Stable Allocations with Network-Based Comparisons*

Chen Cheng Yiqing Xing†

December 27, 2018

Abstract

We consider a model in which each agent cares about her network-based (local) ranking, i.e. the ranking of her allocation among her neighbors' in a network. An allocation is stable if it is not revoked under α-majority voting; that is, there exists no alternative allocation, such that a fraction of at least α of the population have their rankings strictly improved under the alternative. We find a sufficient and necessary condition for a network to permit any α-stable allocation: the network has an independent set of size at least $(1-\alpha)$ of the population. We then characterize the size of the largest independent set for Erdős–Rényi random networks, which reflects how permissive a network is. For large enough population, the level of permissibility solely depends on the expected degree. We provide several comparative statics results: more connected networks, more populated networks (with a fixed link probability), or more homophilous networks are less permissive. We generalize our model to arbitrary blocking coalitions and provide a sufficient and necessary condition for this case. Other extensions of the model include: (1) directed networks and (2) comparisons made only to non-neighbors.

Keywords: Network, social ranking, relative comparison, voting, independent set, stability, group stability, random networks, homophily

*For helpful comments, we thank Francis Bloch, Gabrielle Demange, Ben Golub, Sanjeev Goyal, Matt Jackson, Nicola Persico, Yizhou Xiao, Ning Neil Yu, and participants at Cambridge, Stanford, HUST, Wuhan, 1st NASN Congress, 4th NSF Conference on Network Science in Economics, and 2018 International Conference on Economic Theory.

†Both authors are at Carey Business School, Johns Hopkins University, Baltimore, Maryland 21202 USA. Websites: http://www.chenchengjhu.com/ and http://www.yiqingxing.com/. Emails: chen_cheng@jhu.edu and xingyq@jhu.edu.

1 Introduction

Social comparisons define our subjective well-being. Compelling evidences show that our perceptions of happiness depend on our income in relative rather than absolute terms (Easterlin (1995); Tideman, Frijters and Shields (2008)). Recent empirical findings further show that our comparisons are network-based: we rank ourselves among our neighbours in certain network (Luttmer (2005)). That is, our social comparisons are local.

Our local comparisons in turn, determine what allocations are satisfying. Ancient philosophers have long-recognized the link between social comparisons and societal instability. The conditions under which an allocation is stable when we rank ourselves in a social network, however, is under explored in the literature.

This paper studies the following question: when we care about our local ranking in a social network, which network structures permit stable allocations? Specifically, we consider a model in which agents' payoffs depend solely on their wealth rankings among neighbors in a given social network. A revolution brings about a reshuffling of social rankings by changing allocations, and needs the support of at least α fraction of the population. An agent underwrites a revolution if it strictly betters his or her ranking. We say an allocation is stable if no revolution is possible.

We find the only network feature that matters is the size of independent set[1]. Specifically, a network permits stable allocations *if and only if* the network has an independent set of size at least $1-\alpha$ fraction of the population. For example, when we use simple majority voting, i.e., $\alpha = 1/2$, bipartite networks, star networks, rings with even nodes, core-periphery networks etc., all permit stable allocations. Whereas complete networks, rings with odd nodes, etc., don't permit stable allocations.

Independent set arises in the picture exactly because of network-based comparisons. Since agents care about their ranking among their neighbors, stable allocation requires that there are enough people that have high ranks, and it's only feasible when these people are not connected. The above characterises why the existence of a large enough independent set is necessary. To see why the condition is sufficient, suppose there are n agents in total. When a network has an independent set of size $(1-\alpha)n$, an allocation such that everyone in this independent set ranks the highest among his neighbours is stable. Such an allocation is possible, for example, when we equally divide all the resources in the society among the agents in the independent set, and leave nothing to the rest. In this case, agents in the independent set have no incentive to support any revolution since they have already achieved their best ranking, and the rest do not have enough votes to overthrow the current

[1] An independent set in a network is a set of vertices such that no pair of vertices in the set are connected to each other.

allocation.

This simple sufficient and necessary condition allows us to analyze comparative statics that examine which classes of networks are more *permissive*, in the sense of permitting stable allocations for a larger set of α's. To answer this question, we work with Erdős–Rényi (*E-R*) random networks.[2] For although finding the size of the largest independent set – the independence number – is a classical N-P hard problem for a given network, from the mathematical literature (e.g., Frieze (1990)), we know that one can pin down the independence number almost surely when the network is large enough. The independence number normalized by the population n captures the level of permissibility of a class of random networks. We say the higher the level of permissibility, the more permissive a network is.

We have the following comparative statics results: First, given the linking probability p, larger random networks are less permissive. This is because the size of an *E-R* random network's largest independent set increases at a much slower rate than the expected degree of the network, which is proportional to size of the network n. Second, more connected networks, i.e., networks with a higher p, are less permissive, given the population size n. Intuitively, having more links reduces the size of independent sets by definition. And lastly, the level of permissibility does not vary once the expected degree is fixed.

Above results have interesting implications. For example, the second comparative static result sheds light on how the development of social media, the increase in the transparency of information, etc., would affect the stability of allocations. Specifically, we can interpret links in our model to be informational. That is, an agent can only observe the allocations of her neighbors', and then compare with them. In this case, the development of internet or social media essentially makes the network more connected (i.e., a larger p), and hence less likely to permit any stable allocations.

And the last comparative result mentioned above implies that whether a network is segregated or integrated, won't affect the level of permissibility, if the average degree is the same for the two types of networks. Specifically, we compare an integrated type of Erdős–Rényi random networks $G(n, p)$, to a segregated type networks that are constituted by k isolated subgroups $G(n/k, kp)$, i.e., each of which is densely linked within group but not across. In the above comparison, both networks have the same level of permissibility, for their expected degree is the same $d = np$.

