Dynamic Coordination with Flexible Security Design

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Abstract

Borrowers obtain funding for production by issuing securities backed by the current-period dividend and resale price of a long-lived collateral asset. Borrowers are privately informed about the collateral quality. A higher (lower) resale price lowers (increases) adverse selection and makes the asset a good (lousy) collateral. Conversely, good (lousy) collateral has a high (low) resale price. When only equity is issued, this dynamic feedback between the asset price and collateral quality can lead to multiple equilibria. Optimal flexible security design eliminates multiple equilibria fragility and improves welfare through intertemporal coordination. When the security design is rigid, multiple equilibria reemerge.

Keyword: Liquidity; Dynamic Price Feedback; Intertemporal Coordination; Security Design; Multiple Equilibria; Self-fulfilling Prices; Financial Fragility; Haircut; Repo Runs; Information-Insensitive Securities; Repo; Portfolio Repo; Asset-Backed Security; Collateral; Limited Commitment; Adverse Selection.

JEL classification: G10, G01

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1 Introduction

In this paper, we identify a new source of financial fragility in studying the classical financing problem of borrowers who have access to productive opportunities but lack funding to implement them: a dynamic price feedback loop. In our setting, borrowers face two commonly observed frictions in obtaining funding liquidity for production. First, they cannot fully pledge the productive output to obtain funding.[1] To overcome this non-pledgeability constraint, they borrow against or sell securities backed by a long-lived collateral asset. Second, the quality of the collateral asset is often subject to adverse selection which limits its effectiveness in raising liquidity. Specifically, we consider a dynamic setting where the quality of the collateral asset (captured by the distribution of its dividend payoff) is either high or low and varies period by period. Borrowers are privately informed about the current period quality at the beginning of each period.[2] The key observation is that the resale price of the long-lived asset can ameliorate the resulting adverse selection problem. More importantly, the dependency of the level of adverse selection on the asset price generates a dynamic feedback loop between asset price and liquidity. A higher (lower) resale price lowers (increases) adverse selection and makes the asset good (lousy) collateral. Conversely, good (lousy) collateral has a high (low) resale price. When the set of available financial instruments is restricted, this dynamic feedback loop leads to fragility in liquidity provision which manifests as multiple equilibria and asset price volatility.

To illustrate this new source of fragility, we study a baseline case where in every period borrowers are allowed to sell only asset-backed *equity* to obtain funding for production. Equity is traded in an over-the-counter market and subject to adverse selection. In addition, at the end of each period the collateral asset itself is traded in a frictionless resale market. A higher resale price of the asset lowers adverse selection in the equity market since it allows the borrowers to exchange the asset-backed equity claims for more immediate funding, thereby attracting borrowers with the higher quality collateral asset sell equity claims in the equity market.

When the collateral asset is of either high or low quality, this dynamic price feedback leads to three possible equilibrium regions in this economy. There is a 'separating' region where adverse selection is severe. In this region, high-quality borrowers choose to retain their asset-backed equity claims. Since only low-quality borrowers are selling equity claims and engaging in production, the equity price today

[1] Limited pledgeability may result from non-contractibility of cash flows and lack of commitment of the borrowers to divert cash flows for private consumption.

[2] That is, there is adverse selection about the quality of the collateral asset between borrowers and lenders at the beginning of each period before any borrowing and production takes place. The asymmetric information is short-lived in the model since dividends are independently distributed over time.

is indeed low, the economic output is limited, and the asset resale price is depressed. There is a 'pooling' region where adverse selection is mild. In this region, both types borrow against their equity claims to employ the productive technology, the equity price is high, the output is large and the asset price is booming. There is also a 'multiplicity' region where adverse selection is intermediate, and both separating and pooling equilibria coexist.

Next we turn to optimal security design and allow borrowers to issue securities against the collateral asset, and the design is flexible in the sense that it is updated every period. It is well understood in the literature (e.g., Leland and Pyle (1977); Myers and Majluf (1984a); Nachman and Noe (1994); DeMarzo and Duffie (1995) and many others reviewed later) that in a static economy optimal security design improves liquidity. We find that in a dynamic economy, flexible optimal security design also eliminates multiple equilibria fragility. To state this result more explicitly, let us first describe our notion of liquid vs. illiquid security. We call a security liquid if both borrower types sell it. A liquid security commands a higher price, so more funding can be raised by borrowers to scale up production. Furthermore, liquid securities are information-insensitive in the sense that their prices do not fluctuate much with the underlying quality of the collateral asset. We call a security illiquid if only the low type sells it. An illiquid security has a lower price and is information-sensitive. Our main result on optimal security design shows that there is a *unique* dynamic security design equilibrium where the optimal design involves a *short-term liquid collateralized* debt tranche, and the residual *illiquid* equity tranche.

In the optimal security design, the issuer chooses the face value of the debt as large as possible in order to raise the maximum amount of funding liquidity. As the face value increases, the debt tranche incorporates more of the high dividend states. If the face value is too high, high-quality borrowers, who know that these states are likely, might prefer to retain the debt tranche rather than pool with low-quality borrowers and obtain a discounted price for these states. Hence the security design pushes the face value of the debt up to the point where high-quality borrowers are indifferent between selling at a discount to engage in more productive technology versus retaining the asset-backed debt tranche.[3] A key point is that the tranche always incorporates the resale price of the collateral asset. As the collateral price increases, selling the debt tranche becomes more attractive to the high-quality type, allowing the security designer to increase the face value of the collateralized debt.

The dynamic security design equilibrium Pareto dominates the separating equilibria in the equity-only baseline case and selects the pooling equilibrium in the multiple equilibria range. To see why,

[3]Selling the debt tranche generates value through the technology multiplier. However, selling it is less attractive for the high type as she must pool with the low type and accept a lower price.

suppose a debt tranche backed only by the future resale price is introduced. Since this debt is free of the adverse selection problem, both borrower types will issue it to take advantage of the productive technology. The asset price rises due its collateral role in securing immediate funding. The higher asset price will allow borrowers to increase the face value of debt further by incorporating some of the high dividend states. The face value will increase until high-quality borrowers become indifferent between selling the liquid debt tranche versus retaining it. In the separating equilibrium region of the equity-only baseline case, this process leads to a liquid debt tranche that is traded by both types and improves the welfare of the borrowers. In the multiple equilibria region it selects the pooling equilibrium – that is, issuers sell the entire equity-like "pass-through" debt. Effectively, dynamic coordination on information insensitive securities removes multiplicity. In this unique security design equilibrium, both liquidity and output are higher than the baseline case.

We show that this uniqueness result hinges on the assumption that borrowers have the flexibility to adjust the security design at the beginning of each period. In practice, security contract terms may not be updated frequently because of administrative costs or simply inattention. When contract terms are rigid in the sense that the face value of the contract is not updated at the beginning of every period, a run equilibrium through a negative dynamic price feedback might emerge, and the liquidity of the security market may deteriorate.[4] Essentially during such a run, the asset value and the asset price drop, increasing the severity of adverse selection about the quality of the collateral. This situation makes the previously liquid debt tranche illiquid, which in turn justifies the drop in the asset price. Had the design been flexible, borrowers would redesign the security in this event by lowering the debt threshold to make sure that the debt tranche is liquid. This action will push the asset price up, triggering a positive dynamic price feedback and leading to a full recovery of prices and the debt threshold. However, when the design is rigid, the drop in asset price can be self-fulfilling.

Next, we turn to one implementation of the optimal security design commonly observed in real-world financial markets. The optimal security in our model is liquid, short-term, and collateralized debt which are the key characteristics of short-term repo contracts. Under the repo interpretation, the model generates unique testable predictions on the contract terms such as haircuts and repo rates as well as the collateral asset price.

[4]Runs in our setup are dynamic feedback runs and hence distinct from bank runs as in Diamond and Dybvig (1983), the type of repo runs caused by systemic asset fire sales as in Martin, Skeie, and Von Thadden (2012), due to repo market microstructure features, liquidity need of the lenders as well as the capital position of the borrowers as in Martin, Skeie, and Von Thadden (2014), or the collateral crisis due to the endogenous information production studied in Gorton and Ordonez (2014).

In our model a repo haircut has two main components. The first is related to the productivity of the borrower's technology. This component arises because borrowers, who price the collateral asset, value the liquidity service the asset provides, while lenders, who price the loan, do not value this service. This generates heterogeneous valuation over the collateral assets among agents. As a result, as borrower's become more productive, haircuts increase. This effect is reminiscent of the impact of difference of opinion on haircuts noted by Geanakoplos and Zame (2002), Geanakoplos (2003), Fostel and Geanakoplos (2012), and Simsek (2013).

The second component is the value of the equity tranche relative to the value of the collateral and arises mechanically because equity tranche by definition is excluded from the repo debt. This component responds strongly to the level of information friction. A similar connection between haircuts and information friction also features in Dang, Gorton, and Holmström (2011) and Gorton and Ordonez (2014). In our model, as the quality of the collateral asset deteriorates, the second component may increase or decrease, which is because of two opposing effects. Although the value of the equity tranche might decrease as the value of the underlying collateral falls, it might also increase since the debt threshold is adjusted downwards making the size of the equity tranche larger. A combination of the two effects leads to nonmonotonic impact of information friction on the haircut.

The repo rate in our model is much less sensitive to information friction since repo debt is liquid and both high- and low-quality borrowers participate in this market. Nevertheless, repo debt is risky and the repo rate is determined by the default risk of the repo contacts (which is related to the face value of the repo contract) and the demand for funding liquidity (which is related to the productivity).

Our model also generates predictions on commonly used portfolio repos, which are repo contracts backed by a portfolio of collateral assets. It predicts that when the fraction of safe assets in the collateral pool increases, repo contract terms improve since the level of adverse selection is lowered. Finally we extend the baseline model by allowing asset quality or productivity to be persistent. This extension shows that the dynamic feedback loop creates significant amplification of fundamental shocks in the economy. In one calibration, we find that a one percentage point increase in productivity causes the liquidity premium of the collateral asset to increase by about 15 percentage points.

According to the current understanding, the shadow banking system of overnight repurchase agreements, asset-backed securities, broker-dealers and investments contributed to the financial crisis of 2008-2009 and the runs on the shadow banking system were classic bank runs a la Diamond and Dybvig (1983). However, this popular explanation ignores the fact that most of the securitized products and the short-term funding instruments of these shadow banks are backed by the resale prices of the assets on their balance sheet (in addition to dividend/interest payments). Our model implies that in a dynamic economy, when financial intermediaries can flexibly tranche their assets to create information insensitive securities, the dynamic self-fulfilling price dynamics can be removed and the amount of funding liquidity as well as the real output in the economy will be improved. The culprit of the fragility we have observed is the failure of intertemporal coordination due to the rigidity of securitization contracts. Flexible securitization in fact eliminates fragility.

In addition to optimal security design, our model has implications for asset prices. Asset prices in our model are more than the sum of the discounted future dividends because the collateral asset commands a liquidity price premium. Borrowers pledge the asset's dividend flows and resale price to overcome pledgeability constraints and raise funding for production. The liquidity premium reflects a technology multiplier since the funds are more valuable when the technology is more productive. This connection between productivity and liquidity premium might seem counterintuitive because the long-lived asset is not a direct input in the production technology per se and serves only as collateral to obtain funding liquidity. This theoretical finding about the technology multiplier might speak to the meteoric rise of asset prices during the productivity boom we observed during the mid 2000s and in general procyclical patterns of asset price premium.

The remainder of the paper is structured as follows. In section 2, we review related literature. In section 3, we lay out the basic setup. In section 4, we describe the security design problem. In section 5, we study the baseline case where security design is restricted to equity. In sections 6, we solve for the optimal security design and study its equilibrium properties including uniqueness and runs. In section 7, we explore one of the implementations of the optimal security, short-term repo contracts and associated economic implications. Section 8 concludes and discusses potential applications.

2 Related Literature

The seminal work of Akerlof (1970) started the literature on the lemons market to study the impact of adverse selection on trade volume and efficiency. A long lineage of security design literature has focused on adverse selection including Leland and Pyle (1977); Myers and Majluf (1984a); Nachman and Noe (1994); DeMarzo and Duffie (1995); and DeMarzo and Duffie (1999), who examine informed sellers' incentive to issue optimal security to signal asset quality. As in DeMarzo and Duffie (1999), we model the security-design decision as an ex ante problem faced by the issuer.[5] We contribute to this

[5]Although not crucial for the results we differ from DeMarzo and Duffie (1999) in the way we model securities markets. In our model securities are traded in segmented markets and lenders are unable draw inferences about the asset quality

literature by extending the static setup to a dynamic environment incorporating a price feedback effect. Multiplicity in equilibrium in our model is due to dynamic miscoordination, which differs significantly from the multiplicity in static setting. Furthermore, we discover that security design helps to mitigate the adverse selection problem not only by increasing the amount of liquidity but also by improving dynamic coordination via information-insensitive securities and eliminating dynamic fragility.[6]

Our result that both borrower types issue debt and that debt is liquid is reminiscent of Gorton and Pennacchi (1990), who finds that low-information-intensity (debt-like) securities protect sellers from the risk of selling only high-quality assets when trading with an informed buyer. Boot and Thakor (1993) also find that the optimal security design is implementable by a liquid debt contract and an illiquid equity contract. However, the motivation is to stimulate information production using information-sensitive securities. This literature has now progressed to incorporate endogenous asymmetric information in an optimal security design problem such as Yang (Forthcoming); Dang, Gorton, and Holmström (2013); and Farhi and Tirole (2015). The fact that information friction affects the moneyness of an asset has also been studied by Lester, Postlewaite, and Wright (2012) and Li, Rocheteau, and Weill (2012).

There has also emerged a literature on heterogeneous information and security design such as Ellis, Piccione, and Zhang (2017). Under diverse beliefs, however, there is no fragility under a dynamic environment. There will be speculative premium under diverse beliefs, but it is difficult to investigate financial fragility unless exogenous changes in beliefs are introduced. With adverse selection, as in our model, the changes in market liquidity or "beliefs" can be endogenous.

By studying optimal collateral-backed security design and funding liquidity, our paper is also related to a long line of collateral literature in money and macroeconomics starting with the seminal work of Kiyotaki and Moore (1997) and recent studies on the prevalence of the use of repo contracts in funding financial institutions such as Geanakoplos and Zame (2002), Geanakoplos (2003), Fostel and Geanakoplos (2012), Simsek (2013), and Gottardi, Maurin, and Monnet (2017). Increasingly attempts are made to incorporate financial frictions in macroeconomic models or studying macroeconomic implications of financial friction such as collateral to understand the boom and bust cycles. Recent papers include but are not limited to Gorton and Ordonez (2014); Kuong (2017); Parlatore (Forthcoming); and Miao

from the trading volume in different markets. As a result retaining equity does not signal high-quality. This feature also separates our model from screening models of security design like Biais and Mariotti (2005).

[6]A complementary and extensive security design and financial contracting literature focuses on moral hazard which includes but is not limited to Biais et al. (2007) and Chemla and Hennessy (2014). In the latter paper an originator can put effort privately to improve the probability that the asset is high-quality and privately observes its realization. Their focus is on the role of security design to give the originator "skin-in-the-game," whereas ours is on the dynamic feedback between asset prices and adverse selection and the interaction between financial fragility and security design.

and Wang (2018). Kurlat (2013) and Bigio (2015) study financial frictions that arise endogenously from adverse selection in a dynamic production economy. Our paper demonstrates that designing information-insensitive securities in a dynamic adverse selection environment reduces financial fragility, noting the potential social role of financial intermediaries.

Our paper is closely related to Plantin (2009), Chiu and Koeppl (2016), Donaldson and Piacentino (2017); and Asriyan, Fuchs, and Green (2017), where multiple equilibria are dynamic in nature. Although Asriyan, Fuchs, and Green (2017) focuses on sentiment-driven multiple equilibria and differs from our work in the setup and implications, their insight that asset price and liquidity is closely linked is very close to ours. Interestingly, unlike in their paper, dynamic multiplicity occurs in our setup without any persistent shocks. Furthermore, our finding that optimal security design can coordinate expectations across time and eliminate dynamic fragility is an important contribution to this literature.

3 The Model Setup

In this economy, there are two types of agents. One type has access to a technology that produces an intermediate good. This technology is constant-returns-to-scale and allows the agent to produce one unit of the intermediate good from one unit of labor. However, the intermediate good does not provide direct utility. The other type possesses a "productive technology" that produces a consumption good using the intermediate good through a constant returns-to-scale technology. This technology is highly productive because an input of one unit of intermediate good generates $z > 1$ units of consumption good, and we term it the z-technology. Since the agents who have the ability to produce the intermediate good can be viewed as investors in the z-technology, we refer to them as the type I agents. Since the agents who possess the z-technology borrow the intermediate good from the I agents, we refer to them as the type B agents.

In addition, we assume that all agents have access to a "basic technology" that produces a consumption good. This basic technology produces one unit of the consumption good using one unit of labor. A broad interpretation of the basic technology is that it captures the "outside option" of the agents and its associated benefit is the opportunity cost of undertaking other technologies or investments.

The assumption on the technologies in the economy is made to capture the gain from trade of the intermediate goods between the two types of agents. Intermediate goods can be interpreted as any inputs to the z-technology such as capital, equipment, or intermediate products. The B type agents would like to borrow as many intermediate goods as possible from the I type agents to engage in the productive z-technology. In the theory, type B agents emerge naturally as borrowers and type I agents as lenders because type B agents need funding/intermediate goods from type I agents to take advantage of the productive z-technology. In this way the model parsimoniously captures any situation where it is more efficient to allocate funding to borrowers who might have better investment opportunities (e.g., entrepreneurs or financial institutions), or who have immediate consumption needs (e.g., liquidity constrained consumers or firms).

