Inference on average welfare with high-dimensional state space

Victor Chernozhukov, Whitney Newey, Vira Semenova

August 6, 2019

1 Set-Up

1.1 Motivating examples

We are interested in weighted average welfare

$$\theta_0 = \mathbb{E}w(x)V(x), \tag{1}$$

where $x \in \mathcal{X}$ is the state variable $\mathcal{X} \subset \mathcal{R}^{d_x}$, $w(x) : \mathcal{X} \to \mathcal{R}$ is a known function, and $V(x)$ is the expected value function. There are many interesting objects can be represented as (1). For one example, $w(x) = 1$ corresponds to the average welfare. Another interesting example is the average effect of changing the conditioning variables according to the map $x \to t(x)$. The object of interest is the average policy effect of a counterfactual change of covariate values

$$\theta_0 = \mathbb{E}[V(t(x)) - V(x)] = \int \left(\frac{f_t(x)}{f(x)} - 1 \right) V(x) f(x) dx, \tag{2}$$

where $f_t(x)$ is p.d.f. of $t(x)$ and $w(x) = \frac{f_t(x)}{f(x)} - 1$.

A third example is the average partial effect of changing the subvector $x_1 \subset x$. Assume that x_1 has a conditional density given x_{-1} and $\mathcal{X}$ has bounded support. Then, average partial effect takes the form

$$\mathbb{E}\partial_{x_1} V(x) = \mathbb{E} \left(\frac{\partial_{x_1} f(x_1|x_{-1})}{f(x_1|x_{-1})} \right) V(x), \tag{3}$$

where $w(x) = -\frac{\partial_{x_1} f(x_1|x_{-1})}{f(x_1|x_{-1})}$. A fourth example is the average marginal effect of shifting the distribution of x by vector $c \in \mathcal{R}^{d_x}$

$$\mathbb{E}\partial_c V(x+c) = \mathbb{E} \left(\nabla_c \frac{f_0(x-c)}{f_0(x)} \right) V(x), \tag{4}$$

where $w(x) = \nabla_c \frac{f_0(x-c)}{f_0(x)}$.

Now let us introduce the primitives of the single-agent dynamic discrete choice problem that give rise to the value function $V(x)$. In every period $t \in \mathcal{N}$, the agent observes current value of (x_t, ϵ_t) and chooses an action a_t in a finite choice set $\mathcal{A} = \{1, 2, \dots, J\}$. His utility from action a is equal to $u(x, a) + \epsilon(a)$, where $u(x, a)$ is the structural part that may depend on unknown parameters, and $\epsilon(a)$ is the shock unobserved to the researcher. Under standard assumptions (Assumptions 1,2) of Aguirregabiria and Mira (2002), the maximum ex-ante value at state x is equal to

$$V(x) = \mathbb{E} \max_{a \in \mathcal{A}} v(x, a) := \mathbb{E} \max_{a \in \mathcal{A}} \big[u(x, a) + \epsilon(a) + \beta \mathbb{E}[V(x')|x, a]\big] g(\epsilon) d\epsilon \tag{5}$$

where $\beta < 1$ is the discount factor, $g(\epsilon)$ is the density of the vector $(\epsilon(a))_{a \in \mathcal{A}}$ and

$$v(x, a) := u(x, a) + \beta \int_{x' \in \mathcal{X}} V(x') f(x'|x, a) \tag{6}$$

is the choice-specific value function that is equal to expected value from choosing the action a in the state x. To estimate value function, many methods require the estimate of the transition density $f(x'|x, a), a \in \mathcal{A}$ and the vector of conditional choice probabilities $p(x) = (p(1|x), p(2|x), \dots, p(J|x))$ as a first stage.

The objective of this paper is to find an estimator $\widehat{\theta}$ of the target parameter θ_0 that is asymptotically equivalent to a sample average, while allowing the state space $\mathcal{X}$ to be high-dimensional (i.e., $d_x \geqslant N$) and having the first-stage parameters $f(x'|x, a), p(x)$to be estimated by modern machine learning tools. Specifically, suppose a researcher has an i.i.d sample $(z_i)_{i=1}^N$, where a generic observation $z_i = (x_i, a_i, x_i'), i \in \{1, 2, \dots, N\}$ consists of the current state x, discrete action $a \in \mathcal{A}$, and the future state x'. Our goal is to construct a moment function $m(z; \gamma)$ for θ_0

$$\theta_0 = \mathbb{E} m(z; \gamma_0),$$

such that the estimator $\widehat{\theta} = \frac{1}{N} \sum_{i=1}^N m(z_i; \widehat{\gamma})$ is asymptotically linear:

$$\widehat{\theta} = \frac{1}{N} \sum_{i=1}^N m(z_i, \gamma_0) + O_P(N^{-1/2}). \tag{7}$$

The parameter γ contains the transition density $f(x'|x, a)$ and the vector of CCPs $(p(a|x))_{a \in \mathcal{A}}$, but may contain more unknown functions of x. It will be estimated on an auxiliary sample.

To achieve asymptotic linearity (7), the moment function $m(z_i, \gamma_0)$ must be locally insensitive (or, formally, orthogonal Chernozhukov et al. (2017a) or locally robust Chernozhukov et al. (2017b)) with respect to the biased estimation of $\widehat{\gamma}$. To introduce the condition, let Γ_N be a shrinking neighborhood of γ_0 that contains the first-stage estimate $\widehat{\gamma}$ w.p. $1 - o(1)$. A moment function $m(z;\gamma)$ is locally robust with respect to γ at γ_0 if

$$\partial_r \mathbb{E} m(z; r(\gamma - \gamma_0) + \gamma_0) = 0, \quad \forall \gamma \in \Gamma_N. \tag{8}$$

In Section 1.2, we show that the moment function (1) is already orthogonal with respect to the CCPs for any weighting function $w(x)$. In Section 1.3, we construct the moment function $m(z;\gamma)$ that is orthogonal with respect to the transition density function.

1.2 Orthogonality with respect to the CCP

That the value function is orthogonal with respect to the CCP has been first shown in Aguirregabiria and Mira (2002) for a finite state space $\mathcal{X}$. In this paper, we present an alternative argument that leads to the same conclusion for an arbitrary $\mathcal{X}$.

