# Platform Enterprises:
# Financing, Investment, and Network Growth

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**Abstract**

This paper develops a tractable micro-founded dynamic platform model featuring cross-group network effects. Networks are analogous to capital-assets, and the platform enterprise invests in the networks by making subsidies to users. The paper solves for the entrepreneur's optimal financing and investment strategies, with limited enforcement as the major financial friction. The main results are: 1) making highly aggressive subsidies by using up available funds is constrained-optimal; 2) per-transaction subsidies decrease over time and are followed by increasing fees; 3) the platform with stronger network effects has a propensity to make more subsidies at initial stages and enjoys a higher valuation; 4) staged financing mitigates the limited enforcement problem, and *ceteris paribus,* the number of funding rounds decreases with profitability of the platform and increases with required profits by financiers; 5) the value of funds raised each round increases and the financing frequency decreases over time.

***Key words:*** sharing economy, network effects, platform valuation, dynamic pricing strategy, subsidy, staged financing

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# 1 Introduction

Enterprises that leverage the power of platform business models have grown dramatically in size and scale over the past few years. Prominent examples are Uber, Airbnb, Upwork, etc., through which services are traded; and Amazon, eBay, Taobao, etc., where commodities are traded. A platform enterprise doesn't produce goods by itself. Instead, it creates networks of users who interact and transact through the platform and establishes a platform market. Compared with traditional markets, the platform market is ultimately driven by network effects, which enable the users to benefit more from trading through the platform when the network size of the other group is larger. Usually, the platform enterprise charges spreads between the selling and purchasing prices, and it makes profits from these fees.[1]

The business model of platforms has attracted attention from both industry and academia. While there is copious academic work focusing on competition and fee structure of platforms, studies on the financing and investment of platform enterprises are sparse. Some interesting questions remain unanswered: is it rational for platforms to make highly aggressive subsidies to users at early stages, especially when the platform enterprise is a monopoly with no entry threat? Why does the number of financing rounds vary greatly for different platform enterprises?[2] How would the capital market conditions affect a platform enterprise's financing and investment decisions and its valuation? This paper develops a dynamic theory of platform financing and investment to study these questions.

The platform financing and investment questions are unique in the following aspects. First, a platform is an enterprise but it builds up a market. Therefore, it possesses properties of both a firm and a market. Besides, a platform tends to allocate a large portion of funds for making subsidies to users, even there is no competitor or entry threat. So, subsidies cannot be simply regarded as predatory pricing strategy of the platform enterprise. Instead, subsidies are investment into the networks, and the networks are analogous to capital-assets, which the platform enterprise first invests in and later generates income from. The model in this paper

[1]Fees charged by the platforms can be in different forms. Some platforms directly charge a per-transaction commission from sellers and buyers, e.g. Upwork and Amazon; some set selling and pricing prices for the users and make profits from the price differences, e.g. Uber. Here, the spread includes both of these forms. Some platforms charge membership fees in addition to per-transaction fees. In the model of this paper, the fixed membership fee is set to zero. This simplification assumption does not affect qualitative results.

[2]For example, Amazon raised two rounds of funding in total, while Uber has raised twenty-three rounds till the end of October 2019, according to data from Crunchbase.

captures these distinctive features of a platform and solves for optimal financing and investment decisions of a monopolistic platform enterprise under various capital market conditions.

In the model, there is a massive fully-competitive traditional market and a newly-launched platform market where the same type of good is traded. Network effects in the platform market generate additional surpluses, which can be shared by the platform enterprise and the agents. This is the ultimate reason why the platform enterprise can charge fees on transactions. Once a new agent gets to know the platform market, he will try it out and decide whether to switch to it from the traditional market, with an option to switch back. The traditional market is assumed to be immense and unresponsive to the platform market. The entrepreneurial platform enterprise has the incentive to make subsidies to users to boost network growth and maximize its value. With no internal funds, the entrepreneur has to raise external capital to invest in the networks. Therefore, he needs to decide jointly on the amount and timing of financing and investment in a world with financial frictions and financing costs.