Timing. The economy is set in discrete time and lasts forever. Each period has three dates. At date 1, the intermediate good is produced by agent I. At date 2, consumption good is produced via the z-technology using the intermediate good and/or the basic technology using labor. At date 3, consumption takes place. Any leftover intermediate or consumption good perishes at the end of each period t.[7]

Utilities and discounting. An agent's utility in period t is given by $U_t(x, l) = x - l$ where x is the amount of consumption good consumed and l is the amount of labor supplied by the agent. There is no discounting between dates within a period. Agents discount periods at a rate β, with $0 < \beta < 1$.

Productive Asset and Asymmetric Information. There is an asset in the economy, which pays s units of dividend in terms of consumption good at date 3. The total supply of the asset is A. With probability λ, the dividend of the asset follows distribution $F_L \in \Delta[s_L, s_H]$, with $0 \leq s_L < s_H$. With probability $1 - \lambda$, it follows distribution $F_H \in \Delta[s_L, s_H]$. We assume that F_H first order stochastically dominates F_L. The quality, denoted by $Q \in \{H, L\}$, represented by λ ($Q = L$ with probability λ), is i.i.d. over time. More generally, asset quality could be persistent over time. We will consider that case in a later section.

We assume that agent B's production of the consumption good is not pledgeable. Without this friction, in any period, agent B would borrow unlimited amount of intermediate goods from the I agents at date 1, produce unlimited amount of consumption goods at date 2, and pay back the I agents at date 3. Given the nonpledgeability assumption, agent B cannot promise to pay back at date 3 and hence cannot borrow from the I agents at date 1. The asset provides a way for the B agent to partially overcome this friction by providing liquidity since it can be used as collateral to back up agent B 's promise to pay back. If agent B owns the asset, she can borrow intermediate goods from the I agents at date 1 using both the dividend and the resale value of the asset at date 3 as collateral. If agent B does not fulfill her promise, I agents can seize the collateral asset.

However, the use of this collateral asset for liquidity service is limited by an additional friction in the economy: asymmetric information. We assume that the quality of the collateral asset is privately

[7] The framework of dynamic analysis is borrowed from Lagos and Wright (2005).

observed by agent B at the beginning of the period (i.e., at date 1 of each period). This aspect introduces an adverse selection problem that plays a key role in our analysis. The assumption that agent B is better informed of the collateral asset's quality can be motivated or micro-founded in several ways. As demonstrated later, agent B would purchase all collateral assets in equilibrium because she needs liquidity for the z-technology. Since borrowers hold collateral assets on their balance sheets, they may have an informational advantage about the quality of these assets.[8]

In this asymmetric information environment, the asset provides only a limited amount of liquidity since the amount that agent B can borrow is bounded by the expected dividend and the resale value of the asset. However, agent B can improve liquidity available at the beginning of each period by optimally designing securities which are used to exchange for the intermediate goods at date 1 and deliver consumption good payments at the end of each period. A security is a state-contingent promise at date 1 of consumption good payment at date 3. Denote the payoff from security j at state s to be $y^j(s)$. Because agent B cannot commit to pay, the security must be backed by the dividend and the ex-dividend price of the asset, denoted by ϕ_t. The set of all feasible asset-backed securities at time t for a given price ϕ_t is $\mathcal{I}_t(\phi_t) \subseteq \{y : y(s) \leq s + \phi_t, \forall s \in [s_L, s_H]\}$. The set $\mathcal{I}_t(\phi_t)$ captures any potential exogenous restrictions on the set of feasible securities. One possible set, $\mathcal{I}_t(\phi_t) = \{y : y(s) = s + \phi_t, \forall s \in [s_L, s_H]\}$, consists of only a single "pass-through" security which promises the dividend and resale value of the collateral asset. A second possibility, $\mathcal{I}_t(\phi_t) = \{y : y(s) \text{ increasing in } s, y(s) \leq s + \phi_t, \forall s \in [s_L, s_H]\}$, is the set of all monotone securities backed by the collateral asset. The monotonicity restriction is motivated by realism since the payoff from any loan collateralized directly by the asset or any other collateralized loan is increasing in s.[9]

A security design is a finite selection of securities that are backed by the asset. Next we provide the definition.

Definition 1. Given the asset price ϕ_t, a *security design* consists of a finite set of securities $\mathcal{J}_t(\phi_t) \subseteq \mathcal{I}_t(\phi_t)$.

[8] Empirically, one can motivate the superior information advantage of the asset owner in many ways. By owning the asset, it is easier for asset owners to observe the cashflows of the asset and obtain potentially other cashflow-related information (e.g., governance information). This situation gives them an advantage in valuing the asset. Historically there are incidences where some borrowers, especially when hit by unobservable random negative shocks, debased collateral assets, e.g., by reducing the metallic content of coins below their face value. In recent times, collateral quality has been subject to questioning because of the possibility that borrowers might pledge it multiple times.

[9] In an extension, we relax the monotonicity assumption and allow for Arrow securities to be available for security design.

When $j \in \mathcal{J}_t(\phi_t)$, we say that security j is available.

Trading environment. There are two types of markets in this economy. After state is realized at the end of each period t (i.e., at date 3), a market for the collateral asset opens for trading where the asset price, denoted by ϕ_t, is determined. The important feature of the asset market is that it is not subject to asymmetric information.

In addition, at the beginning of each period t (i.e., at date 1), there is a market associated with each security where borrowers purchase intermediate goods using the security backed by the cashflow of the collateral asset. One can interpret these as markets where lending and borrowing backed by the collateral asset's future cashflows take place. These markets can be thought of as over-the-counter where each security is traded among decentralized counterparties. Specifically, for each available security, there is a submarket where agent B meets at least two randomly chosen I agents to trade asset-backed securities in exchange for intermediate goods. We assume that agent Is simultaneously make price offers per unit of the security. Agent B then observes the price offers and decides the quantity of the security to allocate to each agent I. Since each unit of the security must be backed by one unit of the asset, the total quantity of any given security sold by agent B must be less than or equal to the amount of asset owned by agent B in that period. If agent B decides to sell a positive amount of the security, she allocates the amount of the security that she would like to sell to the agent I who offers the higher price. If several agent Is are tied for the highest offer, agent B equally splits the amount that she would like to sell between them.

The following figure summarizes the time and events in this setup.

4 Security Design Problem

4.1 Defining the security design problem

Agent B chooses the security design $\mathcal{J}_t(\phi_t)$ before the beginning of each period t anticipating the asset price ϕ_t. Given the security design $\mathcal{J}_t(\phi_t)$, we denote the value function of agent $\tau \in \{I, B\}$ in period t by $V_{\tau,t}(a)$ where a is the amount of the asset that agent τ brings into the period.[10] To characterize $V_{\tau,t}(a)$, we first describe agents' optimization problem at date 3 of period t given state s.[11] Agents enter date 3 with some amount of the consumption good (c) that had been produced via the z-technology

[10]We will show that in equilibrium agent B who benefits from the asset as collateral chooses $a = A$ and brings the entire supply of the asset into each period, and agent I chooses $a = 0$.

[11]Recall, that at date 3 agents know the realization of the state s.

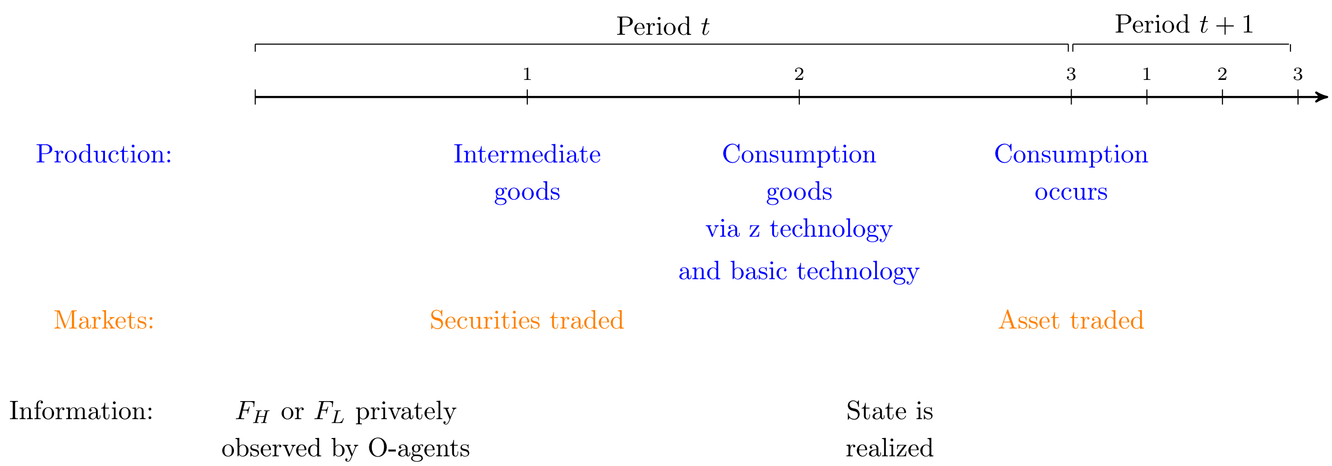

Figure 1: Timeline

(We will explain how c is determined below). Given c and their asset holding a, they decide how much additional consumption good to produce using labor (l), how much to consume (x) and how much asset to carry into the next period ($\tilde{a}$). That is, type $\tau \in \{I, B\}$ agent solves the following optimization problem:

$$W^s_{\tau,t}(c,a) = \max_{x,l,\tilde{a}\geq 0} x - l + \beta V_{\tau,t+1}(\tilde{a}), \tag{1}$$

$$s.t. x + \phi_t \tilde{a} = c + (s + \phi_t)\, a + l.$$

Note that $W^s_{\tau,t}(c,a)$ can be expressed as

$$W^s_{\tau,t}(c,a) = c + (s+\phi_t)\, a + W^s_{\tau,t}(0,0) \tag{2}$$

where $W^s_{\tau,t}(0,0) = \max_{\tilde{a}\geq 0} \beta V_{\tau,t+1}(\tilde{a}) - \phi_t \tilde{a}$.

Next, we describe how much consumption good is produced via the z-technology at date 2. Suppose that security $j \in \mathcal{J}_t(\phi_t)$ pays $y^j_t(s)$ in state s, its price is q^j_t, and agent B of type $Q \in \{L, H\}$ sells $a^Q_t(j)$ units of it. This means that agent B of type Q is able to borrow $\sum_{j\in\mathcal{J}_t(\phi_t)} a^Q_t(j) q^j_t$ of intermediate goods and uses the z-technology to produce $z \sum_{j\in\mathcal{J}_t(\phi_t)} a^Q_t(j) q^j_t$ units of consumption good. Once the state is realized at date 3, she pays out $y^j_t(s)$ to the I agents and is left with the remaining $\sum_{j\in\mathcal{J}_t(\phi_t)} a^Q_t(j) \left[z q^j_t - y^j_t(s) \right]$ units of consumption goods.

Now, we move back to date 1 and describe the equilibrium prices and quantities in security market $j \in \mathcal{J}_t(\phi_t)$. We assume that the expected payoff of security when issued by the high type is weakly more than that by the low type, i.e., $E_L y_t^j(s) \leq E_H y_t^j(s)$.[12] We denote by R_t^j the ratio of the expected value of the security under the low versus the high distribution, i.e., $R_t^j \equiv E_L y_t^j / E_H y_t^j$. As this ratio increases, the expected values of the security under the low versus high distribution become closer, and hence, the adverse selection problem becomes less severe. Hence, we also call R_t^j the adverse selection ratio of security j at period t.

Recall that I agents simultaneously make price offers per unit of the security, agent B observes the price offers, and decides how much of the security to allocate to each agent I.[13] Due to Bertrand competition, I agents make zero surplus in expectation. This means that the equilibrium price, q_t^j, must equal the expected value of a unit of the security given the expectation of I agents about the quantities that will be sold by the two types. These expectations must be incentive compatible: if I agents anticipate that a given type of the B agent sells the security at per-unit price q_t^j, that type must indeed find it profitable to sell the security at price q_t^j. The next proposition characterizes the equilibrium in security market j.

Proposition 1. *If $R_t^j > \zeta \equiv 1 - (z-1)/\lambda z$, in submarket j, the price of the security is $q_t^j = \lambda E_L y_t^j + (1-\lambda) E_H y_t^j$ and $a_{t,L}^j = a_{t,H}^j = a$. If $R_t^j < \zeta$ the price of the security is $q_t^j = E_L y_t^j$ and $a_{t,L}^j = a$ and $a_{t,H}^j = 0$.*[14]

Proposition 1 shows that when R_t^j is above the threshold ζ, the adverse selection problem is not too severe and both types sell a units of the security. In this case, the security price is the pooling price $q_t^j = \lambda E_L y_t^j + (1-\lambda) E_H y_t^j$. When R_t^j is below the threshold, the adverse selection problem is severe and only the low type sells a units of the security. In this case, the security price is the separating price $q_t^j = E_L y_t^j$.

A security that is traded in a pooling equilibrium in the security market generates more liquidity for the borrower than the one that is traded in a separating equilibrium. This is because in a pooling equilibrium, given the higher price, the borrower obtains more intermediate goods in exchange for the security and is able to produce more consumption goods. Hence, we refer to a security that is traded

[12] This assumption is automatically satisfied for monotone securities.

[13] In this formulation agent B has all the bargaining power, but this is not crucial for any of our results.

[14] When $R_t^j = \zeta$ there are multiple equilibria. In particular, both pooling and separating (and even semi-separating) equilibria are possible. To simplify exposition in this knife edge case, we select the pooling equilibrium. To see why there are multiple equilibria, suppose I agents bid $E_L y_t^j$, the low type sells a units and I agents make zero profit. Since $R_t^j = \zeta$, to attract the high type, an I agent must deviate to bidding at least $\lambda E_L y_t^j + (1-\lambda) E_H y_t^j$. However, this deviation is not profitable since by deviating an I agent cannot make a positive surplus. Hence, both $E_L y_t^j$ and $\lambda E_L y_t^j + (1-\lambda) E_H y_t^j$ can be sustained as equilibrium bids.

in a pooling equilibrium as a *liquid security* and one that is traded in a separating equilibrium as an *illiquid security.*

Now we are ready to express agent B's value function given the security design $\mathcal{J}_t(\phi_t)$ and her asset holding a:

$$V_{B,t}(a) = \lambda \int W^s_{B,t}\left(\sum_{j\in\mathcal{J}_t(\phi_t)} a^j_{t,L}\left[zq^j_t - y^j_t(s)\right], a\right) dF_L(s) + (1-\lambda)\int W^s_{B,t}\left(\sum_{j\in\mathcal{J}_t(\phi_t)} a^j_{t,H}\left[zq^j_t - y^j_t(s)\right], a\right) dF_H(s)$$

where

$$q^j_t = \begin{cases} \lambda E_L y^j_t + (1-\lambda)E_H y^j_t, & \text{if } R^j_t \geq \zeta \\ E_L y^j_t, & \text{if } R^j_t < \zeta \end{cases} \tag{3}$$

and

$$a^j_{t,L} = a \text{ and } a^j_{t,H} = \begin{cases} a, & \text{if } R^j_t \geq \zeta \\ 0, & \text{if } R^j_t < \zeta. \end{cases} \tag{4}$$

Prices and quantities in the security markets given by (3) and (4) are the equilibrium outcomes characterized in Proposition 1. Using (2), we can rewrite the value function as:

$$V_{B,t}(a) = \lambda \int \left\{\sum_{j\in\mathcal{J}_t(\phi_t)} a^j_{t,L}\left[zq^j_t - y^j_t(s)\right] + a\,(s+\phi_t)\right\} dF_L(s) + (1-\lambda)\int \left\{\sum_{j\in\mathcal{J}_t(\phi_t)} a^j_{t,H}\left[zq^j_t - y^j_t(s)\right] + a\,(s+\phi_t)\right\} dF_H(s) + W^s_{B,t}(0,0) \tag{5}$$

Let $P_t \subseteq \mathcal{J}_t(\phi_t)$ be the subset of liquid securities for which $R^j_t \geq \zeta$. Using (3) and (4), we can rewrite (5) as:

$$V_{B,t}(a) = a\left[z\left(\lambda E_L s + (1-\lambda)\,E_H s + \phi_t\right) - (1-\lambda)\,(z-1)\sum_{j\in\mathcal{J}_t(\phi_t)\setminus P_t} E_H y^j_t\right] + W^s_{B,t}(0,0). \tag{6}$$

Agent B chooses her asset holdings at date 3 of period t to maximize $\beta V_{B,t+1}(a) - \phi_t a$. We now show that in equilibrium agent B chooses $a_t = A$ at every period and agent I's do not hold any of the asset. To see this, note that the marginal utility that an agent I obtains from holding a unit of the asset is simply $MU_I = \beta\left(\lambda E_L s + (1-\lambda)\,E_H s + \phi_{t+1}\right) - \phi_t$. The marginal utility that agent B obtains from holding a unit of the asset is

$$\begin{aligned} MU_B &= \beta \frac{\partial V_{B,t+1}(a)}{\partial a} - \phi_t \\ &= \beta z \left(\lambda E_L s + (1-\lambda) E_H s + \phi_{t+1}\right) - \beta (1-\lambda)(z-1) \sum_{j \in \mathcal{J}_{t+1}(\phi_{t+1}) \setminus P_{t+1}} E_H y_{t+1}^j - \phi_t . \end{aligned}$$

It is easy to see that $MU_B > MU_I$. In addition, both MU_B and MU_I are independent of a. Hence, if $MU_B > 0$ agent B would demand infinite units of the asset and if $MU_B < 0$ zero units of it. Thus, for the asset market to clear in equilibrium, we must have $MU_B = 0$, and agent B holds the entire supply of the asset.