Let $p(x) = (p(1|x), p(2|x), \dots, p(J|x))$ be a J-vector of the CCPs and let $p_r(x) = r(p(x) - p_0(x)) + p_0(x))$ be a one-dimensional path in the space of J-vector functions; the vector $p_0(x)$ is the vector of true CCPs. Plugging in p_r into (5) and taking the derivative with respect to r, we obtain

$$\partial_r V(x; p_r; f_0)\Big|_{r=0} = \beta \int_{\epsilon \in \mathcal{E}} \int_{x' \in \mathcal{X}} \partial_r V(x'; p_r; f_0)\Big|_{r=0} f_0(x'|x, a^*(\epsilon)) g(\epsilon) dx' d\epsilon,$$

where $a^*(\epsilon) = \arg\max_{a \in \mathcal{A}} (v(x,a) + \epsilon(a))$ is the optimal action as a function of shock ϵ. As shown in Lemma 3, the map $\Gamma : \mathcal{F}_2 \to \mathcal{F}_2$ defined on the space of L_2-integrable functions $\mathcal{F}_2$

$$\Gamma(x, \phi) := \beta \int_{\epsilon \in \mathcal{E}} \int_{x' \in \mathcal{X}} \phi(x'; p; f_0) f(x'|x, a^*(\epsilon)) g(\epsilon) dx' d\epsilon. \tag{9}$$

is a contraction mapping and thus has a unique fixed point. Therefore, $\partial_r V(x; p_r; f_0) = 0 \quad \forall x \in \mathcal{X}$. Therefore, when the nuisance parameter γ consists of the CCPs $p(x)$, the moment equation (1) obeys orthogonality condition (8) with respect to γ.

1.3 Orthogonality with respect to the transition density

ASSUMPTION 1 (Stationarity)**.**

For any positive number $k \geqslant 0$, any sequence $(x_t, x_{t+1}, \dots, x_{t+j}, \dots)$ has the same distribution as $(x_{t+k}, x_{t+1+k}, \dots, x_{t+j+k}, \dots)$.

To derive the bias correction term for the transition density, consider the case $w(x) = 1$. Recall that value function obeys a recursive property (Aguirregabiria and Mira (2002)):

$$V(x; p; f) = \tilde{U}(x; p) + \beta \mathbb{E}_f[V(x'; p; f)|x], \tag{10}$$

where $\tilde{U}(x; p) = \sum_{a \in \mathcal{A}} p(a|x)(u(x, a) + e_x(a; p))$ is the expected current utility and $e_x(a; p)$ is the expected shock conditional on x and a being the optimal action. Consider a one-dimensional parametric submodel $\{f(x'|x, \tau)\}, \tau \geqslant 0$ where $f(x'|x, \tau = \tau_0)$ is the true value of the density. Taking the derivative of (10) w.r.t τ gives

$$\begin{aligned}\partial_\tau V(x; p; f) &= \beta \mathbb{E}[\partial_\tau V(x'; p; f)|x] + \beta \int V(x'; p; f) \partial_\tau f(x'|x; \tau) dx' \\ &= \beta \mathbb{E}_f[\partial_\tau V(x'; p; f)|x] + \beta \mathbb{E} V(x'; p; f) S(x'|x) dx',\end{aligned}$$

where $S(x'|x) = \frac{\partial_\tau f(x'|x,\tau)}{f(x'|x,\tau)}\big|_{\tau=\tau_0}$ is the conditional score. Taking expectations w.r.t x and incurring Assumption 1 gives the expression for the derivative

$$\partial_\tau \mathbb{E} V(x; p; f) = \frac{\beta}{1-\beta} \mathbb{E} V(x'; p; f) S(x'|x) dx'$$

and the expression for the bias correction term is

$$\frac{\beta}{1-\beta} \left(V(x'; p; f) - \ \mathbb{E}_f[V(x'; p; f)|x] \right), \tag{11}$$

where the first-stage parameter $\gamma = \{p(x), f(x'|x, a)\}$ consists of the CCPs $p(x)$, the transition density $f(x'|x, a)$.

Remarkably, we do not require a consistent estimator of the transition density when the weighting function $w(x) = 1$.

Remark 1 (Double Robustness with respect to the transition density)**.**
Here we show that (20) is not only orthogonal to $f(x'|x, a)$, but also robust to its misspecification. Rewriting (10), we express

$$\mathbb{E}_f[V(x')|x] = \frac{1}{\beta} \left(V(x; p; f) - \tilde{U}(x; p) \right) \tag{12}$$

*and note that it holds for **any** $p(x)$ and **any** $f(x'|x, a)$. Plugging (12) into (20) gives an orthogonal moment*

$$m(z; \gamma) = V(x; p; f) + \frac{\beta}{1-\beta} V(x'; p; f) - \frac{V(x) - \tilde{U}(x; p)}{1-\beta}. \tag{13}$$

Let $\Delta[m(z;\gamma)] := m(z;p;f;\lambda_0) - m(z;p;f_0;\lambda_0)$ be the specification error of the transition density $f(x'|x,a)$. Then, specification bias of the transition density is

$$\mathbb{E}\Delta[m(z;\gamma)] = \frac{\beta}{1-\beta}\mathbb{E}[\Delta V(x;p) - \Delta V(x';p)] = 0, \tag{14}$$

where the last equality follows from the stationarity assumption.

Now we present the density correction term for an arbitrary function $w(x)$. Define the function

$$\lambda(x) = \sum_{k \geqslant 0} \beta^k \mathbb{E}[w(x_{-k})|x], \tag{15}$$

where x_{-k} is the k-period lagged realization of x. Alternatively, $\lambda(x)$ can be implicitly defined as a solution to the recursive equation

$$w(x') - \lambda(x') + \beta\mathbb{E}[\lambda(x)|x'] = 0. \tag{16}$$

The bias correction term takes the form

$$\beta\lambda(x)\left(V(x';p;f) - \mathbb{E}_f[V(x';p;f)|x]\right), \tag{17}$$

where the first-stage parameter $\gamma = \{p(x), f(x'|x,a), \lambda(x)\}$ consists of the CCPs $p(x)$, the transition density $f(x'|x,a)$, and $\lambda(x)$. The property (16), which is the generalization of (14), ensures that (20) is doubly robust in $\lambda(x), f(x'|x,a)$.

1.4 Orthogonality with respect to the structural parameter

To derive the bias correction term for the structural parameter, consider the case $w(x) = 1$. Let δ be the structural parameter of the per-period utility function $u_a(x;\delta), a \in \{1, 2, \ldots, J\}$. Taking the derivative of (10) w.r.t δ gives

$$\partial_\delta V(x;p;f) = \sum_{a \in \mathcal{A}} p(a|x)\partial_\delta u_a(x;\delta) + \beta\mathbb{E}[\partial_\delta V(x';p;f)|x].$$

The derivative of $\partial_\delta \mathbb{E}V(x;p;f)$ takes the form

$$\partial_\delta \mathbb{E}V(x;p;f) = \frac{1}{1-\beta}\mathbb{E}\sum_{a \in \mathcal{A}} p(a|x)\partial_\delta u_a(x;\delta).$$

As shown in Chernozhukov et al. (2015), the orthogonal moment takes the form

$$\begin{aligned} m(z;\gamma) &:= \left(1 - \partial_\delta \mathbb{E}V(x;p;f)(\partial_\delta \mathbb{E}V(x;p;f)^\top \partial_\delta \mathbb{E}V(x;p;f))^{-1}\partial_\delta \mathbb{E}V(x;p;f)^\top\right) \\ &\left(w(x)V(x;p;f) + \beta\lambda(x)\left(V(x';p;f) - \mathbb{E}_f[V(x';p;f)|x]\right)\right). \end{aligned}$$

