What Makes Financial Networks Special? Distorted Investment Incentives, Regulation, and Systemic Risk Measurement.

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Abstract

In a model of financial networks with both debt and equity interdependencies, we show that financial organizations have incentives to: choose excessively risky portfolios; overly correlate their portfolios with those of their counterparties; and under-diversify in terms of choosing too few counterparties with whom to share risk. We also provide measures of financial centrality in terms of how a given organization's portfolio affects the values and defaults of other organizations. Additionally, we characterize optimal regulation in terms of the use of reserve requirements versus bailouts and how these relate to financial centrality, and fully characterize the minimum bailouts needed to ensure systemic solvency.

JEL Classification Codes: D85, F15, F34, F36, F65, G15, G32, G33, G38

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1 Introduction

World trade has grown from just under 20 percent of world GDP at the end of the Second World War to over 60 percent.[1] This unprecedented growth in trade has had many benefits from various forms of gains from trade, economies of scope and scale, more efficient

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[1] Imports plus exports over GDP. Detailed data can be found for 1870-1949: Klasing and Milionis (2014), 1950-1959: Penn World Trade Tables Version 8.1, 1960-2015: World Bank World Development Indicators.

investment, and has been accompanied by a comparable growth in the international financial network. For instance, the amount of investment around the world coming from foreign sources went from 26 trillion dollars in 2000 to over 132 trillion dollars in 2016, which today represents more than a third of the total level of world investments.[2] In addition, the financial sector is characterized by strong interdependencies, with capital circulating from financial firm to financial firm. Using administrative data from the US Federal Reserve Bank, Duarte and Jones (2017) estimate that 23% of the assets of bank holding companies come from within the US financial system, as well as 48% of their liabilities - *almost half*.[3,4] Along with the enormous benefits that have accompanied the growing and increasingly inter-connected world economy, have come stronger conduits of shocks and risks of widespread contagion. These are not idle concerns, as we witnessed in 2008 when exposure to a problematic mortgage market led to key insolvencies in the US and elsewhere, and to a broad financial crisis and prolonged recession.[5]

Financial markets differ from textbook efficient markets on several dimensions, and are important to understand since they are fundamental to all businesses and sectors of the economy. Financial markets are ripe with externalities, often subtle but with wide-ranging consequences. At a most basic level, the risk that a counterparty defaults has consequences in a world with some missing markets in which not all risks can be fully hedged.[6,7] Defaults involve substantial inefficiencies and costs, which result from fire sales, early termination of contracts, government bailouts, and legal costs, among others; much of which are born by parties others than those who are responsible for the original default. Even without full cascades, these costs are substantial. For example, estimates of bankruptcy recovery rates are in the 56-57 percent range.[8] Defaults can then lead to potential disasters if left to cascade. These introduce further externalities, since if one organization has poor judgment in its investments, poorly managed business practices, or simply unusually bad luck, this ends up affecting the values of its partners, and their partners, in discontinuous ways. Costs

[2] See IMF publication: Susan Lund and Phlippe Harle "Global Finance Resets" in *Finance and Development*, Dec 2017, volume 54 number 4, pp 1-4.

[3] The large difference reflects the fact that many other types of financial organizations that are not BHCs (e.g., Real Estate Investment Trusts, Insurance Companies, and various sorts of investment funds, etc.) have accounts of cash, money markets, and other deposits held at BHCs.

[4] Also, between 1980 and 2018 there was an enormous consolidation. The number of banks has dropped to a third of what it was, and at the same time banks are managing more than eight times as much in terms of total assets. See Jackson and Pernoud (2019) for more background.

[5] For narratives of the crisis see the US Congressional *Financial Crisis Inquiry Report* of January 2011, as well as Glasserman and Young (2016) and Jackson (2019).

[6] As a poignant example, there are even risks that the organizations offering insurance and hedges default. For instance, a key failure in the financial crisis in 2008 was that AIG was unable to deliver the insurance it had sold on many derivative contracts. Its inability to even meet margin calls on that insurance, and subsequent insolvency, forced a large government intervention.

[7] In terms of relations to textbook general equilibrium and efficient markets, not only are markets incomplete, but they also involve substantial discontinuities that can even preclude existence, as we discuss in Section A.1.

[8] See Branch (2002); Acharya, Bharath and Srinivasan (2007), as well as Davydenko, Strebulaev and Zhao (2012); James (1991).

of avoiding cascades and other externalities can be much less if addressed before they begin. However, this involves having a detailed view of the network of financial inter-dependencies, an understanding of the consequences of those interdependencies, as well as of the incentives that different parties have in choosing their investments and counterparties, and thus their position in the network. Although there is a growing literature on financial networks and their consequences,[9] the incentives of financial organizations to choose their investments are not well-understood.

In this paper, we examine the distortions of financial organizations' incentives to invest due to being embedded in a network. We do this in the context of a new model of financial networks that involves two different forms of contracts: debt and equity. This generalizes existing models used to understand systemic risk that are either built with debt-like inter-dependencies – Eisenberg and Noe (2001), Gai and Kapadia (2010), and Csóka and Herings (2018) – or with equity-based interdependencies – Elliott, Golub and Jackson (2014). Our model is more in the tradition of those that have been used in the accounting and asset valuation literatures (Suzuki (2002); Fischer (2014)) to derive equity prices when firms also issue debt. Our model generalizes those models to include multiple types of contracts, while also allowing for bankruptcies and resulting discontinuities due to bankruptcy costs, since those are central to the inefficiencies and externalities in financial networks. We highlight that accounting for both debt and equity contracts is important, as they lead to very different incentives for investment as well as different probabilities of cascades, for the same initial conditions.[10]

We then examine three basic choices of banks and other sorts of financial organizations: which investments they make, how many counterparties they transact with, and the extent to which they choose to correlate their portfolios with those of their counterparties. We show how externalities result in inefficiencies in all of these choices, and discuss the network consequences. Our first result states that, under general conditions on the network structure, banks choose to take on too much risk as compared to what is socially efficient. This comes from the fact that banks do not account for the negative externalities their default imposes

[9] A partial list of references is Rochet and Tirole (1996), Kiyotaki and Moore (1997), Allen and Gale (2000), Eisenberg and Noe (2001), Upper and Worms (2004), Cifuentes, Ferrucci and Shin (2005), Leitner (2005), Allen and Babus (2009), Lorenz, Battiston, Schweitzer (2009), Gai and Kapadia (2010), Wagner (2010), Billio et al. (2012), Elliott, Golub and Jackson (2014), Demange (2016), Diebold and Yilmaz (2014), Dette, Pauls, and Rockmore (2011), Gai, Haldane, and Kapadia (2011), Greenwood, Landier, and Thesmar (2012), Ibragimov, Jaffee and Walden (2011), Upper (2011), Allen, Babus and Carletti (2012), Cohen-Cole, Patacchini and Zenou (2012), Gouriéroux, Héam and Monfort (2012), Alvarez and Barlevy (2015), Glasserman and Young (2015), Acemoglu, Ozdaglar and Tahbaz-Salehi (2015b), Gofman (2017), Babus (2016), Bardoscia (2017), Cabrales, Gottardi, and Vega-Redondo (2017), Erol and Vohra (2018), Kanik (2018).

[10] Although debt and equity capture only some of the main forms of interdependencies that one observes in practice, they provide a lens into many others, as many swaps and derivatives involve essentially either fixed payments or payments that depend on the realization of the value of some investment of one of the parties. Also, things like syndicated loans and other joint investments have features that are similar to equity. Debt and equity capture the two primary situations: contracts that are fixed in payments and those in which payments depend on value realizations.

on the overall financial system when trading-off the benefits and costs of a risky investment. We use this analysis to examine optimal regulation of portfolio choices in the form of reserve requirements, bailouts, or laissez-faire. Larger potential gains from the risky investment favor laissez-faire, and the use bailouts when network contagion becomes likely, while smaller gains favor reserve requirements. Interestingly, more debt actually favors laissez-faire and bailouts, while greater equity in the network favors reserve requirements, since the opportunity cost of requiring reserves scales with the amount of debt that an organization holds. The optimality of the type of regulation depends on different measures of financial centrality, in a way that we characterize precisely.

In addition, we show that banks have strong incentives to correlate their investments with those of their counterparties, which can be seen as another form of excessive risk-taking. This happens for a slightly different reason, and we refer to it as 'risk-stacking', as it helps them align their solvencies with situations in which they can enjoy better payments from their counterparties, and being insolvent when they expect lower potential payments from their counterparties. On the extensive margin, we show that financial organizations also choose to have fewer counterparties with which to cross-insure that would be socially optimal, again since their owners do not bear the full brunt of their potential bankruptcy costs.

We also discuss how the model can be used to measure systemic risk from a potential shock or default; and we identify the minimum costs and specific interventions a government needs to bail out an insolvent network. Our results fully characterize the conditions needed for solvency under both the best and worst equilibria, as there can exist multiple equilibria. We show that there are multiple equilibria if and only there exist a certain type of dependency cycle in the network. We also show how the minimum bailout payments needed to ensure solvency depend on cycles in the network, and point out the importance of canceling out cycles of debts among banks.

The literature characterizing how financial organizations that are embedded in a network choose their portfolios is scarce. A notable exception is an (independent) study by Shu (2019). He also finds that network externalities between banks' investment decisions, lead them to take more risk than if they were isolated from others and to correlate their investments. His model is complementary to ours as it differs from ours in several important ways. First, he focuses on networks of interbank unsecured debt, and hence abstracts away from countervailing incentives stemming from the interaction between debt and equity claims. His model also lacks bankruptcy costs, and so defaults are not an efficiency concern and our study of the multiplicity of equilibria has no counterpart. Finally, following Acemoglu et al. (2015a), he restricts attention to regular networks in which each bank's total interbank claims and liabilities are equal. In his model, the network externality arises from the fact that debt to depositors has higher seniority than that to other banks. Depositors become residual claimants in case of default, and thus remaining solvent and paying one's debt to the defaulting bank only reduces one's payoff. Our analysis of optimal regulation depends on features not in his model: the centrality of particular financial institutions, different effects of debt and equity, as well as multiple equilibria.

Finally, Elliott, Georg and Hazell (2018) also highlight an incentive of banks to choose partners that have portfolios similar to their own. They examine which network of interbank equity claims and correlation structure of investments arise endogenously in equilibrium when banks act under limited liability. Their main result is that banks have an incentive to partner with counterparties whose portfolios are positively correlated with their own in order to shift losses from shareholders to creditors. Using data on the German banking system, Elliott, Georg and Hazell (2018) also provide strong empirical evidence that banks lend more to other banks with portfolios similar to their own. Thus, such correlations are observed. Similar incentives arise in our model for partly different reasons, and persist even if banks do not have limited liability. In our model, because of financial interdependencies, banks' values depend positively on each other, which induces complementarities in their returns to investments: a high return for a bank is weakly more valuable if its partners have high returns as well, pushing them to correlate their portfolios.[11,12]

Though further away from what we study in this paper, the recent literature on the inefficiency of network formation in financial settings is also worth noting. The equilibrium network often has a core-periphery structure,[13] and is generally socially inefficient, either because it induces excessive systemic risk[14] or too much market power of core organizations.

2 A Model of Financial Interdependencies

Here, we define a model of financial interdependencies that allows us to examine the choices of financial organizations in the network. In the model we include both debt and equity since there are important distinctions in the incentives and systemic risk that they generate on the network. Insolvencies are induced by inability to pay debts, but not by equity. Thus,

[11] Acharya and Yorulmazer (2007) highlight a different channel that can drive banks to correlate their investments: if it is ex-post optimal for the regulator to bail-out banks when many of them fail at once, but to let them rescue each other when only few of them are insolvent, then banks have an incentive to herd in order to capture the bailout subsidies. A similar intuition arises in Arya and Glover (2001) in a principal-agent model, in which agents may want to coordinate on the bad action—e.d. low effort—if this means a higher probability of being bailed-out by the principal. In those papers the inefficiency stems from limited commitment of a principal or regulator, rather than from network externalities. Finally Erol (2019) shows that the anticipation of bailouts leads banks to form a highly concentrated network, which entails greater output volatility and systemic risk.

[12] Duffie and Wang (2016) study whether bilateral bargaining over the terms of contracts between banks achieves efficiency. Assuming away general cross-network externalities—and hence the possibility of default cascades—they propose a bargaining protocol that leads to socially efficient equilibrium contracts between banks. They however highlight that small changes to the proposed protocol may lead to additional inefficient equilibria.

[13] See Soramäki, Bech, Arnold, Glass and Beyeler (2007); Bech and Atalay (2010); Erol (2019); Blasques, Bräuning and Van Lelyveld (2018). There are a variety of reasons to have a core-periphery structure as there are advantages to having a concentration in intermediaries, which can then better manage their inventory and match buyers with sellers (e.g., see Craig and Von Peter (2014); Babus and Hu (2017); Farboodi (2017); Wang (2017)).

[14] For instance, see Elliott, Golub and Jackson (2014).

the relative combination of debt and equity in a bank's portfolio has important implications for incentives, and so it is important to allow for that distinction in the model.

The distinction between debt and equity is not just a theoretical consideration, since both types of securities are needed to capture the balance sheets of some of the most prominent and important types of financial organizations. For example, banks' balance sheets involve substantial portions of deposits, loans, CDOs (collateralized debt obligations), and other sorts of debt-like instruments. In contrast, venture capital firms and many other sorts of investment funds typically hold equity and are either held privately or issue equity. Such funds, and other forms of shadow-banking, are increasingly important as a source of funding for businesses, especially in the tech sector and other growing parts of the economy. Furthermore, some large investment banks are hybrids that involve substantial portions of both types of exposures. Understanding the different incentives that these different forms of organizations have, and the externalities that are present, is thus relevant.

Many forms of contracts, including some swaps, can be approximated as some combination of debt and/or equity. Nonetheless, there are obviously more complex contracts that can also be built into such a model. We also describe a general form of the model for such more complex contracts in an appendix, but the main insights regarding incentives are most crisply analyzed with debt and equity, which captures the main tradeoffs and still remains piecewise-linear and hence tractable.

2.1 Banks, Shadow Banks, CCPs, and other Financial Organizations

Consider a set $N = \{0, 1, \ldots, n\}$ of organizations involved in the network. We treat $\{1, \ldots, n\}$ as the financial organizations, or "banks" for simplicity in terminology. These should be interpreted as a broad variety of financial organizations, including banks, venture capital funds, broker-dealers, CCPs (central counterparties), insurance companies, and many other sorts of shadow banks and institutions that have substantial financial exposures on both sides of their balance sheets. These are organizations that can issue as well as hold debt, buy and sell equity, and make other investments. The broad applicability of the model allows it to be used to assess risks of the evolving roles of CCPs and the large shadow banking sector, and not just traditional banks.

We lump all other actors into 0 as these are entities that either hold debt and equity in the financial organizations (for instance private investors and depositors), or borrow from or raise money from the financial organizations (for instance, most private and public companies). Their balance sheets may be of interest as well, as the defaults on mortgages or other loans could be important triggers of a financial crisis. The important part about 0 is that, although these may be the initial trigger and/or the ultimate bearers of the costs of a financial crisis, they are not organizations that are the dominoes, becoming insolvent and defaulting on payments as a result of defaults on their assets. In aggregate, it may appear that there is debt going both in and out of node 0, but none of the individual private investors that comprise node 0 have debt coming both in and out. [15]

Each organization i has a value V_i, which is the total value of the equity: the value of all investments including those in other organizations, net of all debts owed. This value is shared among private shareholders and institutional shareholders. We now describe these values in greater detail.

2.2 Primitive Assets, Organizations, and Cross-Holdings

Bank portfolios are composed of both investments in primitive assets outside the network and financial contracts within the network. For our purposes the details of investments in primitive assets are not important: suppose they involve some initial investment of capital and then pay off some cash flows over time, often randomly. We call these primitive investment opportunities *assets* – $M = \{1, \ldots m\}$ – and denote by p_k the present value (or market price) of asset $k \in M$. The values of organizations are ultimately based on their investments in these assets. Let $q_{ik} \geq 0$ be the quantity invested in asset k by organization i, and $\mathbf{q}$ the matrix whose (i, k)-th entry is equal to q_{ik}. (Analogous notation is used for all matrices.) The total value of i's direct investments in primitive assets is thus $\sum_k q_{ik} p_k$, or $q_i \cdot p$.

The book or equity value V_i of an organization i equals the value of organization i's primitive assets plus the debts it is owed minus those it owes plus the value of its claims on other organizations. For the sake of transparency and tractability in what follows, we restrict the set of financial contracts to debt and equity. In the appendix (Section A) we discuss how valuations work with fully general contracts across banks, and how existence depends on a monotonicity of those contracts in underlying primitive-asset investments.

If bank i owns an equity share in bank j, it is represented by $S_{ij} V_j$ for some $S_{ij} \in (0, 1)$. A debt contract with a current (net present) value of D_{ij} corresponds to a payment of D_{ij} as long as bank j is solvent, while it will look like an equity share if j becomes insolvent. A call option looks like a value of $D_{ij} = 0$ until V_j exceeds a certain value, and then looks like a claim S_{ij} above that level. In a world with debt and equity, then the value of organization i's investments in other organizations is then $\sum_j D_{ij} + S_{ij} V_j$ and its total debt liability is $\sum_j D_{ji}$. These together with the primitive investments determine the equity value, V_i, which is then owned by private shareholders and other financial organizations through their equity shares in i.

[15] Of course, this is an approximation and there is a spectrum that involves a lot of gray area. For instance, Harvard University invests tens of billions of dollars, including making large loans. At the same time it borrows money and has issued debt of more than five billion dollars. It is far from being a bank, but still has incoming and outgoing debt and other obligations. This is true of many large businesses, some that come closer to resembling banks than others. Also, some companies' solvency could be affected by other bankruptcies and bring down counterparties, especially if they are key players in a supply chain. In that case, those companies would be included in the main $1, \ldots, n$, while companies that are mostly just borrowers or just lenders and not of significant concern as potential dominoes are the actors in 0. It is not so important for us to draw an arbitrary line through this grey area to make the points that we do with our model. Nonetheless, this is something that a regulator does have to take a stand on when trying to assess systemic risk, and in practice may even be dictated by jurisdictional rules.

Let $\mathbf{D}$ the matrix of debt claims and $\mathbf{S}$ the matrix of equity claims, where D_{ij} is what organization j owes to i and S_{ij} is i's equity claim on j. An organization cannot have a debt or equity claim on itself, so that $S_{ii} = D_{ii} = 0$ for all i. Equity is a claim on some portion of a bank's value, and the equity shares sum to one: whatever share is not owned by other financial institutions accrues to some outside investor: $S_{0i} = 1 - \sum_{j \neq 0} S_{ji}$.[16] The one exception is that no shares are held in the outside investors so that $S_{i0} = 0$ for all i (and shares held by banks in private enterprises are modeled via the p_i's).

Finally, in order to ensure that the economy is well-defined, we presume that there exists a directed equity path from every financial institution to some private investor (hence to node 0). This rules out nonsensical cycles where each organization is entirely owned by others in the cycle - but none are owned in any part by any private investor. For instance if A owns all of B and vice versa, then there is no sensible solution to the equity value of those two organizations. A financial network is then a tuple $(N, \mathbf{D}, \mathbf{S})$.

Let

$$D_i^A = \sum_j D_{ij} \quad \text{and} \quad D_i^L = \sum_j D_{ji}$$

denote the total amount of debt owed to i and owed by i, respectively. The former is then i's debt assets, and the later its liabilities.

2.3 Values in a Network of Debt and Equity

Let V_j^+ denote $\max[V_j, 0]$. Given that equity has limited liability, then the value to i of its equity holding in j is $S_{ij}V_j^+$.

The book or equity value V_i of an organization i can then be written as:

$$\begin{aligned} V_i &= \sum_k q_{ik} p_k + \sum_j (D_{ij} - D_{ji}) + \sum_j S_{ij} V_j^+ \\ &= \sum_k q_{ik} p_k + D_i^A - D_i^L + \sum_j S_{ij} V_j^+ \end{aligned} \tag{1}$$

Equation (1) can be written in matrix notation as

$$\mathbf{V} = \mathbf{qp} + \mathbf{D}^A - \mathbf{D}^L + \mathbf{S}\mathbf{V}^+. \tag{2}$$

To ensure a (unique) solution to (2), it is sufficient that there exists at least one private organization (e.g., at least one private citizen who owns some share of some organizations), and that every public organization has some indirect private ownership—i.e. there exists a directed path in equity from every public bank to some private investor. In the case in which

[16]Here, we simply model any fully privately held firm as having some outside investor owning an equity share equal to 1. This has no consequence, but allows us to trace where all values ultimately accrue.

all organizations are solvent, so that $\mathbf{V}^+ = \mathbf{V}$, solving (2)[17] then yields

$$\mathbf{V} = (\mathbf{I} - \mathbf{S})^{-1} \left[\mathbf{qp} + \mathbf{D}^A - \mathbf{D}^L\right]. \tag{3}$$

We handle the case with insolvencies and bankruptcies below.