Now we are ready to state the optimal security design problem. Agent B chooses security design $\mathcal{J}_t(\phi_t) \subseteq \mathcal{I}_t(\phi_t)$ to maximize $V_{B,t}(A)$ subject to the feasibility of security design

$$\sum_{j \in \mathcal{J}_t(\phi_t)} y_t^j(s) \leq s + \phi_t \text{ for all } s \in S. \tag{7}$$

The feasibility constraint ensures that agent B is able to fulfill her promises in every submarket and in all states. Note that the security design is done ex ante, before agent B learns the asset quality. At the security design stage, agent B simply decides which submarkets are open for trading but she cannot commit to quantities that will be traded in a given submarket. In fact, it is easy to see that optimality of the security design requires that the feasibility condition is satisfied with equality for almost all states. Combining the feasibility condition and $MU_B = 0$, we obtain the following Euler equation that determines the asset price:

$$\phi_{t-1} = \beta \left[z \left(\sum_{j \in P_t} q_t^j + \lambda \sum_{j \in \mathcal{J}_t(\phi_t) \setminus P_t} q_t^j \right) + (1-\lambda) \sum_{j \in \mathcal{J}_t(\phi_t) \setminus P_t} E_H y_t^j \right]. \tag{8}$$

Now we state the equilibrium definition for the dynamic security design problem that summarizes the discussion so far.

Definition 2. A dynamic equilibrium consists of asset price ϕ_t and security design $\mathcal{J}_t(\phi_t) \subseteq \mathcal{I}_t(\phi_t)$ at each t such that $\mathcal{J}_t(\phi_t)$ maximizes (5) subject to (7), and ϕ_{t-1} solves the Euler equation given by (8), where security prices q_t^j and quantities $a_{t,L}^j$ and $a_{t,H}^j$ are given by (3) and (4).

According to this definition, the number of securities that can be traded in the securities market is not restricted and optimization problem is time contingent, allowing for security designs that are dynamic and flexible.

5 The Baseline: Fragility of the Dynamic Lemons Market

In this section, we consider a baseline case where only the equity claim to the collateral asset is available to trade in the security market at date 1. We demonstrate that this economy is fragile and exhibits dynamic multiplicity in prices. Specifically, we show that there might be multiple equilibria in the security market at date 1 justified by different prices in the asset market at date 3. The multiple asset prices are themselves justified by the different equilibria in the security market.

For this baseline case, we use the notion of equilibrium in Definition 2 except that we take the equity claim to the collateral asset as the only available security. That is, we set $\mathcal{I}_t(\phi_t) = \{y : y(s) = s + \phi_t, \forall s \in [s_L, s_H]\}$. Choice of security design to maximize (5) becomes trivial since there is only a single feasible security. In equilibrium, the price of the equity claim in the security market must satisfy (3). The payoff of the collateral asset in state s is $s + \phi_t$. Hence, by (3) the price of the equity claim to the collateral asset in the security market is given by $q_t^P = \phi_t + \lambda E_L s + (1-\lambda) E_H s$ if $(E_L s + \phi_t)/(E_H s + \phi_t) \geq \zeta$ and $q_t^S = \phi_t + E_L s$ otherwise. Using (8), we obtain the price of the collateral asset in the asset market as

$$\phi_t = \begin{cases} \beta z q_{t+1}^P, & \text{if } \frac{E_L s + \phi_{t+1}}{E_H s + \phi_{t+1}} \geq \zeta, \\ \beta \left[z \lambda q_{t+1}^S + (1-\lambda)\left(\phi_{t+1} + E_H s\right)\right], & \text{if } \frac{E_L s + \phi_{t+1}}{E_H s + \phi_{t+1}} < \zeta. \end{cases} \tag{9}$$

To characterize the equilibria, we first consider stationary equilibria where the collateral asset is either always traded in a pooling equilibrium, or it is always traded in a separating equilibrium. We then show that, in fact, all equilibria must be stationary.

For the first step, since we are focusing on stationary equilibria we drop the time subscripts. Plugging q^P and q^S into (9) we observe that a pooling equilibrium, in which both types of agent B sell the equity claim in the security market for the intermediate goods, exists if and only if

$$\frac{E_L s + \phi^P}{E_H s + \phi^P} \geq \zeta, \tag{10}$$

where the asset price in the pooling equilibrium is given by

$$\begin{aligned} \phi^P &= \beta z (\phi^P + \lambda E_L s + (1-\lambda) E_H s), \\ \phi^P &= \frac{\beta z \left(\lambda E_L s + (1-\lambda) E_H s\right)}{1 - \beta z}. \end{aligned} \tag{11}$$

Similarly, a separating equilibrium, in which only the low type of agent B sells the equity claim in the security market for the intermediate goods, exists if and only if

$$\frac{E_L s + \phi^S}{E_H s + \phi^S} < \zeta, \tag{12}$$

where the asset price in the separating equilibrium is given by,

$$\begin{aligned} \phi^S &= \beta\left[\lambda z\left(\phi^S + E_L s\right) + (1-\lambda)\left(\phi^S + E_H s\right)\right], \\ \phi^S &= \frac{\beta\left[\lambda z E_L s + (1-\lambda) E_H s\right]}{1-\beta(\lambda z + 1 - \lambda)}. \end{aligned} \tag{13}$$

Note that the pooling price is always higher than the separating price:

$$\phi^P = \frac{\beta z\left[\lambda E_L s + (1-\lambda) E_H s\right]}{1-\beta z} > \phi^S = \frac{\beta\left[\lambda z E_L s + (1-\lambda) E_H s\right]}{1-\beta(\lambda z + 1 - \lambda)}.$$

The discounted value of future dividends is $\beta\left(\lambda E_L s + (1-\lambda) E_H s\right)/(1-\beta)$. Since $z > 1$, the price for the asset is strictly higher than the discounted value of future dividends in both the pooling and the separating scenarios. The difference is justified by the collateral service provided by the asset. The pooling price is higher because the collateral service is more valuable in the pooling equilibrium of the securities market as both types use the collateral to purchase the intermediate goods. Moreover, when z is higher, there is more demand for collateral, which justifies a higher asset price.

Since the pooling price is higher than the separating price, for the same underlying parameters, there may be multiple price equilibria. That is, the separating price ϕ^S may be consistent with a separating equilibrium and the pooling price ϕ^P may be consistent with a pooling equilibrium in the security market. The ratio $\frac{E_L s}{E_H s} \in \left(\frac{s_L}{s_H}, 1\right)$ can be interpreted as a measure of adverse selection. As this ratio increases, the expected dividend with respect to the two distributions becomes closer, and adverse selection declines. The following proposition shows that there is always a range with an intermediate level of adverse selection ratio such that multiple equilibria exist. Moreover, all equilibria must be stationary. To state the proposition we define two cutoffs $\kappa_P < \kappa_S$, where

$$\kappa_S = 1 - \frac{z-1}{z}\frac{1}{\lambda\left(1-\beta+\beta\left(1-\lambda\right)\left(z-1\right)\right)},$$

and

$$\kappa_P = 1 - \frac{z-1}{z}\frac{1}{\lambda\left(1-\beta\right)}.$$

Proposition 2. *In the baseline case, all dynamic equilibria are stationary.*

(i) If $\frac{E_L s}{E_H s} > \kappa_S$*, then there is a unique equilibrium in which the collateral asset is sold in a pooling equilibrium in the security market and its price in the asset market is given by (11).*

(ii) If $\frac{E_L s}{E_H s} < \kappa_P$*, then there is a unique equilibrium in which the collateral asset is sold in a separating equilibrium in the security market and its price in the asset market is given by (13).*

(iii) If

$$\kappa_P \le \frac{E_L s}{E_H s} \le \kappa_S, \tag{14}$$

then both the pooling equilibrium described in (i) and the separating equilibrium described in (ii) exist.

The existence of multiple price equilibria is due to a dynamic price feedback effect. If agents anticipate the asset to be traded in a pooling equilibrium in the security market, the asset price is high. In turn, when the price is high, the high type B agent is willing to pool. Conversely, if agents anticipate the asset to be traded in a separating equilibrium in the security market, the asset price is low. In turn, when the price is low, the high type B agent keeps the asset. The beliefs are self-fulfilling.

To show that all equilibria must be stationary, we first show that if $E_L s/E_H s \geq \kappa_P$ and the collateral asset is traded in a pooling equilibrium in the security market at time $t+1$, then it cannot be traded in a separating equilibrium at time t. This outcome is due to the combination of two factors. When agents anticipate that the collateral asset is traded in a pooling equilibrium at $t+1$ then the price of the asset at time t is relatively high, which lowers adverse selection at time t. Since in the region $E_L s/E_H s \geq \kappa_P$, adverse selection is already relatively low, the asset must also be traded in a pooling equilibrium at time t if it is traded in a pooling equilibrium at time $t+1$. Using a similar logic, we show that if $E_L s/E_H s \leq \kappa_S$ and the collateral asset is traded in a separating equilibrium in the security market at time $t+1$, then it cannot be traded in a pooling equilibrium at time t. Combining these two facts, we see that all equilibria must be stationary.

In the next section, we show that removing the restriction on the set of available securities eliminates the fragility in the economy.

6 Dynamic Flexible Security Design

In this section, we first restriction attention to monotone securities and solve the security design problem given in Definition 2. We show that agent B can use security design to overcome the fragility of the price equilibrium that arises when agents can only trade the underlying asset. We also show the uniqueness of equilibrium does not depend on the restriction of issuing monotone securities. It also obtains when borrowers issue Arrow securities against the dividend payment and the resale value of the asset. In an extension, we show that multiple equilibra might re-emerge when the security design is rigid, that is, when the contract terms of the liquid securities are not updated at the beginning of each period.

6.1 Solving for Optimal Security Design

As a preliminary step, we first show that optimal security design involves at most two securities. One security is always liquid, i.e., traded in a pooling equilibrium, and the other one is illiquid, i.e., traded in a separating equilibrium.

Lemma 1. *If two securities y^j and y^k are both liquid (illiquid), then $y^j + y^k$ is also liquid (illiquid). Moreover, if a security design involves y^j and y^k, replacing the two securities by their combination $y^j + y^k$ is also a feasible security design and provides the same payoff to agent B. Hence, w.l.o.g., we can restrict attention to security design that involves at most two securities, a liquid and an illiquid one.*

Given Lemma 1, we focus on security design with at most one liquid tranche and one illiquid tranche. Given also the fact that the feasibility constraint must be binding, the designer's problem can be simplified into choosing a liquid tranche $y(s)$ and an illiquid tranche $s + \phi - y(s)$. Using (6) the optimal security design simplifies to:

$$\max_{y(s)} (z-1)\left[\lambda(E_L s + \phi) + (1-\lambda)E_H y(s)\right] \tag{15}$$

$$s.t. s + \phi - y(s) \geq 0, \forall s,$$

$$E_L y(s) - \zeta E_H y(s) \geq 0,$$

$$y(s) \text{ is weakly increasing on } [s_L, s_H]. \tag{16}$$

The first constraint above is the feasibility constraint and requires $y(s)$ to be backed by the underlying asset in every state. The second is the pooling constraint and guarantees that the high type agent exchanges the liquid security y for the intermediate good.

Clearly, the liquid tranche in an optimal security design must satisfy $y(s) \geq \phi$ for all $s \in [s_L, s_H]$. Following Ellis, Piccione, and Zhang (2017), we write the monotone security $y(s)$ as:

$$y(s) = \phi + s_L + \int_{s_L}^{s} x(j)dj,$$

where $x(j) \geq 0$ for all $j \in [s_L, s_H]$.[15] Let $\widetilde{F}_Q(s) = 1 - F_Q(s)$ for $Q \in \{L, H\}$ and $s \in [s_L, s_H]$. Then,

$$E_Q y(s) = \phi + s_L + \int_{s_L}^{s_H} \widetilde{F}_Q(j)x(j)dj.$$

[15] In our analysis, we restrict attention to securities that can be written as the sum of an absolutely continuous increasing function and countably many jump points.

Hence, the optimal security design problem (15) is equivalent to the following problem:

$$\arg\max_{x} \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds, \tag{17}$$

$$s.t. \int_{s_L}^{s} x(j)dj \leq s - s_L, \forall s \in [s_L, s_H], \tag{18}$$

$$\int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] x(s)ds + (1-\zeta)\phi \geq 0, \tag{19}$$

$$x(s) \geq 0, \forall s \in [s_L, s_H] \tag{20}$$

In the above problem, (18) corresponds to the feasibility constraint, (19) corresponds to the pooling constraint and (20) guarantees that the security is monotone.

The next proposition shows that, as long as $f_L(s)/f_H(s)$ is decreasing, the optimal liquid tranche is a debt contract with face value $D = \phi + \delta$.

Proposition 3. *Assume that* $\frac{f_L(s)}{f_H(s)}$ *is decreasing in* s*. The optimal liquid security is a standard debt contract* y_D *such that*

$$y_D(s) = \phi + \min(s, \delta),$$

for some $\delta \in (s_L, s_H]$*. The optimal illiquid security is the residual equity tranche* $y_E(s) = \max(0, s - \delta)$*. Moreover,* δ *is unique for a given* ϕ*.*

To prove this proposition, we write the Lagrangian for the optimization problem (17). Letting x^* be the solution to this problem, we then show that there is a unique cutoff $\delta \in (s_L, s_H]$ below which $x^*(s) = 1$ and above which $x^*(s) = 0$. Put differently, $y_D(s) = \phi + s$ when $s \leq \delta$, i.e., it promises the resale price and all of the dividend in the low states, and $y_D(s) = \phi + \delta$ when $s > \delta$, i.e., it promises a flat payoff in the high states. Hence, the optimal liquid security must be a debt contract. We refer to $D = \phi + \delta$ as the face value of the debt contract.

6.2 Characterizing the Optimal Liquid Security

Since $x^*(s) = 1$ for $s < \delta$ and $x^*(s) = 0$ for $s \geq \delta$, the optimization problem in (17) and the associated constraints (18)-(20) can be simplified as

$$\max_{\delta \in [s_L, s_H]} \int_{s_L}^{\delta} \widetilde{F}_H(s)ds \tag{21}$$

subject to

$$(z-1)\left[\phi + s_L + \lambda\int_{s_L}^{\delta} \widetilde{F}_L(s)ds + (1-\lambda)\int_{s_L}^{\delta} \widetilde{F}_H(s)ds\right] \geq \lambda\int_{s_L}^{\delta}\left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right]ds, \tag{22}$$

where the constraint guarantees that B types pool and issue the liquid debt.

To complete the characterization of optimal liquid security, we solve for the equilibrium given in Definition 2. Note that the equilibrium boils down to solving (21) to find the optimal debt threshold level $\delta \in (s_L, s_H]$ given the asset price ϕ and determining the asset price ϕ in the asset market through the corresponding Euler equation:

$$\phi = \beta \left\{ z \left(q_D + \lambda q_E \right) + (1 - \lambda) \int_{\delta}^{s_H} \widetilde{F}_H(s) ds \right\}. \tag{23}$$

Proposition 4. *Assume that* $\frac{f_L(s)}{f_H(s)}$ *is decreasing in* s*. If* $E_L s / E_H s < 1 - (z-1) / (z\lambda (1-\beta))$*, there is a unique equilibrium where the debt threshold* $\delta \in (s_L, s_H)$ *and the asset price* ϕ *are solutions to the following two equations:*

$$\phi = \frac{z}{z-1} \lambda \int_{s_L}^{\delta} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s) \right] ds - \int_{s_L}^{\delta} \widetilde{F}_H(s) ds - s_L \tag{24}$$

$$\phi = \frac{\beta}{1 - \beta z} \left\{ z \left[\lambda E_L s + (1-\lambda) E_H s \right] - (1-\lambda)(z-1) \int_{\delta}^{s_H} \widetilde{F}_H(s) ds \right\} \tag{25}$$

Otherwise, there is a unique equilibrium where $\delta = s_H$ *and* $\phi = \frac{\beta}{1-\beta z} z \left[\lambda E_L s + (1-\lambda) E_H s \right]$*. Moreover, in the former case, the equilibrium of the security design problem strictly Pareto dominates the (unique) separating equilibrium of the baseline case. In the latter case, it strictly Pareto dominates the separating equilibrium of the baseline case and replicates the pooling equilibrium.*

The formal proof of the proposition is in the Appendix. We provide an intuitive discussion of this result and the economic mechanism behind it in the next subsection. The following corollary follows immediately from Proposition 4.

Corollary 1. *Under the welfare improving security design equilibrium, there is nontrivial tranching when* $E_L s / E_H s < 1 - (1-\zeta)/(1-\beta)$.

Note that this condition is the same condition for the left boundary of multiple equilibria region in (14), indicating that security design improves the liquidity of the unique separating regime when only equity is allowed to be traded.

Finally, once δ is pinned down, we can write the prices of the illiquid equity tranche y_E and the liquid debt tranche y_D as:

$$q_E = \int_{\delta}^{s_H} \widetilde{F}_L(s) ds,$$

$$q_D = \phi + s_L + \lambda \int_{s_L}^{\delta} \widetilde{F}_L(s) ds + (1-\lambda) \int_{s_L}^{\delta} \widetilde{F}_H(s) ds.$$

Next, we discuss some implications of these results and present several extensions.

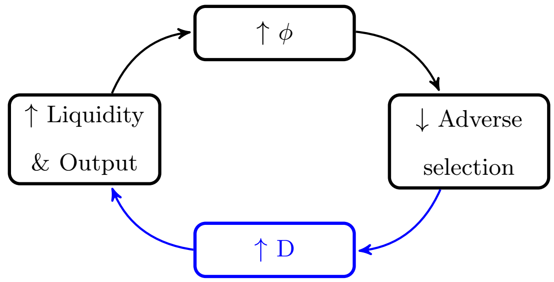

Figure 2: Asset Price ϕ and Liquid Debt Face Value $\phi + D$

6.3 Discussions and Extensions

6.3.1 Dynamic Coordination and Uniqueness with Flexible Design

In this section, we compare the results from Section 5 and the optimal security design problem of Section 6.2 and discuss the underlying mechanism based on dynamic coordination. The discussion also sheds light on the following two differences relative to the baseline case: First, with flexible security design there is nontrivial welfare improving tranching in the separating equilibrium region, and second, the pooling equilibrium is selected as the unique equilibrium in the multiple equilibria region.