For an arbitrary function $w(x)$, define

$$G_\delta := \partial_\delta \mathbb{E} w(x) V(x; p; f) = \frac{1}{1-\beta} \mathbb{E} \lambda(x) \sum_{a \in \mathcal{A}} p(a|x) \partial_\delta u_a(x; \delta).$$

where $\lambda(x)$ is as defined in (15). The orthogonal moment takes the form

$$m(z;\gamma) := \left(1 - G_\delta (G_\delta^\top G_\delta)^{-1} G_\delta^\top\right) \left(w(x) V(x;p;f) + \beta \lambda(x) \left(V(x';p;f) - \mathbb{E}_f[V(x';p;f)|x]\right)\right). \quad (18)$$

2 Asymptotic Theory

ASSUMPTION 2 (Quality of the first-stage parameters)**.** *A There exists a sequence of neighborhoods $\mathcal{T}_N \subset \mathcal{T}$ such that the following conditions hold. (1) The true vector of CCPs $p_0(x) \in \mathcal{T}_N \quad \forall N \geqslant 1$. (2) There exists a sequence $\Delta_N = o(1)$, such that w.p. at least $1 - \Delta_N$, the estimator $\widehat{p}(x) \in \mathcal{T}_N$. (3) There exists a sequence $p_N = o(N^{-1/4})$ such that $\sup_{p \in \mathcal{T}_N} \|p(x) - p_0(x)\|_2 = O(p_N)$.*

B There exists $W < \infty$ and $V < \infty$ such that $\|w(x)\|_\infty \leqslant W$ and $\|V(x)\|_\infty \leqslant V$. There exists $\epsilon > 0$ such that $\epsilon < p(a|x) < 1 - \epsilon < 1, \quad \forall a \in \mathcal{A} \forall x \in \mathcal{X}$. There exists $E < \infty$ such that $\forall x \in \mathcal{X}, \sup_{p \in \mathcal{T}_N} \sup_{x \in \mathcal{X}} \|\partial_{pp} e(x;p)\|_\infty \leqslant E$.

C There exists a sequence of neighborhoods $\Gamma_N \subset \Gamma$ such that the following conditions hold. (1) The true nuisance parameter $\gamma_0 = \{f(x'|x,a), \lambda_0(x)\} \in \Gamma_N \quad \forall N \geqslant 1$. (2) There exists a sequence $\Delta_N = o(1)$, such that w.p. at least $1 - \Delta_N$, the estimator $\widehat{\gamma}(x) \in \Gamma_N$. (3) There exist $p, q > 0 : \quad p + q = 1$ and sequences $\lambda_N = o(1)$ and f_N such that

$$\sup_{(f;\lambda) \in \Gamma} \sup_{a \in \mathcal{A}} \|\lambda(x) - \lambda_0(x)\|_p \|f(x'|x,a) - f_0(x'|x,a)\|_q = O(\lambda_N f_N) = o(N^{-1/2})$$

$$\sup_{(f;\lambda) \in \Gamma} \sup_{a \in \mathcal{A}} \|(\lambda(x) - \lambda_0(x))(f(x'|x,a) - f_0(x'|x,a))\|^2 = O(r_N') = o(N^{-1/2})$$

Theorem 1 (Asymptotic normality with known transition density)**.**
Let the following assumptions hold. (1) The transition function $f(x'|x,a)$ is known. Assumption 1 holds. Assumption 2 (A)-(B) hold. (2) Then, asymptotic linearity 7 holds for the moment function

$$m(z;\gamma) = w(x) V(z;p;f_0). \quad (19)$$

Theorem 2 (Asymptotic theory in the general case)**.**
Let the following assumptions hold. Under Assumption 1 and 2, asymptotic linearity 7 holds for

the moment function

$$m(z;\gamma) := w(x)V(x;p;f) + \beta\lambda(x)\left(V(x';p;f) - \sum_{a\in\mathcal{A}} \mathbb{E}_f[V(x';p;f)|x,a]p(a|x)\right), \tag{20}$$

and $\gamma = \{\{(p(a|x))_{a\in\mathcal{A}}\}, f(x'|x,a), \lambda(x)\}$.

3 Appendix

Lemma 3 (Orthogonality with respect to CCP)**.**

Value function is orthogonal with respect to estimation error of CCP:

$$\partial_r V(x;p_r;f_0) = 0 \quad \forall x \in \mathcal{X}.$$

Proof. Let $\mathcal{F}_k = \{h(x), \|h(x)\|_k \leqslant B\}$ is a subset of functions $h(x)$ that are bounded in the norm k. Throughout the paper, we will focus on two norms: $k = 2$, defined as $\|h(x)\|_2 := (\int_{\mathcal{X}} h^2(x)dx)^{1/2}$ and $\|h(x)\|_\infty := \sup_{x\in\mathcal{X}} |h(x)|$.To prove the theorem, we will show that $\Gamma(\phi) : \mathcal{F}_k \to \mathcal{F}_k$ is a contraction mapping for $k = \infty$. Moreover, if Assumption 1 holds, it is a contraction mapping for $k = 2$. Since $\phi(x) = 0 \quad \forall x \in \mathcal{X}$ is a fixed point of (9), contraction property implies the uniqueness of this solution.

Step 1. Proof for $k = \infty$. First, let us show that for any function $\phi(x) \in \mathcal{F}_\infty$, $\Gamma(\phi) \in \mathcal{F}_\infty$ holds. Indeed,

$$\begin{aligned}
&\|\Gamma(\phi)\|_\infty = \beta \sup_{x\in\mathcal{X}} \Big|\int_{x'\in\mathcal{X}} \int_{\epsilon\in\mathcal{E}} \phi(x')f(x'|x,a^*(\epsilon))g(\epsilon)dx'd\epsilon\Big| \\
&\leqslant \sup_{x\in\mathcal{X}'} |\phi(x')| \int_{x'\in\mathcal{X}} \int_{\epsilon\in\mathcal{E}} f(x'|x,a^*(\epsilon))g(\epsilon)dx'd\epsilon \\
&= \sup_{x\in\mathcal{X}'} |\phi(x')| \underbrace{\int_{\epsilon\in\mathcal{E}} d\epsilon g(\epsilon)}_{=1} \underbrace{\sum_{a\in\mathcal{A}} 1_{[\epsilon(a)+v(x,a)=\arg\max_j \epsilon(j)+v(x,j)]}}_{=1} \underbrace{\int_{x'} f(x'|x,a)dx'}_{=1} \\
&\leqslant \|\phi(x)\|_\infty,
\end{aligned}$$

as long as $\mathcal{X}' \subseteq \mathcal{X}$. Therefore, $\Gamma(\phi) : \mathcal{F}_\infty \to \mathcal{F}_\infty$. Moreover, for two elements ϕ_1 and ϕ_2 from $\mathcal{F}_\infty$

$$\begin{aligned}
\|\Gamma(\phi_1) - \Gamma(\phi_2)\|_\infty &\leqslant \beta \int_{\epsilon\in\mathcal{E}} \int_{x'\in\mathcal{X}} (\phi_1(x') - \phi_2(x'))f(x'|x,a^*(\epsilon))g(\epsilon)dx'd\epsilon \\
&\leqslant \beta\|\phi_1 - \phi_2\|_\infty \int_{\epsilon\in\mathcal{E}} \int_{x'\in\mathcal{X}} f(x'|x,a^*(\epsilon))g(\epsilon)dx'd\epsilon \\
&= \beta\|\phi_1 - \phi_2\|_\infty
\end{aligned}$$

and $\Gamma : \mathcal{F}_\infty \to \mathcal{F}_\infty$ is a contraction mapping.

