Default Risk and the Pricing of U.S. Sovereign Bonds

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Abstract

United States Treasury securities are often viewed in academics and practice as being free of default risk. In principle, nominal outstanding Treasury debt can be inflated away by issuing fiat currency. The same does not hold true, however, for inflation-indexed debt. We examine the relative pricing of nominal and inflation-indexed debt in the presence of risk of default. We show empirically that the breakeven inflation rate between nominal Treasury securities and TIPS is statistically significantly related to the premium paid on U.S. credit default swaps (CDS), controlling for measures of liquidity and slow-moving capital. This evidence motivates us to model the prices of nominal and inflation-protected securities in a no-arbitrage setting. Our model shows that breakeven inflation is related to perceptions of differing rates of recovery in the two markets. The estimated model can simultaneously capture variation in breakeven inflation rates and U.S. Treasury CDS spreads.

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1 Introduction

In both academic literature and practice, United States Treasury securities are often viewed as risk-free securities, in the sense that their nominal payoffs are certain. At least part of the logic behind this treatment is the fact that the United States Treasury can inflate away its debt by issuing fiat currency. As a result, there is no *a priori* reason that the Treasury should default on its obligations denominated in current dollars. Recent history, however, has called this assumption in doubt. Starting in late 2007, as shown in Figure 1, the premium paid to insure United States sovereign debt increased dramatically from 1-2 basis points to nearly 100 basis points. While the spread has since declined, it has remained elevated relative to pre-crisis levels. Moreover, repeated political conflict over the debt ceiling has prompted concern about the possibility of a U.S. Treasury default. In 2011, a debt ceiling crisis led to the downgrading of United States sovereign debt by Standard & Poor's. These crises repeated in 2013 and again, most recently, in the fall of 2017. Public reaction to these crises suggest that there is a perception that there is a non-trivial risk of Treasury default.

While concerns about default have received greater attention in the past decade, the question of whether or not United States Treasury CDS spreads reflect actual fears of default remains a subject of debate, and the Treasury is still able to issue fiat currency to inflate the debt away.[1,2] However, the Treasury cannot inflate away inflation-protected debt. As of December 31, 2017, approximately $1.3 trillion in Treasury Inflation Protected Securities

[1]In general, Arora, Gandhi and Longstaff (2012) show that counterparty credit risk is priced in the CDS market using data covering the height of the 2008 crisis, but the magnitude is trivial resulting from the full collateralization of CDS liabilities.

[2]This point is not without contention, Hilscher, Raviv and Reis (2014) document that increased inflation is unlikely to substantially lower the real liability of the U.S. as a fraction of GDP due to the fact that Treasury debt held by the public is typically short-dated.

(TIPS) were held by the public.[3] While the total outstanding represents only 9% of public debt, it is still an economically significant amount; for example according to the Bank for International Settlements, it is more than Germany's outstanding debt at the end of 2016. Not only are these securities not free of default, but a failure to make payment on inflation-protected securities would also trigger default on nominal Treasury securities. As a result, by issuing inflation protected securities, the Treasury has made all of its debt subject to risk of default.

In this paper, we investigate the degree to which this credit risk plays a role in the pricing of U.S. sovereign debt securities. More specifically, we examine the effects of credit risk on the relative pricing of nominal and inflation-protected securities. We first document that breakeven inflation rates implied in the yields of nominal Treasuries and TIPS are significantly related to premia paid on U.S. Treasury credit default swaps. Over a sample period from late 2007 through 2015, a one standard deviation increase in the Euro-denominated U.S. CDS spread is associated with an approximately 9 basis point increase in hedged breakeven inflation. This is not a phenomenon dominated by the financial crisis; in the sample period from 2010 onward the relation continues to hold. In this latter period, the relation is robust to controlling for variables meant to capture illiquidity and slow-moving capital in the Treasury market. While we cannot completely rule out alternative explanations, our evidence is suggestive of the fact that breakeven inflation co-moves positively with sovereign credit risk.

Motivated by this evidence, we derive a model of nominal and inflation-protected sovereign debt subject to risk of default. Our modeling strategy follows Monfort et al. (2017*a*) in modeling bond yields as affine functions of standard state variables, as well as a default risk variable. The approach departs from the standard Duffie and Singleton (1999) Gaussian

[3] Monthly Statement of the Public Debt of the United States, see www.treasurydirect.gov.

credit risk framework in modeling credit events as Gamma-zero distributed. The advantage to our approach is that it produces closed-form pricing formulas and allows us to avoid the specification of inflation dynamics. We show in this setting that the spread between inflation-linked swaps and breakeven inflation rates (ILS-BEI) is related to the relative rate of recovery anticipated on nominal and inflation-protected securities.

We estimate the parameters of the model using the extended Kalman filter, utilizing the five-year U.S. CDS spreads and five maturities of the ILS-BEI spread as observables. We force the estimation to perfectly match CDS spreads, but capture variation in the ILS-BEI spreads with error. Our results indicate that the model is able to simultaneously capture variation in credit default swap spreads and breakeven rates of inflation. Specifically, while the CDS spread is matched by construction, credit risk factors are able to capture approximately 20% of the total ILS-BEI variation. Further, our estimates indicate that the market perception of the recovery rate on TIPS is about 12 percentage points lower than that of nominal bonds. The results support our conjecture that the pricing of nominal and inflation-protected securities are affected by default risk.

Our paper contributes to at least three broad strands of the fixed income literature. The first is the literature investigating the role of default risk in the pricing of sovereign securities. On the empirical front, Ang and Longstaff (2013) estimate an affine multi-factor model of U.S. and European state and country credit default swaps and conclude that systemic sovereign risk is strongly linked to financial market variables. The authors observe that the estimated U.S. systemic credit risk-neutral default intensities spiked at the beginning of 2009, immediately after the onset of the financial crisis. Similarly, Chernov, Schmid and Schneider (2016) note that CDS spreads of sovereign debt securities in major developed markets rose during the financial crisis and remained elevated in subsequent years. The authors construct a macrofinance model in which CDS premia reflect default probabilities. The authors calibrate their model to the United States macroeconomy, and show that it is able to generate the high premium paid to insure U.S. sovereign debt. Their results suggest that high CDS spreads can arise in an equilibrium framework with default risk. Additionally, Filipovic and Trolle (2013), Monfort et al. (2017*a*), and Augustin, Chernov and Song (2018) investigate the joint pricing of Treasuries and CDS. Our point of departure from this literature is in investigating the role that default risk plays in the differential pricing of nominal and inflation-protected securities.

The second area to which we contribute is the relative pricing of nominal and inflation-protected securities. Fleckenstein, Longstaff and Lustig (2014) document apparent no-arbitrage violations in the pricing of nominal and inflation-protected securities. Specifically, they show that an arbitrage strategy using nominal Treasuries, TIPS and inflation swaps generated large arbitrage profits during the financial crisis, and that these profits were present before and after the crisis period. Their empirical investigation suggests that the arbitrage arises due to slow-moving capital; a lack of arbitrage activity in the Treasury market allows the profits to persist. The ILS-BEI spread utilized in our empirical analysis is closely related to the mispricing that they document. Our results suggest that part of this mispricing is related to credit risk.[4] Other papers investigating the term structure of nominal bonds and TIPS include Buraschi and Jiltsov (2005), Ang, Bekaert and Wei (2008), Chernov and Mueller (2012), Christensen, Lopez and Rudebusch (2012), Haubrich, Pennacchi and Ritchken (2012), Hordahl and Tristani (2012), Abrahams, Adrian, Crump and Moench (2016), ,Campbell, Sunderam and Viceira (2016), Christensen, Lopez and Rude-

[4]The authors note that inflation-protected securities are not necessarily default risk free, but suggest that since CDS do not distinguish between nominal and inflation-protected debt, default risk is unlikely to explain the arbitrage profits.

busch (2014), Roussellet (2017), and Fleckenstein, Longstaff and Lustig (2017). We differ from this literature in explicitly considering the role of default risk in the pricing of these bonds.

A third strand of literature estimates the liquidity premium embedded in TIPS with respect to nominal bonds. Grishchenko and Huang (2012) construct inflation risk premium employing only TIPS yields and controlling for the liquidity premium between TIPS and nominal bonds. Pflueger and Viceira (2016) suggest that there is a large and economically significant liquidity premium that affects the relative pricing of nominal and real bonds. Their evidence includes both U.S. and U.K. nominal and inflation-indexed bond prices. In the same vein, Abrahams, Adrian, Crump, Moench and Yu (2016) decompose real and nominal yields into liquidity, inflation, and real interest rate risk components in an affine term structure model. They conclude that forward breakeven inflation is primarily driven by risk and liquidity premia. D'Amico, Kim and Wei (2018) again propose a substantial liquidity premium as the primary factor driving the wedge between TIPS yields and real riskfree rates, thus causing distortions in the term structure of breakeven inflation. Andreasen, Christensen and Ridell (2017) identify liquidity risk in TIPS with the average deviation across bonds from what a no-arbitrage pricing model would predict. Recent studies also use the ILS-BEI spread as a proxy for liquidity risk only (see e.g. Christensen and Gillan (2018) or Moench and Vladu (2018)). This is because very high liquidity is usually attributed to the swap market in the U.S. (see for instance Driessen, Nijman and Simon (2017) or Camba-Mendez and Werner (2017)). Our evidence suggests that considering the probability of sovereign default further contributes to the understanding of the breakeven spread.

2 Default Exposure in TIPS and Nominal Bonds

In this section, we detail our proposed mechanism to explain the mispricing of TIPS in a stylized fashion. We show that such mispricing can arise whenever the government willingness to pay nominal bond and TIPS holders in case of default is different.

Consider a nominal bond and TIPS issued by the sovereign. These bonds have a one-year maturity, pay coupons at equal rates c semi-annually and are completely risk-free. We denote by $\pi_{1/2}$ and π_1 the realized inflation rate for the first 6 months and the first year respectively. We can always construct a self-financed portfolio by going long the nominal bond and short the TIPS in relative principal amounts such that the price at origination is zero. If the nominal bond and TIPS trade respectively at price $B = 1$ and B^* for a \$1 principal, then a \$1 principal short position in the TIPS allows a long \$$B^*$ principal position in the nominal bond. The payoffs of this strategy are given in Table 1.

Additionally, the sovereign issues zero-coupon nominal bonds and TIPS, whose respective prices D_i and D_i^* mature in $t = i$. It can easily be shown that the term structure of zero-coupon inflation swap rates is given by $D_i^*/iD_i - 1$. The exposure to inflation risks taken in the long-short coupon bonds position can be canceled out by contracting inflation swaps short positions of notional $c/2$ and $1 + c/2$. Because a self-financed position and writing swaps are all costless, the present value of the joint position is still zero although payoffs each periods will not be (see Table 1, joint row). The payoffs can then be canceled out through a self-financing long-short position on nominal zero-coupon bonds.

A crucial point to this example is that the synthetic and true inflation swap positions embed the same risk intrinsically. Let us now assume that the true swap is virtually risk-free whereas the synthetic inflation position is exposed to default risk. The sovereign can default between inception and $t = 1/2$ or during $t \in (1/2, 1]$. We assume that the default event

Table 1: Stylized example: payoffs in a riskless world

Strategy	$t = 1/2$	$t = 1$
BEI (synthetic ILS)	$\frac{c}{2}\left[B^* - (1+\pi_{1/2})\right]$	$\left(1+\frac{c}{2}\right)\left[B^* - (1+\pi_1)\right]$
ILS	$\frac{c}{2}\left(1+\pi_{1/2} - \frac{D^*_{1/2}}{D_{1/2}}\right)$	$\left(1+\frac{c}{2}\right)\left(1+\pi_1 - \frac{D^*_1}{D_1}\right)$
Joint ILS-BEI	$\frac{c}{2}\left(B^* - \frac{D^*_{1/2}}{D_{1/2}}\right)$	$\left(1+\frac{c}{2}\right)\left(B^* - \frac{D^*_1}{D_1}\right)$

terminates both the nominal bond and TIPS contract and provides an immediate payoff given by a fraction ρ_c and ρ_c^* of the (possibly adjusted) principal. Table 2 details the payoffs of the combined strategy for every possible default state. By assuming that the default probability of the sovereign is fixed each period and given by p, we can also compute the present value of the combined ILS-BEI position. This quantity writes:

$$\mathrm{PV} = p\left[D_{1/2}\left(\rho_c B^* - \rho_c^* \frac{D^*_{1/2}}{D_{1/2}}\right) + (1-p)\frac{c}{2}D_{1/2}\left(B^* - \frac{D^*_{1/2}}{D_{1/2}}\right) + (1-p)D_1\left(\rho_c B^* - \rho_c^* \frac{D^*_1}{D_1}\right)\right].$$

We first note that this present value naturally goes to zero if $p = 0$. Then, for any positive default probability, the present value is going to be positive. Indeed, the price of the coupon bond B^* is bigger than the prices of the real zero coupons $D_{1/2}$ and D_1 provided the deflation probability is small enough and $\rho_c > \rho_c^*$. To see this, assume that $\rho_c = \rho_c^* + k$. Simplifying, we obtain:

$$\mathrm{PV} = p\left[D_{1/2}\left(B^* - \frac{D^*_{1/2}}{D_{1/2}}\right)\left(\rho_c^* \frac{2+pc}{2+c} + \frac{c(1-p)}{2}\right) + kB^* \frac{2}{2+c}\left(1 - p + D_{1/2}\right)\right].$$

Empirically, the coupon rate c will be typically close to zero. This leads to the following

Table 2: Stylized example: payoffs in a credit-sensitive world

Default?	$t = 1/2$	$t = 1$
None	$\frac{c}{2}\left(B^* - \frac{D^*_{1/2}}{D_{1/2}}\right)$	$\left(1+\frac{c}{2}\right)\left(B^* - \frac{D^*_1}{D_1}\right)$
At $t < 1/2$	$\frac{c}{2}\left(1+\pi_{1/2} - \frac{D^*_{1/2}}{D_{1/2}}\right) +$ $\rho_c B^* - \rho^*_c\left(1+\pi_{1/2}\right)$	$\left(1+\frac{c}{2}\right)\left(1+\pi_1 - \frac{D^*_1}{D_1}\right)$
At $t \in (1/2, 1]$	$\frac{c}{2}\left(B^* - \frac{D^*_{1/2}}{D_{1/2}}\right)$	$\left(1+\frac{c}{2}\right)\left(1+\pi_1 - \frac{D^*_1}{D_1}\right) +$ $\rho_c B^* - \rho^*_c\left(1+\pi_1\right)$

PV: $p\left[D_{1/2}\left(\rho_c B^* - \rho^*_c \frac{D^*_{1/2}}{D_{1/2}}\right) + (1-p)\frac{c}{2}D_{1/2}\left(B^* - \frac{D^*_{1/2}}{D_{1/2}}\right) + (1-p)D_1\left(\rho_c B^* - \rho^*_c \frac{D^*_1}{D_1}\right)\right]$

approximation:

$$\mathrm{PV} = p\left[\rho^*_c D_{1/2}\left(B^* - \frac{D^*_{1/2}}{D_{1/2}}\right) + kB^*\left(1 - p + D_{1/2}\right)\right].$$

The first term is linear in the difference between the TIPS coupon bond price B^* and the 6-month gross inflation swap rate. These will both be of the order of magnitude of 1, so the difference is small. The term in kB^* thus dominates the expression, and is going to be the primary driver of the present value of the spread. Note that this term is increasing linearly with k and concavely with p. We build on this principle in our empirical analysis.

Note however that in our subsequent analysis, we do not consider cases of partial default for simplification. We will always assume that the sovereign defaults at the same time on nominal bonds and TIPS. The only consequence of this assumption is that we assume that the outstanding sovereign bonds terminate at time of default and do not run for further time periods.

3 Empirical Analysis

In this section we investigate the extent that default risk may influence the relative pricing of nominal and inflation-protected sovereign obligations. Specifically, we test whether CDS spreads are related to the difference between inflation swap rates (ILS) and breakeven inflation (BEI) as represented by nominal less real yields of U.S. sovereign debt. We examine variation in these quantities over the full sample period and a subperiod that does not include the financial crisis of 2007-2009.