Another ubiquitous pattern in social networks is homophily – the phenomena that people link with similar others. It's then natural to ask how would homophily affect the level of permissibility of a social network. A natural way to model homophliy is using the regular ring lattice. We find that homophilous networks are less permissive than Erdős–Rényi random

[2] Erdős–Rényi random networks $G(n, p)$ are such networks that each has n agents, and any pair of agents is linked independently with probability p.

networks that are of the same expected degree.

We also generalize our models in the following directions. The first concerns the blocking coalitions. In our baseline model, they must be of size at least α fraction of the population. We extend our results to arbitrary blocking coalitions and provide a similar necessary and sufficient condition there: the existence of an independent set whose intersection with any blocking coalition must be non-empty. Secondly, we consider what happens if networks are directed, which cover important applications such as Twitter. We obtain a parallel necessary and sufficient condition in this case: a directed network permits stable allocation if and only if it has a subgroup of size at lease $(1-\alpha)n$ that is acyclic. The third extension concerns an alternative problem in which agents' preferences are only based on comparing to strangers instead of to neighbors. In that case, the size of largest clique,[3] rather than independent set, determines whether networks permit stable allocations. Our last extension studies cases in which agents are insensitive to small differences in allocations, such that an agent feels ranked lower than another only if the difference in their allocations exceeds certain threshold.

The rest of the paper is organized as follows. Section 2 discusses related literature. Section 3 presents the model. Section 4 presents the main results: 4.1 characterizes a necessary and sufficient condition for permissive networks, then 4.2 investigates several comparative statics based on random networks. Section 5 generalizes the framework to allow for arbitrary blocking coalitions (defined as "group stability" in Demange (2004)). Section 6 discusses two extensions. Section 7 concludes.

2 Related Literature

Economists have long-recognized the importance of social comparison. Classical discussion goes back to Veblen (1899)'s critique on conspicuous consumption. More recently, Tideman, Frijters and Shields (2008) use relative income concerns to explain the Easterlin puzzle: the observation that average happiness has remained constant over time despite sharp rises in GNP per head. Postlewaite (1998) discusses the socio-economico and evolutionary basis for why people care about relative ranking. Rayo and Becker (2007) show that utility function in relative terms is favored by evolution under certain physical constraints.

There are further evidences imply that people's comparisons are usually local. That is, we compare with a certain reference group: our friends, neighbors, colleagues, ect., rather than comparing with the whole population. See, for instance Festinger (1954); Frank (1985a,b); Van de Stadt, Kapteyn and Van de Geer (1985); Luttmer (2005); Kuhn et al. (2011). Motivated by such findings, there are a few theoretical papers that study the impact of local comparisons. Ghiglino and Goyal (2010), for example, examine what happens to equilibrium

[3] A subset of agents $M \subset N$ consists of a clique of network g, if $g_{ij} = 1$, $\forall i, j \in M$.

price, allocation and welfare when individuals' utility depends negatively on consumption of their neighbors. They find that centrality matters for equilibrium prices and consumption. Immorlica et al. (2017) also study local comparisons, with a focus on "upward looking" comparisons, i.e., individuals are only negatively affected by neighbors whose consumption is higher than theirs. They study how does "upward-looking" comparison affect agents' costly action decision. Bloch and Olckers (2018) studies mechanisms that rank agents based on peer rankings. Although above papers and ours all study ranking and (local) comparisons based on social network settings, our focus is completely different. They are interested in either market equilibrium or strategic decisions, whereas we focus on stable allocations.

Our paper is closely related to the literature that studies stable allocations in social networks. Demange (2004) and Kets et al. (2011) are most related among these papers. We provide a new perspective to this the literature in terms of the meaning of network: in those papers, network captures the structure of communication and/or coordination, whereas in ours the network shapes preferences via defining agents' reference groups.

The seminal papers Erdos and Rényi (1959, 1960) laid the foundation for random graphs. There is a mathematical literature exploring the clique number and chromatic number on random networks (see Grimmett and McDiarmid (1975); Bollobás and Erdös (1976); Bollobás (1988); and surveys by Bomze et al. (1999); Pardalos and Xue (1994)). This literature provides useful tools for our analysis on random networks.

Finally, our "α-stability" is different from the stability concept introduced in the network formation literature (see Jackson (2005); Demange and Wooders (2005); Jackson (2008); Goyal (2007); Mauleon and Vannetelbosch (2015) for useful surveys). Rather, it is related with the core concept in cooperative game theory (e.g., Aumann and Dreze (1974) and Myerson (1977)). The key differences of our paper and the previous are in how we define blocking coalitions and that we study comparison-based preferences.

3 Model

Agents $N = \{1, \ldots, n\}$ are located on a network $g \in G(N)$. The network is undirected and unweighted: $g_{ij} = g_{ji} = 1$ if agents i and j are neighbors, otherwise $g_{ij} = g_{ji} = 0$. We set $g_{ii} = 0$. We also use g to denote the collection of its links, i.e., $g = \{ij \mid i, j \in N, g_{ij} = 1\}$. Let $N_i(g) = \{j \mid g_{ij} = 1\}$ be the set of i's neighbors.