Figure 2 illustrates the feedback loop between the asset price, which depends on future value of the collateral, and the current face value of the debt contract. The face value of the liquid debt, $D = \phi + \delta$, incorporates the resale price, ϕ, in addition to the debt threshold backed by dividend, δ. As the face value increases, more of the dividend states are pledged as collateral, more funds are raised for the productive sector and the real output increases. The feedback loop involves intertemporal coordination since the increase in real output in future periods leads to an increase in today's collateral asset price ϕ. A higher asset price is incorporated into the face value of debt alleviating the adverse selection problem (i.e., the adverse selection ratio for the debt, R, is now lower) and allowing even more dividend to be pledged as collateral.

To understand this mechanism, we revisit the equilibrium construction in the optimal security design equilibrium. Suppose that the security designer sells a liquid debt tranche with a face value $D = \phi + \delta$ and an illiquid equity tranche. Recall that the asset price ϕ in the asset market, after substituting for the prices of the debt and equity tranches q_D and q_E, is given by the Euler equation:

$$\phi = \frac{\beta}{1 - \beta z} \left\{ z \left[\lambda E_L s + (1 - \lambda) E_H s \right] - (1 - \lambda)(z - 1) \int_{\delta}^{s_H} \widetilde{F}_H(s) ds \right\}. \tag{26}$$

According to Equation (26), if security design allows agents to coordinate on a higher debt threshold tomorrow, the asset price today will be higher, since ϕ is increasing in δ.

For any δ, let $\phi(\delta)$ be the asset price in the asset market satisfying (26). Let $\underline{\phi} = \phi(s_L)$ and $\phi^P = \phi(s_H)$. Recall from Section 5 that ϕ^S is the asset price when only the low type sells the asset and high type retains both the resale price and the dividend. In contrast, the asset price calculation in (26) assumes that both types of borrowers sell (liquid) debt claims backed by the future resale price at the minimal as collateral. As a result, $\underline{\phi} > \phi^S$. On the other hand, $\phi(s_H)$ is the same as the pooling price ϕ^P. To see this, note that ϕ^P is calculated assuming that both types use the resale price and the entire dividend of the asset as collateral, which is equivalent to setting the face value of the liquid debt contract to $\phi^P + s_H$. The solid line in Figure 3 depicts the function $\phi(\delta)$.[16]

Next, consider the designer's choice of debt threshold, δ, as a function of the asset price ϕ. Optimal security design chooses δ as large as possible making sure that the debt tranche is liquid. As δ increases, the debt tranche incorporates more of the high dividend states. If δ is too high, the high type, who knows that those states are likely, might prefer to retain the debt tranche rather than pool with the low type. Hence, a flexible optimal security design allows δ be pushed up to the point where the high type is indifferent between selling or retaining the debt. As the asset price increases, selling the debt tranche becomes more attractive to the high type, allowing the security designer to increase δ. Crucially, optimal flexible security design solves the coordination problem that we observed in the baseline case where lenders face strategic uncertainty about the high type's participation in the security market. Optimal security design eliminates this uncertainty by ensuring that both types participate in trading the debt tranche.

The dash dotted line in Figure 3 depicts the function $\delta(\phi)$ for the case $E_L s/E_H s < 1-(1-\zeta)/(1-\beta)$.[17] The figure illustrates that regardless of how low the asset price is, as long as tranching is feasible, optimal security design involves a debt tranche that incorporates some dividend. That is, $\delta\left(\underline{\phi}\right) > s_L$. This is a robust feature of security design that holds regardless of underlying parameters. Also note that in the region depicted, adverse selection is severe, and even when the asset price is as high as possible, the high type prefers to retain the equity tranche. That is, $\delta\left(\phi^P\right) < s_H$.

Using these two curves, $\phi(\delta)$ and $\delta(\phi)$, we can find the equilibrium values (δ^*, ϕ^*). The equilibrium is where the two curves intersect, i.e., when $\phi^* = \phi(\delta^*)$ and $\delta^* = \delta(\phi^*)$. As Figure 3 shows, when $E_L s/E_H s < 1-(1-\zeta)/(1-\beta)$, the equilibrium debt threshold $\delta^* \in (s_L, s_H)$. This explains the first difference relative to the baseline case.

[16] Note that ϕ is strictly increasing for $\delta \in [s_L, s_H)$, $\partial\phi/\partial\delta$ is decreasing and is zero at $\delta = s_H$.

[17] Recall that this is the left boundary of multiple equilibria region in 14. In this region, adverse selection leads to a unique separating equilibrium without security design.

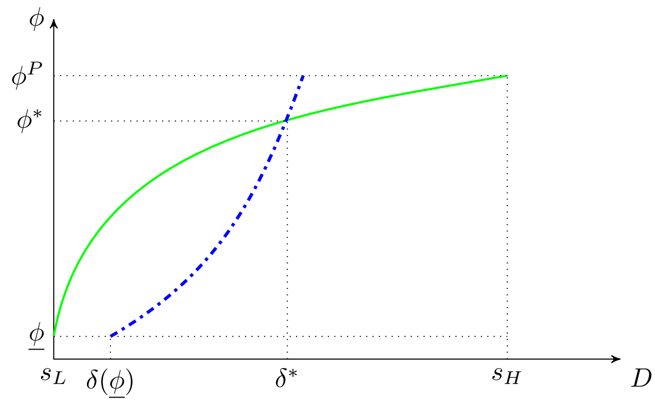

Figure 3: $\phi(\delta)$ and $\delta(\phi)$ when $E_L s/E_H s < 1-(z-1)/(s\lambda(1-\beta))$.

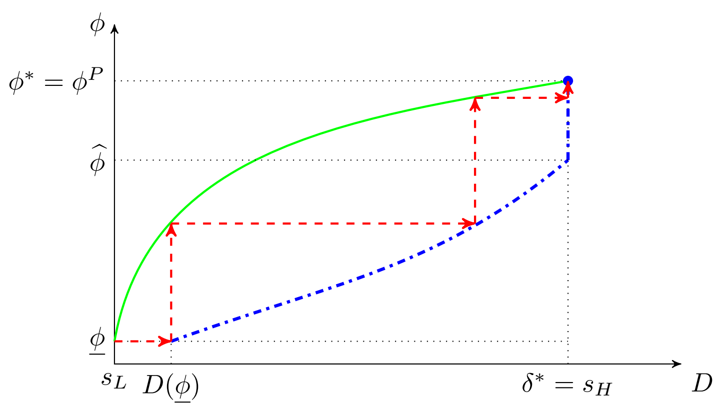

Figure 4: $\phi(\delta)$ and $\delta(\phi)$ when $E_L s/E_H s > 1-(z-1)/(s\lambda(1-\beta))$.

Perhaps more interesting is the case when $E_L s/E_H s > 1-(1-\zeta)/(1-\beta)$ given in Figure 4 where the second difference arises. In this case, adverse selection is less severe and $\delta(\phi)$ function is shifted to the right as the same asset price can sustain a higher face value of the liquid debt. When the asset price is above a threshold denoted by $\widehat{\phi}$, optimal security design incorporates all dividend states s_H to the face value of debt, which is captured by the vertical part of $\delta(\phi)$ function. The two curves intersect only at the upper right corner, $\left(s_H, \overline{\phi}\right)$. As a result, there is a unique equilibrium for the security design problem and it involves setting the debt threshold $\delta^* = s_H$.

The scenario depicted in Figure 4 may seem surprising since, as we illustrated in Section 5, without the possibility of security design there is a coordination problem leading to multiple equilibria in part of this region. Security design solves this coordination problem, and we obtain a unique equilibrium in which agent B sells the entire "pass-through" debt tranche in a pooling equilibrium. To understand this, note that without security design the high type decides among only two options: whether to use the resale price and dividend of the asset as collateral versus retaining both parts. The outcome depends on the asset price. In the good equilibrium $\phi = \phi^P$ and the high type sells the asset. In the bad equilibrium, $\phi = \phi^S$ and the high type retains the asset. The bad equilibrium cannot survive with security design because even if the asset price were ϕ^S, the optimal security design would be able to improve this separating equilibrium by creating a liquid debt tranche with the face value ϕ^S, which in turn would increase the asset price above ϕ^S. Both graphs in Figures 3 and 4 in fact show that the equilibrium asset price in the optimal security design equilibrium is not less than $\underline{\phi} = \phi(s_L) > \phi^S$ (since the face value of the liquid debt is never below $\underline{\phi} + s_L$). Given the increase in the asset price to $\underline{\phi}$ from ϕ^S, the high type's participation constraint is relaxed, which leads to the optimal security design to incorporate more of the dividend into the debt tranche (that is, $\delta > s_L$). A higher δ will increase the asset price ϕ and so on, triggering the dynamic price feedback loop. This unravelling process is illustrated in Figure 4 with the dashed arrows. As the figure shows when the asset price is $\underline{\phi}$, the face value of the debt rises to $\underline{\phi} + \delta\left(\underline{\phi}\right)$. When the face value of the debt increases to $\underline{\phi} + \delta\left(\underline{\phi}\right)$, the asset price further increases, and so on. The process ends when the price rises to ϕ^P.

6.3.2 Extension to Arrow Securities

In this section, we show that the uniqueness of equilibrium does not depend on the restriction to issuing monotone securities. The following proposition shows that if borrowers can issue Arrow securities against the dividend payment and back liquid securities by pledging the resale value of the asset, a unique equilibrium always exists.

Proposition 5. *Assume that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s. The optimal security design under Arrow securities has two tranches, the liquid tranche $y_{1t}(s)$ and illiquid tranche $y_{2t}(s)$.*

$$y_1(s) = \phi + s_L + (s - s_L)\mathbb{I}(s \leq \delta),$$
$$y_2(s) = (s - s_L)\mathbb{I}(s > \delta).$$

If $E_L s/E_H s < 1-(1-\zeta)/(1-\beta)$, there is a unique equilibrium for the optimal design, where $\delta \in (s_L, s_H)$

and the asset price ϕ are solutions to the following two equations:

$$\phi = \frac{\int_{s_L}^{\delta} sdF_H(s) - z\int_{s_L}^{\delta} sdF_\lambda(s)}{z-1} \tag{27}$$

$$\phi = \frac{\beta}{1-\beta z}\left[z\int sdF_\lambda(s) - (z-1)(1-\lambda)\int_{\delta}^{s_H} sdF_H(s)\right] \tag{28}$$

where $F_\lambda(s) = \lambda F_L(s) + (1-\lambda)F_H(s)$. Otherwise, there is a unique equilibrium where $\delta = s_H$ and $\phi = \frac{\beta}{1-\beta z} z\left[\lambda E_L s + (1-\lambda)E_H s\right]$.

As in Proposition 4, in the former case, the equilibrium of the security design problem strictly Pareto dominates all equilibria of the case where only the equity claim of the asset can be used to exchange for the intermediate goods liquidity. In the latter case, the equilibrium of the security design problem strictly Pareto dominates the separating equilibrium of the case where only the asset can be used as collateral and replicates the pooling equilibrium.

6.3.3 Contract Rigidity and Sunspot Runs

Our main model shows that flexible security design eliminates fragility and improves welfare. In this section, we highlight the role of flexibility, that is, the ability of borrowers to design securities and change the terms of the contracts at the beginning of each period, plays in delivering this strong result. Although flexible design is an important benchmark, in practice contract terms may not be updated daily because of associated administrative costs or simply inattention. As we show in this section, the resulting rigidity may be a crucial source of fragility that causes sunspot runs. Moreover, the possibility of runs leads to a more conservative liquid security design in the sense that the liquid tranche is smaller relative to that under the flexible security design.

To capture rigidity, in any period, we allow the economy to be in one of two regimes, 0 or 1. As will become clear later, regime 1 is the run regime and regime 0 is the normal regime. The transition between these two regimes depends on whether security design is flexible or rigid as well as the arrival of a sunspot, which indicates negative sentiment. Specifically, if in the beginning of period t the contract design is rigid and a sunspot occurs, then the economy moves from the normal regime 0 to the run regime 1. This event happens with probability $\chi \geq 0$. Once the economy enters the run regime 1, it returns to the normal regime 0 only if the design is updated. That is, if the economy is in regime 1 at time $t-1$, then it returns to regime 0 if the design becomes flexible in the beginning of period t. This event happens with probability $1-\gamma$ where $\gamma > \chi$. Note that our main model is a special case where $\gamma = 0$.

In this more general model, the value of the asset depends on the regime $i \in \{0, 1\}$ which we denote by v_i. The asset price, which is determined in the asset market at the end of each period, takes into account the possible values of the asset in the following period. Hence, it is distinct from the asset value and depends on the regime. We denote the asset price in regime i by ϕ_i. Similarly, security prices depend on the asset price and hence are also regime dependent. We denote the price of security j in regime i by q_i^j. In addition, even though the security design remains the same in both regimes, because the asset price changes, the adverse selection ratios of securities will also change. We denote the adverse selection ratio of security j in regime i by R_i^j. The formal definition of a sunspot equilibrium which modifies Definition 2 to allow for these regime dependencies is given next.

Definition 3. Let $i \in \{0, 1\}$ denote the regime. A dynamic stationary equilibrium with rigidity consists of asset values and prices v_i and ϕ_i, security design (with monotone securities) $\mathcal{J}(\phi_0) \subseteq \mathcal{I}(\phi_0)$ and security prices q_i^j for each $j \in \mathcal{J}(\phi_0)$ such that

(i) asset prices in states 0 and 1 are given by $\phi_0 = (1-\chi)v_0 + \chi v_1$ and $\phi_1 = \gamma v_1 + (1-\gamma)v_0$,

(ii) $\mathcal{J}(\phi_0)$ solves the security design problem (2),

(iii) security price q_i^j satisfies equation (3) for $i \in \{0, 1\}$, and

(iv) asset value v_i solves the Euler equations given by:

$$v_i = \beta \left[z \left(\sum_{j \in P_i} q_i^j + \lambda \sum_{j \in \mathcal{J}(\phi_0) \setminus P_i} q_i^j \right) + (1-\lambda) \sum_{j \in \mathcal{J}(\phi_0) \setminus P_i} E_H y^j \right], \tag{29}$$

where $j \in P_i \subseteq \mathcal{J}(\phi_0)$ iff $R_i^j \geq \zeta$.

We call such an equilibrium a run equilibrium if $v_0 > v_1$.

Note that agent B designs the securities under the asset price ϕ_0. This is because security design is only possible if the economy is in regime 0 in the beginning of a given period.[18] Under the assumption that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s, following steps similar to the one in the main model, the optimal security is still a standard debt contract. To distinguish the debt cutoff of the contract under rigidity from the one in the main model, we denote it by δ_0.

The equilibrium characterized in Proposition 4 remains an equilibrium even with rigidity. To see why, suppose the asset value is the same under the two regimes, i.e., $v_0 = v_1$, then the asset prices do not depend on the regime, i.e., $\phi_0 = \phi_1$. Consequently, $R_0^j = R_1^j$ and the set of liquid securities are the same in the two states, i.e., $P_0 = P_1$. Specifically, the debt tranche remains liquid under both regimes which in turn justifies the fact that the asset value does not depend on the regime.

[18]By definition the asset price is ϕ_0 at the end of a period in which the economy is in regime 0.

Due to the dynamic price feedback, under rigidity a run equilibrium is also possible. In this equilibrium, the asset value and the asset price drop whenever the economy enters regime 1. As a result, the adverse selection ratio R_i^j of the debt tranche decreases, triggering a run where the debt tranche that was previously liquid becomes illiquid, which in turn justifies the drop in the asset value and the asset price.[19] Had the design been flexible, agent B would redesign the security in this event by lowering the debt threshold to make sure that the debt tranche remains liquid. This action would push the asset price up, and as we discussed before, this process would automatically lead to a full recovery of prices and the debt threshold. However, when the design is rigid, the drop in asset price can be self-fulfilling.

The following proposition characterizes the run equilibrium in which the debt tranche becomes illiquid when economy is in regime 1.

Proposition 6. *There exists a cutoff $\mathit{\Gamma}(\chi,\gamma)$ that is increasing in γ and χ with*

$$\mathit{\Gamma}(\chi,\gamma) > 1 - \frac{z-1}{z\lambda(1-\beta)}$$

such that whenever

$$\frac{E_L s}{E_H s} < \mathit{\Gamma}(\chi,\gamma), \tag{30}$$

(i) There exists a run equilibrium, i.e., $v_0 > v_1$ and $\phi_0 > \phi_1$. (ii) Both values v_0 and v_1, and hence prices ϕ_0 and ϕ_1 are lower than the price ϕ in the equilibrium without a run. (iii) The debt threshold δ_0 in the run equilibrium is strictly lower than δ without a run. Consequently, welfare in the run equilibrium is Pareto dominated by the equilibrium without a run.

The above proposition states that with rigidity, in part of the pooling region of the baseline case, i.e., when $E_L s/E_H s \in (1-(z-1)/(z\lambda(1-\beta)), \mathit{\Gamma}(\chi,\gamma))$, the optimal design is more conservative and pass-through equity is no longer the optimal security. The reason is once again due to dynamic coordination. With rigidity when the sunspot hits in future periods, the optimal security becomes illiquid. This effect leads to a lower current asset price, which increases adverse selection and pushes down the debt threshold.

7 Implementation as Short-Term Repo Contracts

In this section, we describe how the optimal security can be implemented as a repo contract. We define in Section 7.1 the terms of repo contracts in the context of our model. These terms of the contract are endogenous which allows our theory to offer a perspective on how adverse selection affects the terms

[19] During the run, the price of the debt tranche drops to $q_{1D} = \phi_1 + s_L + \int_{s_L}^{\delta_1} \tilde{F}_L(s)ds$.

of short-term repo contracts backed by long-term assets. In Section 7.2, we provide three examples to understand the properties of the repo contracts and discuss its economic implications. The first two examples show that the repo rates reflect mainly the demand for liquidity and the riskiness of the repo debt, whereas repo haircuts reflect mainly the underlying information frictions. In particular, when asset quality deteriorates, somewhat counterintuitively, repo rates may decrease. Haircuts, on the other hand, can sharply increase as expected. The third example provides an extension where asset quality or productivity may be persistent and provides some quantitative predictions about the amplification of shocks resulting from the dynamic feedback loop. Section 7.3 shows that, using the results on sunspot runs from 6.3.3, slow building repo runs can occur when repo contracts have rigidity.