3.1 Credit Risk in Breakeven Inflation Rates

Fleckenstein, Longstaff and Lustig (2014) show that the cash flows of a nominal Treasury bond can be replicated by a portfolio of TIPS, U.S. Treasury STRIPS, and inflation swaps, and that nominal Treasuries trade at a premium to this replicating portfolio. We investigate a measure that is related to their approach, but not subject to exact timing of cash flows: the difference in the inflation level swap rate and breakeven inflation (ILS-BEI). A zero-coupon inflation swap pays cumulative inflation in exchange for a fixed rate determined at initiation of the contract. In its construction, including the reference index of CPI, an inflation swap is comparable to the breakeven inflation rate as defined by equivalent maturity zero-coupon nominal Treasuries less zero-coupon TIPS (also indexed to CPI).[5] Both the ILS and BEI reflect inflation expectations as well as the inflation risk premium. As seen in Figure 2, the

[5]Note that we neglect the deflation floor which is embedded in standard TIPS bonds but not present in inflation-linked swaps. If anything, this will simply reduce the ILS-BEI spread since it will decrease the TIPS yield, which will in turn increase the size of the BEI.

two track rather closely, with a consistent premium attributed to the breakeven rate proxied by the inflation swap derivative contract. Campbell, Shiller and Viceira (2009) suggest the premium is related to the cost of supplying inflation protection and is typical under normal market conditions. ILS-BEI averages 36 basis points over the sample, yet peaks at 210 basis points in late 2008. Both the average and peak spread are roughly comparable to the basis point mispricing between the nominal Treasury bond and the replicating portfolios detailed by Fleckenstein, Longstaff and Lustig (2014).

A divergence in ILS-BEI may be attributable to a series of factors in addition to the average typical cost of supplying inflation protection. We argue that sovereign credit risk contributes to the differential as market participants recognize the non-zero probability of a U.S. sovereign default. Inflation swaps, Treasuries, and TIPS all trade over-the-counter and may be subject to varying liquidity risk. In addition, inflation swaps, although collateral-backed, may incorporate counterparty risk. We attempt to mitigate these potential confounding factors in the analysis.

3.2 Swaps, Breakevens and CDS Spreads

We use Gurkaynak, Sack and Wright (2006) and Gurkaynak, Sack and Wright (2010) (GSW) for nominal and inflation-protected zero coupon bonds respectively. We collect inflation swap data from Bloomberg and EUR-denominated CDS spread data from Markit. While we use EUR denominated U.S. Treasury CDS in the main analysis, our results are qualitatively the same when using USD contracts. Our focus is on the five-year maturity of each security since the five-year maturity is the most liquid CDS tenor. Our data are sampled from January 1, 2008 through September 30, 2015 (full sample). While data on CDS are available prior to this sample period, U.S. Treasury CDS exhibit virtually no variation and volume in the pre-sample period. The quotes are often unchanged for weeks at a time and average between one and two basis points.

We depict the time series of U.S. sovereign credit default swap spreads and ILS-BEI differential in Figure 1. As shown in the figure, CDS spreads are essentially zero until late 2007 and, as documented in Chernov, Schmid and Schneider (2016), soar to 100 basis points in the wake of the Lehman Brothers bankruptcy, timing that is similar to that of the large increase in ILS-BEI. Our conjecture is that this event, and the crisis that followed caused investors to reprice the probability of a U.S. sovereign default and the recovery on Treasury and TIPS in a default scenario. The spread is volatile in 2010-2013 before becoming quiescent from about 2014 onward. Notably, the spread spikes to more than 40 basis points in the days prior to the resolution of the the budget showdown of 2013, which threatened to lead to a U.S. sovereign default.

Summary statistics for these data are provided in Table 3. As shown in the table, over the full sample period, both the ILS-BEI and U.S. CDS spread averaged over 30 basis points (36 and 33 basis points respectively). The ILS-BEI is approximately twice as volatile as the CDS spread, ranging from -1 to 210 basis points. In contrast to the CDS spread, the ILS-BEI declines both on average and in volatility in the post-crisis period, which we define as January 1, 2010 and beyond. Thus, even in the post-crisis period, the Treasury CDS spread averages 34 basis points, considerably greater than its pre-crisis levels.

In addition to possible fears of default risk, the crisis generated considerable fear of counterparty credit risk, a lack of liquidity, increases in perceived quantities and prices of risk, and a deterioration in arbitrage capital available to deploy in financial markets. In order to investigate these other possibilities, we also examine the role of the following variables:

- HPW Noise, the measure of arbitrage capital availability proposed in Hu, Pan and

Wang (2013). This measure is constructed as the root mean squared error in the observed yields of Treasury securities relative to those implied by a Nelson-Siegel-Svensson zero coupon curve across the term structure.[6] The measure takes into account the close relationship between availability of arbitrage capital and liquidity. Fleckenstein, Longstaff and Lustig (2014) posit that the inability of arbitrageurs to immediately eliminate arbitrage may have resulted in the divergence between nominal and inflation-protected securities markets. They suggest that this slow-moving capital hypothesis (Mitchell, Pedersen and Pulvino (2007) and Duffie (2010)) may allow arbitrage profits to persist. Again, the root mean square error, which averages 3.52 basis points, rises to 20.47 basis points during the financial crisis.

- LIBOR-OIS, the spread between LIBOR and the overnight indexed swap rate. As shown in Table 1, this spread, which averages 35 basis points over our sample, rose to 364 basis points during the crisis. This rise has been attributed to an increase in perceived counterparty credit risk in financial markets. Fleckenstein, Longstaff and Lustig (2014) suggest that their arbitrage profits could arise due to counterparty credit risk, especially if nominal Treasuries are viewed as safe haven assets. However, the authors suggest it is an unlikely explanation for their findings due to the collateralization of of swap contracts (Arora, Gandhi and Longstaff (2012)).

- OTR Difference, the yield difference between the 10-year off-the-run GSW par yield and the generic 10-year on-the-run yield from Bloomberg. During periods of stress, market participants may seek the most liquid securities, on-the-run government benchmark bonds, which accordingly often trade at a premium to an equivalent off-the-run bond.

[6] These data are obtained from Jun Pan's webpage, http://www.mit.edu/~junpan/

- VIX, the CBOE volatility index. The VIX is often viewed as a measure of the market's perception of the quantity and/or price of risk in equity markets specifically, and financial markets as a whole. However, Nagel (2012) suggests that an increase in the VIX is associated with a higher premium for liquidity provision, and therefore a reduction in the amount of liquidity in the financial system. The VIX averages 22% over our sample period, with an increase to nearly 81% during the financial crisis.

3.3 Empirical Results

We employ panel regressions for our empirical analysis. We pool ILS-BEI differences across five tenors: 2, 3, 5, 7, and 10 years as the dependent variable. Tables 4, 5, and 6 present regression results for the full sample, the crisis sample, and the post-crisis sample, respectively. The full sample period spans the beginning of 2008 to the third quarter of 2015. We start the sample in 2008 due to the fact that U.S. sovereign CDS contracts were thinly traded prior to the 2008 financial crisis. All regressions contain observations at the daily frequency with week fixed effect. Column (7) in each table shows results from the full regression specification with both week and tenor fixed effects, as well as all the control variables listed in Section 3.2. Finally, in each regression, the U.S. CDS spread is the main explanatory variable, but we include lagged ILS-BEI spread to ensure the persistence of the dependent variable is not driving our results.

We begin by regressing the ILS-BEI differential on U.S. CDS spreads over the full sample January 1, 2008 through September 30, 2015. Results are shown in Table 4. In Column (1), we simply employ the U.S. CDS spread and lagged ILS-BEI spread as explanatory variables. Then, we iteratively add control variables in regressions from Columns (2) to (5) until we arrive at the full specification in Column (6). Lastly, for robustness, we use both week and tenor fixed effects in Column (7).

As shown in the top row Table 4, all coefficient loadings on the U.S. CDS spread are positive and highly significant across the columns. The point estimate of 0.196 in Column (1) suggests that a one basis point increase in U.S. CDS spreads translates into an approximately 0.2 basis point increase in the ILS-BEI differential, and that this effect is statistically significantly different than zero based on a standard error of 0.049. The magnitude of this effect, which translates into a 3 basis point increase in the differential for a unit standard deviation increase in CDS spreads, represents approximately 10% of the mean ILS-BEI differential. This suggests that the results are economically, as well as statistically, significant.

Table 4 also demonstrates the fact that it is essential to include lagged ILS-BEI spread as an explanatory variable since coefficient loadings are positive and highly significant regardless which control variables are used. It should not be surprising that the ILS-BEI spread is highly autocorrelated. Amongst the control variables, the LIBOR-OIS spread in Column (3) and VIX in Column (5) are marginally significant (between 5% and 10% statistical significance) in the panel regression. The OTR Difference is negative and highly significant in Column (4). A higher value of the OTR Difference implies worse liquidity conditions in the Treasury market. This leads to higher TIPS yields, and a narrowing of the BEI, consistent with the finding in Pflueger and Viceira (2016). At the same time, low expected inflation, particularly during the crisis period, causes the ILS spread to decrease as the demand for inflation protection declines. The decrease in ILS spread dominates the the decrease in BEI which leads to the negative loading of ILS-BEI on the liquidity measure.[7] When taken together, in Column (6) of Table 4, only the CDS spread, the lagged ILS-BEI spread, and the OTR Difference are significant in explaining the ILS-BEI differential. The estimated coefficient loading on the

[7]This phenomenon is driven by the crisis period as shown in Column (4) of Table 5. The loading of ILS-BEI on OTR Difference becomes insignificant in Column (5) of Table 6.

CDS spread actually increases from Column (1) to Column (6) when controls are included. Finally, the addition of a tenor fixed effect does not affect the regression outcomes in Columns (6) and (7).

Next, we move to the subsample analysis and focus on the crisis period between 2008 and 2009. We examine the degree to which the crisis influences our conclusions by separating the sample into a crisis period, which we specify as January, 2008 through December, 2009, and a post-crisis period from January, 2010 onward. Results for the crisis period are presented in Table 5. As shown in the first row of the table, there is an observed positive relationship between CDS spreads and the ILS-BEI differential during the crisis. The coefficient of 0.213 is statistically different than zero at the 10% significance level in Column (1), and the coefficient of 0.221 is statistically significant at the 10% level in Column (7) under the full specification. Although their statistical significance are weaker during the crisis period relative to the full sample in Table 4, the point estimates on the U.S. CDS spread are greater, which implies a more pronounced effect between sovereign default risk and the ILS-BEI spread.

In the post-crisis period, the results depicted in Table 6 again suggest a statistically significant impact of the U.S. CDS swap spread on the ILS-BEI differential. The point estimate of 0.171 is statistically significantly different than zero at the 1% level in Column (1) in the absence of control variables. In contrast to the crisis, results in the first row of Table 6 indicate that the CDS spread has even more statistically significant explanatory power for the ILS-BEI differential. The point estimates across all columns are roughly five times greater than their standard errors.

Our interpretation of these results is that determinants of the U.S. CDS spread comove strongly with the difference in ILS and BEI. CDS spreads may be driven by a number of different factors, including actual default risk and liquidity effects. We view the evidence here as sufficiently suggestive to indicate that the IES-BEI differential is influenced by credit risk, and propose a formal model of sovereign nominal and inflation-protected debt and credit default swaps written on these securities.

3.4 Default Risk and Liquidity

Pflueger and Viceira (2016) suggest that much of the spread between nominal and inflation-protected bond yields arises as a premium for liquidity. In their analysis, they find that the portion of breakeven inflation that is related to liquidity rather than inflation expectations accounts on average for 69 basis points of the spread between nominal and inflation-protected securities. While we endeavor to control for liquidity in our earlier analysis, in this section we explicitly examine the contribution of CDS to the liquidity premium that they document.

The authors measure the liquidity premium by breaking the differential in the yield on nominal and inflation-protected securities on a set of liquidity variables and measures of inflation expectations:

$$BEI_t = a_1 + \mathbf{a}_2'\mathbf{X}_t + \mathbf{a}_3'\boldsymbol{\pi}_t^e + \epsilon_t, \tag{1}$$

where $\mathbf{X}_t$ is a vector of liquidity-related variables and $\boldsymbol{\pi}_t^e$ is a vector of measures of inflation expectation. The liquidity premium is measured as

$$\hat{L}_t = -\hat{\mathbf{a}}_2'\mathbf{X}_t. \tag{2}$$

We follow their approach, using the breakeven inflation between 10-year nominal and inflation-protected securities as our dependent variable.

Liquidity is proxied using three variables: the off-the-run spread (OTR), log relative volume in the TIPS and nominal Treasury markets (VOL), and the synthetic-cash spread, which is our variable $ILS - BEI$.[8] The off-the-run spread is the difference between the 10 year off-the-run par yield and the 10-year on-the-run nominal yield from Bloomberg (USGG10YR). Relative volume in the two markets is measured using primary dealers' transaction volume from the New York Federal Reserve FR-2004 survey. Inflation expectations are measured using two variables, the median 10-year CPI forecast from the Survey of Professional Forecasters (CPI^e) and the Chicago Fed National Activity Index ($CFNAI$). The CPI forecast is available quarterly, and the $CFNAI$ is available monthly. We create a daily series using the most recently released data.[9]

Results of the analysis are presented in Table 7. In the first column of Panel A, we present an analysis complementary to that of Pflueger and Viceira (2016). Consistent with their analysis, and with intuition, inflation expectation variables are positively related to breakeven inflation. Our results suggest that the Survey of Professional Forecasters inflation expectation is statistically significantly related to breakeven inflation and that the $CFNAI$ coefficient is marginally statistically significant. This result is slightly different than that found by Pflueger and Viceira (2016), who find that $CFNAI$ is statistically significant but that the Survey of Professional Forecasters is not. However, the statistical significance does not materially impact the interpretation of the results.

All three liquidity variables are statistically significantly related to breakeven inflation. Consistent with. Pflueger and Viceira (2016), the off-the-run spread is negatively related to the BEI, suggesting that periods with especially low breakeven inflation represent episodes

[8] In their main results, Pflueger and Viceira (2016) use the asset swap spread and use the ILS-BEI for robustness. Results using both variables are similar, and we use the ILS-BEI for simplicity and to complement our earlier results.

[9] Our results are similar in terms of signs and magnitude regardless of the data frequency; we also examine weekly and monthly data. However, statistical significance of some of the coefficients declines as we sample at coarser data frequencies.

of flights-to-liquidity, driving down the nominal Treasury yield. The coefficients on relative volume and the ILS-BEI differential suggest a slightly different interpretation than in their results, however. Relative volume is negatively related to the breakeven inflation rate, suggesting that when volume in the TIPS market is relatively high, the breakeven narrows. Closer inspection of the data reveals that relative volume and breakeven inflation co-moved positively during the financial crisis; that is, the BEI narrowed and TIPS volume fell relative to nominal Treasury volume. This is consistent with the liquidity explanation provided by Pflueger and Viceira (2016). However, from 2010 onward, the relation turns negative, largely as a result of a strong upward trend in the volume of TIPS relative to nominal Treasuries.

The coefficient on the ILS-BEI is also worthy of attention. Consistent with the results in Pflueger and Viceira (2016), the coefficient is negative; the authors interpret the result as suggesting that the pronounced decrease in breakeven inflation during the financial crisis reflected security market disruption and constraints on levered investors. The authors find that one cannot reject the hypothesis that the coefficient is equal to negative one. However, in our results, the point estimate of the coefficient is more than two standard errors from one. This result suggests that, consistent with our results above, the ILS-BEI may reflect more than just constraints on levered market participants.