An allocation $w \in \mathbb{R}^n$ assigns each agent i a wealth w_i. Each agent i only cares about his or her local ranking, i.e., how w_i is ranked among the wealth of i's neighbors w_{N_i}. Formally:

Definition 1 (Local ranking) *Given a network g and an allocation w, i's local ranking is*

$$r_i(w) \equiv \#\{j \in N_i(g) | w_j \geq w_i\} + 1$$

Definition 2 (Preference) *Given a network g, a preference vector $\succ \equiv (\succ_1, \ldots, \succ_n)$ is defined as:*

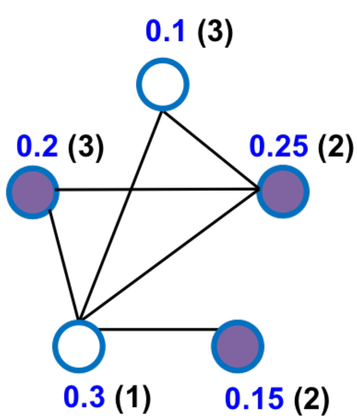
$$w' \succ_i w \quad \textit{iff} \quad r_i(w') < r_i(w)$$

Therefore, an agent i strictly prefers an allocation w' to w if i has a strictly better local ranking under w' than under w.[4]

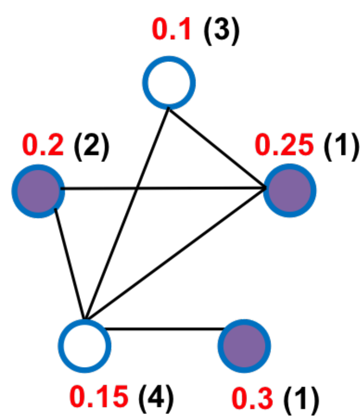
Definition 3 (α-Stable allocations) *An allocation w is stable on network g w.r.t. α-majority voting, or α-stable for short, if there exists no alternative w', such that $w' \succ_i w$ for strictly more than $\alpha \cdot n$ agents.*

We note several things here. First, one can easily imagine there are certain costs involved in voting or rising up for a revolution. This provides justifications for why we say an agent is for a revolution only if his social ranking is *strictly* improved under alternative allocation. Secondly, we don't have to interpret our setting literally as voting on wealth redistribution. In reality, we vote on intermediate policies such as provision of public goods, location of a school, etc., which in the end have payoff consequences.

Example **1** *Consider $\alpha = \frac{1}{2}$, i.e., simple majority voting is employed to revoke a default allocation. Figure 1 presents an allocation w that is not stable on the given network. The three agents colored purple strictly improves their social rankings under the alternative, and hence strictly prefer w' to w. As a result, w is not stable on g.*



(a) Default allocation w



(b) Alternative allocation w'

Figure 1: An example of a network and a default allocation that are not $\frac{1}{2}$-stable. The two panels present default (left) and alternative (right) allocations respectively, with local social rankings, $r_i' s$, in brackets.

[4]A tie with others in the x-th position is viewed strictly worse than being in the x-th position by oneself.

We note that an alternative need not affect everyone's allocation. Instead, it may change only a few agents' allocations, while still have a large impact on relative rankings. In Figure 1, compared to the default allocation w, the alternative w' only change the allocations for the bottom two agents (by swapping their allocations); however, it changes the relative rankings for almost everyone (except the top agent).

Definition 4 (α-Permissive networks) *A network g permits α-stable allocation, or is α-permissive for short, if there exists any α-stable allocation on g.*

We can interpret the networks in our model in various ways. For example:

- Acquaintance networks: in such networks agents compare only with their acquaintances, such as friends, colleagues, neighbors, etc. It could be driven by our preferences, like we only care about those people.
- Cohort networks: agents in these networks compare with people in the same cohort, classified by income, age, education, etc. Sometimes we consider people from the same cohort comparable.
- Information networks: since our information regarding others' income is limited, it also makes sense to compare only with those whose income information is available. In such networks, each agent only observes the allocation(s) of a subset of others, and she only compares to those who she can observe.
- Tournament networks: in these networks, linked agents compete for resources.Allocations may be interpreted as current endowments that are essential for the competition, and the network captures the subset N_i that i competes against.

Some of the above situations can be better captured in the extensions of this paper. For instance, Section 6.1 analyzes directed networks, which may be a better fit to the information based interpretation. Section 6.2 explores the case in which agents do not care about the rankings among neighbors, and only compare to non-friends.

4 Main Results

This section presents main results. We first provide a sufficient and necessary condition that characterizes permissive networks. Armed with such a clean condition, we will then turn to the questions like "how permissive a network is", and "which network structures tend to be more permissive".

4.1 A Sufficient and Necessary Condition for Permissive Networks

Now we characterize permissive networks. We first present the following key definition:

Definition 5 (Independent Set) *A subset of agents $M \subset N$ consists of an independent set of network g, if $g_{ij} = 0$, $\forall i, j \in M$.*

An equivalent definition is that $g|_M$ is an empty network, where $g|_M \equiv \{ij \in g \mid i, j \in M\}$ is the network induced by M.

We observe that having a large enough independent set suffies to deliver a permissive network. This is persented in the following lemma.

Lemma **1 (α-Permissive Networks)** *A network g is α-permissive if g has an independent set of size at least $(1-\alpha)n$.*

Proof of Lemma 1: Suppose M is an independent set of g and $|M| \geq (1-\alpha)n$. The allocation w^* s.t. $w_i^* = 1\{i \in M\}$ is stable on g w.r.t. α-majority voting. This is because any agent $i \in M$ is ranked the 1-st among his/her neighbors, hence none of the agents in M strictly prefer any alternative w' to w^*. As a result, the number of agents that may vote against w^* is at most αn. By Definition 8, w^* is α-stable on g and hence g is α-permissive. ∎

The lemma follows a simple intuition: a large enough independent set implies that there are a large group of people who do not compare with each other. As a result, any allocation that concentrates resources to those people would be stable, as these people would all be happy enough such that they would never vote for any alternative.

While the above condition may seem straightforward, we next show that it is not only sufficient, but necessary.