The special case in which $S_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$ corresponds to the model of Eisenberg and Noe (2001); Gai and Kapadia (2010), and the special case in which $\mathbf{D} = \mathbf{0}$ corresponds to Elliott et al. (2014).

Written this way, the book or equity value of a publicly held organization coincides with its total market value. Indeed as argued by both Brioschi, Buzzacchi, and Colombo (1989) and Fedenia, Hodder, and Triantis (1994), the ultimate (non-inflated) value of an organization to the economy – what we call the "market" value – is well-captured by the equity value of that organization that is held by its *outside* investors – or the *final* shareholders who are private entities that have not issued shares in themselves. This value captures the flow of real assets that accrues to final investors of that organization. This is exactly what is characterized by the above values since summing them up (again, for the case of nonnegative values) gives

$$\begin{aligned}
\sum_{i\neq 0} V_i &= \sum_{i\neq 0}\sum_k q_{ik}p_k + \sum_{i\neq 0} D_i^A - \sum_{i\neq 0} D_i^L + \sum_{i\neq 0}\sum_{j\neq 0} S_{ij}V_j \\
&= \sum_{i\neq 0}\sum_k q_{ik}p_k + D_0^L - D_0^A + \sum_{j\neq 0}(1 - S_{0j})V_j \\
\implies D_0^A - D_0^L + \sum_i S_{0i}V_i &= \sum_i\sum_k q_{ik}p_k
\end{aligned}$$

It is easy to see that the total equity value accruing to all private investors (so value net of debt) equals the total value of primitive investments.[18]

2.4 Discontinuities in Values and Failure Costs

We now introduce bank defaults and their associated costs.

If the value of an organization i's assets falls below the value of its liabilities, then i is

[17]To see that $(I - S)$ is invertible, and to ensure that all of the V's are bounded, it is sufficient that when we examine the directed network defined by positive S_{ij}'s, every node in the network is path connected to some private node - so j that has no public equity. Without this condition, the Vs are indeterminate. Intuitively, without this condition, there are no real owners of some companies - there would be a cycle in which every firm's value is dependent on the other values in the cycle and they all are fully owned in the cycle, and then the values are no longer tied down by the fundamentals. For a more formal argument, see Appendix B.

[18]Note that when debts are zero, this value ends up being the same as that in (3) of Elliott, Golub and Jackson (2014). The difference is that here we explicitly model the outside investors as being part of the network, which enables us to simplify the solution, eliminating the need for tracking the $\widehat{C}$ matrix that was used there.

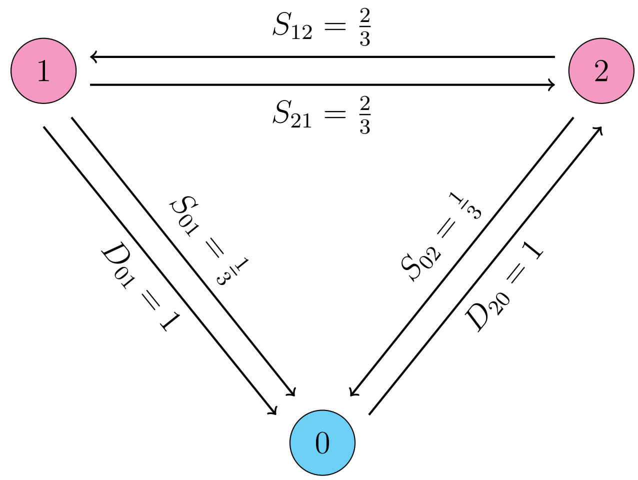


Figure 1: Arrows point in the direction of the flow of claims: to whom value is owed. The banks own shares in each other and the outside investor owns the remaining shares (1/3 in each case). The outside investors have borrowed from Bank 2 and have deposits in Bank 1. Suppose both banks have the same portfolio of primitive investments, of value $p = 1$. Using equation (1), the values of Banks 1 and 2 solve: $V_1 = p + S_{12}V_2 - D_{01}$ and $V_2 = p + S_{21}V_1 + D_{20}$. This yields $V_1 = 2.4$ and $V_2 = 3.6$. Importantly, the value accruing to outside investors is $D_{01} - D_{20} + S_{01}V_1 + S_{02}V_2 = 2 = 2p$, which is exactly the total value of primitive investments.

said to *fail* and incurs failure costs $\beta_i(\mathbf{V}, \mathbf{p})$. These costs could depend on the degree to which i and others are insolvent as well as the value of its various direct investments. In the case of debt and equity, an organization's liabilities are its debt obligations, and its assets include primitive investments and equity it holds in other organizations, combined with the value of debt it is owed by others.[19]

With the possibility of bankruptcy, a debt owed to i by organization j, D_{ij} depends on the value of V_j and thus its solution depends ultimately on the full vector $\mathbf{V}$. To make these interdependencies explicit, we let $D_{ij}(\mathbf{V})$ denote the amount of debt that bank j actually pays back to i.

There are two regimes. If organization j remains solvent, it can repay its creditors in full, and then for all i

$$D_{ij}(\mathbf{V}) = D_{ij}.$$

If instead organization j defaults, then debt holders become the residual claimants in case of insolvency, and

$$D_{ij}(\mathbf{V}) = \frac{D_{ij}}{\sum_h D_{hj}} \max\left(\sum_k q_{jk}p_k + D_j^A(\mathbf{V}) + \sum_h S_{jh}V_h^+ - \beta_j(\mathbf{V}, \mathbf{p}), 0\right). \tag{4}$$

[19]The model extends to allow for other sorts of contracts and thresholds for defaults, by simply having a different rule for when default occurs, for instance in the case of more complicated liabilities.

A simple example of bankruptcy costs corresponds to the case where

$$\beta_j(\mathbf{V},\mathbf{p}) = b + a\left[\sum_k q_{ik}p_k + \sum_j S_{ij}V_j^+ + D_i^A(\mathbf{V})\right]$$

with $b \geq 0$ and $a \in [0,1]$. In that case, bankruptcy costs are some fixed legal or other costs, as well as some share of the value of the bank's assets – for instance, if only recovers some fraction of its assets (e.g., due to a markdown on a firesale of its assets) or has a portion of its legal costs that scale with the size of the enterprise. If the bank is unable to salvage any of its assets, then $D_{ij}(\mathbf{V}) = 0$ whenever j defaults.

In any case, equity holders of organization j do not receive any payment when j is insolvent:

$$S_{ij}V_j^+ = 0.$$

The bankruptcy costs incurred by organization i are then

$$b_i(\mathbf{V},\mathbf{p}) = \begin{cases} 0 & \text{if } \sum_k q_{ik}p_k + \sum_j S_{ij}V_j^+ + D_i^A(\mathbf{V}) \geq D_i^L \\ \beta_i(\mathbf{V},\mathbf{p}) & \text{if } \sum_k q_{ik}p_k + \sum_j S_{ij}V_j^+ + D_i^A(\mathbf{V}) < D_i^L. \end{cases} \tag{5}$$

Note that we have carefully written bankruptcy costs $b_i(\mathbf{V},\mathbf{p})$ as a function of how $\sum_k q_{ik}p_k + \sum_j S_{ij}V_j^+ + D_i^A(\mathbf{V})$ compares to D_i^L instead of as a function of V_i directly. This avoids having bankruptcies driven solely by anticipating bankruptcy costs, even when an organization has more than enough assets, even cash on hand, to cover its liabilities.[20] Our formulation allows for coordination issues, bank runs, and other sorts of multiplicities in equilibria that are of practical interest, while avoiding more trivial self-fulfilling bankruptcies that would just be modeling curiosities.

The valuations in (3) have analogs when we include these discontinuities in value due to failures and bankruptcy costs. The discontinuous drops impose costs directly on organizations' balance sheets. and so the book value of organization i becomes:

$$V_i = \sum_k q_{ik}p_k + \sum_j S_{ij}V_j^+ + D_j^A(\mathbf{V}) - D_i^L - b_i(\mathbf{V},\mathbf{p}),$$

where $b_i(\mathbf{V},\mathbf{p})$ is defined by (5). This leads to a new version of (3):

$$\mathbf{V} = (\mathbf{I} - \mathbf{S}(\mathbf{V}))^{-1}\left(\left[\mathbf{qp} + \mathbf{D}^A(\mathbf{V}) - \mathbf{D}^L\right] - \mathbf{b}(\mathbf{V},\mathbf{p})\right), \tag{6}$$

[20] Such a self-fulfilling bankruptcy would go beyond a bank run, since it would not be due to the organization not having enough cash on hand to pay its debts, but instead due to a fixed point issue that if we presume all the cash is eaten up by paying bankruptcy costs, then indeed the organization can become bankrupt for no other reason. This self-fulfilling problem posed by bankruptcy costs is of less interest to us here, as it is not so much a network issue—it is not a bank-run problem, nor a problem of coordinating payments with some cycle of other banks,— and seems of less practical interest.

where $\mathbf{S}(\mathbf{V})$ reflects the fact that $S_{ij}(\mathbf{V}) = 0$ whenever j defaults, and is S_{ij} otherwise.

It can be that some solutions for the values are negative - as the bankruptcy costs could exceed the assets of the company. We can think of these as real costs, for instance a decaying and polluting plant which is abandoned, or some court costs, etc., which in some cases are absorbed by a government or imposed on the public.

3 The Multiplicity of Bank Values and Minimum Bailouts: the Role of Cycles in the Network

Although the possibility of multiple equilibria in financial networks is well-known, the conditions under which they exist, and their implications are not. In this section we characterize when there exists a multiplicity.

We also characterize the minimum bailout payments needed to ensure solvency in both the best and worst equilibrium. These bailouts are optimal in avoiding the social costs of bankruptcy.

3.1 Existence and a Lattice of Equilibrium Values

Under the assumptions mentioned above, and under the assumption that the bankruptcy costs β are nonincreasing in $(\mathbf{p}, \mathbf{V})$ (so that bankruptcy costs are weakly lower when organizations have greater values), there always exists a solution to equation (6). There can exist multiple solutions to the valuation equation (multiple vectors $\mathbf{V}$ satisfying (6)) in the presence of discontinuities, and in fact, the set of equilibrium values forms a complete lattice.[21] We use the term "equilibrium" to refer to a fixed point satisfying equation (6) to keep with the literature, but note that the term "equilibrium" is also used for situations in which we endogenize the portfolio and partner choices.

As highlighted by Elliott, Golub and Jackson (2014), there are two sources of multiplicity. The first one is due to self-fulfilling bank runs (see classic models such as Diamond and Dybvig (1983)): there can be an equilibrium in which some bank i is solvent and another one in which it defaults even when keeping everything else constant. The second source of multiplicity comes from bank interdependencies: there can exist an equilibrium in which a subset of banks is solvent and another in which they all default. This corresponds to self-fulfilling default cascades.

Since the equilibria form a complete lattice, there exists a "best" equilibrium, as well as a "worst" equilibrium in which the set of defaulting organizations is minimal and maximal, respectively (Elliott, Golub and Jackson (2014)). We note that the worst equilibrium also corresponds to a situation in which no bank makes any partial payments until it has received

[21] This can be seen by an application of Tarski's fixed point theorem, since organizations' values depend monotonically on each other.

all of its payments in.[22] Despite the fact that bank values depend on the details of how they make payments, as well as the rules and timing in bankruptcy proceedings, our results do not.

In this paper, we account for the multiplicity of equilibria, and in some cases distinguish between results for the best and worst equilibria. In particular, studying the worst equilibrium for values after a first failure may be relevant given how financial markets are subjects to runs and freezes: the regulator may want to consider the worst that could happen under such circumstances. Thus, before proceeding, we provide a result outlining the how multiplicity depends on certain kinds of cycles of liabilities, combined with bankruptcy costs.

Eisenberg and Noe (2001) show that, *in the absence of bankruptcy costs* and in a network with only debt liabilities, the values of solvent banks are uniquely determined; and, if a bank has a strictly positive value in one equilibrium, she cannot be defaulting in another. However, with bankruptcy costs, the multiplicity of equilibria comes from the discontinuous drop in value of banks at default, which can create self-fulfilling combinations of defaults. Importantly, this relies on the possibility for the bankruptcy costs to feedback through the financial network, and hence on the presence of cycles, as we now show.

Define a *dependency cycle* to be a directed cycle of organizations, each having either debt or equity on the previous one, that involves *at least some* debt. In particular, a dependency cycle is a sequence $i_0, \dots i_K$, for some $K \geq 1$ such that: $i_0 = i_K$, $D_{i_{\ell+1} i_\ell} > 0$ or $S_{i_{\ell+1} i_\ell} > 0$ for each $\ell < K$, and $D_{i_{\ell+1} i_\ell} > 0$ for at least one $\ell < K$.

The following lemma highlights how equilibrium multiplicity depends on the presence of dependency cycles.

Lemma **1.**

(i) If there is no dependency cycle, then the worst and best equilibria coincide.

(ii) Conversely, if there is a dependency cycle, then there exist bankruptcy costs and values of bank investment portfolios $\mathbf{qp}$ *such that the best and worst equilibria differ.*

Part (i) of Lemma 1 is intuitive: to get a multiplicity requires a feedback between defaults: a cycle of organizations in which each defaulting helps justify the others' defaults, while if they are all solvent then this guarantees their solvency. Without such a feedback cycle, this multiplicity cannot hold.

Part (ii) is more delicate. It is clear that having larger bankruptcy costs makes feedback easier, and lower costs makes it harder. For example, consider a simple debt cycle where bank 1 owes 2 a debt D and 2 owes 1 the same amount D. Without loss of generality, let p_i denote the value of i's outside investments. Since each bank is always able to repay all its liabilities if its counterparty does, the best equilibrium always has both banks being solvent, irrespective of $\mathbf{p}$. The range of primitive investment values (p_is) for which there exists of a

[22] That situation is discussed by Bardoscia, Ferrara, Vause and Yoganayagam (2019). It can also arise as a coordination failure as discussed by Allouch and Jalloul (2017).

(worst) equilibrium in which both banks default depends on the size large bankruptcy costs. For the purposes of illustration, suppose that costs are some fraction a of a bank's assets, and that both banks have a portfolio values $p_1 = p_2 = p$. For which values of p does there exist an equilibrium with mutual bankruptcy? Supposing that such an equilibrium does exist, then the amount that banks pay each other turns out to be some $d < D$ that by (4) solves:

$$d = (1 - a)(p + d),$$

or $d = \frac{1-a}{a}p$. From (5) that we need $p + d < D$ in order to incur bankruptcy costs, which then requires that

$$p + \frac{1-a}{a}p < D$$

or $p < aD$. Thus, as bankruptcy costs fall the set of asset returns $\mathbf{p}$ that can generate self-fulfilling default cycles, and hence multiple equilibria, shrinks.

Interestingly, beyond this example, as we move to more heterogeneous levels of debt and equity dependencies between different banks in a cycle, the potential for such cascades depends on the level of debt imbalances in the network. Indeed, if a bank is a large net creditor – that is, $D_i^A >> D_i^L$ – then it acts as a buffer and makes a feedback cycle harder to find. Informally, self-fulfilling default cycles are a concern only if bankruptcy costs are large enough to offset debt imbalances.

3.2 Minimum Bailouts in both the Worst and Best Equilibria

The presence of a default cycle in a network can be exploited by a regulator who tries to avoid default. Just as defaults cascade, the same operates in reverse and a well-placed bailout can have far-reaching consequences. This problem relates to Demange (2016), who characterizes the optimal cash injection policy in a network of financial liabilities under proportional rationing in case of default. She defines a threat index that identifies banks with highest marginal social value of liquidity, assuming the policy does not change the set of defaulting banks. Here, instead, we examine how much of an injection is needed to change and avoid defaults.

Consider a regulator who can inject liquidity into the network to ensure that some banks remain solvent; i.e., it can bailout a subset of banks $B \subseteq N$ by changing their portfolio values from p_i to some $p_i' > p_i$ by making direct payments to the banks of $p_i' - p_i$. If bank $i \in B$ is sufficiently bailed-out, it pays back its debt to all its creditors, who then may become solvent themselves. We now investigate the necessary costs and how the banks that need to be bailed out depend on the cycles in the network. We also further highlight the difference between the best and worst equilibria.

As should be obvious by now, properly assessing systemic risk involves a holistic view of the network.[23]

[23] Attempting to assess systemic risk without detailed and comprehensive network information is what Jackson (2019) refers to as "flying jets without instruments": operating a complex interactive system without

We first provide an example that illustrates the importance of seeing the structure of the network cycles in order to determine an optimal bailout.

3.2.1 The Necessity of Network Information

An important component of systemic risk assessment is stress testing, which is usually run in a decentralized manner. The main input into many stress tests is balance sheet data, which describes the amount of each type of financial assets and liabilities held by each bank. Depending on the jurisdiction, balance sheet data does not always provide complete, or even partial, information about the identity of one's counterparties, nor their other investments and counterparties, and hence about the network structure. Even if a stress test accounts for direct counterparties, it may miss information about systemic issues such as the presence of cycles which we have shown are so critical to understanding equilibria. "Local" data can also completely miss which banks are most likely to start a default cascade, or be caught up in one. The point is straightforward, but worth emphasizing given its importance, which is illustrated as follows.

For simplicity, consider a network in which banks only have debt contracts between each other. A measure of systemic risk based on local information only depends on $(D_i^A, D_i^L)_{i \in N}$. To show why this is insufficient information, we give an example of financial network in which two banks have *identical* balance sheets, and yet their defaults have significantly different consequences. Hence if the central authority were able to bailout one (and only one) of the two organizations, it could not take the optimal decision based on such local information. Consider the network composed of four banks depicted in Figure 2.

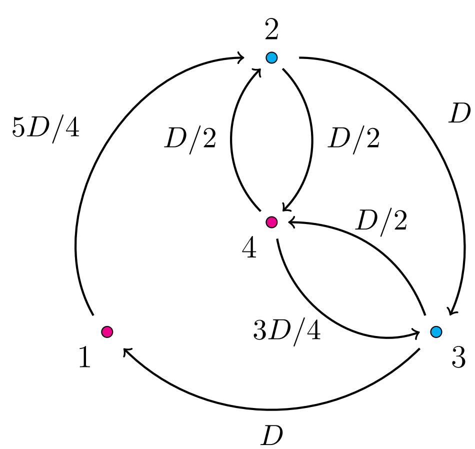


Figure 2: Arrows point in the direction that a debt is owed. Banks 1 and 4 (magenta) have total debt liabilities of $5D/4$ and debt assets of D, and are net debtors. Banks 2 and 3 (blue) have debt liabilities of $3D/2$ and debt assets of $7D/4$, and are net creditors.

the necessary measurements. Even though some stress tests and measures (e.g., S-risk) that work without network information may correlate with more precise full network measures, if those measures are only approximately capturing the real risks, .

Suppose the portfolios of Bank 1 and 4 yield 0, so that they are both insolvent, whereas Banks 2 and 3 earn returns on their investment between $3D/4$ and D. The recovery rate on assets of a defaulting bank is zero. Note that Bank 1 and 4 have the same balance sheet since $D_1^A = D_2^A = D$ and $D_1^L = D_2^L = 5D/4$. However, only Bank 1 induces widespread default contagion if it remains insolvent. Indeed, Banks 2 and 3 have enough buffer to absorb the shock of Bank 4's default, but not that of Bank 1. Hence, bailing out Bank 1 prevents the whole system from insolvency, while bailing out Bank 4 does not change anything and a full systemic failure occurs.

This example also highlights the fact that, without network information, one cannot even identify which banks are at risk of insolvency. For instance, if one examines the books of Bank 3 without knowing that Bank 2 is exposed to Bank 1, even if one knows the portfolio realizations of 3's counterparties, but does not know the looming failure of Bank 1 (which is not one of Bank 3's direct counterparties), it would appear that Bank 3 was free from danger of insolvency.

The first assessments of systemic risk that involve a nontrivial portion of the actual network are beginning to emerge, at least in Europe. For example, the European Central Bank has information on the counterparties involved in the largest exposures of most banks within its jurisdiction. This permits the construction of a network of a portion of the assets and liabilities within the European banking sector, and some pointers to banks outside of Europe, and thus some of the first calculations of systemic risk of a nontrivial part of the network are beginning to emerge (e.g., see Covi, Gorpe and Kok (2018)). Similarly, the Bank of England has regulatory data on bilateral transactions between UK banks, allowing for the analysis of the UK interbank network in different asset classes (see Ferrara et al. (2017); Bardoscia et al. (2018)). This is an important move of the assessment of systemic risk in the right direction, but much more is needed and especially outside of Europe and for the growing shadow banking system which falls outside of most jurisdictions.

3.2.2 Balanced Networks

For the rest of this section, since we are treating portfolio choices as given, we save on notation and just refer to the overall value of $q_i \cdot \mathbf{p}$ as p_i, so without loss of generality for this analysis this can be thought of as treating $\mathbf{q}$ as the identity matrix and $\mathbf{p}$ as being an $n+1$ dimensional vector where the ith entry is i's portfolio value.