In the second column, we add the CDS spread to the regression. Three noteworthy observations emerge. First, the CDS spread is negatively and significantly related to breakeven inflation. To the extent that default risk may have differential impact on nominal and inflation-protected Treasury securities, the negative coefficient suggests that yield spreads on the two securities tighten when default risk increases. This may reflect a flight to the relative safety of nominal Treasuries or a drop in the prices of inflation-protected securities. Second, the coefficients on the remaining variables, with the exception of ILS-BEI, are materially unaffected. Third, after controlling for CDS, one can no longer reject the hypothesis that the coefficient on the ILS-BEI is equal to negative one, consistent with the results in Pflueger and Viceira (2016). Thus, the results indicate that both the BEI and the ILS-BEI reflect co-movement with CDS spreads, perhaps due to credit risk.

Our final analysis of the liquidity premium directly regresses the estimated liquidity premium on the CDS spread. Results are presented in Panel B. As shown in the Table, CDS spreads on Treasury securities are positively and statistically significantly related to the liquidity premium, explaining approximately 9% of its variation. This result suggests that part of the liquidity premium documented in Pflueger and Viceira (2016) may in fact reflect compensation for credit risk. However, the majority of the variation in the premium is unrelated to variation in CDS spreads, indicating that both liquidity and credit risk jointly play a role in understanding the differential pricing of TIPS and nominal Treasury securities.

4 Modeling Nominal and Inflation-Protected Debt with Default Risk

In this section, we discuss the pricing of nominal and inflation-protected sovereign bonds, assuming that there is a possibility of a credit event interrupting the promised payments of the securities. Of particular interest is the spread between inflation-linked swaps and the *breakeven inflation rate*, the risk neutral inflation rate that equates the prices of the nominal and inflation-protected securities.

4.1 The term structure of riskless yields

We consider an economy where the nominal and real term structures are driven by $k_x \times 1$ factors $\mathbf{x_t}$ under the risk-neutral measure $\mathbb{Q}$. More specifically, these factors are partitioned in two blocks such that $\mathbf{x_t} = \left(\mathbf{x_t^{(r)}}^\top, \mathbf{x_t^{(\pi)}}^\top\right)^\top$, where these components are vectors of size $k_x^{(r)}$ and $k_x^{(\pi)}$ respectively. Investors have access to a one-period riskless nominal investment yielding a continuously-compounded interest rate $r_t = -\log\left(D_t^{(1)}\right)$ between t and $t+1$. We assume that the nominal rate risk-neutral dynamics depend on the factors $\mathbf{x_t}$ linearly, such that:

$$r_t = \kappa_0^{(r)} + \boldsymbol{\kappa_r}^\top \mathbf{x_t^{(r)}}.$$

In this economy, investors have also access to real bonds (TIPS) at price $D_t^{*(n)}$ that provide one unit of consumption growth at maturity $t+n$. We assume that inflation π_t as given by the growth rate of the CPI-U index between $t-1$ and t has dynamics given by:

$$\pi_t = \kappa_0^{(\pi)} + \boldsymbol{\kappa_\pi}^\top \mathbf{x_t^{(\pi)}}.$$

We assume that the risk-neutral dynamics of the factors are given by:

$$\mathbf{x_t} = \boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x_{t-1}} + \sqrt{\boldsymbol{\Sigma}^{\mathbb{Q}}}\varepsilon_{\mathbf{t}}^{\mathbb{Q}}, \tag{3}$$

where $\varepsilon_t^{\mathbb{Q}} \overset{i.i.d}{\sim} \mathcal{N}(\mathbf{0}, \mathbf{I_{k_x}})$. It is well-known that the riskless nominal and real bond yields are obtained in closed-form in that setup. We have:

$$D_t^{(n)} = \exp\left(A_n + \mathbf{B_n}^\top \mathbf{x_t}\right) \quad \text{and} \quad D_t^{*(n)} = \exp\left(A_n^* + \mathbf{B_n^*}^\top \mathbf{x_t}\right), \tag{4}$$

where

$$
\begin{aligned}
A_n &= A_{n-1} - \kappa_0^{(r)} + \mathbf{B}_{\mathbf{n-1}}^\top \boldsymbol{\mu}^{\mathbb{Q}} + \frac{1}{2}\mathbf{B}_{\mathbf{n-1}}^\top \boldsymbol{\Sigma}^{\mathbb{Q}} \mathbf{B}_{\mathbf{n-1}} \quad \text{and} \quad \mathbf{B_n} = \boldsymbol{\Phi}^{\mathbb{Q}^\top} \mathbf{B}_{\mathbf{n-1}} - \boldsymbol{\kappa}_{\mathbf{r}}\,, && (5)\\
A_n^* &= A_{n-1}^* - \kappa_0^{(r)} + \kappa_0^{(\pi)} + \left[\mathbf{B}_{\mathbf{n-1}}^* + \begin{pmatrix} \mathbf{0}_{\mathbf{k_r}} \\ \boldsymbol{\kappa_\pi} \end{pmatrix}\right]^\top \left(\boldsymbol{\mu}^{\mathbb{Q}} + \frac{1}{2}\Sigma^{\mathbb{Q}}\left[\mathbf{B}_{\mathbf{n-1}}^* + \begin{pmatrix} \mathbf{0}_{\mathbf{k_r}} \\ \boldsymbol{\kappa_\pi} \end{pmatrix}\right]\right) && (6)\\
\mathbf{B_n^*} &= \boldsymbol{\Phi}^{\mathbb{Q}^\top}\left[\mathbf{B}_{\mathbf{n-1}}^* + \begin{pmatrix} \mathbf{0}_{\mathbf{k_r}} \\ \boldsymbol{\kappa_\pi} \end{pmatrix}\right] - \begin{pmatrix} \boldsymbol{\kappa_r} \\ \mathbf{0}_{\mathbf{k}_\pi} \end{pmatrix}, && (7)
\end{aligned}
$$

starting from initial conditions $A_0 = 0$, $\mathbf{B_0} = \mathbf{0_k}$ and $A_0^* = 0$, $\mathbf{B_0^*} = \mathbf{0_k}$.

4.2 Default and liquidity dynamics

Our modeling framework follows that of Monfort et al. (2017*a*) in modeling risky debt in discrete time. In this framework, sovereign credit events of any kind are represented by the first jump of a non-negative credit-event variable denoted by $\delta_t^{(c)}$. More formally, if τ_c is the default date of the sovereign, we have:

$$\tau_c = \min\{\, t \,|\, \delta_t^{(c)} > 0\}. \tag{8}$$

Our modeling of liquidity events mimics the form employed for credit events, as in e.g. Ericsson and Renault (2006), Monfort and Renne (2013) or Dubecq et al. (2016). We thus assume that liquidity events are represented by the first jump of a liquidity-event variable denoted by $\delta_t^{(\ell)}$, such that the date of the liquidity event τ_ℓ is given by:

$$\tau_\ell = \min\{\, t \,|\, \delta_t^{(\ell)} > 0\}. \tag{9}$$

The risk-neutral dynamics of default and liquidity events are driven by $k_c \times 1$ and $k_\ell \times 1$ vectors of state variables, denoted by $\mathbf{y_t^{(c)}}$ and $\mathbf{y_t^{(\ell)}}$ respectively. We assume that $\mathbf{y_t} = \left(\mathbf{y_t^{(c)\top}}, \mathbf{y_t^{(\ell)\top}}\right)^\top$ follows a vector autoregressive gamma-zero process under the risk-neutral measure.

$$\mathbf{y}_t | \underline{\mathbf{y_{t-1}}} \overset{\mathbb{Q}}{\sim} \Gamma_{\mathbf{0}} \left(\boldsymbol{\alpha}^{\mathbb{Q}} + \boldsymbol{\beta}^{\mathbb{Q}} \mathbf{y}_{t-1}; \mathbf{c}^{\mathbb{Q}}\right) \tag{10}$$

Using the same notations as in Monfort et al. (2017*b*), $\boldsymbol{\alpha}^{\mathbb{Q}} \in \mathbb{R}_+^{k_c+k_\ell}$ and $\boldsymbol{\beta}^{\mathbb{Q}}$ are respectively the intercept vector and the autoregressive matrix with positive components, and the vector of scale parameters $\mathbf{c}^{\mathbb{Q}} \in \mathbb{R}_+^{k_c+k_\ell}$.

Conditionally on the state variables $\mathbf{y}_t$, we assume that the credit and liquidity event variables $\boldsymbol{\delta_t} = \left(\delta_t^{(c)}, \delta_t^{(\ell)}\right)^\top$ are also Gamma-zero distributed,

$$\boldsymbol{\delta_t} \Big| \left(\underline{\mathbf{y}_t}, \underline{\delta_{t-1}}\right) \overset{\mathbb{Q}}{\sim} \Gamma_0 \left(\begin{pmatrix} \boldsymbol{\gamma}_c^\top \mathbf{y_t}^{(\mathbf{c})} \\ \boldsymbol{\gamma}_\ell^\top \mathbf{y_t}^{(\ell)} \end{pmatrix} ; \mathbf{1} \right) \tag{11}$$

where the scaling parameters of the process have been normalized to 1 for identification purposes. Monfort et al. (2017*a*) show that Gamma-zero processes are efficient in representing credit events since they can stay at the value of zero for extended periods of time (no default or liquidity states) and jump to any positive value upon events. Combining the dynamics given by Equations (10) and (11), it can easily be shown that the joint process $(\mathbf{y_t}^\top, \boldsymbol{\delta_t}^\top)^\top$ is affine directly implies that its conditional moment generating function can be written:

$$\varphi_{t-1}(\mathbf{u}, \mathbf{v}) \;=\; \mathbb{E}_{t-1}^{\mathbb{Q}} \left[\exp\left(\mathbf{u}^\top \mathbf{y_t} + \mathbf{v}^\top \boldsymbol{\delta_t}\right)\right] = \exp\left[A(\mathbf{u}, \mathbf{v}) + \mathbf{B}(\mathbf{u}, \mathbf{v})^\top \mathbf{y_{t-1}}\right], \tag{12}$$

for any vector $(\mathbf{u}, \mathbf{v})$ such that the expectation exists. The coefficients $A(\mathbf{u}, \mathbf{v})$ and $\mathbf{B}(\mathbf{u}, \mathbf{v})$

are easily computed as:

$$A(\mathbf{u},\mathbf{v}) \;=\; \boldsymbol{\alpha}^{\mathbb{Q}\top}\frac{\mathbf{c}^{\mathbb{Q}}\left(\mathbf{u}+\tilde{\mathbf{v}}\right)}{\mathbf{1}-\mathbf{c}^{\mathbb{Q}}\left(\mathbf{u}+\tilde{\mathbf{v}}\right)} \tag{13}$$

$$\mathbf{B}(\mathbf{u},\mathbf{v}) \;=\; \boldsymbol{\beta}^{\mathbb{Q}\top}\frac{\mathbf{c}^{\mathbb{Q}}\left(\mathbf{u}+\tilde{\mathbf{v}}\right)}{\mathbf{1}-\mathbf{c}^{\mathbb{Q}}\left(\mathbf{u}+\tilde{\mathbf{v}}\right)}, \tag{14}$$

where the ratio stands for an element-by-element ratio by notation abuse and

$$\tilde{\mathbf{v}} = \left(\frac{v_c}{1-v_c}\boldsymbol{\gamma}_c^\top,\, \frac{v_\ell}{1-v_\ell}\boldsymbol{\gamma}_\ell^\top\right)^\top,$$

4.3 Joint dynamics

The joint conditional moment generating function of the system is given by:

$$\begin{aligned}\psi_{t-1}(\mathbf{u},\mathbf{v},\mathbf{w}) &= \mathbb{E}_{t-1}^{\mathbb{Q}}\left[\exp\left(\mathbf{u}^\top\mathbf{y_t}+\mathbf{v}^\top\boldsymbol{\delta_t}+\mathbf{w}^\top\mathbf{x_t}\right)\right]\\ &= \exp\left[A(\mathbf{u},\mathbf{v})+\mathbf{B}(\mathbf{u},\mathbf{v})^\top\mathbf{y_{t-1}}+\mathbf{w}^\top\left(\mu^{\mathbb{Q}}+\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x_{t-1}}+\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{\Sigma}^{\mathbb{Q}}\mathbf{w}\right)\right]\end{aligned} \tag{15}$$

which is exponential-affine in $\mathbf{z_t} = \left(\mathbf{x_t}^\top,\, \mathbf{y_t}^\top\right)^\top$. We build our pricing results upon a crucial property of Gamma-zero processes. In the following, all nominal quantities of interest to perform pricing take the form:

$$G_t(n,n_c,n_\ell) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j}\right)\mathbb{1}\left\{\sum_{j=0}^{n_c}\delta_{t+j}^{(c)}=0\right\}\mathbb{1}\left\{\sum_{j=0}^{n_\ell}\delta_{t+j}^{(\ell)}=0\right\}\right].$$

The principle is the same for inflation-indexed bonds, by just replacing r_{t+j} by $r_{t+j}-\pi_{t+j+1}$. We denote these by $G_t^*(n,n_c,n_\ell)$. Using the lemma provided in Monfort et al. (2017*a*), we can write:

$$G_t(n, n_c, n_\ell) = \lim_{u \to +\infty} \mathbb{E}_t^{\mathbb{Q}} \left[\exp \left(-\sum_{j=0}^{n-1} r_{t+j} - u \left(\sum_{j=0}^{n_c} \delta_{t+j}^{(c)} + \sum_{j=0}^{n_\ell} \delta_{t+j}^{(\ell)} \right) \right) \right] .$$

$G_t(n, n_c, n_\ell)$ is the limit of the multi-period moment generating function. Because the one-period moment generating function $\psi_{t-1}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is exponential-affine, the multi-period one is also exponential-affine in the factors and we have:

$$G_t(n, n_c, n_\ell) = \exp\left(q_{(n,n_c,n_\ell)} + \mathbf{Q}_{(n,n_c,n_\ell)}^\top \mathbf{z_t}\right) , \quad (16)$$

$$G_t^*(n, n_c, n_\ell) = \exp\left(q_{(n,n_c,n_\ell)}^* + \mathbf{Q}_{(n,n_c,n_\ell)}^{*\top} \mathbf{z_t}\right) . \quad (17)$$

The exact recursions used to obtain the loadings in those equations are detailed in Appendix A.1. Notice that the price of riskless bonds computed above are directly given by $D_t^{(n)} = G_t(n, 0, 0)$ and $D_t^{*(n)} = G_t^*(n, 0, 0)$.

4.4 Risky asset prices

Consider a sovereign state, which issues both nominal and inflation-protected debt with maturity n. With some probability, the bond defaults prior to maturity n. Default happens when, at any time $\tau_c \leq t + h$ where the credit-event variable $\delta_t^{(c)}$ jumps from zero to a positive value (see Equation (8)). In the same fashion, liquidity events happen when the liquidity-event variable $\delta_t^{(\ell)}$ jumps from zero to a positive value.

We assume that nominal bonds of the sovereign are unaffected by liquidity events. In the case of a credit event, nominal bondholders get a recovery payment on their investment,

$\mathcal{P}_c^{(n)}$. The price of a nominal zero-coupon bond is given by:

$$
\begin{aligned}
B_t^{(n)} &= \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{i-1} r_{t+j} \right) \mathcal{P}_c^{(i)} \times \left(\mathbb{1}\left\{ \sum_{j=0}^{i-1} \delta_{t+j}^{(c)} = 0 \right\} - \mathbb{1}\left\{ \sum_{j=0}^{i} \delta_{t+j}^{(c)} = 0 \right\} \right) \right] \\
&+ \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j} \right) \mathbb{1}\left\{ \sum_{j=0}^{n} \delta_{t+j}^{(c)} = 0 \right\} \right]. \qquad (18)
\end{aligned}
$$

Equation (18) simply states that the price of the nominal bond is the sum of discounted recovery payments if default happens between $t+i-1$ and $t+1$, and the discounted principal if no default occurs during the lifespan of the bond. An inflation-indexed bond is priced similarly, with the difference that its payoff is indexed to a reference inflation index, denoted by π_t. We assume that the bond's recovery payment upon default may differ from that of the nominal bond, and designate this recovery payment $\mathcal{P}_c^{*(n)}$. Similarly, the recovery payment in case of liquidity event is given by $\mathcal{P}_\ell^{*(n)}$. The price of the inflation-indexed bond is then given by:

$$
\begin{aligned}
B_t^{*(n)} &= \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \Bigg[\exp\left(-\sum_{j=0}^{i-1} r_{t+j} \right) \left(\mathcal{P}_c^{*(i)} + \mathcal{P}_\ell^{*(i)} \right) \mathbb{1}\left\{ \sum_{j=0}^{i-1} \mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} = 0 \right\} \\
&- \mathcal{P}_c^{*(i)} \exp\left(-\sum_{j=0}^{i-1} r_{t+j} \right) \mathbb{1}\left\{ \sum_{j=0}^{i-1} \mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} + \delta_{t+i}^{(c)} = 0 \right\} \\
&- \mathcal{P}_\ell^{*(i)} \exp\left(-\sum_{j=0}^{i-1} r_{t+j} \right) \mathbb{1}\left\{ \sum_{j=0}^{i-1} \mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} + \delta_{t+i}^{(\ell)} = 0 \right\} \Bigg] \\
&\quad + \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j} - \pi_{t+j+1} \right) \mathbb{1}\left\{ \sum_{j=0}^{n} \mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} = 0 \right\} \right]. \qquad (19)
\end{aligned}
$$

One possible reason that recovery may differ between nominal and inflation-protected bonds is devaluation of the issuing currency. That is, the sovereign may not be able to repay the full real face value even if it is able to fully repay nominal indebtedness.