7.1 Terms of Repo

The optimal security design in our model can be implemented by a one-period repo contract, which is liquid, and an equity-like contract, which is illiquid. In this implementation, the repo borrower pledges the resale price of the collateral as well as any interim cash flow generated by the collateral (e.g., dividend, or accrued interest payment) to obtain repo debt. As in the standard repo contracts, the repo lender returns the collateral at the end of each period if the borrower pays back the repo debt with interest but keeps the collateral if the borrower fails to do so. The borrower can also choose to issue equity by pledging the residual cashflow from the collateral (i.e., the remaining cash flow – if there is any – once repo debt obligations including principal and interest are paid off).

The face value of the repo contract is

$$D = \phi + \delta.$$

The expected value of the repo contract for the lender is

$$q_D = \phi + s_L + \int_{s_L}^{\delta} \left[\lambda \tilde{F}_L(s) + (1-\lambda)\tilde{F}_H(s)\right] ds.$$

The value of collateral underlying the repo contract at the beginning of a period to the productive borrowers is

$$\phi/\beta = z\phi + z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{\delta}^{s_H} \tilde{F}_H(s)ds$$

The last term reflects the loss of value from the illiquid equity tranche.

We are now ready to state the terms of the repo contract, including repo rate, R, and haircut, h. The definition of repo rate is straightforward:

$$R = \frac{\text{face value}}{\text{expected loan value}} - 1 = \frac{D - q_D}{q_D}. \tag{31}$$

From the definition of R, we can observe that the relationship between asset quality and interest rate is not straightforward because asset quality has two opposing effects on the repo rate. When asset quality worsens (improves), the expected value of the repo contract is lower (higher), leading to a high (low) repo rate. At the same time, the face value of the debt might be adjusted down (up), resulting in a lower (higher) repo rate.

The definition of repo haircut in our model is:

$$h = 1 - \frac{\text{expected loan value}}{\text{collateral value}} = \frac{(z-1)q_D + \lambda z q_E + (1-\lambda)\int_{\delta}^{s_H} \tilde{F}_H(s)ds}{\phi/\beta}$$
$$\approx \underbrace{(z-1)}_{\text{productivity}} + \underbrace{\frac{\int_{\delta}^{s_H}\left[\lambda \tilde{F}_L(s) + (1-\lambda)\tilde{F}_H(s)\right]ds}{\phi/\beta}}_{\text{equity/collateral}}, \text{ if } z \text{ is close to 1.} \tag{32}$$

Observe that the repo haircut has two main components: the productivity of the borrower's technology, and the value of the equity tranche relative to the value of the collateral. The first component arises because borrowers, who price the collateral asset, value the liquidity service of the asset, while lenders, who price the loan, do not value it. The term $z-1$ is the net value of the liquidity service; it reflects heterogeneous valuation over the collateral assets between lenders and borrowers in our model. This component is similar to the difference-of-opinion explanation of haircut in Geanakoplos (2003) and Fostel and Geanakoplos (2012). The second is the value of the equity tranche relative to the value of the collateral and arises mechanically because equity tranche by definition is excluded from the repo debt. This component responds strongly to the level of information friction. A similar connection between haircuts and information friction also features in Dang, Gorton, and Holmström (2011) and Gorton and Ordonez (2014).

7.2 Properties of Repo Contracts

In this section, we look at properties of the repo contracts with three examples.

Example 1: Two-point distribution. In this example, we consider simple two point dividend distributions in order to provide closed form solutions to the terms of the repo contract and provide comparative statics.

Suppose that the high (low) quality asset pays one unit of dividend with probability π_H (π_L) and pays zero otherwise. Assume $0 < \pi_L < \pi_H < 1$. In this example, the debt contract becomes very simple.

Regardless of the realization of the dividend, it pays the resale price ϕ. In addition, it pays δ units of the dividend if the dividend is one.

In this case, we can write the expressions for the debt threshold, δ, and the asset price, ϕ, as follows:

$$\delta = \frac{\frac{\beta}{1-\beta z}\left[z\lambda\pi_L + (1-\lambda)\pi_H\right]}{\frac{z}{z-1}\lambda\left(\pi_H - \pi_L\right) - \frac{1-\beta z+\beta(1-\lambda)(z-1)}{1-\beta z}\pi_H},$$

$$\phi = \frac{\beta\left[z\lambda\pi_L + (1-\lambda)\pi_H\right]}{1-\beta z - \beta\frac{(1-\lambda)(z-1)\pi_H}{\frac{z}{z-1}\lambda(\pi_H-\pi_L)-\pi_H}}.$$

Using these expressions, we see that both the resale price ϕ and the debt cutoff δ are decreasing in the probability that the asset is low quality, i.e., $\frac{d\phi}{d\lambda} < 0$ and $\frac{d\delta}{d\lambda} < 0$, and increasing in productivity z, i.e., $\frac{\partial\phi}{\partial z} > 0$ and $\frac{\partial\delta}{\partial z} > 0$. An immediate implication is that more debt is created in good times since $\frac{\partial D}{\partial z} = \frac{\partial(\phi+\delta)}{\partial z} > 0$.

The terms of the repo contract can also be expressed in closed form and allow us to examine the determinants of repo rates and haircuts in this particular example. The repo rate is expressed as

$$R = \left[\frac{1-\pi_H}{\lambda(\pi_H - \pi_L)} + 1\right](z-1)\,. \tag{33}$$

The repo rate is increasing in the productivity of technology z, which measures the demand for liquidity from the productive borrowers. It is also clear that repo rate is decreasing in λ. This result might seem counterintuitive since a worsening (improving) asset quality leads to a lower (higher) repo rate. This effect reflects the fact that the face value of the repo debt decreases (increases) significantly to eliminate (incorporate) risky states. To give another perspective on how the repo rate is related to the riskiness of the cashflow and information frictions, we use the incentive constraint of the high-quality seller of the repo contract, $zq_D = \pi_H\delta + \phi$, to rewrite (33) as,

$$R = \frac{\phi+\delta}{q_D} - 1 = \underbrace{\frac{\phi+\delta}{\phi+\pi_H\delta}}_{\text{Cashflow Riskiness}} \underbrace{z}_{\text{Productivity}} - 1. \tag{34}$$

Taking repo debt face value $\phi+\delta$ as given, (34) implies that the interest rate depends on the riskiness of the high quality assets directly. The degree of information friction plays an indirect role through debt face value. In fact, if the high-quality asset pays dividend for sure, (33) implies that the repo rate R is $z-1$. In this extreme case the repo rate is insensitive to changes in asset quality and driven purely by the productivity of the borrowers, which measures their liquidity demand. This example illustrates that in our model, repo rates depend more on the demand for liquidity and cashflow riskiness of the repo contract and less on the asset quality. This relation is due to the nature of security design: both high- and low-quality borrowers participate in the repo market and repo debts are free from adverse selection.

The repo haircut in this example can be expressed as

$$h = 1 - \beta + \frac{\beta}{1 - \frac{\lambda(\pi_H - \pi_L)}{(z-1)[(1-\lambda)\pi_H + \lambda\pi_L]}}. \tag{35}$$

Suppose z and β are close to 1, from (35) we then have

$$h \simeq \underbrace{(z-1)}_{\text{Productivity}} \left[1 - \underbrace{\frac{\pi_H}{\lambda(\pi_H - \pi_L)}}_{\text{Information Friction}} \right] + 1 - \beta.$$

It demonstrates again the two components in repo haircut highlighted in (32): one is related to the liquidity services of the collateral due to the technology productivity z, and the other is related to the ratio of equity tranche over the collateral asset, which is pinned down by the information friction $\lambda(\pi_H - \pi_L)$. In this example, $\partial h/\partial\lambda = ((z-1)\pi_H) / \left(\lambda^2 (\pi_H - \pi_L)\right) > 0$. That is, as the asset quality deteriorates, haircut monotonically increases. Furthermore, haircut is also increasing in the quality difference between high and low type $\pi_H - \pi_L$, a measure of severity of adverse selection. In general, however, the quality of the collateral asset has a nonmonotonic impact on the haircut. This is because of two opposing effects. First, when the quality of collateral asset deteriorates, the value of the underlying collateral goes down which lowers the value of equity tranche. Second, at the same time, the debt threshold decreases which enlarges the size of the equity tranche and hence increases its value potentially. To summarize, this example again demonstrates that the haircut of a repo contract is a robust indicator of information frictions over the asset quality, while the repo rate reflects the demand for liquidity and the cashflow riskiness of the repo contract.

Example 2: Portfolio repo to improve asset quality In this example, we extend Example 1 to illustrate that pooling safe assets with the collateral asset exposed to information frictions can improve the liquidity of the repo market. Denote the fraction of safe assets in the asset pool as ω. To keep the example tractable, we assume that the asset pool pays 0 or 1 with probability. This simplifying assumption is not essential, and it arises naturally if we think of the asset pool as being held by a special purpose vehicle (SPV) that issues debt contract with face value 1 against the pool. Then, the owners of the debt issued by the SPV face a sequential service constraint when it defaults.

Given ω and the quality of the collateral $Q \in \{L, H\}$, the probability that the pool pays 1 is $\omega + (1-\omega)\pi_Q$. This outcome occurs because with probability π_Q, the risky asset pays 1 and every depositor receives 1 from the pool, and with probability $1 - \pi_Q$, the risky asset pays 0 and each depositor receives 1 with probability ω. Hence, we can modify the expressions for the debt threshold, δ, and the asset price, ϕ, as:

$$\delta = \frac{1}{(1-\omega)\left[\frac{z}{z-1}\lambda\left(\pi_H - \pi_L\right) - \pi_H\right] - \omega}\phi,$$
$$\phi = \frac{\beta\left[\omega\left(z\lambda + (1-\lambda)\right) + (1-\omega)\left(z\lambda\pi_L + (1-\lambda)\pi_H\right)\right]}{1 - \beta z - \beta\frac{(1-\lambda)(z-1)[\omega+(1-\omega)\pi_H]}{(1-\omega)\left[\frac{z}{z-1}\lambda(\pi_H-\pi_L)-\pi_H\right]-\omega}}.$$

From these expressions we see that portfolio repo improves the liquidity of the collateral asset and the asset price since both the asset price and the debt threshold of the asset pool are increasing in ω.

The liquidity improvement also shows up in the term of the repo contract. Let the interest and haircut of the pool be R^{ω} and $h^{\omega}$and those of the standalone collateral asset be R^0 and h^0.

$$R^{\omega} = R^0 = \left[\frac{1-\pi_H}{\lambda(\pi_H - \pi_L)} + 1\right](z-1) \tag{36}$$

It is immediately clear that the interest rate does not respond to ω, which is what we have learned in Example 1, i.e., that the interest rate does not respond to information frictions.

The haircut is

$$h^{\omega} = 1 - \beta + \frac{\beta}{1 - \frac{\lambda(1-\omega)(\pi_H-\pi_L)}{(z-1)[\omega+(1-\omega)((1-\lambda)\pi_H+\lambda\pi_L)]}}.$$

Suppose z and β are close to 1; we show that

$$h^{\omega} \simeq 1 - \beta + (z-1)\left[1 - \frac{\frac{\omega}{1-\omega} + \pi_H}{\lambda\left(\pi_H - \pi_L\right)}\right] = h^0 - \frac{\omega(z-1)}{(1-\omega)\lambda(\pi_H - \pi_L)}. \tag{37}$$

The haircut decreases in ω since pooling the collateral asset with safe assets reduces information friction.

This result is consistent with empirical findings. In particular, Julliard et al. (2018) find that repo contracts backed by a portfolio including AAA rated assets receive (statistically significant) 0.9% to 1.15% lower haircut compared with repo contracts without any AAA rated assets, controlling for counterparty and collateral characteristics.

Example 3: Markov process for asset quality and project productivity In this example, we introduce Markov processes for asset quality and project productivity. Assume that the aggregate state x follows a Markov process, and parameters such as asset quality λ and productivity z are functions of the state. Suppose

$$x_{t+1}\begin{cases} = x_t, & \text{with probability } \rho, \\ \sim G(x), & \text{with probability } 1-\rho. \end{cases}$$

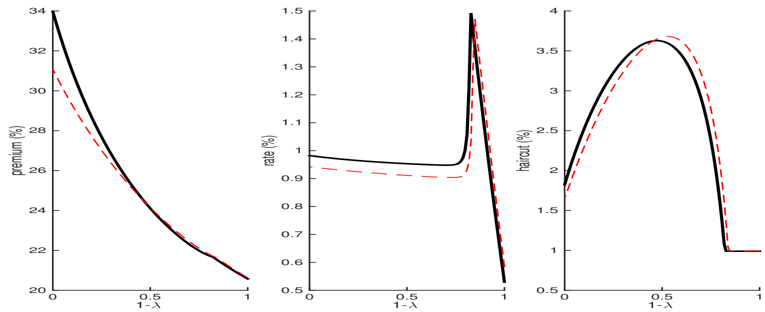

Figure 5: Asset quality, asset price and terms of the repo contract. The parameters for the numerical examples are as follows: high-quality asset dividend follows a beta distribution with $(a, b) = (10, 1)$ and low-quality asset dividend follows a beta distribution with $(a, b) = (0.1, 1)$, $\lambda \sim U[0, 1]$, $\beta = .95$, $z = 1.01$. The solid lines are drawn with $\rho = .95$ and the dashed lines with $\rho = .90$.

We characterize the stationary Markov equilibrium. We assume that period $t+1$ state is publicly revealed after the asset market closes. We denote the value of the asset in state x by v_x. The end of period price of the asset is then given $\phi_x = \rho v_x + (1-\rho)Ev$ where $Ev \equiv \int v_x dG(x)$. The face value of the liquid debt contract is $\phi_x + \delta_x$. In this context, we can write the pooling constraint (24) as

$$\phi_x = \rho v_x + (1-\rho)Ev = \frac{z_x}{z_x - 1}\lambda_x \int_{s_L}^{\delta_x} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds - \int_{s_L}^{\delta_x} \widetilde{F}_H(s) ds - s_L$$

or,

$$v_x = Ev + \frac{1}{\rho}\left\{\frac{z_x}{z_x - 1}\lambda_x \int_{s_L}^{\delta_x} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds - \int_{s_L}^{\delta_x} \widetilde{F}_H(s) ds - s_L - Ev\right\}. \tag{38}$$

From the Euler equation at the end of the period, after the quality of the asset in the period is revealed,

$$v_x = \frac{\beta}{1 - \beta\rho z_x}\left\{z_x\left[\lambda_x E_L s + (1-\lambda_x)E_H s + (1-\rho)Ev\right] - (1-\lambda)(z_x - 1)\int_{\delta_x}^{s_H} \widetilde{F}_H(s) ds\right\} \tag{39}$$

(38) and (39) solve jointly (δ_x, v_x) for all states.

Suppose $z_x = z$ and $\lambda_x = x \in [0, 1]$. Then, with the process, the quality distribution is persistent over time but may change with probability $1-\rho$. When the quality distribution changes, the distribution parameter λ will be drawn from distribution G. We focus on the stationary Markov equilibrium. Security design and the asset price depend on the quality distribution λ. Figure 5 illustrates how the collateral value, interest rate and haircut of the repo contract respond to shocks to quality distribution.

The left subfigure in Figure 5 shows the liquidity premium in the asset price. Let φ_x be the asset value without providing liquidity service:

$$\varphi_x = \frac{\beta\left[(1-\rho)E\varphi + \lambda_x E_L s + (1-\lambda_x)E_H s\right]}{1 - \beta\rho}, \quad E\varphi = \int \varphi_x dG(x).$$

We define liquidity premium as $(v_x - \varphi_x)/\varphi_x$. When λ increases, both v_x and φ_x decrease, that is the value of the collateral decreases whether it provides liquidity services or not. However, because both high- and low-quality assets provide some liquidity service, the value of collateral when it provides liquidity decreases more slowly than when it does not. This is why the liquidity premium decreases in $1 - \lambda$, and the liquidity gain from security design is higher for assets of lower average quality.

Both the haircut (the right subfigure) and the repo rate (the middle subfigure) change nonmonotonically in λ. For the haircut, this is because the value of the equity tranche is nonmonotonic. When the asset quality is on average good (high $1 - \lambda$), the information friction is small enough that there is no need to tranche the cash flow. In this case, no illiquid equity tranche is created. Then, according to (32), the haircut then only reflects heterogeneous valuation over liquidity between borrowers and lenders. When the asset quality is poor on average, the repo tranche is also very likely to default and the value of the equity tranche is small in that case. When the asset quality is in the intermediate range, the adverse selection is severe, and hence, the ratio of equity tranche to the asset is high, resulting in large haircuts.

When there is a non-trivial equity tranche, the interest rate on the repo contract is for the most part decreasing in asset quality, reflecting the declining default probability and loss from default. The uptick in the repo rate reflects the opposing effect of changing asset quality mentioned previously when discussing equation (31): the face value of the debt might increase faster and incorporate more risky dividend states relative to the expected value of repo debt as asset quality improves.

We observe that in this example when adverse selection is strong (high λ and the range where $1 - \lambda$ is low, e.g., $1 - \lambda < 0.6$) the haircut is very sensitive to changes in λ, while the repo rate barely responds. This result is qualitatively consistent with empirical observations during the repo runs where there were rare changes in repo rates but haircuts skyrocketed.