For the sake of Proposition 1, we presume a recovery rate of zero upon default and that there are no cross-equity holdings, and return to describe the equity case later. Furthermore, all of the definitions that follow are relative to some specification of $\mathbf{p}, \mathbf{D}$, and we omit its mention.

The worst equilibrium can then be thought of as the requirement that a bank can pay back its debts if and only if it already has sufficient capital to cover all of its debts based on the amount of incoming debts that have already been paid to it, together with any bailout payments and outside assets. In particular, this rules out partial payments: even if a bank has some money coming in, it cannot use that money to pay some of its debt until it is fully solvent. Such a requirement often applies if all debt claims have equal priority, but makes bailing-out more demanding: the minimum set of banks that need to be bailed-out can be strictly larger under this rule than when partial repayments are allowed.

To characterize optimal bailouts, it is useful to begin by examining some benchmark cases that we refer to as *balanced networks*, as they play key roles in the more general characterization.

We say that a network is *weakly portfolio balanced* if $p_i + D_i^A \geq D_i^L$ for all i. This is a weaker requirement than debt-balance, just requiring that each bank's (non-equity) assets are enough to cover its debt liabilities, presuming its incoming debt assets all are fully valued.

We say that a network is *exactly portfolio balanced* if $p_i + D_i^A = D_i^L$ for all i. Since all debt claims and liabilities cancel out on aggregate, exact balance implies $\sum_i p_i = 0$, and given nonnegative portfolios then implies that $p_i = 0$ for all i. This then also implies a network has exactly balanced-portfolios only if it has balanced debt: $D_i^A = D_i^L$ for all i. Under this requirement, a bank is able to meet its debt obligations going out if and only if it receives all the debt payments it has coming in.

Any of these forms of balance is sufficient for all organizations to be solvent in the best equilibrium. This follows since if all banks but i honor their debt contracts then i can also pay back its debt fully in a weakly balanced network. Essentially, all debts can be canceled out, irrespective of the network structure. Things are different in the worst equilibrium, as we know from Lemma 1.

In cases in which the worst equilibrium differs from the best equilibrium, one can interpret the worst equilibrium as a coordination failure since all banks could have written-off their counterparties' debt without cost so as to avoid a general default of the system. In practice banks are unlikely to be able to coordinate in such a way if debts involve cycles rather than just direct canceling between two counterparties, as it would require all write-offs to be done simultaneously to maintain solvency.[24] Moreover, in practice the debts have different maturities and other covenants and priorities that further complicate any canceling out without an economy-wide renegotiation. This makes looking at the worst equilibrium important, and so we discuss both equilibria in what follows.

When there exist cycles, bailing out a bank on each cycle, by Lemma 1, leads the best and worst equilibrium to coincide. If (and only if) in addition, weak portfolio balance is satisfied, then that ensures that all banks are solvent in all equilibria. Thus, an optimal strategy that guarantees solvency in all equilibria is to use the minimum payments necessary to induce balance and eliminates all cycles in the network, taking advantage of cascades of repayments. We now provide a characterization.

Let us say that a bank i is *unilaterally solvent* if $p_i \geq D_i^L$. This means that regardless of whether any of the other banks pay the debts that they owe to i, i is still able to cover

[24] Some systems for such canceling are emerging, such as enterprises that offer "compression" services, which are essentially canceling out of cycles of contracts (e.g., see D'Errico and Roukny (2019)).

its debts.

We say that a set of banks S is *iteratively strongly solvent* if it consists of a nonempty subset of banks $S_1 \subset S$ that are unilaterally solvent; and then iteratively sets S_k such that the banks $i \in S_k$ are solvent if they receive the debts from all banks in sets $S_1, \ldots S_{k-1}$, but not if they only receive the debts from banks in $S_1, \ldots S_{k-2}$:

$$p_i + \sum_{j \in S_1 \cup \cdots S_{k-1}} D_{ij} \geq D_i^L > p_i + \sum_{j \in S_1 \cup \cdots S_{k-2}} D_{ij}.$$

Note that if N is iteratively strongly solvent, then all organizations are solvent in the worst equilibrium. Proposition 1 provides weaker conditions that are necessary and sufficient for this to hold. This then provides a base to understand minimum bailouts.

PROPOSITION **1.** *All organizations are solvent in the best equilibrium if and only if the network is weakly portfolio balanced.*

All organizations are solvent in the worst equilibrium if and only if the network is weakly portfolio balanced and there exists an iteratively strongly solvent set that intersects each directed (simple) cycle.[25]

An implication of Proposition 1 is that in a weakly balanced network, if one has an iteratively strongly solvent set that intersects each directed (simple) cycle, then that implies that the whole set of banks is iteratively strongly solvent. This is the crux of the proof.

The proposition is less obvious than it appears since an insolvent bank can lie on several cycles at once, and could need all of its incoming debts to be paid before it can pay any out. Solvent banks on different cycles could lie at different distances from an insolvent bank, and showing that each bank eventually gets all of its incoming debts paid before paying any of its outgoing debts is subtle. The proof is based on how directed simple cycles must work in a weakly balanced network and appears in the appendix.

Proposition 1 also implies that if both conditions are satisfied, then there is a unique equilibrium. Conversely, if portfolios are weakly balanced and there is no iteratively strongly solvent set intersecting every cycle, then there are necessarily multiple equilibria. Thus, in a weakly portfolio-balanced network (excluding equity), there is a unique equilibrium *if and only if* there exists an iteratively strongly solvent set that intersects each directed (simple) cycle.

One way to ensure having an iteratively strongly solvent set intersecting each directed cycle is to have at least one unilaterally solvent bank on each cycle, but this is not generally necessary, and so the iterative solvency condition is important. A special case is when the network is exactly portfolio-balanced, such that no bank has a capital buffer: then it becomes necessary to have at least one unilaterally solvent bank on each cycle for the whole system

[25]A simple cycle is one that only contains any organization at most once. If there is a solvent bank on each simple cycle then there is one on every cycle, since every cycle contains a simple cycle. A simple cycle is a list of links i_0i_1, i_1i_2, $\ldots i_Ki_0$ such that $D_{i_ki_{k+1}} > 0$ for all $k = 0, \ldots, K$ (with $K+1 = 0$), and i_0 is the only organization that appears twice.

to clear in the worst equilibrium. That is because in the case of exact balance, each bank is solvent if and only if it receives all the debt payments it is owed. However, note also that with exact balance, no bank that has any debts can be unilaterally solvent since their debts in are just enough to cover their debts out and so they have no buffer to make them solvent without those debt payments coming in, and so in that case bailouts will be necessary to get to solvency.

Proposition 1 highlights both the difference between the best and worst equilibria, as well as the role of cycles in clearing interbank liabilities. It resonates with the increasingly used technique of portfolio compression, which allows banks to eliminate offsetting obligations with other organizations taking part in the process, exploiting cycles in the financial network (e.g., see D'Errico and Roukny (2019)). Portfolio compression then reduces gross interbank exposures while keeping net exposures constant, which not only reduces systemic risk but regulatory requirements of participants as well. Portfolio compression improves the worse equilibrium since less it lowers the debts that banks owe and thus reduces the amounts that they need to be solvent either unilaterally, or once receiving some incoming payments.

3.2.3 Minimum Bailouts and Imbalanced Portfolios

We now investigate the minimum bailouts needed to ensure solvency of all banks in both the best and the worst equilibrium.

The best equilibrium is relatively easy to understand. If the network is not weakly portfolio balanced then some banks must be defaulting, and each bank that is not weakly balanced needs bailouts to be brought back to solvency. It follows then from Proposition 1 that the minimum amount of capital that needs to be injected in the system to ensure its solvency is exactly

$$\sum_i [D_i^L - D_i^A - p_i]^+.$$

The worst equilibrium is more complex, as weak balance is necessary, but not sufficient for solvency. One needs the above minimum payment, but then one also needs to inject enough additional capital into some set of banks to ensure the existence of an iteratively strongly solvent set that intersects each directed cycle in the network.

This is summarized in the follow corollary to Proposition 1.

Corollary **1.** *Consider a possibly imbalanced network with no equity.*

(i) The minimum necessary bailout needed to ensure solvency of the entire network in the best equilibrium is the total net imbalance in the economy $\sum_i [D_i^L - D_i^A - p_i]^+$ *(which is 0 if the network is weakly portfolio balanced).*

(ii) The minimum necessary bailout needed to ensure solvency of the entire network in the worst equilibrium is the total net imbalance in the economy and *injecting the minimum additional capital to generate an iteratively strongly solvent set that intersects each directed cycle in the network.*

(iii) If the network is fully compressed (so that all cycles of debt in the network have been cleared), then the best and worst equilibria coincide and the minimum necessary bailout needed to ensure solvency of the entire network in the best equilibrium is just the total net imbalance in the economy.

To ensure solvency of the entire network, the regulator first has to ensure weak balance of the portfolios of all banks. That is necessary and sufficient to ensure solvency in the best equilibrium. Then we are also back to the logic of the weakly balanced case, and to ensure solvency in the worst equilibrium, the additional capital that is needed is the minimum that then generates an iteratively strongly solvent set intersecting each directed cycle.

We remark that the payments made to regain weak balance, $\sum_i [D_i^L - D_i^A - p_i]^+$, will not be recovered by the government or other entity that intervenes in the bailout. However, all the additional payments made in the case of the worst equilibrium can be recovered. For instance, as we saw in the wheel example, the payments that are made in a cycle all cancel out and can be recovered once the capital has cycled through the entire network. This logic holds more generally, even in more complex networks. Indeed, once necessary payments to regain weak balance has been made, each bank's balance sheet satisfies $p_i + D_i^A \geq D_i^L$. To ensure solvency in the worst equilibrium, an appropriate set of banks then needs to bailed-out, that is an additional $D_i^L - p_i$ needs to be injected in a subset of banks. However, since this guarantees solvency of the whole network, we know that in the end these banks will get their debt payments $D_i^A \geq D_i^L - p_i$ in full. Hence the regulator will be able to recover the additional capital it had to inject to generate the iteratively strongly solvent set.

Figure 3 illustrates some of these ideas. In the imbalanced case on the right, the net payment that is needed to reach weak balance is to give d to Bank 3, which is never recovered, and that is enough to ensure full solvency in the best equilibrium. Then paying an additional d to Bank 3 is the minimum additional bailout needed to ensure solvency in the worst equilibrium, as that is enough to get Bank 3 to make its payments, which then makes Bank 2 solvent, which then makes Bank 1 solvent.[26] Bank 2 being solvent means it is then able to repay its debt of d to Bank 3, which can be recovered by the entity that intervened. Hence net bailout costs here equal d, which is the payment needed to make the network weakly balanced.

This problem of generating an iteratively strongly solvent set that intersects each directed cycle in the network is well-defined, but finding a minimum-cost set can be computationally hard. This is true both in terms of requiring a lot of information on the network structure, as well as the computational complexity of finding the minimum combination of banks to bailout to have at least one on each cycle. For instance, many banks will sit on multiple cycles, and even simply identifying all of the cycles in a network is challenging. This problem is in fact NP-hard.[27]

[26] Paying $d/2$ to 1 would not be enough to get 2 to be solvent since it still gets no payments from 3 in the worst equilibrium. One would need to pay $3d/2$ to 2 in order to ensure that it would be unilaterally solvent. So, 3 is the cheapest option.

[27] Indeed, when looking for all simple cycles in a graph, one also solves the Hamiltonian cycle problem,

Figure 3: Suppose all banks have $p_i = 0$. Ensuring solvency of the full network in the worst equilibrium in the balanced case (left panel), requires bailing out at least one bank per cycle, which can be done by either bailing out banks 1 and 3, or just bank 2. In the imbalanced case (right panel) the whole system clears if bank 3 is bailed out, and hence bailing out one bank per cycle is not necessary, and $\{3, 2, 1\}$ (in that order) form an iteratively strongly solvent set (once 3 gets an bailout injection of $2d$).

In practice, the problem is simplified for two reasons. First, many organizations will already be solvent, and it may only be a subset that are problematic—attention can then be concentrated on a subnetwork and some key organizations. Second, a core-periphery network (which many financial networks resemble) has a large amount of structure to it that makes its cycles easier to identify: it consists of a core clique together with a bunch of extra links to banks that have few connections to the core.

It can be significantly costlier to ensure solvency of an imbalanced than a balanced financial network, since the regulator necessarily has to inject the amount of net imbalance of all net borrowers. In reality, this imbalance can be large as many organizations have some debt contracts with partners who are private individuals and who are not otherwise involved in the network: for instance they have loans out as mortgages, or deposits that can be treated like debt for our purposes (e.g., demand deposits, certificates of deposit, overnight loans, money market accounts, etc.). These debts do not recycle into the network and so cannot be canceled out. Even though debts fully balance in aggregate—for each lender there is a borrower—some organizations or individuals are net lenders and others are net borrowers. Any organization belonging to the later category can be a first failure, and start a default cascade. They can also propagate failures, and are hence the critical organizations. On the contrary net lenders can never be the first to fail, but can be brought to bankruptcy as well if enough of their counterparties default.

Proposition 1 and Corollary 1 do not address the case in which banks hold equity in each other. That further complicates the calculations, since instead of just $[D_i^L - D_i^A - p_i]^+$, the imbalance of a bank now also depends on the value of its equity holdings in other banks, which then depends on who is solvent.

The general program to ensure solvency at minimum cost can be written as

$$\min_{p' \geq p : \mathbf{V}(\mathbf{p}') \geq 0} ||\mathbf{p}' - \mathbf{p}||,$$

which is known to be NP-complete. For an early algorithm to find all cycles, see Johnson (1975).

where the $\mathbf{V}$ is chosen to be either the best or worst equilibrium, depending on which is of interest. Again, this requires that the imbalance is at most 0 for all organizations, but now this includes equity values and so has to be solved as a fixed point.

The algorithm for finding the amount needed to remove the net imbalance in the case of the best equilibrium is straightforward to describe. It is as follows.

Let

$$p_i^n = p_i + D_i^A - D_i^L.$$

Now, calculate the best equilibrium values associated with asset returns $\mathbf{p}^n$, equity holdings $\mathbf{S}$ (noting that equity values in negative-valued enterprises are 0), and no debt $\mathbf{D} = \mathbf{0}$. The opposite of the total sum of the valuations of the organizations with negative values (ignoring bankruptcy costs) is the minimum bailout that is needed. Effectively, we know that all debts will be repaid in a bailout that ensures full solvency, and then the resulting bank values will be the basis on which equity values accrue. Organizations that are still negative, including all of their equity positions are the ones that will require bailout payments.

In the case of the worst equilibrium, the same logic applies, but then the base values are associated with the worst equilibrium. Then once those payments are made, one *recalculates* the worst equilibrium values given those payments, but with the original $\mathbf{D}$. By doing this, one identifies banks that are then unilaterally solvent (after the initial bailout payments), as well as any resulting iteratively strongly solvent set by consequence of those unilateral solvencies. If these are not enough to intersect each directed cycle, then additional bailouts will be needed, and an algorithm needs to be run to find the cheapest set. Note that those bailouts might not even be used to generate unilateral solvencies, but might just be enough to generate secondary solvencies given the unilateral solvencies, which eventually generate more solvencies. This is the analog of the problem without equity, but just augmented by additional value calculations that include equity of the resulting solvent organizations for each possible configuration of bailouts that is considered and the corresponding worst equilibria. Here, part (iii) of the Corollary becomes particularly important, since it means that if one can compress the network, then the issues with the worst equilibrium are avoided and one only has to deal with the initial bailouts needed to restore weak balance, which are necessary in any case.

Since many of the results apply to both the best and worst equilibria (as well as intermediate selections), as long as one is consistent in using the same equilibrium throughout the analysis,[28] we are only explicit about which equilibrium applies when it becomes necessary.

[28]We rule out that a bank believes that the selection of equilibrium changes as a function of its decisions – for instance that the best equilibrium applies for some decisions and the worst for others, which could alter the incentives arbitrarily. For more discussion of the coordination game underlying defaults, see Allouch and Jalloul (2017).

4 Distorted Incentives and Optimal Regulation

The previous section focused on characterizing values of banks, taking as given their portfolio and the structure of the financial network. We now model banks' investment decisions, and the potential inefficiencies that arise from network externalities.

4.1 Overly Risky Investment: The Intensive Margin

Because of interdependencies between organizations in the financial network, a bank's investment decision not only affects its value but also those of others. This relates to the standard agency problem between the manager of a firm and its shareholders highlighted in Jensen and Meckling (1976), in which there are differing objectives of the manager and shareholders.[29]

In this section, we study how financial contracts between organizations distort investment incentives, because of associated externalities in insolvencies and bankruptcy costs. The distortions in incentives are easy to understand, but are important to document because of their extremity and also because this analysis provides the base for an analysis of regulation and bailouts, as well as how incentives depend on whether interdependencies are based on debt or equity.

Let us begin by examining the incentive problems of a single bank that takes as given investments made by other financial organizations. Without loss of generality, the bank has a unit of capital to invest either in a risk-free asset with net return r or in some portfolio that pays a random p_i. One can think of this as a standard two-fund separation setting, with the main decision of the investor being how much risk to take.

We take the bank owners to be risk neutral and choose the portfolio that maximizes their expected returns from their investments. This allows us to abstract away from results linked to risk tolerance and to analyze solely systemic and structural externalities. We comment below on how the results extend to the case of risk-aversion. We also presume that the decision-makers in an organization are maximizing its profits and not the profits of other financial organizations.

The main incentive issues that we examine are between the shareholders (owners) of a firm and the other agents who are impacted by the firm's solvency. The decisions that we examine are the choices of portfolios and partnerships, which are observable and contractible, and so we abstract away from agency problems between the shareholders and managers of the firms. Those can also be an issue in some circumstances, but are well-studied elsewhere, and so we focus on the less-studied inefficiencies that arise beyond standard moral hazard problems.

[29] For more on this point, see Admati and Hellwig (2013). For some analysis in different network settings see Brusco and Castiglionesi (2007); Galeotti and Ghiglino (2019), and for a more general discussion about agency problems of excessive risk taking in the presence of externalities see Hirshleifer and Teoh (2009).

Suppose the bank's outstanding debt D_i^L is low enough, such that it could be paid back entirely were the bank to only invest in the safe asset: $D_i^L \leq (1+r)$. Insolvency happens if the value of the portfolio falls below the bank's liability D_i^L, in which case a cost of $b \leq D_i^L$ is incurred. Under limited liability, the shareholders get a payoff of zero in case of insolvency, and the bankruptcy cost b is born by whomever is holding debt, or a government or other enterprise that steps in.

The bank solves

$$\max_{q_i \in [0,1]} (1-s)\mathbb{E}\left[\left(q_i p_i + (1-q_i)(1+r) + \sum_{j \neq i} S_{ij} V_j(\mathbf{p}, q_i, q_{-i})^+ + D_i^A(\mathbf{p}, q_i, q_{-i}) - D_i^L\right)^+\right].$$

Here we do not allow for short sales (of either asset), which limits $q_i \in [0,1]$. As will be clear below, the analysis extends to short sales: the bank would choose to short the risk-free asset and leverage its investment in the risky asset.

We allow $V_j(\mathbf{p}, q_i, q_{-i})$ and $D_i^A(\mathbf{p}, q_i, q_{-i})$ to depend on the vector of portfolio values $\mathbf{p}$, as well as full vector of investment decisions (q_i, q_{-i}). We give sufficient conditions on the network such that it is a strictly dominant strategy for i to fully invest in the risky asset. Intuitively, this is the case as soon as such i's risky investments cannot feedback to i in discontinuous ways because of insolvencies of others that it triggers even without being insolvent itself.

We say that i is "*at risk of discontinuous feedback*," if i is part of a dependency cycle that involves *both* debt and equity, and that begins with a bank having an equity claim on i that is large enough to cause it to default on its debts.[30]

Proposition 2. *Suppose that a bank i has an opportunity to invest in a risky portfolio with an expected value that exceeds the risk-free rate of return, $\mathbb{E}[p_i] > 1 + r$. If bank i is not at risk of discontinuous feedback then it invests fully in that risky portfolio.*

What the discontinuous feedback condition precludes are cases in which, by making a safer investment, bank i can prevent another bank's default, which triggers debt repayments that benefit i. Thus, this requires that i's outcome can lead another bank to default while i is still solvent, and such that the other bank owes i a sizeable sum. So, incentives to take (excessively) risky positions can be mitigated by interdependencies in some settings. However, this requires a somewhat extreme feedback effect in which there *must* be a nontrivial chance of driving a counterparty in whom i has a large stake into bankruptcy *without* having i become bankrupt. We now present an example in which Proposition 2 does not hold, and discuss how this example corresponds to the case in which a bank has the least incentives to take risks.

[30] $\exists i_0, \ldots i_K$ for some K such that: $i_0 = i_K = i$, $S_{i_1 i_0} > 0$, and $D_{i_l i_{l-1}} > 0$ or $S_{i_l i_{l-1}} > 0$ for each $l \geq 1$ with at least one l such that $D_{i_l i_{l-1}} > 0$.