Following Duffie and Singleton (1999) and to be consistent with the CDSs, we assume the recovery of face value assumption (RFV), which states that in case of default, recovery payments are proportional to the face value of the bond, by a factor equal to the recovery rate. For TIPS, we assume that the principal is continuously adjusted by the realized inflation. The total recovery payments write:

$$\mathcal{P}_c^{(n)} = \rho_c \,, \quad \mathcal{P}_c^{*(n)} = \rho_c^* \exp\left(\sum_{j=0}^{n-1} \pi_{t+j+1}\right) \,, \quad \text{and} \quad \mathcal{P}_\ell^{*(n)} = \rho_\ell^* \exp\left(\sum_{j=0}^{n-1} \pi_{t+j+1}\right)$$

We show in the Appendix that under these assumptions, using equation (18)-(19), bond prices at time t can be approximated by:

$$B_t^{(n)} = \exp\left[\mathcal{A}_n(\rho_c) + \mathcal{B}_n(\rho_c)^\top \mathbf{z_t}\right] \tag{20}$$

$$B_t^{*(n)} = \exp\left[\mathcal{A}_n^*(\rho_c^*, \rho_\ell^*) + \mathcal{B}_n(\rho_c^*,\, \rho_\ell^*)^\top \mathbf{z_t}\right] \,, \tag{21}$$

where $\mathbf{z_t} = \left(\mathbf{x_t}^\top,\, \mathbf{y_t}^\top\right)^\top$, and assuming that the bond has survived to time t, and where the loadings can be obtained as:

$$\begin{aligned} \mathcal{A}_n(\rho_c) &= q_{(n,n,0)} + \rho_c \sum_{i=1}^{n} \left(q_{(i,i-1,0)} - q_{(i,i,0)}\right) \\ \mathcal{B}_n(\rho_c) &= \mathbf{Q}_{(n,n,0)} + \rho_c \sum_{i=1}^{n} \left(\mathbf{Q}_{(i,i-1,0)} - \mathbf{Q}_{(i,i,0)}\right) \,, \end{aligned}$$

and,

$$\begin{aligned}
\mathcal{A}_n^*(\rho_c^*,\, \rho_\ell^*) &= q^*_{(n,n,0)} + \sum_{i=1}^{n} \left((\rho_c^* + \rho_\ell^*) q_{(i,i-1,i-1)} - \rho_c^* q_{(i,i,0)} - \rho_\ell^* q_{(i,i,0)} \right) \\
\mathcal{B}_n^*(\rho_c^*,\, \rho_\ell^*) &= \mathbf{Q}^*_{(n,n,0)} + \sum_{i=1}^{n} \left((\rho_c^* + \rho_\ell^*) \mathbf{Q}_{(i,i-1,i-1)} - \rho_c^* \mathbf{Q}_{(i,i,0)} - \rho_\ell^* \mathbf{Q}_{(i,i,0)} \right) .
\end{aligned}$$

The recursions to obtain these loadings are again detailed in Appendix A.1. An immediate object of interest is the breakeven inflation rate $\mathrm{BEI}(t, h)$, that is the spread between nominal and TIPS yields. Building on our assumed risk-neutral dynamics, we can write:

$$\mathrm{BEI}_t^{(n)} = \frac{1}{n} \left(\mathcal{A}_n^*(\rho_c^*, \rho_\ell^*) - \mathcal{A}_n(\rho_c) + \left[\mathcal{B}_n(\rho_c^*,\, \rho_\ell^*) - \mathcal{B}_n(\rho_c) \right]^\top \mathbf{z_t} \right) \tag{22}$$

$\mathrm{ILS}_t^{(n)}$ is the inflation-linked swap rate, a (virtually) risk-free equivalent of the breakeven inflation rate, such that:

$$\mathrm{ILS}_t^{(n)} = \frac{1}{n} \left(A_n^* - A_n + \left[\mathbf{B_n^*} - \mathbf{B_n} \right]^\top \mathbf{x_t} \right) . \tag{23}$$

The spread between ILS and BEI is therefore given by:

$$\mathrm{ILS}_t^{(n)} - \mathrm{BEI}_t^{(n)} = \frac{1}{n} \left(A_n^* - \mathcal{A}_n^*(\rho_c^*, \rho_\ell^*) + \mathcal{A}_n(\rho_c) - A_n + \left[\begin{pmatrix} \mathbf{B_n^*} - \mathbf{B_n} \\ \mathbf{0}_{k_c + k_\ell} \end{pmatrix} - \mathcal{B}_n(\rho_c^*,\, \rho_\ell^*) + \mathcal{B}_n(\rho_c) \right]^\top \begin{bmatrix} \mathbf{x_t} \\ \mathbf{y_t} \end{bmatrix} \right) . \tag{24}$$

For our risk-neutral dynamics, we can show that the spread between ILS and BEI almost depends entirely on the credit and liquidity factors $\mathbf{y_t}$. This provides us with a clean credit-liquidity decomposition of the term structure of ILS-BEI.

4.5 CDS pricing

We assume that a buyer of protection makes periodic payments from time t to maturity n to protect against default on the underlying nominal sovereign bond. The cash flow payment at time $t+i$ conditional on no default is designated as $s_t^{(n)}$. The present value of the stream of cash flows paid by the protection buyer is:

$$\mathrm{PB}_t^{(n)} = s_t^{(n)} \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{i-1} r_{t+j} \right) \mathbb{1}\left\{ \sum_{j=0}^{i} \delta_{t+j}^{(c)} = 0 \right\} \right]$$

If the sovereign defaults at time $t+i$, we assume that the protection seller pays the buyer a payment of $1-\rho_c$. The present value of the protection sold is:

$$\mathrm{PS}_t^{(n)} = \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{i-1} r_{t+j} \right) (1-\rho_c) \left(\mathbb{1}\left\{ \sum_{j=0}^{i-1} \delta_{t+j}^{(c)} = 0 \right\} - \mathbb{1}\left\{ \sum_{j=0}^{i} \delta_{t+j}^{(c)} = 0 \right\} \right) \right] .$$

No arbitrage pricing requires that the present value of the protection bought is equal to the present value of the protection sold. Equating both legs at inception, and solving for the swap spread yields:

$$s_t^{(n)} = \frac{\sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{i-1} r_{t+j} \right) (1-\rho_c) \left(\mathbb{1}\left\{ \sum_{j=0}^{i-1} \delta_{t+j}^{(c)} = 0 \right\} - \mathbb{1}\left\{ \sum_{j=0}^{i} \delta_{t+j}^{(c)} = 0 \right\} \right) \right]}{\sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{i-1} r_{t+j} \right) \mathbb{1}\left\{ \sum_{j=0}^{i} \delta_{t+j}^{(c)} = 0 \right\} \right]}$$

Using the risk-neutral dynamics of Section 4.2 we can rewrite the previous expression as:

$$s_t^{(n)} = \frac{(1-\rho_c)\sum_{i=1}^{n}\left[\exp\left(q_{(i,i-1,0)} + \mathbf{Q}_{(i,i-1,0)}^{\top}\mathbf{z_t}\right) - \exp\left(q_{(i,i)} + \mathbf{Q}_{(i,i,0)}^{\top}\mathbf{z_t}\right)\right]}{\sum_{i=1}^{n}\exp\left(q_{(i,i,0)} + \mathbf{Q}_{(i,i,0)}^{\top}\mathbf{z_t}\right)} \tag{25}$$

where all loadings can be computed through closed-form recursions presented in Appendix A.1. As usual, even if the model is affine, the swap rate ends up a closed-form non-affine function of the state variables.

5 Model Estimation

5.1 Data and estimation method

We consider different types of observable variables at the monthly frequency. First, the spreads between breakevens and inflation-linked swaps are constructed in two steps. We use Gurkaynak, Sack and Wright (2006) and Gurkaynak, Sack and Wright (2010) for nominal and inflation-protected zero coupon bonds respectively. The BEI variable is the difference between the former and the latter. On the other hand, zero coupon inflation-linked swaps are directly available from Bloomberg. ILS-BEI spreads are calculated for 2-, 5-, and 10-years to maturity. In the empirical application, we fit both the term structure of ILS and ILS-BEI spreads. We also consider the 5-year U.S. sovereign credit default swap spread from Markit.[10] We add inflation data computed as the log-change of the CPI-U index provided by the Bureau of Labor Statistics (BLS). The input sample period for the model estimation

[10]We are well-aware of potential data issues with USD denominated CDS spreads (see for instance Chernov, Schmid and Schneider (2016)). However, using directly USD denominated spreads allows us to avoid the modeling of the exchange rate if we were to consider EUR denominated CDS.

spans July of 2004 to March of 2015.

In our baseline specification, we consider one riskless factor ($k_x^{(r)} = 1$), one inflation factor ($k_x^{(\pi)} = 1$), two credit factors ($k_c = 2$) and one liquidity factor ($k_\ell = 1$). Our estimation only consists in month-by-month curve fitting and is thus only concerned with risk-neutral parameters. Our estimation algorithm relies on non-linear least squares techniques. We first fix a set of parameters θ such that:

$$\theta = \left\{ \kappa_0^{(r)},\, \kappa_r,\, \kappa_0^{(\pi)},\, \kappa_\pi,\, \boldsymbol{\mu}^{\mathbb{Q}},\, \boldsymbol{\Phi}^{\mathbb{Q}},\, \boldsymbol{\Sigma}^{\mathbb{Q}},\, \boldsymbol{\alpha}^{\mathbb{Q}},\, \boldsymbol{\beta}^{\mathbb{Q}},\, \boldsymbol{c}^{\mathbb{Q}},\, \boldsymbol{\gamma}_c,\, \gamma_\ell,\, \rho_c,\, \rho_c^*,\, \rho_\ell^* \right\} .$$

Then, for each month, gather all our 8 observable variables (3 ILS, 3 ILS-BEI spreads, inflation and CDS spread) in a vector and estimate the factors by minimizing the weighted sum of squared residuals using Equations (23), (24) and (25).[11] Note that because the CDS pricing formula is non-linear in the factors, we use numerical optimization methods. Since our main interest is to know how much of the ILS-BEI spreads can be explained by credit factors, we subordinate the factor estimation for each date: we first minimize the squared residuals in all factors but liquidity, and then perform a second optimization to find the best liquidity factor. Note that since there is no default or liquidity event throughout the sample, we fix $\delta_t^{(c)} = \delta_t(\ell) = 0$ for all dates.

For parsimony and identification purposes, we impose that $\kappa_r = \kappa_\pi = 1$, $\boldsymbol{\Sigma}^{\mathbb{Q}}$ is a diagonal matrix, $\boldsymbol{\beta}^{\mathbb{Q}}$ is lower triangular, $\boldsymbol{c}^{\mathbb{Q}} = \mathbf{1}_{k_r + k_\ell}$. We end up with a total of 25 parameters that we estimate by minimizing the sum of all dates squared residuals.

[11]For the weights, we normalize all sum of squared residuals by the sample variance of the respective observable variable. We then multiply the sum of squared CDS residuals by 10^4, the sum of squared ILS and ILS-BEI residuals by 10^2. These values are sufficient to impose that the CDS spread is perfectly fitted and that the term structures are fitted as much as possible.

5.2 Estimation results

Table 8 presents the parameter estimates of the model. All parameters are highly significant with the exception of $\kappa_0^{(r)}$. The feedback from the liquidity to credit factors is also significant, such that higher liquidity issues result in higher (risk-neutral) probability of sovereign default. The third panel of Table 8 presents the estimated recovery rates for both nominal and real bonds. We see that our assumption of a differential recovery is supported by the data, and nominal bonds and TIPS show recovery rates of 77% and 69% respectively. Liquidity issues are on average more severe, since the recovery rate associated with liquidity is only 48%. Note that this value is very close to the 50% recovery usually assumed in reduced-form term structure models.

[Insert Table 8 about here.]

The three latent factors, $\mathbf{y}_t$, filtered by our model are plotted in Figure 3. The first two factors jointly determine the likelihood of default while the thrid factor is the liquidity factor. The credit factors allow us to perfectly track the 5y CDS spread. As for the CDS data, most of the movements can be observed after the outbreak of the financial crisis. In contrast, the liquidity factor increases gradually before the crisis and spikes up in 2008. Movements in the liquidity factor are much less pronounced post crisis.

[Insert Figure 3 about here.]

Next, we plot the fit of the model on Figures 4-6 and the corresponding R-square values are presented in Table 9. Our model produces a perfect fit of the CDS spread mainly by construction (see Figure 5). For the ILS in Figure 4, the model does a tremendous job in capturing the term structure. We obtain R-squares ranging from 91% to 95%. The intuition behind this result is straightforward: because our second riskless factors is used to fit the inflation rate, the first factor is used solely to fit the term structure of inflation swaps. As usual in the term structure literature, one factor is usually sufficient to capture the bulk of fluctuations of the different maturities. For the risky equivalents, the model does a reasonable job in capturing the ILS-BEI spreads (see Figure 6). Because the credit factors are imposed to fit the CDS and the liquidity factor is always positive, the model struggles a bit more for the long maturity. We obtain R-squares of 96% and 88% for the 2y and 5y maturities respectively, while the fitted values for the 10y explain only 34% of the variance of the ILS-BEI spreads. Further inspection on the time-series fir shows that most of this lack of power comes from a poor fit during the pre-crisis period. This is probably linked to our requirement to fit the CDS data perfectly. Indeed, there is little movement in the two default factors during this period, and it is hard for the model to explain the entire term structure with the liquidity factor only.

[Insert Figures 4-6 and Table 9 about here.]

Overall, the performance of the model is satisfactory. It achieves good fit along four dimensions of data: CDS, inflation, ILS, and BEI. As a result, the estimated model can explain significant variations in the ILS-BEI spread, especially during and following the financial crisis. Earlier in the paper, panel regression results in Subsection 3.3 show that the U.S. CDS spread is positive and significant in explaining ILS-BEI spreads across tenors in our sample spanning 2008 to 2015, in the presence of controls including liquidity. With the aid of the model, we can further decompose the ILS-BEI spread into its credit and liquidity components to examine the dynamic contribution of the credit factors during the after the crisis.

5.3 Decomposition of ILS-BEI spreads

To understand the relative importance of default risk in driving a wedge between ILS and BEI, we fit the ILS-BEI curves at maturities of 2-, 5-, and 10-years by employing only the credit risk factors. Then, we contrast the fitted curves with only the credit risk factors against the fitted ILS-BEI curves in Figure 6 (red, dashed lines). The results are plotted in Figure 7.

[Insert Figures 7-8 about here.]