The first key finding of the model is that in face of financial constraints, it is constrained-optimal for the platform to make highly aggressive subsidies by using up available funds early on. This result highlights the importance of networks as intangible assets of the platform enterprise. Timely and sufficient investment in the networks is crucial for the success of a platform.[3] Financial constraints are endogenized in this paper: the entrepreneur may rationally choose not to invest up to the unconstrained-optimal level because of financing costs. When the financial constraint is binding, it's suboptimal to use the funds gradually and smoothly. On the contrary, it is rational to make heavy subsidies early on and charge zero spread[4] subsequently for a certain period.

The model explicitly solves for the (constrained) optimal dynamic pricing strategies of the platform enterprise and the corresponding network growth paths. It finds that the spread charged by the platform is non-decreasing over time. That is, the

[3]A prominent example of failure due to inadequate subsidies and slow network growth is SideCar, a ride-sharing company similar to Uber. According to an article on Havard Business Review: *it deliberately pursued innovation and a conservative slow-growth strategy in order to be financially responsible. The fatal flaw was not recognizing the importance of attracting both sides of the platform. Sidecar also raised much less venture capital than Uber and Lyft, and was unable to attract enough drivers and riders to survive much beyond the startup phase.* (Yoffie, Gawer and Cusumano, 2019).

[4]The model assumes a zero maintenance cost for simplicity. Intuitively, the platform may retain some funds and charge a maintenance-level commission if the maintenance costs are positive.

per-transaction subsidies decrease over time and are followed by weakly increasing fees. The model also predicts that *ceteris paribus*, stronger network effects, defined as more user benefits from the same network size, lead to more aggressive subsidies early on and hence a higher platform valuation and faster network growth.

In terms of financing patterns, this paper argues that staged financing can be a natural choice to mitigate incentive problems. Intuitively, staging allows the financiers to abandon the project in case of misconduct, and fewer available funds in hand reduce the entrepreneur's incentives of misconduct. The type of potential misconduct this paper considers is limited enforcement – the entrepreneur could abscond with funds in hand. This is an extreme case of fund embezzlement, where the entrepreneur could embezzle all the funds he had just raised. In practice, fund embezzlement is indeed a severe problem for startups, especially for platform enterprises which need enormous investment before earning profits.[5] This paper finds that, with potential embezzlement, each round the entrepreneur cannot raise more funds than his expected profits from successfully managing the platform enterprise. This leads to an increased value of funds raised each round over time and decreased financing frequency. The model simultaneously endogenizes the number of rounds, the financing frequency, and the value of funds raised.

The paper also analyzes how the number of financing rounds is affected by the profitability of the platform as well as the capital market conditions. All things equal, higher profitability leads to fewer rounds of financing; a more competitive capital market leads to fewer rounds of financing. When the capital market is not fully competitive, the more profits the financiers require, the less investment the entrepreneur tends to make, and the more rounds of financing he has to raise. However, an excessively high cost of financing would lead to no financing and no investment in the networks. This paper characterizes the interaction between discrete financing choices and continuous investing decisions under different scenarios.

### ***Relation to Literature***

Rochet and Tirole (2003, 2006) and Armstrong (2006) provide pioneering work on platform markets. Their work involves the type of platforms where price non-

[5]Many failed platform enterprises have been accused of entrepreneur fund embezzlement. For example, Kongkonghu, a platform trading second-hand goods in China, was reported the CEO embezzlement of funds for private usage. In this paper, I assume that fund embezzlement can be detected before the next round of financing, because of investors' monitoring efforts or due diligence undertaken before each new round of financing.

neutrality of the two sides is a key feature. They define the platform markets with this non-neutral price structure as two-sided markets. Rochet and Tirole (2006) summarize that factors making a market two-sided are: absence or limits on the bilateral pricing setting, platform-imposed constraints on pricing and membership fixed costs or fixed fees. Examples of those markets are credit card markets, newspapers, Videogame platforms, etc. Rochet and Tirole (2003, 2006) and Armstrong (2006) develop static models on the two-sided markets to discuss competition and price structure of platforms in various cases. This paper, in contrast, involves platforms which either allow bilateral pricing by end-users (e.g. Amazon, Deliveroo, Upwork, etc.) or charge no membership fees and optimally set prices satisfying the market clearing condition to maximize profits (e.g. Uber). Thus, in this paper, equilibrium transaction amounts and prices are endogenously determined by market clearing, and only the level of fees charged by the platform matters in equilibrium. This simple price structure allows me to explore the dynamics of the platform and find closed-form expressions. To my knowledge, this paper provides the first attempt to describe the optimal dynamic pricing strategy and network growth path of the platform market based on a fully micro-founded model.