4.1.1 An Example with Countervailing Incentives

There are situations with discontinuous feedback in which debt can offer a countervailing incentive that leads to less risky investments, if there is a special sort of feedback. As suggested by Proposition 2, such feedback effect only arise of the bank is part of a cycle that involves *both* equity claims and debt claims. We illustrate this effect in the following example. Essentially, it has to be that a low portfolio return would not directly cause the bank to become insolvent, but would instead lead one of its equity-holding counterparties to become insolvent, which would then lead to a loss of a payment back to the bank itself.

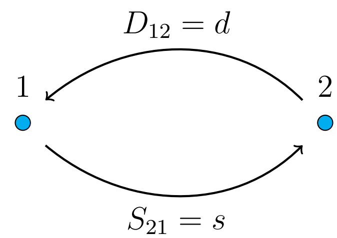


Figure 4: A network in which Proposition 2 does not apply. Bank 2 owes d to bank 1, and owns an equity share s on bank 1. Each bank can either invest in the safe asset, on in her own risky asset that pays a positive return R with probability θ. Suppose $sR > d$ such that bank 1 can prevent bank 2's default if it has a high portfolio realization. We want to show that for some parameter values, both bank fully investing in the risky asset—$q_1 = q_2 = 1$—is not an equilibrium. Suppose $q_2 = 1$. Choosing $q_1 = 1$ yields an expected payoff to bank 1 of

$$\mathbb{E}[V_1(q_1 = 1, q_2 = 1)] = \theta^2 R + \theta(1-\theta)(R+d) + \theta(1-\theta)d = \theta[R + (2-\theta)d].$$

However, by choosing a safer portfolio, bank 1 could prevent 2's default whenever none of the risky assets pays off. This requires choosing $q_1 = q_1^*$ such that $s(1-q_1^*)(1+r) = d$. Bank 1's expected payoff is then

$$\mathbb{E}[V_1(q_1 = q_1^*, q_2 = 1)] = (1-q_1^*)(1+r) + d + \theta q_1^* R = \theta R + d - \frac{d}{s(1+r)}[R - (1+r)].$$

Such safer portfolio is better than fully investing in the risky asset for bank 1 as soon as

$$\mathbb{E}[V_1(q_1 = q_1^*, q_2 = 1)] > \mathbb{E}[V_1(q_1 = 1, q_2 = 1)] \iff 1 + s(1-\theta)^2 > \frac{\theta R}{1+r}.$$

If the risk premium is not too high, it is optimal for bank 1 to choose a safer portfolio so as to prevent it's debtor's default, and get its payment of d back with certainty. This is however only possible when bank 2 has some equity claim on 1.

This example illustrates some of the nuances of financial interdependencies. Which mix of debt and equity is best for incentives depends on multiple features. Both debt and equity generally incentivize banks to fully invest in the risky portfolio. For them to choose safer investments, it is necessary to have a mix of debt and equity that allows for discontinuous indirect effects of one's return on its own value through the network.

Interestingly, the network depicted in Figure 4 has the largest potential for such feedback effects. Indeed, this sort of feedback starts with some bank j – here bank 2 – having some (direct or indirect) equity claim on another i – here bank 1. If this claim is large, then the latter bank need not invest a lot in the safe asset to prevent the former bank's default. Hence the larger this claim, the lower the cost of preventing someone's default. Note that an indirect equity claim can always be replicated by a direct claim that is at least as big; hence a single bank having large direct equity claim on another, as it is the case in the example, is most likely to generate countervailing incentives. This is, however, the case only if the losses induced by the bank's default can feedback to i. This is true for instance if j owes some debt to i, or if i has a claim on one of j's creditors. Again, the largest feedback can always be induced by i having a direct, high enough, debt claim on j.

Figure 4 then represents the network in which a bank has the least incentives to invest in the risky asset. Thus, it provides, a sufficient condition for banks to fully invest in the risk asset even when at risk of discontinuous feedback. If no single bank has a total (direct or indirect) equity claim of more than $\frac{\theta R-(1+r)}{(1-\theta)^2(1+r)}$ on another, then investing fully in the risky asset is the only equilibrium outcome.

4.1.2 Inefficiency, Risk Aversion, and Costs of Capital

Fully investing in the risky portfolio is often socially inefficient, since a bank's decision also affects the rest of the financial network. First, a bank does not account for default/bankruptcy costs when deciding its investment: the above maximization decision is independent of b. Indeed, under limited liability, the bank shareholders only consider returns earned when solvent and completely disregard what happens under insolvency. Second, a bank's investment decision impacts others through financial contracts and cross-holdings. In particular, if i defaults it will not honor its debt liabilities and its creditors may be driven to insolvency, causing bankruptcy costs to add up. Because of these, a planner would often prefer less risky investments.

The intuition behind this is straightforward. The the bank has incentives to maximize

$$\mathbb{E}[V_i|V_i > 0]\,\Pr[V_i > 0],$$

while the full impact on society is

$$\mathbb{E}[V_i|V_i > 0]\,\Pr[V_i > 0] + \mathbb{E}[V_i|V_i < 0]\,\Pr[V_i < 0] - [\text{Contagion}\ \ \text{Cost}|V_i < 0]\,\Pr[V_i < 0].$$

The second and third terms in the latter expression are negative and generally become more negative in the amount of risk taken in the investment, q_i. Thus, Proposition 2 implies that inefficiently risky investments are made.

This expression also makes it clear that, although the results may be attenuated, they will extend to the case of risk averse investors – there are still negative externalities on others that are not taken into account by those choosing the investment.

4.1.3 Investment Incentives Under Debt vs. Equity

Banks have lower default thresholds when their liabilities are in the form of equity rather than debt. In the above analysis, including Proposition 2, if none of the liabilities were in the form of debt, then there would never be any bankruptcy costs or inefficiencies. Keeping incentives constant, equity is then more efficient since it reduces the probability of default, and hence the expected bankruptcy costs that the system could have to pay. The contrast between debt and equity also implies that the systemic risk roles of organizations like banks whose balance sheets have large amounts of debt on both sides, differ from that of venture capital and other funds whose balance sheets are more equity-like.

Obviously, however, there reasons for using debt: to pay workers who may have their own fixed bills to pay, to account for the risk aversion of investors, and to handle short term loans and demand deposits, etc. For instance a risk-averse investor would prefer to have a fixed payment than a random one, for the same expected value.[31]

4.2 Measuring and Regulating Risk-Taking

Network externalities lead banks to take on too much risk, as they generally do not internalize how their investment decisions affect other organizations in the system. This sort of incentive problem is reminiscent of other settings with externalities, but we now explore more deeply what its network aspect implies for optimal regulation.

4.2.1 Financial Centrality

We provide a network-based measure of financial impact of a given organization. Conceptually, given our approach, there is a unique and clear way to assess financial impact. What limits its implementation is a lack of regulation requiring all counterparties to be revealed to a central bank or other oversight agency. The actual computation of the measures below can be demanding in practice, but it is feasible to accurately approximate, especially once a monitoring system is in place and constantly updated.

The obvious way to define a bank's impact on the rest of the economy following a change in its portfolio is to calculate its net impact on the overall value in the economy, as follows.

[31] Once bankruptcy costs are involved, this would no longer be the case, since a fixed payment would also be variable and have a lower expected value, although also a lower variance. The optimal contract in the face of a risk-averse investor and bankruptcy costs, could turn out to be a hybrid of debt and equity, or all one or the other, depending on the bankruptcy costs and variability of the investment portfolio and level of risk-aversion.

Let the *financial centrality* of i, given a network $(\mathbf{D}, \mathbf{S})$ and vector of investments $\mathbf{q}$, from a change from investment choices $\mathbf{q}_i$ to $\mathbf{q}'_i$ for i be[32]

$$FC_i(\mathbf{q}, \mathbf{q}'_i; \mathbf{D}, \mathbf{S}) = \mathbb{E}_{\mathbf{p}} \left[V_0(\mathbf{q}, \mathbf{q}'_i, \mathbf{p}; \mathbf{D}, \mathbf{S}) - V_0(\mathbf{q}, \mathbf{q}'_i, \mathbf{p}; \mathbf{D}, \mathbf{S}) \right].$$

This is the total impact on the economy – captured via the value to final shareholders in node 0 – that comes from a change in i's investment strategy. Changes in the values of public companies eventually all indirectly accrue to private equity and debt holders, and so including any of the public values would amount to double counting. Note that this is equivalent to

$$FC_i(\mathbf{q}, \mathbf{q}'_i; \mathbf{D}, \mathbf{S}) = \mathbb{E}_{\mathbf{p}} \left[\mathbf{q}'_i \cdot \mathbf{p} - \mathbf{q}_i \cdot \mathbf{p} - \sum_j \left(b_j(\mathbf{V}(\mathbf{q}_{-i}, \mathbf{q}'_i), \mathbf{p}) - b_j(\mathbf{V}(\mathbf{q}), \mathbf{p}) \right) \right],$$

which counts the total change in the portfolio and the total incidence of all changes in bankruptcy costs.

We define the *net financial centrality* of i as

$$NFC_i(\mathbf{q}, \mathbf{q}'_i; \mathbf{D}, \mathbf{S}) = FC_i(\mathbf{q}, \mathbf{q}'_i; \mathbf{D}, \mathbf{S}) - \mathbb{E}_{\mathbf{p}} \left[\mathbf{q}'_i \cdot \mathbf{p} - \mathbf{q}_i \cdot \mathbf{p} \right]$$

This is the impact beyond the direct change in i's portfolio value. Hence this captures all of the bankruptcy costs that i causes in the economy from a change in its investments. If there are no changes in bankruptcy costs, then the net financial centrality of i is 0.

Another important concept is the impact of guaranteeing to bailout a particular bank. Define a bank's *bailout centrality* to be

$$BC_i(\mathbf{q}; \mathbf{D}, \mathbf{S}) = -\mathbb{E}_{\mathbf{p}} \left[\sum_j b_j \left(\mathbf{V}_{-i}, V_i^+, \mathbf{p} \right) - b_j \left(\mathbf{V}, \mathbf{p} \right) \right],$$

where $\mathbf{V}_{-i}$ in the first set of bankruptcy costs are calculated presuming that i does not default on any payments and has value V_i^+. This is the total difference in overall bankruptcy costs if a firm is insured by the government and bailed out whenever it becomes insolvent compared to a world in which it is left to fail.

In a network with only debt between banks, these measures capture chains of cascading defaults. Some banks can stop such cascades if they have enough value and/or small enough debt liabilities compared to debt assets, $D_i^A \geq D_i^L$. Note also that these "chains" could hit some banks multiple times and so intersect. With equity, cascades do not follow direct

[32] Implicit in defining financial centrality, one has to take a stand on which equilibrium set of values is being used since those define the values $V(\mathbf{p})$ and thus the bankruptcy costs $b(\mathbf{V}(\mathbf{p}), \mathbf{p})$. Typically we are interested in either the best or worst equilibrium, but one could make other choices, or change from best to worst if one anticipates a freezing of payments in response to the failure of some organization(s). For some discussion about the importance of uncertainty about which equilibrium applies, see Roukny et al. (2018), and for strategic choices of defaults see Allouch and Jalloul (2017).

default chains but can skip a bank – that is, bank k could own j who owns i. Even if j does not default, its value could go down if i defaults, which could indirectly cause k to default. Hence with equity we can have indirect failures, whereas with debt there must only be chains of direct failures.

Regardless of the presence of equity, debts are still a key ingredient since they drive defaults. For example, in an exactly balanced network, in which debt assets exactly compensate debt liabilities on every bank's balance sheet, no bank has any net financial centrality when considering the best equilibrium. Indeed, a bank is always able to repay its debt assuming all its counterparties are solvent and portfolios are non-negative. However such a network could still be very fragile. For instance, consider a balanced network with no contracts other than debt. One can change banks on paths away from i to have D_j^L slightly higher than D_j^A. If debts are large relative to the p's, then the resulting default centralities can be very large. More generally, this implies that (net) financial centralities are discontinuous and can be very sensitive, especially to the debt structure in the network.

4.2.2 Optimal Regulation: Bailouts versus Deposit Requirements

We consider two ways the regulator can intervene to reduce the inefficiency of banks' investments: she can use reserve requirements to limit the amount invested in the risky asset, and can bail-out insolvent banks at some cost. We first study when reserve requirements are improving on laissez-faire, and then consider bailouts.

We consider a regulator who is deciding whether to regulate the best way to regulate bank i's investments, taking as given the financial network $g = (\mathbf{D}, \mathbf{S})$ and other banks' equilibrium (possibly regulated) investments $\mathbf{q}_{-i}$. We consider the setting of Section 4.1 and a bank i that does not face a large discontinuous feedback so that without regulation it would fully invest in the risky portfolio.

A reserve requirement is then simply an upper bound $\bar{q}_i$ on the share of its portfolio that i can invest in the risky asset. Proposition 2 implies that the reserve requirement for bank i will be binding. The optimal reserve requirement for i balances the gain from the risk premium with the *overall societal* expected bankruptcy costs, which is a solution to

$$\max_{q_i} \; q_i \mathbb{E}[p_i] + (1 - q_i)(1 + r) - \mathbb{E}_{\mathbf{p}} \left[\sum_j b_j(\mathbf{V}(q_i, \mathbf{q}_{-i}), \mathbf{p}) \right].$$

The first part of the objective function captures the expected return of the portfolio, which is linearly increasing in q_i because of the excess return to the risky portfolio. The second term, capturing expected bankruptcy costs, jumps down discontinuously at some values of q_i where marginally changing i's investment changes the set of defaulting banks in some states of the world. Because of these discontinuities, there are several levels of investment that can be optimal and need to be compared to each other. A natural one is to not regulate bank i, and to let it choose $q_i^* = 1$. Another one is the critical level of investment in the risky asset $\bar{q}_i$ under which bank i remains solvent irrespective of the realization of the risky asset's return. This threshold solves

$$(1-\bar{q}_i)(1+r) = D_i^L \quad \text{or} \quad \bar{q}_i = 1 - \frac{D_i^L}{1+r}.$$

It then follows that imposing a reserve requirement of $\bar{q}_i = 1 - \frac{D_i^L}{1+r}$ on bank i improves on laissez-faire if and only if

$$\left[\frac{\mathbb{E}[p_i]-(1+r)}{1+r}\right] D_i^L < NFC_i((\mathbf{q}_{-i}, q_i^*), \bar{q}_i; \mathbf{D}, \mathbf{S}).$$

Thus, preventing i's default by imposing reserves is beneficial if the opportunity cost of doing so – the loss in risk premium needed to ensure i's solvency – is less than the expected reduction in bankruptcy costs. A social welfare maximizing regulator will impose a reserve requirement when the risk premium is below threshold and when a bank's net financial centrality is above a threshold. Interestingly, the opportunity cost of reserves (the left hand side of the inequality above) is increasing in the bank's outstanding debt. The intuition behind this is that as debt increases, a greater investment in the safe asset is required to avoid default, which is increasingly costly in terms of expected returns to investment. However, the net financial centrality of i can also be increasing in D_i^L, and so whether reserve requirements are more or less likely as debt is increased is ambiguous and depends on details of the network structure and bankruptcy costs.

Next, suppose that the regulator also has the possibility of bailing out bank i when it gets the low return and is insolvent, but this involves some expected cost $c_i > 0$ (above any capital injection, which is just a transfer). Then bailing out bank i whenever it is insolvent is preferred to imposing a reserve requirement of $\bar{q}_i$ whenever

$$\left[\frac{\mathbb{E}[p_i]-(1+r)}{1+r}\right] D_i^L > \mathbb{E}_{\mathbf{p}}\left[\sum_j b_j(V_i(\mathbf{q}_{-i}, q_i^*)^+, V_{-i}(\mathbf{q}_{-i}, q_i^*), \mathbf{p})\right] - \mathbb{E}_{\mathbf{p}}\left[\sum_j b_j(\mathbf{V}(\mathbf{q}_{-i}, \bar{q}_i), \mathbf{p})\right] + c_i$$

which is equivalent to

$$\left[\frac{\mathbb{E}[p_i]-(1+r)}{1+r}\right] D_i^L > c_i - [BC_i(\mathbf{q}_{-i}, q_i^*) - NFC_i((\mathbf{q}_{-i}, q_i^*), \bar{q}_i)],$$

where we omit the $\mathbf{D}, \mathbf{S}$ from the centrality notation since they are fixed.

Imposing high enough reserves or providing bailouts each imply that a bank never defaults. Hence both bailout and reserve centrality capture how preventing i from ever defaulting affects the overall expected bankruptcy costs. This effect can be quite large if, for instance, the bank's solvency triggers a repayment cascade. In a network with only debt, these two measures follow chains of potential default cascades and thus always coincide.

As soon, however, as some bank has a sizeable equity claim on i, they can differ substantially from each other. Indeed, reserve centrality captures the effect of making i's portfolio safer, whereas bailout centrality the effect of truncating bad realizations of its risky portfolio. Hence, in the latter, bank i still enjoys high realizations of its risky portfolio, which may prevent others from defaulting through equity claims. If however equity claims are small relative to debt claims, then the two centrality measures should be fairly close as well.

These notions of centrality allow us to characterize the optimal way to regulate a bank.

Proposition **3.**

- *If $c_i \geq BC_i(\mathbf{q}^*)$, then bailouts are never optimal. In this case, reserve requirements improve on laissez-faire if and only if*

$$\left[\frac{\mathbb{E}[p_i] - (1+r)}{1+r}\right] D_i^L \leq NFC_i((\mathbf{q}_{-i}, q_i^*), \bar{q}_i).$$

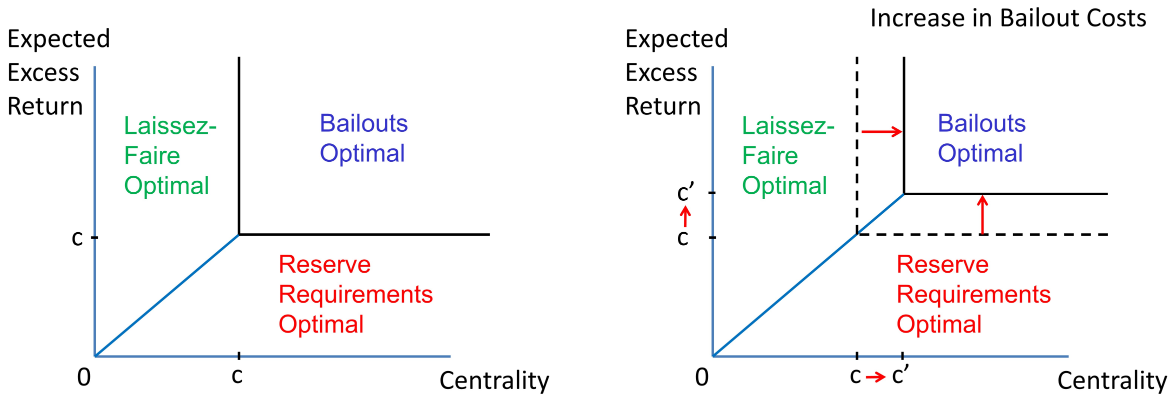
- *If $c_i < BC_i(\mathbf{q}^*)$, then laissez-faire is never optimal. In this case, reserve requirements improve on bailouts if and only if*

$$\left[\frac{\mathbb{E}[p_i] - (1+r)}{1+r}\right] D_i^L \leq c_i - [BC_i(\mathbf{q}_{-i}, q_i^*) - NFC_i((\mathbf{q}_{-i}, q_i^*), \bar{q}_i)].$$

Proposition 3 is layered, so it helps to illustrate it in a figure. In Figure 5.[33] we see that bailouts are optimal when both the expected excess returns from the risky investment and bank i's centrality are high. The attractiveness of the expected returns means that one wants to take advantage of those returns, but then the high centrality also means that bailouts are preferred to laissez fair. If returns are lower, but centrality is still high, then reserves are preferable to laissez faire. In contrast, if centrality is low enough, but returns are high, then laissez faire becomes optimal. In short:

- laissez-faire is best when centrality is low and excess returns are relatively high;
- reserve requirements become optimal when excess returns are low and centrality is relatively high; and
- bailouts are optimal when excess returns are high and so is centrality.

[33]Since there are three variables in question, we also offer a different depiction in Figure 10 in the appendix.



(a) Optimal Regulation

(b) An Increase in Bailout Costs

Figure 5: The optimal regulation of a bank as a function of the expected excess return of the bank's available risky investments and the centrality of the bank. The second panel examines an increase in the regulator's bailout costs.

