Cheap TIPS or Expensive Inflation Swaps? Mispricing in Real Asset Markets *

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Abstract

Inflation risks are explicit in either (i) the nominal pricing of real payoffs in which prices are denominated in dollars, or (ii) the real pricing of nominal payoffs in which prices are denominated in consumption baskets. While the former involves over-the-counter inflation-indexed contracts of real asset market, the latter involves exchange-traded and highly liquid contracts of nominal asset market. We employ a parametric pricing model to investigate the asymmetry between these two markets. The model obtains a liquidity-free distribution of future inflation using new price data of T-note futures in nominal asset market, and implies liquidity risk premia separately for any traded contract in real asset market. These premia indicate both an underpricing for TIPS and an overpricing for inflation swaps, whose significance increases with the tenor of these assets. Such a mispricing in inflation swaps helps temper a severe implied mispricing of TIPS needed to match the puzzling trade profit on the nominal-TIPS yield spread. While yields on TIPS still command a liquidity component, this finding implicates less pronounced borrowing costs to the U.S. government in issuing TIPS.

JEL-Classification: E31, G12, G13.

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1 Introduction

Inflation is a central factor that influences the economy, the financial market, and their constituents: producers, consumers, traders, and policy makers. As a result, the forecast and pricing of inflation risk are of great interests to market participants, and traditionally are obtained using direct surveys, statistical analysis, or both. The advent of *inflation-indexed* bonds and their derivatives has brought a new market-based input to enhance the inflation forecast and pricing, as their inflation-sensitive prices reflect market's inflation expectation. However, the market nature of inflation-indexed assets also exposes them to market imperfections, namely liquidity and segmentation. As a result, the observed prices of inflation-indexed contracts (hereafter, real assets) reflect not only inflation, but also these imperfections. The employment of real asset price data hence may confound premia of various non-inflation risks and potentially skew the market-based forecast and pricing of inflation risk.

In this paper, we first decouple the liquidity and other market imperfection concerns from the estimation of the U.S. future inflation by employing proprietary price data of the most liquid and exchange-traded contracts on treasury bonds (hereafter, nominal assets), namely T-note futures, together with professional forecasts of the price index. We then employ the resulting pure-inflation estimates to price individually every real asset out-of-sample. We compare these model-implied prices with the corresponding observed prices to determine the "mispricing" of each real asset. By construction, this mispricing is benchmarked against liquid nominal assets (T-note futures) in an inflation-risk pricing model. As a result, the mispricing is due to non-inflation (hereafter, generally referred to as "liquidity") factors in real asset market that are not priced by the model. Such a mispricing is relative to our inflation risk pricing model, and would vanish in more comprehensive pricing models that take into account risk factors beyond the inflation risk and are able to fit TIPS and inflation swap prices.

Recent literature has found significant spreads between the yields of nominal bonds and Treasury Inflation-Protected Securities (TIPS). These findings hint at important TIPS liquidity premia (e.g., D'Amico et al. (2018)) or highly profitable strategies involving TIPS, inflation swaps, and nominal bonds (Fleckenstein et al. (2014)). The mispricing magnitudes obtained in the current paper show significant non-inflation premia for both TIPS and inflation swaps, and indicate their individual contributions to the profit of strategies on the nominal-TIPS yield spread. The mispricing in inflation swaps lessens the extent of TIPS mispricing needed to accommodate the above profit.

This complementarity between inflation swap and TIPS mispricings lowers the implied borrowing cost to U.S. government in issuing secured real debts, albeit such a cost still subjects the government to pay for a liquidity premium of real asset market.

We find that (i) markets generally expect a higher, yet less uncertain, annualized long-run inflation, which is quantified by a future inflation distributions of higher mean and lower variance as the estimation horizon increases, (ii) markets expect a lower probability, yet higher price, of the deflation risk at longer horizons, and the time-series fluctuation of the Arrow-Debreu price of the deflation state arises mainly from movements in the deflation distribution, but not in its price of risk, (iii) both TIPS and inflation swaps appear mispriced, and more significantly so for longer tenors: TIPS appear consistently underpriced and inflation swaps consistently overpriced (to fixed rate payors) for contracts of 10 years or longer maturities. These mispricings also vary with time. Overall, the mispricing pattern of TIPS and inflation swaps indicates time-varying returns on the nominal-real yield spread of short terms (less than 10 years), and consistent profits on the same spread of longer-term yields (10 years or more).[1]

To arrive at these results, our paper adopts an empirical strategy using price data of nominal assets and inflation surveys to estimate the inflation process in conjunction with a real pricing model, in which prices are denominated in units of consumption baskets. Specifically, we specify inflation and real stochastic discount factor (SDF) processes in terms of latent state variables, which in turn have affine dynamics in the data-generating (physical) probability measure. A Kalman filter, which employs data of inflation consensus forecasts, T-note futures prices, and a short-term real interest rate proxy, is constructed to jointly estimate the SDF and future inflation distribution parametrically.

The mechanism by which the price data of nominal assets, such as T-note futures, help to determine the market's expected inflation is as follows. Nominal asset payoffs, when denominated in the real term (i.e., in units of consumption baskets), explicitly reflect future inflation. A joint specification of inflation and real SDF processes translates inflation consensus forecasts into current prices of nominal assets in an asset pricing framework. Therefore, data of inflation consensus and nominal asset prices help to back out the best-fit parameters of the underlying joint specification in an estimation framework. The advantage of this inflation estimation is in the fact that nominal

[1] Returns on the nominal-real yield spreads are returns on the strategy that takes a long position in nominal bonds and a short position in TIPS of similar maturities (tenors), i.e., earning nominal and paying real yields. We will also consider a related arbitrage strategy by adding positions in inflation swaps (paying fixed, receiving floating rates) to fully hedge the inflation risk in the above nominal-real yield spread strategy.

asset market is well developed, and nominal asset prices are subject the least to liquidity and other frictions. This inflation estimation is on the flipped side of the approach in the literature that employs inflation consensus data and real asset prices. Such an approach jointly specifies inflation and nominal SDF processes, and exploits the sensitivity of dollar (i.e., nominal) prices of real assets to the inflation risk. However, as real asset market tends to be less liquid, real asset prices are potentially subject to liquidity and other market imperfections that might hinder the estimation in such an approach.

Our inflation estimation described above is therefore motivated and dictated by a practical consideration of market data. In fact, the market size and transaction volume of U.S. nominal assets (Treasury bonds and their derivatives) are substantially larger than those of U.S. real assets (TIPS, inflation swaps, inflation options and their derivatives), indicating superior liquidity for the former. We acquire and employ T-note futures price data provided by the Chicago Mercantile Exchange (CME). These futures contracts are standardized and exchange-traded, featuring transaction transparencies, large average daily trading volume (of approximately 300 billion dollars), and matured markets. In comparison, TIPS and inflation swaps are mostly traded in OTC markets, with significantly lower average daily trading volumes (of approximately 20 and 1 billion dollars, respectively) and less matured markets.[2] In this regard, while it appears that nominal yields might also substitute for T-notes futures prices in the inflation estimation process, the employment of nominal yield data is not straightforward. This is because nominal yields are typically derived from coupon bond prices, so their inputs do not directly match the closed-form zero-coupon yields derived from the estimation model. The construction of zero-coupon yields for all horizons needed in our estimation from available coupon bond prices is intricate, and is beyond the scope of the current paper.[3]

We make a specification assumption that nominal asset market is liquid and free of other frictions. Therefore, by employing only nominal asset price data, our estimation model only concerns inflation risk (but not liquidity or other frictions). Post estimation, given the obtained future inflation distribution and pricing kernel, the model is able to price real assets individually and out-

[2]T-notes futures were first traded in CME in 1976. TIPS and inflation swaps were first introduced in 1997 and 2002, and their price data are available to us starting in 2003 and 2004 respectively. In notional amounts, in 2017, average daily trading volumes are almost 300 billion dollars for T-note futures, and 1 billion dollars for inflation swaps. See, e.g., Fleming and Krishnan (2012) and Fleming and Sporn (2013) for further description and discussion of TIPS and inflation swap markets.

[3]Gurkaynak et al. (2006) and Le and Singleton (2013) discuss and present such a construction of constant-maturity zero-coupon Treasury yields.

of-sample. These model-implied prices capture only the exposure to inflation risk of real assets, and evidently do not match the corresponding prices observed in real asset market. Their differences represent the model-implied mispricing, which reflects the liquidity and other (non-inflation) market frictions inherent in real assets. The mispricing provides a quantitative assessment of the integration and relative liquidity between the real and the much larger nominal asset market over time. By not employing real asset price data, our estimation also indicates a venue to forecast the inflation in developing economies, wherein real asset market is either underdeveloped or non-existent.

It is important to observe that while prices of liquid nominal assets constitute a high quality data source, they alone are insufficient to estimate the future inflation distribution. From an economic aspect, we recall that the pricing of risky assets only reflects risk premia, i.e., expected asset returns in excess of the short-term risk-free rate (short rate). Intuitively, that the inflation-protected (resp., nominal) assets are expensive when the market's expectation about the future price index is high (resp., low) is a relative notion. This expensiveness is only quantifiable with respect to an appropriate baseline, which is the nominal (resp., real) bond in the setting. As a result, quantifying the sensitivity of asset prices to the inflation risk necessitates the knowledge of the corresponding baseline, namely, the nominal (resp., real) short rate. That is, the short rate inputs help to pin down the baseline, restore the full expected asset returns. Follows from which an estimation of the distribution and pricing of the inflation risk. From a pricing aspect, in the difference with a pure statistical estimation, a pricing model specification is essential in establishing a relationship between the inflation distribution and asset prices, forming the basis of a market-based estimation of the future inflation from current prices. In the model, asset prices therefore mingle and reflect parameters of both the SDF and inflation distribution. To disentangle these parameters, two additional data inputs, namely inflation surveys (which pertain only to inflation distribution parameters) and short-term real interest rate (which pertains only to the real SDF parameters) are also employed in our estimation.[4] In our estimation, the inflation consensus forecast input is sourced from the Blue Chip Economic Indicators (BCEI), and short-term real interest rate input is proxied by the difference between short-term nominal interest rates and short-term professional inflation forecasts.[5]

Related Literature: Our paper contributes to a vibrant literature of the market-based estimation

[4]In principle, we just need one of these two data inputs to complement the nominal asset price data in the estimation.

[5]This proxy is built upon the Fisher equation and motivated by the fact that, historically, the short-term future inflation is forecastable by market professionals.

of the inflation. Various price data sources on TIPS, inflation swaps, and nominal yields are employed in Christensen et al. (2010), Gurkaynak et al. (2010), Chernov and Mueller (2012), Haubrich et al. (2012), Grishchenko and Huang (2013), Fleckenstein et al. (2017). While TIPS (resp., nominal yields) are sensitive to the future inflation distribution in the dollar (resp., consumption basket) denomination, inflation swaps are sensitive to the inflation in both denominations. Hence papers using inflation swap data, such as Haubrich et al. (2012) and Fleckenstein et al. (2017), examine the inflation risk pricing and estimation in a joint nominal and real perspective. However, inflation swap data and market size are limited. Our paper employs T-note futures prices as a new data source, which is associated with a much larger and more liquid market.

The link between nominal price data (in particular, nominal yields) and expected inflation are analyzed further within term structure settings in above-mentioned and other papers. Ang et al. (2008) estimate a term structure model of nominal and real interest rates, with inflation as a state variable switching between regimes, employing parametric restrictions and nominal yields but no survey data. D'Amico et al. (2018) estimate another term structure model, employing CPI, nominal and TIPS yields, and inflation survey data. Duffee (2018) studies how shocks in the inflation expectation impact nominal yields. The construction of zero-coupon yields of various (especially, long) maturities from a cross-section of Treasury coupon bonds is intricate as seen in Le and Singleton (2013). Our paper differs from these works in that we directly employ price data of the exchange-traded T-note futures – hence circumvent the construction of nominal zero-coupon yields – as well as inflation surveys, but no TIPS nor inflation swap data.

The use of survey data in forecasting inflation is pioneered by Pennacchi (1991). Ang et al. (2007), Faust and Wright (2013) and Bauer and McCarthy (2015) find that inflation surveys tend to outperform market-based forecasts of the U.S. inflation using inflation-indexed asset prices. Ehling et al. (2018) further show that subjective and heterogeneous beliefs about inflation have impacts on nominal interest rates. Our inflation estimation relies importantly on the BCEI consensus forecasts, but also employs (nominal) price data. Price data offers a much richer variety of tenors than survey data, and hence, is important in estimating entire distributions of future inflation at various horizons.

Potential arbitrage opportunities between real and nominal bonds have been investigated in Fleckenstein et al. (2014), who find significant pricing anomalies in TIPS markets for the period of 2004-2009. This finding follows from documenting (i) a difference in prices of TIPS and their replicating portfolios of inflation swaps and nominal bonds, and (ii) no significant difference in prices of several corporate-issued real bonds and their replicating portfolios of inflation swaps and corporate-issued nominal bonds. Our inflation estimation employs only the most liquid exchange-traded nominal assets, whose price data is available from 1982 onward. We treat both TIPS and inflation swaps as out-of-sample assets to the inflation estimation, which enables the pricing of the inflation risk separately for TIPS and inflation swaps. As a result, our paper quantifies and attributes pricing anomalies between nominal and real bonds down separately to an underpricing in TIPS and an overpricing in inflation swaps.[6] We find that, these mispricings vary significantly with time, and increase with assets' tenor.

The importance of determining inflation expectation from policy making perspectives is discussed in Dudley et al. (2009), Bernanke (2012) and Bullard (2016). Our findings indicate the mispricing also in the smaller inflation swap market, which is integrally important to assess the mispricing and borrowing cost of TIPS issuance. Neely and Rapach (2011) and Grishchenko et al. (2017) further investigate the co-movement and anchoring of inflation in international settings. Our inflation estimation makes use of only nominal asset prices and inflation surveys, hence can be extended to other economies in which real asset markets are either underdeveloped or nonexistent.

Liquidity in real asset market is studied in various recent papers. Campbell et al. (2009) and Pflueger and Viceira (2011) examine the liquidity component of these anomalies. Driessen et al. (2017) include an explicit liquidity factor, estimate its price in a Fama-MacBeth regression framework, and find that liquidity explains an important part of the spread between the nominal bond and the its replicating portfolio of TIPS and inflation swap. In event studies, D'Amico and King (2013) and Christensen and Gillan (2018) examine effects of a large purchase of T-bonds and TIPS on the pricing and liquidity of nominal and real assets. D'Amico et al. (2018) emphasize the relative illiquidity of TIPS over nominal Treasury bonds and the information distortion caused by ignoring the liquidity factor in anomalous yield spreads on these assets. These papers extend the vibrant literature investigating pricing anomalies in risky debts, e.g., Chen et al. (2007), Bao et al. (2011), Huang and Huang (2012) to "safe" debts. Our paper concurs with a significant mispricing in real asset market, but is agnostic about the specific nature of non-inflation factors (liquidity and other market imperfections) impacting real assets.

The paper is structured as follows. Section 2 describes data sources and provides key intuitions

[6] We implement estimations for the periods of 2003-2017, and 1982-2017, which also differ from the period of 2004-2009 in Fleckenstein et al. (2014).

underlying the estimation. Section 3 specifies the actual estimation model and procedure. Section 4 presents estimation results. Section 5 discusses the mispricing in real asset markets. Section 6 concludes. Appendices A, B and C present robustness results and technical derivations omitted in the main text.

2 Preliminaries: Data and Inflation Risk Pricing

In this section, we describe data sources and their relevant features. We also discuss the basics of the inflation risk pricing as well as non-inflation premia (i.e., the mispricing) in out setup. This discussion demonstrates all intuitions of the introduction section, as well as elucidates the essence of the full estimation model of Section 3.

2.1 Data

Our estimation employs monthly data of T-note futures prices and BCEI inflation consensus forecasts, both are available from early 1980. Post estimation, to determine the mispricing in inflation swaps, we compare model-implied inflation swap rates with the (transacted) inflation swap rate given by Bloomberg. To determine the mispricing in TIPS, we compare model-implied TIPS yields with the observed TIPS yields given by Fed's inflation-indexed constant maturity yields. We note that Fed's inflation-indexed constant maturity yields indicate, but are not necessarily exactly equal to, TIPS yields transacted in markets. Therefore, the mispricing in TIPS indicates, but is not necessarily exactly equal to, the arbitrage in trading TIPS (against nominal bonds and inflation swaps).

2.1.1 Data used in the estimation

T-note futures prices: Treasury Bond futures (also referred to as T-note futures here) were introduced on the Chicago Board of Trade (CBOT) in 1977, augmented over the years by the introduction of 10-year, 5-year, 2-year T-note futures, and are subject to the rules and regulations of the CBOT. Our data is acquired and sourced from Chicago Mercantile Exchange (CME), from 1982 to the end of 2017. T-note face value in these futures is \$100,000 (except for 2-year and 3-year T-note futures, for which the face is \$200,000 USD). The normal commercial round-lot is \$1 million face value. T-note futures permit the delivery of any U.S. Treasury security provided it matures within an eligible period (deliverable grade). Due to flexibility, T-note futures employ a "conversion factor" invoicing system to reflect the value of the security that is delivered by reference to the 6% futures contract standard. The intent of the conversion factor invoicing system is to make the delivery of any eligible securities fair (though, not perfectly). The principal invoice amount paid from long to short upon delivery may be identified as the futures settlement price multiplied by the conversion factor multiplied by $1,000. (or $2,000 of 2-year and 3-year T-note futures). The T-note futures market is among the most liquid asset markets, featuring average daily trading volume of almost 3 millions contracts (or, face values of $300 billions daily). Prices of options on T-note futures are also a quality data source employed to estimate the pricing kernel non-parametrically as in Bakshi et al. (2018).

BCEI inflation consensus forecasts: Since 1976, each month Blue Chip Economic Indicators has polled approximately 50 business economists for future changes in inflation (and 14 other important economic indicators). Our data is acquired from a global provider of professional information Wolters Kluwer. We employ the data series in which surveys conducted monthly (12 surveys per year) from 1982 to the end of 2017. Each survey contains inflation forecasts for several coming quarters, starting with the current one, on to the last quarter of the next year.[7] In the survey, BCEI collects professional forecasts of the percent change in the U.S. seasonally adjusted consumer price index for all urban consumers (CPI-U) from the prior quarter expressed at an annual rate. BCEI compiles estimates into a consensus average forecast published each month based on responses, along with averages of the 10 highest and 10 lowest forecasts, and a median forecast to eliminate the effects of extremes on the consensus.

Yields on inflation-indexed securities at "constant maturity": These yields are a part of the Selected Interest Rates (H.15) statistical release compiled by Board of Governors of the Federal Reserve System. Yields on inflation-indexed securities at "constant maturity" are interpolated from the daily yield curve for TIPS in the over-the-counter market. The inflation-indexed constant maturity yields are read from this yield curve at fixed maturities of 5, 7, 10, 20, and 30 years. We employ monthly data series from 2003 to the end of 2017.

Zero-coupon inflation swaps: Zero-coupon inflation swaps are the simplest and most traded among all inflation swaps, in which parties settles cashflows only at the swap's maturity (details

[7] So in a survey (month), there are at least 5 forecasts (if the month is at the end of a year), and at most 8 forecasts (if the month is at the beginning of a year). This relatively large number of forecast data points is the main reason we choose BCEI the inflation consensus forecast inputs for our estimation.

are in Section 3.3). The inflation swap rate data are from Bloomberg. Maturities of inflation swaps range from 1 up to 55 years. Inflation swap data are available starting from 2004. We employ the mid-quotes on the inflation swap rates, from 2004 (available date for most maturities) to the end of 2017.

2.2 Inflation Risk Pricing and Mispricing

Mispricing: Before presenting ingredients of the inflation estimation procedure, it is instructive to quantify and discuss the concept of mispricing in our approach. By design, the current paper's joint estimation of the inflation process and pricing characteristics (SDF) employs only price data of liquid nominal assets (T-note futures) and inflation consensus forecasts. To the extent that these nominal assets are not subject to non-inflation risks and other market imperfections such as liquidity and defaults,[8] our estimation concerns and reveals only the inflation risk and its pricing. Because this estimation does not employ input data from real asset market, TIPS and inflation swaps are out-of-sample assets with respect to the estimation. Therefore, the estimated pricing model, when applied on TIPS and inflation swaps, produces prices for these assets that capture the compensation for their exposure only to the inflation risk. As a result, the difference between these model-implied and observed prices of real assets necessarily reflects premia on non-inflation risks borne by them. Our paper is agnostic about the specific nature of the non-inflation risks and other market imperfections impacting real assets that are responsible for such premia.[9] Accordingly, we refer to the difference between model-implied and observed prices and returns of real assets broadly as *mispricing* throughout the paper. It is possible that this mispricing is priced in other pricing models, those account for non-inflation risks and market imperfections not modeled in our estimation.

We now turn to the pricing of the inflation risk, starting with basic notations. Let I_t be the spot price in dollars of the consumption basket at time t. In data, I_t is the consumer price index (CPI). Without loss of generality, we can set $I_0 = 1$ at initial time $t = 0$. An inflation is realized from time t to $T > t$ if the consumption basket price increases during that period, $I_T \geq I_t$. Otherwise, a deflation is realized when ($I_T < I_t$). Any nominal payoff (or price) D_{NT} in units of dollars at

[8] This identification assumption is in relative sense and innocuous. Our estimation treats liquid nominal contracts (T-bond futures) as benchmark assets, against which other assets are evaluated. This does not require that T-bond futures need be perfectly liquid in absolute sense.

[9] Literature has found that liquidity is an important factor that differentiates real (TIPS and inflation swaps) from nominal (T-bonds) assets. See D'Amico et al. (2018) and the references therein.

time T can be contemporaneously converted to the real payoff of $\frac{D_{NT}}{I_T}$ in units of consumption baskets. A high inflation (larger $\frac{I_T}{I_t}$) depresses the real value of the payoff and vice versa. At time $t < T$, in the nominal term (in which prices are denominated in dollars), risks are associated with fluctuations in the realized nominal payoff D_{NT}. In the real term (in which prices are denominated in consumption baskets), inflation risks are inherent in the realized real payoff $\frac{D_{NT}}{I_T}$, which reflects the movements in both D_{NT} and inflation I_T. Let M_{Nt} be the nominal stochastic discount factor (SDF) process which prices assets in the dollar denomination. Similarly, let M_{Rt} be the real SDF, which price assets in the consumption basket denomination.

Non-parametric consideration: To gain preliminary intuitions about the inflation estimation, we first discuss the pricing of inflation risks in a non-parametric setting, in which SDFs follow diffusion processes,

$$\frac{dM_{Rt}}{M_{Rt}} = -r_{Rt}dt - \eta'_{Rt}dZ_t, \qquad \frac{dM_{Nt}}{M_{Nt}} = -r_{Nt}dt - \eta'_{Nt}dZ_t,$$

where r_R, r_N are real and nominal interest rate, and η_{Rt}, η_{Nt} are real and nominal prices of risks. By definition, the two SDFs are related by the multiplicative factor of the price index,

$$I_t = \frac{M_{Rt}}{M_{Nt}}, \qquad \forall t. \tag{1}$$

This relationship assures that the pricing of a future payoff is consistent across real and nominal denominations. Indeed, the price P_t (in spot dollars at t) of a nominal future payoff D_{NT} can be priced by either the nominal SDF M_N or the real SDF M_R,[10]

$$P_t = E_t\left[\frac{M_{NT}}{M_{Nt}}D_{NT}\right], \qquad P_t = I_t E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{D_{NT}}{I_T}\right], \qquad \forall t, T. \tag{2}$$

Following from this non-parametric setup, several preliminary observations are in order.