Another strand of literature studies competition with same-side network effects, following the work by Katz and Shapiro (1985), and Farrell and Saloner (1985, 1986). Some more recent work of this literature addresses the issue of dynamic competition and growth. Mitchell and Skrzypacz (2006) derive the Markov perfect equilibrium of an infinite-period game with network effects. They assume the consumer's utility to be an increasing function of the network size at the time of trade. I make similar assumptions in this paper. That is, the agents are assumed to be myopic and not forward looking. Cabral (2011) considers dynamic pricing competition between two proprietary networks with forward-looking agents. My work is distinguished from this strand of literature in the following aspects. First, I model the cross-group network effects of the platform market. Second, I focus on the effect of network itself on the dynamic pricing decisions without competition or entry threat. Third, I introduce financial frictions in the model and study the joint decisions of dynamic financing and pricing. The topic on dynamic platform competition and its interaction with financing issues is a good direction for future research.

For the financing part, this paper is related to the literature on venture capital (VC) staging. Admati and Pfleiderer (1994) model the staging as a way to mitigate agency problems such as asymmetric information and overinvestment. Wang and

Zhou (2004) investigate the cases with moral hazard and uncertainty and find that staged financing can control risk and mitigate moral hazard. Most work on VC staging has assumed either fixed investment levels or a fixed number of stages. This paper endogenizes the financing and investment levels, the number of stages, and frequency of financing simultaneously, without resort to uncertainty.

This paper is also related to work on continuous-time models of the firm's financing and investment decisions and the impact of external financing costs on investment. Decamps et al. (2011) explore a model where a firm has a fixed-size investment project and generate random cash flows, and they study the impact of external financing costs on equity issuance and stock prices. Bolton, Chen, and Wang (2011) study the case of flexible firm size and the dynamic patterns of corporate investment. Demarzo et al. (2012) study the dynamic investment theory with dynamic optimal incentive contracting, and endogenize financial constraints. Section 3 of this paper also endogenizes financial constraints, but this paper involves no uncertainty. Limited enforcement along with financing costs leads to endogenous financing level, staging, and pricing decisions in this paper. Besides, the literature consider either fixed-size investment or AK production technology, while this paper endogenizes the cash flows by modelling the unique business patterns of the platform enterprise.

The remainder of this paper proceeds as follows. Section 2 sets up and solves for the framework of the dynamics of the platform market, and compares it with the traditional markets and firms. Section 3 discusses the financing and investment issues with financial frictions, and characterizes the constrained-optimal financing and investment patterns. Section 4 concludes. Proofs appear in the Appendix.

## 2 Framework of the Platform Market

To understand the strategies of the platform enterprise, we first need to know how the platform market works. In this section, I build a framework to analyze the dynamics of the platform market, and the corresponding pricing strategies of the platform enterprise.

The investment strategy of the entrepreneur who runs the platform enterprise is highlighted in this section, assuming no financial frictions or financing costs, or equivalently, the entrepreneur has sufficient internal funds to make the investment.

Discussion for financial frictions and the interaction between financing and investment decisions are deferred until next section.

## 2.1 The Model

Time is continuous and the discount rate in this economy is $r > 0$.

In this economy, there is a massive fully-competitive traditional market trading one good, with no network effects. All agents originally trade in this traditional market. At time $t = 0$, a monopolistic platform is launched by an entrepreneur to trade the same good. The platform enterprise creates networks of users, which in turn generate network effects. The initial network size is normalized to be one on each side of the platform market. An agent can switch frictionlessly between the traditional market and the platform market once he gets to know both. Hence, a user of the platform always compares the utility gained from trading in the platform market with his reservation utility when trading in the traditional market, and decides whether to stay in or exit the platform market. There is a fixed cost $F_0$ to launch the platform, which can be ignored in this section, since it is a sunk cost and does not affect the entrepreneur's investment decisions thereafter.