Step 2. Proof for $k = 2$. First, let us show that for any function $\phi(x) \in \mathcal{F}_2$, $\Gamma(\phi) \in \mathcal{F}_2$ holds.

$$\|\Gamma(\phi)\|_2 = \beta\|\mathbb{E}[\phi(x')|x]\|_2 \leqslant^i \beta\|\mathbb{E}\phi(x')\|_2 =^{ii} \beta\|\mathbb{E}\phi(x)\|_2,$$

where i is by the property of conditional expectation and ii is by stationarity. Therefore, $\Gamma(\phi) : \mathcal{F}_2 \to \mathcal{F}_2$. Moreover, for two elements ϕ_1 and ϕ_2 from $\mathcal{F}_\infty$

$$\|\Gamma(\phi_1) - \Gamma(\phi_2)\|_2 \leqslant \beta\|\phi_1 - \phi_2\|_2,$$

and $\Gamma : \mathcal{F}_2 \to \mathcal{F}_2$ is a contraction mapping.

■

Define the following operators that map $\mathcal{F}_k \to \mathcal{F}_k$:

$$A\phi := \phi - \beta \int_{\mathcal{X}'} \phi(x')f(x'|x,a)dx' \sum_{a\in\mathcal{A}} p(a|x) \tag{21}$$

and

$$\widehat{A}\phi := \phi - \beta \int_{\mathcal{X}'} \phi(x')f(x'|x,a)dx' \sum_{a\in\mathcal{A}} \widehat{p}(a|x). \tag{22}$$

Then, $V(x;\widehat{p};f_0)$ solves the integral equation of the second kind:

$$\widehat{A}V(x;\widehat{p};f_0) = \tilde{U}(x;\widehat{p})$$

and $V(x;p_0;f_0)$ solves

$$AV(x;p_0;f_0) = \tilde{U}(x;p_0).$$

Lemma 4 and 5 show that $\|V(x;\widehat{p};f_0) - V(x;p_0;f_0)\|_k = O(\sum_{a\in\mathcal{A}} \|\widehat{p}(a|x) - p(a|x)\|_k)$.

Lemma 4 (Verification of the regularity conditions)**.**
The following statements hold. (1) Either $k = \infty$ and $\mathcal{X}' \subset \mathcal{X}$ or Assumption 1 holds with $k = 2$. (2) Assumptions 2 [A], [B] hold.

1. $\|A^{-1}\|_k \leqslant \frac{1}{1-\|I-A\|_k} \leqslant \frac{1}{1-\beta}$.
2. $\|A^{-1}(\widehat{A} - A)\|_k = o(1)$

Proof. Step 1. Proof of (1). Let us show that $\forall k \in \{2,\infty\} \quad \|(I-A)\|_k \leqslant \beta < 1$. Then, A^{-1} is the sum of geometric series $A^{-1} = \sum_{l\geqslant 0}(I-A)^l$ and has a bounded norm: $\|A^{-1}\| \leqslant \frac{1}{1-\|I-A\|} \leqslant \frac{1}{1-\beta}$.

- Case $k = \infty$.For any $\phi \in \mathcal{F}_\infty$, $\|(I - A)\phi\| = \beta\|\mathbb{E}[\phi(x')|x]\| \leqslant \beta \sup_{x' \in \mathcal{X}'} \|\phi(x')\| \leqslant \beta\|\phi\|$.
- Case $k = 2$. Suppose Assumption 1 holds. For any $\phi \in \mathcal{F}_2$,

$$\|(I - A)\phi\| = \beta\|\mathbb{E}[\phi(x')|x]\| \leqslant \beta\|\mathbb{E}[\phi(x')]\| = \|\mathbb{E}[\phi(x)]\|.$$

Proof of (2): Fix $\phi(x) \in \mathcal{F}_\infty$. Fix an action $1 \in \mathcal{A} = \{1, 2, \ldots, J\}$. We plug $p(1|x) := 1 - \sum_{a=2}^J p(a|x)$ and $\widehat{p}(1|x) := 1 - \sum_{a=2}^J \widehat{p}(a|x)$ into (21) and (22).

$$i := (\widehat{A} - A)\phi(x) = \beta \sum_{a=2}^{J} (\widehat{p}(a|x) - p(a|x)) \int \phi(x')(f(x'|x, a) - f(x'|x, 1))dx'.$$

Case $k = \infty$.

$$\begin{aligned}
\|i\| &\leqslant \beta \sum_{a=2}^{J} \sup_{x \in \mathcal{X}} |\widehat{p}(a|x) - p(a|x)| \sup_{x \in \mathcal{X}} |\int \phi(x')(f(x'|x, a) - f(x'|x, 1))dx'| \\
&\leqslant \beta \sum_{a=2}^{J} \sup_{x \in \mathcal{X}} |\widehat{p}(a|x) - p(a|x)| \sup_{x \in \mathcal{X}'} |\phi(x')|| \sup_{x \in \mathcal{X}} \int |(f(x'|x, a) - f(x'|x, 1))|dx' \\
&= \beta \sum_{a=2}^{J} \sup_{x \in \mathcal{X}} |\widehat{p}(a|x) - p(a|x)| \|\phi\| \sup_{x \in \mathcal{X}} \int |(f(x'|x, a) - f(x'|x, 1))|dx' = o(1)
\end{aligned}$$

Case $k = 2$.

$$\begin{aligned}
\|(\widehat{A} - A)\phi(x)\| &\leqslant^{i} J\beta \sum_{a=2}^{J} \|(\widehat{p}(a|x) - p(a|x)) \int \phi(x')(f(x'|x, a) - f(x'|x, 1))\|_2 \\
&\leqslant^{ii} J\beta \sum_{a=2}^{J} \|(\widehat{p}(a|x) - p(a|x))\|_2 \| \int \phi(x')(f(x'|x, a) - f(x'|x, 1))\|_2 \\
&\leqslant^{iii} J\beta \|\phi(x')\|_2 \sum_{a=2}^{J} \|(\widehat{p}(a|x) - p(a|x))\|_2 \|(f(x'|x, a) - f(x'|x, 1))\|_2 \\
&\leqslant^{iv} \|\phi(x)\|_2 \big[\beta J \sum_{a=2}^{J} \|(\widehat{p}(a|x) - p(a|x))\|_2 \|(f(x'|x, a) - f(x'|x, 1))\|_2\big] = o(1),
\end{aligned}$$

where *i*-*iii* is by Cauchy-Scwartz, and *iv* $\|\phi(x')\|_2 = \|\phi(x)\|_2$ is by Assumption 1.