First, the inflation $\frac{I_T}{I_t}$ is explicit in the cross-denomination pricing equation (second equation in (2)), in which a nominal payoff is priced in the real term by SDF M_R. Fluctuations in the inflation $\frac{I_T}{I_t}$ lead to fluctuations in the real value $\frac{I_t}{I_T}D_{NT}$ of the payoff. To employ input price data of nominal assets, the estimation models the real SDF M_R to pick on, and price, these fluctuations, e.g., by a specification that relates M_{Rt} with I_t (Section 3). Hence, when D_{NT} is a nominal

[10]In the latter case, first the nominal payoff of D_T dollars is converted into a real payoff of $\frac{D_T}{I_T}$ to be priced by M_R. The resulting real price is converted back into dollars to obtain the spot price at time t.

fixed-income (fixed) payoff or the model is agnostic about possible relationship between D_{NT} and the price index I_T, the pricing in the real term of nominal assets remains sensitive to the future inflation distribution due to the explicit appearance of the inflation $\frac{I_T}{I_t}$. Price data of nominal assets (together with inflation surveys and short-term real rates) then help estimate inflation distribution. In this real pricing perspective, it is natural to adopt the consumption basket denomination and the associated real SDF M_R specification in the inflation estimation. Symmetrically, the inflation rate is also explicit in another (cross-denomination) pricing equation in the nominal term (associated with M_N) of a real payoff.[11] In principle, prices of real assets (together with inflation surveys and short-term nominal rates) could also help estimate inflation distribution. However, real assets are less liquid and their price inputs to the estimation may compound liquidity with inflation premia and skew the the latter's forecast.[12]

Second, an identical probability distribution is associated with the real SDF M_R and nominal SDF M_N, and is also the physical probability measure. We recall that that M_N (M_R) is the marginal utility of a nominal (real) representative agent to whom the risk-free asset pays surely one dollar (one consumption basket) next period. However, this difference in the risk-free concept associated with real and nominal pricing does not distort the associated probability measure because dollar and consumption basket are just two alternative numeraires of the same market. To invoke a metaphor from international finance: while U.S. and U.K. investors perceive different risk-free bonds (Treasury bonds vs. Gilts), they may share identical probability distribution of the future state of the world economy. Therefore, the real pricing perspective does not interdict a consistent estimation of future inflation distribution in the physical measure.

Third, the inflation estimation requires additional input data other than nominal asset prices. To see this, we examine the limit of nominal asset price data in the risk premium of short-term

[11] Similar to (2), the nominal and real pricing equations of a real payoff D_{RT} (denominated in consumption baskets) are, $\frac{1}{I_t}E_t\left[\frac{M_{NT}}{M_{Nt}}D_{RT}I_T\right] = E_t\left[\frac{M_{RT}}{M_{Rt}}D_{RT}\right]$.

[12] A priori, the inflation factor is not explicit in the same-denomination pricing, i.e., either the pricing of a nominal payoff in the nominal term $E_t\left[\frac{M_{NT}}{M_{Nt}}D_{NT}\right]$, or the pricing of a real payoff in the real term $E_t\left[\frac{M_{RT}}{M_{Rt}}D_{RT}\right]$. They can be converted into cross-denomination pricing equations after the inflation is specified.

nominal assets,[13]

$$\mu_{ND} - (\mu_I + r_R) = (\eta_R' + \sigma_I')(\sigma_{ND} - \sigma_I). \tag{3}$$

Generally, the estimation procedure takes as inputs the moments $\{\mu_{ND}, \sigma_{ND}\}$ of asset returns (provided by nominal asset price data), and generates joint output estimates $\{\mu_I, \sigma_I; r_R, \eta_R\}$ of the inflation distribution and real pricing kernel.[14] Clearly, these output estimates are mixed in the risk premium (3). In particular, μ_I and r_R influence this nominal risk premium through their sum $\mu_I + r_R$. This implies that nominal asset price data (e.g., nominal risk premium) can only help estimate the mean inflation μ_I and short-term real interest rate r_R up to a linear combination. This identification issue indicates the need for additional data that involve separately inflation parameters (i.e., inflation surveys) and pricing kernels (i.e., short-term interest rates).

Other output estimates (σ_I and η_R) also mingle in the risk premium (3) but to a less extent. Such a mixing does not lead to an identification issue and may be addressed by employing a varieties of nominal assets of different maturities. However, these nominal asset price data are associated with the inflation and pricing kernel characteristics of different time horizons, and might affect the estimation efficiency. This complication motivates a parametric setting which systematically maps prices of various nominal assets into few basic parameters of the pricing model and inflation, before the estimation is carried out. Such a term-structure analysis is beyond the scope of the current non-parametric consideration, and is discussed next.

Parametric consideration: To illustrate the use and integration of nominal asset price data associated with diverse maturities in the estimation, we consider a toy dynamic term-structure model. Let the state variable X_t have affine dynamics in a diffusion setting, i.e., $dX_t = \mu_X dt + \sigma_X dZ_t$, with linear drift and variance,

$$\mu_X = K_0 + K_1 X_t, \qquad \left(\sigma_X \sigma_X^T\right)_{ij} = H_{0,ij} + H_{1,ij}^T X_t \equiv \left(H_0 + [H_1 \cdot X_t]\right)_{ij}. \tag{4}$$

[13]The risk premium (3) arises from substituting the inflation process $\frac{dI_t}{I_t} = \mu_I dt + \sigma_I' dZ_t$, and the short-term nominal asset return $\frac{dP_t + D_t dt}{P_t} = \mu_{ND} dt + \sigma_{ND}' dZ_t$ into the real pricing equation (second equation in (2)),

$$1 = E_t\left[\frac{M_{Rt+dt}}{M_{Rt}} \frac{(P_{t+dt} + D_t dt)/P_t}{I_{t+dt}/I_t}\right] = E_t\left[(1 - r_R dt - \eta_R' dZ_t)\frac{1 + \mu_{ND} dt + \sigma_{ND}' dZ_t}{1 + \mu_I dt + \sigma_I' dZ_t}\right].$$
$$= 1 + \left\{\mu_{ND} - (\mu_I + r_R) - (\eta_R' + \sigma_I')(\sigma_{ND} - \sigma_I)\right\} dt.$$

Note that (3) is the usual risk premium in the real term: $\mu_{RD} - r_R = \eta_R' \sigma_{RD}$, where $\mu_{RD} = \mu_{ND} - \mu_I - \sigma_I'(\sigma_{ND} - \sigma_I)$ and $\sigma_{RD} = \sigma_{ND} - \sigma_I$ are mean and volatility of the real return $\frac{(P_{t+dt} + D_t dt)/P_t}{I_{t+dt}/I_t}$.

[14]Nominal pricing characteristics then arise from (1), $M_{Nt} = \frac{M_{Rt}}{I_t}$.

Let both the price index and real SDF be exponential affine functions of the state variable,

$$I_t = I_0 e^{iX_t}, \qquad M_{Rt} = M_{R0} e^{mX_t}, \quad \forall t \geq 0. \tag{5}$$

Given this specification, to estimate the inflation, one needs to estimate both (i) the parameters $\{K_0, K_1, H_0, H_1\}$ to determine the state variable dynamics, and (ii) inflation parameter i (and SDF parameter m) to relate the inflation (and the pricing kernel) to the state variable.

Consider a nominal zero-coupon bond that pays one dollar at maturity T, or equivalently $\frac{1}{I_T}$ units of consumption baskets. The above parametric (affine) setting yields a tractable bond price $P_{t,T}$, which is exponential affine in the state variable,[15]

$$P_{t,T} = I_t E_t \left[\frac{M_{RT}}{M_{Rt}} \frac{1}{I_T} \right] = e^{p_{0t} + p_{1t} X_t},$$

with coefficients p_{0t}, p_{1t} satisfying a system of Riccati's differential equations,

$$\begin{aligned} \tfrac{dp_{1t}}{dt} &= -K_1 p_{1t} - \tfrac{1}{2}(p_{1t} H_1 p_{1t}), \qquad p_{1T} = m - i, \\ \tfrac{dp_{0t}}{dt} &= -K_0 p_{1t} - \tfrac{1}{2}(p_{1t} H_0 p_{1t}), \qquad p_{0T} = 0, \end{aligned} \tag{6}$$

where K's and H's characterize the state variable dynamic (4). Following from this parametric setup, several important observations are in order.

First, nominal asset prices (and nominal yields) are sensitive to the inflation dynamics through the influence of the parameter i on the coefficients p_{0t}, p_{1t}. In contrast, in the same parametric setting, real asset prices (and real yields) do not reflect the inflation dynamics when priced in the real term (i.e., by M_{Rt}). This is because while these data are sensitive to the distribution of state variable X, the Riccati's equations of real bond prices do not contain the key parameter i (5) that links the state variable to inflation.[16] Hence, the need to employ liquid nominal assets and data in the inflation estimation motivates the adoption of real pricing specification M_R. Moreover, nominal assets of different maturities T are systematically integrated into, and reinforce, the estimation because the same parameter set $\{K, H, m, i\}$ drives all these asset prices in this parametric setting.

Second, while the parameter i of the inflation does appear in the real pricing of nominal bonds, it

[15]The nominal bond price arises from (2) with the nominal payoff payoff $D_T = \mathbb{1}_T$.

[16]Indeed, the real pricing of real bonds, $P_{t,T} = E_t \left[\frac{M_{RT}}{M_{Rt}} \mathbb{1}_T \right] = \exp\left(p_{R0t} + p_{R1t} X_t\right)$, is established by the Riccati's equations on p_{R0t}, p_{R1t}: $\frac{dp_{R1t}}{dt} = -K_1 p_{1t} - \frac{1}{2}(p_{1t} H_1 p_{1t})$, $\frac{dp_{0t}}{dt} = -K_0 p_{1t} - \frac{1}{2}(p_{1t} H_0 p_{1t})$, with terminal conditions $p_{R1T} = m$, $p_{R0T} = 0$. Clearly, this equation system does not involve the inflation parameter i explicitly.

only influences their prices through the combination $(m - i)$ in the Riccati's system (6). As a result, these bond prices alone can only implicate the difference between the pricing (SDF) parameter m and the inflation parameter i, but not m or i separately. Note that all derivative contracts (e.g., options) on nominal bonds feature the same combination $(m - i)$ because they share the conversion of nominal (dollar) payoffs to consumption baskets. In principle, more sophisticated derivatives contracted directly on the inflation (e.g., inflation swaps) may enrich the estimation because their inflation-contingent payoffs possibly entail other combinations of m and i. In practice, inflation derivative markets are much smaller than nominal bond derivative markets and may subject to a similar liquidity concern of TIPS in adverse market conditions. For this reason, we also look beyond asset markets for additional inputs to enrich the inflation estimation.

Third, the discussion above points to the need of estimation inputs that are sensitive to SDF and inflation parameters (other than their difference) and are free of potential asset market liquidity concerns. Following the literature, we consider two such inputs, namely (i) the inflation surveys and (ii) short-term real interest rate. With regard to (i), inflation surveys by market professionals are an important source of forecasts that have long been employed and shown to outperform inflation forecasts by other measures (Pennacchi (1991), and Ang et al. (2007)). With regard to (ii), because the inflation is forecastable at short terms, short-term real interest rate proxies exist and equal the difference between short-term nominal interest rates and short-term inflation forecasts as suggested by the Fisher equation. In the current parametric setting, the inflation surveys map into the inflation parameter i, the real interest rates map into the real SDF parameter m.[17] Therefore, these quantities supplement nominal asset price data with needed inputs to estimate inflation and pricing parameters separately.

Looking back, in the real pricing perspective, the inflation estimation employs three sources of inputs: inflation surveys (which implicate inflation parameter i), short-term real interest rates (which implicate pricing parameter m), and prices of nominal assets (which implicate $m - i$). In theory, inflation surveys alone suffice to estimate the inflation parameter i as in a pure statistical approach (and short-term interest rates suffice to estimate the pricing parameter m). In practice, these estimates are separate and hence may lose efficiency. Nominal asset price data contain both parameters and their inputs to the estimation connect the above two separate estimates and improve the overall efficiency. In comparison, the inflation estimation in the literature, e.g., Chernov and

[17]The interest rate is the drift term of the SDF (5), $r_R = -\frac{1}{dt} E_t\left[\frac{dM_{Rt}}{M_{Rt}}\right] = -m\,\mu_X - \frac{1}{2} m^2 \sigma_X^2$.

Mueller (2012), Haubrich et al. (2012), and Kitsul and Wright (2013), employs inflation surveys, short-term nominal interest rates, and prices of real assets (TIPS or inflation derivatives). The inclusion of inflation derivatives, e.g., inflation swaps, to the inflation estimation is also possible in the real pricing perspective of the current paper. However, we restrain from employing these inflation derivative inputs at the onset to eliminate potential liquidity concerns associated with these assets. In doing so, our estimation is able to price these derivatives as out-of-sample assets and quantifies their model-implied liquidity premia.

3 The Model

3.1 Model Specification

State Variables: We fix a probability space $(\Omega, \mathcal{F}, P)$ associated with a physical (data-generating) probability measure $P : \mathcal{F} \to [0, 1]$ and an information filtration $\mathcal{F}_t$. We consider a setting with n state variables stacked into $n \times 1$ vector X_t. While we use the notation n throughout, the actual estimation employs $n = 3$ state variables. For tractability, we assume that X_t is a continuous-time autoregressive Gaussian process taking values in a state space $D \subset \mathbb{R}^n$. This process belongs to the class of affine dynamics (with linear conditional expected growths and constant conditional volatilities), highly tractable asset prices and return distributions. Specifically, X_t satisfies the stochastic differential equation (SDE),

$$dX_t = \mathcal{K} (\Theta - X_t) \, dt + \left[\sqrt{S}\right] dW_t, \tag{7}$$

where W_t is a n-dimensional standard Brownian motion adapted to $\mathcal{F}_t$, $\left[\sqrt{S}\right]$ is the $n \times n$ constant volatility matrix, $\mathcal{K}$ and Θ are $n \times n$ and $n \times 1$ constant matrices characterizing the mean reversion of state variables. State variables can also be decomposed into proper modes associated with (separate) mean-reverting and volatility dynamics by the respective orthogonalization,

$$\mathcal{K} = V \, \mathrm{Diag} \, [\mathcal{K}] \; V^{-1}, \qquad \sqrt{S} = \Sigma \mathrm{Diag} \left[\sqrt{S}\right]. \tag{8}$$

Above, the $n \times n$ invertible matrix V prescribes an orthogonalization of state variables' mean reversion rates, $V^{-1} \mathcal{K} V = \mathrm{Diag} \, [\mathcal{K}] = \mathrm{Diag} \, [\kappa_i]$, $i \in \{1, \ldots, n\}$, with κ_i denoting the rate of the i-th orthogonalized mean reversion mode. The $n \times n$ matrix Σ prescribes an orthogonalization of state variables' conditional covariance matrix, $\Sigma' Cov_t(dX_t)\Sigma = \text{Diag}[S] = \text{Diag}[S_i]$, $i \in \{1,\ldots,n\}$, with S_i denoting the conditional variance of the i-th orthogonalized state variable, and notation $'$ denoting the matrix transpose. Note that the real symmetric covariance matrix is diagonalized by an orthogonal matrix Σ (i.e., $\Sigma\Sigma' = \Sigma'\Sigma = \mathbb{1}_{n\times n}$).

In special cases in which the two orthogonalizations can be reconciled ($\Sigma = V$), every orthogonalized state variable is mean reverting autonomously. However, in general cases, the two orthogonalizations are distinct ($\Sigma \neq V$), offering richer state variable dynamics, in which orthogonalized mean reversion modes correlate and orthogonalized state variables are not autonomous. Our specification does not preclude these general cases a priori. The distribution of the Gaussian state variable X_t is characterized by the first two unconditional moments of X_t,

$$E_0[X_t] = e^{-t\mathcal{K}}X_0 + \left(1 - e^{-t\mathcal{K}}\right)\Theta, \qquad Var_0[X_t] = \int_0^t e^{-2(t-s)\mathcal{K}}\Sigma\text{Diag}[S]\,\Sigma' ds. \tag{9}$$

Inflation: Let I_t denote the price of the consumption basket in spot dollars at time t. We specify an exponential affine process for the price index I_t,

$$I_t(X_t) = \exp\left(i_1' X_t\right), \tag{10}$$

where the sign $'$ denotes the matrix transpose, and i_1 is a (time-independent) $n \times 1$ vector of parameters. The economic interpretation of the basket price is that I_t identifies with a measure of the consumer price index (CPI) in the economy.

Real SDF: We specify an exponential affine real SDF process M_{Rt}, which prices financial assets in real term, i.e., asset prices generated by M_{Rt} are in units of consumption baskets.

$$M_{Rt}(X_t) = \exp\left(m_{0t} + m_{1t}' X_t\right), \tag{11}$$

where m_{0t} and m_{1t} are respectively scalar and $n \times 1$ vector. In the estimation of the model, we employ further parametric specifications in which $m_{0t} = -\beta t$, and $m_{1t} = m_1$ is a constant vector. The economic interpretation of the real SDF is that M_{Rt} identifies with the representative agent's marginal utility of consuming a basket at time t.[18] Thus the constant parameter $\beta \in \mathbb{R}^+$ characterizes the time discount factor of the economy's real representative agent.

[18]The intertemporal marginal rate of substitution (IMRS) in basket consumptions is the growth $\frac{M_{Rt+dt}(X_{t+dt})}{M_{Rt}(X_t)}$.

3.2 Model Implications

The above parametric specification of state variables, price index, and real SDF implies the parametrization of other relevant quantities of the pricing model, namely inflation moments, the real interest rate and real prices of risks, the nominal SDF (and hence the nominal interest rate and nominal prices of risks), and risk-neutral distributions of state variables. We present these quantities below, and relegate derivations to Appendix B.

Inflation Moments: The inflation follows from the log price index (10), $\log I_t = i_1' X_t$, is linear in state variables X_t, and hence is conditional Gaussian. The conditional mean and variance of the inflation follow from those of state variables derived in Appendix (B.1), equations (53), (54),

$$E_t[\log I_T] = i_1'\Theta + i_1' V \text{Diag}\left[e^{(t-T)\mathcal{K}}\right] V^{-1}(X_t - \Theta), \qquad Var_t[\log I_T] = i_1' V\,\mathcal{S}_{tT}\,V' i_1, \tag{12}$$

where the $n \times n$ symmetric matrix $\mathcal{S}_{tT}$ is such that its jh-th element is (see also Lemma 1),

$$[\mathcal{S}_{tT}]_{jh} \equiv \frac{1 - e^{(t-T)(\kappa_j + \kappa_h)}}{\kappa_j + \kappa_h}\left(V^{-1}\Sigma \text{Diag}[S]\Sigma' V'^{-1}\right)_{jh}, \quad \forall j, h \in \{1, \ldots, n\}. \tag{13}$$

Similarly follows the log of the expected growth of price index (Lemma 1),

$$\log \overline{I}_{tT} \equiv \log E_t\left[\frac{I_T}{I_t}\right] = i_1'\mathcal{U}_{t-T,0}\Theta + \frac{1}{2} i_1' V \mathcal{S}_{tT} V' i_1 - i_1'\mathcal{U}_{t-T,0} X_t, \tag{14}$$

where matrix $\mathcal{S}_{tT}$ is as in (13), and the $n \times n$ matrix $\mathcal{U}_{t_1 t_2}$ is defined as

$$\mathcal{U}_{t_1 t_2} \equiv -V \text{Diag}\left[e^{t_1\mathcal{K}} - e^{t_2\mathcal{K}}\right] V^{-1} = -\left[e^{t_1\mathcal{K}} - e^{t_2\mathcal{K}}\right], \quad \forall t_1, t_2. \tag{15}$$

We can also characterize inflation as a stochastic process. The state variable specification (7) and the price index (10) imply an SDE for the inflation process (via Itô's lemma),

$$\begin{gathered} \tfrac{dI_t}{I_t} = \tfrac{I_{t+dt}}{I_t} - 1 = \mu_{It} dt + \sigma_{It}' dW_t, \\ \mu_{It} \equiv i_1' V \text{Diag}[\mathcal{K}] V^{-1}(\Theta - X_t) + \tfrac{1}{2} i_1'\Sigma \text{Diag}[S]\Sigma' i_1, \quad \sigma_{It} \equiv \text{Diag}\left[\sqrt{S}\right]\Sigma' i_1. \end{gathered} \tag{16}$$

The inflation process $\frac{dI_t}{I_t}$ has constant conditional volatility, and hence is conditional Gaussian.

Real Interest Rate and Prices of Risks: Similarly, the state variable specification (7) and the real SDF (11) imply the real SDF growth process,

$$\frac{dM_{Rt}}{M_{Rt}} = \frac{M_{Rt+dt}}{M_{Rt}} - 1 = -r_{Rt}dt - \eta'_{Rt}dW_t, \tag{17}$$

with the affine real interest rate, and $n \times 1$ real prices of risks,

$$r_{Rt} = \rho_{R0t} + \rho'_{R1t}X_t, \qquad \eta_{Rt} = -\text{Diag}[\sqrt{S}]\Sigma' m_{1t}. \tag{18}$$

$$\text{with} \begin{cases} \rho_{R0t} \equiv -\frac{dm_{0t}}{dt} - (V'm_{1t})'\text{Diag}[\mathcal{K}]V^{-1}\Theta - \frac{1}{2}m'_{1t}\Sigma\text{Diag}[S]\Sigma' m_{1t}, \\ \rho_{R1t} \equiv -\frac{dm_{1t}}{dt} + V'^{-1}\text{Diag}[\mathcal{K}]V'm_{1t}. \end{cases}$$

Nominal SDF, Interest Rate and Prices of Risks: The nominal SDF prices asset in the numeraire of spot dollars. It is related to the real SDF by the factor of price index, and hence is also an exponential affine function of state variables,

$$M_{Nt} = \frac{M_{Rt}}{I_t} = \exp\left[m_{0t} + (m'_{1t} - i'_1)X_t\right]. \tag{19}$$

From this relationship follows the nominal SDF growth $\frac{dM_{Nt}}{M_{Nt}} = -r_{Nt}dt - \eta'_{Nt}dW_t$, which in turn implies the nominal interest rate and $n \times 1$ nominal prices of risks,

$$\eta_{Nt} = \eta_{Rt} + \sigma_I = \text{Diag}[\sqrt{S}]\Sigma'\left(i_1 - m_{1t}\right), \qquad r_{Nt} = r_{Rt} + (\mu_I - \sigma'_I\sigma_I) - \eta'_{Rt}\sigma_I. \tag{20}$$

Clearly, nominal prices of risks account for real prices of risks as well as the volatility of inflation. As a result, from a real pricing perspective, the nominal bond is risky. It offers expected return $r_N - (\mu_I - \sigma'_I\sigma_I)$, and expected excess return $r_N - (\mu_I - \sigma'_I\sigma_I) - r_R = -\eta'_{Rt}\sigma_I$ as a compensation for bearing inflation risk σ_I.[19] The substitution of the inflation (16) and real interest rate (18) into above expression yields an affine nominal interest rate,

$$r_{Nt} = \rho_{N0t} + \rho'_{N1t}X_t, \tag{21}$$

$$\text{with} \begin{cases} \rho_{N0t} \equiv -\frac{dm_{0t}}{dt} - (m_{1t} - i_1)'V\text{Diag}[\mathcal{K}]V^{-1}\Theta - \frac{1}{2}(m_{1t} - i_1)'\Sigma\text{Diag}[S]\Sigma'(m_{1t} - i_1), \\ \rho_{N1t} \equiv -\frac{dm_{1t}}{dt} + V'^{-1}\text{Diag}[\mathcal{K}]V'(m_{1t} - i_1). \end{cases}$$

The comparison of (18) and (21) shows that the inflation slope factor i_1 drives the wedge between

[19] Expected returns on nominal bonds in the real term follow from Euler equation $E_t\left[\frac{M_{Rt+dt}}{M_{Rt}}\frac{I_t}{I_{t+dt}}\right] = 1$.

the real and nominal interest rate dynamics.[20]

Real ($\mathbb{Q}_R$) and Nominal ($\mathbb{Q}_N$) Risk-Neutral State Dynamics: We note that because risk-free bonds are different in real and nominal terms, the risk-neutrality concept also varies with the pricing denomination (i.e., in either real consumption baskets or nominal dollars). Accordingly, let W_t, $W_{\mathbb{Q}_R t}$, and $W_{\mathbb{Q}_N t}$ respectively denote the standard n-dimensional Brownian motions in the physical measure $\mathbb{P}$, the real risk-neutral measure $\mathbb{Q}_R$, and the nominal risk-neutral measure $\mathbb{Q}_N$. They are related with one another through the real and nominal prices of risks η_{Rt} (18), η_{Nt} (20),

$$dW_t = dW_{\mathbb{Q}_R t} - \eta_{Rt}dt = dW_{\mathbb{Q}_N t} - \eta_{Nt}dt.$$

Substituting these relationships into (7) yields the the state variable dynamics in the real and nominal risk-neutral measures,

$$\begin{aligned} dX_t &= V\text{Diag}[\mathcal{K}]V^{-1}\left(\Theta_{\mathbb{Q}_R} - X_t\right)dt + \Sigma\text{Diag}[\sqrt{S}]dW_{\mathbb{Q}_R t} \\ &= V\text{Diag}[\mathcal{K}]V^{-1}\left(\Theta_{\mathbb{Q}_N} - X_t\right)dt + \Sigma\text{Diag}[\sqrt{S}]dW_{\mathbb{Q}_N t}, \end{aligned} \tag{22}$$

with state variable's long-term mean vectors $\Theta_{\mathbb{Q}_R} \equiv \Theta + V\text{Diag}[\mathcal{K}^{-1}]V^{-1}\Sigma\text{Diag}[S]\Sigma' m_{1t}$ and $\Theta_{\mathbb{Q}_N} \equiv \Theta + V\text{Diag}[\mathcal{K}^{-1}]V^{-1}\Sigma\text{Diag}[S]\Sigma'(m_{1t} - i_1)$. Clearly, the state variable remains affine in either real or nominal risk-neutral measures by model's (complete affine) construction.