Three observations can be discerned from Figure 7. First, the credit component of the ILS-BEI spread contributes nothing to the overall fit of the curves prior to 2008. This is, again, due to the minimal premium on U.S. CDS contracts before the financial crisis as seen in Figure 5. As a results, the liquidity factor explains the entirety of the ILS-BEI spreads before the crisis across maturities. Second, in the middle of the crisis around September of 2008, the peak of the ILS-BEI spread is mostly driven by the liqudity factor. Going back to Figure 6, the liquidity factor is the dominate component that drives good fit of the model-implied ILS-BEI curves against those in the data during the crisis. Third, the credit component of the ILS-BEI spread is the dominate factor in capturing the variability of the ILS-BEI curves in the data in the post-crisis period. Figure 7 shows this is especially so at long maturities as evidenced by the 10-year ILS-BEI curves with and without the liquidity component.

To put the decomposition in more precise terms, Figure 8 plots the percent contribution of the credit component relative to the whole fitted ILS-BEI curves at the same three maturities. The same three implcations from Figure 7 can be visualized here. Namely, the credit component contributes little prior to the 2008 crisis, its contribution is overshadowed by the liquidity component at the height of the crisis, and lastly, it is the main explanatory variable of the ILS-BEI spread in the post-crisis sample up to 2015.

Our structural model validates the results of our panel regressions in Subsection 3.3. In particular, over the full sample between 2008 and 2015, the U.S. CDS spread driven by the credit factors has positive and significant explanatory power of the ILS-BEI spreads after controlling for liquidity. Moreover, contrasting Column (7) in Tables 5 and 6, we see that the explanatory power of the credit component (proxied by the CDS spread) is statistically weaker (t-statistic of 1.84) during the crisis period and much stronger (t-statistic of 4.46) in the post-crisis period. Whereas the opposite is true for the liquidity factor (proxied by the OTR Difference) with t-statistic of 3.97 in Table 5 and t-statistic of 0.52 in Table 6. This is consistent with the decomposition of the fitted ILS-BEI curves shown in Figures 7 and 8.

5.4 Panel Regressions with Credit Factors

As the last exercise to tie our estimated credit factors to the data, we perform panel regressions similar to those used in Subsection 3.3, where ILS-BEI spreads across tenors are projected onto the two credit factors and the liquidity factor constructed here. As before, we estimate the coefficient loading for the sample between 2008 and 2015, as well as the two subsamples covering the crisis period and the post-crisis period. The results are shown in Table 10. Column (1) is for the entire sample, Column (2) is for the post-crisis sample, and Column (3) is for during the crisis. Following the previous empirical specification, we control for lagged ILS-BEI spreads.

[Insert Table 10 about here.]

In all three sample periods, while controlling for liquidity, the second credit factor is always positive and significant in explaining the ILS-BEI spread, consistent with the idea that higher default risk drives a wedge between nominal and real Treasury yields that is beyond inflation risk. Moreover, we notice the first credit factor is positive and significant in Columns (1) and (2) but not in Column (3), which is during the financial crisis. This is consistent with the observation that, in the estimated model, the liquidity factor is the primary driver of the ILS-BEI dynamics during the crisis, whereas in the post-crisis sample, the importance of the credit factors is accentuated. Overall, the panel regressions with credit and liquidity factors reconfirms our results using CDS spreads.

6 Conclusion

In this paper, we explore the relative pricing of nominal and real U.S. sovereign securities in the presence of credit risk. In fact, we argue that while most of the previous studies attribute the mispricing of TIPS to liquidity factors or slowly moving capital, credit risk can also represent a significant driver of deviations oftentimes interpreted as violations of no-arbitrage. Our study shows that in the presence of credit risk, the spreads between inflation-linked swaps and breakeven inflation rates can reflect differences in propensity of the sovereign to reimburse nominal and real bonds in case of default, that is a difference in recovery rates. Our first empirical approach that U.S. CDS spreads are positively correlated with the ILS-BEI spreads after the financial crisis, even controlling for liquidity and potential alternative explanations. We then conduct more formal empirical analysis through an intensity-based affine asset pricing model. We show that credit risk factors extracted from the CDS are able to explain 20% of the ILS-BEI yield curve and up to 30% for some maturities. Our model estimates confirms the existence of a lower recovery rate for TIPS than for nominal bonds by about 13 percentage points.

References

Abrahams, Michael, Tobias Adrian, Richard Crump and Emanuel Moench. 2016. "Decomposing Real and Nominal Yield Curve." *Journal of Monetary Economics (forthcoming)* .

Abrahams, Michael, Tobias Adrian, Richard Crump, Emanuel Moench and Rui Yu. 2016. "Decomposing Real and Nominal Yield Curves." *Journal of Monetary Economics* 84:182–200.

Andreasen, Martin, Jens H.E. Christensen and Simon Ridell. 2017. The TIPS Liquidity Premium. Technical report FRBSF.

Ang, Andrew and Francis A. Longstaff. 2013. "Systemic Sovereign Credit Risk: Lessons from the U.S. and Europe." *Journal of Monetary Economics* 60:493–510.

Ang, Andrew, Geert Bekaert and Min Wei. 2008. "The Term Structure of Real Rates and Expected Inflation." *Journal of Finance* 63(2):797–849.

Arora, Navneet, Priyank Gandhi and Francis A. Longstaff. 2012. "Counterparty Credit Risk and the Credit Default Swap Market." *Journal of Financial Economics* 103:208–293.

Augustin, Patrick, Mike Chernov and Dongho Song. 2018. Sovereign credit risk and exchange rates: Evidence from CDS quanto spreads. Technical report McGill University.

Buraschi, Adrea and Alexei Jiltsov. 2005. "Inflation Risk Premia and the Expectations Hypothesis." *Journal of Financial Economics* 75:429–490.

Camba-Mendez, Gonzalo and Thomas Werner. 2017. The inflation risk premium in the post-Lehman period. Technical report European Central Bank.

Campbell, John Y., Adi Sunderam and Luis M. Viceira. 2016. "Inflation Bets or Deflation Hedges? The Changing Risk of Nominal Bonds." *Critical Finance Review* .

Campbell, John Y., Robert J. Shiller and Luis M. Viceira. 2009. "Understanding Inflation-Indexed Bond Markets." *Brookings Papers on Economic Activity* 40(1 (Spring)):79–138.

Chernov, Mikhail, Lukas Schmid and Andres Schneider. 2016. "A macrofinance view of US Sovereign CDS premiums." unpublished manuscript, University of California, Los Angeles.

Chernov, Mikhail and Philippe Mueller. 2012. "The Term Structure of Inflation Expectations." *Journal of Financial Economics* 1(106):367–394.

Christensen, J. H., J. A. Lopez and G. D. Rudebusch. 2012. "Extracting Deflation Probability Forecasts from Treasury Yields." *International Journal of Central Banking* 8(4):21–60.

Christensen, Jens H.E. and James Gillan. 2018. Does Quantitative Easing Affect Market Liquidity? Technical report FRBSF.

Christensen, Jens H.E., Jose A. Lopez and Glenn D. Rudebusch. 2014. Can spanned term structure factors drive stochastic yield volatility? Working paper series Federal Reserve Bank of San Francisco.

D'Amico, Stefania, Don H. Kim and Min Wei. 2018. "Tips from TIPS: The Information Content of Treasury Inflation-Protected Security Prices." *Journal of Financial and Quantitative Analysis* 53(1):395–436.

Driessen, Joost, Theo Nijman and Zorka Simon. 2017. The Missing Piece of the Puzzle: Liquidity Premiums in Inflation-Indexed Markets. Technical report Tilburg University.

Dubecq, Simon, Alain Monfort, Jean-Paul Renne and Guillaume Roussellet. 2016. "Credit and liquidity in interbank rates: A quadratic approach." *Journal of Banking & Finance* .

Duffie, Darrell. 2010. "Asset price dynamics with slow-moving capital." *Journal of Finance* 65:1238–1268.

Duffie, Darrell and Kenneth J. Singleton. 1999. "Modeling term structures of defaultable bond yields." *Review of Financial Studies* 12:687–720.

Ericsson, Jan and Olivier Renault. 2006. "Liquidity and Credit Risk." *The Journal of Finance* 61(5):2219–2250.

Filipovic, Damir and Anders B. Trolle. 2013. "The term structure of interbank risk." *Journal of Financial Economics* 109(3):707–733.

Fleckenstein, M., F. A. Longstaff and H. Lustig. 2017. "Deflation Risk." 30(1):2719–2760.

Fleckenstein, Matthias, Francis A. Longstaff and Hanno Lustig. 2014. "The TIPS Treasury Bond Puzzle." *Journal of Finance* 69:2151–2197.

Grishchenko, Olesya and Jing-Zhi Huang. 2012. "The Inflation Risk Premium: Evidence from the TIPS Market." unpublished manuscript, Federal Reserve Board.

Gurkaynak, Refet S., Brian Sack and Jonathan H. Wright. 2006. The U.S. Treasury yield curve: 1961 to the present. Finance and Economics Discussion Series 2006-28 Board of Governors of the Federal Reserve System (U.S.).

Gurkaynak, Refet S., Brian Sack and Jonathan H. Wright. 2010. "The TIPS Yield Curve and Inflation Compensation." *American Economic Journal: Macroeconomics* 2(1):70–92.

Haubrich, Joseph, George Pennacchi and Peter Ritchken. 2012. "Inflation Expectations, Real Rates, and Risk Premia: Evidence from Inflation Swaps." *Review of Financial Studies* 25(5).

Hilscher, Jens, Alon Raviv and Ricardo Reis. 2014. "Inflating Away the Public Debt? An Empirical Assessment." unpublished manuscript, NBER.

Hordahl, Peter and Oreste Tristani. 2012. "Inflation Risk Premia in the Term Structure of Interest Rates." *Journal of the European Economic Association* 10(3):634–657.

Hu, Grace Xing, Jun Pan and Jiang Wang. 2013. "Noise as information for illiquidity." *Journal of Finance* 68:2341–2382.

Mitchell, Mark, Lasse Heje Pedersen and Todd Pulvino. 2007. "Slow moving capital." *American Economic Review, Papers and Proceedings* 97:215–220.

Moench, Emanuel and Andreea Vladu. 2018. A Fine Model For Nominal and Real Bonds. Technical report Bundesbank.

Monfort, Alain, Fulvio Pegoraro, Jean-Paul Renne and Guillaume Roussellet. 2017*a*. "Affine Modeling of Credit Risk, Pricing of Credit Events and Contagion." unpublished manuscript, McGill University.

Monfort, Alain, Fulvio Pegoraro, Jean-Paul Renne and Guillaume Roussellet. 2017*b*. "Staying at zero with affine processes: An application to term structure modelling." *Journal of Econometrics* 201(2):348 – 366.

Monfort, Alain and Jean-Paul Renne. 2013. "Default, liquidity and crises: an econometric framework." *Journal of Financial Econometrics* 11(2):221–262.

Nagel, Stefan. 2012. "Evaporating liquidity." *Review of Financial Studies* 25:2005–2039.

Pflueger, Carolin and Luis Viceira. 2016. "Return Predictability in the Treasury Market: Real Rates, Inflation, and Liquidity." *Handbook of Fixed-Income Security* John Wiley and Sons (ed P. Veronesi).

Roussellet, Guillaume. 2017. Affine Term Structure Modeling and Macroeconomic Risks at the Zero Lower Bound. Technical report McGill University.

Tables

Table 3: **Summary Statistics**

Table 3 provides summary statistics for the variables used in the regression analysis. Panel A includes the full sample period from January 1, 2008 through September 30, 2015. Panel B is the post crisis subsample from January 1, 2010 through September 30, 2015. $ILS - BEI$ is the difference in the 5-year inflation swap rate and the 5-year breakeven inflation rate (Treasury-TIPS). Both $Tsy\ ZC\ Yield$ and $TIPS\ ZC\ Yield$ are for the 5-year maturity. 5-year $US\ CDS$ spreads are denominated in EUR. $LIBOR - OIS$ is the difference in the London Inter-bank Offered Rate and the overnight indexed swap rate. $HPWNoise$ follows Hu, Pan and Wang (2013). VIX denotes the CBOE Volatility Index. $RepoFails$ is the total of weekly failed deliveries and receipts.

Panel A			**Full Sample**		
	Mean	SD	Min	Max	N
ILS-BEI (bps)	36	30	-1	210	1908
Infl Swap Rate	2.04	0.49	-0.57	3.31	1985
Tsy ZC Yield	1.74	0.70	0.59	3.76	1910
TIPS ZC Yield	0.06	1.04	-1.72	3.88	1910
US CDS (bps)	33	16	6	100	1988
LIBOR-OIS	0.35	0.43	0.06	3.64	1989
HPW Noise	3.52	3.54	0.72	20.47	1999
VIX	22.01	10.50	10.32	80.86	1919
Panel B			**Post Crisis**		
	Mean	SD	Min	Max	N
ILS-BEI (bps)	23	10	-1	59	1409
Infl Swap Rate	2.09	0.29	1.24	2.71	1465
Tsy ZC Yield	1.45	0.51	0.59	2.79	1409
TIPS ZC Yield	-0.41	0.62	-1.72	0.83	1409
US CDS (bps)	34	12	14	63	1465
LIBOR-OIS	0.19	0.09	0.06	0.50	1466
HPW Noise	1.99	0.73	0.72	4.58	1476
VIX	18.42	6.15	10.32	48.00	1414

Table 4: **ILS-BEI - January 2008 to September 2015**

Table 4 shows the results from a panel regression of ILS-BEI on US CDS spreads and various controls using daily observations. The sample period is from January 2008 to October 2015. $ILS - BEI$ is the difference in the inflation swap rate and the breakeven inflation rate (Treasury-TIPS) for 2-, 3-. 5-, 7-, and 10-year tenors. $US\ CDS$ spreads are for the 5-year tenor. $HPWNoise$ follows Hu, Pan and Wang (2013). $LIBOR - OIS$ is the difference in the London Inter-bank Offered Rate and the overnight indexed swap rate. $OTR\ Difference$ is the difference in 10-year Treasury par yield from Gurkaynak, Sack and Wright (2006) less the on-the-run 10-year Treasury yield from Bloomberg. VIX denotes the CBOE Volatility Index. Standard errors are reported in parentheses.

Dep Var: ILS-BEI Spread	(1)	(2)	(3)	(4)	(5)	(6)	(7)
US CDS	0.196***	0.195***	0.205***	0.186***	0.218***	0.200***	0.200***
	(0.049)	(0.049)	(0.050)	(0.049)	(0.051)	(0.051)	(0.051)
ILS-BEI$_{t-1}$	0.822***	0.822***	0.822***	0.822***	0.822***	0.822***	0.816***
	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)
HPW Noise		0.111				0.348	0.346
		(0.283)				(0.287)	(0.286)
LIBOR-OIS			−3.303*			−2.079	−2.076
			(1.826)			(1.899)	(1.895)
OTR Difference				−24.81***		−24.29***	−24.31***
				(4.816)		(4.949)	(4.940)
VIX					−0.065*	−0.025	−0.024
					(0.041)	(0.041)	(0.041)
Week	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Tenor	No	No	No	No	No	No	Yes
Observations	9147	9147	9147	9147	9127	9127	9127
R^2	0.955	0.955	0.955	0.955	0.955	0.955	0.955

*,**,*** represent statistical significance at the 10%, 5%, and 1% critical threshold, respectively.

Table 5: **ILS-BEI - January 2008 to December 2009**

Table 5 shows the results from a panel regression of ILS-BEI on US CDS spreads and various controls using daily observations. The sample period is from January 2008 through December 2009. $ILS - BEI$ is the difference in the inflation swap rate and the breakeven inflation rate (Treasury-TIPS) for 2-, 3-. 5-, 7-, and 10-year tenors. $US\,CDS$ spreads are for the 5-year tenor. $HPW Noise$ follows Hu, Pan and Wang (2013). $LIBOR-OIS$ is the difference in the London Inter-bank Offered Rate and the overnight indexed swap rate. $OTR\ Difference$ is the difference in the on-the-run 10-year U.S. Treasury and the off-the-run 9-year U.S. Treasury from the Bloomberg on/off-the-run U.S. Treasury curve. VIX denotes the CBOE Volatility Index. Standard errors are reported in parentheses.