The red dashed lines correspond to lower persistence of the Markov process. The left subfigure shows that when the quality distribution is less persistent (the red dashed line), the collateral value/ liquidity premium is less responsive to the current quality relative to the persistent case (the solid line). The right subfigure shows that for the high-quality states, if the quality is less persistent (the red dashed line), adverse selection becomes more severe in those states, the haircut increases relative to the more persistent case (the solid line). The middle subfigure highlights again that repo rates are less sensitive to quality changes and the low persistent case has a lower interest rate for the most part (e.g., $1 - \lambda < 0.6$), which might reflect lower face values of these repo debts and hence less embedded default risks.

Alternatively, suppose $\lambda_x = \lambda$ and $z_x = (1 - x)z_L + xz_H$. Figure 6 illustrates that when firms are more productive, the liquidity premium is higher, more repo contracts are issued, repo rate and

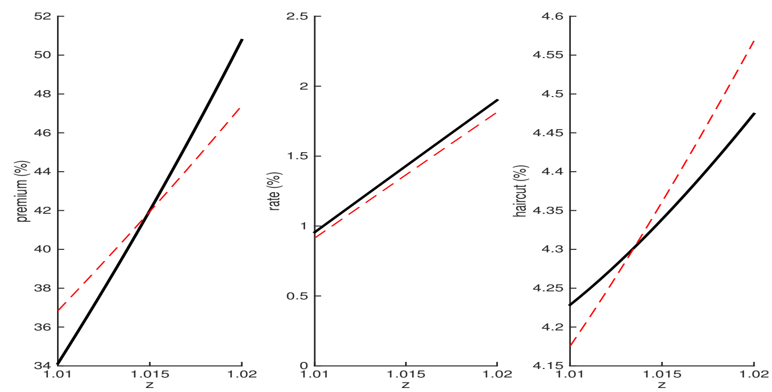

Figure 6: Productivity, asset price and terms of the repo contract. The parameters for the numerical examples are as follows: high-quality asset dividend follows a beta distribution with $(a, b) = (10, 1)$ and low-quality asset dividend follows a beta distribution with $(a, b) = (0.1, 1)$, $\lambda = 0.99$, $\beta = .95$, $z \sim U[1.01, 1.02]$. The solid lines are drawn with $\rho = .95$ and the dashed lines with $\rho = .90$.

haircut increase. As before, the red dashed lines correspond to lower persistence of the Markov process but in productivity rather than asset quality. Explanations for the patterns of the different persistent distributions in productivity are similar to those for the asset quality.

Note that a percentage point increase in productivity causes the liquidity premium of the collateral asset to increase by about 15 percentage points. This amplification of productivity shocks reflects the dynamic feedback effect between the future collateral value and liquidity of the current market. Future collateral value increases in future productivity. This reduces adverse selection in the current market, further increasing the collateral value.

7.3 Repo Runs

In section 6.3.3, we discussed how contract rigidity may be a crucial source of fragility. Take the overnight repo market as an example. Our results imply that when borrowers are able to update the terms of overnight repo contracts each day, the market is robust to run. In practice, the haircut of a repo contract is determined by the value-at-risk assessment of the collateral assets. This assessment is not performed continuously for the bank's risk management team to revise the haircut frequently, which introduces some amount of rigidity in the updating of the contracts. Hence, proposition 6 implies that repo market might be susceptible to runs.

Recall from 6.3.3 that with rigidity there are two possible regimes 0 and 1. Under the repo interpre-

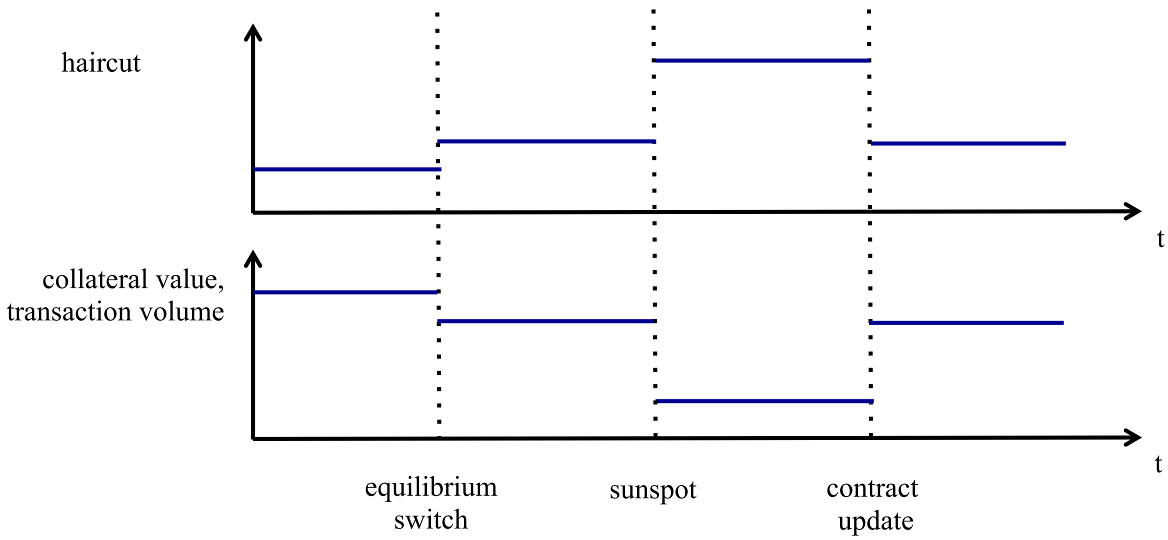

Figure 7: Dynamics of Security run.

tation, the terms of the repo contract are rigid in the sense that the book value of the repo contract, D, does not depend on the regime. However, because in a run the asset price decreases to a lower level, ϕ_1, the effective debt threshold increases from δ_0 to $\delta_1 \equiv \min(s_H, \delta_0 + \phi_0 - \phi_1)$.[20] When investors receive a sunspot during an episode of rigidity, the repo contract becomes illiquid. Only owners with low-quality assets trade the securities. In this scenario, denote the effective interest rate and haircut as R_1^S and h_1^S, respectively, and the price of the debt tranche in regime i as q_{iD}. Note that

$$R_1 = \frac{D - q_{1D}}{q_{1D}}, \tag{40}$$

$$h_1 = 1 - \frac{q_{1D}}{\phi_1/\beta}, \tag{41}$$

and

$$q_{1D} = \phi_1 + s_L + \int_{s_L}^{\delta_1} \tilde{F}_L(s)ds.$$

In the appendix, we show $q_{1D} < q_{0D}$. Then, the repo rate increases when the sunspot arrives, $R_1 > R_0$. The response of the haircut is indeterminate because both $\phi_1 < \phi_0$ and $q_{1D} < q_{0D}$.

The dynamics in a typical repo run is illustrated in Figure 7 where we consider the case in which haircuts increase when a sunspot arrives. In the figure before the moment marked by the equilibrium switch the economy is a "good" equilibrium in which agents expect that the repo tranche will be liquid and

[20]By part (i) of proposition 6, $\phi_0 > \phi_1$ so, $\delta_1 > \delta_0$.

the asset price will be high even when the security design is rigid. After the switch, a repo run typically takes two stages. First, the equilibrium switches to the sunspot equilibrium described in Proposition 6. Once the economy enters a sunspot equilibrium, the haircut of the repo contract immediately increases because investors anticipate that the repo contract will be illiquid when a sunspot hits the economy. At the same time, the asset price and the repo volume decrease. When the sunspot actually hits the economy, asset price and the repo volume decrease further. The repo rate increases further, while the repo haircut may also increase. These effects occur despite the fact that the face value of repo debt remains unchanged due the contract rigidity. The drop in repo volume and the asset price is higher when the sunspot hits because the repo backed by high-quality collateral stops circulating entirely. When the contract terms are updated, the update restores investors' sentiment about the liquidity of the repo market, the price and the volume recover partially to the levels right after the equilibrium switch. The fluctuation driven by sunspots may take place repeatedly as long as the economy remains in the sunspot equilibrium.

Note that the equilibrium switch can be triggered either by a switch of self-fulfilling beliefs from the equilibrium without repo run to an equilibrium with repo run, or by a small shift in the fundamental. Suppose the fundamental of the economy, represented by asset quality λ or productivity z, is initially such that condition (30) does not hold. As the fundamental deteriorates and condition (30) holds, even if the change in fundamentals is very small, a sunspot equilibrium might emerge, leading to a discontinuous drop in market liquidity and the asset price.

8 Conclusion

Our paper studies optimal flexible security design in a dynamic lemons setup. We show that the implementation of optimal security design involves short-term liquid collateralized debt. Because optimal security design helps coordinate investors' intertemporal decisions, the dynamic lemons market under optimal security design is robust to multiple-equilibrium fragility induced by intertemporal miscoordination. We show a dynamic run might occur when contract terms update infrequently. We also explore economic implications of an implementation of optimal security, short term repos, and derive dynamic equilibrium properties of repo rates, haircuts and volume, and aggregate funding liquidity over the productivity and the asset quality cycles.

We conclude by discussing a few potential applications of this framework of dynamic price feedback with security design to highlight its generality. One immediate application could be on security lending.

In the setup of the baseline model, there is a gain to trade since the cash borrowers can use cash to generate more output (i.e., at a multiplier z) than the cash lenders. We can modify this setup and assume also that borrowers value collateral at a discount, u, relative to the cash lenders. Now, there is another gain from trade: the cash borrowers have an incentive to lend out the asset at a lower price due to their low private valuation. In this case, the haircut on the collateral asset might turn negative. The fluctuation of multiplier z and value discount u might explain a time-varying haircut for some firms.[21] Another potential application is on the dynamic pecking order of financing. The classical pecking order theory of Myers and Majluf (1984b) is in a static environment where security issuers are more informed about the future dividend states and can only pledge the dividend when issuing securities. By allowing issuers to also pledge future resale price of the security as in our dynamic price feedback framework, the adverse selection environment might change, which could potentially reverse the static pecking order of financing. In fact, equity might emerge as the most liquid and desired form of financing when the borrowing firm has a highly productive project and suffers relatively less adverse selection regarding its interim dividend cashflows.[22] Finally, the insight that creating a liquid tranche from collateral assets exposed to adverse selection will trigger the dynamic feedback of higher asset prices; hence, a larger liquid tranche could offer new perspectives on how to implement effective monetary policies. For example, quantitative easing, instead of being viewed as a plain liquidity injection, could act as a catalyst in this dynamic price feedback and create more pledgeable liquid tranches from illiquid collateral assets, i.e., a liquid collateral multiplier, when the economy suffers severe adverse selection. We leave these applications for future research.

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[21] During the financial crisis of 2018-2019, AIG changed from charging a haircut to paying a haircut Peirve (2017). According to our theory, this change might be due to AIG's time-varying need for external cash relative to securities.

[22] Fulghieri, Garćia, and Hackbarth (2016) have also extended the static pecking order model to incorporate dynamic considerations by giving the issuing firm access to a growth option for dynamic considerations and have generalized the distribution assumption of dividend cashflows. Our dynamic price feedback framework will add an additional dimension of endogenous security price to this line of the literature.

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A Appendix

A.1 Proof of Proposition (1)

Proof. Let $\overline{q}^j = \lambda E_L y_t^j + (1-\lambda) E_H y_t^j$. Note that $z\overline{q}^j - E_H y_t^j \gtreqless 0$ iff $R_t^j \gtreqless \zeta$.

Consider the case $R_t^j > \zeta$. Suppose that the equilibrium price q_t^j is strictly less than $\overline{q}$. In this case an I agent can deviate and bid $\overline{q} - \epsilon$ where $\epsilon > 0$. For ϵ small enough, $z(\overline{q} - \epsilon) - E_H y_t^j > 0$. Hence at this price both types sell a units of the security and the deviation generates strictly positive surplus. This means that the equilibrium price must be at least $\overline{q}$. At price $\overline{q}$ or above both types will sell a units of the security, hence the only price that is consistent with zero profit condition is $q_t^j = \overline{q}$.

Now consider the case $R_t^j < \zeta$. In this case high type will sell the security only if q_t^j is sufficiently larger than $\overline{q}$. However, at prices above $\overline{q}$, I agents make negative profit. Hence equilibrium price must be below $\overline{q}$. If q_t^j is below $\left(E_L y_t^j\right)/z$ then neither type sells the security. In this case, one of the I agents can deviate and bid $E_L y_t^j - \epsilon$ where $\epsilon > 0$. For ϵ small enough, $z\left(E_L y_t^j - \epsilon\right) - E_L y_t^j > 0$ so the low type sells the security and the deviating agent makes strictly positive surplus. If q_t^j is at least $\left(E_L y_t^j\right)/z$ but less than $E_L y_t^j$ then the low type sells the security to the I agents who bid that price. In this case, one of the I agents who bids $E_L y_t^j$ or less can deviate and bid slightly above q_t^j. This agent then buys the security alone and increases her surplus. At prices greater than equal to $E_L y_t^j$ (and below $\overline{q}$), the low type alone sells a units of the security. Hence the only price that is consistent with zero profit condition is $q_t^j = E_L y_t^j$. □

A.2 Proof of Proposition 2

We use the following lemma to prove the proposition.

Lemma 2. *If $\frac{E_L s}{E_H s} > \kappa_P$ then the security market for the collateral asset cannot be in a separating equilibrium at time t, and a pooling equilibrium at time $t+1$. Conversely, if $\frac{E_L s}{E_H s} < \kappa_S$ then the security market for the collateral asset cannot be in a pooling equilibrium at time t, and a separating equilibrium at time $t+1$.*

Proof. Plugging in for q_{t+1}^P and q_{t+1}^S into (9) we obtain:

$$\phi_t = \begin{cases} \beta z\left[(\lambda E_L s + (1-\lambda) E_H s) + \phi_{t+1}\right], & \text{if } \frac{E_L s + \phi_{t+1}}{E_H s + \phi_{t+1}} \geq \zeta, \\ \beta\left[(z\lambda E_L s + (1-\lambda) E_H s) + \left[(1-\lambda) + \lambda z\right]\phi_{t+1}\right], & \text{if } \frac{E_L s + \phi_{t+1}}{E_H s + \phi_{t+1}} < \zeta. \end{cases}$$

Hence if at time $t+1$ security market is in a pooling equilibrium,

$$\phi_t - \beta z \phi_{t+1} = \beta z \left(\lambda E_L s + (1-\lambda) E_H s\right). \tag{A.1}$$

And if at time $t+1$ security market is in a separating equilibrium,

$$\phi_t - \left[(1-\lambda) + \lambda z\right] \phi_{t+1} = \beta \left(\lambda z E_L s + (1-\lambda) E_H s\right). \tag{A.2}$$

Moreover, if at time t security market is in a pooling equilibrium then

$$\frac{E_L s + \phi_t}{E_H s + \phi_t} \geq \zeta \Leftrightarrow \phi_t \geq \frac{\zeta E_H s - E_L s}{1-\zeta}. \tag{A.3}$$

Similarly, if at time t security market is in a separating equilibrium then

$$\phi_t \leq \frac{\zeta E_H s - E_L s}{1-\zeta}. \tag{A.4}$$

Suppose the security market for the collateral asset is in a separating equilibrium at time t, and a pooling equilibrium at time $t+1$. From (A.3) and (A.4) we have:

$$\phi_t - \beta z \phi_{t+1} \leq \left(\frac{\zeta E_H s - E_L s}{1-\zeta}\right)(1-\beta z). \tag{A.5}$$

Using (A.1) we can write (A.5) as:

$$\beta z \left(\lambda E_L s + (1-\lambda) E_H s\right) \leq \left(\frac{\zeta E_H s - E_L s}{1-\zeta}\right)(1-\beta z).$$

Plugging in for ζ and rearranging we can rewrite above inequality as $\frac{E_L s}{E_H s} \leq \kappa_P$. This proves the first statement in the lemma.

To prove the second statement suppose the security market for the collateral asset is in a pooling equilibrium at time t, and a separating equilibrium at time $t+1$. From (A.3) and (A.4) we have:

$$\phi_t - \left[(1-\lambda) + \lambda z\right] \phi_{t+1} \geq \left(\frac{\zeta E_H s - E_L s}{1-\zeta}\right)\left(1 - \left[(1-\lambda) + \lambda z\right]\right).$$

Plugging in for ζ and rearranging we can rewrite above inequality as $\frac{E_L s}{E_H s} \geq \kappa_S$. This proves the second statement in the lemma. □

Now we are ready to complete the proof of Proposition 2.

Proof. By plugging the asset price in a stationary pooling equilibrium given in (11) into (10) we see that a stationary pooling equilibrium exists if and only if $\frac{E_L s}{E_H s} \geq \kappa_P$. Similarly, by plugging the asset price in a stationary separating equilibrium given in (13) into (12) we see that a stationary separating equilibrium exists if and only if $\frac{E_L s}{E_H s} \leq \kappa_S$. Hence, (i)-(iii) hold if we restrict attention to stationary equilibria.

Next we show that all equilibria are stationary by looking at three cases.

Case 1: Suppose $\frac{E_L s}{E_H s} < \kappa_P$. By Lemma 2, in this region there is no switch from pooling to separating. Hence the only possible non-stationary equilibria are those that remain separating for a finite number of periods, switch to pooling and remain in pooling. However, this is not possible since after the switch the equilibria is stationary pooling equilibrium but in this region unique stationary equilibrium is separating.

Case 2: Suppose $\frac{E_L s}{E_H s} > \kappa_S$. By Lemma 2, in this region there is no switch from separating to pooling. Hence the only possible non-stationary equilibria are those that remain pooling for a finite number of periods, switch to separating and remain in separating. However, this is not possible since after the switch the equilibria is stationary separating equilibrium but in this region unique stationary equilibrium is pooling.