3.3 Model Pricing

The current model features closed-form prices for nominal and real assets, including nominal bonds, T-note futures, real bonds, TIPS, and inflation swaps, which facilitate the model's estimation in subsequent sections. We describe these assets and present their prices below, and relegate derivations to Appendices B.2, B.3.

Nominal Bond Prices: Consider a nominal zero coupon bond that pays one dollar (or equivalently, $\frac{1}{I_T}$ units of consumption baskets) at maturity T. The current price at time t of this bond in consumption baskets is $E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{1}{I_T}\right]$. Given the model specifications (11), (10), the nominal zero coupon bond price in spot dollar at time t is exponential affine in state variables,

$$B_{tT} = I_t E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{1}{I_T}\right] = E_t\left[\frac{e^{(m_{0T}-i_0)+(m'_{1T}-i'_1)X_T}}{e^{(m_{0t}-i_0)+(m'_{1t}-i'_1)X_t}}\right] = e^{(m_{0T}-m_{0t}+b_{0tT})+(i'_1-m'_{1t}+b'_{1tT})X_t}, \tag{23}$$

[20]This wedge results from the difference between real and nominal SDFs (11), (20), which are related by the substitution $m_{1t} \leftrightarrow m_{1t} - i_1$.

where time-dependent coefficients $b_{0tT} \in \mathbb{R}$ and $b_{1tT} \in \mathbb{R}^n$ solve the the conditional expectation, $E_t\left[e^{(m'_{1T}-i'_1)X_T}\right] = e^{b_{0tT}+b'_{1tT}X_t}$, and are obtained from Lemma 1 (Appendix B.1),

$$b_{1tT} = V'^{-1}\mathrm{Diag}\left[e^{(t-T)\mathcal{K}}\right]V'(m_{1T}-i_1), \tag{24}$$

$$b_{0tT} = (m_{1T}-i_1)'V\mathrm{Diag}\left[\mathbb{1}-e^{(t-T)\mathcal{K}}\right]V^{-1}\Theta + \frac{1}{2}(m_{1T}-i_1)'V\mathcal{S}_{tT}V'(m_{1T}-i_1),$$

where the $n \times n$ symmetric matrix $\mathcal{S}_{tT}$ is defined in (13).

Futures on Nominal Bonds: Let us consider a futures contract initiated at current time t that delivers at time $T > t$ a nominal bond of one dollar face value and maturity $\tau > T$. The futures price $F_{tT\tau}$ (contracted at the initiation time t) is derived in (57) (Appendix B.2),

$$F_{tT\tau} = \frac{B_{t,\tau}}{B_{t,T}} = \frac{e^{(m_{0\tau}+b_{0t\tau})+b'_{1t\tau}X_t}}{e^{(m_{0T}+b_{0tT})+b'_{1tT}X_t}} = e^{(m_{0\tau}-m_{0T}+b_{0t\tau}-b_{0tT})+(b'_{1t\tau}-b'_{1tT})X_t}, \tag{25}$$

where the second equality has employed the expression (23) for bond prices. In the case in which m_{1t} is time-independent, the substitution of (24) into the above expression yields an explicit solution for the log futures price,

$$\begin{aligned} f_{T-t,\tau-t} \equiv \log F_{tT\tau} = m_{0\tau} - m_{0T} + (m_1-i_1)'\mathcal{U}_{t-T,t-\tau}X_t \\ -(m_1-i_1)'\mathcal{U}_{t-T,t-\tau}\Theta + \tfrac{1}{2}(m_1-i_1)'V\left[\mathcal{S}_{t\tau}-\mathcal{S}_{tT}\right]V'(m_1-i_1), \end{aligned} \tag{26}$$

where $\mathcal{U}_{t-T,t-\tau}$ is defined in (15).

Real Bond Prices: Let us consider first a stylized real zero-coupon bond that is issued at time t_0 and matures at T_i. The bond offers an inflation-indexed payoff of $\frac{I_{T_i-\delta}}{I_{t_0-\delta}}$ in spot dollars at maturity T_i, where δ is the indexation lag of three months.[21] This nominal payoff at T_i is equivalent to $\frac{1}{I_{T_i}}\frac{I_{T_i-\delta}}{I_{t_0-\delta}}$ units of consumption baskets. Therefore, the price $B^{R,t_0}_{tT_i}$ at t ($t_0 \le t \le T_i$) of this bond is $E_t\left[\frac{M_{RT_i}}{M_{Rt}}\frac{1}{I_{T_i}}\frac{I_{T_i-\delta}}{I_{t_0-\delta}}\right]$ in units of consumption baskets, and $I_tE_t\left[\frac{M_{RT_i}}{M_{Rt}}\frac{1}{I_{T_i}}\frac{I_{T_i-\delta}}{I_{t_0-\delta}}\right]$ in spot dollars at t. Appendix B.3 derives the following expression for the price of this bond (in spot dollars at t),

$$B^{R,t_0}_{tT_i} = \exp\left\{(m_{0T_i} - m_{0t} + b^R_{0tT_i}) + (i'_1 - m'_{1t} + {b^R_{1tT_i}}')X_t - i'_1X_{t_0-\delta}\right\}, \tag{27}$$

[21]This stylized real zero-coupon bond represents a real stripped coupon due at T_i of a TIPS issued at t_0. In practice, the official realized inflation level is published with a time lag δ of three months. Therefore, at the bond maturity T_i, $I_{T_i-\delta}$ is the most recent official realized inflation available, and is employed to determine the real bond's nominal payoffs. The valuation of real stripped coupons helps to price TIPS.

where coefficients $b^R_{0tT_i} \in \mathbb{R}$ and $b^R_{1tT_i} \in \mathbb{R}^n$ are given in equation (59). The real yield associated with the real zero-coupon bond is,

$$y^{R,t_0}_{tT_i} = \frac{-1}{T_i - t} \left\{ (m_{0T_i} - m_{0t} + b^R_{0tT_i}) + (i_1' - m_{1t}' + {b^R_{1tT_i}}') X_t - i_1' X_{t_0-\delta} \right\}.$$

TIPS Prices: Let us consider a TIPS contract issued at time t_0 of a unit notional face value, the coupon rate k, and the maturity T. The TIPS cash flows consist of a series of stripped coupons payable at times $\{T_i\}$, and the final payment at T of the last coupon and the principal. All cash flow settlements (coupon and principal payments) of TIPS are indexed to the spot inflation at respective payment times. In addition, the principal settlement at maturity T is guaranteed to be no less than the TIPS notional face value at issuance. This floor option protect the principal payment against the deflation. Therefore the TIPS price in spot dollars at t is composed of the valuation of (i) real stripped coupons, (ii) the real principal, and (iii) the floor option on the principal,

$$P^{\text{TIPS},t_0}_{tT} = \left(k \sum_{i:T_i \in [t,T]} B^{R,t_0}_{tT_i} \right) + B^{R,t_0}_{tT} + \mathcal{C}^{R,t_0}_t (X_t). \tag{28}$$

Zero-coupon real bond prices $B^{R,t_0}_{tT_i}$ and B^{R,t_0}_{tT} are given in (27), and option price $\mathcal{C}^{R,t_0}_t(X_t)$ are derived in (61), Appendix B.3.

Inflation Swap Rates: Let us consider a zero-coupon inflation swap contract initiated at time t_0 of a unit notional value and the maturity T. The fixed-rate payor in the swap contract pays a constant inflation swap rate h_{t_0T} and receives a floating rate indexed to the available inflation. As in zero-coupon contracts, these payments are settled at the maturity of contract and based on the notional value of the contract. At time $t \in [t_0, T]$, the floating rate is indexed to the inflation rate $\frac{I_{t-\delta}}{I_{t_0-\delta}}$ due to the same inflation indexation lag δ of three months discussed above. Given a unit notional value, the fixed-rate payor pays $e^{h_{t_0T}(T-t_0)}$ and receives $\frac{I_{T-\delta}}{I_{t_0-\delta}}$ in spot dollars at maturity T. Equivalently, these payments are $\frac{1}{I_T} e^{h_{t_0T}(T-t_0)}$ and $\frac{1}{I_T} \frac{I_{T-\delta}}{I_{t_0-\delta}}$ in units of consumption baskets. At the initiation time, the swap contract is fair to both parties, i.e., the net value to the fixed-rate payor is zero,

$$0 = E_{t_0} \left[\frac{M_{RT}}{M_{Rt_0}} \frac{1}{I_T} \left(\frac{I_{T-\delta}}{I_{t_0-\delta}} - e^{h_{t_0T}(T-t_0)} \right) \right]. \tag{29}$$

From this follows the inflation swap rate (see (63), Appendix B.3),

$$h_{t_0T} = \frac{1}{T - t_0} \left[b^R_{0t_0T} - b_{0t_0T} + ({b^R_{1t_0T}}' - b'_{1t_0T})X_{t_0} - i'_1 X_{t_0-\delta} \right], \tag{30}$$

where coefficients $b_0, b^R_0 \in \mathbb{R}$, and $b_1, b^R_1 \in \mathbb{R}^n$ are given in (24) and (59) respectively.

4 Estimation and Results

In order to estimate the future inflation distribution in physical measure, we need to estimate both (i) state dynamics parameters that govern the state variable distribution in $\mathbb{P}$, and (ii) parameters $\{i_{1j}\}$, $(j \in \{1, \ldots, n\})$, that connect the inflation process to state variables. We employ Kalman filter to jointly estimates all parameters in the model. Therefore, apart from the future inflation distribution, we also simultaneously estimate risk pricing (SDF) parameters.

4.1 Maximum likelihood and the Kalman Filter Estimation

State-space Setup

In the Gaussian setting of our paper, the maximum likelihood based on the Kalman filter estimator is both consistent and optimal in the sense of achieving the least mean square errors.[22] We summarize below the main formula of the estimator, and relegate underlying details to Appendix C.1.

The Kalman filter works with the state space formulation of state equation (concerning state variables X_t) and observation equations (concerning observable quantities y_t),

$$X_{t+1} = A + BX_t + \nu_{t+1}, \tag{31}$$

$$y_t = a + bX_t + \epsilon_t, \tag{32}$$

where ν_{t+1} and ϵ_t are normally distributed. In our model, the state equation is given in (7), and we will discuss the choice of the observable variables in the next section.

Our maximum likelihood estimator is based on the likelihood of observing y_t. For our Gaussian

[22] Therefore, estimators generated by the Kalman filter are optimal among all estimators that are linear in past estimators and observations.

setting, the log likelihood function is

$$\mathcal{L}(y_t|\mathcal{P}) = \sum_{t=0}^{T-1} \left\{ -\frac{1}{2} \log \mathrm{Det}[\widehat{V}_{\Delta y t+1|t}] - \frac{1}{2} \Delta y'_{t+1} \left(\widehat{V}_{\Delta y t+1|t} \right)^{-1} \Delta y_{t+1} \right\}, \tag{33}$$

where $\mathcal{P}$ is the full set of the model's parameters, vector $\Delta y_{t+1} \equiv y_{t+1} - \widehat{y}_{t+1|t}$ denotes innovations to observations y_t, and matrix $\widehat{V}_{\Delta y t+1|t} \equiv E\left[(\Delta y_{t+1})^2 \right]$ denotes their covariances. The observation equation (32) gives the predicted values $\widehat{y}_{t+1|t}$ and $\widehat{V}_{\Delta y t+1|t}$ in terms of $\widehat{X}_{t+1|t}$ and $\widehat{V}_{Xt|t-1}$, which in turns are provided by the Kalman filter.

The recursive Kalman procedure starts with initial estimates of state variables and their covariance matrix

$$\widehat{X}_{1|0} = \Theta, \qquad \widehat{V}_{X1|0} = \frac{1}{4} \left(\Sigma \mathrm{Diag}[S] \Sigma' \mathcal{K}^{-1} + \mathcal{K}'^{-1} \Sigma \mathrm{Diag}[S] \Sigma' \right).$$

At time period t, given new observations y_t, state variable estimates and their covariance matrix are recursively updated as

$$\widehat{X}_{t+1|t} = (A + B\widehat{X}_{t|t-1}) + B\widehat{V}_{Xt|t-1} b' \left(b\widehat{V}_{Xt|t-1} b' + \Sigma_\epsilon \right)^{-1} (y_t - \widehat{y}_{t|t-1}),$$

$$\widehat{V}_{Xt+1|t} = B \left[\widehat{V}_{Xt|t-1} - \widehat{V}_{Xt|t-1} b' \left(b\widehat{V}_{Xt|t-1} b' + \Sigma_\epsilon \right)^{-1} b\widehat{V}_{Xt|t-1} \right] B' + S,$$

where $\widehat{y}_{t|t-1} = a + b\widehat{X}_{t|t-1}$, and S and Σ_ϵ respectively are covariance matrices of innovations ν_{t+1} in the state equation and observation errors ϵ_t in the observation equation.

Observable variables

We take as inputs the data from three groups of price data, survey data, and real interest rate (see also Section 2.1). Therefore the observation equations are (26), (14), and (18).

First, price data is composed of futures prices $F_{tT\tau}$ (25) associated with CME T-note futures. In the model, futures prices $F_{tT\tau}$ are functions of parameter differential $m_1 - i_1$. Hence, T-note futures prices as input data help estimate the joint parametric model of inflation and pricing (SDF). Second, survey data is composed of the expected inflation (14) associated with the BCEI inflation consensus forecasts for different horizons (varying from one-quarter up to two-year). In the model, the inflation expectations are functions of the inflation slope parameter i_1. Hence, inflation expectations as input data also help estimate the model. Finally, while real interest rates are not directly observed in markets, their proxies exist for the short-term horizon. We consider such an observable proxy for the short-term real interest rate r_R, namely the difference between the one-month nominal interest rate r_N (as yield on one-month Treasury bills from the Federal Reserve's H.15 release) and one-month expected inflation $\overline{I}_e$ (as interpolated BCEI inflation consensus),

$$\tilde{r}_{Rt} = r_{Nt} - \overline{I}_{et}. \tag{34}$$

This proxy reflects the essence of the Fisher equation.[23] In the model, the short-term real interest rate r_R (18) is function of the pricing parameter m_1. Hence, as an input data, the real interest rate (34) complements data on futures prices and inflation surveys in the estimation of inflation distribution and risk pricing characteristics.

The estimation procedure is implemented using monthly input data above. At each current month t, we collect q futures prices associated with different futures settlement times T and different bond maturity times τ, q_{inf} BCEI inflation consensus forecasts associated with various survey horizons T_{inf}, and the short-term nominal interest rate. Depending on particular month t, there are from 4 to 11 futures contracts ($4 \leq q \leq 11$), and from 5 to 8 inflation consensus forecasts ($5 \leq q_{inf} \leq 8$).[24] Post estimation, we also use the inflation-indexed constant maturity yields compiled and provided by the Fed as observed TIPS yields, and inflation swap rates provided by Bloomberg as observed swap rates.[25] We calculate the difference between model-implied and observed TIPS yields (as well as inflation swap rates) to determine the mispricing of these real assets.

Model's parameters

We estimate the following parameters of the model.

1. *State variable parameters*: Our actual estimation employs three state variables, i.e., $n = 3$ hereafter. The state variable parameters to be estimated are,

- n^2 parameters $\mathcal{K}_{ij}$, $i, j \in \{1, \ldots, n\}$, of the mean reversion matrix $\mathcal{K}$,
- n elements $\{\Theta_1, \ldots, \Theta_n\}$ of the long-term mean vector $\Theta \in \mathbb{R}^n$,
- $(n^2 + n)/2$ elements of the covariance matrix S. Without loss of generality, we work with

[23] At short-term horizons, inflation risk is small, the Fisher equation holds approximately. In the model, the proxy (34) equals the real interest rate when we omit second- and higher-order terms.

[24] BCEI gives inflation consensus forecasts for future quarters, up to two years ahead.

[25] See also the discussion at the beginning of Section 2.1.

the orthogonalized form $\sqrt{S} = \Sigma \mathrm{Diag}\left[\sqrt{S}\right]$. For $n = 3$, these elements are (i) 3 diagonal variances $\{S_1, S_2, S_3\}$, which are strictly positive entries of the diagonal matrix Diag$[S]$,[26] and (ii) 3 Euler angles $\{a_1, a_2, a_3\}$ that fully parametrize the 3×3 orthogonal matrix Σ (a.k.a., Euler rotation matrix),

$$\Sigma = \begin{bmatrix} c_2 & -c_3 s_2 & s_2 s_3 \\ c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 \\ s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix},$$

where $s_i \equiv \sin a_i$, $c_i \equiv \cos a_i$, for $i \in \{1, 2, 3\}$.

2. *Inflation parameters*: n slope parameters $\{i_{11}, \ldots, i_{1n}\}$ (which are components of the inflation slope vector $i_1 \in \mathbb{R}^n$).

3. *Real-pricing parameters*: For the simplicity of the estimation, we assume a simple (log linear) time dependence of the real SDF (11),

$$m_{0t} = -\beta\, t, \qquad m_{1t} \text{ is constant vector in } \mathbb{R}^n. \tag{35}$$

Intuitively, the constant parameter $\beta \in \mathbb{R}^+$ characterizes the time discount factor of the economy's real representative agent. We constrain this discount factor to be in the range of 0-5%. Therefore, there are $n + 1$ real pricing parameters to be estimated: 1 time preference $\tilde{\beta} \in \mathbb{R}$, where $\tilde{\beta} \equiv -\log(\frac{5\%}{\beta} - 1)$, and n slope parameters $\{m_{11}, \ldots, m_{1n}\}$ (as components of the vector $m_1 \in \mathbb{R}^n$).

4. *Volatility parameters*: We assume that the observation errors ϵ_t are uncorrelated in the cross section, i.e., $(q + q_{inf} + 1) \times (q + q_{inf} + 1)$ covariance matrix Σ_ϵ is diagonal. For simplicity, we further assume that

$$\Sigma_\epsilon = \begin{bmatrix} \Sigma_{\epsilon 0} & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & 0 & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \Sigma_{\epsilon 0} & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & \Sigma_{\epsilon I} & \ldots & 0 & 0 \\ \vdots & \ddots & 0 & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & \Sigma_{\epsilon I} & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 & \Sigma_{\epsilon r} \end{bmatrix} \equiv \mathrm{Diag}[\Sigma_{\epsilon 0}, \ldots, \Sigma_{\epsilon 0}, \Sigma_{\epsilon I}, \ldots, \Sigma_{\epsilon I}, \Sigma_{\epsilon r}]. \tag{36}$$

[26] In the actual estimation procedure, we employ the associated transformed parameters $s_j \equiv \log S_j$ for $j \in \{1, 2, 3\}$.

Hence, there are 3 volatility parameters ($\Sigma_{\epsilon 0}$, $\Sigma_{\epsilon I}$, $\Sigma_{\epsilon r}$). In the actual estimation procedure, we employ the associated transformed parameters $\sigma_j \equiv \log \Sigma_j$ for $j \in \{\epsilon 0, \epsilon I, \epsilon r\}$.

Altogether, there are $n^2 + 5n + 4 = 28$ model parameters to be estimated,

$$\mathcal{P} = \{\kappa_{ij}\,;\, \Theta_i\,;\, s_i, a_i\,;\, i_{1i}\,;\, \tilde{\beta}, m_{1i}\,;\, \sigma_q\}, \tag{37}$$

where $i, j \in \{1, \ldots, n\}, q \in \{\epsilon 0, \epsilon I, \epsilon r\}, n = 3$.

4.2 Estimation Results

This section presents the estimation of future inflation and deflation distributions for various horizons. The estimation is at monthly frequency and employs data for the period of 2003-2017, chosen to be contemporaneous with the availability of TIPS and inflation swap data.[27] For the robustness, Appendix A presents the inflation estimation for the entire period 1982-2017. We first list the parameter estimates of the state dynamic and the underlying pricing model of Section 3.1.

Table 1 reports the estimates and standard errors of the model's parameters, which are obtained from the maximization of the log likelihood function (82) with a Kalman filter. The estimation employs monthly data of CME T-note futures prices and BCEI inflation consensus forecasts, for the period of 2003-2017. We note that $m_j \times i_j > 0$ for $j \in \{1, 2\}$ in Table 1. As a result, state variables X_j, for $j \in \{1, 2\}$, influence the real SDF growth and price index growth in the same direction. That is, a change in X_j increases (resp. decreases) the price index while also pushes the real SDF M_R higher (resp. lower), for $j \in \{1, 2\}$. These estimates indicate that state variables X_j, $j \in \{1, 2\}$, are responsible for the counter-cyclical behavior of price index as high SDF signifies a bad state of the economy. Hence they represent the risk characteristic of the inflation process as seen from the real pricing perspective.

Future Inflation: Distribution and Risk Pricing

Given the process (9) for the state variable X_t, the scalar $i_1' X_t$ has a normal distribution, whose conditional mean and and variance are derived respectively in (53) and (54). As a result, $i_1'(X_T - X_t)$

[27] Post estimation, we are to price TIPS and inflation swaps (out of sample) and compare results with their observed prices in markets. Running the estimation of the period contemporaneous to the availability of these assets' observed prices aims to assure that the pricing parameter estimates are also contemporaneous and facilitate the comparison.

Table 1: **Model Parameter Estimates**

Parameter	Estimate	Standard Error
$\mathcal{K}_{11}$	0.0629	0.00001
$\mathcal{K}_{12}$	0.0152	0.00009
$\mathcal{K}_{13}$	0.0337	0.00001
$\mathcal{K}_{21}$	0.0179	0.00004
$\mathcal{K}_{22}$	0.1388	0.00005
$\mathcal{K}_{23}$	0.0487	0.00003
$\mathcal{K}_{31}$	-0.0250	0.00001
$\mathcal{K}_{32}$	0.0113	0.00004
$\mathcal{K}_{33}$	0.0120	0.00001
Θ_1	-23.138	2.04979
Θ_2	156.54	2.02552
Θ_3	-293.49	1.51887
s_1	-8.9840	0.00040
s_2	-6.1923	0.00023
s_3	-7.6114	0.00032
$\tilde{\beta}$	-10.642	2.10622
m_1	0.7718	0.00198
m_2	-1.0910	0.00269
m_3	0.5434	0.00220
i_1	2.1992	0.00086
i_2	-4.0436	0.00063
i_3	-2.0264	0.00086
$\sigma_{\epsilon 0}$	-4.0686	0.03261
$\sigma_{\epsilon I}$	-13.349	0.02952
$\sigma_{\epsilon r}$	-8.6890	0.05633

Notes: Panel A shows the maximum likelihood estimates of the parameters of the inflation pricing model of (7), (10), (11) (Section 3.1) and the associated standard errors. The model is estimated using a Kalman filter and data at monthly frequency for the period from 2003 to 2017.

has a time t-conditional Gaussian distribution.[28] After rescaling, the following random variable has a conditional standard normal distribution at time t,

$$\frac{i_1'(X_T - X_t) - E_t\left[i_1'(X_T - X_t)\right]}{\sqrt{\mathrm{Var}_t\left[i_1' X_T\right]}} \sim \mathcal{N}_t(0, 1).$$

Therefore, the inflation from t to T (10) $\frac{I_T(X_T)}{I_t(X_t)} = \exp\left(i_1'[X_T - X_t]\right)$ then is a log normal random variable and has the the conditional probability density function,

$$\begin{aligned}\rho_{I_T/I_t}(x) &= \frac{1}{x\sqrt{2\pi \mathrm{Var}_t\left[i_1'(X_T - X_t)\right]}} \exp\left\{-\frac{1}{2}\frac{\left([\log x] - E_t\left[i_1'(X_T - X_t)\right]\right)^2}{\mathrm{Var}_t\left[i_1'(X_T - X_t)\right]}\right\} \\ &= \frac{1}{x\sqrt{2\pi i_1' V \mathcal{S}_{tT} V' i_1}} \exp\left\{-\frac{1}{2}\frac{\left([\log x] + i_1'\left[\mathcal{U}_{t-T,0}\left(X_t - \Theta\right)\right]\right)^2}{i_1' V \mathcal{S}_{tT} V' i_1}\right\},\end{aligned} \tag{38}$$

wherein $\mathcal{S}$ and $\mathcal{U}$ are given in (13), (15).

