Coarse Revealed Preference

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Revealed preference theory

- Pioneered by *Samuelson (1938)*
- Consumer theory: *Afriat (1967)*
- General equilibrium theory: *Brown and Matzkin (1996)*
- Industrial organization: *Carvajal et al. (2013)*
- Matching theory: *Echenique et al. (2013)*
- among many others.

Research question in its simplest form

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Investigate the observable restriction of economic models including

- Rational choice with imperfect observation
- Multiple preferences
- Monotone multiple preferences
- Minimax regret

Outline

Model

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B_i is a nonempty subset of A_i for each i.

To simplify the statements below, we write

$C_i := A_i \setminus B_i$,

$A(\mathcal{O}') := \cup_{(A_i,B_i)\in\mathcal{O}'} A_i$,

$C(\mathcal{O}') := \cup_{(A_i,B_i)\in\mathcal{O}'} C_i$.

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Definition

A coarse data set $\mathcal{O}$ is coarsely rationalizable by a linear order if $\exists$ P such that

$$\max(A_i, P) \in B_i$$

for all i.

Example

A coarse data set including four observations:

$$A_1 = \{x_1, x_2, x_3, x_4\}, B_1 = \{x_1, x_2\};$$

$$A_2 = \{x_2, x_3, x_4, x_5\}, B_2 = \{x_2, x_3\};$$

$$A_3 = \{x_3, x_4, x_5, x_1\}, B_3 = \{x_3, x_4\};$$

$$A_4 = \{x_4, x_5, x_1, x_2\}, B_4 = \{x_4, x_5\}.$$

Example

Suppose that the data set is rationalizable by a linear order.

$$A_1 = \{x_1, x_2, x_3, x_4\}, B_1 = \{x_1, x_2\};$$

$$\implies \text{(1a) } x_1 P^* x_2,\ x_1 P^* x_3,\ x_1 P^* x_4, \text{ or (1b) } x_2 P^* x_1,\ x_2 P^* x_3,\ x_2 P^* x_4;$$

$$A_2 = \{x_2, x_3, x_4, x_5\}, B_2 = \{x_2, x_3\};$$

$$A_3 = \{x_3, x_4, x_5, x_1\}, B_3 = \{x_3, x_4\};$$

$$A_4 = \{x_4, x_5, x_1, x_2\}, B_4 = \{x_4, x_5\}.$$

Example

Suppose that the data set is coarsely rationalizable by a linear order.

(1a) $x_1 P^* x_2$, $x_1 P^* x_3$, $x_1 P^* x_4$, or (1b) $x_2 P^* x_1$, $x_2 P^* x_3$, $x_2 P^* x_4$;

(2a) $x_2 P^* x_3$, $x_2 P^* x_4$, $x_2 P^* x_5$, or (2b) $x_3 P^* x_2$, $x_3 P^* x_4$, $x_3 P^* x_5$;

(3a) $x_3 P^* x_1$, $x_3 P^* x_4$, $x_3 P^* x_5$, or (3b) $x_4 P^* x_1$, $x_4 P^* x_3$, $x_4 P^* x_5$;

(4a) $x_4 P^* x_1$, $x_4 P^* x_2$, $x_4 P^* x_5$, or (4b) $x_5 P^* x_1$, $x_5 P^* x_2$, $x_5 P^* x_4$.

Example

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Considering all $2^{|\mathcal{O}|}$ possible combinations?

Suppose that $\mathcal{O}$ is coarsely rationalizable by a linear order.

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Consider any nonempty subcollection $\mathcal{O}' = \{(A_{k_j}, B_{k_j})\}_{j=1}^{m}$ of $\mathcal{O}$.

$\implies$ the maximal element in A_{k_j} is not contained in C_{k_j}, $\forall j$

$\implies$ the maximal element in $A(\mathcal{O}')$ is not contained in $C(\mathcal{O}')$.

Necessary condition for coarse rationalizability that we call Coarse SARP:

Coarse SARP. For any $\emptyset \neq \mathcal{O}' \subseteq \mathcal{O}$, $A(\mathcal{O}') \setminus C(\mathcal{O}') \neq \emptyset$.

Example

A coarse data set including four observations:

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$$A_2 = \{x_2, x_3, x_4, x_5\}, B_2 = \{x_2, x_3\}, C_2 = \{x_4, x_5\};$$

$$A_3 = \{x_3, x_4, x_5, x_1\}, B_3 = \{x_3, x_4\}, C_3 = \{x_5, x_1\};$$

$$A_4 = \{x_4, x_5, x_1, x_2\}, B_4 = \{x_4, x_5\}, C_4 = \{x_1, x_2\}.$$

$A(\mathcal{O}) \setminus C(\mathcal{O}) = \emptyset$

$\implies$ Violation of Coarse SARP

$\implies$ Not coarsely rationalizable by a linear order.

Coarse SARP is also a sufficient condition.

Theorem

A coarse data set is coarsely rationalizable by a linear order

if and only if

it satisfies the Coarse SARP property.

Illustrating the proof using an example

Example

Consider the following coarse data set including five observations:

	A_i	B_i	C_i
$i = 1$	$\{x_1, x_2, x_3, x_4\}$	$\{x_1, x_2\}$	$\{x_3, x_4\}$
$i = 2$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3\}$	$\{x_4, x_5\}$
$i = 3$	$\{x_3, x_4, x_5, x_6\}$	$\{x_3, x_4\}$	$\{x_5, x_6\}$
$i = 4$	$\{x_4, x_5, x_6, x_7\}$	$\{x_4, x_5\}$	$\{x_6, x_7\}$
$i = 5$	$\{x_5, x_6, x_7, x_1\}$	$\{x_5, x_6, x_7\}$	$\{x_1\}$

Let $\mathcal{O}_1 := \mathcal{O}$.