■

Lemma 5 (Second-order effect of CCPs)**.**
The following statements hold. (1) Either $k = \infty$ and $\mathcal{X}' \subset \mathcal{X}$ or Assumption 1 holds with $k = 2$. (2) Assumptions 2 [A], [B] hold. (3) Either $J = 2$ (binary case) or the unobserved shock $\epsilon(a), a \in \mathcal{A}$ has i.i.d. extreme value distribution. Then, the following bounds hold:

$$\|V(x; \widehat{p}; f_0) - V(x; p_0; f_0)\|_k = O(\sum_{a \in \mathcal{A}} \|\widehat{p}(a|x) - p(a|x)\|_k^2) \tag{23}$$

Proof. We apply Theorem 9 with A defined in (21), $\widehat{A}$ defined in (22), $\widehat{\xi} = \tilde{U}(x;\widehat{p})$ and $\xi = \tilde{U}(x;p)$. The conditions of Theorem 9 are verified in Lemma 4.

$$
\begin{aligned}
(\widehat{A} - A)V(x) + \widehat{\xi} - \xi &= \sum_{a=2}^{J} \Big[\big[\beta(\mathbb{E}[V(x')|x,a] - \mathbb{E}[V(x')|x,1]) + u(x;a) - u(x;1)\big](\widehat{p}(a|x) - p(a|x)) \\
&\quad + (e_x(a;\widehat{p}) - e_x(1;\widehat{p}))\widehat{p}(a|x) - (e_x(a;p) - e_x(1;p))p(a|x) \Big] \\
&= \sum_{a=2}^{J} (v(a,x) - v(1,x))(\widehat{p}(a|x) - p(a|x)) + \sum_{a\in\mathcal{A}} e_x(a;\widehat{p})\widehat{p}(a|x) - e_x(a;p)p(a|x)
\end{aligned}
$$

where i is by definition of $v(x,a)$ in (6). By Assumption 2[B], for each $a \in \mathcal{A}, e_x(a;p)$ is a continuous infinitely differentiable function of the vector $p(\cdot|x)$ with bounded derivatives. Thus, it suffices to show that for each action $a \in \{2,\dots,J\}$, for each $x \in \mathcal{X}$,

$$
\begin{aligned}
\partial_{p(a|x)} e_x(a;p)p(a|x) - \partial_{p(a|x)} e_x(1;p)(1 - \sum_{a=2}^{J} p(a|x)) + e_x(a;p) - e_x(1;p) \\
+ v(a,x) - v(1,x) = 0
\end{aligned} \tag{24}
$$

■

Lemma 6 (Derivatives of $e_x(a;p)$)**.**
Equation (24) holds if either of the following statements hold: (a) (Binary case) $J = 2$ or (b) (Logistic case).

Proof. Case (a). Binary case.

Case (b). Logistic case. $e_x(a;p) = \gamma - \log p(a|x)$ and $v(a,x) - v(1,x) = \log \frac{p(a|x)}{p(1|x)}$. Plugging these quantities into (24), we obtain

$$
\begin{aligned}
&v(a,x) - v(1,x) + \partial_{p(a|x)} e_x(a;p)(a|x) - \partial_{p(a|x)} e_x(1;p)(1 - \sum_{a=2}^{J} p(a|x)) + e_x(a;p) - e_x(1;p) \\
&= \log\frac{p(a|x)}{p(1|x)} - \frac{p(a|x)}{p(a|x)} + \frac{p(1|x)}{1 - \sum_{a=2}^{J} p(a|x)} - \log\frac{p(a|x)}{p(1|x)} = 0.
\end{aligned}
$$

■

Lemma 7 (Adjustment term for the transition density)**.**
Equation (20) is an orthogonal moment for the transition density $f(x'|x,a)$.

Proof. Now we describe the adjustment term for the transition function $f(x'|x) = \sum_{a\in\mathcal{A}} f(x'|x,a)p_0(a|x)$, where the vector of CCP $p(x)$ is fixed at the true value $p_0(x)$. We calculate the adjustment term for $\mathbb{E}w(x)V(x;\tau)$ as the limit of Gateuax derivatives as described in Ichimura and Newey (2018). Let $f_0(x',x)$ be true joint p.d.f of the future and current state. Let $h(x',x)$ be another joint p.d.f. Consider the sequence $(1-\tau)f_0(x',x)+\tau h(x',x), \tau\to 0$. Then, the adjustment term $\alpha(x)$ can be obtained from the representation

$$\partial_\tau \mathbb{E}w(x)V(x,\tau) = \int \alpha(x)h(x,x')dx'dx$$

We find $\alpha(x)$ in the three steps.

Step 1. We obtain a closed-form expression for $\partial_\tau V(x,\tau)$. Recursive equation (10) at $p_0(x)$ takes the form

$$V(x;\tau) = \tilde{U}(x;p_0) + \beta\int V(x';\tau)f(x'|x;\tau)dx' \tag{25}$$

Taking the derivative w.r.t τ gives

$$\begin{aligned}\partial_\tau V(x;\tau)\Big|_{\tau=0} &= \beta\int V(x')\partial_\tau \log f(x'|x;\tau)f(x'|x)dx' + \beta\int \partial_\tau V(x';\tau)f(x'|x)dx' \\ &= \beta\mathbb{E}[V(x')S(x'|x)|x] + \beta\mathbb{E}[\partial_\tau V(x';\tau)|x] \\ &=: \beta g(x) + \beta\mathbb{E}[\partial_\tau V(x';\tau)|x] \qquad (26)\end{aligned}$$

where $S(x'|x) = \partial_\tau \log f(x'|x,\tau)$ is the conditional score of x' given x. Plugging x' into (25) and taking expectation $\mathbb{E}_{x'}[\cdot|x]$ gives

$$\beta\mathbb{E}[\partial_\tau V(x';\tau)|x] = \beta\mathbb{E}[g(x')|x] + \beta^2\mathbb{E}[\partial_\tau V(x'';\tau)|x] \tag{27}$$

Adding (25) and (26) and iterating gives

$$\partial_\tau V(x;\tau) = \sum_{k\geqslant 0}\beta^k\mathbb{E}[g(x_k)|x]. \tag{28}$$

Step 2. The expression (28) is hard to work with since it involves the k-th period forward realization of the state variable. Using Assumption 1, we will simplify it as follows

$$\begin{aligned}\partial_\tau \mathbb{E}w(x)V(x;\tau) &= \mathbb{E}w(x)\partial_\tau V(x;\tau) \\ &=^{i} \mathbb{E}w(x)\left(\sum_{k\geqslant 0}\beta^k\mathbb{E}[g(x_k)|x]\right) = \sum_{k\geqslant 0}\beta^k\mathbb{E}w(x)g(x_k) \\ &=^{ii} \sum_{k\geqslant 0}\beta^k\mathbb{E}w(x_{-k})g(x) \qquad \text{(Stationarity)} \\ &=^{iii} \mathbb{E}\big[\sum_{k\geqslant 0}\beta^k\mathbb{E}[w(x_{-k})|x]\big]g(x) = \mathbb{E}\lambda(x)g(x) \qquad \text{(Equation 15)}\end{aligned}$$