The information of the platform market is disseminated in the economy by "word of mouth": current users of the platform will constantly tell his or her friends of it. An agent who has just heard of the platform market will transact one unit of the good and compare the utility with his corresponding reservation utility to decide whether to switch to the platform market. To simplify the analysis, I assume there is no cost to commence, keep, or terminate the platform membership. The platform generates profits by charging spreads between the purchasing and selling prices. The platform's objective is to maximize all discounted future profits.

The remainder of Subsection 2.1 set up the model in details. Part A describes the instantaneous utilities of platform users; Part B then solves for the instantaneous supply and demand functions of the platform market and the market-clearing equilibrium; Part C introduces the law of motion of the platform market; Part D defines the problem of the platform.

### A. Instantaneous Utilities of Platform Users

The good is indivisible. On each side of the market, agents are homogeneous. Their utilities are assumed to be quasilinear. A buyer has the standard decreasing marginal utility and a seller experiences an increasing marginal cost. A key setup factor in this model is the cross-group network effects in agents' utilities when trading through the platform. That is, the net utility of an agent not only depends on the amount of goods/services he consumes or sells, but is also related to the network size of the group on the other side of the market.

The utility $U_D$ of a buyer on the demand side and the utility $U_S$ of a seller on the supply side are, respectively,

$$U_D(x_D) = \int_0^{x_D} (b_1 + \epsilon_i - N_S^{-\eta} x)dx - p_D x_D, \tag{1}$$

$$U_S(x_S) = p_S x_S - \int_0^{x_S} (b_2 + \epsilon_i + N_D^{-\eta} x)dx. \tag{2}$$

For a buyer, the term $(b_1 + \epsilon_i - N_S^{-\eta} x)$ is the utility he gains from consuming an infinitesimal unit of the good at consumption level $x$. $b_1$ denotes the average product quality. The zero-mean random variable $\epsilon_i$ is an identical and mutually independent shock component in the quality of each infinitesimal unit of the good, where $i$ is an index for the ordinal value of the units. $-N_S^{-\eta} x$ exhibits a decreasing marginal utility. The component $N_S^{-\eta}$ represents the cross-group network effect, where $\eta$ is a non-negative parameter representing the strength of the network effect.[6] $p_D$ is the per-unit purchasing price of the good, and $x_D$ is the level of consumption.

The above setting assumes that the cross-group network size directly affects the pace of decline in the marginal utility, but not the quality of the good. This is a novel way of specifying network effects. It is a reasonable assumption because the benefits of cross-group network effects usually hail from product variety, lower searching costs or better matching results, which slow down the decrease in the agent's marginal utility rather than affecting the quality of the good. For example, a buyer on Amazon enjoys slower utility decrease if there are more sellers and thus product differentiation; the utility of a Uber-rider also decreases more slowly if there are more drivers and hence higher matching frequency. More broadly, network effects

[6] When $\eta = 0$, there is no network effect. As will be shown later, the platform can never charge a positive commission in this case, so the optimal choice is not to launch the platform if there is no network effect.

can be understood as benefits of convenience generated by the platform market, which do not exist in the traditional market – to purchase more differentiated goods, the potential buyer has to go to different stores in the traditional market; to take more rides, the traveller has to book taxis several more times and wait longer in the traditional market; while with platforms like Amazon and Uber, they don't have to. Thus, users of the platform tend to experience a slower decrease in the marginal utility and potentially consume more.

A seller's utility is the profits he gets from selling $x_S$ units of the good, since his utility is also quasilinear. $p_S$ is the selling price of the good. $p_S$ may be different from $p_D$ because the platform can charge a spread between these two prices. This spread can be either positive or negative. The term $(b_2 + \epsilon_i + N_D^{-\eta} x)$ is the cost of providing an infinitesimal unit of the good at level $x$. Similar to the demand side, $\epsilon_i$ is a shock in the cost of each infinitesimal unit of good produced, $N_D^{-\eta} x$ demonstrates the increasing marginal cost, and $N_D^{-\eta}$ measures the cross-group network effect. The larger the network size of the demand side, the more slowly the marginal cost increases. This setting is most suitable for the suppliers in the "sharing economy" or "gig economy", where the economies of scales generally do not apply to an individual supplier, and they encounter increasing marginal costs. For example, a freelancer of Upwork or a room-provider of Airbnb usually experience increasing marginal costs from more work or longer waiting time in between, but the larger network size of the demand side can mitigate the increase.