We first focus on the first moment of the annualized inflation. The conditional expectation of the inflation follows from (14),

$$E_t\left[\frac{I_T(X_T)}{I_t(X_t)}\right] = E_t\left[e^{i_1'(X_T - X_t)}\right] = \exp\left\{-i_1'\mathcal{U}_{t-T,0}(X_t - \Theta) + \frac{1}{2} i_1' V \mathcal{S}_{tT} V' i_1\right\}. \tag{39}$$

Figure 1 plots the conditional expectation of the annualized inflation, or $\frac{1}{T-t}\log E_t\left[\frac{I_T(X_T)}{I_t(X_t)}\right]$, for various horizons in a time series of spot time t. In the cross section, the annualized inflation expectation tends to be slightly lower at shorter horizons as observed earlier. In the time series, the annualized inflation expectation at all horizons dipped in the end of 2008 as a result of the financial crisis. These patterns broadly agree with the inflation expectation estimated by Haubrich et al. (2012) and Fleckenstein et al. (2017). However, these papers employ price data on different (inflation-indexed) assets from the the current paper's nominal (T-note futures) assets. As a result, the detailed differences in the time-series estimates of the expected future inflation reflect the differences between real and nominal asset prices. We pursue a detailed analysis on the differential pricing of real and nominal assets in Section 5 below.

To have an overall picture of the estimated future inflation, we examine its distribution. Figure 2 plots the conditional probability density function (38) of the annualized inflation for various horizons (of 1, 2, 5 and 10 years) in a time series of spot times t. Two features stand out in this

[28] The mean and variance of $i_1'(X_T - X_t)$ are respectively, $E_t\left[i_1'(X_T - X_t)\right] = -i_1'\left[\mathcal{U}_{t-T,0}\left(X_t - \Theta\right)\right]$, and $\mathrm{Var}_t\left[i_1'(X_T - X_t)\right] = \mathrm{Var}_t\left[i_1' X_T\right] = i_1' V \mathcal{S}_{tT} V' i_1$, with $\mathcal{U}$ (15), $\mathcal{S}$ (13).

Expected Inflation (2003-2017)

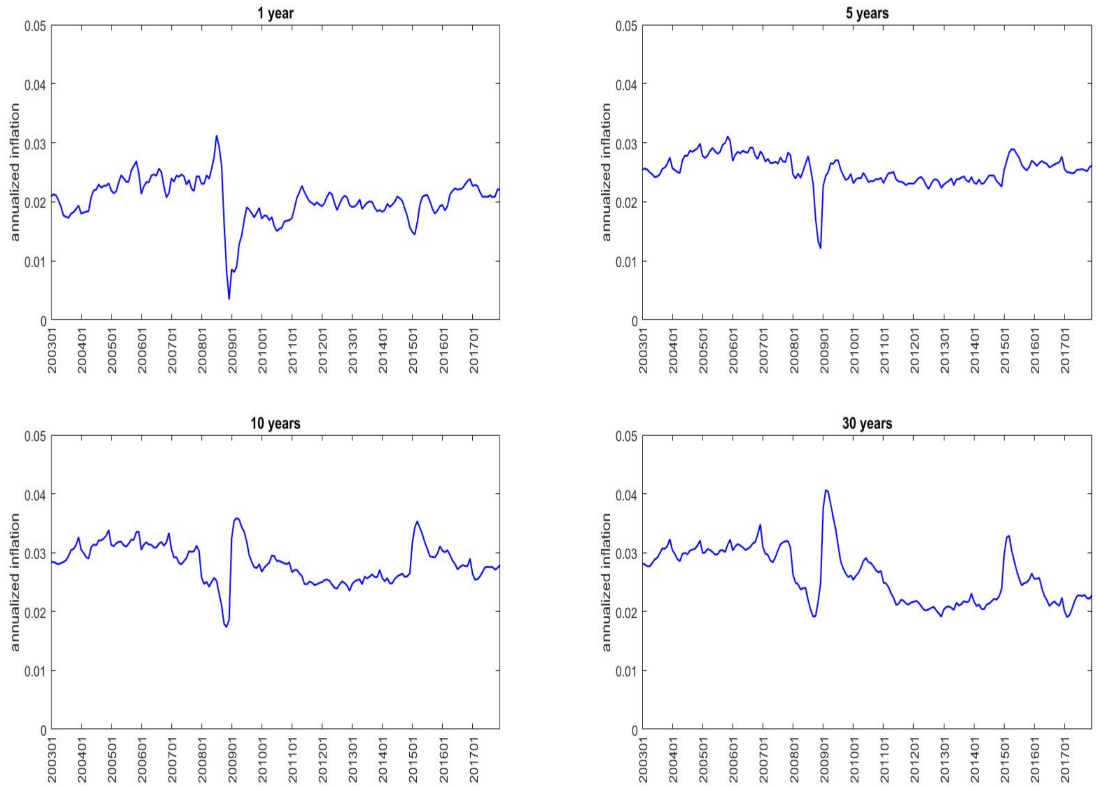

Figure 1: Expected annualized inflation estimated for various horizons. Values are annualized. The estimation is based on data of 2003-2017 period.

figure. First, in a cross-sectional aspect, the probability distribution of the annualized inflation exhibits a rightward shift (as well as becomes more concentrated) as the horizon increases (fixing current time t). This cross-sectional pattern indicates that asset markets reflect a slightly higher annualized inflation prospect in the longer run. Second, in a time-series aspect, the probability distribution of the inflation exhibits a leftward shift in the years of 2008-2009 (relatively to other years) for each horizon. This time-series pattern is consistent with an downward revision by market participants about the prospect of the future (annualized) inflation at different horizons, given that the economy was experiencing the great recession starting in the later half of 2008. Compared to the distributions of future inflation estimated from price data of real assets (Fleckenstein et al. (2017)), Figure 2 indicate flatter distributions estimated from T-note futures price data, which place relatively higher chances on inflationary scenarios. This pattern is consistent with the presence of a liquidity component in real asset markets, which reduces real asset prices, and hence, produces a weaker outlook of inflationary scenarios from price data of these real assets.

Probability Density Function of the Future Inflation (2003-2017)

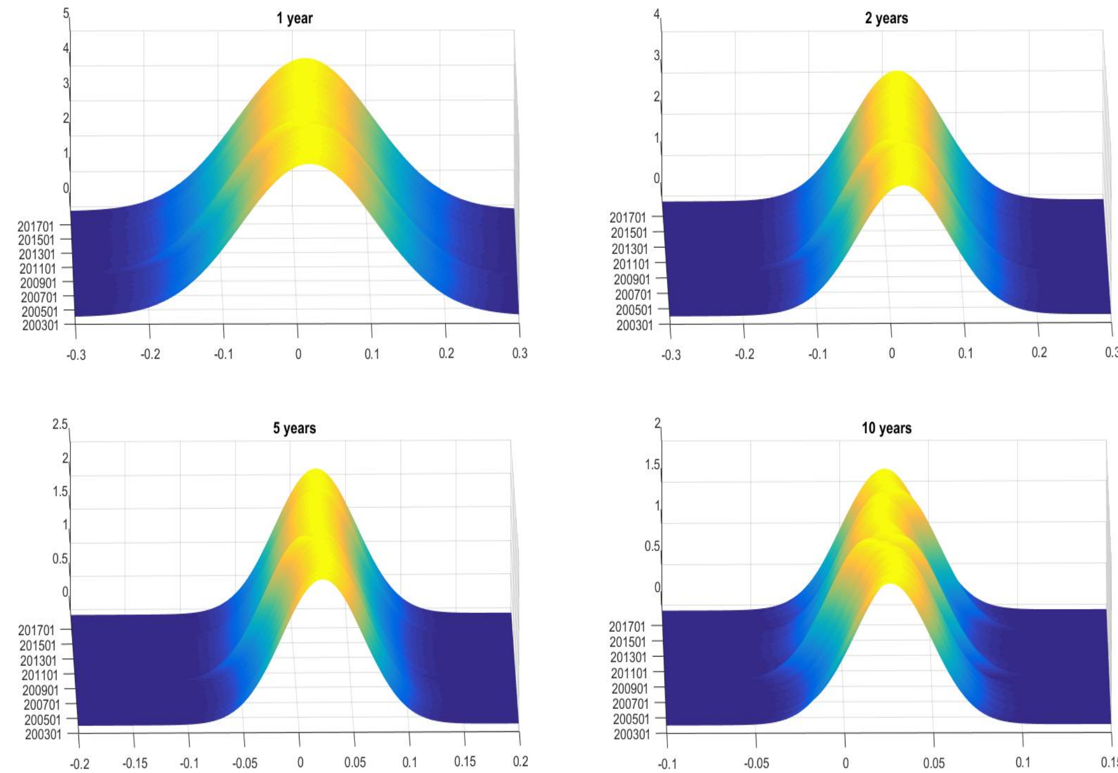

Figure 2: Probability density function of the future annualized inflation estimated for various horizons. Values are annualized. The estimation is based on data of 2003-2017 period.

Given above inflation distribution perceived in asset markets, we now discuss the price of inflation risk. From the nominal perspective, the pricing of inflation risk is given by the covariation between the price index I_t and the nominal SDF M_{Nt}. The (annualized) nominal inflation risk premium for the time period from t to T is,

$$\overline{\pi}^N_{t,T} = \frac{1}{T-t} Cov_t \left(\frac{M_{NT}}{M_{Nt}}, \frac{I_T}{I_t} \right). \tag{40}$$

An explicit expression of this premium is derived in (64) (Appendix B.4). In the theory, a positive premium $\overline{\pi}^N_{t,T}$, i.e., the nominal SDF M_{Nt} tends to be high when the price index is high, quantifies that inflation is a risk to the nominal agent in the economy.[29] In this case, $\overline{\pi}^N_{t,T}$ is the premium

[29]That is, the price index is counter-cyclical from the nominal perspective: I_t is high in the bad nominal state (high M_{Nt}), and vice versa.

required by the nominal agent to bear inflation risk. Whereas, a negative premium $\overline{\pi}_{t,T}^{N}$ signifies that the price index is pro-cyclical from the nominal perspective. We recall that the nominal SDF M_{Nt} represents the nominal agent's marginal utility of consuming an extra dollar worth of consumption good (but not an extra consumption basket). Therefore, the nominal agent does not represent a consumption-based economic agent of the economy, and as a result, the sign of $\overline{\pi}_{t,T}^{N}$ does not necessarily reflect the consumption-based risk pricing.[30]

In the estimation, Figures 3 plots the nominal price of the inflation risk (40). In a cross-sectional aspect, $\overline{\pi}_{t,T}^{N}$ is negative across (but increasing with) maturities. In a time-series aspect, the nominal inflation risk premium slightly increases (i.e., becomes least negative) around 2008-2009 for all horizons. Practically, the magnitude (in the absolute value) of the nominal price of inflation risk is relatively small across the board.

Prices of Future Inflation Risks (Nominal Perspective, 2003-2017)

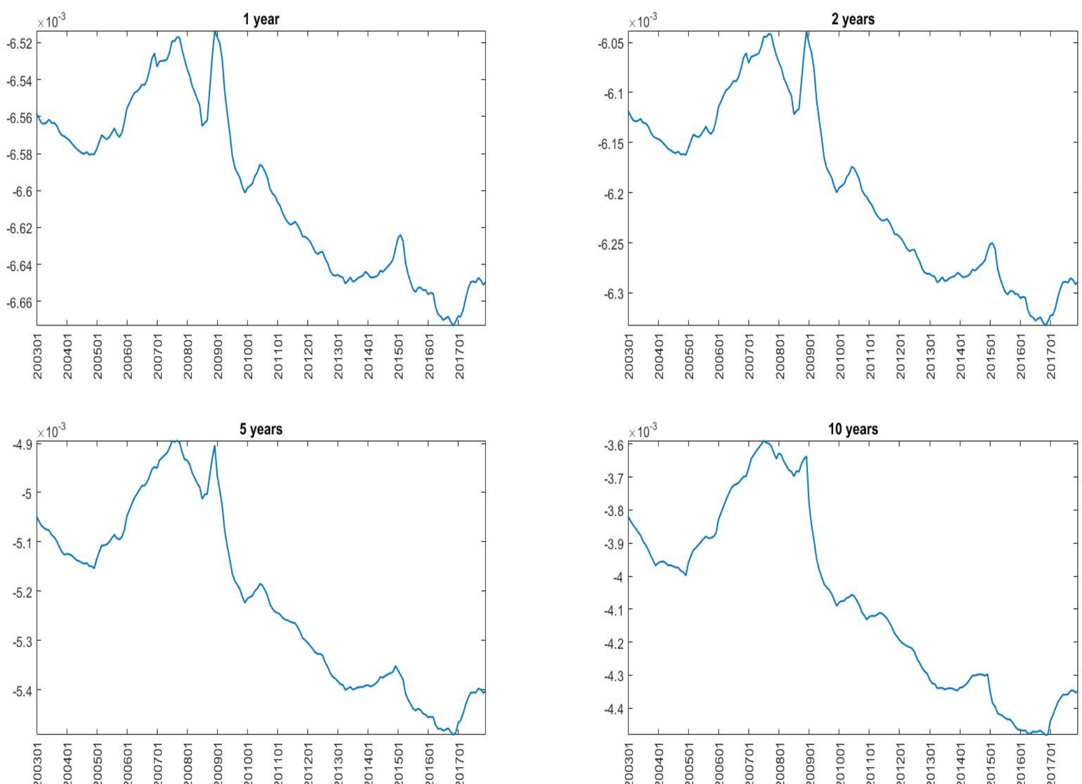

Figure 3: Annualized prices of future inflation risks estimated for various horizons. The estimation is based on data of 2003-2017 period.

[30] McCown and Shaw (2010) elaborate on a view that governments may also be subject to a risk premium on nominal bonds, as a result of which they are willing to issue and pay high TIPS yields.

Given the possible ambiguity in the economic (consumption-based) interpretation of the nominal price of inflation risk, we examine its real counterpart. The real price of inflation risk arises from the covariation between the price index and the real SDF M_{Rt}, and is characterized by the (annualized) real inflation risk premium for the time period from t to T,

$$\overline{\pi}_{t,T}^{R} = -\frac{1}{T-t} Cov_t \left(\frac{M_{RT}}{M_{Rt}}, \frac{I_t}{I_T} \right). \tag{41}$$

An explicit expression of this premium is derived in (65) (Appendix B.4). A positive premium $\overline{\pi}_{t,T}^{R}$, i.e., the price index is counter-cyclical from the real perspective, quantifies that inflation is a risk to the real agent. In this case, $\overline{\pi}_{t,T}^{R}$ is the premium required by the real agent to bear inflation risk. We recall that the real SDF M_{Rt} represents the real agent's marginal utility of consuming an extra consumption basket, who is the economic agent in the consumption-based asset pricing framework. Figures 4 plots the real price of the inflation risk (40). In a cross-sectional aspect, $\overline{\pi}_{t,T}^{R}$ is positive (in agreement with Haubrich et al. (2012)) and decreasing with maturities. This pattern signifies the risk characteristic of inflation, whose prices feature a decreasing term structure from the perspective of the real agent in the economy. In a time-series aspect, the real inflation risk premium dips during the period of 2008-2009, but has since gradually increases for all horizons. The average of the (annualized) real price of inflation risk is about 15 basis points, which is relatively small and of similar magnitude with the inflation risk premium estimated by Buraschi and Jiltsov (2005), Haubrich et al. (2012) and Fleckenstein et al. (2017).[31]

Future Deflation: Distribution and Risk Pricing

This section presents an estimate of the deflation perceived in asset markets. Following Fleckenstein et al. (2017), the prospect of deflation for time horizon T is quantified by the conditional probability $Prob_t\left(\frac{I_T}{I_t} \leq 1\right)$ of the event that the price index at T drops bellow the current index. Given that the inflation $\frac{I_T}{I_t}$ has a conditional log normal distribution (38), the conditional probability of deflation is,

$$Prob_t \left(\frac{I_T}{I_t} \leq 1 \right) = Prob_t \left(i_1'(X_T - X_t) \leq 0 \right) = CDF_{\mathcal{N}} \left(\frac{i_1' \left[\mathcal{U}_{t-T,0} \left(X_t - \Theta \right) \right]}{\sqrt{i_1' V \mathcal{S}_{tT} V' i_1}} \right), \tag{42}$$

where $CDF_{\mathcal{N}}(\cdot)$ denotes the standard cumulative normal distribution function. Figure 5 plots this

[31]Note that some papers in the literature adopt the definition of inflation risk premium in which it is equal to the difference between the inflation swap rate and the expected inflation. Omitting second-order (convexity) terms, this difference coincides with the real price of inflation risk $\overline{\pi}_{t,T}^{R}$ (41).

Prices of Future Inflation Risks (Real Perspective, 2003-2017)

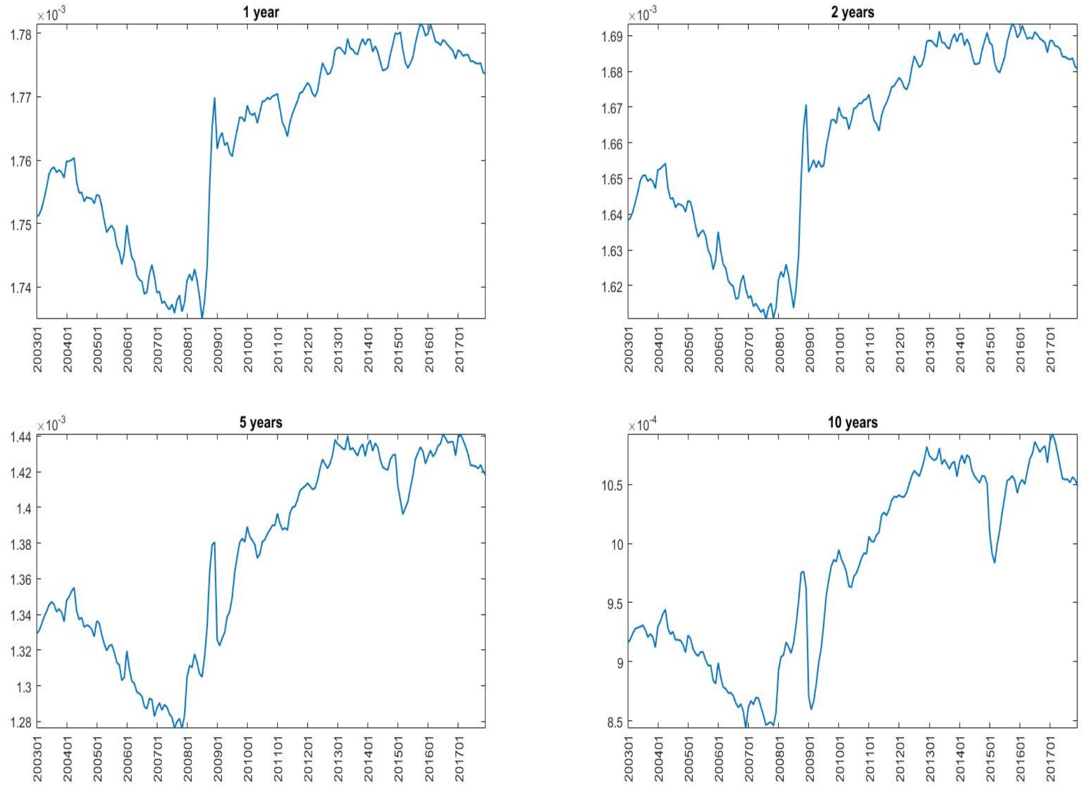

Figure 4: Annualized prices of future inflation risks estimated for various horizons. The estimation is based on data of 2003-2017 period.

probability of deflation estimated for various horizons of 1, 2, 5, and 10 years in a time series of spot time t. In the cross section, the deflation probability tends to decrease as the horizon increases. This pattern reflects the fact that, on average, the price index tends to increase as time progresses.[32] In the time series, the deflation probability at all horizons spiked in the end of 2008 as the financial crisis unfolded.

To assess the pricing effect of the deflation risk perceived in asset market, we follow the tail risk literature to consider the ratio of deflation probabilities in risk-neutral and physical measures. Technically, the pricing content of this ratio reflects in the fact that it captures the pricing kernel of the deflation state.[33] Intuitively, because the risk-neutral discount rate (i.e. risk-free rate) is

[32] Note that the underlying function $Prob_t\left(\frac{I_T}{I_t} \leq 1\right)$ (42) measures the probability of the event that the price index I_T at a specific future time T is below the current index I_t, but not the event that the future price index I_s, $s \in (t, T]$, ever drops below I_t before time T).

