Destructive Bidding in All-Pay Auctions

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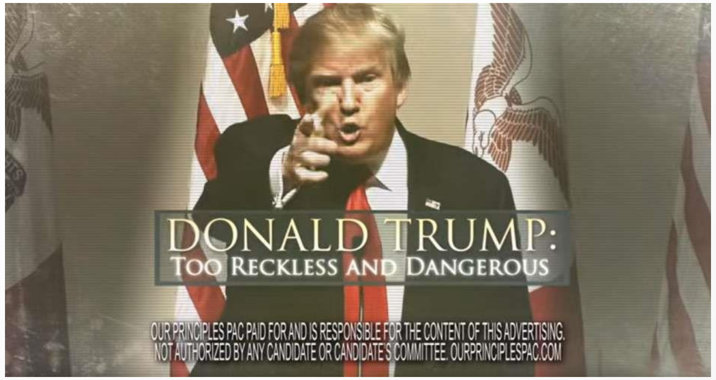
All-Pay Auctions

- Any activity in which make a non-recoverable investment towards winning a contest.
  - R&D: Patent Races
  - Military Conflicts, Arms Races
  - In Politics
    - Campaigns
    - Lobbying
  - Lotteries

Destructive Investments in All-Pay Auctions

- Reduces the value of the prize for one or more contestants
- Examples:
  - Negative Advertising in Political Campaigning
  - Military Actions which destroy infrastructure
  - Comparative Advertising by Firms



Overview of All-Pay Auctions

Model Set-up

- N risk-neutral contestants have common valuation, v, for a prize.
- Each bidder simultaneously submits a bid, b_i.
- The highest bid wins the prize (ties broken randomly)

Nash Equilibrium Behavior

- No pure strategy Nash Equilibrium
- Symmetric Equilibrium behavior is to mix one's bid according to the following cumulative distribution function

$$b_i \sim F(b) = \left(\frac{b}{v}\right)^{\frac{1}{N-1}} \text{ on } [0, v]$$

Common Valuation All-Pay Auction with Destructive Bidding

- Structure of Auction
  - *N Risk-Neutral Bidders*
  - *Common Valuation v*
  - Bids b_i reduce value of the prize by γb_i
  - Final prize value $\tilde{v} = v - \gamma \sum_{i=1}^{N} b_i$
  - Highest bidder wins prize
  - All bidders pay their bid

Common Valuation All-Pay Auction with Destructive Bidding

- Nash Equilibrium Bidding Behavior
  - *No Pure Strategy Nash Equilibrium Bidding Behavior Exists*
    - *Any bids* $\max_{j\neq i} b_j < v$ *has a best response* $b_i^{br} = \max_{j\neq i} b_j + \varepsilon$
    - *Any bids* $\max_{j\neq i} b_j = v$ *has best response* $b_i^{br} = 0.$
  - *Symmetric Mixed Strategy Equilibrium*
    - *Assume all other players play* $b_j \sim f(b)$ *expected surplus for player* i

$$EU_i(b_i) = \Big(v - \underbrace{\Big(\gamma b_i + (N-1)\gamma \int_0^{b_i} \frac{bf(b)}{F(b_i)}\,db\Big)}_{\text{Expected Destruction Conditional on Winning}} - b_i\Big) \underbrace{F^{N-1}(b_i)}_{\Pr(\text{Win})} - b_i \underbrace{\left(1 - F^{N-1}(b_i)\right)}_{\Pr(\text{Lose})}$$

Common Valuation All-Pay Auction with Destructive Bidding

- *Indifference Principle implies* $EU_i(b_i) = 0$

$$b_i = F^{N-1}(b_i)(v - (\gamma b_i + \frac{(N-1)\gamma}{F(b_i)} \int_0^{b_i} bf(b)\, db)) = E[Prize]$$

- ***Result:*** *Bid per Standard All Pay Auction reduced by Expected Destruction*
- *dwrt* b_i *and solve for* $f(b_i)$ *yields*