In this model, I assume $\epsilon_i \sim U[-\frac{b}{2}, \frac{b}{2}]$, *i.i.d.*, where $b = b_1 - b_2$. More explanations of this assumption will be given in Section 2.1.C.

Integrate (1) and (2), and by the law of large numbers, the agents' utilities are in the quadratic quasilinear form:

$$U_D(x_D) = (b_1 x_D - \frac{1}{2} N_S^{-\eta} x_D^2) - p_D x_D, \tag{3}$$

$$U_S(x_S) = p_S x_S - (b_2 x_S + \frac{1}{2} N_D^{-\eta} x_S^2). \tag{4}$$

### B. Demand, Supply, and Platform Market Equilibrium

Knowing agents' utilities, we can solve for the demand and supply functions as well as the market equilibrium, taken the spread charged by the platform as given.[7]

[7]In the model, I assume the platform will impose the market-clearing equilibrium instead of rationing equilibrium, because the former maximizes the platform's profits and also generates the

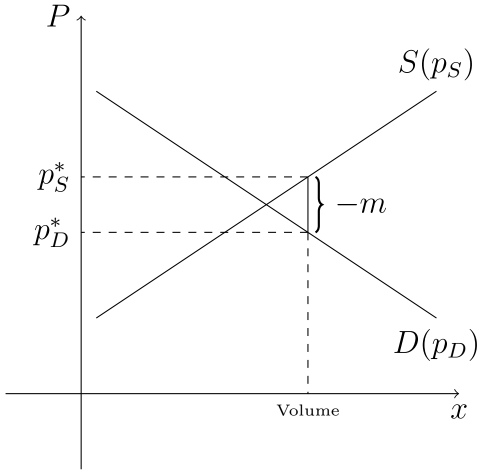


**Figure 1**
Example: the platform market equilibrium with a negative spread ($m < 0$).

**Lemma 1** ***(Market equilibrium with Spread*** $m$***)***
*Denote $m = p_D - p_S$, $b = b_1 - b_2$, and assume $N_S = N_D = N$, then the instantaneous equilibrium results of the platform market are: $p_S^* = \frac{b_1+b_2-m}{2}$, $p_D^* = \frac{b_1+b_2+m}{2}$, $x_S^* = \frac{b-m}{2}N^\eta$, $x_D^* = \frac{b-m}{2}N^\eta$, $x_S^* N_S^* = x_D^* N_D^* = \frac{b-m}{2}N^{1+\eta}$, $U_S^* = \frac{(b-m)^2}{8}N^\eta$, $U_D^* = \frac{(b-m)^2}{8}N^\eta$.*

$m$ is the spread charged by the platform. For simplicity, here I solve for a symmetric equilibrium where $N_S = N_D = N$. In this paper, the initial network sizes are assumed to be symmetric on the supply and demand sides, and the growth speeds are also symmetric, as will been shown in Section 2.1.C. Thus, $N_S = N_D = N$ throughout time in the model. The model best suits the normative "sharing economy" or "gig economy", where the participants are both goods/services providers and consumers. In this case, the supply side and the demand side are composed of the same group of people and thus symmetric.

Proof of Lemma 1 is direct. By utility maximization, we derive the individual demand functions: $x_D = (b_1 - p_D)N_S^\eta$, $x_S = (p_S - b_2)N_D^\eta$. Apply the market-clearing condition $N_D x_D = N_S x_S$, we can solve for the equilibrium prices and quantities, as well as agents' utilities.

maximum transaction amounts. In practice, platforms do apply dynamic pricing strategies and follow a market-clearing equilibrium. For example, Uber uses a dynamic pricing model that matches fares to the rider-to-driver ratio. They call it "surge pricing". For platforms where end-users can set the prices, e.g. Upwork, Amazon, market-clearing equilibrium is even more prevailing because the sellers can adjust the prices according to the demand.

Figure 1 depicts the market demand function, $D(p_D) = (b_1 - p_D)N_S^{\eta}N_D$, the market supply function, $S(p_S) = (p_S - b_2)N_D^{\eta}N_S$, and the equilibrium prices and trading volumes under a negative spread $m$. With a negative $m$, the trading volume is stimulated and agents trade more because of the subsidies. If instead $m$ is positive, the platform makes profits and the equilibrium trading volume is lower than the zero-spread volume.