Step 3. To obtain the adjustment term, we take the derivative w.r.t. τ inside the function $g(x)$:

$$\begin{aligned}
\frac{1}{\beta}\mathbb{E}\lambda(x)g(x) &=^{i} \mathbb{E}\lambda(x)V(x')S(x'|x) \\
&=^{ii} \partial_\tau \int \lambda(x)V(x')\frac{(1-\tau)f_0(x',x)+\tau h(x',x)}{(1-\tau)f_0(x)+\tau h(x)}\bigg|_{\tau=0} f_0(x)dx'dx \\
&= \int \lambda(x)V(x')\left(\frac{h(x',x)-f_0(x',x)}{f_0(x)} - \frac{h(x)-f_0(x)}{f_0(x)}f_0(x'|x)f_0(x)\right)dx'dx \\
&= \int \lambda(x)V(x')\big(h(x',x)-h(x)f_0(x'|x)\big)dx'dx \\
&=^{iii} \int \lambda(x)\big[V(x')-\mathbb{E}[V(x')|x]\big]h(x',x)dx'dx,
\end{aligned}$$

where i is by (26), ii is by definition of $S(x'|x) = \frac{\partial_\tau f(x'|x)}{f(x|x)}$ and iii is by definition of marginal density $h(x) = \int h(x',x)dx'$. Therefore, the adjustment term $\alpha(x)$ is given by

$$\alpha(x) = \beta\lambda(x)\big[V(x')-\mathbb{E}[V(x')|x]\big] \tag{29}$$

Combining Steps 1-3, we get

$$\partial_\tau \mathbb{E}w(x)V(x;\tau) =^{i} \mathbb{E}\lambda(x)g(x) =^{ii} \beta\int \lambda(x)\big[V(x')-\mathbb{E}[V(x')|x]\big]h(x',x)dx'dx,$$

where i is by Steps 1 and 2, and ii is by Step 3. By Ichimura and Newey (2018), the adjustment term takes the form $\beta\lambda(x)\big[V(x')-\mathbb{E}[V(x')|x]\big]$.

■

Remark 2.

Adjustment term for $w(x)=1$ Let $w(x)=1$. Then, $\lambda(x)=\frac{1}{1-\beta}$ and the adjustment term (29) takes the form

$$\alpha(x) = \frac{\beta}{1-\beta}\big[V(x')-\mathbb{E}[V(x')|x]\big]$$

Lemma 8 (Lipshitzness of $V(x;p;f)$ in transition density)**.**

Bellman equation implies

$$\begin{aligned}
\|V(x;p;f)-V(x;p;f_0)\|_\infty &\leqslant \beta\max_{a\in\mathcal{A}}\int |V(x')(f(x'|x,a)-f_0(x'|x,a))|dx' \\
&\beta\max_{a\in\mathcal{A}}\|f(x'|x,a)-f_0(x'|x,a)\|_\infty\|V(x')\|_1.
\end{aligned}$$

$$\begin{aligned}
\|V(x;p;f)-V(x;p;f_0)\|_2 &\leqslant \beta\max_{a\in\mathcal{A}}\int |V(x')(f(x'|x,a)-f_0(x'|x,a))|dx' \\
&\leqslant \beta\max_{a\in\mathcal{A}}\|f(x'|x,a)-f_0(x'|x,a)\|_2\|V(x')\|_2.
\end{aligned}$$

Proof of Theorem 1. Here we present the proof for the estimator $\widehat{p}(x)$ obtained by cross-fitting with K-folds $(I_k)_{k=1}^K$. Let $\mathcal{E}_N$ be the event that $\widehat{p}_k(x) \in \mathcal{T}_N, \quad \forall k \in \{1, 2, \ldots, K\}$. Let $\{P_N\}_{N \geqslant 1}$ be a sequence of d.g.p. such that $P_N \in \mathcal{P}_N$ for all $N \geqslant 1$. By Assumption 2 and union bound, $P_{P_N}(\mathcal{E}_N) \geqslant 1 - K\Delta_N = 1 - o(1)$.

Step 1. On the event $\mathcal{E}_N$,

$$\left| \frac{1}{n} \sum_{i \in I_k} w(x_i) V(x_i; \widehat{p}) - \frac{1}{n} \sum_{i \in I_k} w(x_i) V(x_i; p_0) \right| \leqslant \frac{\mathcal{I}_{1,k} + \mathcal{I}_{2,k}}{\sqrt{n}},$$

where

$$\begin{aligned} \mathcal{I}_{1,k} &= \mathbb{G}_{n,k}[w(x_i)V(x_i; \widehat{p}) - w(x_i)V(x_i; p_0)] \\ \mathcal{I}_{2,k} &= \sqrt{n}\mathbb{E}_{P_N}[w(x_i)V(x_i; \widehat{p}) | I_k^c] - \mathbb{E}_{P_N}[w(x_i)V(x_i; p_0)]. \end{aligned}$$

Step 2. On the event $\mathcal{E}_N$ conditionally on I_k^c,

$$\begin{aligned} \mathbb{E}[\mathcal{I}_{1,k}^2 | I_k^c] &\leqslant \mathbb{E}_{P_N}\big[(w(x_i)(V(x_i; \widehat{p}) - V(x_i; p_0))^2 | I_k^c\big] \leqslant W^2 \sup_{p \in \mathcal{T}_N} \mathbb{E}(V(x_i; p) - V(x_i; p_0))^2 \\ &\leqslant^i W^2 \sup_{p \in \mathcal{T}_N} \|\partial_{pp} e_x(a; p)\|_\infty^2 J \sup_{p \in \mathcal{T}_N} \sum_{a \in \mathcal{A}} \|p(a|x) - p_0(a|x)\|^2 \\ &\leqslant^{ii} W^2 E^2 J p_N^2, \end{aligned}$$

where i is by Lemma 5 and ii is by Assumption 2. Therefore, $\mathcal{I}_{1,k} = O_{P_N}(p_N)$ by Lemma 6.1 in Chernozhukov et al. (2017a).

Step 3.

$$\begin{aligned} |\mathcal{I}_{2,k}| &\leqslant \sup_{p \in \mathcal{T}_N} \mathbb{E}\big|w(x)(V(x; p) - V(x; p_0))\big| \leqslant^i \|w(x)\|_2 \sup_{p \in \mathcal{T}_N} \|V(x; p) - V(x; p_0)\|_2 \\ &\leqslant^{ii} \|w(x)\|_2 \sup_{p \in \mathcal{T}_N} \|\partial_{pp} e_x(a; p)\|_\infty \sup_{p \in \mathcal{T}_N} (\sum_{a \in \mathcal{A}} \|p(a|x) - p_0(x)\|_2^2) \\ &\leqslant^{iii} WBJp_N^2, \end{aligned}$$

where i is by Cauchy-Scwartz, ii is by Lemma 5 and iii is by Assumption 2.