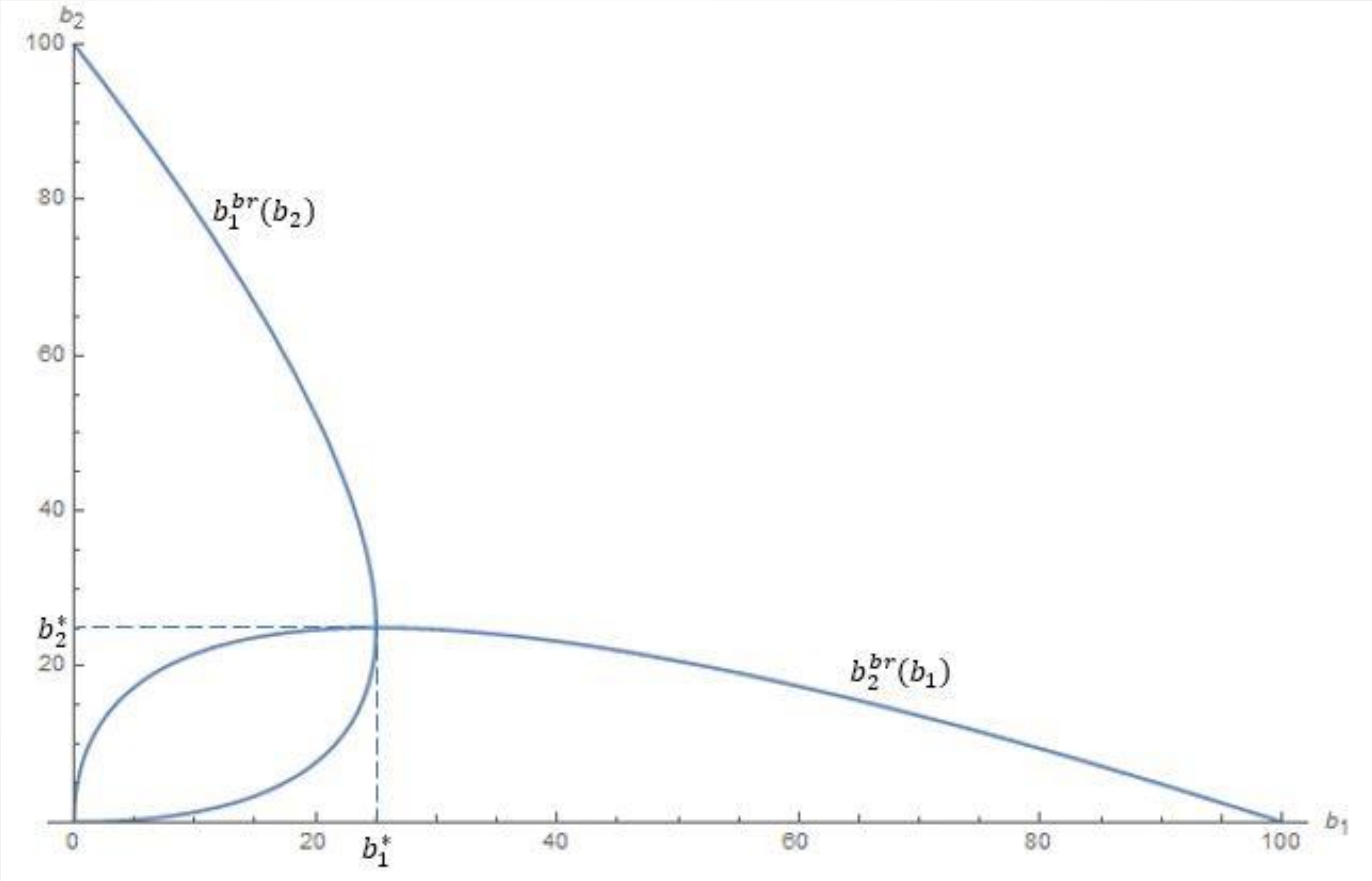
$$f(b_i) = \frac{F(b_i) + \gamma F^N(b_i)}{(v - \gamma N\, b_i)F^{N-1}(b_i) + (N-2)b_i}$$

- **Result:** $f(b_i)$ *is increasing thus* $F(b_i)$ *is convex as* $b_i \rightarrow \bar{b}$. *Mixed strategy bidding is weighted towards higher bids. i.e. Compete to win!*