The above result characterizes the optimal way to regulate bank i taking as given other banks' investments. More generally, the full optimal regulation would also tackle the problem of regulating all banks *jointly*. For instance, optimal reserve requirements for each bank $(\bar{q}_i)_i$ then solve

$$\max_{(\bar{q}_i)_i} \sum_i \bar{q}_i \mathbb{E}[p_i] + (1-\bar{q}_i)(1+r) - \mathbb{E}_{\mathbf{p}}\left[\sum_i b_j(V(\bar{q}), \mathbf{p})\right].$$

More generally a regulator might want to impose some reserve requirements and also do some ex post bailouts. This is a significantly harder problem to solve as now a bank's centrality depends on how other banks in the network are regulated. It is then necessary to consider all different subsets of banks to regulate, and compare the overall societal value in each case. Still, note that Proposition 3 must hold at an optimal regulation, for each i - but the joint problem solution determines what the $\mathbf{q}_{-i}$ and values are in the characterization from Proposition 3.

For the sake of tractability, we characterize the optimal regulation for a symmetric, core-periphery network. This is enough to highlight how centrality determines which banks should have their investments regulated, and which banks should be allowed to invest freely and be bailed-out ex post when necessary.

A Core-Periphery Example Consider the core-periphery network of debt claims depicted in Figure 6. Banks can either invest in the safe asset, or in their own proprietary asset. Banks' assets are identically and independently distributed, yielding return R with probability θ and zero otherwise, with $\theta R > 1+r$. Suppose $1+r \geq R+\bar{d}$ such that investing fully in the safe asset ensures solvency of any bank. Finally, defaulting induces a bankruptcy cost of $b > 0$. Importantly, this cost is incurred by someone even if the bank has limited liability, so it will be whomever those costs are owed to or has to step in (e.g., government, courts, other creditors, etc.). In what follows, we focus on the best-case equilibrium for bank values.

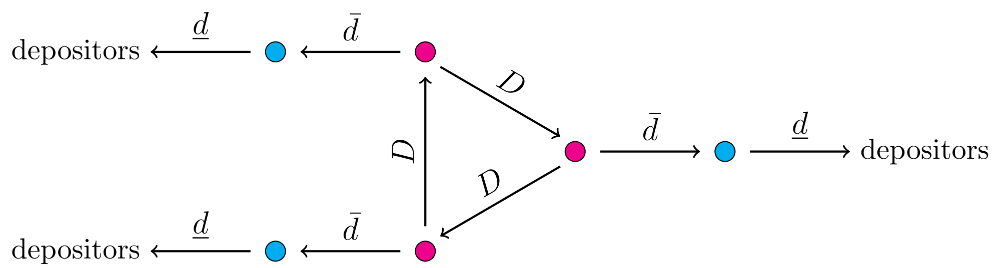


Figure 6: Arrows point in the direction that debts are owed. There are three core banks (pink) linked together via debt claims of D on one another. There are three periphery banks (blue), each having a debt claim $\bar{d}$ on a single core bank. Periphery banks act as intermediaries between outside investors (depositors) and core banks. They each have a liability of $\underline{d} < \bar{d}$ towards depositors.

Proposition 2 applies in this example, and if left unregulated, all banks invest fully in the risky asset. Core banks are then solvent if and only if their risky asset pays off. A periphery bank can however remain solvent even when its portfolio yields zero as long as its debt claim on the core bank is repaid. Hence the overall social value of the unregulated network is

$$6\theta R - b[3(1-\theta) + 3(1-\theta)^2].$$

A higher social value may however be achieved by imposing reserve requirements on some banks. Several observations simplify the problem of finding the optimal way to regulate this network. First note that imposing reserves on a bank can only beneficial insofar as it prevents it from defaulting when $p_i = 0$. More importantly, there are gains from regulating all core banks together as opposed to just one, or two, of them. Indeed, if they are all regulated and hence never default, then they always get their debt D repaid by their core counter-party, and a reserve requirement of $1 - \frac{\bar{d}}{1+r}$ is enough to ensure solvency of all three. If not all three are regulated, then a higher reserve requirement of $1 - \frac{\bar{d}+D}{1+r}$ is necessary to prevent bankruptcy. By symmetry, the gains from preventing a core bank's default is the same across core banks. Hence it is either optimal to regulate all three and set a reserve of $1 - \frac{\bar{d}}{1+r}$ on all of them, or to regulate none of them. If core banks are regulated, periphery banks never defaults and hence need not be regulated. If core banks are not regulated, then it might be optimal to impose a reserve requirement of $1 - \frac{\underline{d}}{1+r}$ on the latter's investments. By symmetry and because periphery banks are not linked (directly or indirectly) to each other, if such a restriction is optimal for one periphery bank then it is optimal for all of them.

All in all, there are three possible optimal regulations: (i) not to regulate anyone, (ii) to impose a reserve requirement of $1-\frac{\bar{d}}{1+r}$ on core banks and leave periphery banks unregulated, and (iii) to impose a reserve requirement of $1 - \frac{\underline{d}}{1+r}$ on periphery banks and leave core banks unregulated. Regulating core banks yields an overall social value of

$$3\theta R + 3[1 - \frac{\bar{d}}{1+r}]\theta R + 3\bar{d} = 6\theta R - 3\bar{d}\left[\frac{\theta R - (1+r)}{1+r}\right]$$

and ensures that no one ever defaults. Regulating periphery banks yields an overall social value of

$$3\theta R + 3[1 - \frac{\underline{d}}{1+r}]\theta R + 3\underline{d} - 3b(1-\theta) = 6\theta R - 3\underline{d}\left[\frac{\theta R - (1+r)}{1+r}\right] - 3b(1-\theta),$$

and tolerates default of core banks. Regulating the core of the network is then optimal if

$$(\bar{d} - \underline{d})\left[\frac{\theta R - (1+r)}{1+r}\right] < b(1-\theta) \quad \text{and} \quad \bar{d}\left[\frac{\theta R - (1+r)}{1+r}\right] < b(1-\theta)(2-\theta),$$

which holds as soon as bankruptcy costs are high enough. However, if core banks are much more indebted than periphery ones, such that $\bar{d} >> \underline{d}$, then it may be too costly to prevent their default via reserve requirements. It can then be optimal to only impose reserves on the periphery, and let core banks invest freely.

Finally, if bailouts are possible at a cost $c < b$, then they are always optimal ex post. In this example, this means that imposing reserves on the periphery is never optimal: core banks will be bailed-out if insolvent, ensuring the solvency of periphery banks as well irrespective of their investments. Hence the optimal regulation is then either to impose reserves on the core of the network—if the opportunity cost of doing so is below c—or to opt for laissez-faire and bail out the core when necessary.

4.3 Correlated Investments: Popcorn *and* Dominoes

The metaphor of "popcorn or dominoes" was made by Eddie Lazear, the chairman of the council of economic advisors under Bush during the financial crisis. The question was whether there really was any issue of potential contagion and "dominoes", or whether much of the crisis was instead simply due to all banks "boiling in the same hot oil" - i.e. all having extensive exposure to an under-performing mortgage market. The answer is that both were true. Banks had very correlated portfolios and all had dangerously low values in their investments at the same time, and hence most were either barely solvent, or even insolvent. Nonetheless, they also had large exposures to each others' debts, as well as to derivatives from AIG, who could not even manage margin payments, as well as securities issued by Fannie Mae and Freddie Mac, which were both insolvent. This made it clear that a large cascade would occur without intervention.[34]

This highlights the fact that correlation of investments across banks matters for financial

[34]For discussion of this see Jackson (2019), as well as the extensive analysis and data in the *Financial Crisis Inquiry Report*, commissioned by an act of the US congress.

contagion: the fact that many organizations held directly or indirectly similar subprime mortgages made the whole system substantially more fragile. This can also happen through things like syndicated loans and other partnerships, swaps, and general incentives to hold the same assets. Correlation of investments affects systemic risk in two ways: it makes the network more prone to contagion conditional on a first failure, but can also change the probability of a first failure. Indeed, under correlated investments, as discussed above, if a bank gets a low or negative return on its investment then it is likely that its counterparties are in a similar situation. Equity claim and/or debt claims then may also pay weakly less, which increases the probability that the bank defaults. If it does default, its counterparties are also more likely to become insolvent since they also face insufficiently high enough asset returns to absorb the shock.

The point that banks can *prefer* to be partnered with other organizations that have similar portfolios was first made by Elliott, Georg and Hazell (2018), in which banks choose to lend to organizations with similar exposures so as to correlate their defaults. This also appears in a particularly robust and simple form in our model, and so we illustrate it now. We discuss how this result relates to theirs below.

We model a bank's choice of portfolio as a choice of states in which to get high returns, and allow banks to arbitrarily correlate their returns by choosing how much the sets of states in which they get high returns overlap - the flip side of choosing to whom to lend.

The idea behind our result is simple, which is why it turns out to be so robust: financial organizations prefer to be solvent when their counterparties earn highest returns in order to enjoy part of those returns, and prefer to be insolvent when their counterparties are insolvent since then there are then lower returns coming in indirectly.

The importance of these results is that, regardless of whether one believes there is any serious contagion across banks, their incentives to correlate investments lead to coordinated failures and large losses for the economy at the same time.

4.3.1 An Example with Two Banks

Consider two banks – each have debt d to some outside investors and a (net) share of s in each other. Suppose there exists two independently distributed risky assets yielding a return of R_1 and R_2, respectively, with same probability θ, where $R_i > d$ for each i. Each bank can choose in which portfolio of these two assets it wants to invest.

To understand the equilibrium, we analyze two cases. The first is such that a bank that gets 0 becomes insolvent, but that a bank that earns a positive return stays solvent. The second is such that both banks are solvent if either gets a positive return, and then both are insolvent only when they both get 0 returns. There is a third case in which if either gets a 0 then both become insolvent, and it is straightforward that they prefer to correlate their portfolios then, so we do not do the incentive calculations for that case.

We start with the first case in which a bank that gets 0 becomes insolvent, but that a bank that earns a positive return stays solvent. Suppose, for now, that bank 1 is fully invested in asset 1. Then bank 2 wants to choose the same portfolio if

$$\theta \frac{(1+s)\,[R_1 - d]}{(1-s^2)} > \theta^2 \frac{[R_2 + sR_1 - (1+s)d]}{(1-s^2)} + \theta(1-\theta)\,[R_2 - d]\,.$$

This simplifies to

$$R_1[1 + s(1-\theta)] > R_2[1 - s^2(1-\theta)] + s(1+s)(1-\theta)d.$$

This is always true when $R_1 \geq R_2$,[35] and holds even if $R_1 < R_2$, if d is sufficiently small.

Next, consider the second case in which both banks are solvent if either gets a positive return, and then both are insolvent only when they both get 0 returns. Suppose, for now, that bank 1 is fully invested in asset 1. Then bank 2 wants to choose the same portfolio if

$$\theta \frac{(1+s)\,[R_1 - d]}{1-s^2} > \theta^2 \frac{[R_2 + sR_1 - (1+s)d]}{1-s^2} + \theta(1-\theta)\frac{[R_2 - (1+s)d]}{1-s^2} + (1-\theta)\theta\left[\frac{s}{1-s^2}\,(R_1 - (1+s)d) - d\right].$$

This simplifies to

$$R_1 > R_2 - (1+s)(1-\theta)d.$$

Again, this is always true when $R_1 \geq R_2$, and holds even if $R_1 < R_2$, this time for large enough $(1+s)(1-\theta)d$.

Thus, there exist equilibria in a variety of settings in which both banks fully invest in a risky asset that is first order stochastically dominated by another because of the incentive to correlate their investment. There also always exist equilibria in which they both invest in the asset that pays the highest payoff, even if that fully correlates their portfolios.

More generally, the above analysis implies that if they can invest in different portfolios that have the marginal payoff distributions, and can choose whether to correlate them or not, then they will strictly prefer to correlate them. In such cases, correlation is the unique equilibrium.

If we look at the social value of investments, no costs of bankruptcy are born if the banks are both solvent, and so having the banks choose independent portfolios rather than highly correlated ones is generally preferable. Indeed, in the second case of the above example, the social optimum whenever R_2 and R_1 are close to each other is to have one bank invest in one asset and the other bank invest in the other (or to have banks hold both R_1 and R_2). Instead the banks prefer to hold all of the same investment. Note that this misalignment between bank incentives to correlate and what is socially optimal depends on how correlation affects the expected number of defaults in the network. In the second case, a single asset paying off is enough to ensure solvency of all banks, which in this setting is equivalent to assuming

[35] Note that $R_1 \geq R_2 > d$, so the right hand side is less than $R_2[1 - s^2(1-\theta) + s(1+s)(1-\theta)]$ which is $R_2[1 + s(1-\theta)]$, directly comparable to the left hand side.

independent portfolios strictly decrease systemic risk. If on the contrary we look at the first case that assumes $s(R_i - d) < d$ for $i = 1, 2$—i.e., a bank defaults as soon as its own portfolio pays zero irrespective of the realization of its counterparty's—then correlated portfolios are no longer inefficient since they actually reduce systemic risk.

4.3.2 A General Result on Correlation and 'Risk Stacking'

To see how the above example generalizes, we now consider a set $N = \{1, \ldots, n\}$ of banks, whose financial interdependencies are summarized in the matrix of equity holdings $\mathbf{S}$ and debt holdings $\mathbf{D}$. Each bank has a proprietary investment opportunity that yields a return $R_i > 0$ with probability θ and 0 otherwise. We examine how they would choose to correlate their returns.

To examine the correlation in full generality, we model the world as having a large number of equally likely states of nature, and each bank can choose in which of those states they get $p_i = R_i$ and in which they get $p_i = 0$, subject to having a total probability of θ of getting $p_i = R_i$. We model this by introducing a set of K equally likely primitive payoff-irrelevant states, and a set of K Arrow-Debreu securities, each paying off in exactly one state. Each bank can then choose which states it gets 0 in and which ones it gets R_i in, as long as it maintains an expected return equal to θR_i; i.e., each bank chooses a fraction θ of the K states that it wants to get R_i in.[36] An equilibrium is a state-contingent portfolio return for each bank that is feasible and optimal given equilibrium strategies of others in the financial network.[37]

Thus, if banks want perfectly correlated portfolios, they will all choose to get 0 in the same states, while to have independent portfolios they will all choose to get their respective 0's in a pattern that corresponds to a binomial distribution. They could also choose to get their 0's only when all others get $\mathbf{R}_{-i}$, and thus negatively correlate their portfolios, and so forth.

Note that in this world, we can write the V_is as a function of the vector of 0s and Rs that are realized. Let $\mathbf{p_{-i}} = \mathbf{R_{-i}}$ denote that all banks other than i have received R_js, and $\mathbf{p_{-i}} = \mathbf{0}$ denote that all other banks have gotten 0s.

Proposition **4.** *Suppose that the value of a high portfolio realization is higher when all other banks also have a high portfolio realization than when they all have zero returns—i.e.*

$$V_i^+(p_i = R_i, \mathbf{p}_{-i} = \mathbf{R}_{-i}) - V_i^+(p_i = 0, \mathbf{p}_{-i} = \mathbf{R}_{-i}) \geq V_i^+(p_i = R_i, \mathbf{p}_{-i} = \mathbf{0}) - V_i^+(p_i = 0, \mathbf{p}_{-i} = \mathbf{0})$$

[36] Here we cap how much that can invest in any state. Without that requirement, there are even more extreme equilibria in which banks fully correlate and invest even more in the risky asset - in fact they earn the highest return by putting all of their investments in just one state, which then minimizes the probability of having to pay any debt.

[37] We take θ to be a rational number, and K to be large enough so that $K\theta^n$ is an integer. Banks must choose θK different states to get R_i in, so they cannot, for instance, choose to get $2R_i$ in some states. As will become clear in the proof, the ideas generalize.

for all banks, with a strict inequality for at least some i. Then there is no equilibrium of the investment game in which portfolios are independent across banks, but there exists an equilibrium in which they are perfectly correlated.

Proposition 4 shows that fairly weak conditions are sufficient to ensure that full correlation is always an equilibrium and that independent portfolios are not part of any equilibrium. Intuitively the latter result comes from the fact that independence puts positive probability on states in which all banks but one get a zero portfolio realization. If high returns are complement, then a bank will prefer to move its high portfolio realization from that state to a more favorable one.

Note that $V_i^+(p_i = 0, \mathbf{p}_{-i} = \mathbf{0}) = 0$, since a bank cannot get positive market value when none of the assets pays off. Hence there is an incentive to correlate as soon as the benefits from a high return when other banks get high returns as well is larger than the value a bank gets when it is the only one with a non-zero portfolio in the network. A sufficient condition for this to hold is if banks depend sufficiently on each other so that, in the extreme case where all other banks in the whole economy get no return, the only bank with a positive return cannot survive. Then the bank would want to move its high portfolio realization to a state in which it would not be dragged down to insolvency by others.

We refer to this incentive to correlate as '*risk stacking*' rather than risk shifting. Risk shifting is a different phenomenon in which an investor has an incentive to arrange a portfolio so that the risk falls on other investors, while here the phenomenon is to explicitly 'stack' all of the risk of all organizations into the same states. The result in Elliott, Georg and Hazell (2018) is an example of shifting, as banks want to correlate their assets to shift losses from states in which the bank is solvent—in which case the loss is incurred by *shareholders*—to states in which the bank defaults—in which case it is incurred by *debtholders*. The intuition behind our result is more general, as our result holds even if we relax limited liability. The incentive to correlate here comes from the fact that high portfolio realizations are complements: a bank generally gains more by remaining solvent and getting $p_i = R_i$ when others also have high portfolio realizations since its own value depends positively on others' through financial interdependencies. In particular, as long as there exists an equity cycle in the financial network, then a bank's positive return gets magnified when others on the cycle remain solvent, which incentivizes correlation of portfolios even if banks do not act under limited liability. This intuition holds generally regardless of how bankruptcies are resolved or how large those costs are, and in particular without assuming that a bank would bear the costs of its counterparty's bankruptcy.

For a bank not to want to correlate its portfolio, it has to be that by getting a positive return it can prevent the default of some of its debtors when $\mathbf{p}_{-i} = \mathbf{0}$. This requires the bank to be part of a cycle involving both debt and equity,[38] and hence to be at risk of

[38] To prevent the default of some of its debtors when $\mathbf{p}_{-i} = \mathbf{0}$, the latter must have a high enough equity claim on bank i and on its portfolio realization so as to remain solvent despite all other assets paying zero. This can be beneficial for i if it means getting a net debt coming in in such states. Then perfectly

discontinuous feedback. Financial networks in which banks want to correlate their portfolios are also those in which they want to take on as much risk as possible (Proposition 2), and vice versa.

Risk Aversion Although we have worked with risk-neutral banks to keep the analysis uncluttered and to emphasize the impact of financial interdependencies, it should again be apparent that, just as in the risky investment case of Section 4.1, the above results extend to the case in which investors are risk averse. If others have correlated investments and when they become insolvent so will a given bank, then it is in that bank's interest to correlate its investments with those of the others regardless of its risk aversion. Thus, perfect correlation remains an equilibrium regardless of risk tolerances. In addition, the result that full independence is not an equilibrium holds for exactly the same reasons, regardless of risk tolerance.

4.3.3 The Inefficiency of Full Correlation

In terms of efficiency, maximizing the total value of all private investors in the economy is equivalent to minimizing the expected number of defaults. Indeed, the correlation structure of investments across banks does not change the expected aggregate portfolio value $\sum_i p_i = \theta \sum_i R_i$, but it does impact the set of defaulting banks and hence the amount of bankruptcy costs incurred. Correlated investments across banks are then socially efficient if and only if they induce a lower expected number of defaults than some other configuration. This holds if any bank that gets $p_i = 0$ always becomes insolvent irrespective of what else happens to other portfolios: independent investments do not attenuate systemic risk since high portfolio realizations from some banks can never prevent another from defaulting. In that extreme case, correlated investments are socially efficient, and banks' incentives are aligned with that of the social planner. However, as soon as correlation worsens contagion risk, the equilibrium is generally not socially optimal. This will be true in many cases of interest, such as when θ is high and so full independence would make it rare for banks to get low returns together and bankruptcies would be much rarer under independence than under full correlation. In general, full correlation is the worst possible case for bankruptcy costs since all banks are insolvent whenever their return is 0, and so every time someone gets a 0 outcome, they incur bankruptcy costs. If instead, one changes the correlation structure so that there are states in which some banks get 0 returns and do not become insolvent (and also so that the only insolvent banks are ones with 0 returns), then one decreases the bankruptcy costs. As long as the total frequency of overall insolvencies is less without perfect correlation, so that there are fewer bankruptcies than 0's on average (and presuming symmetry in bankruptcy costs),

correlated portfolios would not be an equilibrium, but independent portfolios need not be either. Indeed, for independence to be an equilibrium, such incentives must hold for all banks. This is impossible since banks that are net borrowers cannot benefit from such feedback effect, and will always prefer defaulting as well when $\mathbf{p}_{-i} = \mathbf{0}$.

then bankruptcy costs drop when one moves away from perfect correlation. Thus, in most cases the decentralized equilibrium is socially inefficient, since Proposition 4 highlights that there does not exist an equilibrium in which banks choose independent portfolios, but there exists one in which they choose perfect correlation, irrespective of how correlation affects systemic risk.