[33] To see this, consider a one-period setting for simplicity. The price of Arrow-Debreu asset paying off in the deflation

Deflation Probability (2003-2017)

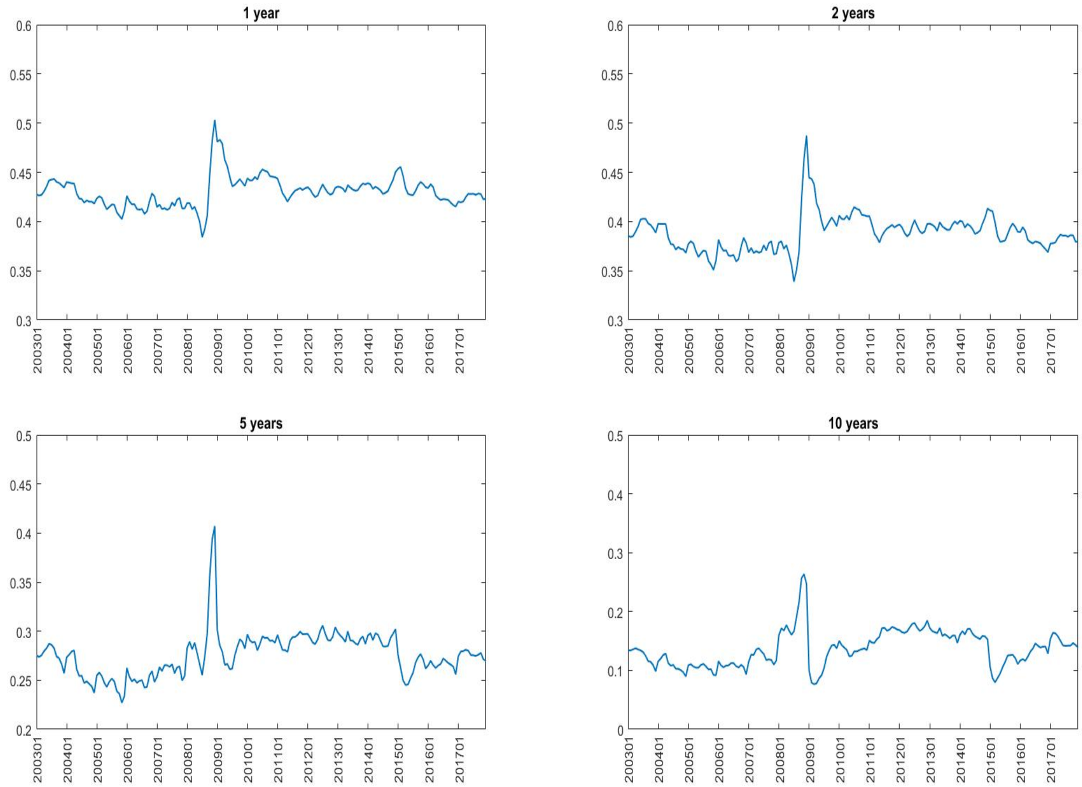

Figure 5: Conditional probability of the future deflation estimated for various horizons. The estimation is based on data of 2003-2017 period.

state-independent, the relative magnitude of the risk-neutral probability of a state signifies the price of risk of that state. In the model, there are two risk-neutral measures $\mathbb{Q}_N$ and $\mathbb{Q}_R$ associated with nominal and real pricing perspectives (22). Therefore, there are two corresponding probability ratios of type (42), namely,

$$\begin{aligned}
\frac{Prob_t^{\mathbb{Q}_N}\left(\frac{I_T}{I_t}\leq 1\right)}{Prob_t\left(\frac{I_T}{I_t}\leq 1\right)} &= CDF_{\mathcal{N}}\left(\frac{i_1'\left[\mathcal{U}_{t-T,0}\left(X_t-\Theta_{\mathbb{Q}_N}\right)\right]}{\sqrt{i_1' V \mathcal{S}_{tT} V' i_1}}\right), \\
\frac{Prob_t^{\mathbb{Q}_R}\left(\frac{I_T}{I_t}\leq 1\right)}{Prob_t\left(\frac{I_T}{I_t}\leq 1\right)} &= CDF_{\mathcal{N}}\left(\frac{i_1'\left[\mathcal{U}_{t-T,0}\left(X_t-\Theta_{\mathbb{Q}_R}\right)\right]}{\sqrt{i_1' V \mathcal{S}_{tT} V' i_1}}\right),
\end{aligned} \tag{43}$$

where $\Theta_{\mathbb{Q}_N}$ and $\Theta_{\mathbb{Q}_R}$, given below (22), represent the long-term mean of state variables in nominal

state d can be computed in either physical or risk-neutral measures: $AD(d) = prob(d)M(d) = \frac{prob^Q(d)}{1+r_f}$, where $M(d)$ is the pricing kernel of the deflation state and r_f the risk-free rate. This implies a proportional relationship between the ratio of probability and the pricing kernel, $\frac{prob^Q(d)}{prob(d)} = (1+r_f)M(d)$.

and real risk-neutral measures. Figures 6 and 7 plot the probability ratios (43), which respectively

Prices of Future Deflation Risks (Nominal Perspective, 2003-2017)

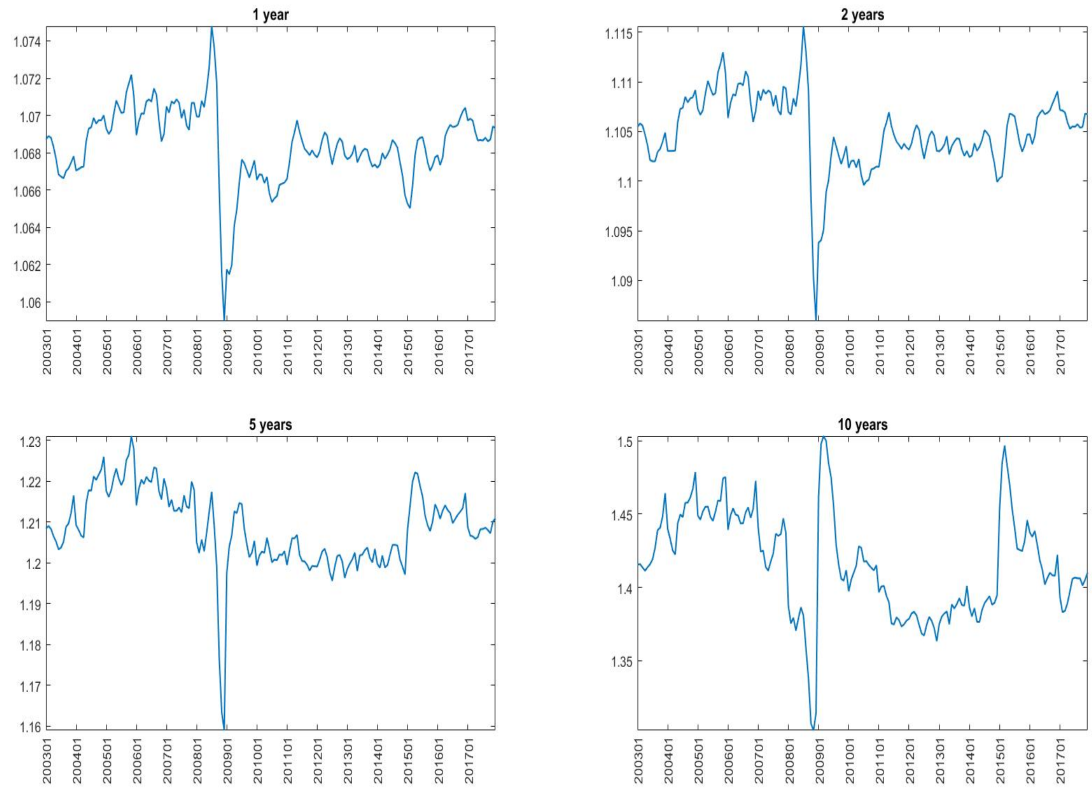

Figure 6: Prices of future deflation risks estimated for various horizons. The estimation is based on data of 2003-2017 period.

characterize the price of the deflation risk under the nominal and real pricing perspective. Several features stand out in these figures. In the cross section, the price of the deflation risk tends to increase with the horizon, from both nominal and real pricing perspectives. In the time series, the deflation prices are stable overall.[34] These prices exhibit more notable movements during the financial crisis of 2008-2009, but only for longer (10-year) horizon. Combined with an earlier observation concerning the deflation distribution (Figures 5), our estimation indicates that (i) markets place a lower probability, yet higher price, of the deflation risk at longer horizons, and (ii) the time-series movements of the Arrow-Debreu price of deflation state the deflation state price arise mainly from their distributions, but not their prices of risks.[35] Furthermore, markets perceive

[34] Numerical values of deflation risk prices, or the ratio of deflation probabilities in risk-neutral and physical measures (on vertical axis of Figures 6), vary by few percentage points for 1-, 2-, and 5-year horizons, and 20 percentage points for 10-yr horizon.

[35] The Arrow-Debreu price of deflation state, i.e., the deflation state price, characterizes the cost to insure against

Prices of Deflation Risks (Real Perspective, 2003-2017)

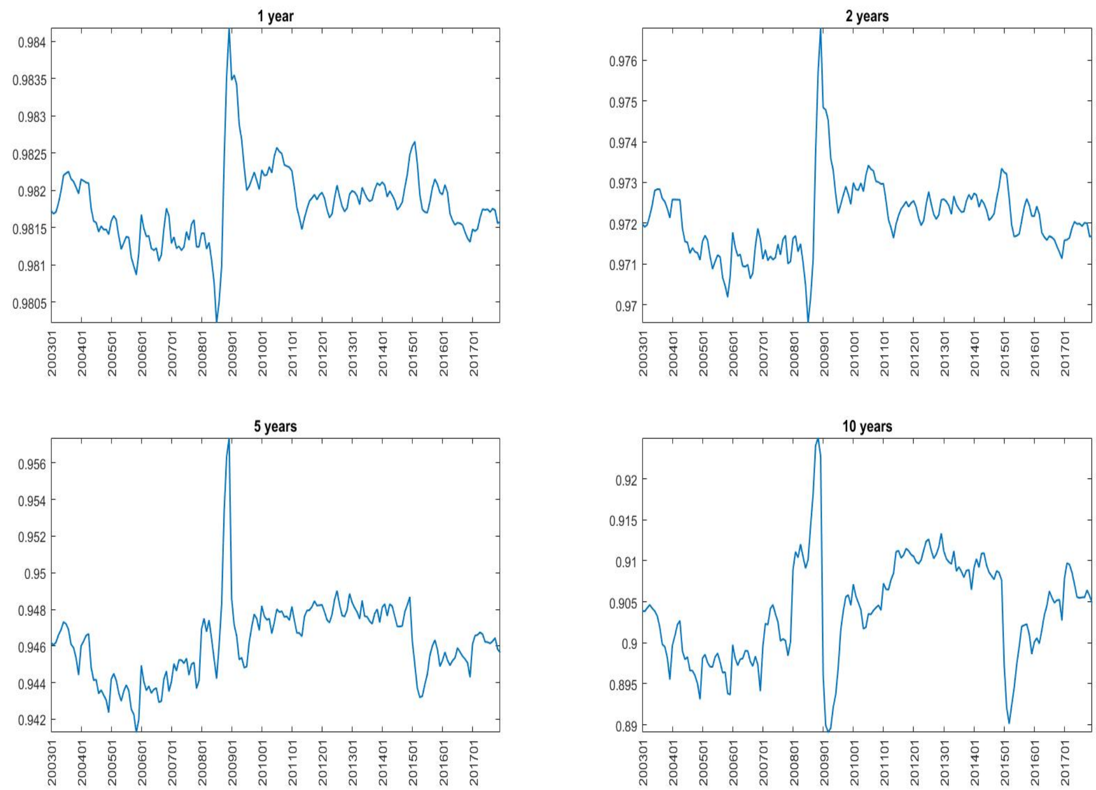

Figure 7: Prices of future deflation risks estimated for various horizons. The estimation is based on data of 2003-2017 period.

a lower price of deflation risks in the real pricing perspective (Figure 7) than in the nominal pricing perspective (Figure 6). This is because the real representative agent denominates all payoffs and prices in consumption baskets. and the real pricing kernel growth is the product of the nominal pricing kernel growth and the inflation (19). In the deflation state, the inflation $\frac{I_T}{I_t}$ is less than unit. Therefore, the real price of the deflation risk is dominated by its nominal price.

5 Mispricing in Real Asset Markets

In this section we compute the difference between model-implied and observed prices of real assets, namely TIPS and inflation swaps. As we discuss at the beginning of Section 2.2, this difference captures the compensation for the exposure of real assets to non-inflation risks and other market

future deflation. It equals the product of deflation probability and the price of deflation risk.

imperfections, so is also referred to as a mispricing with respect to the estimation model.

Mispricing in TIPS Markets

Specifically, we define the mispricing of TIPS as the difference in yields on model-implied TIPS and observed TIPS,

$$\Delta y_{tT}^{TIPS} = \frac{-1}{T-t} \log P_{tT}^{\mathrm{TIPS},t} \; - \; \tilde{y}_{tT}^{TIPS}, \tag{44}$$

where $P_{tT}^{\mathrm{TIPS},t}$ is the price of TIPS (28) maturing at T implied from the model, and $\tilde{y}_{t,T}^{TIPS}$ is the observed yield at time t on the same TIPS provided by the Fed (see Section 2.1). A negative mispricing, $\Delta y_{tT}^{TIPS} < 0$, indicates that the model-implied price of TIPS is higher than the price observed in markets (i.e., according to the model, TIPS is undervalued, or "cheap", in markets), and vice versa. Figure 8 plots the time series of mispricing in TIPS markets, or the yield differential

Mispricing in TIPS Markets (2003-2017)

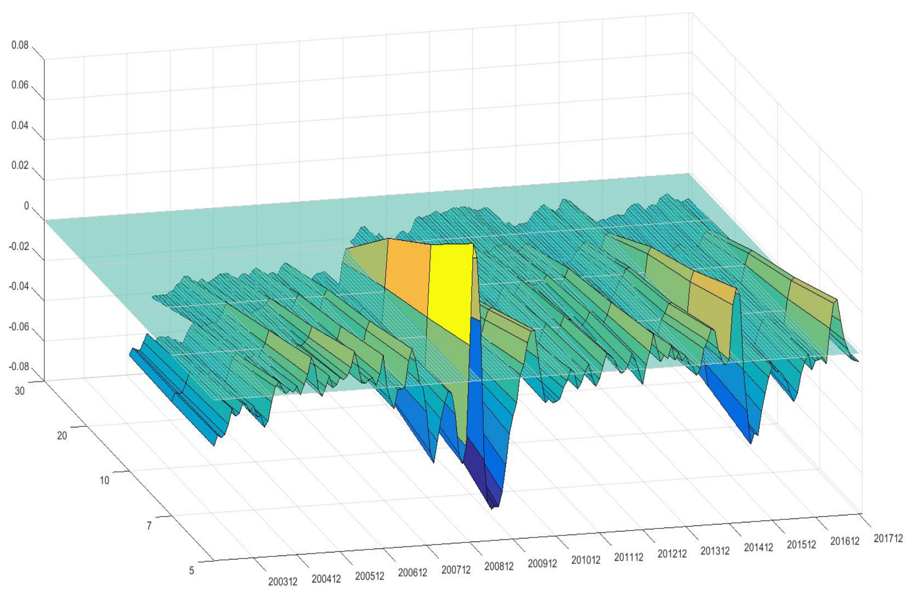

Figure 8: Time series of the yield differential Δy_{tT}^{TIPS} (44) for maturities of 5, 7, 10, 20, and 30 years. Values are annualized. The estimation is based on data of 2003-2017 period.

Δy_{tT}^{TIPS} (44), for various maturities. For a larger part of the period 2003-2017 and specially for longer horizons, this yield differential has a negative value (shaded area, below the grid plane at zero altitude in Figure 8), or TIPS are observed in markets at prices lower than implied by the model. Given that the model's estimation features only the inflation risk, this underpricing of observed

TIPS indicates that investors do perceive and price non-inflation risks and other imperfections inherent in TIPS markets. In the cross section, TIPS of longer maturities tend to be underpriced more consistently, suggesting that non-pricing risks, e.g., liquidity issues, tend to be more important and consistent in longer-maturity TIPS. In the time series, the underpricing of TIPS is most notable is the periods of before 2005, 2007-2010, and 2014-2016, though the mispricing (overpriced TIPS) turned positive briefly at the onset of the 2008 financial crisis for shorter horizons. More generally, the time series pattern of underpricing ($\Delta y_{tT}^{TIPS} < 0$) and overpricing ($\Delta y_{tT}^{TIPS} > 0$) tend to be more volatile for shorter-maturity TIPS (see also Figure 10 below). Before discussing further implications of the mispricing of TIPS on the divergence and long–short strategies on nominal and real asset markets, we turn to the pricing of inflation swaps.

Mispricing in Inflation Swap Markets

We recall that the inflation swap rate is the fixed rate specified in the swap contract, at which rate the fixed-rate payor pays (in exchange for receiving a floating rate equal to the spot inflation at settlement dates). Similar to (44), we define the mispricing of inflation swaps as the difference in the model-implied and observed inflation swap rates,

$$\Delta h_{tT} = h_{tT} \; - \; \tilde{h}_{tT}, \tag{45}$$

where h_{tT} is the model-implied inflation swap rate (30) of a swap contract initiated at t that matures at T, and $\tilde{h}_{tT}$ is the observed swap rate on the same inflation swap provided by Bloomberg. Adopting the perspective of the fixed rate payor, a negative mispricing, $\Delta h_{tT} < 0$, indicates that the model-implied swap rate is lower than the rate observed in the market, (i.e., according to the model, the inflation swap is overvalued, or "expensive", to the fixed-rate payor in the market), and vice versa. This pricing characterization is purely conventional. Its possible economic content is clearly tied to the practical feature that which economic agents are the fixed rate payor in the inflation swap market. Apparently, this finding indicates there are more buyers than sellers in the inflation swap market. To illustrate, assume that pension funds have long-term real liabilities and want to hedge them using inflation swaps. They would buy inflation swap contracts of commensurate durations because as fixed-rate payors in these contracts, pension funds receive floating cashflows to offset inflation movements and match their real liabilities. Figure 9 plots the time series of mispricing in inflation swap markets, or the rate differential Δh_{tT} (45), for various maturities from

Mispricing in Inflation Swap Markets (2003-2017)

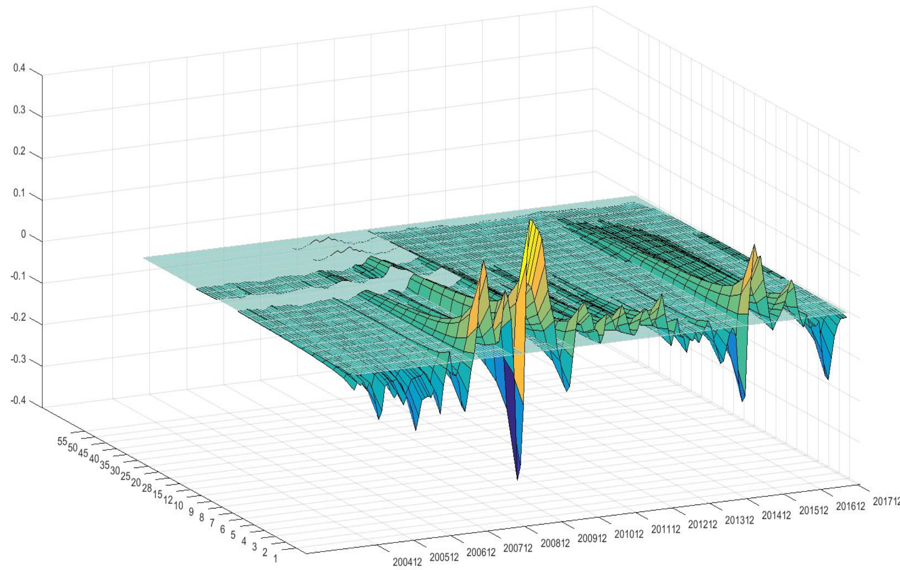

Figure 9: Time series of the rate differential Δh_{tT} (45) for maturities from 1 year out to 55 years. Values are annualized. The estimation is based on data of 2003-2017 period.

1 year to 55 years. In the time series, this rate differential has a negative value (shaded area, below grid plane at zero altitude in Figure 9) for the period before 2007. After that, the rate differential fluctuates between negative and positive values, indicating inflation swaps can either overvalued or undervalued, i.e., “expensive” or “cheap,” from the fixed rate payor’s perspective. Around the financial crisis, during the period of 2008-2010, this rate differential is mostly positive (undervalued). In the cross section, similar to TIPS, inflation swaps of longer maturities tend to be overpriced more consistently, suggesting that non-pricing risks tend to be more consistent in longer-maturity inflation swaps. Analyzing proprietary data limited to June-August 2010 period, Fleming and Sporn (2013) document that trading in the inflation swap market concentrates in certain tenors, in particular, inflation swaps of 10-year and shorter maturities are more actively traded. This trading pattern might be responsible for the cross sectional heterogeneity in the liquidity and mispricing seen in Figure 9. To the extent that pension funds participate more in longer-maturity inflation swaps (as fixed rate payors) to hedge their long-term real liabilities, they appear to overpay for these swap contracts.

Mispricing in TIPS and Inflation Swap Markets

To relate the mispricing in real asset markets with the profitable trade on the nominal-TIPS yield spread, we recall that a portfolio of TIPS and zero-coupon inflation swaps can replicate a nominal Treasury bond in theory.[36] In the absence of arbitrages and other frictions, the return on the replication portfolio is equal to the nominal interest rate r_{Nt} (21). Empirically, however, the replicating portfolio tends to be cheaper than the respective nominal bond and offers higher yield, $\tilde{h}_{tT} + \tilde{y}_{tT}^{\text{TIPS},t} > \tilde{r}_{Nt}$, where the tilde notation denotes observed quantities (44), (45) in markets. In the literature, this "underpricing" of the replicating portfolio is the basis of the profitable trade on the nominal-TIPS yield spread (e.g., Fleckenstein et al. (2014)) and can be attributed to the presence of liquidity, frictions, or arbitrages in markets. These market imperfections are beyond the inflation risk framework, and hence, can also be responsible for the mispricing of TIPS and inflation swaps found in our pure-inflation risk estimation. In particular, the possibility of "cheap" TIPS and "expensive" inflation swaps (i.e., the observed TIPS yield $\tilde{h}_{tT}$ and inflation swap rate $\tilde{h}_{tT}$ are in excess of their model-implied values) lend supports to the underpricing of the replicating portfolio documented in the literature.

To examine this mispricing possibility of TIPS and inflation swaps within our estimation, and their individual contributions to the profitable trade on the nominal-TIPS yield spread, Figure 10 plots the time series of the TIPS mispricing (44), inflation swap mispricing (45), and their total, for different maturities. In the cross section, the total mispricing (depicted by the continuous blue line) decreases with maturities, suggesting higher profits for the trade on the nominal-TIPS yield spread on longer-maturity bonds. For shorter maturities (10 year or less), the individual mispricing of TIPS and inflation swap alternates in signs, suggesting that their contributions to the trade vary as each can be either undervalued or overvalued in our estimation model. For longer maturities (20 and 30 years), the mispricing of both TIPS and inflation swaps is negative, suggesting that both mispricings contribute to the profit of the trade on the nominal-TIPS yield spread.

[36] Specifically, at initial time t, the replicating portfolio consists of (i) long position in one unit of TIPS of one basket face value that matures at T, and (ii) entering the inflation swap of notional value of \$1 that also matures at T as the floating rate payor (receiving fixed swap rate h_{tT} (30)). Because it is costless to enter swap contracts, the initial value of the portfolio is $P_{tT}^{\text{TIPS},t}$ (28) in spot dollars at t. At T, the payoff of the portfolio consists of (i) one consumption basket valued at $\frac{I_T}{I_t}$ in dollars at T from TIPS position, (ii) the net cashflow of $e^{(T-t)h_{tT}} - \frac{I_T}{I_t}$ also in dollars at T as settlement from the swap contract. In summary, the portfolio invests $P_{tT}^{\text{TIPS},t}$ at t to receive a fixed payoff of $e^{(T-t)h_{tT}}$ at T, hence replicating a nominal bond. As a result, the return on replicating portfolio equals nominal interest rate,

$$\frac{e^{(T-t)h_{tT}}}{P_{tT}^{\text{TIPS},t}} = e^{(T-t)r_{Nt}} \implies h_{tT} \; + \; \frac{-1}{T-t} \log P_{tT}^{\text{TIPS},t} = r_{Nt}.$$

Mispricing in TIPS and Inflation Swap Markets (2003-2017)

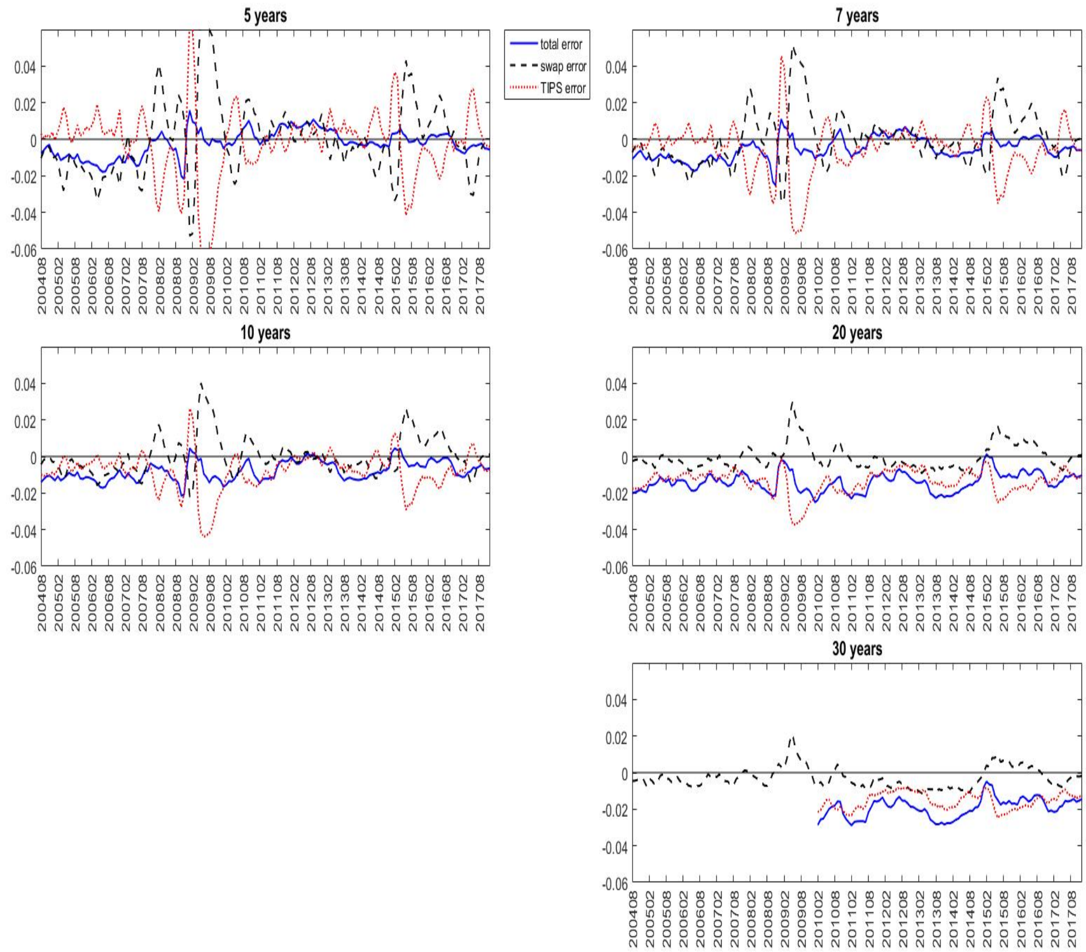

Figure 10: Time series of the TIPS mispricing $\Delta y_{tT}^{\text{TIPS},t}$ (44) (red dotted line), the inflation swap mispricing Δh_{tT} (45) (black dashed line), and their sum (blue continuous line) for various maturities. Values are annualized. The estimation is based on data of 2003-2017 period. Data on TIPS of 30-year tenor is not available before 2010.