### C. Law of Motion of the Platform Market

The law of motion guarantees how this platform market dynamically evolves. It consists of three parts - dissemination of news, the joining decision of a newcomer, and exit decision of an existing user.

To understand the joining and exit decisions of the agents, we need to know their outside opportunities. In this economy, there is a traditional competitive market trading the same good with the same quality, but with no network effects. Agents can always trade in this traditional market. The selling and purchasing prices are equal in the traditional market. Thus, utility parameters $b_1$ and $b_2$ remain the same in the traditional market; with no network effects, $\eta = 0$; with no price difference, $p_S = p_D = p$. Namely, agents' instantaneous utilities in the traditional market are:

$$U_D(x_D) = (b_1 x_D - \frac{1}{2}x_D^2) - p \cdot x_D, \quad U_S(x_S) = p \cdot x_S - (b_2 x_S + \frac{1}{2}x_S^2). \tag{5}$$

Therefore, the equilibrium price of the traditional market is $p^* = \frac{b_1+b_2}{2}$, and the equilibrium utility of a buyer or seller is $U_R^* = \frac{b^2}{8}$.

For the dissemination of news on the platform market, assume a member of the platform tells $\lambda$ fraction of his friends about the platform each unit of time. That means, the arrival rate of newcomers on each side is $\lambda N$.

A newly arrived agent will choose to "have a taste" of the network to gather information. To be specific, his decision rule is as follows: trade an infinitesimal unit of the good through this platform and compare the utility gained with the corresponding reservation utility in the traditional market. The latter is $b/2$. [8]

[8]It is actually the marginal utility at $x = 0$ in the traditional market, which can be derived from (5). There is no shock to the quality in the traditional market, because the buyers and sellers usually meet and examine the goods before the transaction is made. In the platform market, however, transactions are usually made before or even without personal contact between a buyer and a seller, which leads to shocks in the realization of the costs and the good quality.

If he gains a higher utility than the corresponding reservation utility, he will join the network and switch from the traditional market to the platform market. With some abuse of notation, I denote the utility from trading an infinitesimal unit in the platform market by the marginal utility, $MU$:

$$MU_D|_{x=0} = (b_1 + \epsilon_i) - p_D = \frac{b-m}{2} + \epsilon_i, \tag{6}$$

$$MU_S|_{x=0} = p_S - (b_2 - \epsilon_i) = \frac{b-m}{2} + \epsilon_i. \tag{7}$$

Therefore, the probability of a newcomer to join the network is

$$Pr(\frac{b-m}{2} + \epsilon_i \geqslant \frac{b}{2}) = Pr(\epsilon_i \geqslant \frac{m}{2}) = \frac{b-m}{2b}, \tag{8}$$

because we have assumed that $\epsilon_i \sim U[-\frac{b}{2}, \frac{b}{2}]$, *i.i.d.* in Section 2.1.A.

Once joining the platform market, the user will optimally choose to trade the equilibrium amount as shown in Lemma 1. If the utility he gets is lower than the equilibrium reservation utility in the traditional market, $U_R^* = \frac{b^2}{8}$, he will immediately exit the platform. Because agents are homogeneous, the platform would collapse in this case. So, the platform will always guarantee $U_S^* = U_D^* \geqslant U_R^*$, i.e. $(b-m)^2 N^\eta \geqslant b^2$. I name this constraint "no-exit condition".

Combining all the three parts above, and assuming the potential size of the platform market is $\bar{N}$, the law of motion for each side of the platform market is [9]

$$\dot{N} = \begin{cases} \frac{\lambda}{2b}(b-m)N, & if \quad N < \bar{N}, \\ 0, & if \quad N = \bar{N}, \end{cases} \tag{9}$$

where $m$ and $N$ must always satisfy the no-exit condition: $(b-m)^2 N^\eta \geqslant b^2$.