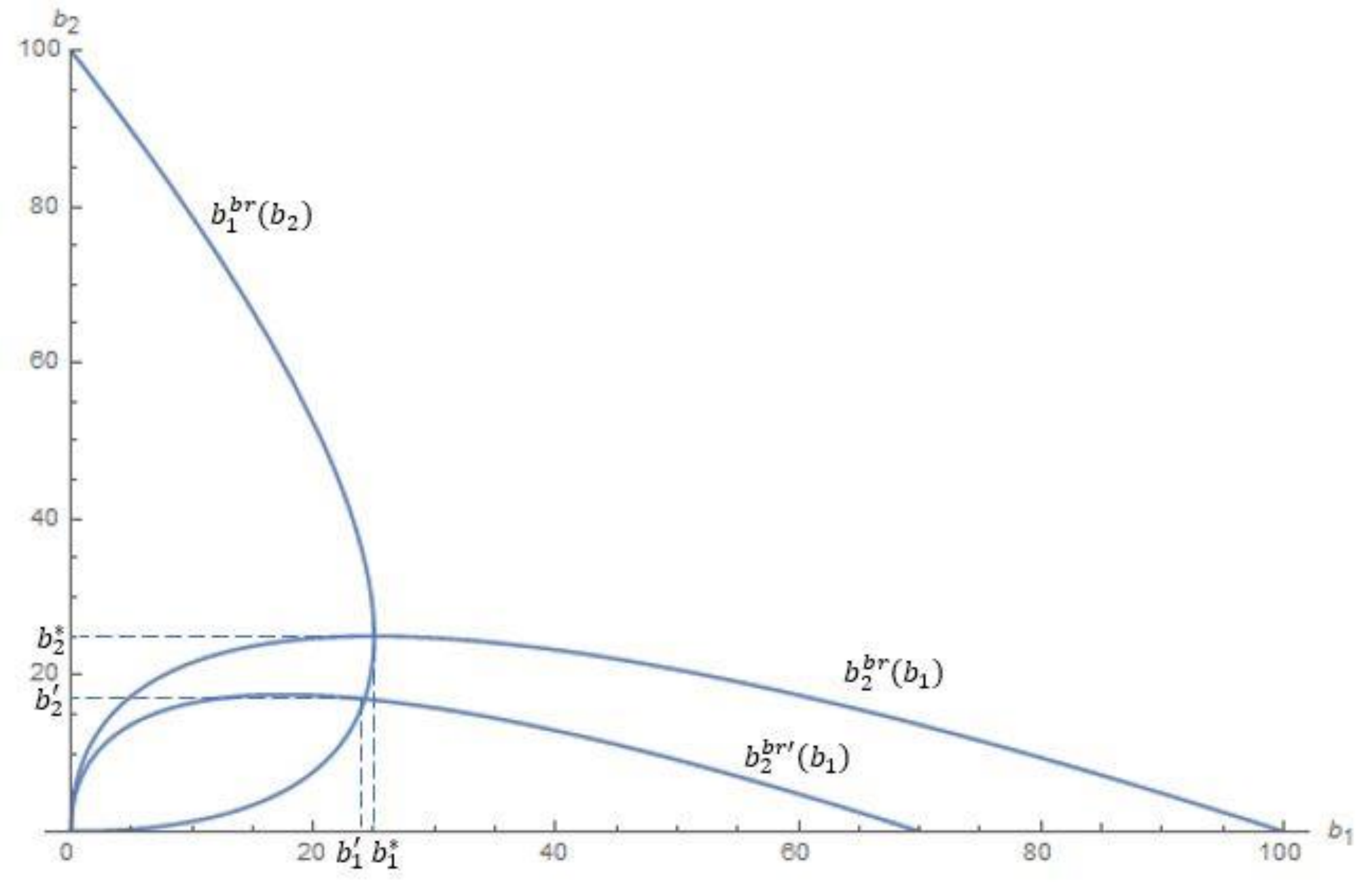
Model of Destructive Investment in an All-Pay Contest with Stochastic Winner

- N Risk-Neutral Contestants Play Game in Two Rounds
- Round 1: Destructive Investment
  - Contestants simultaneously choose Destructive Investment d_i
  - Valuation of each contestant is $v_i(d_i, d_{-i})$
    - $\frac{\partial v_i}{\partial d_j} < \frac{\partial v_i}{\partial d_i} \leq 0$ (Reduction of opponent's valuation larger than on own's valuation)
- Round 2: Bidding
  - Contestants simultaneously choose bid b_i
  - Probability of winning the prize is $\rho_i(b_i, b_{-i})$
    - $\frac{\partial \rho_i}{\partial b_i} > 0, \frac{\partial \rho_i}{\partial b_j} < 0$
  - All contestants pay cost $c_i(b_i, d_i) = b_i + c_i(d_i)$ where $c_i''(d_i) > 0$
- Objective Function: $\max_{b_i, d_i} \rho_i(b_i, b_{-i}) v_i(d_i, d_{-i}) - c_i(b_i, d_i)$

Bidding Round Best Response and Nash Equilibrium



Effect of Destructive Investment



Optimal Destructive Investment

$$\underbrace{-\left(\frac{\partial b_i^*}{\partial v_i}\frac{\partial v_i}{\partial d_i}+\sum_{j\neq i}\frac{\partial b_i^*}{\partial v_j}\frac{\partial v_j}{\partial d_i}\right)+\sum_{j\neq i}\frac{\partial \rho_i}{\partial b_j}\left(\sum_{k=1}^{N}\frac{\partial b_j^*}{\partial v_k}\frac{\partial v_k}{\partial d_i}\right)v_i}_{\text{Marginal Benefit of Destructive Investment}} = \underbrace{\frac{\partial c_i}{\partial d_i}-\frac{\partial \rho_i}{\partial b_i}\left(\sum_{k=1}^{N}\frac{\partial b_i^*}{\partial v_k}\frac{\partial v_k}{\partial d_i}\right)v_i-\rho_i\frac{\partial v_i}{\partial d_i}}_{\text{Marginal Cost of Destructive Investment}}$$

Optimal Destructive Investment

$$-\left(\frac{\partial b_i^*}{\partial v_i}\frac{\partial v_i}{\partial d_i} + \sum_{j\neq i}\frac{\partial b_i^*}{\partial v_j}\frac{\partial v_j}{\partial d_i}\right) + \sum_{j\neq i}\frac{\partial \rho_i}{\partial b_j}\left(\sum_{k=1}^{N}\frac{\partial b_j^*}{\partial v_k}\frac{\partial v_k}{\partial d_i}\right)v_i = \frac{\partial c_i}{\partial d_i} - \frac{\partial \rho_i}{\partial b_i}\left(\sum_{k=1}^{N}\frac{\partial b_i^*}{\partial v_k}\frac{\partial v_k}{\partial d_i}\right)v_i - \rho_i\frac{\partial v_i}{\partial d_i}$$