In the time series, the mispricing in both TIPS and inflation swaps exhibits significant more variations (specially for shorter maturities). However, movements in the mispricing tend to offset one another, resulting in more stable and mostly negative total mispricing (specially for longer maturities), and indicating a profitable trade on the nominal-TIPS yield spread. In the pre-financial crisis period before 2008, the total mispricing is negative and stable for all maturities, which is consistent with Fleckenstein et al. (2014)'s finding that the trade on the nominal-TIPS yield spread is profitable consistently across different tenors, based on their data of 2004-2009. In the period of 2008-2010, TIPS mispricing is mostly negative while inflation swap mispricing is mostly positive for maturities of 10 year or less. This pattern indicates that during crisis, shorter-maturity TIPS and shorter-maturity inflation swaps (from fixed rate payor's perspective) both appear to be underpriced. The vigorous time series movements in the mispricing of real asset market during and after crisis are possibly due to deterioration and improvement in market conditions. Christensen and Gillan (2018) find that the quantitative easing, i.e., government injecting cash into markets by conducting large purchases of government bonds and other assets between 11/2010 and 06/2011, improves liquidity and decreases mispricing in real asset markets.[37] Event studies lie beyond the scope of the current paper, but are important to understand the economic forces behind the time variation of the mispricing in real asset markets.

6 Conclusion

This paper employs new price data of T-note futures from CME (and BCEI inflation consensus forecasts) to estimate the distribution of the U.S. future inflation in a real pricing model. The estimation is based on the time series data of the most liquid and exchange-traded nominal assets. We then use these estimates to price real assets out of sample and obtain the model-implied mispricing separately for TIPS and inflation swaps. Our findings indicate that the well documented profitable trade on the nominal-TIPS yield spread owes to the mispricing of both TIPS (mostly underpriced) and inflation swaps (mostly overpriced, to fixed rate payors). Our paper is agnostic about the nature (e.g., liquidity, and other market imperfections) of the mispricing. We leave this topic, and an extension of this inflation estimation to international settings for future research.

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[37]Note that their total liquidity premium of TIPS and inflation swaps, (which is the difference between market-observed and theoretical values), is opposite to our total mispricing definition (which is the difference between theoretical and observed values).

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Appendices

A Estimates for 1982-2017 Period

This appendix presents the inflation estimation using monthly data of T-note futures prices and BCEI inflation consensus forecasts for the longer period of 1982-2017. Figure A.1 plots the conditional expectation (39) of the annualized inflation for various horizons (corresponding to Figure 1 in the main text). Figure A.2 plots the conditional probability density function of the annualized inflation growth for various horizons (corresponding to Figure 2 for 2003-2017 in the main text). Figure A.3 and A.4 plot the (annualized) nominal and real price of inflation risk for various horizons (corresponding to Figures 3 and 4 for 2003-2017 in the main text). Figure A.5 plots the probability of deflation estimated for various horizons (corresponding to Figure 5 in the main text). Figures A.6 and A.7 plot the probability ratios (43), which characterize the price of the deflation risk under the nominal and real pricing perspective (corresponding to Figures 6 and 7 in the main text).

The estimation and these figures show that the expected inflation is significantly higher in the earlier period (before 2000), while the expected deflation (except for the financial crisis of 2008) is significantly lower. Prices of inflation risk remain negative to the nominal agent, and positive to the real agent (these signs are the same as those obtained from the data of 2003-2017 in the main text). Prices of deflation risk in nominal (resp. real) perspective are higher (resp. lower) in the the earlier period. Broadly, these patterns are similar to the earlier estimation results using more recent data of 2003-2017.

Expected Inflation (1982-2017)

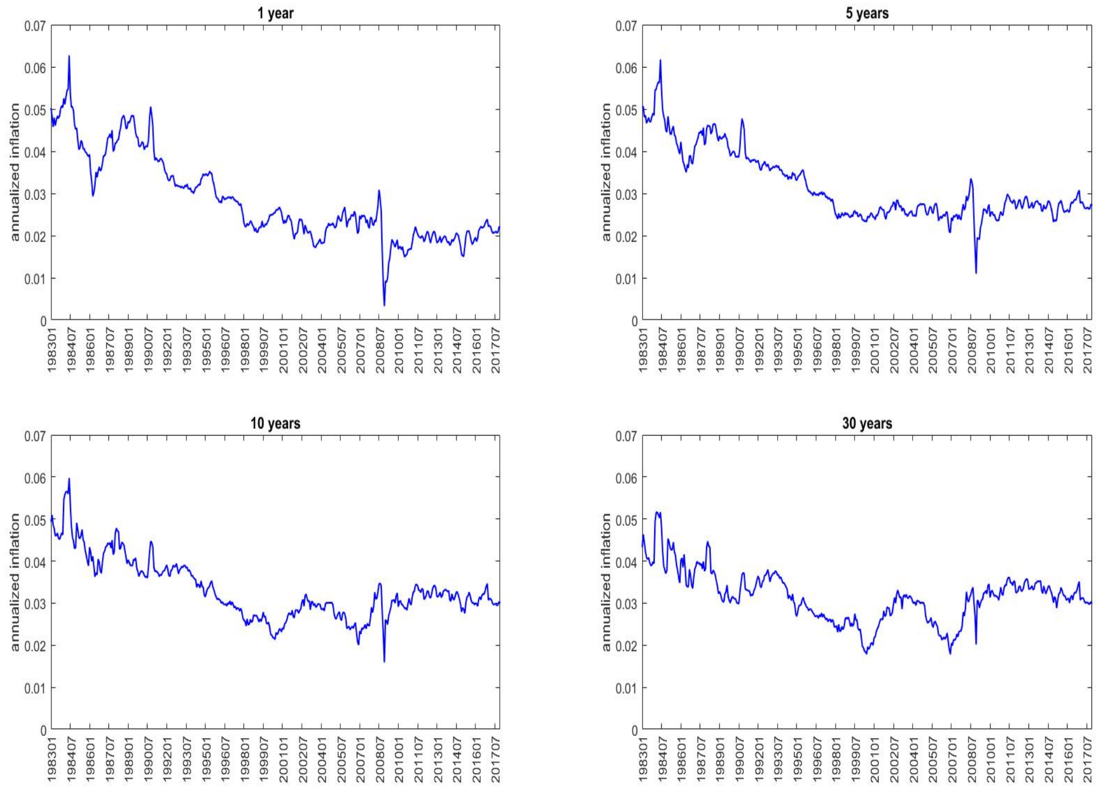

Figure A.1: Expected annualized inflation estimated for various horizons. Values are annualized. The estimation is based on data of 1982-2017 period.

B Derivations

B.1 Conditional Distribution

We first discuss the notation. Given that $\{\kappa_i\}$ are the eigenvalues of matrix $\mathcal{K}$, the corresponding diagonalization is $V^{-1}\mathcal{K}V = \text{Diag}[\mathcal{K}] = \text{Diag}[\kappa_1, \ldots, \kappa_n]$. We also employ a general diagonal matrix notation $\text{Diag}[e^{u\mathcal{K}}]$, $\forall u \in \mathbb{R}$, to denotes the following explicit matrix throughout,

$$V^{-1}e^{u\mathcal{K}}V = e^{uV^{-1}\mathcal{K}V} = e^{u\text{Diag}[\mathcal{K}]} = \text{Diag}[e^{u\mathcal{K}}] = \begin{pmatrix} e^{u\,\kappa_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{u\,\kappa_n} \end{pmatrix}, \quad \forall u \in \mathbb{R}. \tag{46}$$

Given the Markovian state variable dynamics (7) in physical measure $\mathbb{P}$, we want to characterize

Probability Density Function of the Future Inflation (1982-2017)

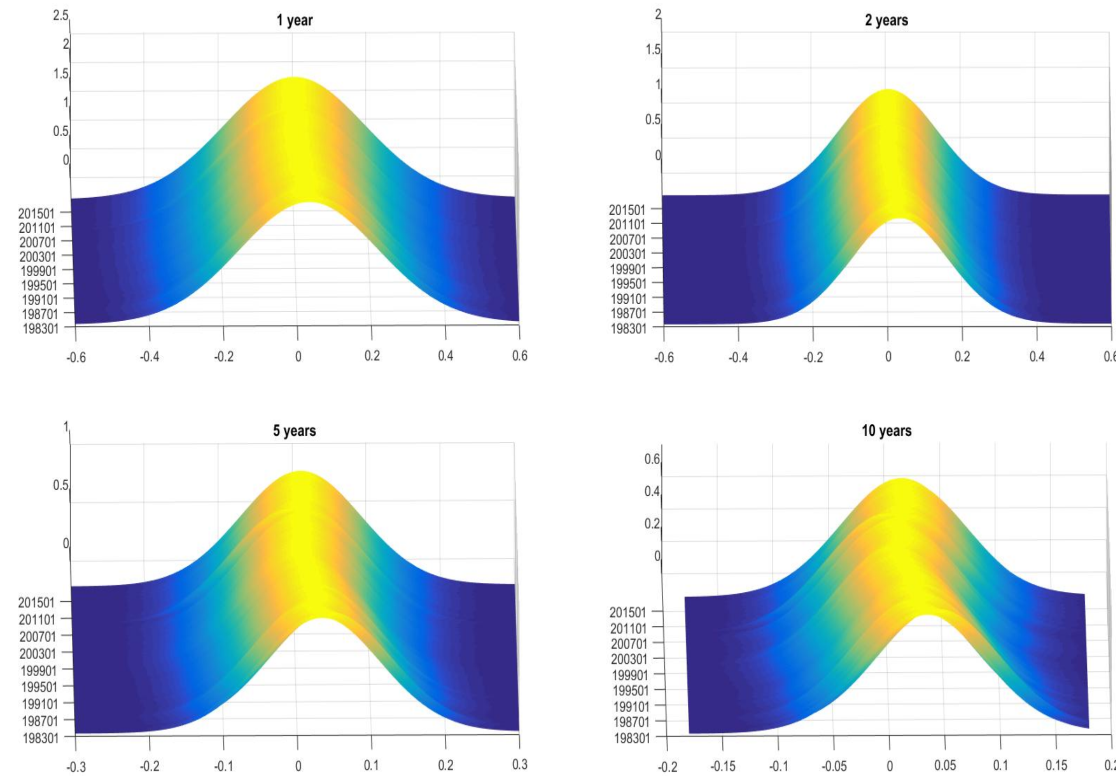

Figure A.2: Probability density function of the future annualized inflation growth estimated for various horizons. Values are annualized. The estimation is based on data of 1982-2017 period.

their conditional distribution and determine certain conditional expectations needed for pricing assets in the model.

Conditional Expectation: In affine settings, a quantity of interest is the conditional expectation of an exponential affine function of state variables X_t. For conveniences, the following result recapitulates a known analytical expression for this conditional expectation when X_t has a conditional Gaussian distribution.

Lemma 1 *Given the state variable dynamics (7), a $n \times 1$ parameter vector A_T that may vary with terminal time T, and assuming standard regularity conditions such that the conditional expectation $E_t\left[e^{A_T' X_T}\right]$ is well defined, then*

$$L_t(X_t) \equiv E_t\left[e^{A_T' X_T}\right] = e^{l_{0t;A_T} + l_{1t;A_T}' X_t}, \tag{47}$$

Prices of Future Inflation Risks (Nominal Perspective, 1982-2017) ($\times 10^{-4}$)

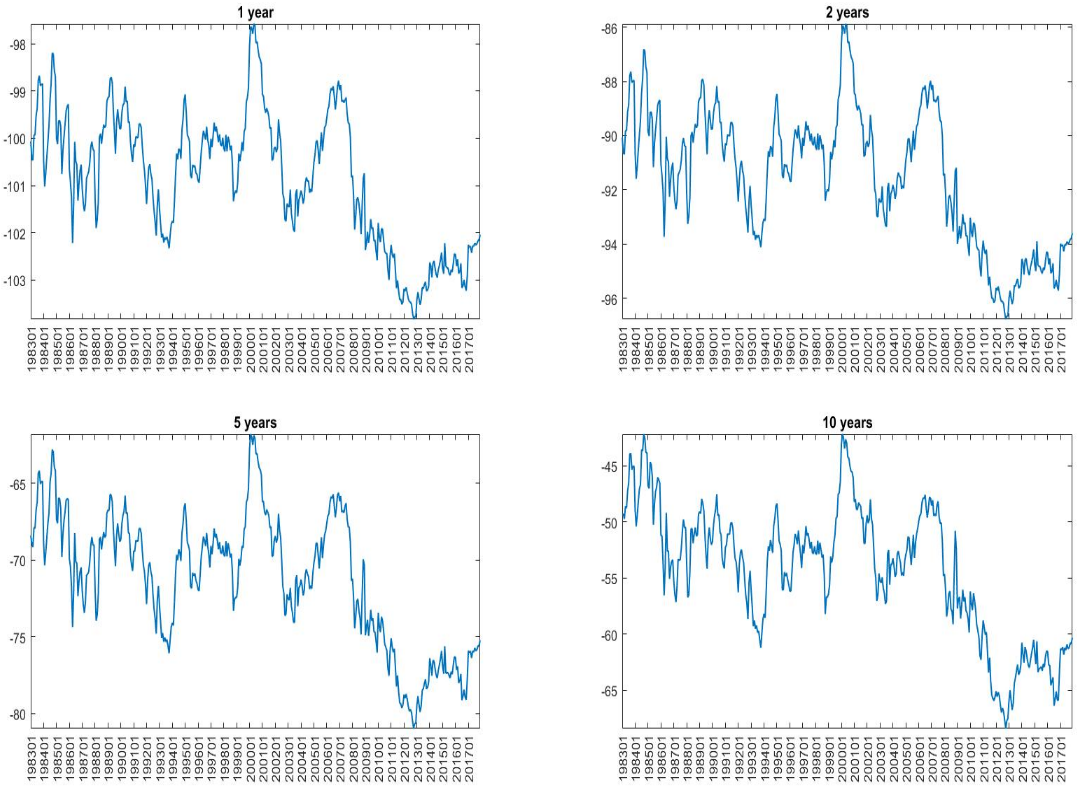

Figure A.3: Annualized prices of future inflation risks (in basis points) estimated for various horizons. The estimation is based on data of 1982-2017 period.

with

$$l'_{1t;A_T} = A'_T V \, Diag\left[e^{(t-T)\mathcal{K}}\right] V^{-1}, \tag{48}$$

and

$$l_{0t;A_T} = A'_T V \, Diag\left[\mathbb{1} - e^{(t-T)\mathcal{K}}\right] V^{-1}\Theta + \frac{1}{2} A'_T V \mathcal{S}_{tT} V' A_T, \tag{49}$$

where $n \times n$ symmetric matrix $\mathcal{S}_{tT}$ is defined such that its jh-element is (see also (13)),

$$[\mathcal{S}_{tT}]_{jh} \equiv \frac{1 - e^{(t-T)(\kappa_j + \kappa_h)}}{\kappa_j + \kappa_h} \left(V^{-1}\Sigma Diag[S]\Sigma' V'^{-1}\right)_{jh}, \qquad \forall j, h \in \{1, \ldots, n\},$$

and $Diag\left[e^{(t-T)\mathcal{K}}\right]$ denotes a diagonal matrix in the notation of (46).

We employ explicit notations $l_{0t;A_T}$, $l_{1t;A_T}$ to signify the dependence of the expectation solution on the given parameter vector A_T. In this way, the Lemma's results (47), (48), (49) apply generally for

Prices of Future Inflation Risks (Real Perspective, 1982-2017) ($\times 10^{-4}$)

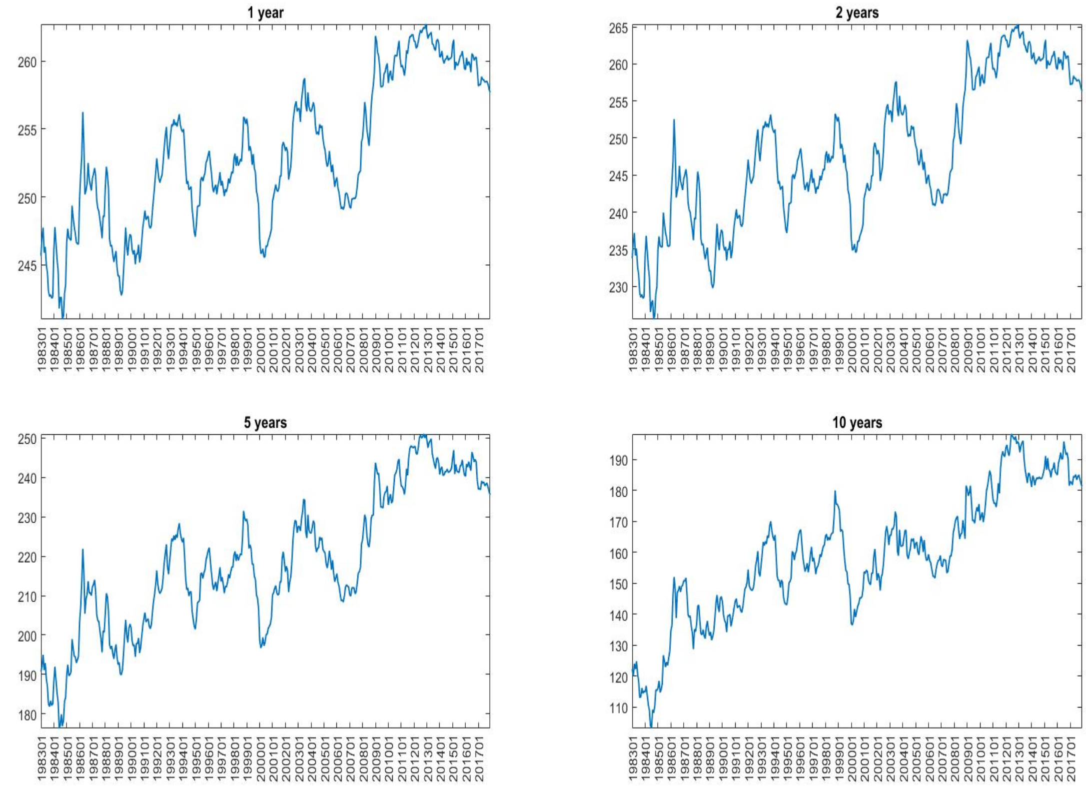

Figure A.4: Annualized prices of future inflation risks (in basis points) estimated for various horizons. The estimation is based on data of 1982-2017 period.

any terminal parameter vector A_T. To further understand the formulation and notation of Lemma 1, it is instructive to examine its derivation.

Proof: Under the assumed standard regularity conditions, $L_t(X_t)$ is a $\mathbb{P}$-martingale and has zero drift, from which follows the differential equation,

$$\frac{\partial L_t(X_t)}{\partial t} + \sum_{j}^{n} \frac{\partial L_t(X_t)}{\partial X_{jt}} \mu_{jt}(X_t) + \frac{1}{2} \sum_{j,h}^{n} \frac{\partial^2 L_t(X_t)}{\partial X_t^2} \sigma_{jt}(X_t) \sigma_{ht}(X_t) = 0.$$

Substituting the expression (47) for $L_t(X_t)$ and the state variable specification (7) into the above differential equation implies further equations for the time-dependent $n \times 1$ vector $l_{1t;A_T}$ and scalar

Deflation Probability (1982-2017)

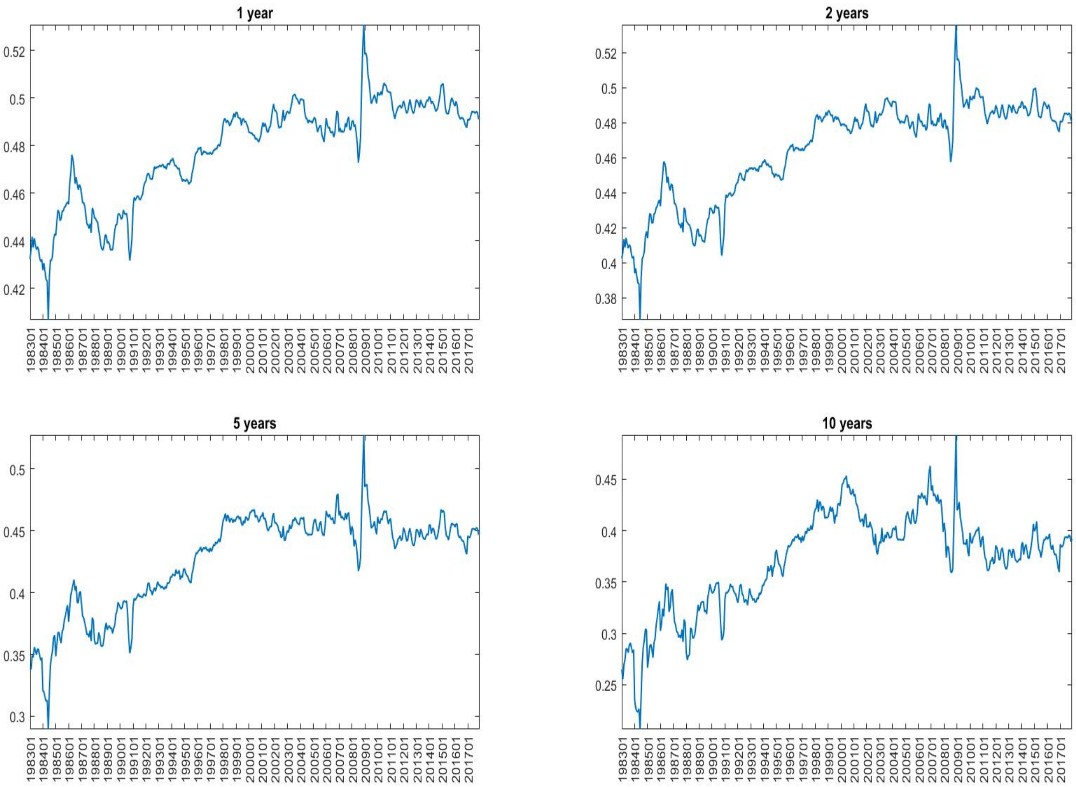

Figure A.5: Conditional probability of the future deflation estimated for various horizons. The estimation is based on data of 1982-2017 period.