Case 3: Suppose $\kappa_P \leq \frac{E_L s}{E_H s} \leq \kappa_S$. By Lemma 2, in this region there is no switching from separating to pooling or from pooling to separating. Hence all equilibria must be stationary. □

A.3 Proof of Lemma 1

Proof. If two securities, y^j and y^k, are both liquid, $E_L y^j \geq \zeta E_H y^j$ and $E_L y^k \geq \zeta E_H y^k$. Then combining the two security retains liquidity. Similarly, combining two illiquid securities results in an illiquid security. To see the second statement, first note that replacing the two securities with their combination is clearly feasible. In addition, when y^j, y^k and $y^j + y^k$ all trade in a pooling (separating) equilibrium, q^{jk}, the price of $y^j + y^k$, is the sum of q^j and q^k, the prices of y^a and y^b. Now consider the liquid case. Ignoring the irrelevant terms, agent B's payoff when the two securities are separate is:

$$\lambda \int \left\{ a\left[zq^j - y^j(s)\right] + a\left[zq^k - y^k(s)\right] \right\} dF_L(s) + (1-\lambda) \int \left\{ a\left[zq^j - y^j(s)\right] + a\left[zq^k - y^k(s)\right] \right\} dF_H(s)$$

and when they are combined is:

$$\lambda \int \left\{ a\left[zq^{jk} - \left(y^j(s) + y^k(s)\right)\right] \right\} dF_L(s) + (1-\lambda) \int \left\{ a\left[zq^{jk} - \left(y^j(s) + y^k(s)\right)\right] \right\} dF_H(s).$$

Since $q^{jk} = q^j + q^k$, when the liquid securities are combined agent B's payoff is unchanged.

Next consider the illiquid case. Once again ignoring the irrelevant terms, agent B's payoff when the two securities are separate is:

$$\lambda \int \left\{ a\left[zq^j - y^j(s)\right] + a\left[zq^k - y^k(s)\right] \right\} dF_L(s) + (1-\lambda) \int \left\{ ay^j(s) + ay^k(s) \right\} dF_H(s)$$

and when they are combined is:

$$\lambda \int \left\{ a \left[zq^{jk} - \left(y^{j}(s) + y^{k}(s) \right) \right] \right\} dF_L(s) + (1-\lambda) \int \left\{ a \left(y^{j}(s) + y^{k}(s) \right) \right\} dF_H(s).$$

Once again, when the illiquid securities are combined agent B's payoff is unchanged. □

A.4 Proof of Proposition 3

Proof. First note that the feasible set is compact, convex and nonempty so the optimization problem must have a solution. Moreover, since the objective function is bounded above, the solution must be finite. The Lagrangian of the optimization problem is

$$\begin{aligned} \mathcal{L}(x;\gamma,\mu,\mu_x) = & \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds + \int_{s_L}^{s_H} \gamma(s) \left[s - s_L - \int_{s_L}^{s} x(j)dj \right] ds \\ & + \mu \left\{ \int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] x(s)ds + (1-\zeta)\phi \right\} + \int_{s_L}^{s_H} \mu_x(s)x(s)ds. \end{aligned}$$

Note that for any feasible x and for $\gamma \geq 0$, $\mu \geq 0$ and $\mu_x \geq 0$ we have

$$\mathcal{L}(x;\gamma,\mu,\mu_x) \geq \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds.$$

Let $\mathcal{L}(\gamma,\mu,\mu_x) = \max_x \mathcal{L}(x;\gamma,\mu,\mu_x)$. Let $\mathcal{L}^* = \min_{\gamma\geq 0,\mu\geq 0,\mu_x\geq 0} \mathcal{L}(\gamma,\mu,\mu_x)$. Note that $\mathcal{L}^*$ is the value of the original optimization problem. We can rewrite $\mathcal{L}(x;\gamma,\mu,\mu_x)$ as

$$\begin{aligned} \mathcal{L}(x;\gamma,\mu,\mu_x) = & \int_{s_L}^{s_H} \left\{ \widetilde{F}_H(s) + \mu \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] - \int_{s}^{s_H} \gamma(j)dj + \mu_x(s) \right\} x(s)ds \\ & + \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \left(\int_{s}^{s_H} \gamma(j)dj \right) ds. \end{aligned}$$

Let $\eta(s) = \int_{s}^{s_H} \gamma(j)dj$. We can rewrite the problem as:

$$\begin{aligned} \mathcal{L}(x;\eta,\mu,\mu_x) = & \int_{s_L}^{s_H} \left\{ \widetilde{F}_H(s) + \mu \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] - \eta(s) + \mu_x(s) \right\} x(s)ds \\ & + \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \eta(s)\, ds. \end{aligned}$$

Now note that the quantity inside the curly brackets must be zero or otherwise the value of the optimization problem would be infinite. Consider the following dual problem of the optimization problem,

$$\begin{aligned} \min_{\mu\geq 0} \quad \min\nolimits_{\eta\geq 0,\mu_x\geq 0} \quad & \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \eta(s)\, ds \\ s.t. \quad & \widetilde{F}_H(s) + \mu \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right] - \eta(s) + \mu_x(s) = 0. \end{aligned}$$

Note that the value of this problem is $\mathcal{L}^*$. Let $H_\mu(s) = \widetilde{F}_H(s) + \mu \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) \right]$. We can rewrite the above problem one more time as:

$$\begin{aligned} \min_{\mu\geq 0} \quad \min\nolimits_{\eta\geq 0} \quad & \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \eta(s)\, ds \\ s.t. \quad & \eta(s) \geq H_\mu(s), \end{aligned}$$

and the constraint that $\eta(s)$ is a decreasing function in s. Note, $h_\mu(s) \equiv \frac{\partial H_\mu(s)}{\partial s} = -f_H(s)\left[1 + \mu\left(\frac{f_L(s)}{f_H(s)} - \zeta\right)\right]$. Clearly $H_\mu(s_L) > 0$ and $H_\mu(s_H) = 0$. Since $\mu > 0$ we must have $h_\mu(s_L) < 0$. To see this suppose $h_\mu(s_L) \geq 0$. Then it must be the case that $1 + \mu\left(\frac{f_L(s)}{f_H(s)} - \zeta\right) \leq 0$. Since $\frac{f_L(s)}{f_H(s)}$ is decreasing, this implies that $h_\mu(s) > 0$ for all $s \in (s_L, s_H]$ contradicting that $H_\mu(s_H) = 0$.

Since $\frac{f_L(s)}{f_H(s)}$ is decreasing in s one of the following must be true:

(i) There exists a unique cutoff $\widehat{s}_\mu \in (s_L, s_H)$ such that $h_\mu(s) < 0$ for $s < \widehat{s}_\mu$ and $h_\mu(s) > 0$ for $s > \widehat{s}_\mu$,

(ii) $h_\mu(s) < 0$ for all $s \in (s_L, s_H)$.

In case (i) the function $H_\mu(s)$ crosses from positive to negative once, eventually increasing to zero at s_H. In case (ii) $H_\mu(s)$ decreases to zero at s_H. Let $s^*_\mu \in (s_L, s_H)$ be the unique s for which $H_\mu(s) = 0$ if it exists, otherwise let $s^*_\mu = s_H$.

Note that for given $\mu \geq 0$ optimal η_μ is given by:

$$\eta_\mu(s) = \begin{cases} H_\mu(s) & \text{if} \quad s \leq s^*_\mu \\ 0 & \text{if} \quad s > s^*_\mu \end{cases}.$$

Plugging this into the minimization problem we get:

$$\min_{\mu \geq 0} \mu(1-\zeta)\phi + \int_{s_L}^{s^*_\mu} \left(\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right]\right) ds.$$

The first order condition for this problem is:

$$(1-\zeta)\phi + \int_{s_L}^{s^*_\mu} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] ds + \frac{\partial s^*_\mu}{\partial \mu} H_\mu\left(s^*_\mu\right) \geq 0$$

Because $H_\mu\left(s^*_\mu\right) = 0$,

$$(1-\zeta)\phi + \int_{s_L}^{s^*_\mu} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] ds \geq 0$$

with complementary slackness.

Let $s^* \in (s_L, s_H]$ be the unique s for which

$$(1-\zeta)\phi + \int_{s_L}^{s^*} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] ds = 0$$

if it exists. If

$$(1-\zeta)\phi + \int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] ds > 0$$

for all $s \in [s_L, s_H]$, then $s^* = s_H$.

If $s^* < s_H$ then $\mu > 0$, $s^*_\mu = s^*$, and

$$\mathcal{L}^* = \mu(1-\zeta)\phi + \int_{s_L}^{s^*} \left(\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right]\right) ds = \int_{s_L}^{s^*} \widetilde{F}_H(s)ds.$$

If $s^* = s_H$ then $\mu = 0$, $s^*_\mu = s_H$, and

$$\mathcal{L}^* = \int_{s_L}^{s_H} \widetilde{F}_H(s)ds.$$

To complete the proof, let $\delta = s^*$ and note that $x(s) = 1$ for $s \in [s_L, \delta)$ and $x(s) = 0$ for $s \in [\delta, s_H]$ achieves the value $\mathcal{L}^*$ and it is feasible, and must be optimal for the original problem. □

A.5 Proof of Proposition 4

Proof. Observe that to maximize (21) agent B must set δ as large as possible subject to satisfying the constraint (22). We first show that either there is a unique δ that satisfies (22) with equality, or (22) is not binding. Let

$$\begin{aligned}\mathcal{T}(x) &\equiv (z-1)\left[\phi + s_L + \lambda\int_{s_L}^{x} \widetilde{F}_L(s)ds + (1-\lambda)\int_{s_L}^{x} \widetilde{F}_H(s)ds\right] - \lambda\int_{s_L}^{x}\left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds \\ &= (z-1)\left[\phi + s_L + \int_{s_L}^{x} \widetilde{F}_H(s)ds\right] - z\lambda\int_{s_L}^{x}\left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds.\end{aligned}$$

Observe that,

$$\mathcal{T}(s_L) = (z-1)(\phi + s_L) > 0, \qquad \mathcal{T}'(x) = (z-1)\widetilde{F}_H(x) - z\lambda\left[\widetilde{F}_H(x) - \widetilde{F}_L(x)\right],$$

$$\mathcal{T}'(s_L) = z - 1 > 0, \qquad \mathcal{T}'(s_H) = 0,$$

$$\mathcal{T}''(x) = -(z-1)f_H(x) + z\lambda\left[f_H(x) - f_L(x)\right] = f_H(x)\left[z(\lambda-1) + 1 - z\lambda\frac{f_L(x)}{f_H(x)}\right].$$

When $\frac{f_L(x)}{f_H(x)}$ is monotonically decreasing in s, $\mathcal{T}(x)$ is quasi-concave with $\mathcal{T}(s_L) > 0$. So, there is either a unique δ that satisfies $\mathcal{T}(\delta) = 0$ or $\mathcal{T}(x) > 0$ for all $x \in [s_L, s_H]$.

Case (i): Constraint (22) is binding. In this case the face value of the debt contract that solves the security design problem is given by:

$$\phi = \frac{z}{z-1}\lambda\int_{s_L}^{\delta}\left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds - \int_{s_L}^{\delta} \widetilde{F}_H(s)ds - s_L. \tag{A.6}$$

In addition, the asset price ϕ satisfies (23). Substituting for q_D and q_E we rewrite (23) as:

$$\phi = \frac{\beta}{1-\beta z}\left\{z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{\delta}^{s_H} \widetilde{F}_H(s)ds\right\}. \tag{A.7}$$

Substituting ϕ in (A.6) using (A.7), the equilibrium can be solved by a single equation of δ, $\Gamma(\delta) = 0$, where

$$\begin{aligned}\Gamma(\delta) = &\frac{\beta}{1-\beta z}\left\{z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_\delta^{s_H} \widetilde{F}_H(s)ds\right\}\\ &- \frac{z}{z-1}\lambda\int_{s_L}^{\delta}\left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right]ds + \int_{s_L}^{\delta}\widetilde{F}_H(s)ds + s_L\end{aligned}$$

Observe that:

$$\begin{aligned}\Gamma'(\delta) &= \frac{\beta}{1-\beta z}(1-\lambda)(z-1)\widetilde{F}_H(\delta) - \frac{z}{z-1}\lambda\left[\widetilde{F}_H(\delta) - \widetilde{F}_L(\delta)\right] + \widetilde{F}_H(\delta)\\ &= \left[\frac{\beta}{1-\beta z}(1-\lambda)(z-1) + 1 - \frac{z}{z-1}\lambda\right]\widetilde{F}_H(\delta) + \frac{z}{z-1}\lambda\widetilde{F}_L(\delta).\\ \Gamma''(\delta) &= -\left[\frac{\beta}{1-\beta z}(1-\lambda)(z-1) + 1 - \frac{z}{z-1}\lambda\right]f_H(\delta) - \frac{z}{z-1}\lambda f_L(\delta)\\ &= f_H(\delta)\left\{\frac{z}{z-1}\lambda\left[1 - \frac{f_L(\delta)}{f_H(\delta)}\right] - \frac{\beta}{1-\beta z}(1-\lambda)(z-1) - 1\right\}\\ \Gamma(s_L) &= s_L\left[1 + \frac{\beta}{1-\beta z}(1-\lambda)(z-1)\right] + \frac{\beta}{1-\beta z}\left[z\lambda E_L s + (1-\lambda)E_H s\right] > 0\\ \Gamma'(s_L) &= \frac{\beta}{1-\beta z}(1-\lambda)(z-1) + 1 > 0\\ \Gamma'(s_H) &= 0.\end{aligned}$$

Once again $\Gamma(s)$ is quasi-concave if $\frac{f_L(\delta)}{f_H(\delta)}$ is monotonically decreasing in D. Because $\Gamma(s_L) > 0$, there is a unique equilibrium. The constraint (22) is binding iff $\Gamma(s_H) < 0$. We rewrite $\Gamma(s_H)$ as:

$$\begin{aligned}\Gamma(s_H) &= \frac{\beta z}{1-\beta z}\left[\lambda E_L s + (1-\lambda)E_H s\right] - \frac{z}{z-1}\lambda\int_{s_L}^{s_H}\left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right]ds + \int_{s_L}^{s_H}\widetilde{F}_H(s)ds + s_L\\ &= \frac{E_H s}{(1-\beta z)(z-1)}\left[\lambda z(1-\beta)\left(\frac{E_L s}{E_H s} - 1\right) + z - 1\right].\end{aligned}$$

Hence, $\Gamma(s_H) < 0$ iff

$$\frac{E_L s}{E_H s} < 1 - \frac{z-1}{z\lambda(1-\beta)}.$$

Case (ii): Constraint (22) is not binding. □

A.6 Proof of Proposition 5

Claim 1. Assume that $\frac{f_L(s)}{f_H(s)}$ is decreasing in s. The optimal securities are

$$\begin{aligned}y_{1t}(s) &= \phi + s_L + (s - s_L)\mathbb{I}(s \le \delta),\\ y_{2t}(s) &= (s - s_L)\mathbb{I}(s > \delta).\end{aligned}$$

for some $\delta \in (s_L, s_H]$.

Proof. The maximization

$$\arg\max_{x,m} \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds, \tag{A.8}$$

$$s.t. \int_{s_L}^{s} x(j)dj \leq s - s_L, \forall s \in [s_L, s_H], \tag{A.9}$$

$$\int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s)\right] x(s)ds + (1-\zeta)\phi \geq 0, \tag{A.10}$$

$$\int_{s_L}^{s} x(j)dj \geq 0, \forall s \in [s_L, s_H] \tag{A.11}$$

First note that the feasible set is compact, convex and nonempty so the optimization problem must have a solution. Moreover, since the objective function is bounded above, the solution must be finite. The Lagrangian of the optimization problem is

$$\mathcal{L}(x;\gamma,\mu,\nu) = \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds + \int_{s_L}^{s_H} \gamma(s)\left[s - s_L - \int_{s_L}^{s} x(j)dj\right] ds$$
$$+ \mu\left\{\int_{s_L}^{s_H} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] x(s)ds + (1-\zeta)\phi\right\} + \int_{s_L}^{s_H} \nu(s)\left[\int_{s_L}^{s} x(j)dj\right] ds.$$

Note that for any feasible x and for $\gamma \geq 0$, $\mu \geq 0$ and $\nu \geq 0$ we have

$$\mathcal{L}(x;\gamma,\mu,\nu) \geq \int_{s_L}^{s_H} \widetilde{F}_H(s)x(s)ds.$$

Let $\mathcal{L}(\gamma,\mu,\nu) = \max_x \mathcal{L}(x;\gamma,\mu,\nu)$. Let $\mathcal{L}^* = \min_{\gamma\geq 0,\mu\geq 0,\mu_x\geq 0} \mathcal{L}(\gamma,\mu,\nu)$. Note that $\mathcal{L}^*$ is the value of the original optimization problem. We can rewrite $\mathcal{L}(x;\gamma,\mu,\nu)$ as

$$\mathcal{L} = \int_{s_L}^{s_H} \left\{\widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] + \int_{s}^{s_H} [\nu(j) - \gamma(j)]\, dj\right\} x(s)ds$$
$$+ \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \left(\int_{s}^{s_H} \gamma(j)dj\right) ds$$

Now note that the quantity inside the curly brackets must be zero or otherwise the value of the optimization problem would be infinite. Consider the following dual problem of the optimization problem,

$$\begin{aligned} \min_{\mu\geq 0} \quad \min\nolimits_{\gamma\geq 0,\nu\geq 0} \quad & \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \left(\int_{s}^{s_H} \gamma(j)dj\right) ds \\ s.t. \quad & \widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right] + \int_{s}^{s_H} [\nu(j) - \gamma(j)]\, dj = 0. \end{aligned}$$

Note that the value of this problem is $\mathcal{L}^*$. Let $H_\mu(s) = \widetilde{F}_H(s) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s)\right]$. Let $\eta(s) = \int_s^{s_H} \gamma(j)dj$, $\xi(s) = \int_s^{s_H} \nu(j)dj$. We can rewrite the above problem one more time as:

$$\begin{aligned} \min_{\mu\geq 0} \quad \min\nolimits_{\eta,\xi\geq 0} \quad & \mu(1-\zeta)\phi + \int_{s_L}^{s_H} \eta(s)ds \\ s.t. \quad & H_\mu(s) + \xi(s) - \eta(s) = 0. \end{aligned}$$

and the constraints that $\eta(s)$ and $\xi(s)$ are decreasing functions in s.