- Marginal Benefits of Destructive Investment
  - Lower own bid due to destroying own value
  - Lower own bid due to opponents lowering their bids due to destroyed value
  - Increased probability of winning as opponents bid less
- Marginal Costs of Destructive Investment
  - Direct cost of destructive investment
  - Lower valuations reduce own bid reducing probability of winning
  - Lower value of prize due to destroying own value

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Symmetric Model Solution

- Two Contestants, $N = 2$
- Probability of Winning: $\rho_i = \frac{b_i}{b_i+b_j}$
- Destructive Investment only reduces opponent's valuation:

$$v_i(d_i, d_j) = \bar{v} - \gamma_{own} d_i - \gamma_{opp} d_j$$

- Cost of bid and destructive investment: $c_i(b_i, d_i) = b_i + d_i^2$
- Risk Neutral Contestants Maximize Expected Surplus

$$E[u_i] = \rho_i v_i - c_i$$

Sample Model Solution

$$d^* = \max\left\{\frac{1}{8}\left(\gamma_{opp} - 2\,\gamma_{own}\right), 0\right\}$$

$$v^* = \max\left\{\frac{1}{8}\left(8\,\bar{v} + 2\gamma_{own}^2 + \gamma_{own}\gamma_{opp} - \gamma_{opp}\right), 0\right\}$$

$$b^* = \frac{v^*}{4}$$

$$Eu^* = \max\left\{\frac{1}{64}\left(16\,\bar{v} - 3\gamma_{opp}\left(\gamma_{opp} - 2\gamma_{own}\right)\right), 0\right\}$$

Key Results

- **Result 1:** For destructive investment to occur, the investment decision must not be simultaneous with the bidding decision.
- **Result 2a:** The equilibrium size of the destructive investment depends on the effect on opponents relative to one's own value destruction.
- **Result 2b:** If the destructive investment does not affect opponents' valuations more than it affects one's own valuation, investment will not occur.
- **Result 2c:** If the effect of destructive investment on opponent's valuations is large enough, no pure strategy Nash equilibrium exists.
- **Result 3:** Destructive investments reduce Nash Equilibrium surplus for all contestants. Contestants have an incentive to disallow destructive investments whenever they provide insufficient direct offsetting value.

Asymmetric Probabilities of Winning

- Two Contestants, $N = 2$
- Probability of Winning: $\rho_1 = \frac{\alpha\, b_1}{\alpha\, b_1 + b_2}, \rho_2 = \frac{b_2}{\alpha\, b_1 + b_2}$ for $\alpha \geq 1$
- Destructive Investment:

$$v_i(d_i, d_j) = \bar{v} - \gamma_{own} d_i - \gamma_{opp} d_j$$

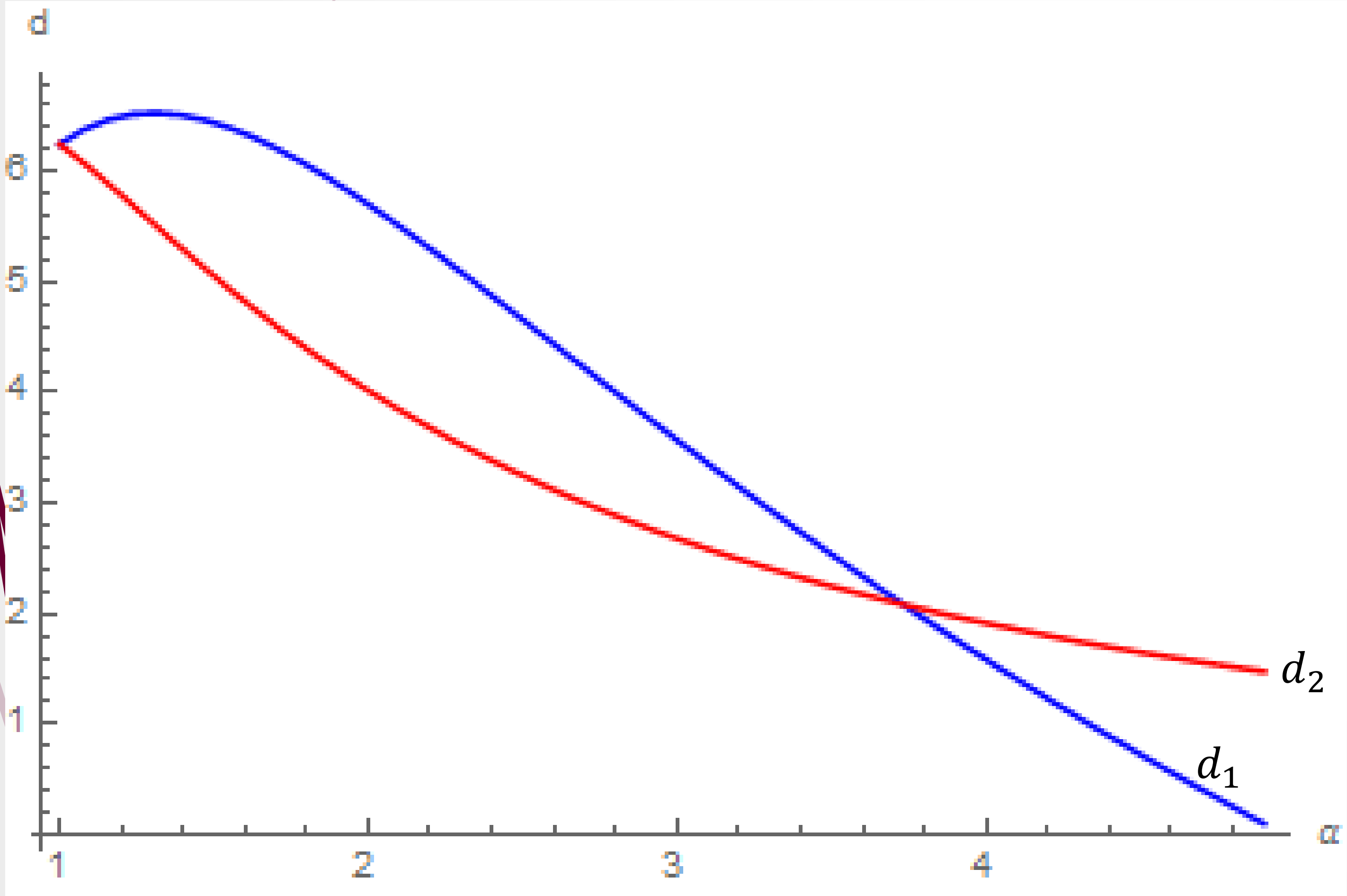
- Separable Cost of bid and destructive investment:

$$c_i(b_i, d_i) = b_i + \chi d_i^2$$

- Risk-Neutral Contestants Maximize Expected Surplus

$$E[u] = \rho_i v_i - c_i$$

Destructive Investment when Contestant 1 is Advantaged

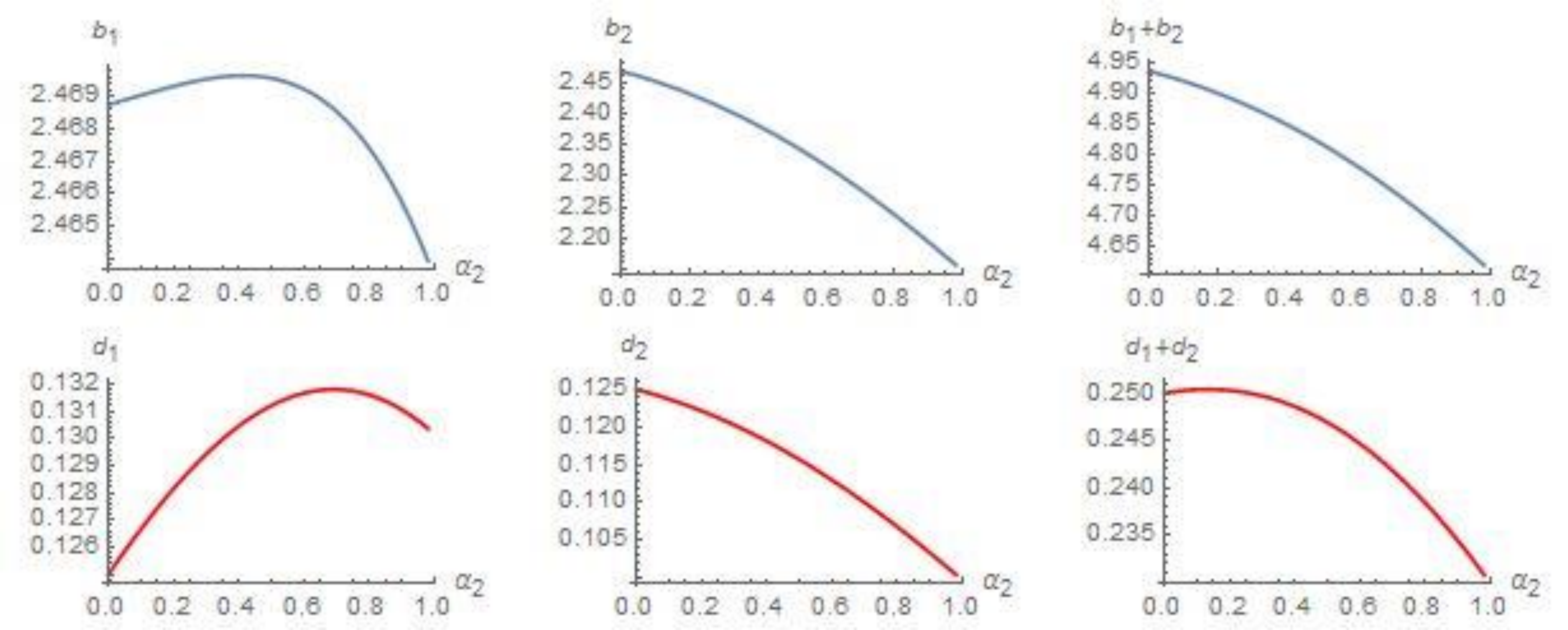


- Small advantage will lead Contestant 1 to be more willing to destroy value.
- As victory is more assured this declines.
- As victory is near certainty, Contestant 2 is more willing to destroy value as Contestant 1 is unable to increase her likelihood of victory through value destruction.