Dep Var: ILS-BEI Spread	(1)	(2)	(3)	(4)	(5)	(6)	(7)
US CDS	0.213*	0.209*	0.232*	0.199*	0.258**	0.220*	0.221*
	(0.120)	(0.121)	(0.122)	(0.120)	(0.124)	(0.125)	(0.120)
ILS-BEI$_{t-1}$	0.795***	0.795***	0.795***	0.795***	0.796***	0.796***	0.696***
	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.011)	(0.013)
HPW Noise		0.364				0.837	0.731
		(0.631)				(0.645)	(0.621)
LIBOR-OIS			−3.374			−0.890	−0.913
			(3.501)			(3.372)	(3.370)
OTR Difference				−44.46***		−44.03***	−44.29***
				(11.19)		(11.58)	(11.15)
VIX					−0.151	−0.082	−0.063
					(0.098)	(0.106)	(0.102)
Week	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Tenor	No	No	No	No	No	No	Yes
Observations	2387	2387	2387	2387	2387	2387	2387
R^2	0.915	0.915	0.915	0.915	0.915	0.915	0.915

*,**,*** represent statistical significance at the 10%, 5%, and 1% critical threshold, respectively.

Table 6: **ILS-BEI - January 2010 to September 2015**

Table 6 shows the results from a panel regression of ILS-BEI on US CDS spreads and various controls using daily observations. The sample period is from January 2010 to October 2015. $ILS - BEI$ is the difference in the inflation swap rate and the breakeven inflation rate (Treasury-TIPS) for 2-, 3-. 5-, 7-, and 10-year tenors. $US\ CDS$ spreads are for the 5-year tenor. $HPWNoise$ follows Hu, Pan and Wang (2013). $LIBOR - OIS$ is the difference in the London Inter-bank Offered Rate and the overnight indexed swap rate. $OTR\ Difference$ is the difference in 10-year Treasury par yield from Gurkaynak, Sack and Wright (2006) less the on-the-run 10-year Treasury yield from Bloomberg. VIX denotes the CBOE Volatility Index. Standard errors are reported in parentheses.

Dep Var: ILS-BEI Spread	(1)	(2)	(3)	(4)	(5)	(6)	(7)
US CDS	0.171***	0.170***	0.171***	0.172***	0.164***	0.164***	0.165***
	(0.035)	(0.035)	(0.035)	(0.035)	(0.037)	(0.037)	(0.037)
ILS-BEI$_{t-1}$	0.905***	0.905***	0.905***	0.905***	0.904***	0.904***	0.888***
	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.006)
HPW Noise		−0.396*				−0.393*	−0.388*
		(0.230)				(0.231)	(0.230)
LIBOR-OIS			−4.591			−4.156	−3.924
			(7.704)			(7.756)	(7.725)
OTR Difference				1.563		1.985	1.922
				(3.667)		(3.737)	(3.722)
VIX					0.022	0.020	0.021
					(0.028)	(0.028)	(0.028)
Week	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Tenor	No	No	No	No	No	No	Yes
Observations	6755	6755	6755	6755	6735	6735	6735
R^2	0.927	0.927	0.927	0.927	0.927	0.926	0.927

*,**,*** represent statistical significance at the 10%, 5%, and 1% critical threshold, respectively.

Table 7: **Liquidity Premia and CDS**

Table 7 presents results of an analysis of liquidity premia. In Panel A, we present results from regressions

$$\begin{aligned} BEI_t &= a_1 + a_2 OTR_t + a_3 VOL_t + a_4 ILS-BEI_t + a_5 CPI_t^e + a_6 CFNAI_t + \epsilon_{1t} \\ BEI_t &= b_1 + b_2 OTR_t + b_3 VOL_t + b_4 ILS-BEI_t + b_5 CPI_t^e + b_6 CFNAI_t + b_7 US\ CDS_t + \epsilon_{2t}, \end{aligned}$$

where the dependent variable is breakeven inflation, and the independent variables are OTR, the on-the-run 10-Year Treasury Spread, VOL, the log ratio of volume in the TIPS market to the nominal Treasury market, $ILS-BEI$, the inflation swap-adjusted BEI, CPI^e, the median forecast of 10-year CPI inflation from the Survey of Professional Forecasters, $CFNAI$, the Chicago Fed National Activity Index, and $US\ CDS$, the 5-year credit default swap spread for U.S. Treasury securities. In Panel B, the estimated liquidity premium from Panel A is regressed on the U.S. CDS spread. The liquidity premium is measured as

$$\hat{L}_t = -\left(\hat{a}_2 OTR_t + \hat{a}_3 VOL_t + \hat{a}_4 ILS-BEI_t\right).$$

Newey-West standard errors are reported in parentheses.

Panel A: Breakeven Inflation

Dep Var: BEI	(1)	(2)
OTR	-1.143^{**}	-1.218^{***}
	(0.153)	(0.155)
VOL	-0.438^{***}	-0.472^{***}
	(0.058)	(0.060)
$ILS-BEI$	-1.284^{***}	-1.158^{***}
	(0.103)	(0.109)
CPI^e	0.960^{***}	0.997^{***}
	(0.109)	(0.106)
$CFNAI$	0.027^{*}	0.031^{**}
	(0.014)	(0.014)
$US\ CDS$		-0.214^{***}
		(0.075)
R^2	0.651	0.656

Panel B: Liquidity Premium

Dep Var: $\hat{L}$	
$US\ CDS$	0.608^{***}
	(0.126)
R^2	0.086

Notes: *,**,*** represent statistical significance at the 10%, 5%, and 1% critical threshold, respectively.

Table 8: Model parameters

	$\boldsymbol{\mu}^{\mathbb{Q}}$	$\boldsymbol{\Phi}^{\mathbb{Q}}$		$\boldsymbol{\Sigma}^{\mathbb{Q}}$
$x_t^{(r)}$	0.208729	0.900778	0.018697	0.140503
	(0.004679)	(0.002899)	(0.002346)	(0.003519)
$x_t^{(\pi)}$	−0.11378	0.274697	0.783866	0.134388
	(0.005167)	(0.005581)	(0.004235)	(0.004247)

	$\boldsymbol{\alpha}^{\mathbb{Q}}$	$\boldsymbol{\beta}^{\mathbb{Q}}$			$\boldsymbol{\gamma}$
$y_{1,t}^{(c)}$	0.000727	1.094302	0	0	$1.27 \cdot 10^{-10}$
	$(2.67 \cdot 10^{-6})$	(0.00394)	–	–	$(1.87 \cdot 10^{-12})$
$y_{2,t}^{(c)}$	0.000805	0.342226	0.837805	0	$2.65 \cdot 10^{-6}$
	$(2.79 \cdot 10^{-6})$	(0.00162)	(0.004407)	–	$(4.17 \cdot 10^{-8})$
$y_t^{(\ell)}$	0.00108	0.351114	0.362457	0.956864	$9.36 \cdot 10^{-10}$
	$(6.08 \cdot 10^{-6})$	(0.001375)	(0.001791)	(0.004784)	$(1.62 \cdot 10^{-11})$

ρ_c	ρ_c^*	ρ_ℓ^*	$\kappa_0^{(r)}$	$\kappa_0^{(\pi)}$
0.76618	0.685344	0.480105	$-8.2 \cdot 10^{-6}$	$-1.3 \cdot 10^{-5}$
(0.000714)	(0.000702)	(0.001158)	$(4.21 \cdot 10^{-6})$	$(4.6 \cdot 10^{-6})$

Notes: This table presents the parameter estimates from the model of Section 4. All parameters associated with the vector autoregressive gamma process are imposed positive. Standard deviations are obtained through the cross-product approximation. We compute numerical derivatives with the symmetric difference quotient with a step size of 10^{-5}.

Table 9: R-squared values from term structure model

	CDS	Inflation	ILS			ILS-BEI spreads		
Maturity	5y		2y	5y	10y	2y	5y	10y
R^2	1	0.893	0.949	0.989	0.912	0.957	0.877	0.342

Table 10: **ILS-BEI - Model Credit Factors**

Table 10 shows the results from a panel regression of monthly ILS-BEI on model-generated credit and liquidity factors. Column (1) includes the sample period from January 2008 through March 2015. Column (2) includes the sample period from January 2010 through March 2015. Column (3) includes the sample period from January 2008 through December 2009. $ILS - BEI$ is the difference in the inflation swap rate and the breakeven inflation rate (Treasury-TIPS) for 2-, 3-. 5-, 7-, and 10-year tenors.

Dep Var: ILS-BEI Spread	(1)	(2)	(3)
ILS-BEI$_{t-1}$	0.426***	0.377***	0.445***
	(0.030)	(0.044)	(0.057)
Credit Factor(1)	1.057***	1.524***	1.039
	(0.338)	(0.519)	(0.660)
Credit Factor(2)	0.002***	0.002***	0.002**
	(0.000)	(0.000)	(0.001)
Liquidity Factor ($\times 10^{-4}$)	0.148***	0.190***	0.148***
	(0.008)	(0.016)	(0.016)
Constant	−0.191	−2.626	−1.390
	(1.343)	(2.329)	(3.825)
R^2	0.878	0.617	0.820
Observations	405	285	115

*,**,*** represent statistical significance at the 10%, 5%, and 1% critical threshold, respectively.

Figures

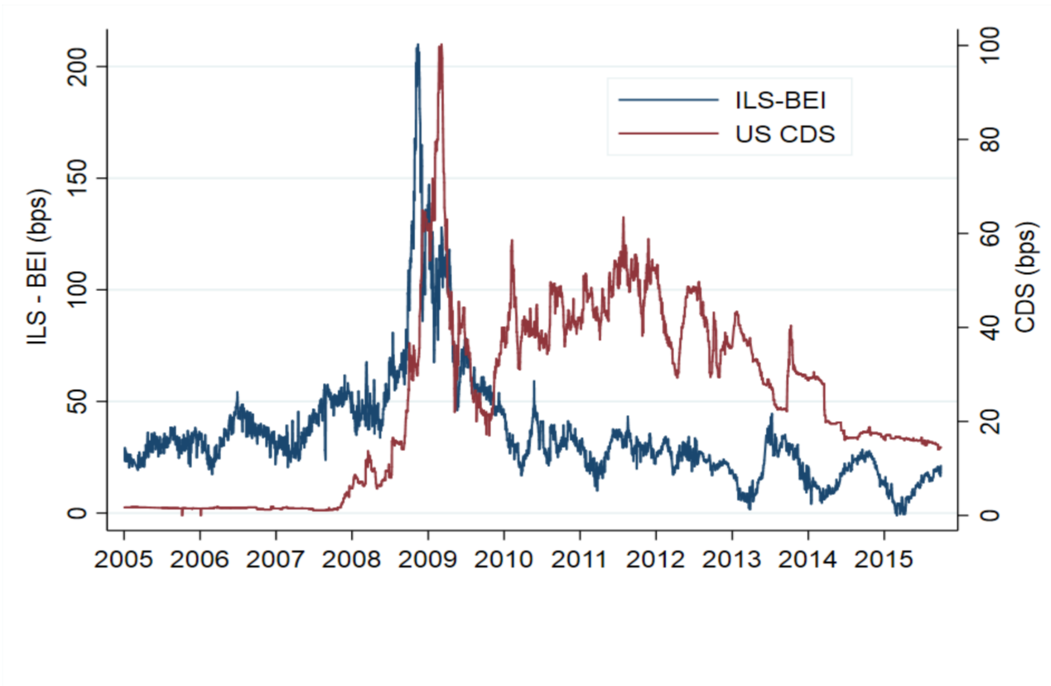


Figure 1: Figure 1 shows the time series of differential in ILS and BEI rates (LHS) and US CDS spreads (RHS)

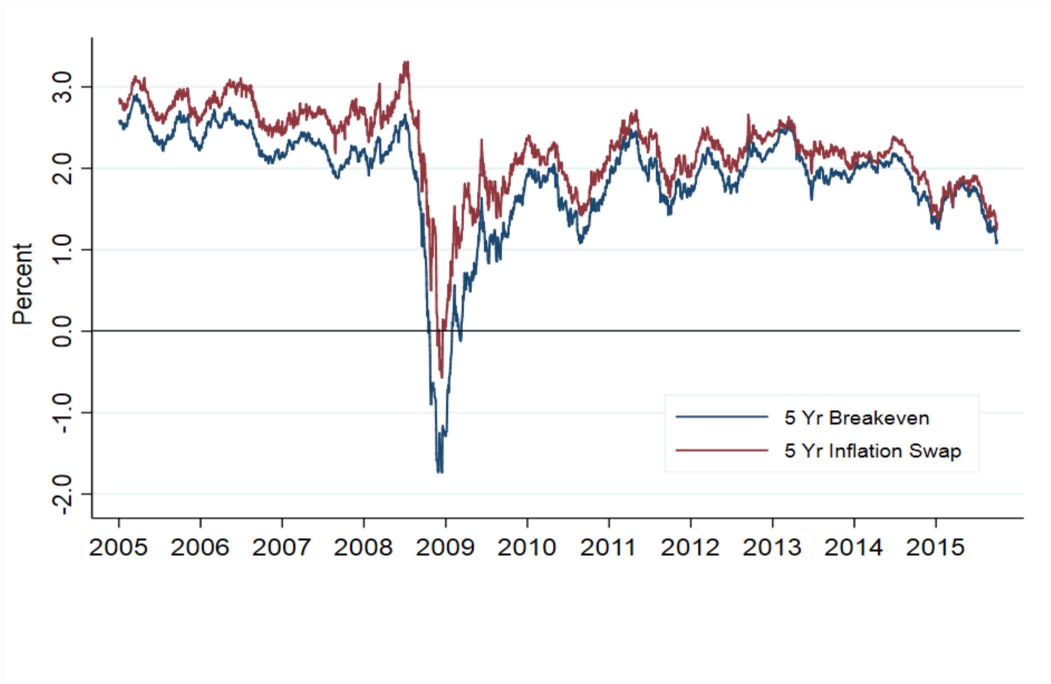
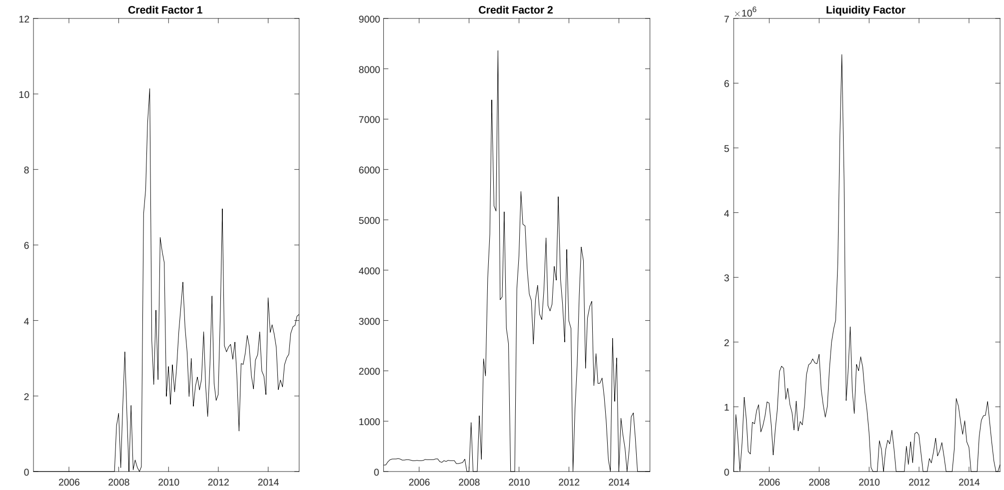


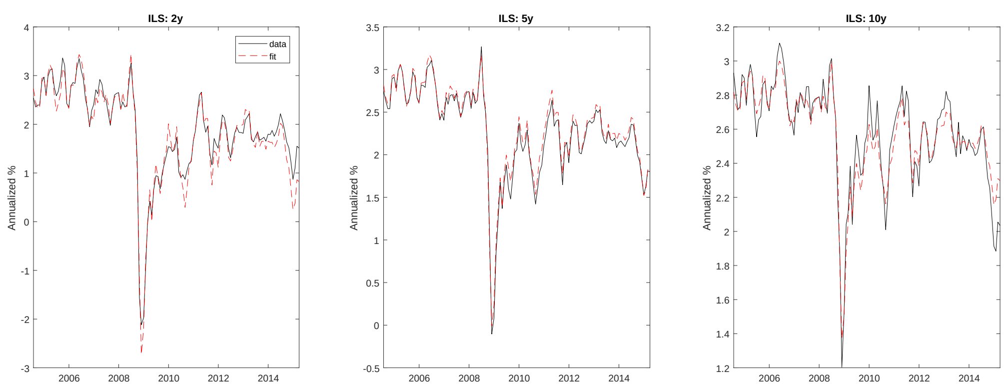
Figure 2: Figure 2 displays the time series of 5-year inflation swap rates (ILS) and breakeven inflation rates (BEI) as denoted by the difference in the 5-year zero coupon Treasury yield and the 5-year zero coupon TIPS yield

Figure 3: Factors extracted from the NLS estimation in the term structure model



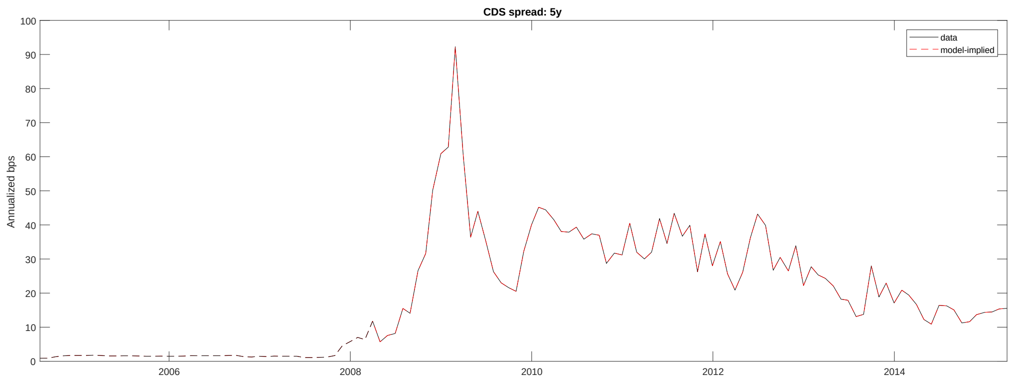
Notes: Factors 1 and 2 correspond to credit risk factors while factor 3 is the liquidity factor.