4.3.4 Uniqueness of the Full Correlation Equilibrium

Proposition 4 provides insight into two of the most natural types of correlation, but does not address all possible correlation structures. Generally, banks have incentives to line up their positive returns, so as not to be dragged down by other banks. Nonetheless, one does need stronger conditions to ensure that all banks want to perfectly align their returns in *all* equilibria - such a strong form of uniqueness requires ruling out equilibria in which the network is partitioned into subsets of banks that correlate their portfolios within each subset, but choose (partially) uncorrelated portfolios across subsets. As we show next, under stronger conditions perfect correlation of portfolios is the unique equilibrium.

PROPOSITION **5.** *If the value from a high portfolio realization when others receive p_{-i} is increasing in the number of high portfolio realizations among other banks—i.e.*

$$V_i(p_i = R_i, p_{-i}) - V_i(p_i = 0, p_{-i}) > V_i(p_i = R_i, p'_{-i}) - V_i(p_i = 0, p'_{-i})$$

for each i, p_{-i}, p'_{-i} such that $|\{j \neq i : p_j = R_j\}| > |\{j \neq i : p'_j = R_j\}|$, then there is a unique equilibrium *out of all possible portfolio configurations and it involves perfect correlation.*

The sufficient condition that we give for perfect correlation to be the unique equilibrium is that the marginal gain in market value from a high realization of one's own portfolio is strictly increasing in the number of other banks that also have a high portfolio realization. This holds under a symmetry assumption on the underlying financial network, such that no bank would prefer to correlate with a particular counterparty as opposed to some other larger set of banks.

Without such a condition, there can exist other equilibria in which there is partial correlation of portfolios. Figure 7 gives an example of a financial network in which this condition does not hold, and describes an equilibrium in which banks correlate their portfolios within subgroups.

Note that even when there exist other equilibria with partial correlation, the banks each get their highest possible payoff in the full correlation equilibrium.

4.3.5 Oversight and Combating Incentives for Correlation

Given the result that at least some form of correlation in portfolios is to be expected in all equilibria of the financial system, then running stress tests for each bank separately overlooks a significant source of systemic risk. Indeed, without detailed information on the overall

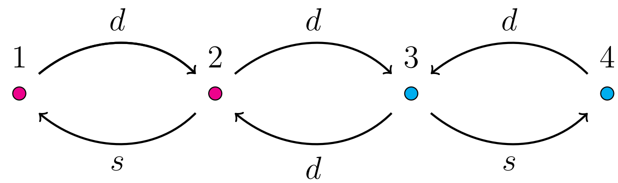


Figure 7: A network in which Proposition 5 does not apply. Bank 2 and 3 owe debt d to each other; bank 1 (4) owes d to 2 (3) and owns an equity share s of bank 2 (3) as well. Let $R < d < (1+s)R$ such that bank 1 (4) remains solvent if and only if both bank 1 (4) and 2 (3) have high portfolio realizations. Perfect correlation of portfolios is an equilibrium, but it is not the only one. There also exists an equilibrium in which banks in pink perfectly correlate their portfolio with each other, but do not correlate with the blue banks—that is, in the θK states in which $p_1 = p_2 = R$, $p_3 = p_4 = 0$ and reciprocally.

network, a bank-specific stress test does not capture the fact that a decrease in a bank's direct asset holdings is also likely to depress other banks' values, and hence also depress the value of its inter-bank assets. Going back to the example of the last financial crisis, a single bank stress-testing its own portfolio would underestimate the imminent collapse, as it would depress parts of the portfolio but not fully account for the fact that many other assets would be dropping in value at the same time due to the interlinkages, firesales, and multitude of network feedbacks. Better oversight and network-based stress testing could identify correlation patterns before they become catastrophic, given the many pressures for inter-linked financial organizations to invest in the same assets.

4.4 Too Few Partners: The Extensive Margin

In the previous sections, we looked at the intensive margin of investment choices, and highlighted the incentives to take excessively risky investments and to choose correlated investments. We now turn to the extensive margin and study how many counterparties a bank chooses. We find an under-diversification of bank portfolios in terms of number of partners.

The above results presume that banks have access to similar investments. There are various reasons, including regional presence, international boundaries, as well as proprietary advantages, that some banks might have access to investments that others do not. These, together with dynamic variations in banks' portfolios, can induce them to contract with each other–as evidenced by the large inter-financial interdependencies mentioned in the introduction.

As also mentioned in the introduction, one of the many things that make financial networks special is that financial contagion depends non-monotonically on the average degree (Elliott, Golub and Jackson (2014)). A higher average number of counterparties facilitates contagion conditional on a first failure as more organizations can be reached from the first failure. However, it also leads bank interdependencies to become more diversified through lower exposure to any single counterparty (holding fixed the total amount of overall exposure), reducing the risk of a first-failure and its probability of leading others into bankruptcy. Beyond a certain level of financial integration, this diversification effect dominates. There is thus a critical number of counterparties at which systemic risk is maximal, what Elliott, Golub and Jackson (2014) call the sweetspot. We now examine how banks choose the number of their counterparties.

4.4.1 Syndicated Investments

Let's consider a bank's choice between two different regimes: one in which it makes joint investments with $m+1$ other banks and another where it makes joint investments with m other banks. Each bank has debt liability $D_i^L = d$ to outside investors and no other contracts—all banks are hence symmetric. Without loss of generality, we write the problem from bank 1's perspective and consider partnering with the first m banks. It costs c to contract with each other bank.

A bank prefers to invest (equally) with m other banks if and only if

$$\mathbb{E}\left[\sum_{i=1}^{m+1}\frac{p_i}{m+1}-d\middle|\sum_{i=1}^{m+1}\frac{p_i}{m+1}\geq d\right]\Pr\left[\sum_{i=1}^{m+1}\frac{p_i}{m+1}\geq d\right]$$

$$-\mathbb{E}\left[\sum_{i=1}^{m}\frac{p_i}{m}-d\middle|\sum_{i=1}^{m}\frac{p_i}{m}\geq d\right]\Pr\left[\sum_{i=1}^{m}\frac{p_i}{m}\geq d\right]\geq c.$$

If the p_i's are perfectly correlated, then the left-hand-side expression is 0, and so there is no potential benefit to syndication. So, consider the case in which the p_i's are less than perfectly correlated.

The distribution $\sum_{i=1}^{m}\frac{p_i}{m}$ is a mean-preserving spread of $\sum_{i=1}^{m+1}\frac{p_i}{m+1}$, and so the expectation of a nondecreasing and convex function of these variables will be higher under m than $m+1$ (e.g., see Hadar and Russel (1971)). Since $\left[\sum_{i=1}^{m}\frac{p_i}{m}-d\right]^+$ is nondecreasing and convex, it follows that

$$\mathbb{E}\left[\sum_{i=1}^{m}\frac{p_i}{m}-d\middle|\sum_{i=1}^{m}\frac{p_i}{m}\geq d\right]\Pr\left[\sum_{i=1}^{m}\frac{p_i}{m}\geq d\right]$$

is decreasing in m, and so the banks prefer not to be involved in syndicates, and choose $m=0$.

A planner who values the total value of all returns and costs in a society is not concerned with debt repayments which are simply transfers, nor the expected returns which are realized regardless of the solvencies. The planner is concerned with the contracting and bankruptcy costs. Noting that all of the banks would go bankrupt at the same time in this case, the planner's problem would then prefer swaps of size $m+1$ to m if and only if:

$$b\left(\Pr\left[\sum_{i=1}^{m+1}\frac{p_i}{m+1}\geq d\right]-\Pr\left[\sum_{i=1}^{m}\frac{p_i}{m}\geq d\right]\right)\geq c.$$

The sign of

$$\Pr\left[\sum_{i=1}^{m+1}\frac{p_i}{m+1}\geq d\right]-\Pr\left[\sum_{i=1}^{m}\frac{p_i}{m}\geq d\right]$$

depends on the size of d relative to the distribution of the p_i's. If sensible-enough investments from debt-holders are to be expected, returns should on expectation be enough to cover liabilities $\mathbb{E}[p_i]\geq d$. Then taking an average over m compared to $m+1$ observations is a mean preserving spread and lowers the probability that the average is above d. Thus, in most cases of interest we would expect $\Pr\left[\sum_{i=1}^{m}\frac{p_i}{m}\geq d\right]$ to be increasing and concave in m, and correspondingly the difference to be positive and decreasing in m.[39]

If the size of bankruptcy costs relative to partnering costs b/c is nontrivial, then the planner will prefer to have some partnering, while the banks will prefer to remain isolated—and so they underconnect relative to what is socially optimal.

The effect of asset correlation on the optimal number of partners is to reduce the benefits of partnering.

The above analysis focuses on the incentives for complete sharing. However, in that situation there are no possibilities of contagion. Next we turn to a setting with possibilities of defaults on payments, or drops in equity values of one bank, and its potential effect on others, which introduces additional externalities.

4.4.2 Equity Shares

Consider the problem of symmetric banks that can acquire a total equity share s in other banks, and must decide between how many banks to split this investment. Each bank has debt liability $D_i^L=d$ to outside investors, and no other financial contracts. For the sake of tractability we examine *cliques*, where a clique is a set of $m+1$ banks in which each bank owns a share s/m in every other member for some $m\geq 1$. For instance, banks could choose to form cliques of 3 banks, in which each acquires a share $s/2$ in the two others. Focusing on cliques is with loss of generality, but greatly simplifies the analysis and can be interpreted as a stylized representation of clustered networks.[40] Finally suppose that direct asset holdings are i.i.d. across banks, with $\mathbb{E}[p_i]>d$, so that there is a possibility of solvency. If a bank defaults, suppose that a bankruptcy cost of b is incurred.

Partnering with more banks decreases the variance in a bank's portfolio, which can

[39] This depends on the distribution. For instance, for the normal distribution and many other continuous unimodal distributions, this is true. But it can fail for some m for multimodal distributions. For instance, take a binomial distribution with equal likelihoods of 1 and 0's, and then set $d=2/5$. the probability that the average of 1 draw is below d is $1/2$, for 2 draws it is $1/4$, but for 3 draws it is $1/2$ again, then for 4 draws it is $5/16$. It converges to 0, but has some nonmonotonicities in its convergence.

[40] Clique-based networks can arise endogenously when banks are concerned with second-order counterparty risk—see Erol (2019) for a model of financial network formation in which equilibrium networks in the absence of regulation are composed of disjoint cliques.

improve its probability of solvency. When choosing the size of a clique, bank i solves

$$\max_{m} \quad \mathbb{E}[V_i(m)|V_i(m) \geq 0] \Pr[V_i(m) \geq 0] - cm.$$

Here

$$V_i(m) = \max[0, p_i - d + \frac{s}{m} \sum_{j \neq i} V_j(m)]$$

So, again, since averaging over a smaller number of V_j's will often be a mean preserving spread, and the max function is convex, this leads banks to prefer $m = 0$ and have lower expected values as m increases in this setting as well. Here, however, things are slightly more complicated given that the V_j's now have distributions that are correlated, and it is possible for the distribution not to be ordered by second order stochastic dominance as m changes - so there can be some nonmonotonicities. Thus, we provide some illustrative calculations for specific distributions below, and still find them to be decreasing in m and for banks to prefer $m = 0$.

Regarding the socially optimal level of m, given that the total value of all investments – the p_i's – are all realized regardless of any contracting and which banks are solvent, and all of the equity and debt are only transfers, the only part of the total value in a society that changes with m are bankruptcy and connection costs. Given the symmetry, the social optimum is the m that maximizes

$$-b \Pr[V_i(m) < 0] - cm.$$

This is equivalent to maximizing

$$b - b \Pr[V_i(m) < 0] - cm$$

or

$$b \Pr[V_i(m) \geq 0] - cm$$

We thus again find that the banks tend to prefer to have no partners, while for nontrivial values of b/c, the social optimum is some positive m.

We provide some calculations that illustrate the differences for various distributions and parameter values. Figure 8 depicts the objective of an individual bank and of the planner for different debt levels, assuming that bank portfolios are uniformly distributed. As argued above, it is always optimal for banks to choose not to hold any shares in each other. Indeed, although more counterparties do increase the probability that a bank remains solvent, that is not not enough to compensate the decrease in its expected value conditional on being solvent. The planner however prefers some diversification and incentives are misaligned.

Figure 9 shows how the objective functions vary with the distribution of the p_i's, which affects the variance and tails of the distributions, and hence the probabilities of bankruptcies and those associated costs.

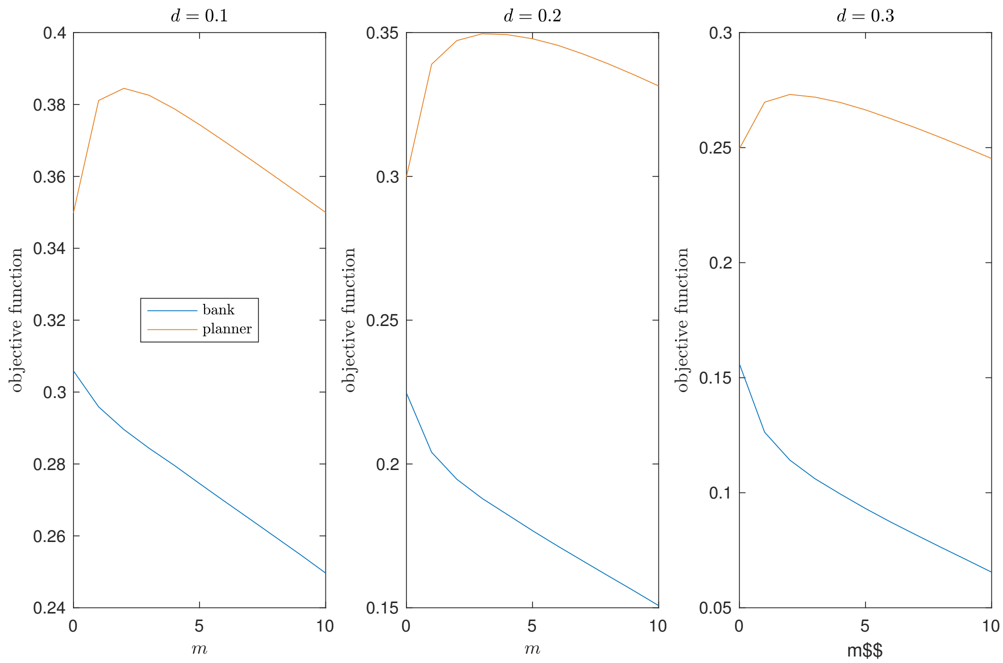


Figure 8: The bank and planner's objectives as functions of the number of partners of a given bank. The parameters for these calculations are $p_i \sim U(0, 0.8)$, $s = 0.5$, $b = 0.4$, $c = 0.005$, and three different levels of d are presented.

4.4.3 Discussion of Banks' Under-Investment in Partnerships

The intuition behind the inefficiently low number of partners chosen by banks can be seen as follows. In the above analyses, banks do not bear any of their own bankruptcy costs, nor do they bear each other's bankruptcy costs. In that context, consider adding some contract – either syndication or some other claims on each other – that would make a bank solvent when otherwise it would be insolvent. The injection of capital from other banks is costly to them, and all it does is then save an enterprise that ends up paying debts to outside investors that exceed the value of its own investments. So, the other banks essentially are paying something to add a net negative value to the overall value of all the banks. On average this *has* to be a net negative for the total value of all the banks. Thus, the results above are not special to a symmetric case: in any setting in which banks begin by only having debt liabilities to outsiders, additional partnering between banks that sometimes prevents the default of some must end up decreasing the total expected value of all banks involved. Even with asymmetries, some of the banks prefer not to have such partnering, and thus it would be blocked. Essentially, the banks do not bear the brunt of their bankruptcy costs, and so from an ex ante perspective, they prefer to fail when they are insolvent: spreading payoffs around to keep each other solvent only decreases their total expected value.

Sufficient risk aversion would change the incentives of the banks to partner with each

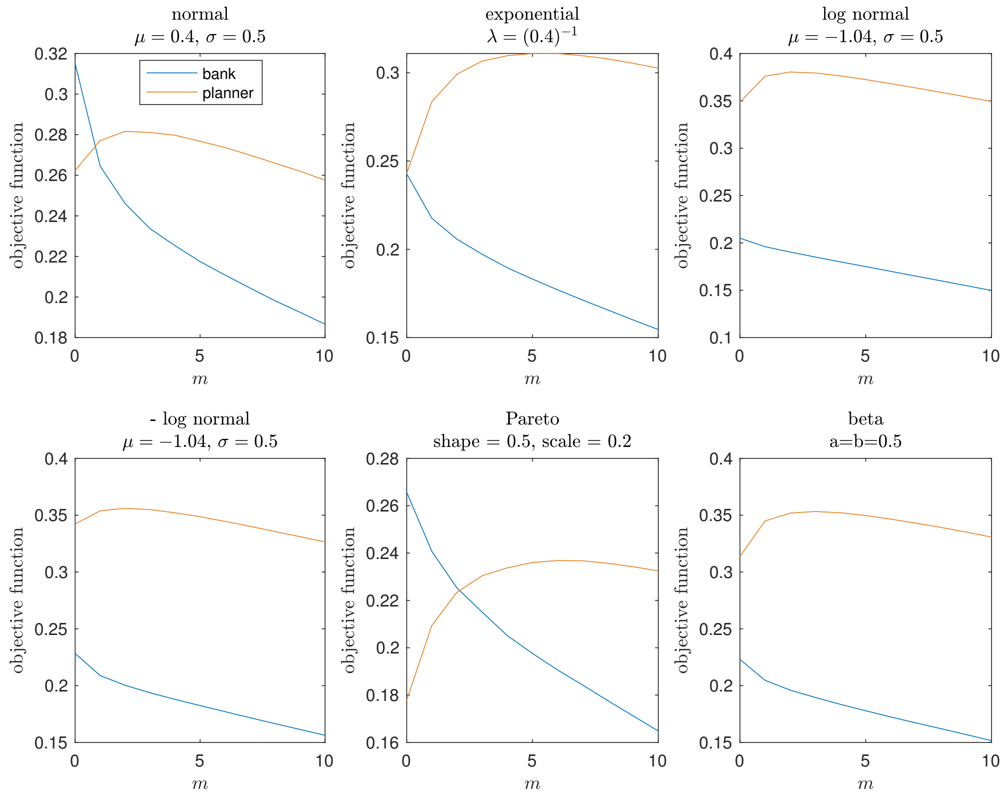


Figure 9: The bank and planner's objectives as functions of the number of partners of a given bank, for debt level $d = 0.2$, equity share $s = 0.5$, bankruptcy cost $b = 0.4$, and partnering cost $c = 0.005$. The results are presented for several different distributions of p_i, which are parametrized so as to keep the expected value of the asset constant to $\mathbb{E}[p_i] = 0.4$ (for the mirror-image log normal and beta distributions, this requires shifting the distribution appropriately).

other. With high enough levels of risk aversion, bank owners would be willing to shift some of their payoffs from situations in which they are wealthy to some of the situations in which they are insolvent. Then partnering with other banks would allow them to cross-insure, smoothing their expected payoffs across different portfolio realizations, and increasing their expected utility for high levels of risk aversion. Although sufficiently high levels of risk aversion would enhance bank owners' incentives for partnering, it would still not lead to full efficiency. These risk aversion effects would then also enter a social planner's calculations - who would evaluate the value of these cross payments in a similar manner since the planner values the expected utilities of the banks' owners. However, there would remain a key difference: the social optimum would still account for bankruptcy costs and lost debt payments to outside investors, while the banks' owners would not. Thus, they would still have lower incentives to avoid insolvencies than what would be socially optimal.

If banks have some cross debt holdings – so for some other reason they end up having debt in each other (e.g., for meeting short-term reserve requirements, etc.), then they do bear some of each other's bankruptcy costs and have more incentives to cross insure. Nonetheless, as long as some of their debts are to outsiders, they are not facing the full costs of their insolvencies. So, as long as they take deposits and have debts to outside investors, their calculations of the values of their portfolios omit key bankruptcy costs that impact other investors, and so they have incentives to take on too much risk, whether it be via excessively risky portfolios, under-insuring via partnering with each other, and over-correlating their investments whenever they do have partnerships.

5 Concluding Remarks

We have highlighted two main points.

One is that to properly assess systemic risk one needs detailed network data. This one is "easy" to fix, as once one has counterparty information, although data-intensive, the way in which one should assess systemic risk is straightforward. We put "easy" in quotes since, although what is needed is simple and obvious, it may be politically difficult to get. Financial organizations, for a variety of reasons, would prefer to keep their detailed investment information private. It also opens questions of how public one makes the outcomes of such stress tests and how one acts upon the information. Nonetheless, it is clear that operating without such information is just asking for another financial crisis to happen, or else requires having excessively onerous regulation to ensure solvency regardless of the network conditions.