Simplified Model with Risk Aversion

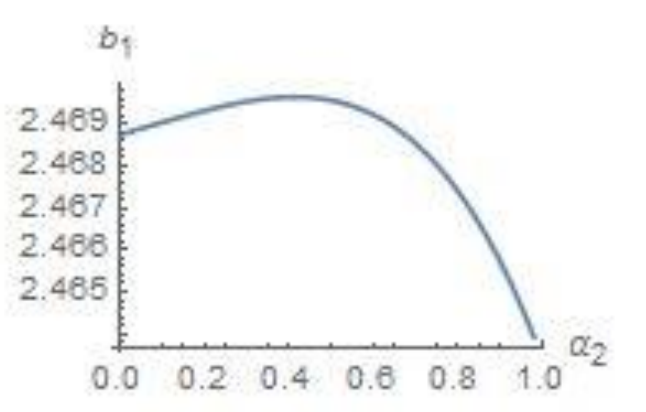
- Two Contestants, $N = 2$
- Probability of Winning: $\rho_i = \frac{b_i}{b_i + b_j}$
- Destructive Investment only reduces opponent's valuation:
  $$v_i(d_j) = \bar{v} - \gamma d_j$$
- Separable Cost of bid and destructive investment:
  $$c_i(b_i, d_i) = b_i + d_i^2$$
- Utility is Constant Coefficient of Relative Risk Aversion:
  $$u_i(w_i) = w_i^{1-\alpha_i}, CRRA = \alpha_i \in [0,1)$$
- Maximize Expected Utility
  $$E[u] = \rho_i u_i(\overline{w} + v_i - c_i) + (1 - \rho_i) u_i(\overline{w} - c_i)$$

Numerically Estimated Equilibrium Responses to Increasing Risk Aversion by Contestant 2



Increasing Risk Aversion Results

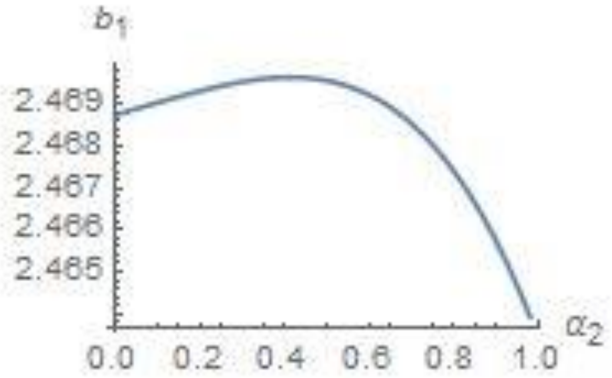
$\uparrow CRRA_2 \rightarrow \uparrow bid_1$

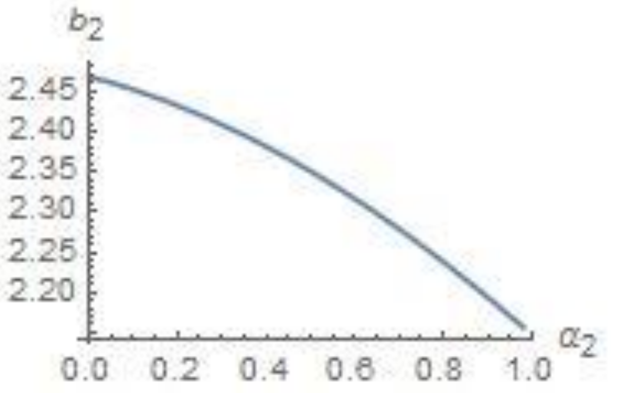


Increasing Risk Aversion Results

$\uparrow CRRA_2 \rightarrow \uparrow bid_1$

$\uparrow CRRA_2 \rightarrow \downarrow bid_2 \rightarrow \downarrow bid_1$





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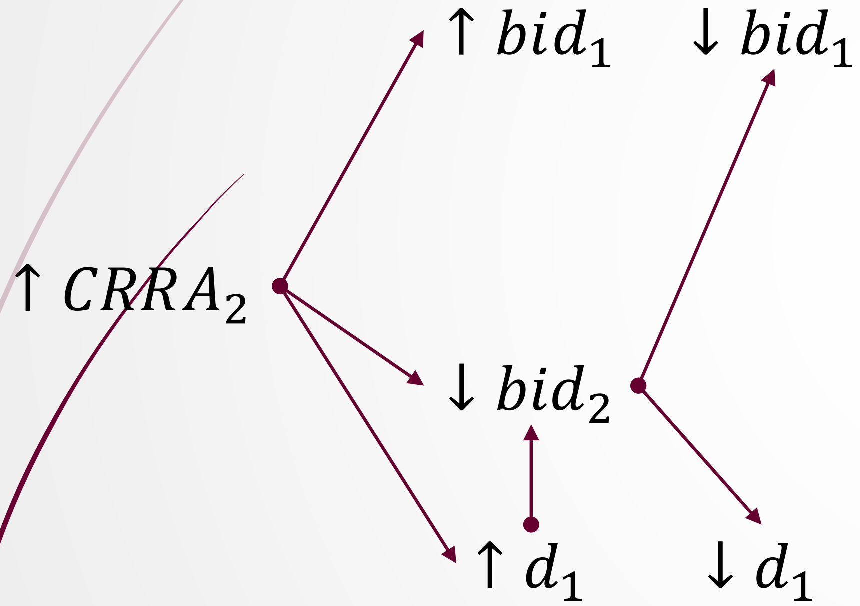