Figure 4: Observable inflation linked swaps and model-implied values



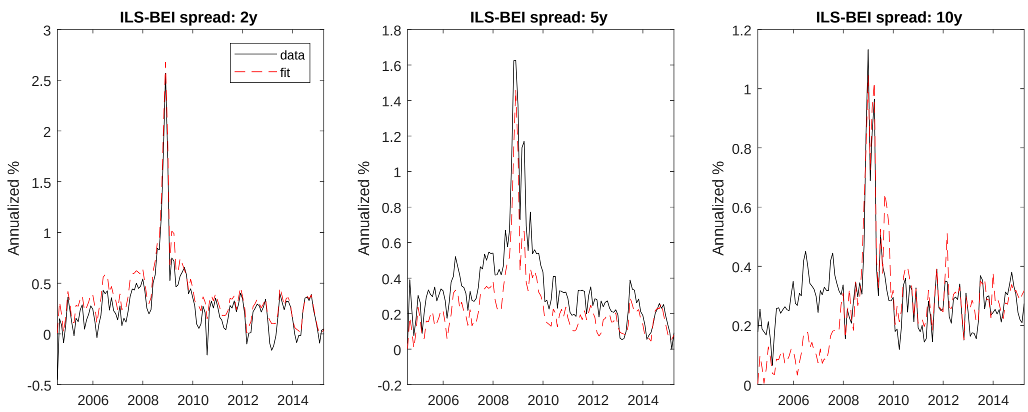
Notes: Black solid lines represent the observable variables while the red dashed lines are model-implied. All values are in annualized percentage points.

Figure 5: Observable CDS spread and model-implied values



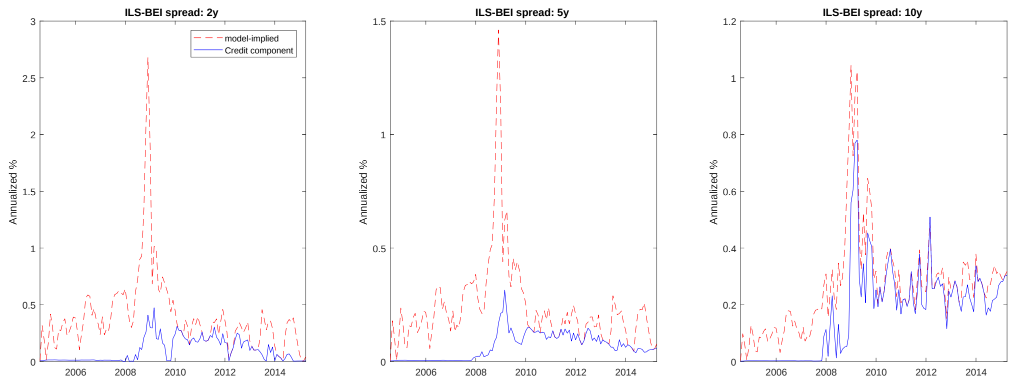
Notes: Black solid lines represent the observable variable while the red dashed line is model-implied. All values are in bps.

Figure 6: Observable ILS-BEI spreads and model-implied values



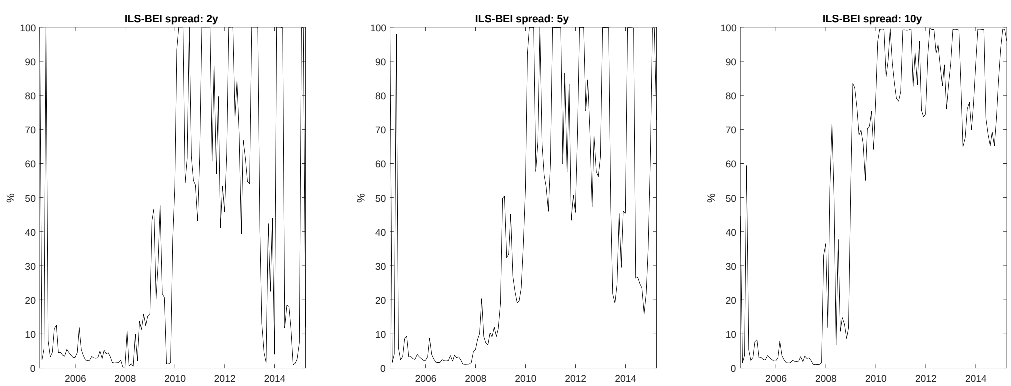
Notes: Black solid lines represent the observable variables while the red dashed lines are model-implied. All values are in annualized percentage points.

Figure 7: Decomposition of ILS-BEI spreads



Notes: Red dashed lines represent the fitted variables while the blue solid lines represent the component explained by the credit factors. The remaining of the spread is explained by liquidity. All values are in annualized percentage points.

Figure 8: Proportion of ILS-BEI spreads explained by credit



Notes: Black solid lines represent the proportion of ILS-BEI spreads explained by the credit factors. The proportion is in ratio of the values fitted by the model. The remaining of the spread is explained by liquidity.

A Appendix

A.1 Recursive formulas

We detail in this Appendix the computations necessary to obtain the recursive pricing formulas.

Riskless yield curves: Remember our distributional assumptions:

$$\begin{aligned} r_t &= \kappa_0^{(r)} + \boldsymbol{\kappa_r}^\top \mathbf{x_t^{(r)}} & (26)\\ \pi_t &= \kappa_0^{(\pi)} + \boldsymbol{\kappa_\pi}^\top \mathbf{x_t}^{(\pi)} & (27)\\ \mathbf{x_t} &= \boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Phi}^{\mathbb{Q}} \mathbf{x_{t-1}} + \sqrt{\boldsymbol{\Sigma}^{\mathbb{Q}}} \varepsilon_{\mathbf{t}}^{\mathbb{Q}}, & (28) \end{aligned}$$

where $\varepsilon_t^{\mathbb{Q}} \overset{i.i.d}{\sim} \mathcal{N}(\mathbf{0}, \mathbf{I_{k_x}})$. Let us assume that

$$D_t^{(n)} = \exp\left(A_n + \mathbf{B_n}^\top \mathbf{x_t}\right) \quad \text{and} \quad D_t^{*(n)} = \exp\left(A_n^* + \mathbf{B_n^*}^\top \mathbf{x_t}\right). \tag{29}$$

By no-arbitrage, we have:

$$\begin{aligned} D_t^{(n)} &= \mathbb{E}_t^{\mathbb{Q}}\left(e^{-r_t} D_{t+1}^{(n-1)}\right)\\ &= \exp\left(-\kappa_0^{(r)} - \boldsymbol{\kappa}_r{}^\top \mathbf{x_t^{(r)}}\right) \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(A_{n-1} + \mathbf{B_{n-1}}^\top \mathbf{x_{t+1}}\right)\right]\\ &= \exp\left[-\kappa_0^{(r)} - \boldsymbol{\kappa}_r{}^\top \mathbf{x_t^{(r)}} + A_{n-1} + \mathbf{B_{n-1}}^\top\left(\boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x_t}\right) + \frac{1}{2}\mathbf{B_{n-1}}^\top \boldsymbol{\Sigma}^{\mathbb{Q}} \mathbf{B_{n-1}}\right] \end{aligned}$$

Thus we directly have:

$$A_n = A_{n-1} - \kappa_0^{(r)} + \mathbf{B}_{\mathbf{n-1}}^\top \boldsymbol{\mu}^{\mathbb{Q}} + \frac{1}{2}\mathbf{B}_{\mathbf{n-1}}^\top \boldsymbol{\Sigma}^{\mathbb{Q}} \mathbf{B}_{\mathbf{n-1}} \quad \text{and} \quad \mathbf{B_n} = \boldsymbol{\Phi}^{\mathbb{Q}^\top} \mathbf{B}_{\mathbf{n-1}} - \begin{pmatrix} \boldsymbol{\kappa_r} \\ \mathbf{0}_{\mathbf{k}_\pi} \end{pmatrix}.$$

For real bonds, using the same reasoning, we have:

$$\begin{aligned} D_t^{*(n)} &= \mathbb{E}_t^{\mathbb{Q}}\left(e^{-r_t+\pi_{t+1}} D_{t+1}^{*(n-1)}\right) \\ &= \exp\left(-\kappa_0^{(r)} - \boldsymbol{\kappa_r}^\top \mathbf{x_t^{(r)}}\right) \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(A_{n-1}^* + \kappa_0^{(\pi)} + \boldsymbol{\kappa_\pi}\, \mathbf{x_{t+1}^{(\pi)}} + \mathbf{B}_{\mathbf{n-1}}^{*^\top} \mathbf{x_{t+1}}\right)\right] \\ &= \exp\left(-\kappa_0^{(r)} + A_{n-1}^* + \kappa_0^{(\pi)} - \boldsymbol{\kappa_r}^\top \mathbf{x_t^{(r)}}\right) \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\left(\mathbf{B}_{\mathbf{n-1}}^{*^\top} + \left[\mathbf{0}_{\mathbf{k_r}}{}^\top, \boldsymbol{\kappa_\pi}^\top\right]\right) \mathbf{x_{t+1}}\right)\right] \end{aligned}$$

so we directly obtain:

$$\begin{aligned} A_n^* &= A_{n-1}^* - \kappa_0^{(r)} + \kappa_0^{(\pi)} + \left[\mathbf{B}_{\mathbf{n-1}}^* + \begin{pmatrix} \mathbf{0}_{\mathbf{k_r}} \\ \boldsymbol{\kappa}_\pi \end{pmatrix}\right]^\top \left(\boldsymbol{\mu}^{\mathbb{Q}} + \frac{1}{2}\boldsymbol{\Sigma}^{\mathbb{Q}}\left[\mathbf{B}_{\mathbf{n-1}}^* + \begin{pmatrix} \mathbf{0}_{\mathbf{k_r}} \\ \boldsymbol{\kappa_\pi} \end{pmatrix}\right]\right) \\ \mathbf{B}_{\mathbf{n}}^* &= \boldsymbol{\Phi}^{\mathbb{Q}^\top}\left[\mathbf{B}_{\mathbf{n-1}}^* + \begin{pmatrix} \mathbf{0}_{\mathbf{k_r}} \\ \boldsymbol{\kappa_\pi} \end{pmatrix}\right] - \begin{pmatrix} \boldsymbol{\kappa_r} \\ \mathbf{0}_{\mathbf{k}_\pi} \end{pmatrix}, \end{aligned}$$

All these recursions are starting from initial conditions $A_0 = 0$, $\mathbf{B_0} = \mathbf{0_k}$ and $A_0^* = 0$, $\mathbf{B_0^*} = \mathbf{0_k}$.

Multi-horizon conditional MGF: Let us remember our notations here:

$$
\begin{aligned}
\psi_{t-1}(\mathbf{u},\mathbf{v},\mathbf{w}) &= \mathbb{E}^{\mathbb{Q}}_{t-1}\left[\exp\left(\mathbf{u}^\top \mathbf{y_t} + \mathbf{v}^\top \boldsymbol{\delta_t} + \mathbf{w}^\top \mathbf{x_t}\right)\right] \\
&= \exp\left[A(\mathbf{u},\mathbf{v}) + \mathbf{B}(\mathbf{u},\mathbf{v})^\top \mathbf{y_{t-1}} + \mathbf{w}^\top\left(\mu^{\mathbb{Q}} + \mathbf{\Phi}^{\mathbb{Q}}\mathbf{x_{t-1}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{\Sigma}^{\mathbb{Q}}\mathbf{w}\right)\right] \\
&= \exp\left[\widetilde{A}(\mathbf{u},\mathbf{v},\mathbf{w}) + \mathbf{B}(\mathbf{u},\mathbf{v})^\top \mathbf{y_{t-1}} + \mathbf{C}\left(\mathbf{w}\right)^\top \mathbf{x_{t-1}}\right] \\
G_t(n,n_c,n_\ell) &= \mathbb{E}^{\mathbb{Q}}_t\left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j}\right)\mathbb{1}\left\{\sum_{j=0}^{n_c}\delta^{(c)}_{t+j}=0\right\}\mathbb{1}\left\{\sum_{j=0}^{n_\ell}\delta^{(\ell)}_{t+j}=0\right\}\right] \\
G^*_t(n,n_c,n_\ell) &= \mathbb{E}^{\mathbb{Q}}_t\left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j} - \pi_{t+j+1}\right)\mathbb{1}\left\{\sum_{j=0}^{n_c}\delta^{(c)}_{t+j}=0\right\}\mathbb{1}\left\{\sum_{j=0}^{n_\ell}\delta^{(\ell)}_{t+j}=0\right\}\right].
\end{aligned}
$$

Using the lemma provided in Monfort et al. (2017*a*), we can write:

$$
\begin{aligned}
G_t(n,n_c,n_\ell) &= \lim_{u\to+\infty}\mathbb{E}^{\mathbb{Q}}_t\left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j} - u\left(\sum_{j=0}^{n_c}\delta^{(c)}_{t+j} + \sum_{j=0}^{n_\ell}\delta^{(\ell)}_{t+j}\right)\right)\right] \\
G^*_t(n,n_c,n_\ell) &= \lim_{u\to+\infty}\mathbb{E}^{\mathbb{Q}}_t\left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j} - \pi_{t+j+1} - u\left(\sum_{j=0}^{n_c}\delta^{(c)}_{t+j} + \sum_{j=0}^{n_\ell}\delta^{(\ell)}_{t+j}\right)\right)\right]
\end{aligned}
$$