$l_{0t;A_T}$ (by matching separately the term associated with X_t and the free term),

$$\begin{aligned} \frac{dl_{1t;A_T}}{dt} &= V'^{-1}\mathrm{Diag}[\mathcal{K}]V'l_{1t;A_T}, && l_{1T;A_T} = A_T, \\ \frac{dl_{0t;A_T}}{dt} &= -(V^{-1}\Theta)'\mathrm{Diag}[\mathcal{K}]V'l_{1t;A_T} - \tfrac{1}{2}l'_{1t;A_T}\Sigma\mathrm{Diag}[S]\Sigma'l_{1t;A_T}, && l_{0T;A_T} = 0. \end{aligned} \tag{50}$$

Multiplying V' to the left of both sides of the equation on $l_{1t;A_T}$ yields a simple (decoupled) differential equation on $V'l_{1t;A_T}$, and consequently its explicit solution (matching the terminal condition $l_{1T;A_T} = A_T$)

$$\frac{d}{dt}\left(V'l_{1t;A_T}\right) = \mathrm{Diag}[\mathcal{K}]\left(V'l_{1t;A_T}\right) \Longrightarrow V'l_{1t;A_T} = \mathrm{Diag}[e^{(t-T)\mathcal{K}}]V'A_T,$$

where $\mathrm{Diag}[e^{(t-T)\mathcal{K}}]$ denotes the explicit diagonal matrix (46) (with u therein replaced by $t-T$). Multiplying V'^{-1} to the left of both sides of the above equation yields the unique solution (48) of

Prices of Future Deflation Risks (Nominal Perspective, 1982-2017)

Figure A.6: Prices of future deflation risks estimated for various horizons. The estimation is based on data of 1982-2017 period.

$l_{1t;A_T}$.

Next, substituting the above solutions for $V'l_{1t;A_T}$ and $l_{1t;A_T}$ into the equation (50) on $l_{0t;A_T}$ transforms it into,

$$\begin{aligned} \frac{dl_{0t;A_T}}{dt} &= -(V^{-1}\Theta)'\mathrm{Diag}[\mathcal{K}e^{(t-T)\mathcal{K}}]V'A_T \\ -\tfrac{1}{2}(V'A_T)'\mathrm{Diag}[e^{(t-T)\mathcal{K}}]\, V^{-1}\Sigma\mathrm{Diag}[S]\Sigma'V'^{-1}\, &\mathrm{Diag}[e^{(t-T)\mathcal{K}}]V'A_T, \qquad l_{0T;A_T} = 0. \end{aligned} \tag{51}$$

In particular, the second term on the right-hand side can be written explicitly as,

$$-\frac{1}{2}\sum_{j,h=1}^{n}(V'A_T)_j e^{(t-T)\kappa_j}\left[V^{-1}\Sigma\mathrm{Diag}[S]\Sigma'V'^{-1}\right]_{jh} e^{(t-T)\kappa_h}(V'A_T)_h.$$

Integrating differential equation (51) over the time dimension while matching the terminal condition $l_{0T;A_T} = 0$ yields the unique solution (49) of $l_{0t;A_T}$ ■

Prices of Deflation Risks (Real Perspective, 1982-2017)

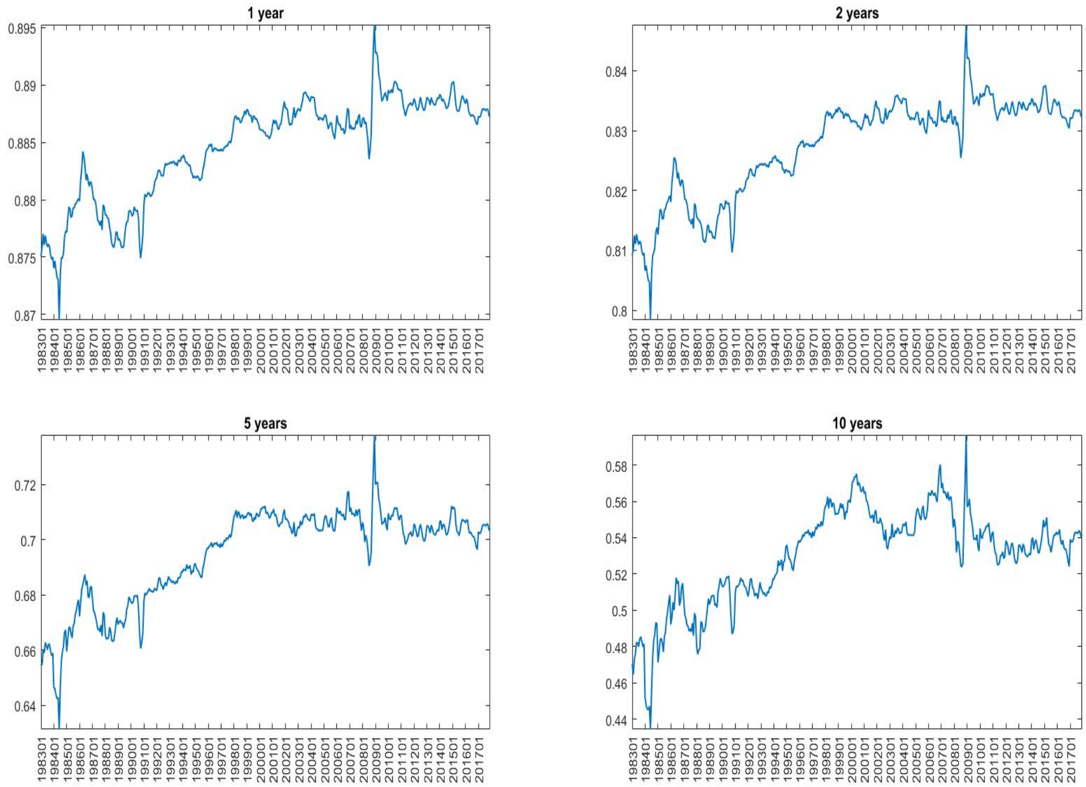

Figure A.7: Prices of future deflation risks estimated for various horizons. The estimation is based on data of 1982-2017 period.

Characteristic Function and Moments of State Variable Distribution: The conditional characteristic function of state variables X (7) is a version of the conditional expectation (47) (in which the parameter vector A_T is constant). Specifically, given a n vector C of constant parameters, an application of Lemma 1 yields the conditional characteristic function,

$$\mathcal{X}_t(C) \equiv E_t\left[e^{C'X_T}\right] = e^{c_{0t;C}+c'_{1t;C}X_t}, \qquad \text{with,} \qquad c_{1t;C} = V'^{-1}\text{Diag}\left[e^{(t-T)\mathcal{K}}\right]V'C$$
$$c_{0t;C} = (V^{-1}\Theta)'\text{Diag}\left[\mathbb{1} - e^{(t-T)\mathcal{K}}\right]V'C + \tfrac{1}{2}\left(V'C\right)'\mathcal{S}_{tT}V'C, \tag{52}$$

where $n \times n$ matrix $\mathcal{S}_{tT}$is defined in (13). Recall that state variables X specified in (7) have conditional Gaussian distribution, which is fully characterized by the first two moments, namely the conditional mean and the conditional covariance matrix. These conditional moments are obtained by valuing the derivatives of the characteristic function with respective to parameters in C at their zeros. In this regard, we observe that $c_{1t;C}$ and $c_{0t;C}$ are homogeneous of degree one and two in C respectively.

Taking the first-order derivative of (52) yields the $n \times 1$ conditional mean vector of state variables,

$$E_t[X_T] = \left.\frac{\partial \mathcal{X}}{\partial C}\right|_{C=0} = \Theta + V\text{Diag}\left[e^{(t-T)\mathcal{K}}\right] V^{-1}\left(X_t - \Theta\right). \tag{53}$$

Taking the second-order derivative of (52) yields the $n \times n$ conditional covariance matrix of state variables,

$$\text{Var}_t\left[X_T\right] = E_t[X_T X_T'] - E_t[X_T]E_t[X_T'] = \left.\frac{\partial^2 \mathcal{X}}{\partial C^2}\right|_{C=0} - \left.\frac{\partial \mathcal{X}}{\partial C}\right|_{C=0}\left.\frac{\partial \mathcal{X}'}{\partial C}\right|_{C=0} = V\mathcal{S}_{tT}V', \tag{54}$$

where $n \times n$ matrix $\mathcal{S}_{tT}$ is defined in (13). Hence, the conditional distribution of state variables is normal $\mathcal{N}\left(E_t[X_T], \text{Var}_t\left[X_T\right]\right)$ (53), (54), and described by the following conditional probability density function,

$$f_t(X_T) = \frac{\exp\left\{-\frac{1}{2}\left(X_T - E_t[X_T]\right)' \ \text{Var}_t^{-1}\left[X_T\right] \ \left(X_T - E_t[X_T]\right)\right\}}{\sqrt{(2\pi)^n \text{Det}\left(\text{Var}_t\left[X_T\right]\right)}}. \tag{55}$$

In the short-term limit ($T = t + dt$), the above conditional mean, variance and probability density become,

$$\begin{aligned} E_t[X_{t+dt}] &= X_t + V\text{Diag}\left[\mathcal{K}\right] V^{-1}(\Theta - X_t)dt, \\ Var_t[X_{t+dt}] &= dt\, \Sigma\text{Diag}\left[S\right]\Sigma', \\ f_t(X_t + dt) &= \frac{\exp\left\{-\frac{1}{2}\left(X_{t+dt} - E_t[X_{t+dt}]\right)' \ \text{Var}_t^{-1}\left[X_T\right] \ \left(X_{t+dt} - E_t[X_{t+dt}]\right)\right\}}{\sqrt{(2\pi)^n \text{Det}\left(\text{Var}_t\left[X_{t+dt}\right]\right)}}. \end{aligned} \tag{56}$$

B.2 Pricing Nominal Bond Derivatives

Pricing Futures on Nominal Bonds: At a time t_1, we consider a futures contract that delivers at time $t_2 \geq t_1$ a nominal bond of one dollar face value and maturity $t_3 \geq t_2$. Let F_{t_1} denote the contractual futures price of this contract at t_1, and B_{t_2,t_3} the price of the underlying nominal bond at t_2. At the futures delivery date t_2, the realized payoff to a long position of the futures then is $B_{t_2,t_3} - F_{t_1}$ in spot dollars, or equivalently in $\frac{B_{t_2,t_3} - F_{t_1}}{I_{t_2}}$ in consumption baskets. As in all zero-net (at the initiation date) financial contracts the futures price F_{t_1} is settled at time t_1 such that, given the information set at t_1, the contract has a fair (zero) value to both long and short parties. In real pricing (i.e., in consumption baskets), this valuation is,

$$E_{t_1}\left[\frac{M_{Rt_2}}{M_{Rt_1}}\frac{B_{t_2,t_3}-F_{t_1}}{I_{t_2}}\right]=0$$

which implies the futures price,

$$F_{t_1}=\frac{I_{t_1}E_{t_1}\left[\frac{M_{Rt_2}}{M_{Rt_1}}\frac{B_{t_2,t_3}}{I_{t_2}}\right]}{I_{t_1}E_{t_1}\left[\frac{M_{Rt_2}}{M_{Rt_1}}\frac{1}{I_{t_2}}\right]}=\frac{B_{t_1,t_3}}{B_{t_1,t_2}}, \tag{57}$$

where in the last equation we have used the nominal bond price (23). Reassuringly, the futures price satisfies the standard forward parity.[38]

B.3 Pricing Real Bond Derivatives

Pricing Real Zero-coupon Bonds: The price $B_{tT_i}^{R,t_0}$ of the real zero-coupon bond (27) in spot dollars at t ($t_0\le t\le T_i$) can be written explicitly as,

$$B_{tT_i}^{R,t_0}=\frac{I_t}{I_{t_0-\delta}}\times E_t\left[\frac{M_{RT_i}}{M_{Rt}}\frac{I_{T_i-\delta}}{I_{T_i}}\right]=e^{i_1'\left(X_t-X_{t_0-\delta}\right)}e^{m_{0T_i}-m_{0t}-m_{1t}'X_t}E_t\left[e^{m_{1T_i}'X_{T_i}+i_1'\left(X_{T_i-\delta}-X_{T_i}\right)}\right]. \tag{58}$$

We first compute the conditional expectation,

$$E_t\left[e^{m_{1T_i}'X_{T_i}+i_1'\left(X_{T_i-\delta}-X_{T_i}\right)}\right]=E_t\left[e^{i_1'X_{T_i-\delta}}E_{T_i-\delta}\left[e^{(m_{1T_i}'-i_1')X_{T_i}}\right]\right]$$

$$=E_t\left[e^{i_1'X_{T_i-\delta}}e^{b_{0T_i-\delta,T_i}+b_{1T_i-\delta,T_i}'X_{T_i-\delta}}\right]=e^{b_{0T_i-\delta,T_i}}\,E_t\left[e^{(i_1'+b_{1T_i-\delta,T_i}')X_{T_i-\delta}}\right],$$

where coefficients $b_{0T_i-\delta,T_i}\in\mathbb{R}$, $b_{1T_i-\delta,T_i}\in\mathbb{R}^n$ have been computed in (24), with the replacements of t by $T_i-\delta$, and T by T_i. Similarly, an application of Lemma 1 (Appendix B.1) again yields an explicit expression for the above conditional expectation,

$$e^{b_{0T_i-\delta,T_i}}\,E_t\left[e^{(i_1'+b_{1T_i-\delta,T_i}')X_{T_i-\delta}}\right]=e^{b_{0tT_i}^R+b_{1tT_i}^{R}{}'X_t},$$

$$\text{with}\quad b_{1tT_i}^R=V'^{-1}\mathrm{Diag}\left[e^{(t-T_i+\delta)\mathcal{K}}\right]V'(i_1+b_{1T_i-\delta,T_i}), \tag{59}$$

$$b_{0tT_i}^R=b_{0T_i-\delta,T_i}+(i_1'+b_{1T_i-\delta,T_i}')V\mathrm{Diag}\left[\mathbb{1}-e^{(t-T_i+\delta)\mathcal{K}}\right]V^{-1}\Theta+\frac{1}{2}(i_1'+b_{1T_i-\delta,T_i}')V\mathcal{S}_{t,T_i-\delta}V'(i_1+b_{1T_i-\delta,T_i}),$$

[38] The forward parity reads $F_te^{-y(T-t)}=S_t$, where S_t is the spot price of underlying asset (B_{t_1,t_3} in the current case), and $e^{-y(T-t)}$ is the risk-neutral discount (B_{t_1,t_2} in the current case).

where the $n \times n$ symmetric matrix $\mathcal{S}_{tT}$ is defined in (13), and $b_{0T_i-\delta,T_i} \in \mathbb{R}$, $b_{1T_i-\delta,T_i} \in \mathbb{R}^n$ are in (24). Substituting these results into (58) yields the real zero-coupon bond price,

$$B_{tT_i}^{R,t_0} = \exp\left\{(m_{0T_i} - m_{0t} + b_{0tT_i}^R) + (i_1' - m_{1t}' + b_{1tT_i}^R{}')X_t - i_1' X_{t_0-\delta}\right\}.$$

The real yield associated with the real zero-coupon bond is,

$$y_{tT_i}^{R,t_0} = \frac{-1}{T_i - t}\left\{(m_{0T_i} - m_{0t} + b_{0tT_i}^R) + (i_1' - m_{1t}' + b_{1tT_i}^R{}')X_t - i_1' X_{t_0-\delta}\right\}.$$

In the special case of zero lag indexation, $\delta = 0$, (24) implies that $b_{0T_i-\delta,T_i} = 0$, $b_{1T_i-\delta,T_i} = m_{1T_i} - i_1$, and the real zero-coupon bond price and the associated real yield reduce to,

$$B_{tT_i}^{R,t_0} = \exp\left\{(m_{0T_i} - m_{0t} + b_{0tT_i}^R) + (i_1' - m_{1t}' + b_{1tT_i}^R{}')X_t - i_1' X_{t_0}\right\},$$

$$y_{tT_i}^{R,t_0} = \frac{-1}{T_i - t}\left\{(m_{0T_i} - m_{0t} + b_{0tT_i}^R) + (i_1' - m_{1t}' + b_{1tT_i}^R{}')X_t - i_1' X_{t_0}\right\},$$

where coefficients $b_{0tT_i}^R$ and $b_{1tT_i}^R$ solve,[39]

$$b_{1tT_i}^R = V'^{-1}\text{Diag}\left[e^{(t-T_i)\mathcal{K}}\right]V' m_{1T_i}, \tag{60}$$

$$b_{0tT_i}^R = m_{1T_i}' V \text{Diag}\left[\mathbb{1} - e^{(t-T_i)\mathcal{K}}\right]V^{-1}\Theta + \frac{1}{2} m_{1T_i}' V \mathcal{S}_{tT_i} V' m_{1T_i},$$

where the $n \times n$ symmetric matrix $\mathcal{S}_{tT}$ is defined in (13).

Pricing Floor Option on TIPS Principal: Let us consider the TIPS that is issued at t_0, matures at T, and has a unit notional face value and coupon rate k, as in (28). The floor option associated with the TIPS principal protects investors against deflation, and pays off only when the indexed inflation $\frac{I_{T-\delta}}{I_{t_0-\delta}}$ at maturity is less then unit. Hence, the TIPS terminal payoff is $1+\left(\frac{I_{T-\delta}}{I_{t_0-\delta}} - 1\right)^+$ in spot dollars at maturity T, or equivalently $\frac{1}{I_T}\left[1 + \left(\frac{I_{T-\delta}}{I_{t_0-\delta}} - 1\right)^+\right]$ in consumption baskets. Therefore the price in spot dollars at time t of this terminal payoff has two components

[39]When lag indexation $\delta = 0$, (24) implies that $b_{0T_i-\delta,T_i} = 0$, $b_{1T_i-\delta,T_i} = m_{1T_i} - i_1$. Then the system (59) reduces to (60).

(associated with a real payoff and a floor option),

$$I_t E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{1}{I_T}\left[1+\left(\frac{I_{T-\delta}}{I_{t_0-\delta}}-1\right)^+\right]\right]=\underbrace{E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{I_t}{I_T}\right]}_{B^{R,t_0}_{t,T}}+\underbrace{E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{I_t}{I_T}\left(\frac{I_{T-\delta}}{I_{t_0-\delta}}-1\right)^+\right]}_{\mathcal{C}^{R,t_0}_{t,T}}.$$

The real bond price $B^{R,t_0}_{t,T}$ is given in (58). We decompose option price $\mathcal{C}^{R,t_0}_{t,T}$ into two terms,

$$\mathcal{C}^{R,t_0}_{t,T}=E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{I_t}{I_T}\frac{I_{T-\delta}}{I_{t_0-\delta}}\mathbb{1}_{\{I_{T-\delta}\geq I_{t_0-\delta}\}}\right]-E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{I_t}{I_T}\mathbb{1}_{\{I_{T-\delta}\geq I_{t_0-\delta}\}}\right],$$

and compute each term separately. First,

$$E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{I_t}{I_T}\frac{I_{T-\delta}}{I_{t_0-\delta}}\mathbb{1}_{\{I_{T-\delta}\geq I_{t_0-\delta}\}}\right]=E_t\left[E_{T-\delta}\left[\frac{M_{RT}}{I_T}\right]\frac{I_t}{M_{Rt}}\frac{I_{T-\delta}}{I_{t_0-\delta}}\mathbb{1}_{\{I_{T-\delta}\geq I_{t_0-\delta}\}}\right]$$

$$=e^{m_{0T}-m_{0t}-(m'_{1t}-i'_1)X_t-i'_1X_{t_0-\delta}}\times E_t\left[E_{T-\delta}\left[e^{(m'_{1T}-i'_1)X_T}\right]e^{i'_1X_{T-\delta}}\mathbb{1}_{\{i'_1X_{T-\delta}\geq i'_1X_{t_0-\delta}\}}\right]$$

$$=e^{m_{0T}-m_{0t}+b_{0T-\delta,T}-(m'_{1t}-i'_1)X_t-i'_1X_{t_0-\delta}}E_t\left[e^{(b'_{1T-\delta,T}+i'_1)X_{T-\delta}}\mathbb{1}_{\{i'_1X_{T-\delta}\geq i'_1X_{t_0-\delta}\}}\right],$$

where $E_{T-\delta}\left[e^{(m'_{1T}-i'_1)X_T}\right]=e^{b_{0T-\delta,T}+b'_{1T-\delta,T}X_{T-\delta}}$ is given in (24). Second,

$$E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{I_t}{I_T}\mathbb{1}_{\{I_{T-\delta}\geq I_{t_0-\delta}\}}\right]=E_t\left[E_{T-\delta}\left[\frac{M_{RT}}{I_T}\right]\frac{I_t}{M_{Rt}}\mathbb{1}_{\{I_{T-\delta}\geq I_{t_0-\delta}\}}\right]$$

$$=e^{m_{0T}-m_{0t}-(m'_{1t}-i'_1)X_t}\times E_t\left[E_{T-\delta}\left[e^{(m'_{1T}-i'_1)X_T}\right]\mathbb{1}_{\{i'_1X_{T-\delta}\geq i'_1X_{t_0-\delta}\}}\right]$$

$$=e^{m_{0T}-m_{0t}+b_{0T-\delta,T}-(m'_{1t}-i'_1)X_t}E_t\left[e^{b'_{1T-\delta,T}X_{T-\delta}}\mathbb{1}_{\{i'_1X_{T-\delta}\geq i'_1X_{t_0-\delta}\}}\right],$$

Combining these two conditional expectation yields the price of the floor option,

$$\mathcal{C}^{R,t_0}_{t,T}=e^{m_{0T}-m_{0t}+b_{0T-\delta,T}-(m'_{1t}-i'_1)X_t} \tag{61}$$

$$\times\left(e^{-i'_1X_{t_0-\delta}}E_t\left[e^{(b'_{1T-\delta,T}+i'_1)X_{T-\delta}}\mathbb{1}_{\{i'_1X_{T-\delta}\geq i'_1X_{t_0-\delta}\}}\right]-E_t\left[e^{b'_{1T-\delta,T}X_{T-\delta}}\mathbb{1}_{\{i'_1X_{T-\delta}\geq i'_1X_{t_0-\delta}\}}\right]\right).$$

Finally, an application of Lemma 2 below yields an expression for the conditional expectations involving the indicator function $\mathbb{1}_{\{i'_1X_{T-\delta}\geq i'_1X_{t_0-\delta}\}}$ in the floor option price above.

Lemma 2 (Duffie et al. (2000)) *Assume the affine dynamics (7) of state variables X_t, two constant vectors $p,q\in\mathbb{R}^n$, and a scalar parameter $k\in\mathbb{R}$. Then the following identity concerning a* *conditional expectation of exponential affine and indicator functions hold,*

$$E_t\left[e^{p'X_T}\mathbb{1}_{\{q'X_T\geq k\}}\right]=\frac{1}{2}e^{l_{0t;p}+l'_{01;p}X_t}-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{v}Im\left[e^{l_{0t;p-\iota vq}+l'_{01;p-\iota vq}X_t}e^{\iota vk}\right]dv, \tag{62}$$

where $Im[A]$ retains only the imaginative component of express A, and coefficients l_{0t}, l_{1t} are given in (47).

This lemma constitutes a special result from Duffie et al. (2000), which is reproduced here for the self-sufficiency of this appendix.

Inflation Swap Rate: We consider a zero-coupon inflation swap contract initiated at time t_0 of a unit notional value and the maturity T. The zero net value to the fixed rate payor (29) at the initiation of this contract implies the swap rate,

$$h_{t_0T}=\frac{1}{T-t_0}\left(\ln E_{t_0}\left[\frac{M_{RT}}{M_{Rt_0}}\frac{1}{I_T}\frac{I_{T-\delta}}{I_{t_0-\delta}}\right]-\ln E_{t_0}\left[\frac{M_{RT}}{M_{Rt_0}}\frac{1}{I_T}\right]\right)=\frac{1}{T-t_0}\left(\ln B^{R,t_0}_{t_0,T}-\ln B_{t_0,T}\right), \tag{63}$$

where $B_{t_0,T}$ (23) and $B^{R,t_0}_{t_0,T}$ (27) are zero-coupon nominal and real bond prices. From this follows the inflation swap rate (30).