Theorem **1 (α-Permissive Networks)** *A network g is α-permissive if and only if g has an independent set of size at least $(1-\alpha)n$.*

Proof of Theorem 1:
"If" was presented as Lemma 1.
"Only if": Suppose g is α-permissive so that there exists some allocation w that is α-stable on g. Now we label the nodes as follows:

0. Let $l_i = null$ for all i.

1. Pick $i \in \arg\max_{j \text{ s.t. } l_j = null} w_j$, let $l_i = 1$.

2. Let $l_i = 0$ if $l_i = null$ and i is a neighbor of some j with $l_j = 1$.

3. Repeat steps 1 and 2 until all agents are labeled with 0 or 1.

Denote $M_k = \{j \in N \mid l_j = k\}, k = 0, 1$, we have $M_1 \bigcup M_0 = N$ and $M_1 \bigcap M_0 = \emptyset$. Consider an alternative allocation w' s.t. $w'_i = w_i$ if $i \in M_0$ and $w'_i = \underline{w}$ if $i \in M_1$, for some $\underline{w} < min_{i \in N} w_i$. By construction, all agents in M_0 strictly prefers w' to w: under w, every $i \in M_0$ has a neighbor $j \in M_1$ such that $w_i \leq w_j$; whereas under w', the relative ranking among agents in M_0 remain the same, and everyone in M_0 has a strictly higher allocation than everyone in M_1.

Therefore, the fact that w is α-stable on g implies $|M_0| \leq \alpha n$, and hence $|M_1| \geq (1-\alpha)n$. Finally notice that M_1 is an independent set of g: by construction, any $i \in M_1$, all i's neighbors are labeled with 0. Hence any pair of agents in M_1 are not neighbors to each other.

We have constructed M_1, an independent set of g and its size is at least $(1 - \alpha)n$. ∎

The above necessary and sufficient condition provides a clean characterization of permissive networks. Armed with this clean condition, we next explore "how permissive a network is", i.e., the level of permissibility, and compare across networks to see which network is more permissive than another.

Definition 6 (Level of Permissibility) *The level of permissibility $s(g)$ for a network $g \in G(N)$ is defined as following:*

$$s(g) \equiv \frac{S(g)}{n}$$

To see why the above is a proper definition for the level of permissibility, it follows from Theorem 1 that a network is α-permissive for any $\alpha \geq 1 - s(g)$, and therefore a larger $s(g)$ implies that the network is α-permissive for a larger set of α's.

The following corollary then follows immediately from Theorem 1:

Corollary **1 (Monotonicity)** *Consider $g, g' \in G(N)$ s.t. $g' \subset g$, then $s(g') \geq s(g)$. Moreover, $\forall \alpha \in (0, 1)$:*

- *If g is α-permissive, then g' is α-permissive;*
- *If g' is not α-permissive, then g is not α-permissive.*

Example **2** *Consider $\alpha = \frac{1}{2}$, i.e., simple majority voting is employed to revoke a default allocation. We present several stylized networks that are permissive, and that are not permissive, respectively.*

- *The following networks are $\frac{1}{2}$-permissive: bipartite networks, stars, rings with even nodes, networks that are included by any of the above.*

- *The following networks are not $\frac{1}{2}$-permissive: complete networks, rings with odd population n, and networks that includes any of the above.*

Up to now, we have shown that the only network feature that matters for stable allocations is the size of independent set. This implies that to explore the question of whether a network permits stable allocations, it suffices to study the size of its largest independent set(s), i.e., the independence number. However, one challenge we face is that for an arbitrary network, finding its independence number is a classical NP-hard problem.[5] Fortunately, for random networks, we can pin down their independence numbers almost surely.

4.2 Comparative statics: which networks are more permissive?

In this part, we are interested in the following questions: are larger networks more permissive than smaller networks? How about connected networks compared to sparse networks? How would segregated networks compare with integrated networks? And lastly, are more homophilous networks more permissive? We provide answers to these questions for a basic set of networks: Erdős–Rényi random networks.

Erdős–Rényi random networks.

Consider a class of random networks, $G(n, p)$, such that there are n agents and any link ij appears with a probability of p, *i.i.d.* across links. We aim to explore which classes of random networks are more permissive than another. We first provide an approximation for the size of the largest independent set of random graphs $G(n, p)$. This result is proved by Frieze (1990), and for convenience, we state the result below as Lemma 2.

Let $S(n, p)$ denote the independence number of a random graph $G(n, p)$, we have the following:

Lemma **2 (Frieze (1990))** *Let $d = np$ and $\varepsilon > 0$ be fixed. Suppose $d_\epsilon \leq d = o(n)$ for some sufficiently large d_ϵ. Then*

$$\left| S(n,p) - \frac{2n}{d}(\ln d - \ln\ln d - \ln 2 + 1) \right| \leq \frac{\varepsilon n}{d}$$

with probability going to 1 as $n \to \infty$.

Therefore, we can define the level of permissibility of *E-R* random graphs based on the above approximation and Definition 6. Specifically, it is the ratio between the independence number of $G(n, p)$ and the size of the graph.

[5] See, for instance, Bomze et al. (1999) and papers cited therein.

Definition 7 *The level of permissibility of E-R random graphs $G(n,p)$ is*

$$\tilde{s}(n,p) \equiv \frac{2}{d}(\ln d - \ln\ln d - \ln 2 + 1),$$

in which $d = np$.

We are now ready to answer the questions posted at the beginning of this part. In particular, we will provide a series of comparative statics based on the above approximation and definition.

Proposition **1 (Comparative Statics)** *For E-R random networks $G(n,p)$, the level of permissibility $\tilde{s}(n,p)$:*

1. *decreases in n, given any $p \in (0,1)$;*
2. *decreases in p, given any n;*
3. *is constant in n or p, given the expected degree $d = np$.*

The above results come directly from Definition 7 and we omit the proof here. The first point implies that for given p, the larger the network is, the less likely that it will permit stable allocations. This is because the size of an *E-R* random network's largest independent set increases at a much slower rate than the expected degree of the network, which is proportional to size of population n. Also, from the second point, the more connected a network is, the less permissive it becomes. Intuitively, as p increases, the network becomes more connected, and the size of the largest independent set would decrease. This can be viewed as a counterpart to Corollary 1 for random graphs. Lastly, when the expected degree d is fixed, the size or the connectedness of a network does not affect its level of permissibility $\tilde{s}(n,p)$, because in this case, $\tilde{s}(n,p)$ will only depend on the expected degree.