∎

Proof of Theorem 2.

$$\begin{aligned} \mathbb{E}_{n,k}[m(z_i; \widehat{\gamma}) - m(z_i; \gamma_0)] &= \mathbb{E}_{n,k}[m(z_i; \widehat{\gamma}) - m(z_i; f_0; \widehat{p}; \widehat{\lambda})] \\ &\quad + \mathbb{E}_{n,k}[m(z_i; f_0; \widehat{p}; \widehat{\lambda}) - m(z_i; f_0; \widehat{p}; \lambda_0)] \\ &\quad + \mathbb{E}_{n,k}[m(z_i; f_0; \widehat{p}; \lambda_0) - m(z_i; f_0; p_0; \lambda_0)] \\ &=: R_{1,k} + R_{2,k} + R_{3,k}. \end{aligned}$$

On the event $\mathcal{E}_N$, for each $j \in \{1, 2, 3\}$ $|R_{j,k}| \leqslant \frac{\mathcal{I}^j_{1,k} + \mathcal{I}^j_{2,k}}{\sqrt{n}}$ where

$$\mathcal{I}^j_{1,k} = \sqrt{n}(R_{j,k} - \mathbb{E}_{P_N}[R_{j,k}|I^c_k])$$
$$\mathcal{I}^j_{2,k} = \sqrt{n}\mathbb{E}_{P_N}[R_{j,k}|I^c_k].$$

Below we construct bounds for $\mathcal{I}^j_{1,k}$ and $\mathcal{I}^j_{2,k}$ for $j \in \{1, 2, 3\}$.

Step 0. Let us prove (1) for an arbitrary $w(x)$. The specification bias of the transition density is

$$\mathbb{E}\Delta[m(z; \gamma)] = \mathbb{E}[(w(x) - \lambda(x))\Delta V(x; p)] + \mathbb{E}[\lambda(x)\Delta V(x'; p)] = i + ii$$

By Law of Iterated Expectations,

$$ii = \beta\mathbb{E}_{x'}[\mathbb{E}[\lambda(x)|x']\Delta V(x'; p)].$$

Assumption 1 implies

$$i = \mathbb{E}[(w(x') - \lambda(x'))\Delta V(x'; p)].$$

Summing i and ii yields follows by the definition of $\lambda(x)$ (16) :

$$i + ii = \mathbb{E}[(w(x') - \lambda(x') + \beta\mathbb{E}[\lambda(x)|x'])\Delta V(x'; p)] = 0.$$

Therefore, the specification error $f(x'|x, a) - f_0(x'|x, a)$ creates zero bias in (20). Thus, the bias of specification error is proportional to

$$\left|\mathbb{E}(\lambda(x) - \lambda_0(x))(V(x; p; f) - V(x; p; f_0))\right| \leqslant \beta\|V(x)\|_p \sup_{a\in\mathcal{A}} \|f(x'|x, a) - f_0(x'|x, a)\|_q,$$

where $p, q \geqslant 0 : p + q = 1$. Therefore, (20) is doubly robust with respect to transition density $f(x'|x, a)$ and $\lambda(x)$.

Step 1. Bound on $\mathcal{I}^1_{2,k}$. On the event $\mathcal{E}_N$, $|\mathcal{I}^1_{2,k}| \leqslant \sup_{\gamma\in\Gamma_N} |\mathbb{E}_{P_N}[m(z_i; \gamma) - m(z_i; p; f_0; \lambda)]|$. Let $\Delta V(x'_i; p) = V(x'_i; p; f) - V(x'_i; p; f_0)$.

$$\begin{aligned}
\mathbb{E}_{P_N}[m(z_i; \gamma) - m(z_i; p; f_0; \lambda)] &=^i \mathbb{E}_{P_N}[\Delta V(x'_i; p) \left(w(x'_i) - \lambda_0(x'_i) + \mathbb{E}[\lambda_0(x_i)|x'_i]\right)] \\
&\quad + \mathbb{E}_{P_N}[\Delta V(x'_i; p) \left(\lambda_0(x'_i) - \lambda(x_i)\right) + \mathbb{E}[\lambda_0(x_i) - \lambda(x_i)|x'_i]\big)] \\
&\leqslant^{ii} 0 + \mathbb{E}_{P_N}[\Delta V(x'_i; p) \left(\lambda_0(x'_i) - \lambda(x_i) + \mathbb{E}[\lambda_0(x_i) - \lambda(x_i)|x'_i)\right]] \\
&\leqslant^{iii} \|\lambda(x) - \lambda_0(x)\|_2\|\Delta V(x; p)\|_2 + \|\mathbb{E}[\lambda_0(x)|x'] - \mathbb{E}[\lambda(x)|x']\|_2\|\Delta V(x; p)\|_2 \\
&\leqslant^{iv} 2\lambda_N\delta_N
\end{aligned}$$

where i, ii follows from Remark 1, iii is by stationarity and Cauchy-Scwartz, and iv is by Assumption 2.

Step 2. Bound on $I^1_{1,k}$. First, let us establish the bound on

$$\mathbb{E}_{P_N}[m(z_i;\gamma) - m(z_i;p;f_0;\lambda)]^2 \leqslant \sup_{p \in \mathcal{T}_N} \mathbb{E}_{P_N} \Delta^2 V(x_i';p)(\lambda_0(x_i') - \lambda(x_i') + \mathbb{E}[\lambda_0(x_i) - \lambda(x_i)|x_i'])^2$$
$$\leqslant 4 \sup_{p \in \mathcal{T}_N} \mathbb{E}_{P_N} \Delta^2 V(x_i';p)(\lambda_0(x_i') - \lambda(x_i'))^2 = O(r_N'^2)$$

Therefore, $I^1_{1,k} = O_{P_N}(r'_N)$ conditionally on $\mathcal{E}_N$. By Lemma 6.1 of Chernozhukov et al. (2017a), $I^1_{1,k} = O_{P_N}(r'_N)$.

Step 3. Bound on $\mathcal{I}^2_{2,k}$. On the event $\mathcal{E}_N$, $|\mathcal{I}^2_{2,k}| \leqslant \sup_{\gamma \in \Gamma_N} |\mathbb{E}_{P_N}[m(z_i;p;f_0;\lambda) - m(z_i;p;f_0;\lambda_0)]|$.

$$\mathbb{E}_{P_N}[m(z_i;p;f_0;\lambda) - m(z_i;p;f_0;\lambda_0)] = \mathbb{E}_{P_N}(\lambda(x) - \lambda_0(x))(V(x';p;f_0) - \mathbb{E}[V(x';p;f_0)|x]) = 0.$$

Step 4. Bound on $\mathcal{I}^2_{1,k}$. First, let us establish a bound on

$$\mathbb{E}_{P_N}[m(z_i;p;f_0;\lambda) - m(z_i;p;f_0;\lambda_0)]^2 \leqslant \sup_{\gamma \in \Gamma_N} \mathbb{E}[\lambda(x) - \lambda_0(x)]^2[V(x';p;f_0) - \mathbb{E}[V(x';p;f_0)|x]]^2$$
$$\leqslant 4V^2\lambda_N^2$$

Therefore, $\mathcal{I}^2_{1,k} = O_{P_N}(2V\lambda_N)$.