	A_i	B_i	C_i
$i=1$	$\{x_1, x_2, x_3, x_4\}$	$\{x_1, x_2\}$	$\{x_3, x_4\}$
$i=2$	$\{x_2, x_3, x_4, x_5\}$	$\{x_2, x_3\}$	$\{x_4, x_5\}$
$i=3$	$\{x_3, x_4, x_5, x_6\}$	$\{x_3, x_4\}$	$\{x_5, x_6\}$
$i=4$	$\{x_4, x_5, x_6, x_7\}$	$\{x_4, x_5\}$	$\{x_6, x_7\}$
$i=5$	$\{x_5, x_6, x_7, x_1\}$	$\{x_5, x_6, x_7\}$	$\{x_1\}$

Then $A(\mathcal{O}_1) \setminus C(\mathcal{O}_1) = \{x_2\}$.

Let $P_1 := A(\mathcal{O}_1) \setminus C(\mathcal{O}_1) = \{x_2\}$.

Rank x above y if $x \in P_1$ and $y \in A(\mathcal{O}) \setminus P_1$.

Let $\mathcal{O}_2 := \{(A_i, B_i) \in \mathcal{O}_1 : A_i \cap P_1 = \emptyset\}$.

	A_i	B_i	C_i
$i = 3$	$\{x_3, x_4, x_5, x_6\}$	$\{x_3, x_4\}$	$\{x_5, x_6\}$
$i = 4$	$\{x_4, x_5, x_6, x_7\}$	$\{x_4, x_5\}$	$\{x_6, x_7\}$
$i = 5$	$\{x_5, x_6, x_7, x_1\}$	$\{x_5, x_6, x_7\}$	$\{x_1\}$

Repeat this logic...

$\mathcal{O}$ is finite...

Strict partial order $\rightarrow$ linear order.

Coarse SARP and the classical SARP

In the special case that B_i is a singleton set for each i,

Coarse SARP reduces to the classcial SARP.

Both directions are easy to verify.

Application 1: Rational choice with imperfect observation

We represent the observed behavior of the DM by (Σ, f), where

$\Sigma \subset \mathcal{X}$,

$f(A)$ is superset of the choice of the DM in $A \in \Sigma$.

Application 2: Multiple preferences

The DM has a set $\triangleright$ of strict preferences, and she chooses

$$f_{\triangleright}(A) := \{x \in A : x = \max(A, \succ) \text{ for some } \succ \in \triangleright\}$$

from each feasible set A.

See, for example, *Salant and Rubinstein (2008)*.

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We say that (Σ, f) is rationalizable by multiple preferences if there exists a set $\triangleright$ of strict preferences such that

$$f_{\triangleright}(A) = f(A)$$

for all $A \in \Sigma$.

Divide and conquer

For each $A \in \Sigma$ and $x \in f(A)$, we construct a coarse data set $\mathcal{O}_{A,x}$ indexed by (A, x) as follows:

$$\mathcal{O}_{A,x} := \{(A', f(A'))\}_{A' \in \Sigma, A' \neq A} \cup (A, x).$$

Let

$$\mathfrak{D} := \{\mathcal{O}_{A,x}\}_{A \in \Sigma,\, x \in f(A)}.$$

A necessary condition for the data set (Σ, f) to be rationalizable by multiple preferences is that each $\mathcal{O}_{A,x}$ constructed in this way is rationalizable by a linear order.

Theorem

(Σ, f) *is rationalizable by multiple preferences*

if and only if

each $\mathcal{O}_{A,x}$ *in* $\mathfrak{D}$ *is rationalizable by a linear order.*

Application 3: Minimax regret

Let $u : X \to R$ be a utility function for the DM.

Under u, the regret of choosing x instead of y is $u(y) - u(x)$.

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Given a finite set of utility functions $\mathcal{U}$, the worst-case regret of choosing x from $A \in \mathcal{X}$ is

$$\max_{y \in A} \max_{u \in \mathcal{U}} \left[u(y) - u(x) \right].$$

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The DM has a finite set of utility functions $\mathcal{U}$ defined on X and she chooses

$$\min_{x \in A} \left\{ \max_{y \in A} \max_{u \in \mathcal{U}} \left[u(y) - u(x) \right] \right\}.$$

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We say that (Σ, f) is rationalizable under the minimax regret model if there is a finite set of utility functions $\mathcal{U}$ such that

$$f(A) = \arg\min_{x \in A} \left\{ \max_{y \in A} \max_{u \in \mathcal{U}} [u(y) - u(x)] \right\}.$$

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for each $A \in \Sigma$.

For simplicity, write $\phi(x, y) = \max_{u \in \mathcal{U}} [u(y) - u(x)]$.

Suppose that (Σ, f) includes the following observation $f(\{x, y, z\}) = x$.

It must be the case that

$$\max\{\phi(y, x), \phi(y, z)\} > \max\{\phi(x, y), \phi(x, z)\}$$

and

$$\max\{\phi(z, x), \phi(z, y)\} > \max\{\phi(x, y), \phi(x, z)\}.$$

Construct a corresponding coarse data set...

Related Literature

Fishburn (1976)

- Partial congruence axiom

de Clippel and Rosen (2018)

- Bounded rationality theories under incomplete data
- Enumeration procedure

Hu et al. (2018)

- Explore related ideas in different settings
- Weak order