Segregation and permissibility We have seen how would the size and connectedness of a network affect its level of permissibility. The next natural question is whether more segregated networks are more permissive or not. To answer this question, consider the following two types of random networks, both of which have a total size n and expected degree $d = np$:

- G_1: *Integrated networks*, with one group of size n: $G_1 = G(n, d/n)$.
- G_2: *Segregated networks*, with K isolated groups, whose sizes are $n_1, \dots, n_K$ such that $n_1 + \ldots + n_K = n$. Only within group pairs will be linked, with i.i.d. probability $p_k = d/n_k$ among members in group k. That is, G_2 consists of the union of K random networks $\{G(n_k, d/n_k)\}_{k=1,\dots,K}$. Assume all n_k's are big enough.

In other words, we consider an extreme pattern of segregation such that a network consists of several isolated subgroups, each of which is more connected within. We always compare networks that have the same expected degree. However, once we have the expected degree fixed, we can see immediately from point 3 of Proposition 1 that the segregation pattern won't affect how permissive a network is.

PROPOSITION **2 (Irrelevance of Segregation)** *Let the expected degree d be fixed. For large enough $n = n_1 + \ldots + n_K$, the integrated networks and the segregated networks have the same level of permissibility.*

That is, if the integrated and segregated networks have the same expected degree, then the level of permissibility will not be affected by the number of segregated groups or the way those groups are divided, as long as each group has large enough population.

However, if one would think that the segregation patterns would also affect the expected degree, then how permissive a network is would not be immune to integration/segregation. For example, if segregated networks also have a higher expected degree, then we would have the result that the more segregated networks are less permissive than integrated networks are.

Homophily and permissibility Homophily is the pattern that individuals tend to interact more with others that are similar to themselves. It is an epidemic in social networks and is of great social and economic importance (see McPherson, Smith-Lovin and Cook (2001) for an overview). We explore the following question in this part: are homophilous networks more permissive than say random networks? In an attempt to answer this question, we consider the following example.

EXAMPLE **3 (Regular Ring Lattices (Watts and Strogatz (1998)))** *n individuals are placed on a ring, in the order of $1, 2, \ldots, n$. Suppose d is even and each individual is linked to the d closest others: $d/2$ on each side. The size of the largest independent set(s) is approximately $n/(0.5d + 1)$ and therefore:*

$$s^{regular\text{-}ring\text{-}lattice}(d) \approx 1/(0.5d + 1)$$

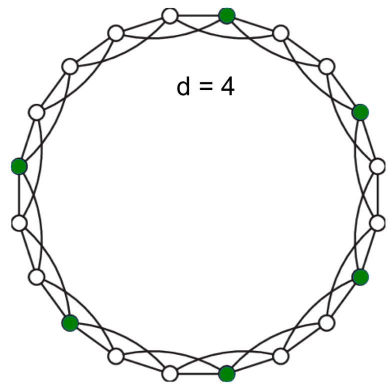


Figure 2: A regular ring lattice with $d = 4$. Solid green dots consist of one (largest) independent set.

PROPOSITION **3 (Homophilous Networks are Less Permissive)** *As n approaches infinity, for large enough even d, a regular ring lattice with degree d is less permissive than Erdős–Rényi random networks $G(n, p)$ with $d = np$.*

Proof of Proposition 3 We aim to show that $\frac{2}{d}(\ln d - \ln\ln d - \ln 2 + 1) > 1/(0.5d + 1)$. $LHS - RHS = \frac{2}{d}(f(d) + \frac{2}{d+2}) > \frac{2}{d}f(d)$, in which $f(d) \equiv \ln d - \ln\ln d - \ln 2$ is always positive for $d > 2$ because f increases in d and $f(2) = -\ln\ln 2 \approx 0.36 > 0$ ∎

5 Group Stability: General Sets of Blocking Coalitions

In our baseline model, each agent has one vote, and any coalition that has at least αn votes could overthrow current allocation. In reality, agents might have heterogenous (political) power. Moreover, the restriction on blocking coalition might be more general than its size. For example, suppose only connected agents are able to coordinate their votes/revolt. To capture above situations, we need a more general definition of blocking coalition.

Specifically, we call a subset of agents $C \subset N$ a *potential blocking coalition* if the agents are powerful enough to propose any alternatives to replace the default allocations. Let $\mathcal{C} \subset 2^N$ be the collection of all potential blocking coalitions. We define the stability notion accordingly:[6]

DEFINITION 8 ($\mathcal{C}$-STABLE ALLOCATIONS) *An allocation w is $\mathcal{C}$-stable on network g, if there exists no potential blocking coalition $C \in \mathcal{C}$ and alternative allocation w', such that*

$$w' \succ_i w, \quad \forall i \in C$$

[6]This concept was introduced as "group-stability" in Demange (2004), who focuses on super-additive games. Our setting can be viewed as non-transferable-utility (NTU) games without super-additivity. We thank Gabrielle Demange for pointing out this linkage.

Definition 9 ($\mathcal{C}$-Permissive networks) *A network g permits $\mathcal{C}$-stable allocation, or is $\mathcal{C}$-permissive for short, if there exists any $\mathcal{C}$-stable allocation on g.*

The definitions of α-stability/permissibility in previous sections are special cases of the above notions. To see this, let $\mathcal{C}_\alpha \equiv \{C \mid |C| \geq \alpha n\}$ be the collection of all coalitions that each includes at least αn agents, then $\mathcal{C}_\alpha$-stability is equivalent to α-stability.