Step 5 and 6. On the event $\mathcal{E}_N$, $|\mathcal{I}^2_{2,k}| \leqslant \sup_{\gamma \in \Gamma_N} |\mathbb{E}_{P_N}[m(z_i;p;f_0;\lambda_0) - m(z_i;\gamma_0)]|$.

$$\begin{aligned}
\mathbb{E}_{n,k} m(z_i;p;f_0;\lambda_0) - m(z_i;\gamma_0) &= \underbrace{\mathbb{E}_{n,k} w(x_i)(V(x_i;p;f_0) - V(x_i;p_0;f_0))}_{\mathcal{J}_{1,k}} \\
&+ \beta \mathbb{E}_{n,k} \lambda_0(x_i)(V(x_i';p;f_0) \sum_{a \in \mathcal{A}} p(a|x) - V(x_i';p_0;f_0) \sum_{a \in \mathcal{A}} p_0(a|x)) \\
&- \beta \mathbb{E}_{n,k} \lambda_0(x_i)(\mathbb{E}_{f_0}[V(x_i';p;f_0)|x_i,a] \sum_{a \in \mathcal{A}} p(a|x_i) \\
&- \mathbb{E}_{f_0}[V(x_i';p_0;f_0)|x_i,a] \sum_{a \in \mathcal{A}} p_0(a|x_i)) = \mathcal{J}_{1,k} + \mathcal{J}_{2,k}.
\end{aligned}$$

On the event $\mathcal{E}_N$, for each $j \in \{1,2\}$ $|\mathcal{J}_{j,k}| \leqslant \frac{\mathcal{J}^j_{1,k} + \mathcal{J}^j_{2,k}}{\sqrt{n}}$ where

$$\mathcal{J}^j_{1,k} = \sqrt{n}(R_{j,k} - \mathbb{E}_{P_N}[R_{j,k}|I^c_k])$$
$$\mathcal{J}^j_{2,k} = \sqrt{n}\mathbb{E}_{P_N}[R_{j,k}|I^c_k].$$

Assumption 2 implies the bound

$$\mathcal{J}^1_{2,k} \leqslant W \sup_{p \in \mathcal{T}_N} \|V(x_i;p;f_0) - V(x_i;p_0;f_0)\|_2 = O_{P_N}(WBJp_N^2)$$

To bound $\mathcal{J}^1_{1,k}$, consider the bound on

$$\mathbb{E}_{P_N}[w(x_i)^2(V(x_i;\widehat{p};f_0)-V(x_i;p_0;f_0))^2|I_k^c] \leqslant W^2 \sup_{p\in\mathcal{T}_N} \mathbb{E}_{P_N}(V(x_i;p;f_0)-V(x_i;p_0;f_0))^2 \leqslant W^2 p_N^2.$$

Therefore, $\mathcal{J}^1_{1,k} = O_{P_N}(Wp_N)$.

Define $R(x;p;a) := V(x;p;f_0) - \mathbb{E}[V(x';p;f_0)|x,a]$. Then,

$$\begin{aligned}
\mathcal{J}^1_{2,k} + \mathcal{J}^2_{2,k} &= \mathbb{E}_{n,k}\lambda_0(x_i)\sum_{a\in\mathcal{A}} R(x_i;p;a)p(a|x) - \mathbb{E}_{n,k}\sum_{a\in\mathcal{A}}\lambda_0(x_i)R(x_i;p_0;a)p_0(a|x) \\
&= \underbrace{\mathbb{E}_{n,k}\sum_{a\in\mathcal{A}}\lambda_0(x_i)R(x_i;p;a)(p(a|x)-p_0(a|x))}_{i} \\
&+ \underbrace{\mathbb{E}_{n,k}\sum_{a\in\mathcal{A}}\lambda_0(x_i)(R(x_i;p;a)-R(x_i;p_0;a))p_0(a|x)}_{ii}.
\end{aligned}$$

Since $\mathbb{E}[R(x_i;p;a)|x_i,a] = 0$, $\mathbb{E}[i|I_k^c] = 0$ and $\mathbb{E}[ii|I_k^c] = 0$ conditionally on I_k^c. To see that $i = o_P(p_N)$, recognize that

$$\mathbb{E}_{P_N}[i^2|I_k^c] = \sup_{p\in\mathcal{T}_N}\mathbb{E}_{P_N}[(\sum_{a\in\mathcal{A}}\lambda_0(x_i)R(x_i;p;a)(p(a|x_i)-p_0(a|x_i)))^2|I_k^c] \leqslant V^2 J p_N^2.$$

For every $a \in \mathcal{A}$,

$$\begin{aligned}
\sup_{p\in\mathcal{T}_N}\mathbb{E}(R(x_i;p;a)-R(x_i;p_0;a))^2 \quad &\leqslant \sup_{p\in\mathcal{T}_N}\mathbb{E}(V(x_i';p;f_0)-V(x_i';p_0;f_0))^2 = o(Ep_N^2) \\
\mathbb{E}_{P_N}[ii^2|I_k^c] = W^2E^2p_N^2,
\end{aligned}$$

and $\mathcal{J}^2_{1,k} = O(Vp_N + WEp_N) = o(1)$.

■

4 Auxiliary statements

Theorem 9 (Convergence)**.**

Let $A : X \to Y$ be a bounded linear operator. Suppose A has a bounded inverse A^{-1}. Let $\widehat{\phi}$ solve $\widehat{A}\widehat{\phi} = \widehat{\xi}$ and ϕ solve $A\phi = \xi$. Then, for all $\widehat{A}$ such that $\|A^{-1}(\widehat{A}-A)\| < 1$, the inverse operators $\widehat{A}^{-1}$ exist and are bounded, there holds the error estimate

$$\|\widehat{\phi}-\phi\| \leqslant \frac{\|A^{-1}\|}{1-\|A^{-1}(\widehat{A}-A)\|}\left(\|(\widehat{A}-A)\phi + \widehat{\xi}-\xi\|\right).$$

Proof. See the proof of Theorem 10.1 from Kress (1989). ■

References

Aguirregabiria, V. and Mira, P. (2002). Swapping the nested fixed point algorithm: A class of estimators for discrete markov decision models. *Econometrica*, 70(4):1519–1543.

Chernozhukov, V., Chetverikov, D., Demirer, M., Duflo, E., Hansen, C., Newey, W., and Robins, J. (2017a). Double/debiased machine learning for treatment and causal parameters.

Chernozhukov, V., Escanciano, J. C., Ichimura, H., and Newey, W. (2017b). Locally robust semiparametric estimation.

Chernozhukov, V., Hansen, C., and Spindler, M. (2015). Valid post-selection and post-regularization inference: An elementary, general approach. *Annual Review of Economics*, 7:649–688.

Ichimura, H. and Newey, W. (2018). The influence function of semiparametric estimators.

Kress, R. (1989). *Linear Integral Equations*. Springer.