The second point is that the externalities in financial networks lead to several incentive problems:[41] organizations have incentives to take overly risky positions, to involve too few counterparties, and to overly correlate their portfolios with those of their counterparties. These are harder to fix. Excessive risk can be partly, but imperfectly, addressed by reserve requirements and/or bailouts as we have shown. The imperfection relates to the fact that such reserves are generally only imposed based on a portion of the liabilities and only for a subset of financial organizations (e.g., missing much of the shadow banking system). Incentives to take on too few counterparties and to overly correlate portfolios are also issues that have been ignored by policymakers, and not ones for which there are easy policies. Requiring that some markets have Central Counterparty Clearing Houses – CCPs – can be thought of as part of a solution to these issues. These pass all transactions through a central intermediary, or a few, which can monitor positions and impose margin requirements. One then has to worry about providing the CCPs with appropriate incentives and worry about their size.[42] Large government-sponsored enterprises that process huge amounts of securities have an uneven history of success, especially if one examines Fannie Mae and Freddie

[41] There are other endogeneity issues that we have not discussed, for instance, whether two organizations wish to merge, or how large they become. In Appendix C we briefly discuss how banks' size interacts with systemic risk and investment incentives; but we leave further analysis to future research.

[42] See for instance, Duffie and Zhu (2011).

Mac's failures in the 2008 crisis. Moreover, although it can mitigate some of the systemic ramifications of the inefficiencies, it still does not eliminate the excessively correlated and risky portfolios that are induced, and hence the individual bankruptcy costs that are still not incorporated.

Regardless of the precise policy that one undertakes, developing and maintaining a more complete picture of the network, and the portfolios of banks together with those of their counterparties, is a necessary first step both to improving crisis management and to better understanding and monitoring incentive distortions.

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Appendix A General Contracts Between Financial Organizations

We here discuss bank values when contracts are not restricted to debt and equity. In full generality a contract between organizations i and j is denoted by $f_{ij}(\mathbf{V}, \mathbf{p})$ and can not only depend on the value of organization j, but also on the value of other organizations. This represents some stream of payments that j owes to i in exchange for some good, payment, or investment that has been given from i to j.

A.1 Values with more General Contracting

In the case in which contracts are not restricted to equity and debt holdings, the value V_i of an organization i is then

$$V_i = \sum_k q_{ik} p_k + \sum_j f_{ij}(\mathbf{V}, \mathbf{p}) - \left[\sum_j f_{ji}(\mathbf{V}, \mathbf{p}) - S_{ji}(\mathbf{V}) V_i^+ \right] - b_i(\mathbf{V}, \mathbf{p}), \qquad (7)$$

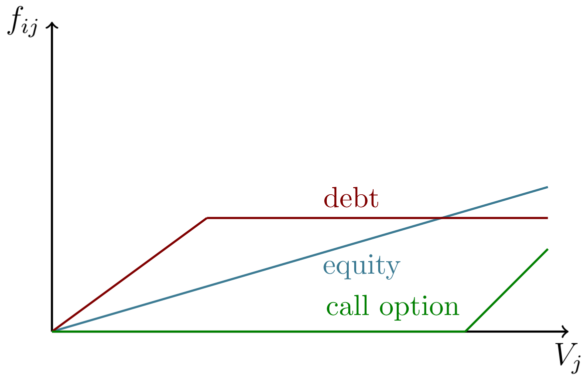
where $f_{ji}(\mathbf{V}, \mathbf{p}) - S_{ji}(\mathbf{V}) V_i^+$ accounts for the fact that debt and contracts other than equity are included as liabilities in a book value calculation.[43]

If $b_i(\mathbf{V}, \mathbf{p})$s are nonincreasing in $\mathbf{V}$ and bounded (supposing that the costs cannot exceed some total level), then if each $f_{ij}(\mathbf{V}, \mathbf{p})$ is a nondecreasing function of $\mathbf{V}$, $\sum_j f_{ji}(\mathbf{V}, \mathbf{p}) - S_{ji}(\mathbf{V}) V_i$ is nonincreasing in V, and either f is bounded or possible values of V are bounded, (using the usual Euclidean partial order) then again there exists a fixed point by Tarski's fixed point theorem for each $\mathbf{p}$. They again comprise a complete lattice. Discontinuities, which come from bankruptcy costs and potentially the financial contracts themselves, can lead to multiple solutions for organizations' values.

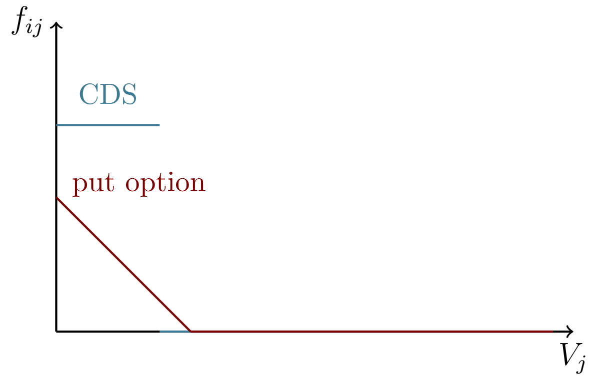
When financial contracts are not increasing functions of the values of organizations, $\mathbf{V}$, there may not exist an solution for the values of the V_is. For instance, as soon as some banks insure themselves against the default of a counterparty or bet on the failure of another, simple accounting rules may not yield consistent values for all organizations in the financial network. We illustrate this in the following example.

Example of Non-Existence of a Solution for V: Credit Default Swaps. Consider a financial network composed of $n = 3$ organizations, each of which owns a proprietary asset $\mathbf{q}$ is the identity matrix. For simplicity all assets $k \in \{1, 2, 3\}$ have the same value $p_k = 2$. The

[43] This more general model also embeds that of Barucca, Bardoscia, Caccioli, D'Errico, Visentin, Battiston and Caldarelli (2016) in which banks hold debt on each other, but these debt claims are not valued under full information: they allow for uncertainty regarding banks' external assets and ability to honor their interbank liabilities, whose face value may then be discounted depending on available information. Financial contracts as defined here can capture this kind of uncertainty if f_{ij} equals the expected payment from j to i given some information—e.g. a subset of known bank values or primitive asset values.



(a) Non-Decreasing Financial Contracts



(b) Non-Increasing Financial Contracts

values of organizations are linked to each other through the following financial contracts: organization 2 holds debt from 1 with face value $D_{21} = 1$; 2 is fully insured against 1's default through a CDS with organization 3 in exchange of payment $r = 0.4$; finally 1 holds a contract with 3 that is linearly decreasing in 3's value. Suppose an organization defaults if and only if its book value falls below its interbank liabilities, in which case it incurs a cost $\beta = 0.1$. Formally, the contracts are

$$\begin{aligned} f_{21}(\mathbf{V}) &= D_{21}\mathbb{1}_{V_1 \geq 0} \\ f_{23}(\mathbf{V}) &= D_{21}\mathbb{1}_{V_1 < 0} \\ f_{32}(\mathbf{V}) &= r\mathbb{1}_{V_1 \geq 0} \\ f_{13}(\mathbf{V}) &= -0.5V_3. \end{aligned}$$

Note that organization 2 and 3 never default: the former's value is always at least $2 - r > 0$ and the latter 's is at least $2 - D_{21} > 0$. We then check that there is no solution in which organization 1 is solvent. In such a case, $V_3 = 2 + r$ and $V_1 = 2 - 0.5V_3 - D_{21} = -0.2 < 0$: but then bank 1 defaults, which is a contradiction. Finally suppose that 1 defaults. Then $V_3 = 2 - D_{21}$ and $V_1 = 2 - 0.5V_3 - \beta = 1.4 > 0$, another contradiction.

Appendix B Proofs

Proof of Lemma 1: (i) Let $\bar{V}$ and $\underline{V}$ be the best and worst equilibrium values of bank respectively. Since there is no dependency cycle, there are three (non necessarily exclusive) types of banks:

1. those that are part of an equity cycle—let C be the set of such banks;
2. those that have no value coming from a cycle—that is there is no directed path from a bank in C to them;

3. and those that have no value going to a cycle—that is no directed path from them to a bank in C.

Note that a bank could belong to both category 2. and 3., if it is part of a component that has no link to C—e.g. a string.

Step 1—we show that $\bar{V}_i = \underline{V}_i$ for all bank in category 2. Since banks in 2 have no value coming from a cycle, it has to be that a subset of them X_0 only derive value from their outside investments: for all $i \in X_0$, $D_i^A = 0$ and $S_{ij} = 0$ for all j. The value of these banks is then pinned down solely by their investments, and importantly is independent of the value of other banks:

$$V_i = \sum_k p_k q_{ik} - D_i^L - \left[b + a \sum_k p_k q_{ik}\right] \mathbb{1}\{\sum_k p_k q_{ik} < D_i^L\} \quad \forall i \in X_0.$$

Hence $\bar{V}_i = \underline{V}_i$ for all such banks. Now let X_1 be the set of banks in category 2 that have value coming from—i.e. debt claim, or equity claim on—banks in X_0 only. Their value is independent of that of banks outside X_1 by construction. Hence

$$\begin{aligned} V_i = &\sum_k p_k q_{ik} + \sum_{j \in X_0} D_{ij} + S_{ij} V_j^+ - D_i^L \\ &- \left[b + a\left(\sum_k p_k q_{ik} + \sum_{j \in X_0} D_{ij} + S_{ij} V_j^+\right)\right] \mathbb{1}\{\sum_k p_k q_{ik} + \sum_{j \in X_0} D_{ij} + S_{ij} V_j^+ < D_i^L\} \quad \forall i \in X_1. \end{aligned}$$

Since $\bar{V}_i = \underline{V}_i$ for all $i \in X_0$, best and worst equilibrium values of banks in X_1 are also the same: $\bar{V}_i = \underline{V}_i$ for all $i \in X_1$. We can more generally construct X_k as the set of banks in category 2 that have value coming from banks in X_{k-1} only. If best and worst equilibrium values for banks in X_{k-1} coincide, then the same holds for banks in X_k. For some finite K, we have that $\cup_{k=0}^K X_k$ covers all banks in category 2. Hence by iterating this process, we get that $\bar{V}_i = \underline{V}_i$ for all bank in category 2.

Step 2—we show that $\bar{V}_i = \underline{V}_i$ for all bank in category 1. By construction, the value of banks in category 1 is independent of that of banks in category 3, but can depend on the value of banks in category 2. For each $i \in C$, let z_i be the value that is directly accruing to i from banks in category 2—because i has some claim on at least one bank in category 2. Importantly, since the values of the latter is the same in the best and worst equilibria, z_i is also the same in all equilibria. Then the values of banks in C solve

$$\begin{aligned} V_i = &\sum_k p_k q_{ik} + z_i + \sum_{j \in C} S_{ij} V_j^+ - D_i^L \\ &- \left[b + a\left(\sum_k p_k q_{ik} + z_i + \sum_{j \in C} S_{ij} V_j^+\right)\right] \mathbb{1}\{\sum_k p_k q_{ik} + z_i + \sum_{j \in C} S_{ij} V_j^+ < D_i^L\} \quad \forall i \in C. \end{aligned}$$

Abusing notation, and letting $\mathbf{S}$ be the $|C| \times |C|$ matrix of equity claims between banks in C, this is equivalent to

$$\mathbf{V} = \mathbf{qp} + \mathbf{z} - \mathbf{D}^L + \mathbf{SV} - \mathbf{I}_D \left[b + a(\mathbf{qp} + \mathbf{z} + \mathbf{SV}) \right],$$

where $\mathbf{I}_D$ is an $|C| \times |C|$ matrix indicating whether each bank is defaulting.[44] Because $(\mathbf{I} - \mathbf{S})$ is invertible, $(\mathbf{I} - (\mathbf{I} - a\mathbf{I}_D)\mathbf{S})$ is invertible as well, and this system of equations has a unique solution: $\bar{V}_i = \underline{V}_i$ for all bank in category 1.

Step 3—we show that $\bar{V}_i = \underline{V}_i$ for all bank in category 3. There are two possible cases: either all banks in category 3 are also in category 2, in which case the proposition is proven; or not. If not, similarly as in step 1, let Y_0 be the set of banks in category 3 that only have value coming from banks in C. Since whatever value they get from banks in C is the same in the best and worst equilibria, $\bar{V}_i = \underline{V}_i$ for all $i \in Y_0$. We can then iteratively construct Y_k as the set of banks in category 3 that have value coming from banks in Y_{k-1} only. If best and worst equilibrium values for banks in Y_{k-1} coincide, then the same holds for banks in Y_k. For some finite K, we have that $\cup_{k=0}^{K} Y_k$ covers all banks in category 3. Hence by iterating this process, we get that $\bar{V}_i = \underline{V}_i$ for all bank in category 3.

(ii) We now prove that if there is a dependency cycle in the network, then there exists $\mathbf{qp}$ and bankruptcy costs such that $\bar{V} \neq \underline{V}$. In particular, we show that under full bankruptcy costs—that is when $a = 1$—and proprietary assets—$\mathbf{q} = \mathbf{I}$—then there exists returns $\mathbf{p}$ such that this is true.

Let C be the set of all banks that belong to a dependency cycle. All other banks can either (i) have value (directly or indirectly) flowing in a dependency cycle; (ii) get value from a dependency cycle; or (iii) none of the above. Equilibrium values of banks in C is independent from that of banks in categories (ii) and (iii), because there is no directed path from such banks to C. The values of banks in (i) do impact those of banks in C, but we can set the return of their assets to zero such that banks in C actually get no value from them. Then equilibrium values of banks in C depend only on asset returns and values of banks in C, and at least one of them has some debt liability to another.

Redefine $D_i^A \equiv \sum_{j \in C} D_{ij}$ for all $i \in C$. We want to set portfolio values $(p_i)_{i \in C}$ as low as possible, while ensuring all banks in C remain solvent in the best equilibrium. Abusing notation, let $\mathbf{S}$ be the $|C| \times |C|$ matrix of equity claims between banks in C, and define $\mathbf{A} \equiv (\mathbf{I} - \mathbf{S})^{-1}$. When all banks in C are solvent, their equilibrium values write

$$V_i = A_{ii}[p_i + D_i^A - D_i^L] + \sum_{j \in C} A_{ij}[p_j + D_j^A - D_i^L].$$

Hence a smallest $(p_i)_{i \in C}$ that ensures they all remain solvent in the best equilibrium must satisfy (i) $p_i = 0$ if $D_i^A \geq D_i^L$, and (ii) $p_i = \left(A_{ii}[D_i^L - D_i^A] + \sum_{j \in C} A_{ij}[D_j^L - D_i^A - p_j] \right)^+$ if

[44] All off-diagonal entries are, and the i's diagonal entry is equal to 1 if i is defaulting and to 0 otherwise.

$D_i^A < D_i^L$. By construction, all banks in C are solvent in the best equilibrium, such that $\bar{V}_i \geq 0$ for all $i \in C$.

Now consider what happens in the worst equilibrium, under full bankruptcy costs—that is when the recovery rate on a defaulting bank's assets is zero. Suppose all banks in C with some debt liability to another in C—i.e. with $\sum_{j \in C} D_{ji} > 0$—default. This not only mean that banks in C do not get their debt payments from other banks in C, but also that they get no value from their equity claims. Indeed, equity claims are either on banks that have $D_j^L > 0$ and are defaulting by assumption, or on banks with no outstanding debt $D_j^L = 0$ but with zero asset return $p_j = 0$. In either case, $V_j^+ = 0$. Then

$$V_i = p_i + \sum_{j \in C^+} S_{ij} V_j^+ - D_i^L = p_i - D_i^L \quad \forall i \in C.$$

Since all banks that have zero outstanding debt must be net creditors, $p_i = 0$ for them, and $V_i = 0$ as well. Banks that have some outstanding debt either have $p_i = 0$ (if their claims on net creditors are large enough to compensate for their debt imbalance) or $p_i = A_{ii}[D_i^L - D_i^A] + \sum_{j \in C} A_{ij}[D_j^L - D_i^A - p_j]$. In the first case, their value equals $V_i = -D_i^L < 0$ and they default. In the second case, their value in the best equilibrium was precisely zero. Since they are part of a dependency cycle, they must have a (potentially indirect) claim on a debt payment within the cycle—that is, there exists j such that $A_{ij} > 0$ and $D_j^A > 0$. As we are starting from the assumption that none of the debt is repaid in the cycle, the value of bank i must be strictly lower than in the best equilibrium: $\underline{V}_i < \bar{V}_i = 0$. Hence a bank in this second case defaults as well. Assuming all banks with some debt due to someone in C is then self-fulfilling: it indeed leads all banks in C with $D_i^L > 0$ to default. The best and worst equilibria differ. ∎

Proof of Proposition 2: Let μ be the measure on $\mathbf{p}$, the vector of all portfolio values, and let

$$A(q_i) = \{\mathbf{p} \mid q_i p_i + (1 - q_i)(1 + r) + \sum_{j \neq i} S_{ij} V_j(\mathbf{p}, q_i)^+ + D_i^A(\mathbf{p}, q_i) \geq D_i^L\}.$$

Note that $\mu(A(1)) > 0$ by Chebychev's inequality since $\mathbf{p}$ is bounded and $\mathbb{E}[p_i] > D_i^L$ and all other variables are nonnegative. This implies that $\mu(A(q_i)) > 0$ for any possible optimizing level of q_i.

Consider any $q_i < 1$ for which $\mu(A(q_i)) > 0$ (which are the only possible optimizers), and let us examine the gain in utility that results from increasing q_i to $q_i + \varepsilon$. We show that for any such q_i there is an $\varepsilon > 0$ for which there is a gain in the expected value, and this then implies that the optimizer is 1.

Note that

$$\int_{A(q_i+\varepsilon)} V_i(\mathbf{p}, q_i+\varepsilon) d\mu(p) - \int_{A(q_i)} V_i(\mathbf{p}, q_i) d\mu(\mathbf{p}) \geq \int_{A(q_i+\varepsilon)\cap A(q_i)} \left[V_i(\mathbf{p}, q_i+\varepsilon) - V_i(\mathbf{p}, q_i)\right] d\mu(\mathbf{p}) - \int_{A(q_i)\setminus A(q_i+\varepsilon)} V_i(\mathbf{p}, q_i) d\mu(\mathbf{p}).$$

Next, since i is not at risk of discontinuous feedback, $D_i^A(\mathbf{p}, q_i+\varepsilon) = D_i^A(\mathbf{p}, q_i)$ for all $\mathbf{p} \in A(q_i+\varepsilon) \cap A(q_i)$. Similarly

$$\sum_{j\neq i} S_{ij}\left(V_j(\mathbf{p}, q_i+\varepsilon)^+ - V_j(\mathbf{p}, q_i)^+\right) = c\left(q_i p_i + (1-q_i)(1+r)\right)$$

for some $c \geq 0$ (which follows since the V_j's depend on q_i only via linear functions of V_i) whenever $\mathbf{p} \in A(q_i+\varepsilon) \cap A(q_i)$. That is because bank i cannot trigger discontinuous losses that will feedback to itself while remaining solvent.

Also, for $\mathbf{p} \in A(q_i) \setminus A(q_i+\varepsilon)$, it must be that $V_i(\mathbf{p}, q_i+\varepsilon') = 0$ for some $\varepsilon' < \varepsilon$ and that $V_i(\mathbf{p}, q_i+\varepsilon'') > 0$, for all $\varepsilon'' \in [0, \varepsilon')$. Thus, for all $\mathbf{p} \in A(q_i) \setminus A(q_i+\varepsilon)$

$$V_i(\mathbf{p}, q_i) \leq (1+c)\varepsilon'(1+r) \leq (1+c)\varepsilon(1+r).$$

Then

$$\int_{A(q_i+\varepsilon)} V_i(\mathbf{p}, q_i+\varepsilon) d\mu(p) - \int_{A(q_i)} V_i(\mathbf{p}, q_i) d\mu(p)$$
$$\geq \int_{A(q_i+\varepsilon)\cap A(q_i)} (1+c)\varepsilon\left(p_i - (1+r)\right) d\mu(p) - (1+c)\varepsilon(1+r)\int_{A(q_i)\setminus A(q_i+\varepsilon)} d\mu(p).$$

Claim 1. *If $\mu(A(q_i)) > 0$, then as $\varepsilon \longrightarrow 0$ $\mu(A(q_i) \setminus A(q_i+\varepsilon)) \longrightarrow 0$ while $\mu(A(q_i) \cap A(q_i+\varepsilon)) \longrightarrow \mu(A(q_i))$.*

Proof of Claim 1. Since no bank is at risk of discontinuous feedback, a marginal change in q_i cannot trigger the default of a bank from whom there is a directed path to i, that is from whom value is accruing to i either directly or indirectly. This implies that for all $\mathbf{p} \in A(q_i) \setminus A(q_i+\varepsilon)$ bank i gets the same debt payments $D_i^A(\mathbf{p})$, and gets a fraction c_{ij} of other banks' portfolio value and net debt through equity claims. This fraction c_{ij} is independent of ε for ε small enough since the set of defaulting banks from whom i gets some value is unchanged. Thus we can write

$$A(q_i) \setminus A(q_i+\varepsilon) = \{\mathbf{p} : D_i^L - q_i p_i - (1-q_i)(1+r) - D_i^A(\mathbf{p}) - \sum_j c_{ij}[D_j^A(\mathbf{p}) - D_j^L] \leq \sum_j c_{ij} p_j$$
$$\text{and } \sum_j c_{ij} p_j < D_i^L - (q_i+\varepsilon)p_i - (1-q_i-\varepsilon)(1+r) - D_i^A(\mathbf{p}) - \sum_j c_{ij}[D_j^A(\mathbf{p}) - D_j^L]\}$$

where the RHS of the second inequality converges to the LHS of the first inequality as $\varepsilon \longrightarrow 0$. Hence $A(q_i) \setminus A(q_i + \varepsilon)$ converges to the set of $\mathbf{p}$ such that $q_i p_i + \sum_j c_{ij} p_j$ is equal to a constant, which has measure zero since the price vector $\mathbf{p}$ has an atomless distribution: $\mu(A(q_i) \setminus A(q_i + \varepsilon)) \longrightarrow 0$.