Theorem **2 ($\mathcal{C}$-Permissive Networks)** *Given a collection of potential blocking coalitions $\mathcal{C} \subset 2^N$, a network g is $\mathcal{C}$-permissive if and only if there exists an independent set $M(g)$ on g, such that*

$$M(g) \bigcap C \neq \emptyset, \quad \forall C \in \mathcal{C}.$$

[See Appendix for proof.]

The above theorem shows that the key to $\mathcal{C}$-permissivity is the existence of an *expansive* enough independent set, so that it intersects every potential blocking coalition $C \in \mathcal{C}$. As a special case, when $\mathcal{C}_\alpha \equiv \{C \mid |C| \geq \alpha n\}$, the existence of a big enough independent set is sufficient to permit stable allocations, because the number of supporters is the only thing that matters in this case.[7]

6 Two Extensions

In this section, we consider two extensions of our previous model. The first extension generalizes our model to directed networks. The second extension concerns comparisons with strangers instead of friends. We derive similar necessary and sufficient conditions for permissive networks for each of the two extensions.

6.1 Directed Networks

Up till now, we have studied undirected networks only. However, there are situations that are better captured by directed networks. For instance, people need not follow their followers in Twitter. Also, an agents' wealth is not observable to all, and this relationship of observing each other's wealth needs not be reciprocal.

So in this part, we extend our model to directed network and characterize a similar necessary and sufficient condition. Specifically, let $g_{ij} = 1$ if i compares to j's wealth, and 0 otherwise. A network g is directed such that $g_{ij} = 1$ does not imply $g_{ji} = 1$. Each agent i ranks herself among her (outgoing) "neighbors" $N_i^{out}(g) \equiv \{j \mid g_{ij} = 1\}$, i.e., whose wealth i

[7] Notice that the independent set $I(g)$ mentioned above need not be the largest independent set. In addition, given the same network g, it could be different independent sets for different $\mathcal{C}$.

pays attention to. Whereas $N_i^{in}(g) \equiv \{j \mid g_{ji} = 1\}$ is the set of i's incoming neighbors, who compares to i. The stability and permissibility notions are defined similarly as before, with the modified concept of social ranking (based on outgoing neighbors).

Definition 10 (Path) *A (directed) path in a network g is a sequence of links $i_1i_2, ..., i_{K-1}i_K$ such that $i_ki_{k+1} \in g$ for each $k \in \{1, ..., K-1\}$.*

Definition 11 (Cycle) *A (directed) cycle in a network g is a path $i_1i_2, ..., i_{K-1}i_K$ such that $i_1 = i_K$. A network is acyclic if it contains no cycles.*

Theorem **3 (α-Permissivity for Directed Networks)** *A network g is α-permissive (or permissive w.r.t. α-majority voting) if and only if there exists M s.t. $|M| \geq (1-\alpha)n$ and $g|_M$ is acyclic.*

6.2 Comparing to Non-Neighbors

We have assumed that agents only compare with their neighbors on the network. This part explores the opposite case: what if agents only rank themselves against those to whom they are not linked, i.e., strangers? For example, this could happen when one doesn't care about their ranking against their friends, for they have internalized friends' payoff to some extent.

Our model can be readily applied to this setting. In particular, i's *local ranking* is redefined as following:

$$\widetilde{r}_i(w) \equiv \#\{j \notin N_i(g) | w_j \geq w_i\} + 1,$$

so that preference and α-permissibility are defined based on $\widetilde{r}$.

The following theorem characterizes the sufficient and necessary condition for α-permissive networks when agents compare with non-neighbors. Note that the only modification from previous condition is that it's the size of the largest clique[8] that matters, rather than the set of the largest independent set.

Theorem **4** *When agents compare to non-neighbors, i.e., ranking is defined as $\widetilde{r}$, network g is α-permissive if and only if g has a clique of size at least $(1-\alpha)n$.*

To see this, we first note that the network in which agents compare to strangers is exactly the *complement* (definition see below) of the network in which agents compare to friends. Then finding an independent set in a network g is equivalent to finding a clique in its complement network g^C. Formally, we define complement network below:

Definition 12 (Complement network) *A network g's complement network is $g^C \equiv \{ij \mid ij \notin g,\ i,j \in N\}$.*

[8] A subset of agents $M \subset N$ consists of a clique of network g, if $g_{ij} = 1$, $\forall i,j \in M$.

Lemma 3 **(Independent Set and Clique)** *M is an independent set of a network $g \in G(N)$, if and only if $N \setminus M$ is a clique of the complement network g^C.*

For E-R random networks, the size of the largest independent set in a network $g \in G(n,p)$ is equal to the size of the largest clique in its complement $g^C \in G(n, 1-p)$.

This Lemma make Theorem 4 an immediate corollary to Theorem 1.

7 Conclusion

In this paper we discuss the following question: when people care about their wealth ranking among neighbors, what networks allow for stable allocations? We say an allocation is α-stable if there are not enough agents to overthrow it, i.e., there exists an alternative which makes more than $1-\alpha$ fraction of the population to strictly improve their local rankings. We provide a sufficient and necessary condition for networks that permit any α-stable allocations: there exists an independent set that involve at least $1-\alpha$ fraction of the population.

Our comparative statics results may help us understand some reality and policy debate. For example, one result shows that denser networks are less likely to permit stable allocations. When we interpret the link in our model to be informational, i.e., we only compare with a group of people because our knowledge regarding others' wealth is limited. Then when information becomes more transparent, e.g., with the development of social media, our prediction is that our society may become less stable. In addition, we show that networks are less likely to permit any stable allocation when they become more homophilous. Our result also shows that once the expected degree of a random graph is given, then whether a network is segregated or integrated would not affect whether the network permits stable allocations.

There are many interesting and important questions that are left to future work. For example, one could use our framework to incorporate strategic decisions such as investment, effort or consumption decisions. Moreover, one can endogenize the network formation. In our current setting, one's reference group is given, an interesting question is when people care about local ranking, how would they choose their reference group.

