p_0$, very different preference types t elect A than when voters hold a common posterior $p_1 \leq p_0$.

Impossibility of Public Persuasion with Exogeneous Private Signals: When voters receive exogeneous private signals and the preferences of voters are monotone, as in Section 4, it follows from the Condorcet Jury Theorem 0 that persuasion of large electorates with public signals is not possible; compare to the discussion at the end of Section 2.

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