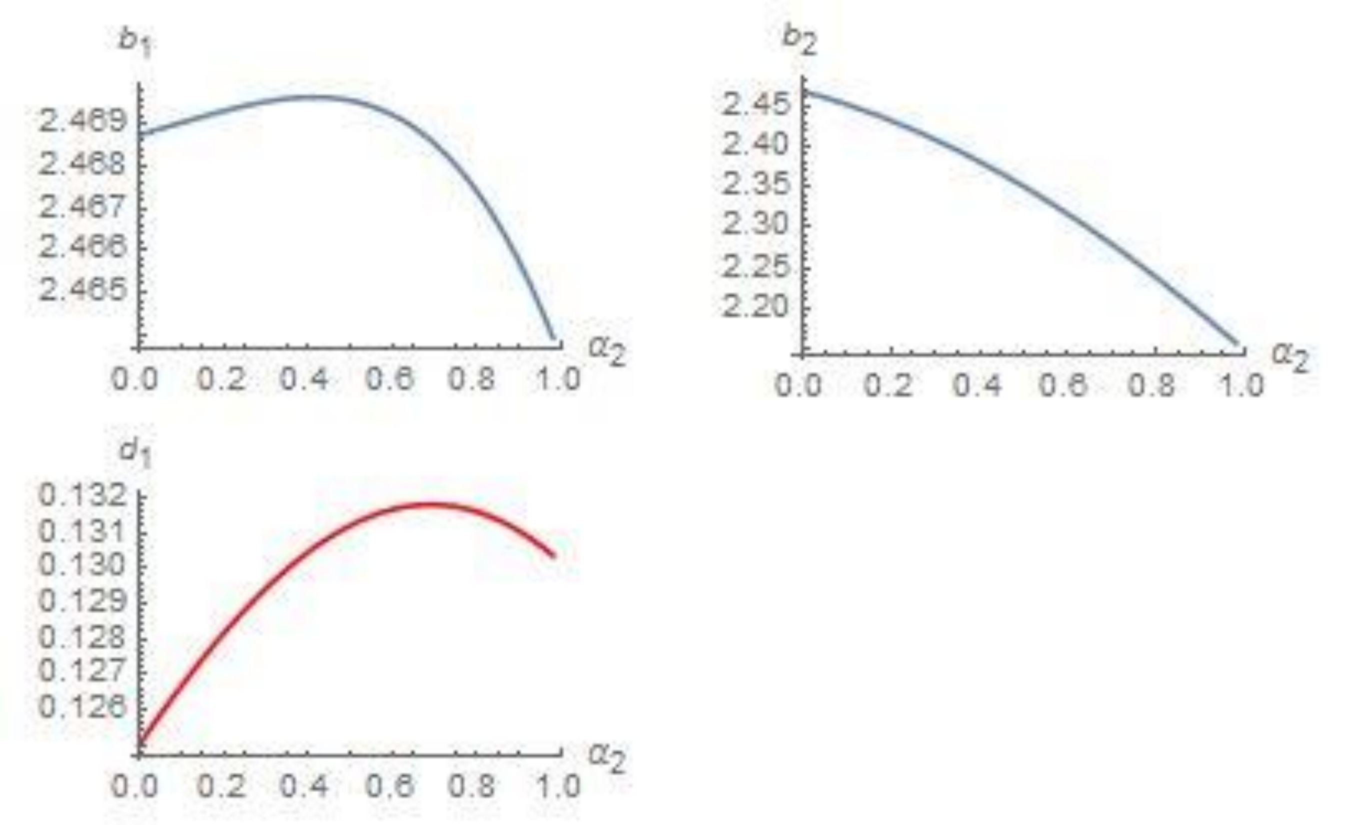

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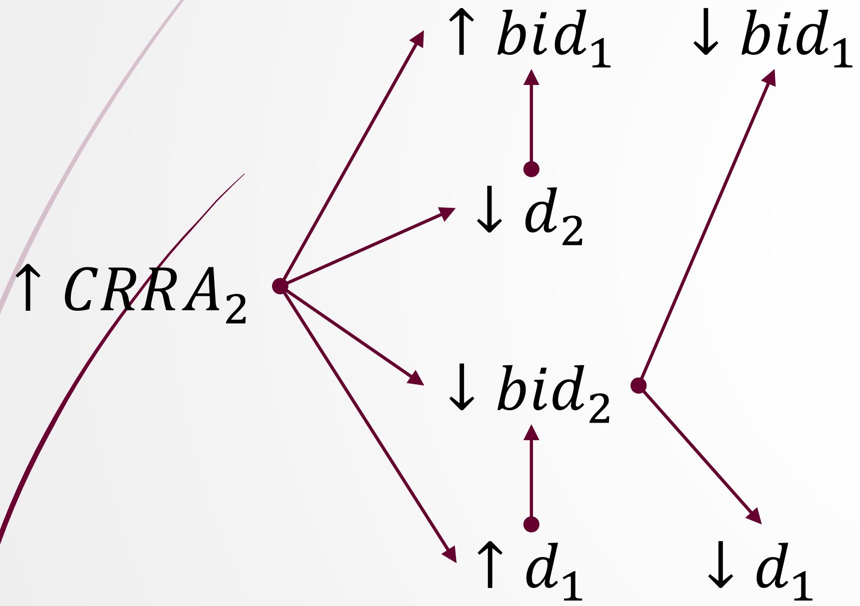

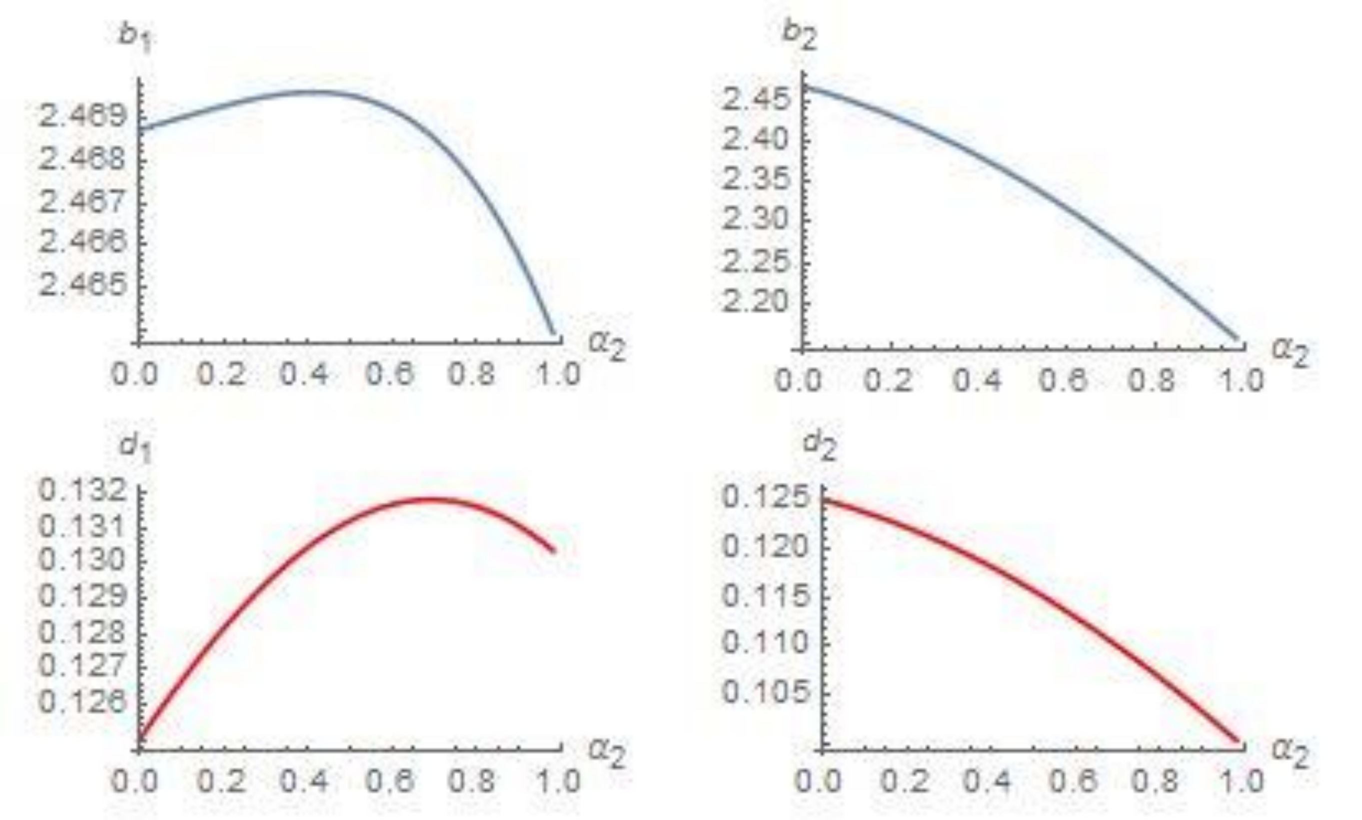

Generalized Asymmetry Results

- Small advantages increase destructive investment by the advantaged party amplifying the advantage.
- As the probability of victory is sufficiently increased, willingness to destroy value declines.
- The disadvantaged party reduces their destructive investment as the disadvantage grows.
- The disadvantaged party may, for sufficiently large disparities, have a stronger destructive investment than the advantaged party who is nearly assured of victory.

Comments and Question

- raisa1sr@cmich.edu
- Looking for coauthors in Auction Theory, Pricing Theory, and/or Behavior under Uncertainty.