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A Appendix: Omitted Proofs

A.1 Proofs of Theorem 2.

"If": Suppose M is an independent set of g and $M \bigcap C \neq \emptyset, \forall C \in \mathcal{C}$. Any allocation w^* s.t. $w_i^* = 1\{i \in M\}$ is $\mathcal{C}$-stable on g. This is because any agent $i \in M$ is ranked the 1-st among his/her neighbors, and therefore would not (strictly) prefer any alternative allocation. Formally, for any $C \in \mathcal{C}$, there is always some non-empty set, such that $w' \succ_i w$ does not hold for any $i \in M \bigcap C$ and any w'. By Definition 8, w^* is $\mathcal{C}$-stable on g and hence g is $\mathcal{C}$-permissive.

"Only if": Suppose g is $\mathcal{C}$-permissive so that there exists some allocation w that is $\mathcal{C}$-stable on g. Now we label the nodes as follows:

0. Let $l_i = null$ for all i.

1. Pick $i \in \arg\max_{j \text{ s.t. } l_j = null} w_j$, let $l_i = 1$.

2. Let $l_i = 0$ if $l_i = null$ and i is a neighbor of some j such that $l_j = 1$.

3. Repeat steps 1 and 2 until all agents are labeled with 0 or 1.

Denote $M_k = \{j \in N \mid l_j = k\}, k = 0, 1$, we get a partition the population: $M_1 \bigcup M_0 = N$ and $M_1 \bigcap M_0 = \emptyset$. Consider an alternative allocation w' s.t. $w'_i = w_i$ if $i \in M_0$ and $w'_i = \underline{w}$ if $i \in M_1$, for some $\underline{w} < min_{i \in N} w_i$. By construction, all agents in M_0 strictly prefers w' to w: under w, every $i \in M_0$ has a neighbor $j \in M_1$ such that $w_i \leq w_j$; whereas under w', the relative ranking among agents in M_0 remain the same, and everyone in M_0 has a strictly higher allocation than everyone in M_1.

Therefore, the fact that w is $\mathcal{C}$-stable on g implies that for every $C \in \mathcal{C}$, $C \nsubseteq M_0$, i.e. $C \bigcap M_1 \neq \emptyset$ as $M_1 = N \setminus M_0$. Finally notice that M_1 is an independent set of g: by construction, any $i \in M_1$, all i's neighbors are labeled with 0; hence any pair of agents in M_1 are not neighbors to each other.

We have constructed M_1, an independent set of g and it intersects every $C \in \mathcal{C}$. ∎

A.2 Proofs of Theorem 3.

"If": Suppose there exists $M \subset N$ such that $g|_M$ is acyclic and $|M| \geq (1 - \alpha)n$. We aim to find an allocation w^* such that $r_i(w^*) = 1, \forall i \in M$.

Recall $N_i^{out}(g)$ $[N_i^{in}(g)]$ consists of i's direct outgoing [incoming] neighbors. Let $\widetilde{N}_i^{out}(g)$ $[\widetilde{N}_i^{in}(g)]$ be the sets of i's indirect outgoing [incoming] neighbors. Formally,

$$\widetilde{N}_i^{out}(g) \equiv \{j \mid \text{there is a path from } i \text{ to } j \text{ on } g\}; \text{ and } \widetilde{N}_i^{in}(g) \equiv \{j \mid \text{there is a path from } j \text{ to } i \text{ on } g\}.$$

The following observation is a key to the proof:

$$g \text{ is acyclic } \implies \widetilde{N}_i^{out}(g) \bigcap \widetilde{N}_i^{in}(g) = \emptyset, \forall i.$$

Otherwise, if $j \in \widetilde{N}_i^{out}(g) \bigcap \widetilde{N}_i^{in}(g)$, there exists a path from i to j and a path from j to i, and they combined together is a path from i to j then back to i, i.e., a cycle.

Now we are ready to assign values (allocations) to agents, first to agents in set M, then for those outside.

1. For every $i \in M$ that is isolated on $g|_M$, let $w^*(i) = 0.5$.

2. For every $i \in M$ s.t. $\widetilde{N}_i^{out}(g|_M) = \emptyset$ and $w^*(i) \neq 0$, let $w^*(i) = 1$.

3. For every $i \in M$ s.t. $\widetilde{N}_i^{in}(g|_M) = \emptyset$ and $w^*(i) \neq 0$, let $w^*(i) = 2$.

4. Pick any agent $i \in M$ that is not assigned a value yet. Let

$$B(i) \equiv \{j \in M \mid w^*(j) \text{ is defined before } w^*(i)\},$$

$$w^*(i) \equiv 0.5 \left(\max_{j \in \widetilde{N}_i^{out}(g|_M) \bigcap B(i)} w^*(j) + \min_{j \in \widetilde{N}_i^{in}(g|_M) \bigcap B(i)} w^*(j) \right),$$

in which the max and min are calculated based on the j's whose values are already assigned.[9]

5. Repeat Step # 4 until all agents in M are assigned a value under w^*.

6. Finally, for every $i \notin M$, let $w^*(i) = 0$.

Now we aim to show a key property: for every $i \in M$ whose value is defined in above step # 4 (or 5)

$$\max_{j \in \widetilde{N}_i^{out}(g|_M) \bigcap B(i)} w^*(j) < \min_{j \in \widetilde{N}_i^{in}(g|_M) \bigcap B(i)} w^*(j). \quad (1)$$

[9] Notice that the max and min are well defined given $g|_M$ is acyclic: there exist $j_1 \in \widetilde{N}_i^{out}(g|_M)$ and $j_2 \in \widetilde{N}_i^{in}(g|_M)$, s.t. $w^*(j_1) = 1$ and $w^*(j_2) = 2$.