### D. Problem of the Entrepreneur

In Section 2, I assume the entrepreneur who runs the platform enterprise is financially unconstrained. Therefore, his objective is to maximize the discounted

[9] For the law of motion in (9) to hold, $m \in [-b, b]$, because the probability $\frac{b-m}{2b} \in [-1, 1]$. The upper bound never binds since the static monopolistic price is $\frac{b}{2}$; the lower bound may bind. In this paper, I only consider the set of parameters $\{\eta, \lambda, r, \bar{N}\}$ where the lower bound of $m$ does not bind, which is realistic, since platforms in general will not make extreme subsidies. Even if this lower bound binds, it won't affect the qualitative results of this paper, but only generates a segment of $(-b)$ level of $m$ at initial stages of a platform's dynamic pricing strategy.

cash flows generated by the platform enterprise, subject to the law of motion and the no-exit condition:

$$
\begin{aligned}
&\max_{m(t)} \int_0^\infty e^{-rt}\, \frac{m(t)[b-m(t)]}{2}\, N(t)^{(1+\eta)}\, dt \\
&s.t. \qquad \dot{N}(t) = \begin{cases} \dfrac{\lambda}{2b}[b-m(t)]N(t), & if\ N(t) < \bar{N}, \\ 0, & if\ N(t) = \bar{N}, \end{cases} \\
&\& \qquad [b-m(t)]^2 N(t)^\eta \geqslant b^2,\ \forall t.
\end{aligned}
\tag{P1}
$$

## 2.2 Model Solutions and Analysis

In this subsection, I present the closed-form solutions for the entrepreneur's problem (P1). Problem (P1) is a standard dynamic optimization problem, with $m(t)$ as the choice variable and $N(t)$ as the state variable. Denote the optimal policy function by $m^*(t)$ and the corresponding optimal state function by $N^*(t)$. Denote value function by $\Pi^{m^*}(t)$. [10] Then the optimized objective function in Problem (P1) is $\Pi^{m^*}(0)$.

We know that the static monopolistic price is $\frac{b}{2}$. Here, I first show that if the platform can indeed charge this monopolistic price when the network reaches the maximum size $\bar{N}$ , and the optimal strategy is indeed to make subsidies at the very beginning ($m^*(0) < 0$), then the no-exit constraint will never bind all along the optimal path. Put differently, if the no-exit constraint does not bind on the initial point and the endpoint, then it does not bind on the whole path.

**Lemma 2** *If $m^*(0) < 0$ and $\bar{N} \geqslant 4^{\frac{1}{\eta}}$, the no-exit constraint $[b-m(t)]^2 N(t)^\eta \geqslant b^2$ will not bind on the optimal path.*

Proof of Lemma 2 is in Appendix. In this paper, I will only consider the cases where $m^*(0) < 0$, because this paper focuses on the discussion of the optimal investment strategy and its interaction with financing decisions. Besides, as will be shown later, the cases where the platform optimally makes no investment at the very beginning ($m^*(0) > 0$) are less profitable cases. With a high launch cost $F_0$, these cases are naturally ruled out because the profits of the platform enterprise are not enough to cover the launch costs.

[10] The value function can also be expressed as a function of the state variable, i.e. $\Pi^{m^*}(t) = \Pi(N^*(t))$.

The closed-form solutions of the entrepreneur's problem (P1) is presented in Proposition 1. Please see Appendix for detailed derivation and proof.

**Proposition 1** ***(Solutions to P1)***
*Let $\frac{\lambda(1+\eta)}{2r} = a > 1$, and $\bar{N} \geqslant 4^{\frac{1}{\eta}}$. The optimal policy function $m^*(t)$, network growth $N^*(t)$, and discounted future profits $\Pi^{m^*}(t)$, are as follows:*

$$m^*(t) = \begin{cases} \frac{b}{a}\sqrt{a-1}\tan(\frac{r}{2}\sqrt{a-1}\cdot t + c_1) + \frac{b}{a}(a-1) & , \textit{if } t < T^* \\ b/2 & , \textit{if } t \geqslant T^* \end{cases}$$

$$N^*(t) = \begin{cases} e^{\frac{r}{1+\eta}t + c_2} \cdot [\cos(\frac{r}{2}\sqrt{a-1}\cdot t + c_1)]^{\frac{2}{1+\eta}} & , \textit{if } t < T^* \\ \bar{N} & , \textit{if } t \geqslant T^* \end{cases}$$

$$\Pi^{m^*}(t) = \frac{1}{2r}N^*(t)^{(1+\eta)}[b - m^*(t)]^2,$$

*if $c_1 < \arctan(-\sqrt{a-1})$. The exogenous parameters are $(b, r, \eta, \lambda, \bar{N})$. Constants $c_1$ and $c_2$, and the market growth termination time $T^*$ are determined by the end points: $N^*(0) = 1$, $N^*(T^*) = \bar{N}$, $m^*(T^*) = \frac{b}{2}$.*