Let us denote by $\psi^{(n)}_t(\mathbf{u}_1,\mathbf{v_1},\mathbf{w_1},\ldots,\mathbf{u_n},\mathbf{v_n},\mathbf{w_n})$ the multi-period conditional MGF of our state variables, i.e.:

$$
\psi^{(n)}_t(\mathbf{u}_1,\mathbf{v_1},\mathbf{w_1},\ldots,\mathbf{u_n},\mathbf{v_n},\mathbf{w_n}) = \mathbb{E}^{\mathbb{Q}}_{\mathbf{t-1}}\left[\exp\left(\sum_{\mathbf{j=1}}^{\mathbf{n}}\mathbf{u_j}^\top\mathbf{y_{t+j}} + \mathbf{v_j}^\top\boldsymbol{\delta_{t+j}} + \mathbf{w_j}^\top\mathbf{x_{t+j}}\right)\right].
$$

A first natural property resulting from the affine formulation of the joint process is that the multi-period conditional MGF is an exponential-affine function. It is easy to show that:

$$
\begin{aligned}
\psi_t^{(n)}(\mathbf{u}_1, \mathbf{v_1}, \mathbf{w_1}, \ldots, \mathbf{u_n}, \mathbf{v_n}, \mathbf{w_n}) &= \exp\Bigg[\Psi_0^{(n)}(\mathbf{u}_1, \mathbf{v_1}, \mathbf{w_1}, \ldots, \mathbf{u_n}, \mathbf{v_n}, \mathbf{w_n}) \\
&+ \Psi_y^{(n)}(\mathbf{u}_1, \mathbf{v_1}, \mathbf{w_1}, \ldots, \mathbf{u_n}, \mathbf{v_n}, \mathbf{w_n})^\top \mathbf{y_t} \\
&+ \Psi_x^{(n)}(\mathbf{u}_1, \mathbf{v_1}, \mathbf{w_1}, \ldots, \mathbf{u_n}, \mathbf{v_n}, \mathbf{w_n})^\top \mathbf{x_t}\Bigg],
\end{aligned}
$$

where:

$$
\begin{aligned}
&\Psi_0^{(k)}(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n) = \Psi_0^{(k-1)}(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n) \\
+ \;\; &\widetilde{\mathbf{A}}\left[\mathbf{u}_{n-k+1} + \Psi_y^{(k-1)}(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n),\, \mathbf{v}_{n-k+1},\, \mathbf{w}_{n-k+1} + \Psi_x^{(k-1)}(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n)\right] \\
&\Psi_y^{(k)}(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n) = \mathbf{B}\left[\Psi_y^{(k-1)}\left(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n\right) + \mathbf{u}_{n-k+1},\, \mathbf{v}_{n-k+1}\right] \\
&\Psi_x^{(k)}(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n) = \mathbf{C}\left[\Psi_x^{(k-1)}\left(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n, \mathbf{w}_n\right) + \mathbf{w}_{n-k+1}\right]
\end{aligned}
$$

Let us denote by $n_{max} = \max(n+1,\, n_c,\, n_\ell)$. Then, $G_t(n,\, n_c,\, n_\ell)$ and $G_t^*(n,\, n_c,\, n_\ell)$ can be rewritten with the multi-horizon conditional MGF:

$$
\begin{aligned}
G_t(n,\, n_c,\, n_\ell) &= e^{-r_t-(n-1)\kappa_0^{(r)}} \psi_t^{(n)}(\mathbf{u}_1, \mathbf{v_1}, \mathbf{w_1}, \ldots, \mathbf{u_{n_{max}}}, \mathbf{v_{n_{max}}}, \mathbf{w_{n_{max}}}) \\
G_t^*(n,\, n_c,\, n_\ell) &= e^{-r_t-(n-1)\kappa_0^{(r)}+n\kappa_0^{(\pi)}} \psi_t^{(n)}(\mathbf{u}_1^*, \mathbf{v_1}^*, \mathbf{w_1}^*, \ldots, \mathbf{u}_{\mathbf{n_{max}}}^*, \mathbf{v}_{\mathbf{n_{max}}}^*, \mathbf{w}_{\mathbf{n_{max}}}^*)
\end{aligned}
$$

where:

$$
\begin{aligned}
\mathbf{u}_j &= \mathbf{0}_{k_c+k_\ell} \\
\mathbf{v}_j &= \begin{cases} -v\,(1,\,1)^\top & \text{if} \quad j \leqslant \min(n_c,\,n_\ell) \\ -v\,(\mathbb{1}\{n_c > n_\ell\},\,\mathbb{1}\{n_\ell > n_c\})^\top & \text{if} \quad j \in (\min(n_c,\,n_\ell), \max(n_c,\,n_\ell)] \\ 0 & \text{if} \quad j \in (\max(n_c,\,n_\ell), n_{max}] \end{cases} \\
\mathbf{w}_j &= \begin{cases} \left(-\kappa_r^\top,\,\mathbf{0}_{k_\pi}^\top\right)^\top & \text{if} \quad j \leqslant n-1 \\ 0 & \text{if} \quad j \in [n, n_{max}] \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{u}_j^* &= \mathbf{0}_{k_c+k_\ell} \\
\mathbf{v}_j^* &= \mathbf{v}_j \\
\mathbf{w}_j^* &= \begin{cases} \left(-\kappa_r^\top,\,-\kappa_\pi^\top\right)^\top & \text{if} \quad j \leqslant n-1 \\ \left(\mathbf{0}_{k_x}^\top,\,-\kappa_\pi^\top\right)^\top & \text{if} \quad j = n \\ 0 & \text{if} \quad j \in [n, n_{max}] \end{cases}
\end{aligned}
$$

where v is a scalar tending to infinity. By a continuity argument, we obtain that both G_t and G_t^* are exponential-affine functions such that:

$$
\begin{aligned}
G_t(n, n_c, n_\ell) &= \exp\left(q_{(n,n_c,n_\ell)} + \mathbf{Q}_{(n,n_c,n_\ell)}^\top \mathbf{z_t}\right)\,, \\
G_t^*(n, n_c, n_\ell) &= \exp\left(q_{(n,n_c,n_\ell)}^* + \mathbf{Q}_{(n,n_c,n_\ell)}^{*\top} \mathbf{z_t}\right)\,,
\end{aligned}
$$

and the loadings are obtained through the recursions defined earlier.

Let us use this result to compute the price of defaultable bonds.

Defaultable bond pricing: Let us first consider nominal bonds.

$$\begin{aligned} B_t^{(n)} &= \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{i-1} r_{t+j} \right) \mathcal{P}_c^{(i)} \times \left(\mathbb{1} \left\{ \sum_{j=0}^{i-1} \delta_{t+j}^{(c)} = 0 \right\} - \mathbb{1} \left\{ \sum_{j=0}^{i} \delta_{t+j}^{(c)} = 0 \right\} \right) \right] \\ &+ \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j} \right) \mathbb{1} \left\{ \sum_{j=0}^{n} \delta_{t+j}^{(c)} = 0 \right\} \right] \\ &= \lim_{v \to +\infty} \rho_c \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{i-1} r_{t+j} \right) \times \left(\exp\left(-v \sum_{j=0}^{i-1} \delta_{t+j}^{(c)} \right) - \exp\left(-v \sum_{j=0}^{i} \delta_{t+j}^{(c)} \right) \right) \right] \\ &+ \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j} \right) \mathbb{1} \exp\left(-v \sum_{j=0}^{n} \delta_{t+j}^{(c)} \right) \right] \end{aligned}$$

Conditionally on no default at date t, we have:

$$\begin{aligned} B_t^{(n)} &= \rho_c \sum_{i=1}^{n} \left[G_t(i,\, i-1,\, 0) - G_t(i,\, i,\, 0) \right] + G_t(n,\, n,\, 0) \\ &= \rho_c \sum_{i=1}^{n} \left[\exp\left(q_{(i,\, i-1,\, 0)} + \mathbf{Q}_{(i,\, i-1,\, 0)}^{\top} \mathbf{z_t} \right) - \exp\left(q_{(i,\, i,\, 0)} + \mathbf{Q}_{(i,\, i,\, 0)}^{\top} \mathbf{z_t} \right) \right] + \exp\left(q_{(n,\, n,\, 0)} + \mathbf{Q}_{(\mathbf{n},\, \mathbf{n},\, \mathbf{0})}^{\top} \mathbf{z_t} \right) \end{aligned}$$

Let us consider a first order Taylor expansion of the previous expression:

$$\begin{aligned} B_t^{(n)} &\simeq \rho_c \sum_{i=1}^{n} \left[q_{(i,\, i-1,\, 0)} - q_{(i,\, i,\, 0)} + \left(\mathbf{Q}_{(i,\, i-1,\, 0)} - \mathbf{Q}_{(i,\, i,\, 0)} \right)^{\top} \mathbf{z_t} \right] + 1 + q_{(n,\, n,\, 0)} + \mathbf{Q}_{(n,\, n,\, 0)}^{\top} \mathbf{z_t} \\ &\simeq \exp\left\{ \rho_c \sum_{i=1}^{n} \left[q_{(i,\, i-1,\, 0)} - q_{(i,\, i,\, 0)} \right] + q_{(n,\, n,\, 0)} + \left(\rho_c \sum_{i=1}^{n} \left[\mathbf{Q}_{(i,\, i-1,\, 0)} - \mathbf{Q}_{(i,\, i,\, 0)} \right] + \mathbf{Q}_{(n,\, n,\, 0)} \right)^{\top} \mathbf{z_t} \right\} \end{aligned}$$

Thus, we immediately have:

$$
\begin{aligned}
\mathcal{A}_n(\rho_c) &= q_{(n,n,0)} + \rho_c \sum_{i=1}^{n} \left[q_{(i,i-1,0)} - q_{(i,i,0)} \right] \\
\mathcal{B}_n(\rho_c) &= \mathbf{Q}_{(n,n,0)} + \rho_c \sum_{i=1}^{n} \left[\mathbf{Q}_{(i,i-1,0)} - \mathbf{Q}_{(i,i,0)} \right]
\end{aligned}
$$

The proof for the inflation-indexed bonds is similar in spirit:

$$
\begin{aligned}
B_t^{*(n)} &= \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \Bigg[\exp\left(-\sum_{j=0}^{i-1} r_{t+j} - \pi_{t+j+1} \right) (\rho_c^* + \rho_\ell^*)\, \mathbb{1}\left\{ \sum_{j=0}^{i-1} \mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} = 0 \right\} \\
&- \rho_c^* \exp\left(-\sum_{j=0}^{i-1} r_{t+j} - \pi_{t+j+1} \right) \mathbb{1}\left\{ \sum_{j=0}^{i-1} \mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} + \delta_{t+i}^{(c)} = 0 \right\} \\
&- \rho_\ell^* \exp\left(-\sum_{j=0}^{i-1} r_{t+j} - \pi_{t+j+1} \right) \mathbb{1}\left\{ \sum_{j=0}^{i-1} \mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} + \delta_{t+i}^{(\ell)} = 0 \right\} \Bigg] \\
&\quad + \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j} - \pi_{t+j+1} \right) \mathbb{1}\left\{ \sum_{j=0}^{n} \mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} = 0 \right\} \right] \\
&= \sum_{i=1}^{n} \mathbb{E}_t^{\mathbb{Q}} \Bigg[\exp\left(-\sum_{j=0}^{i-1} r_{t+j} - \pi_{t+j+1} + v\mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} \right) (\rho_c^* + \rho_\ell^*) \\
&- \rho_c^* \exp\left(-\sum_{j=0}^{i-1} r_{t+j} - \pi_{t+j+1} + v\mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} \right) \exp\left(-v\delta_{t+i}^{(c)} \right) \\
&- \rho_\ell^* \exp\left(-\sum_{j=0}^{i-1} r_{t+j} - \pi_{t+j+1} + v\mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} \right) \exp\left(-v\delta_{t+i}^{(\ell)} \right) \Bigg] \\
&\quad + \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\sum_{j=0}^{n-1} r_{t+j} - \pi_{t+j+1} + v\mathbf{e}_2^\top \boldsymbol{\delta}_{t+j} \right) \exp\left(-v\mathbf{e}_2^\top \boldsymbol{\delta}_{t+n} \right) \right]
\end{aligned}
$$

Assuming no default or liquidity event happened at date t, we obtain:

$$
\begin{aligned}
B_t^{*(n)} &= \sum_{i=1}^{n} \left[(\rho_c^* + \rho_\ell^*) G_t^*\,(i,\, i-1,\, i-1) - \rho_c^* G_t^*\,(i,\, i,\, i-1) - \rho_\ell^* G_t^*\,(i,\, i-1,\, i) \right] \\
&\quad + \quad G_t^*\,(n,\, n,\, n) \\
&= \sum_{i=1}^{n} \Bigg[(\rho_c^* + \rho_\ell^*) \exp\left(q^*_{(i,\, i-1,\, i-1)} + \mathbf{Q}^{*\top}_{(i,\, i-1,\, i-1)} \mathbf{z_t} \right) - \rho_c^* \exp\left(q^*_{(i,\, i,\, i-1)} + \mathbf{Q}^{*\top}_{(i,\, i,\, i-1)} \mathbf{z_t} \right) \\
&\quad - \quad \rho_\ell^* \exp\left(q^*_{(i,\, i-1,\, i)} + \mathbf{Q}^{*\top}_{(i,\, i-1,\, i)} \mathbf{z_t} \right) \Bigg] + \exp\left(q^*_{(n,\, n,\, n)} + \mathbf{Q}^{*\top}_{(n,\, n,\, n)} \mathbf{z_t} \right)
\end{aligned}
$$

Again, considering a first order Taylor expansion:

$$
\begin{aligned}
B_t^{*(n)} &\simeq \sum_{i=1}^{n} \Bigg[(\rho_c^* + \rho_\ell^*) \left(q^*_{(i,\, i-1,\, i-1)} + \mathbf{Q}^{*\top}_{(i,\, i-1,\, i-1)} \mathbf{z_t} \right) - \rho_c^* \left(q^*_{(i,\, i,\, i-1)} + \mathbf{Q}^{*\top}_{(i,\, i,\, i-1)} \mathbf{z_t} \right) \\
&\quad - \quad \rho_\ell^* \left(q^*_{(i,\, i-1,\, i)} + \mathbf{Q}^{*\top}_{(i,\, i-1,\, i)} \mathbf{z_t} \right) \Bigg] + 1 + q^*_{(n,\, n,\, n)} + \mathbf{Q}^{*\top}_{(n,\, n,\, n)} \mathbf{z_t} \\
&\simeq \exp \Bigg\{ \sum_{i=1}^{n} \left[(\rho_c^* + \rho_\ell^*) q^*_{(i,\, i-1,\, i-1)} - \rho_c^* q^*_{(i,\, i,\, i-1)} - \rho_\ell^* q^*_{(i,\, i-1,\, i)} \right] + q^*_{(n,\, n,\, n)} \\
&\quad + \quad \left[\sum_{i=1}^{n} \left[(\rho_c^* + \rho_\ell^*) \mathbf{Q}^*_{(i,\, i-1,\, i-1)} - \rho_c^* \mathbf{Q}^*_{(i,\, i,\, i-1)} - \rho_\ell^* \mathbf{Q}^*_{(i,\, i-1,\, i)} \right] + \mathbf{Q}^*_{(n,\, n,\, n)} \right]^\top \mathbf{z_t} \Bigg\}
\end{aligned}
$$

so we obtain:

$$
\begin{aligned}
\mathcal{A}_n^*(\rho_c^*,\, \rho_\ell^*) &= \sum_{i=1}^{n} \left[(\rho_c^* + \rho_\ell^*) q^*_{(i,\, i-1,\, i-1)} - \rho_c^* q^*_{(i,\, i,\, i-1)} - \rho_\ell^* q^*_{(i,\, i-1,\, i)} \right] + q^*_{(n,\, n,\, n)} \\
\mathcal{B}_n^*(\rho_c^*,\, \rho_\ell^*) &= \sum_{i=1}^{n} \left[(\rho_c^* + \rho_\ell^*) \mathbf{Q}^*_{(i,\, i-1,\, i-1)} - \rho_c^* \mathbf{Q}^*_{(i,\, i,\, i-1)} - \rho_\ell^* \mathbf{Q}^*_{(i,\, i-1,\, i)} \right] + \mathbf{Q}^*_{(n,\, n,\, n)}
\end{aligned}
$$