B.4 Inflation Risk Premium

Nominal Perspective: Recall that the Euler pricing equation of a zero-coupon real bond (which is risky to nominal agent) is,

$$e^{-y^{R,t}_{tT}(T-t)}\equiv B^{R,t}_{tT}=E_t\left[\frac{M_{NT}}{M_{Nt}}\frac{I_T}{I_t}\right]=Cov_t\left(\frac{M_{NT}}{M_{Nt}},\frac{I_T}{I_t}\right)+E_t\left[\frac{M_{NT}}{M_{Nt}}\right]E_t\left[\frac{I_T}{I_t}\right].$$

From this follows the definition of the (annualized) nominal inflation risk premium $\pi^N_{t,T}$ (40),

$$\begin{aligned}\pi^N_{t,T}&=\frac{1}{T-t}Cov_t\left(\frac{M_{NT}}{M_{Nt}},\frac{I_T}{I_t}\right)=\frac{1}{T-t}\left(E_t\left[\frac{M_{NT}}{M_{Nt}}\frac{I_T}{I_t}\right]-E_t\left[\frac{M_{NT}}{M_{Nt}}\right]E_t\left[\frac{I_T}{I_t}\right]\right)\\&=\frac{1}{T-t}\left(B^{R,t}_{tT}-B_{tT}E_t\left[\frac{I_T}{I_t}\right]\right)=\frac{1}{T-t}\left(e^{-y^{R,t}_{tT}(T-t)}-e^{-y^{N,t}_{tT}(T-t)}E_t\left[\frac{I_T}{I_t}\right]\right).\end{aligned}$$

Now, applying $e^{-y^{R,t}_{tT}(T-t)}$ from (27) (for $T_i=T$, and no indexation lag $\delta=0$), $e^{-y^{N,t}_{tT}(T-t)}\equiv B_{tT}$ and B_{tT} from (23), $E_t\left[\frac{I_T}{I_t}\right]$ from (14), yields the (annualized) inflation risk premium from the nominal perspective,

$$\overline{\pi}^N_{t,T} = \frac{1}{T-t}\times \tag{64}$$

$$\times\left(e^{(m_{0T}-m_{0t}+b^R_{0tT})+(-m'_{1t}+b^R_{1tT}{}')X_t} - e^{(m_{0T}-m_{0t}+b_{0tT})+(i'_1-m'_{1t}+b'_{1tT})X_t}e^{-i'_1\mathcal{U}_{t-T,0}(X_t-\Theta)+\frac{1}{2}i'_1V\mathcal{S}_{tT}V'i_1}\right),$$

where coefficients $b_0, b^R_0 \in \mathbb{R}$, and $b_1, b^R_1 \in \mathbb{R}^n$ are given in (24); $b^R_{0tT} \in \mathbb{R}$ and $b^R_{1tT} \in \mathbb{R}^n$ are given in (60) (note $t_0 \equiv t$), $n \times n$ symmetric matrix $\mathcal{S}_{tT}$ is defined in (13), and $\mathcal{U}_{t-T,0}$ in (15).

Real Perspective: Symmetrically, from a real pricing perspective, the Euler pricing equation of a zero-coupon nominal bond (which is risky to real agent) is,

$$e^{-y^{N,t}_{tT}(T-t)} \equiv B_{tT} = E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{I_t}{I_T}\right] = Cov_t\left(\frac{M_{RT}}{M_{Rt}},\frac{I_t}{I_I}\right) + E_t\left[\frac{M_{RT}}{M_{Rt}}\right]E_t\left[\frac{I_t}{I_T}\right].$$

From this follows the definition of the (annualized) real inflation risk premium $\overline{\pi}^R_{t,T}$ (41),

$$\overline{\pi}^R_{t,T} = -\frac{1}{T-t}Cov_t\left(\frac{M_{RT}}{M_{Rt}},\frac{I_t}{I_T}\right) = -\frac{1}{T-t}\left(E_t\left[\frac{M_{RT}}{M_{Rt}}\frac{I_t}{I_T}\right] - E_t\left[\frac{M_{RT}}{M_{Rt}}\right]E_t\left[\frac{I_t}{I_T}\right]\right)$$

$$= -\frac{1}{T-t}\left(B_{tT} - B^{R,t}_{tT}E_t\left[\frac{I_t}{I_T}\right]\right) = -\frac{1}{T-t}\left(e^{-y^{N,t}_{tT}(T-t)} - e^{-y^{R,t}_{tT}(T-t)}E_t\left[\frac{I_t}{I_T}\right]\right).$$

Note that similar to (14),

$$E_t\left[\frac{I_t}{I_T}\right] = e^{i'_1X_t}E_t\left[e^{-i'_1X_T}\right] = \exp\left\{i'_1\mathcal{U}_{t-T,0}(X_t-\Theta) + \frac{1}{2}i'_1V\mathcal{S}_{tT}V'i_1\right\},$$

where we have applied Lemma 1 (in which $A_T \equiv -i_1$) to obtain the last expression. Substituting this conditional expectation, $e^{-y^{R,t}_{tT}(T-t)}$ from (27) (with $T_i = T$, and $\delta = 0$), $e^{-y^{N,t}_{tT}(T-t)} \equiv B_{tT}$ and B_{tT} from (23) into the above equation for $\overline{\pi}^R_{t,T}$ yields the (annualized) real inflation risk premium,

$$\overline{\pi}^R_{t,T} = -\frac{1}{T-t}\times \tag{65}$$

$$\times\left(e^{(m_{0T}-m_{0t}+b_{0tT})+(i'_1-m'_{1t}+b'_{1tT})X_t} - e^{(m_{0T}-m_{0t}+b^R_{0tT})+(-m'_{1t}+b^R_{1tT}{}')X_t}e^{i'_1\mathcal{U}_{t-T,0}(X_t-\Theta)+\frac{1}{2}i'_1V\mathcal{S}_{tT}V'i_1}\right),$$

where coefficients $b_0, b^R_0 \in \mathbb{R}$, and $b_1, b^R_1 \in \mathbb{R}^n$ are given in (24); $b^R_{0tT} \in \mathbb{R}$ and $b^R_{1tT} \in \mathbb{R}^n$ are given in (60) (note $t_0 \equiv t$), $n \times n$ symmetric matrix $\mathcal{S}_{tT}$ is defined in (13), and $\mathcal{U}_{t-T,0}$ in (15).

C Estimations

C.1 Kalman Filter

First, given a set of model's parameters $\mathcal{P}$ (37), the Kalman filter generates estimates of state variables and their covariance matrix recursively as more price data are observed and employed in the estimation procedure over time. The current-step estimates are linear in previous-step estimates and updates from latest data observation to minimize mean square errors (MSE). In the Gaussian setting of our paper, Kalman filter estimators are both consistent and (MSE) optimal. Second, a log likelihood function is maximized to estimate model's parameters $\mathcal{P}$.

Recursive Estimation Procedure

The Kalman filter of the inflation dynamics starts with (i) the state space representation of the model specification (7) (i.e., state equations), and (ii) model-implied relationships between observable quantities and state variables (i.e., observation equations), all in discrete time. The observable quantities include futures prices on nominal bonds and the inflation expectation.

State Equations:

$$X_{t+1} = A + BX_t + \nu_{t+1}, \tag{66}$$

where $n \times 1$ vector ν_{t+1} denotes normally distributed innovations to state variables of zero mean and covariance matrix S, and $n \times 1$ vector A and $n \times n$ matrix B are,

$$\nu_{t+1} \in \mathcal{N}(0, S), \qquad A = \mathcal{K}\Theta, \qquad B = \mathbb{1}_{n\times n} - \mathcal{K}, \tag{67}$$

with the diagonalization (8), $\text{Diag}[\mathcal{K}] = V^{-1}\mathcal{K}V$, $\text{Diag}[S] = \Sigma'\mathcal{K}\Sigma$. Note that the above distribution for ν_{t+1} and expressions for A and B follow from a discrete version of the state variable dynamics (7).[40]

Observation Equations:

$$y_t = a + bX_t + \epsilon_t, \quad \epsilon_t \in \mathcal{N}(0, \Sigma_\epsilon), \qquad \text{or in matrix notation,} \tag{68}$$

[40]The discrete version of (7) reads, $X_{t+1} = X_t + \mathcal{K}(\Theta - X_t) + \left[\sqrt{S}\right]\omega_t$, $\omega_t \in \mathcal{N}(0, \mathbb{1}_{n\times n})$.

$$
y_t = \begin{bmatrix} f_{tT_1\tau_1} \\ \vdots \\ f_{tT_q\tau_q} \\ \log \overline{I}_{tT_1} \\ \vdots \\ \log \overline{I}_{tT_{q_{inf}}} \\ \tilde{r}_{Rt} \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_q \\ a_{I_1} \\ \vdots \\ a_{I_{q_{inf}}} \\ a_r \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{q1} & \dots & b_{qn} \\ b_{I_1 1} & \dots & b_{I_1 n} \\ \vdots & \ddots & \vdots \\ b_{I_{q_{inf}} 1} & \dots & b_{I_{q_{inf}} n} \\ b_{r1} & \dots & b_{rn} \end{bmatrix} \begin{bmatrix} X_{1t} \\ \vdots \\ X_{nt} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \vdots \\ \epsilon_{qt} \\ \epsilon_{I_1 t} \\ \vdots \\ \epsilon_{I_{q_{inf}} t} \\ \epsilon_{rt} \end{bmatrix}
$$

where $(q + q_{inf} + 1) \times 1$ vector y_t contains the $(q + q_{inf} + 1)$ observable quantities employed as inputs in the estimation. They are: (i) q log T-note futures prices $f_{tT_i\tau_i}$ (26) corresponding to q different maturities $T_i\tau_i$, (ii) q_{inf} (surveyed) inflation expectations $\log \overline{I}_{tT_i}$ associated with q_{inf} different horizons T_i (14), and (iii) one short-term real interest rate proxy $\tilde{r}_{Rt}$ (34). We assume that observable quantities y_t are observed with normally distributed errors ϵ_t.

Therefore, $(q + q_{inf} + 1) \times 1$ vector a and $(q + q_{inf} + 1) \times n$ matrix b are determined from the model-implied relationships between the $(q + q_{inf} + 1)$ observable quantities and state variables. Specifically, the model-implied coefficients are (see (26), (14), (18), and using specification (35)),

$$
a = \begin{bmatrix} a_1 \\ \vdots \\ a_q \\ a_{I_1} \\ \vdots \\ a_{I_{q_{inf}}} \\ a_r \end{bmatrix} = \begin{bmatrix} -\beta(\tau_1 - T_1) - (m_1 - i_1)' \mathcal{U}_{t-T_1, t-\tau_1} \Theta + \frac{1}{2}(m_1 - i_1)' V \left[\mathcal{S}_{t\tau_1} - \mathcal{S}_{tT_1} \right] V'(m_1 - i_1) \\ \vdots \\ -\beta(\tau_q - T_q) - (m_1 - i_1)' \mathcal{U}_{t-T_q, t-\tau_q} \Theta + \frac{1}{2}(m_1 - i_1)' V \left[\mathcal{S}_{t\tau_q} - \mathcal{S}_{tT_q} \right] V'(m_1 - i_1) \\ i_1' \mathcal{U}_{t-T_1,0} \Theta + \frac{1}{2} i_1' V \mathcal{S}_{tT_1} V' i_1 \\ \vdots \\ i_1' \mathcal{U}_{t-T_{q_{inf}},0} \Theta + \frac{1}{2} i_1' V \mathcal{S}_{tT_{q_{inf}}} V' i_1 \\ \beta - m_{1t}' V \mathrm{Diag}[\mathcal{K}] V^{-1} \Theta - \frac{1}{2} m_1' \Sigma \mathrm{Diag}[S] \Sigma' m_1 \end{bmatrix} \tag{69}
$$

$$
b = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{q1} & \dots & b_{qn} \\ b_{I_1 1} & \dots & b_{I_1 n} \\ \vdots & \ddots & \vdots \\ b_{I_{q_{inf}} 1} & \dots & b_{I_{q_{inf}} n} \\ b_{r1} & \dots & b_{rn} \end{bmatrix} = \begin{bmatrix} (m_1 - i_1)' \mathcal{U}_{t-T_1, t-\tau_1} \\ \vdots \\ (m_1 - i_1)' \mathcal{U}_{t-T_q, t-\tau_q} \\ -i_1' \mathcal{U}_{t-T_1,0} \\ \vdots \\ -i_1' \mathcal{U}_{t-T_{q_{inf}},0} \\ m_{1t}' V \mathrm{Diag}[\mathcal{K}] V^{-1} \end{bmatrix}, \tag{70}
$$

where the $n\times n$ matrix $\mathcal{S}_{tT}$ is defined in (13) and the $n\times n$ matrix $\mathcal{U}_{t_1t_2}$ in (15) ($\mathcal{U}_{t_1t_2}\equiv V\mathrm{Diag}[e^{t_2\mathcal{K}}-e^{t_1\mathcal{K}}]V^{-1}$, $\forall t_1,t_2$).

Initiation: The recursive estimation starts with the initial estimates for state variables and their covariance matrix which are their long-run (unconditional) expected values (taking the limit of moments in (9) as $t\in\infty$),

$$\widehat{X}_{1|0}=\Theta,\qquad \widehat{V}_{X1|0}=\frac{1}{4}\left(\Sigma\mathrm{Diag}[S]\Sigma'\mathcal{K}^{-1}+\mathcal{K}'^{-1}\Sigma\mathrm{Diag}[S]\Sigma'\right). \tag{71}$$

Updating: At time $t-1$, we are endowed with the latest estimates of state variables and their covariance matrix,

$$\widehat{X}_{t|t-1},\qquad \widehat{V}_{Xt|t-1}=E[(X_t-\widehat{X}_{t|t-1})^2], \tag{72}$$

where for notational simplicity, given a $k\times 1$ vector H, $E[H^2]$ denotes the $k\times k$ matrix $E[HH']$ throughout. The model-implied relationship (68) gives the estimate for the observable quantities and their covariance matrix,

$$\widehat{y}_{t|t-1}=a+b\widehat{X}_{t|t-1},\qquad \widehat{V}_{yt|t-1}=b\widehat{V}_{Xt|t-1}b'. \tag{73}$$

Now with newly observed data y_t, the perceived innovations in observable quantities are $y_t-\widehat{y}_{t|t-1}=y_t-(a+b\widehat{X}_{t|t-1})$. The Kalman filter then "linearly" updates the state variable estimates with the above innovations,

$$\widehat{X}_{t|t}=\widehat{X}_{t|t-1}+\Gamma_t(y_t-\widehat{y}_{t|t-1}), \tag{74}$$

in such a way that minimizes the square error of the estimate. That is, the optimal $n\times(q+q_{inf}+1)$ weight matrix Γ_t solves,

$$\Gamma_t=\arg\min E\left[\left(X_t-\widehat{X}_{t|t}\right)^2\right]=\arg\min E\left[\left((X_t-\widehat{X}_{t|t-1})-\Gamma(y_t-\widehat{y}_{t|t-1})\right)^2\right].$$

The optimal Γ_t then follows from the associated FOC,

$$\Gamma_t=E\left[(X_t-\widehat{X}_{t|t-1})(y_t-\widehat{y}_{t|t-1})'\right]\left[E\left[\left(y_t-\widehat{y}_{t|t}\right)^2\right]\right]^{-1}.$$

By virtue of the observation equation (68) and the estimate (73), $y_t-\widehat{y}_{t|t-1}=b(X_t-\widehat{X}_{t|t-1})+\epsilon_t$, and therefore

$$\Gamma_t = \widehat{V}_{Xt|t-1} b' \left(b\widehat{V}_{Xt|t-1} b' + \Sigma_\epsilon \right)^{-1},$$

where $\widehat{V}_{Xt|t-1}$ (72) and Σ_ϵ (68) are the covariance matrix of the estimate $\widehat{X}_{t|t-1}$ and observation errors in the data of y_t.

Substituting this optimal weight matrix Γ_t back into (74), the updated estimates of state variables and its covariance matrix (associated with the lowest mean square error) are

$$\widehat{X}_{t|t} = \widehat{X}_{t|t-1} + \widehat{V}_{Xt|t-1} b' \left(b\widehat{V}_{Xt|t-1} b' + \Sigma_\epsilon \right)^{-1} (y_t - \widehat{y}_{t|t-1}), \tag{75}$$

$$E\left[\left(X_t - \widehat{X}_{t|t} \right)^2 \right] = \widehat{V}_{Xt|t-1} - \widehat{V}_{Xt|t-1} b' \left(b\widehat{V}_{Xt|t-1} b' + \Sigma_\epsilon \right)^{-1} b\widehat{V}_{Xt|t-1}. \tag{76}$$

Forecasting: Employing the updated estimates $\widehat{X}_{t|t}$ (75) in the state equation (66), we obtain the new estimates of state variables,[41]

$$\widehat{X}_{t+1|t} = A + B\widehat{X}_{t|t} = (A + B\widehat{X}_{t|t-1}) + B\widehat{V}_{Xt|t-1} b' \left(b\widehat{V}_{Xt|t-1} b' + \Sigma_\epsilon \right)^{-1} (y_t - \widehat{y}_{t|t-1}), \tag{77}$$

where the second term represents the optimal ("Kalman gain") update from the latest observation innovation $(y_t - \widehat{y}_{t|t-1})$. Then follows the covariance matrix of the new state variable estimates,

$$\begin{aligned} \widehat{V}_{Xt+1|t} &= E[(X_{t+1} - \widehat{X}_{t+1|t})^2] = E\left[\left(B(X_t - \widehat{X}_{t|t}) + \nu_{t+1} \right)^2 \right] \\ &= B\left[\widehat{V}_{Xt|t-1} - \widehat{V}_{Xt|t-1} b' \left(b\widehat{V}_{Xt|t-1} b' + \Sigma_\epsilon \right)^{-1} b\widehat{V}_{Xt|t-1} \right] B' + S, \end{aligned} \tag{78}$$

where we have used (76). The latest estimates (77), (78) replace the previous ones (72) in the recursive estimation procedure.

C.2 Log Likelihood Function Associated with the Kalman Filter

The log likelihood function is constructed from the model-implied probability density function of the observable quantities y_t,[42] whose estimators and variances are obtained in the Kalman filtering at every time step.

At every time period in the recursive process, the $(q + q_{inf} + 1) \times 1$ innovations Δy_{t+1} in

[41] In Kalman filter, estimates are (minimum-variance) linear projector on the previous-step estimates and current data observations. Therefore, innovation ν_{t+1} in the state equation (66) drops out from $\widehat{X}_{t+1|t}$ (77).

[42] Since state variables X_t are not observable in the model, to confront the data, we do not employ the log likelihood function constructed from the model-implied probability density function of state variables X_t.

observable quantities and their $(q + q_{inf} + 1) \times (q + q_{inf} + 1)$ covariance matrices are implied from the state variable estimates $\widehat{X}_{t+1|t}$ and their covariance matrix $\widehat{V}_{Xt+1|t}$ as follows,

$$\Delta y_{t+1} \equiv y_{t+1} - \widehat{y}_{t+1|t} = y_{t+1} - a - b\widehat{X}_{t+1|t} = b(X_{t+1} - \widehat{X}_{t+1|t}) + \epsilon_{t+1}, \tag{79}$$

$$\widehat{V}_{\Delta yt+1|t} \equiv E\left[(\Delta y_{t+1})^2\right] = b\widehat{V}_{Xt+1|t}b' + \Sigma_\epsilon. \tag{80}$$

In particular, state variables' initial values (71) then imply the following initial values,

$$\Delta y_1 \equiv y_1 - \widehat{y}_{1|0} = y_1 - a - b\widehat{X}_{1|0} = y_1 - a - b\Theta,$$

$$\widehat{V}_{\Delta y1|0} \equiv E\left[(\Delta y_1)^2\right] = \frac{1}{4}b\left(\Sigma \mathrm{Diag}[S]\Sigma'\mathcal{K}^{-1} + \mathcal{K}'^{-1}\Sigma \mathrm{Diag}[S]\Sigma'\right)b' + \Sigma_\epsilon \tag{81}$$

$$= \frac{1}{4}b\left(\Sigma \mathrm{Diag}[S]\Sigma' V \frac{1}{\mathrm{Dial}[\mathcal{K}]}V^{-1} + (V')^{-1}\frac{1}{\mathrm{Dial}[\mathcal{K}]}V'\Sigma \mathrm{Diag}[S]\Sigma'\right)b' + \Sigma_\epsilon.$$

In the model, these quantities are functions of the model's parameters $\mathcal{P}$ (37). Hence, the log likelihood function of the model's parameters reads,

$$\mathcal{L}_{\{\Delta y_t\}}(\mathcal{P}) = \sum_{t=0}^{T-1} \mathcal{L}_t(\mathcal{P}) = \sum_{t=0}^{T-1}\left\{-\frac{1}{2}\log \mathrm{Det}[\widehat{V}_{\Delta yt+1|t}] - \frac{1}{2}\Delta y'_{t+1}\left(\widehat{V}_{\Delta yt+1|t}\right)^{-1}\Delta y_{t+1}\right\} \tag{82}$$

$$= \sum_{t=0}^{T-1}\left\{-\frac{1}{2}\log \mathrm{Det}[b\widehat{V}_{Xt+1|t}b' + \Sigma_\epsilon] - \frac{1}{2}(y'_{t+1} - a' - \widehat{X}'_{t+1|t}b')\left(\widehat{V}_{\Delta yt+1|t}\right)^{-1}(y_{t+1} - a - b\widehat{X}_{t+1|t})\right\}.$$

Note that, given the model's parameters $\mathcal{P}$ (37), estimates $\widehat{X}_{t+1|t}$ (77) and $\widehat{V}_{Xt+1|t}$ (78) are determined in the Kalman filter estimation above.

These model's parameters $\mathcal{P}$ then are obtained by maximizing this log likelihood function, $\mathcal{P} = \arg\max \mathcal{L}_{\{\Delta y_t\}}(\mathcal{P})$. The first-order condition associated with the variation of the parameter $p_j \in \mathcal{P}$ at optimality is,

$$\frac{\partial \mathcal{L}_{\{\Delta y_t\}}(\mathcal{P})}{\partial p_j} = -\sum_{t=0}^{T-1}\frac{1}{2}\mathrm{Tr}\left[\widehat{V}^{-1}_{\Delta yt+1|t}\frac{\partial \widehat{V}_{\Delta yt+1|t}}{\partial p_j}\right]$$

$$-\sum_{t=0}^{T-1}\frac{\partial \Delta y'_{t+1}}{\partial p_j}\widehat{V}^{-1}_{\Delta yt+1|t}\Delta y_{t+1} - \sum_{t=0}^{T-1}\frac{1}{2}\Delta y'_{t+1}\widehat{V}^{-1}_{\Delta yt+1|t}\frac{\partial \widehat{V}_{\Delta yt+1|t}}{\partial p_j}\widehat{V}^{-1}_{\Delta yt+1|t}\Delta y_{t+1}.$$

Recall from (37) that p_j is one of 28 model parameters in the set

$$\mathcal{P} = \{\kappa_{ij}\,;\,\Theta_i\,;\,s_i, a_i\,;\,i_{1i}\,;\,\tilde{\beta}, m_{1i}\,;\,\sigma_q\},$$

Note that the $(q + q_{inf} + 1) \times 1$ vector $\frac{\partial \Delta y_{t+1}}{\partial p_j}$, is as follows (see (79)),

$$\frac{\partial \Delta y_{t+1}}{\partial p_j} = -\frac{\partial a}{\partial p_j} - \frac{\partial b}{\partial p_j}\widehat{X}_{t+1|t} - b\frac{\partial \widehat{X}_{t+1|t}}{\partial p_j}, \qquad \forall p_j \in \mathcal{P}. \tag{83}$$

The $(q + q_{inf} + 1) \times (q + q_{inf} + 1)$ matrix $\frac{\widehat{V}_{\Delta y t+1|t}}{\partial p_j}$, is as follows (see (80)),

$$\frac{\partial \widehat{V}_{\Delta y t+1|t}}{\partial p_j} = \frac{\partial b}{\partial p_j}\widehat{V}_{Xt+1|t}b' + b\frac{\partial \widehat{V}_{Xt+1|t}}{\partial p_j}b' + b\widehat{V}_{Xt+1|t}\frac{\partial b'}{\partial p_j}, \qquad \forall p_j \in \mathcal{P}. \tag{84}$$

In principle, therefore, to compute $\frac{\partial \Delta y_{t+1}}{\partial p_j}$, and $\frac{\partial \widehat{V}_{\Delta y t+1|t}}{\partial p_j}$, for every parameter $p_j \in \mathcal{P}$ (37) (which appear in the FOC of the maximum log likelihood function), we need to compute $\frac{\partial \widehat{X}_{t+1|t}}{\partial p_j}$, and $\frac{\partial \widehat{V}_{Xt+1|t}}{\partial p_j}$, $\forall p_j \in \mathcal{P}$. All of these values can be calculated analytically and the details are available from the authors upon request.