To see this: i is not assigned a value yet implies that $\widetilde{N}_i^{out}(g|_M) \neq \emptyset$ and $\widetilde{N}_i^{in}(g|_M) \neq \emptyset$. The non-emptyness for indirect neighbors implies the same for direct neighbors, and therefore there exist i' and i'' s.t. $i'i, ii''$ is a path on $g|_M$. It follows from the acyclicity of $g|_M$ that extending the path $i'i, ii''$ forward [backward] will eventually reach some agent j_1 [j_2] with $\widetilde{N}_{j_1}^{out}(g|_M) = \emptyset$ [$\widetilde{N}_{j_1}^{in}(g|_M) = \emptyset$]. By construction j_1 and j_2 are not isolated on $g|_M$, so their values are 1 and 2 respectively.

And as a result,

$$w^*(i) \in \left(\max_{j \in \widetilde{N}_i^{out}(g|_M) \bigcap B(i)} w^*(j), \min_{j \in \widetilde{N}_i^{in}(g|_M) \bigcap B(i)} w^*(j) \right) \subset (1, 2) \tag{1a}$$

We prove (1) using the mathematical induction, w.r.t. the order in which i is defined.

For the first i defined in step # 4, $\max_{j \in \widetilde{N}_i^{out}(g|_M) \bigcap B(i)} w^*(j) = 1$ because all those $w^*(j)$'s are defined in step #2, and $\max_{j \in \widetilde{N}_i^{out}(g|_M) \bigcap B(i)} w^*(j) = 2$ because all those $w^*(j)$'s are defined in step #3. Therefore, eq. (1) holds.

In addition, pick any i, suppose eq. (1) holds for all j's whose values are defined before i. We aim to show that (1) also holds for i. Suppose not, then there would exist $j_1, j_2 \in B(i)$, such that $j_1 \in \widetilde{N}_i^{out}(g|_M)$, $j_2 \in \widetilde{N}_i^{in}(g|_M)$, and $w^*(j_1) \geq w^*(j_2)$. It follows the definitions of $\widetilde{N}_i^{out}$ and $\widetilde{N}_i^{in}$ that there exist a path from j_2 to i and a path from i to j_1, which combined together becomes a path from j_2 to j_1 (all on $g|_M$). Hence $j_1 \in \widetilde{N}_{j_2}^{out}(g|_M)$ and $j_2 \in \widetilde{N}_{j_1}^{in}(g|_M)$, and therefore $w^*(j_1) \geq w^*(j_2)$ implies that (1) is violated for either j_1 or j_2 (the one whose w^* is defined earlier). A contradiction.

Thus we concluded the proof of eq. (1).

The last piece of proof of the "if" part is to show that for every $i \in M$, $r_i(w^*) = 1$. Notice that if $i \in M$ and $j \notin M$, then $w^*(i) \geq 0.5\ w^*(i) \geq 0.5 > 0 = w^*(j)$.

Therefore it suffices to show for all $i, j \in M$ s.t. $j \in \widetilde{N}_i^{out}(g)$

$$w^*(j) < w^*(i) \tag{2}$$

If $w^*(i)$ is defined in steps #1 or #2, then $\widetilde{N}_i^{out}(g|_M) = \emptyset$, so that (2) automatically holds.

If $w^*(i)$ is defined in step #3, then $j \in \widetilde{N}_i^{out}(g)$ is defined in steps 4 or 5, and therefore (2) is implied by (1a).

Finally, consider the case that $w^*(i)$ is defined in step #4 (or 5). If $j \in B(i)$ i.e. $w^*(j)$ is defined before $w^*(i)$, then it follows from eq. (1a) (for i) that

$$w^*(j) \leq \max_{j' \in \widetilde{N}_i^{out}(g|_M) \bigcap B(i)} w^*(j') < w^*(i)$$

Otherwise, $w^*(j)$ is defined after $w^*(i)$, so also in step#4 (or 5). Then it follows from eq. (1a) (for j) that

$$w^*(j) < \min_{j' \in \widetilde{N}_j^{in}(g|_M) \bigcap B(j)} w^*(j') < w^*(i)$$

This concludes the proof of the "if" part.

"Only if": Suppose g is α-permissive so that there exists some allocation w that is α-stable on g. Now we label the nodes as follows:

0. Let $l_i = null$ for all i.

1. Pick $i \in \arg\max_{j \text{ s.t. } l_j=null} w_j$, let $l_i = 1$.

2. Let $l_i = 0$ if $l_i = null$ and $i \in N_j^{out}$ for some $l_j = 1$.

3. Repeat steps 1 and 2 until all agents are labeled with 0 or 1.

Denote $M_k = \{j \in N \mid l_j = k\}, k = 0, 1$, we have $M_1 \bigcup M_0 = N$ and $M_1 \bigcap M_0 = \emptyset$. Consider an alternative allocation w' s.t. $w'_i = w_i$ if $i \in M_0$ and $w'_i = \underline{w}$ if $i \in M_1$, for some $\underline{w} < min_{i \in N} w_i$. By construction, all agents in M_0 strictly prefers w' to w: under w, every $i \in M_0$ has a neighbor $j \in M_1$ such that $w_i \leq w_j$; whereas under w', the relative ranking among agents in M_0 remain the same, and everyone in M_0 has a strictly higher allocation than everyone in M_1.

Therefore, the fact that w is α-stable on g implies $|M_0| \leq \alpha n$, and hence $|M_1| \geq (1-\alpha)n$. Finally notice that $g|M_1$ is acyclic: by construction, $\forall i_1, i_2 \in M_1$ such that i_1 is labeled before i_2, we have $i_2 \notin N_{i_1}^{out}$. That is to say, $\forall i, j \in M_1$,$g_{ij} = 1$ implies that j is labeled before i, and hence $g_{ji} = 0$.

We have constructed a set M_1, whose size is at least $(1 - \alpha)n$, and $g|M_1$ is acyclic. ∎