We now show that $\mu(A(q_i) \cap A(q_i + \varepsilon)) \longrightarrow \mu(A(q_i))$ as $\varepsilon \longrightarrow 0$. Since bank i is still solvent for $\mathbf{p} \in A(q_i + \varepsilon)$ and is at least as dependent upon its own portfolio as others, this marginal increase in its risky investment cannot induce another's default. Hence it gets the same debt payments and linear claims on others' value. Since these linear claims are bounded, $V_i(\mathbf{p}, q_i + \varepsilon) \longrightarrow V_i(\mathbf{p}, q_i)$ as $\varepsilon \longrightarrow 0$ and $\mu(A(q_i) \cap A(q_i + \varepsilon)) \longrightarrow \mu(A(q_i))$.

□

Therefore, for any $\delta > 0$, for all small enough ε the gain in utility is at least

$$\varepsilon \left[[\mu(A(q_i)) - \delta] \mathbb{E} \left[p_i - (1+r) \,|A(q_i) \right] - [\mu(A(q_i) \setminus A(q + \varepsilon))](1+r) \right],$$

which is at least

$$\varepsilon[\mu(A(q_i)) - \delta] \mathbb{E} \left[p_i - (1+r) \right] - \varepsilon(1+r)\mu(A(q_i) \setminus A(q_i + \varepsilon)),$$

which is strictly positive for small enough δ and ε, establishing the result. ▮

Proof of Proposition 4: We first show by contradiction that there cannot be an equilibrium in which banks choose independent portfolios. Suppose such equilibrium exists and consider the problem faced by bank i. Independent portfolios require the existence of at least one state of the world in which $p_i = R_i$ but $p_j = 0$ for all $j \neq i$, and similarly of at least one state in which $p_i = 0$ but $p_j = R_j$ for all $j \neq i$. We however show that bank i would be strictly better off if it were to switch its portfolio realization between two such states. Since these states are equally likely, such deviation is profitable for i as soon as

$$V_i(p_i = R_i, p_{-i} = \mathbf{R_{-i}})^+ + V_i(p_i = 0, p_{-i} = \mathbf{0})^+ > V_i(p_i = 0, p_{-i} = \mathbf{R_{-i}})^+ + V_i(p_i = R_i, p_{-i} = \mathbf{0})^+.$$

First note that

$$V_i(p_i = 0, p_{-i} = \mathbf{0})^+ = V_i(p_i = R_i, p_{-i} = \mathbf{0})^+ = 0$$

by the assumption that any bank is insolvent if all other organizations are. Thus the previous inequality becomes

$$V_i(p_i = R_i, p_{-i} = \mathbf{R_{-i}})^+ > V_i(p_i = 0, p_{-i} = \mathbf{R_{-i}})^+,$$

which is satisfied by assumption that at least one bank sees some positive value from its portfolio returns.

We now show that there exists an equilibrium with correlated assets. Given that all other banks chose correlated portfolios—i.e chose the exact same θK states in which to receive the nonzero return—we look at the problem faced by i. Note that, similarly as before, if i decides not to perfectly correlate its portfolio then there must be at least one state in which its portfolio pays off but none of the others does, and at least one state in which its portfolio does not pay off but all the others do. Hence correlation is an equilibrium as soon as the above same inequality holds *weakly* for all banks. By assumption, no bank can remain solvent if it is the only one with a positive portfolio realization, hence $V_i(p_i = R_i, p_{-i} = \mathbf{0}) = 0$ for all i. Again, the incentive condition boils down to

$$V_i(p_i = R_i, p_{-i} = \mathbf{R_{-i}})^+ \geq V_i(p_i = 0, p_{-i} = \mathbf{R_{-i}})^+,$$

which is true since V is weakly increasing in $\mathbf{p}$. Hence all banks choosing perfectly correlated portfolios is an equilibrium. ▮

Proof of Proposition 5: We first show that, unless portfolios are perfectly correlated across banks, there always exists a bank i that gets $p_i = R_i$ when others get p_{-i}, and $p_i = 0$ when others get p'_{-i} such that $|\{j \neq i : p'_j = R_j\}| > |\{j \neq i : p_j = R_j\}|$. In words, we first show that there always exists a bank i that could move one of her high portfolio realizations to a state in which there are more other banks with high portfolio realizations. Then the sufficient condition in the proposition implies directly that doing so is a profitable deviation.

Let $\Omega = \{\omega_k\}_{k=1}^K$ be the set of K states, and p_i^k the portfolio realization of bank i in state k. Let $k^* \in \arg\max_k \sum_i \mathbb{1}\{p_i^k = R_i\}$ be one of the states with highest number of high portfolio realizations. There are two cases:

(i) $k^* < n$, then there is a bank i that gets zero in state k^*: $p_i^{k^*} = 0$. Such bank could move one of her high portfolio realizations from any other state k in which $\sum_{j \neq i} \mathbb{1}\{p_j^k = R_j\} < k^*$ (by definition of k^*) to state k^*. Given the condition in the proposition, this is a profitable deviation.

(ii) $k^* = n$, then ignore all states that have k^* high portfolio realizations. Since portfolios are not perfectly correlated, this leaves at least 2 states k with $0 < \sum_i \mathbb{1}\{p_i^k = R_i\} < n$, for which the reasoning in case (i) applies.

Proof of Proposition 1:

The characterization of solvency in the best equilibrium derives directly from the definition of the best equilibrium, and the algorithm used to compute it. Recall that the first step of this algorithm is to compute each bank's value assuming all the others remain solvent and pay back their debt. Since we are focusing on networks without equity cross-holdings, these are $V_i = p_i - D_i^L + D_i^A \geq 0$ for all i if and only if the network is weakly portfolio-balanced. If a bank does not have a weakly balanced portfolio, then it must be defaulting in the best equilibrium.

The characterization of solvency in the worst equilibrium requires more work. First, since it cannot be that some banks default in the best equilibrium but not in the worst, weak balancedness of the network is also a necessary condition to have full solvency in the worst equilibrium. It is however no longer sufficient, except in the special case of a network that involves no cycles. Indeed in such a case the network is solely composed of disjoint strings, and weak balancedness is both necessary and sufficient for full solvency: it guarantees that the first bank of each string—that only has debt liabilities but no debt assets,—is unilaterally solvent which triggers a repayment cascade down the string. Hence for the following, we suppose there is at least one cycle in the network. Finally, since it is without loss of generality to prove the claim for a connected component, we suppose the network is connected.

We first show that having an iteratively strongly solvent set that intersects each directed cycle in addition to weak portfolio-balancedness implies all organizations are solvent. Let G be the set of directed edges in the financial network such that $ij \in G$ if and only if $D_{ji} > 0$. An edge from i to j means that bank i owes some debt to bank j.

First note that N can be partitioned into three sets of banks: banks that are part of at least one cycle, banks that are part of no cycle but belong to a string that eventually points to some bank in a cycle (in-going strings)[45], and banks that are part of no cycle but belong to a string that is pointed to by an organization on a cycle (out-going strings)[46]. Following the same argument as above, weak-balancedness is enough to guarantee solvency of banks on in-going strings. Furthermore, note that banks on out-going strings do not have liabilities towards banks on cycles. Hence debt payments that still (after accounting for in-going strings) have to be made to banks on cycles can only come from banks that also lie on a cycle. Now suppose there is an iteratively strongly solvent set that intersects each directed cycle, and call it B. By definition, all banks in B must be solvent in the worst case equilibrium, which means that there is at least one solvent bank on each cycle. We prove by induction on the number of banks n that this implies all banks in the network are solvent.

It clearly holds for $n = 2$, since the assumption of at least on cycle implies a single possible network configuration. One bank being solvent means that the other gets all of its incoming debts and, by weak-balancedness, can pay the full amount out.

Now suppose the claim holds for a network of size up to $n-1$, we show it holds for $n \geq 3$. Pick any bank i_0 that *is* solvent and lies on a cycle in the network with n nodes.

Let X_1^{in} be the set of nodes that only point at i_0 and X_1^{out} the set of nodes that are only pointed at by i_0 (which could be empty.) Iteratively, define X_t^{in} to be the set of nodes that only point at members of X_{t-1}^{in}, and similarly for X_t^{out}. Since the network is finite this process must terminate. Importantly note that all nodes in $X^{in} \equiv \cup_t X_t^{in}$ only point at nodes in $X^{in} \cup i_0$, and all nodes in $X^{out} \equiv \cup_t X_t^{out}$ are only pointed at by nodes in $X^{out} \cup i_0$. Either

[45]Formally an in-going string is a set of nodes $X \subseteq N$ that can be partitioned into K elements $X = X_1 \cup X_2 \cup \cdots \cup X_K$ such that nodes in X_1 are not pointed to by anyone ($D_i^A = 0$ for $i \in X_1$) and nodes in X_k are only pointed at by nodes in $X_1 \cup X_2 \cup \cdots \cup X_{k-1}$.

[46]Formally an in-going string is a set of nodes $X \subseteq N$ that can be partitioned into K elements $X = X_1 \cup X_2 \cup \cdots \cup X_K$ such that nodes in X_K point at anyone ($D_i^L = 0$ for $i \in X_K$) and nodes in X_k only point at by nodes in $X_{k+1} \cup X_2 \cup \cdots \cup X_K$.

or both of these sets could be empty.

There are two possible cases:

1. The subgraph found by removing $X^{out} \cup X^{in} \cup \{i_0\}$ contains no cycle. In that case i_0 being solvent clears the entire system: if i_0 is solvent then all banks in X_1^{out} are solvent as well. If X_t^{out} is solvent then all banks in X_{t+1}^{out} are solvent as well, and more generally all banks in X^{out} are solvent. Recall that in-going strings are always solvent by above argument. Since all banks in X^{out} are solvent, they repay their debts in full, and the first organizations in any remaining out-going string must receive their debt payments in full. By weak-balancedness, they are then solvent as well, and this cascades down the string: the system clears, and the claim holds.

2. The subgraph found by removing $X^{out} \cup X^{in} \cup \{i_0\}$ contains at least one cycle. Any isolated string that is generated by removing $X^{out} \cup X^{in} \cup \{i_0\}$ must be solvent by weak-balancedness, and argument in 1. Any other component contains at least one cycle. We claim that having i_0 as well as one bank per cycle in the remaining subnetwork is enough to ensure solvency of the original network. First, recall that i_0 solvent means that all banks in X^{out} are solvent, and hence that all debt payments from banks in the removed $X^{out} \cup X^{in} \cup \{i_0\}$ to banks in the remaining subnetwork are made in full. This ensures that the remaining subnetwork is still weakly balanced. Now by assumption, having one bank per simple cycle in this subnetwork is enough to ensure its full solvency: all debt is paid back within the subnetwork. The last banks added to X^{in} have debt coming from outside of X^{in} only, and thus they are solvent. This spreads through X^{in} and eventually reaches i_0. Hence the system clears, and the claim holds: weak-balancedness as well as having one solvent bank per cycle guarantees systemic solvency.

Finally, we prove that if there does not exist an iteratively strongly solvent set that intersects every cycle, then some banks default in the worst equilibrium. First note that the union of iteratively strongly solvent sets is also an iteratively strongly solvent set: hence there exists a maximal one which, by assumption, does not intersect every cycle and hence does not include all banks. We claim that the maximal iteratively strongly solvent set is then actually the set of solvent banks in the worst equilibrium, and that this comes directly from the definition of the algorithm used to derive the worst equilibrium. Indeed, assuming no bankruptcy costs, the algorithm to compute the worst equilibrium first assumes no debt is repaid. This entails only unilaterally solvent banks remain solvent. Iterating on this, only banks in the maximal iteratively strongly solvent set remain solvent in the worst equilibrium. If such set is not equal to N, then some banks must be defaulting. ∎

Proof of Corollary 1: Proposition 1 gives necessary condition to have systemic solvency if the best and worst equilibrium. Hence the minimum necessary bailout needed to ensure solvency in each case are such that the resulting network satisfies these necessary conditions. For the best equilibrium, this only requires making the network weakly portfolio balanced, which means rebalancing each bank's portfolio by injecting $[D_i^L - D_i^A - p_i]^+$ in each bank i.

For the worst equilibrium, it also requires enough capital to generate an iteratively strongly solvent set that intersect each directed cycle.

To prove (iii), first note that once the network has been cleared of all cycles, it is only composed of strings. Suppose the claim is not true, such that the best and worst equilibria differ. For them to differ, it has to be that at least some bank i defaults in the worst equilibrium, but not in the best. For this to be true, it has to be that $p_i + D_i^A \geq D_i^L > p_i$, and that it remains solvent in the best equilibrium because once of its debtor j is solvent in the best equilibrium and repays its debt to i, but defaults in the worst equilibrium. Iterating the same reasoning one step up the string, bank j is solvent in the best equilibrium but defaults in the worst one only if one of its own predecessor does as well. Iterating up until one reaches the beginning of the string, this implies that one of the banks at the origin of the string defaults in the worst equilibrium, but remains solvent in the best one. This is however impossible since such bank has no liability, and is either unilaterally solvent, or defaults in *every* equilibrium.

The Existence of Values Satisfying Equation (3): $(\mathbf{I} - \mathbf{S})$ is invertible if (and only if) the matrix power series $\sum_{k=0}^{\infty} \mathbf{S}^k$ converges, which is equivalent to the largest eigenvalue of $\mathbf{S}$, in magnitude, being strictly below one. Let us treat the case in which $\mathbf{S} \neq 0$, as otherwise the result is obvious. Denote by λ the largest eigenvalue in magnitude, and w the associated eigenvector. From the Perron Frobenius theorem, we know that $\lambda \geq 0$ and w is nonnegative and nonzero.

By contradiction, suppose that $\lambda \geq 1$. Then $\sum_{j \neq 0} S_{ij} w_j = \lambda v_i$ for each i[47] implies that $\sum_{j \neq 0} w_j \sum_{i \neq 0} S_{ij} \geq \sum_{i \neq 0} w_i$. Since $\sum_{i \neq 0} S_{ij} \leq 1$ for all j, this is equivalent to $\sum_{j \neq 0} w_j \sum_{i \neq 0} S_{ij} = \sum_{i \neq 0} w_i$. We rewrite the indices on the left side $\sum_{i \neq 0} w_i \sum_{j \neq 0} S_{ji} = \sum_{i \neq 0} w_i$.

This requires that if $w_i > 0$, then $\sum_{j \neq 0} S_{ji} = 1$. Since the eigenvector is not all zeros, we know there exists at least one bank i with $w_i > 0$. If i is such that $S_{0i} = 1 - \sum_{j \neq 0} S_{ji} > 0$, then we get a contradiction directly. If instead i's equity value is entirely owned by other financial institutions, there must exist another bank j with $S_{ji} > 0$. This implies $w_j > 0$. Same argument applies: either j is partly owned by outside investors, in which case we directly get a contradiction, or we can move back the equity path to, yet another, bank. Since there must exist an equity path from outside investors to any bank, this process must terminate to some bank j' that is, at least partly owned by outside investors, such that it has $\sum_{j \neq 0} S_{jj'} < 1$ and yet $w_{j'} > 0$. This is a contradiction of the inequality we started from, and hence $\lambda < 1$.

[47] Recall that $S_{i0} = 0$ for all i.

Appendix C Additional Discussions

C.1 Inefficient Bank Size

Mergers can affect the interdependencies in the financial network in complex ways. Here we examine banks' incentives to merge at an ex ante stage, before a cascade. This complements an analysis by Kanik (2018) who discusses banks' incentives to merge to save themselves from failure in a cascade.

The analysis in Section 4.4.1 can be seen as a sort of merger analysis - as all banks in those syndicates have identical outcomes.

More generally, we can look at the effect of a merger between bank i and j into a larger organization k such that $D_{k\ell} = D_{i\ell} + D_{j\ell}$ and $D_{\ell k} = D_{\ell i} + D_{\ell j}$ for all $\ell \neq i, j$, and similarly for equity shares.

How bank size interacts with the fact that banks choose overly risky investments is ambiguous. Indeed a merger can affect a bank's choice of risky investment in either direction: it can lead the merged organization to invest a larger or smaller share of its portfolio in the risky asset. It has generally no effect since most organizations invest fully in the risky asset irrespective of the network structure (Proposition 2). It is however possible to find examples in which size, i.e. a merger, matters. For instance consider another version of the example in Section 3.2 with $n = 3$ banks. Bank 3 can only invest in the safe asset; it has equity share s in bank 1 and debt claim $\underline{d}$ in bank 2. It owes $\overline{d} > \underline{d}$ to bank 2 as well. Suppose $1 + r + \underline{d} < \overline{d}$ such that bank 3 defaults if its equity claim on bank 1 does not yield anything. In the only decentralized equilibrium, both bank 1 and 2 invest fully in the risky asset.

Now suppose bank 1 and 2 merge, and call this new organization bank 4. Bank 4 can always prevent the default of bank 1 by investing a minimum amount in the safe asset. If this required safe investment is small enough—i.e. if $\varepsilon := \overline{d} - \underline{d} - 1 - r$ is small enough—doing so can be optimal: bank 4 may optimally set $q_4^* < 1$, and the merger may reduce investment in the risky asset. Finally suppose bank 3 and 4 merge. Then there only remains a single bank, whose optimal portfolio must have all its capital invested in the risky asset. So in general a merger can change incentives in either direction.

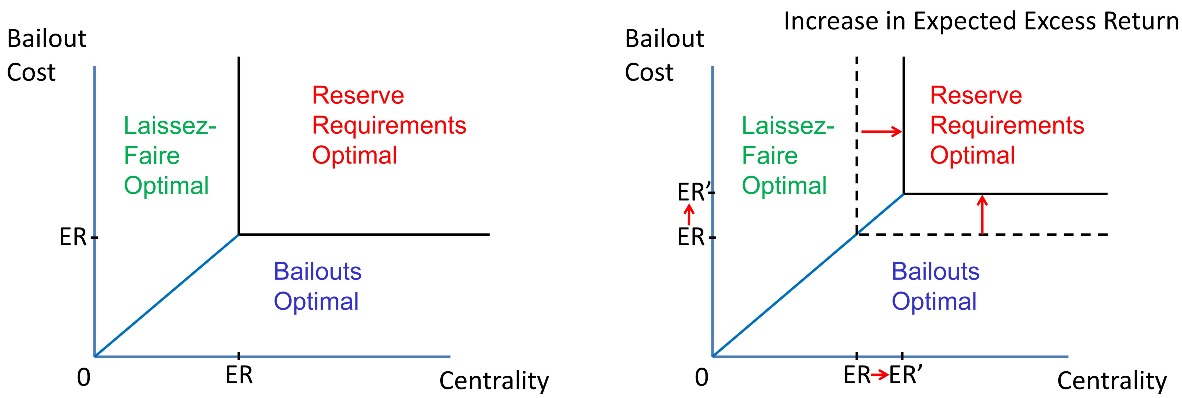
Mergers can however mitigate some of the inefficiencies coming from over-correlation of bank portfolios (highlighted in section 4.3). Indeed the incentive to correlate investments straight-forwardly disappears if the two banks merge: the larger organization then only invests in the asset with highest expected return.

Finally size of banks matters when analyzing contagion: larger banks can serve as buffers and stop default cascades, or on the contrary be brought to insolvency by one of their smaller branches. If the two merging banks have debt claims on each other then the merger also decreases their insolvency threshold, which reduces the likelihood of default all else equal. A merger between bank i and j increases the set of defaulting banks if and only if one of the two banks—say bank i—would have remained solvent had the merger not happened whereas they both default once merged. This requires $X_i - D_{ji} \geq 0$ but $X_j + D_{ji} - D_{ij} < 0$, where X_i

is the net value of i's asset excluding its debt contracts with j, and similarly for X_j. Bank j brings i to insolvency during the merger if

$$X_i + X_j < 0 \implies X_j < -D_{ji}$$

that is, if the net debt that j owes to other banks is at least as large as what it used to owe to bank j. Hence in general, a merger can either mitigate or worsen a default cascade.



(a) Optimal Regulation

(b) An Increase in Bailout Costs

Figure 10: The optimal regulation of a bank as a function of the regulator's bailout cost and the centrality of the bank. ER is the bank's expected excess return of it's available risky investments. The second panel examines an increase in that excess return.