Note, $h_\mu(s) \equiv \frac{\partial H_\mu(s)}{\partial s} = -f_H(s)\left[1+\mu\left(\frac{f_L(s)}{f_H(s)}-\zeta\right)\right]$. Clearly $H_\mu(s_L) > 0$ and $H_\mu(s_H) = 0$. Since $\mu > 0$ we must have $h_\mu(s_L) < 0$. To see this suppose $h_\mu(s_L) \geq 0$. Then it must be the case that $1+\mu\left(\frac{f_L(s)}{f_H(s)}-\zeta\right) \leq 0$. Since $\frac{f_L(s)}{f_H(s)}$ is decreasing, this implies that $h_\mu(s) > 0$ for all $s \in (s_L, s_H]$ contradicting that $H_\mu(s_H) = 0$.

Since $\frac{f_L(s)}{f_H(s)}$ is decreasing in s one of the following must be true:

(i) There exists a unique cutoff $\widehat{s}_\mu \in (s_L, s_H)$ such that $h_\mu(s) < 0$ for $s < \widehat{s}_\mu$ and $h_\mu(s) > 0$ for $s > \widehat{s}_\mu$,

(ii) $h_\mu(s) < 0$ for all $s \in (s_L, s_H)$.

In case (i) the function $H_\mu(s)$ crosses from positive to negative once, eventually increasing to zero at s_H. In case (ii) $H_\mu(s)$ decreases to zero at s_H.

Note that for given $\mu \geq 0$ optimal η_μ and ξ_μ are given by:

$$\xi_\mu(s) = \begin{cases} -H_\mu(\hat{s}_\mu) & \text{if} \quad s \leq \hat{s}_\mu, \\ -H_\mu(s) & \text{if} \quad s > \hat{s}_\mu. \end{cases}$$

$$\eta_\mu(s) = \begin{cases} H_\mu(s) - H_\mu(\hat{s}_\mu) & \text{if} \quad s \leq \hat{s}_\mu, \\ 0 & \text{if} \quad s > \hat{s}_\mu. \end{cases}$$

This is because ξ_μ and η_μ must be decreasing in s. When $s > \hat{s}_\mu$, $H_\mu(s)$ is increasing. So it is feasible to let $\eta_\mu(s) = 0$ and $\xi_\mu(s) = -H_\mu(s)$ in this region. When $s < \hat{s}_\mu$, $H_\mu(s)$ is decreasing in s. The optimal η and ξ would be $\xi_\mu(s) = -H_\mu(\hat{s}_\mu)$ and $\eta_\mu(s) = H_\mu(s) - H_\mu(\hat{s}_\mu)$. Plugging this into the minimization problem we get:

$$\begin{aligned} &\min_{\mu\geq 0} \mu(1-\zeta)\phi + \int_{s_L}^{\hat{s}_\mu} \left(H_\mu(s) - H_\mu(\hat{s}_\mu)\right) ds \\ =&\min_{\mu\geq 0} \mu(1-\zeta)\phi + \int_{s_L}^{\hat{s}_\mu} \left\{\widetilde{F}_H(s) - \widetilde{F}_H(\hat{s}_\mu) + \mu\left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s) - \left(\widetilde{F}_L(\hat{s}_\mu) - \zeta\widetilde{F}_H(\hat{s}_\mu)\right)\right]\right\} ds \end{aligned}$$

The first order condition for this problem is:

$$\Gamma(\hat{s}_\mu) \equiv (1-\zeta)\phi + \int_{s_L}^{\hat{s}_\mu} \left[\widetilde{F}_L(s) - \zeta\widetilde{F}_H(s) - \left(\widetilde{F}_L(\hat{s}_\mu) - \zeta\widetilde{F}_H(\hat{s}_\mu)\right)\right] ds \geq 0$$

with complementary slackness.

$$\frac{\partial\Gamma(s^*)}{\partial s^*} = (s^* - s_L) f_H(s^*)\left(\frac{f_L(s^*)}{f_H(s^*)} - \zeta\right).$$

By definition of $\hat{s}_\mu$, $\frac{f_L(s^*)}{f_H(s^*)} - \zeta = -\frac{1}{\mu}$. So, $\frac{\partial \Gamma(s^*)}{\partial s^*} = -(s^* - s_L)\frac{f_H(s^*)}{\mu} < 0$. And $\Gamma(s_L) = (1-\zeta)\phi > 0$. Then, if there exists a solution for $\Gamma(s^*) = 0$, the solution is unique and it satisfies

$$(1-\zeta)\phi + \int_{s_L}^{s^*} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) - \left(\widetilde{F}_L(s^*) - \zeta \widetilde{F}_H(s^*)\right)\right] ds = 0$$

Otherwise,

$$(1-\zeta)\phi + \int_{s_L}^{s^*} \left[\widetilde{F}_L(s) - \zeta \widetilde{F}_H(s) - \left(\widetilde{F}_L(s^*) - \zeta \widetilde{F}_H(s^*)\right)\right] ds > 0$$

for all $s \in [s_L, s_H]$ and $s^* = s_H$.

If $s^* < s_H$ then $\mu > 0$, $\hat{s}_\mu = s^*$, and

$$\mathcal{L}^* = \int_{s_L}^{s^*} \left[\widetilde{F}_H(s) - \widetilde{F}_H(s^*)\right] ds.$$

If $s^* = s_H$ then $\mu = 0$, $s^*_\mu = s_H$, and

$$\mathcal{L}^* = \int_{s_L}^{s_H} \widetilde{F}_H(s) ds.$$

To complete the proof note that $\int_{s_L}^{s} x(j)dj = s - s_L$ for $s \in [s_L, s^*]$ and $\int_{s_L}^{s} x(j)dj = 0$ for $s \in [s^*, s_H]$ achieves the value $\mathcal{L}^*$ and it is feasible, and must be optimal for the original problem. □

Proof. Given Claim 1 that the optimal security design under Arrow securities has two tranches, the liquid tranche $y_{1t}(s)$ and illiquid tranche $y_{2t}(s)$.

$$y_{1t}(s) = \phi + s_L + (s - s_L)\mathbb{I}(s \le \delta),$$
$$y_{2t}(s) = (s - s_L)\mathbb{I}(s > \delta).$$

The equilibrium is solved by the following two equations, representing the incentive constraint of an owner with high quality collateral and the Euler equation for the asset price. In the incentive constraint,

$$z\left[\phi + s_L + \int_{s_L}^{\delta} (s - s_L)\, dF_\lambda(s)\right] - \left[\phi + s_L + \int_{s_L}^{\delta} (s - s_L)\, dF_H(s)\right] \ge 0$$

One can easily verify that the left hand side of the incentive constraint is decreasing in δ as long as the monotone likelihood ratio assumption holds. This confirms the conjecture of the optimal security design.

The Euler equation for the asset price is

$$\phi = \frac{\beta}{1-\beta z}\left[z\left(s_L + \int (s - s_L) dF_\lambda(s)\right) - (z-1)(1-\lambda)\int_{\delta}^{s_H} (s - s_L)\, dF_H(s)\right]$$

The equilibrium value of δ is determined by

$$0 = \Gamma(\delta) = \frac{\beta}{1-\beta z}\left[z\left(s_L + \int (s - s_L) dF_\lambda(s)\right) - (z-1)(1-\lambda)\int_{\delta}^{s_H} (s - s_L)\, dF_H(s)\right]$$
$$- \frac{s_L + \int_{s_L}^{\delta}(s - s_L) dF_H(s) - z\left[s_L + \int_{s_L}^{\delta}(s - s_L) dF_\lambda(s)\right]}{z-1}$$

$$\Gamma(s_L) = \frac{\beta}{1-\beta z}\left[z\left(s_L + \int (s-s_L)dF_\lambda(s)\right) - (z-1)(1-\lambda)\int_{s_L}^{s_H}(s-s_L)\,dF_H(s)\right] + s_L > 0$$

$$\begin{aligned}
\Gamma'(\delta) &= \frac{\beta(z-1)(1-\lambda)\delta f_H(\delta)}{1-\beta z} - \frac{\delta f_H(\delta) - z\delta\left[\lambda f_L(\delta) + (1-\lambda)f_H(\delta)\right]}{z-1} \\
&= \delta f_H(\delta)\left\{\frac{\beta(z-1)(1-\lambda)}{1-\beta z} - \frac{1 - z\left[\lambda f_L(\delta)/f_H(\delta) + (1-\lambda)\right]}{z-1}\right\} \\
&= \delta f_H(\delta)\left\{\frac{\beta(z-1)(1-\lambda)}{1-\beta z} - \frac{1-z(1-\lambda)}{z-1} + \frac{z\lambda f_L(\delta)/f_H(\delta)}{z-1}\right\}
\end{aligned}$$

If $\frac{\beta(z-1)(1-\lambda)}{1-\beta z} - \frac{1-z(1-\lambda)}{z-1} < 0$, there exists a unique δ^* such that $\Gamma'(\delta) > 0$ if and only if $\delta < \delta^*$. There exists a most one solution for the equation $\Gamma(\delta) = 0$.

$$\begin{aligned}
\Gamma(s_H) &= \frac{\beta z}{1-\beta z}\left[(1-\lambda)\mathbb{E}_H s + \lambda \mathbb{E}_L s\right] - \frac{\left[1-(1-\lambda)z\right]\mathbb{E}_H s - z\lambda\mathbb{E}_L s}{z-1} \\
&= \mathbb{E}_H s\left[\frac{\beta z}{1-\beta z}(1-\lambda) - \frac{1-(1-\lambda)z}{z-1} + \left(\frac{\beta z}{1-\beta z}\lambda + \frac{z\lambda}{z-1}\right)\frac{\mathbb{E}_L s}{\mathbb{E}_H s}\right].
\end{aligned}$$

The condition for there to be a unique $\delta \in (s_L, s_H)$ in equilibrium is

$$\begin{aligned}
\frac{\mathbb{E}_L s}{\mathbb{E}_H s} &\leq -\frac{\left(\frac{\beta z}{1-\beta z} + \frac{z}{z-1}\right)(1-\lambda) - \frac{1}{z-1}}{\left(\frac{\beta z}{1-\beta z} + \frac{z}{z-1}\right)\lambda} \\
&= 1 - \frac{z-1}{\lambda z\,(1-\beta)}.
\end{aligned}$$

□

A.7 Proof of Proposition 6

Proof. Using definition 3 and following similar steps leading to Proposition 4 we observe that the optimal design involves a debt tranche with face value $D = \phi_0 + \delta_0$ and the residual equity tranche. By design the debt tranche is traded in a pooling equilibrium in regime 0. Let's suppose that in regime 1 the debt tranche is traded by only the low type. In this case, the Euler equations given by ((29)) for the value of the asset in regimes 0 and 1 can be rewritten as:

$$v_0 = \beta\left\{z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{\delta_0}^{s_H}\widetilde{F}_H(s)ds + z\phi_0\right\}, \tag{A.12}$$

and

$$v_1 = \beta\left\{z\lambda E_L s + (1-\lambda)E_H s + \left(z\lambda + (1-\lambda)\right)\phi_1\right\} \tag{A.13}$$

Letting $C_0(\delta_0) = z\left[\lambda E_L s + (1-\lambda)E_H s\right] - (1-\lambda)(z-1)\int_{\delta_0}^{s_H}\widetilde{F}_H(s)ds$ and $C_1 = z\lambda E_L s + (1-\lambda)E_H s$ we can rewrite (A.12) and (A.13) as:

$$v_0 = \beta\left\{C_0(\delta_0) + z\phi_0\right\}, \tag{A.14}$$

and

$$v_1 = \beta \left\{ C_1 + (z\lambda + (1-\lambda))\, \phi_1 \right\}. \tag{A.15}$$

Note that $C_0(\delta_0) > C_1$ for $\delta_0 \in (s_L, s_H]$. The incentive constraint of owners of high quality collateral is analogous to (24):

$$\phi_0 = \frac{z}{z-1}\lambda \int_{s_L}^{\delta_0} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds - \int_{s_L}^{\delta_0} \widetilde{F}_H(s) ds - s_L, \tag{A.16}$$

From equations (A.12) and (A.13) we get:

$$\phi_0 = \frac{\beta \left\{ C_0(\delta) + z\chi\, (\phi_1 - \phi_0) \right\}}{1-\beta z}$$

$$\phi_1 = \frac{\beta \left\{ C_1 + z'\gamma\, (\phi_1 - \phi_0) \right\}}{1-\beta z'}$$

$$\phi_1 - \phi_0 = \frac{\beta \left[(1-\beta z)\, C_1 - (1-\beta z')\, C_0(\delta)\right]}{(1-\beta z)(1-\beta z') - \beta(1-\beta z)\, z'\gamma + \beta(1-\beta z')\, z\chi},$$

where $z' = z\lambda + (1-\lambda)$. From the above equations it is immediate that $\phi_0 > \phi_1$ proving part of claim (i). Letting

$$\Lambda = \frac{((1-\chi) - \beta(\gamma - \chi)\, z)}{1 - \beta(\gamma-\chi)\, z}$$

we get ϕ_0. Substituting into (A.16) we see that equilibrium can be solved by a single equation of δ, $\Gamma(\delta) = 0$, where

$$\begin{aligned}\Gamma(\delta) =& \frac{\beta}{1-\beta z}\left[z\lambda E_L s + ((1-\Lambda) + \Lambda z)(1-\lambda) E_H s - \Lambda(1-\lambda)(z-1)\int_{\delta}^{s_H} \widetilde{F}_H(s) ds\right] \\ & - \frac{z}{z-1}\lambda \int_{s_L}^{\delta} \left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right] ds + \int_{s_L}^{\delta} \widetilde{F}_H(s) ds + s_L\end{aligned}$$

Observe that:

$$\begin{aligned}\Gamma'(\delta) &= \frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1)\widetilde{F}_H(\delta) - \frac{z}{z-1}\lambda\left[\widetilde{F}_H(\delta) - \widetilde{F}_L(\delta)\right] + \widetilde{F}_H(\delta) \\ &= \left[\frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1) + 1 - \frac{z}{z-1}\lambda\right]\widetilde{F}_H(\delta) + \frac{z}{z-1}\lambda \widetilde{F}_L(\delta). \\ \Gamma''(\delta) &= -\left[\frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1) + 1 - \frac{z}{z-1}\lambda\right] f_H(\delta) - \frac{z}{z-1}\lambda f_L(\delta) \\ &= f_H(\delta)\left\{\frac{z}{z-1}\lambda\left[1 - \frac{f_L(\delta)}{f_H(\delta)}\right] - \frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1) - 1\right\} \\ \Gamma(s_L) &= s_L\left[1 + \frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1)\right] + \frac{\beta}{1-\beta z}\left[z\lambda E_L s + (1-\lambda)E_H s\right] > 0 \\ \Gamma'(s_L) &= \left[\frac{\beta}{1-\beta z}\Lambda(1-\lambda)(z-1) + 1\right] > 0 \\ \Gamma'(s_H) &= 0.\end{aligned}$$

$\Gamma(s)$ is quasi-concave if $\frac{f_L(\delta)}{f_H(\delta)}$ is monotonically decreasing in δ. Because $\Gamma(s_L) > 0$, there is a unique equilibrium. In this unique equilibrium $\delta < s_H$ iff $\Gamma(s_H) < 0$. We write $\Gamma(s_H)$ as:

$$\Gamma(s_H) = \frac{\beta}{1-\beta z}\left[z\lambda E_L s + ((1-\Lambda)+\Lambda z)(1-\lambda)E_H s\right] - \frac{z}{z-1}\lambda\int_{s_L}^{s_H}\left[\widetilde{F}_H(s) - \widetilde{F}_L(s)\right]ds + \int_{s_L}^{s_H}\widetilde{F}_H(s)ds + s_L, \text{ or,}$$

$$\Gamma(s_H) = \frac{E_H s}{(1-\beta z)(z-1)}\left[(\beta(z-1)((1-\Lambda)+\Lambda z)(1-\lambda) - z(1-\beta z)\lambda + (1-\beta z)(z-1)) + (1-\beta)\lambda z\frac{E_L s}{E_H s}\right]$$

Hence $\Gamma(s_H) < 0$ iff

$$\frac{E_L s}{E_H s} < \frac{(1-\beta z)(1-z(1-\lambda)) - \beta(z-1)((1-\Lambda)+\Lambda z)(1-\lambda)}{(1-\beta)\lambda z} \equiv \mathit{\Gamma}(\gamma,\chi).$$

It is easy to see that $\mathit{\Gamma}(\gamma,\chi) > 1-(z-1)/(z\lambda(1-\beta))$ iff $z > 1$. Note that:

$$\frac{\partial\Gamma}{\partial\Lambda} = \frac{\beta(z-1)(1-\lambda)}{1-\beta z}\left[E_H s - \int_\delta^{s_H}\widetilde{F}_H(s)ds\right] > 0.$$

This means that as Λ increases, Γ shifts up. As a result when (30), Γ crosses zero at a higher value, implying that the maximum interest payment δ is increasing in Λ.

Since no repo run equilibrium corresponds to $\Lambda = 1$, we see that debt threshold D is strictly lower with repo run than without (proving (ii)). As χ approaches one, Λ approaches zero and the unique equilibrium approaches the illiquid equilibrium when only equity is available as collateral asset (proving iv). Moreover,

$$\frac{\partial\Lambda}{\partial\gamma} = \frac{-\beta z\chi}{(1-\beta(\gamma-\chi)z)^2} < 0, \text{ and}$$
$$\frac{\partial\Lambda}{\partial\chi} = \frac{\beta\gamma z - 1}{(1-\beta(\gamma-\chi)z)^2} < 0.$$

Hence, if the probability of rigidity or sunspot increases, δ decreases (proving (iii)). $\square$