The exogenous parameters $b$ and $r$ are properties of the economy, and $\eta$, $\lambda$, and $\bar{N}$ are specific for the platform enterprise. Here, the inequality $c_1 < \arctan(-\sqrt{a-1})$ guarantees that the optimal strategy of the platform is indeed to make investment (subsidies) at initial stages; Restriction on the potential market size, $\bar{N} \geqslant 4^{\frac{1}{\eta}}$, guarantees the platform can charge the monopolistic price $\frac{b}{2}$, when the market reaches its maximum size. For the range of $a$, I will show in Appendix that if $a \leqslant 1$, $m^*(0) > 0$. Therefore, this paper focuses on $a > 1$. A visual example of Proposition 1 is given in Figure 2.

**Remark 1.1** *The optimal pricing strategy $m^*(t)$ is monotonically increasing.*

Remark 1.1 emphasizes that it is efficient for the platform to make subsidies early on and charge a commission later. It is never optimal to undulate this spread, switching between subsidies and commission. As will be shown in Section 3, this monotone-increasing property of the pricing strategy holds in more general cases with financial frictions.

**Remark 1.2** *Let the economy has constant $r$ and $b$. Then for the platform enterprise, a stronger network effect $\eta$, a faster information dissemination speed $\lambda$, or a large* *maximum potential network size $\bar{N}$, will result in more subsidies at initial stages, and meanwhile, a higher valuation of the platform.*

Remark 1.2 predicts that with constant $r$ and $b$, if we observe that a platform makes heavier subsidies at initial stages, then it tends to enjoy a higher valuation. This result can be got directly from Proposition 1 by plugging $t = 0$ into $\Pi^{m^*}(t)$: $\Pi^{m^*}(0) = \frac{[b-m^*(0)]^2}{2r}$. That is, a more negative $m^*(0)$ leads to a higher $\Pi^{m^*}(0)$. Besides, since a higher $\eta$, $\lambda$, or $\bar{N}$ leads to a higher valuation of the platform, [11] each of them must lead to more subsidies at the beginning.

Figure 2 depicts how $m^*(t)$, $N^*(t)$ and $\Pi^{m^*}(t)$ evolve over time, and also flows of investment (subsidies) in the time-zero value.[12] This example demonstrates that a stronger network effect leads to a steeper optimal pricing path, faster network growth, and a higher platform value. Here, $\Pi^{m^*}(0) = 15.68$ if $\eta = 1$, and $\Pi^{m^*}(0) = 8.35$ if $\eta = 0.9$. Figure 2(D) shows that the time-zero value of investment flows keep decreasing over time in this example.

### A. Comparison with the Traditional Market

The main distinction between the platform market and the traditional market is the existence of cross-group network effects in the former.

Take Deliveroo as an example. It is a British online food ordering and delivering platform linking the restaurants and the consumers. Consumers can order online from a wide range of restaurants which may not be within walking distance, and wait at home for the food to be delivered. The traditional market counterpart is the decentralized local restaurant market. Through Deliveroo, consumers have access to a large group of restaurants. Hence, they tend to consume more because of the richer product variety brought by the platform, compared to limited choices of local restaurants near home. Deliveroo charges a commission from the restaurant which in turn affects the prices of food listed on the platform by the restaurants. The story is similar for Amazon. Uber is also similar if we regard the traditional market as the taxi-booking market. [13]

[11] A heuristic proof is as follows. With higher $\lambda$, keep the path for $N(t)$ the same as before. Then the new path $m(t) > m^*(t)\ \forall t$, which results a higher $\Pi(0)$. With higher $\eta$, keep the path for $N(t)^{(1+\eta)}$ as before, and this leads to a higher $m(t)$ and higher instantaneous profits for all $t$, thus a higher $\Pi(0)$. With a higher $\bar{N}$, keeping the original path can lead to a higher $\Pi(0)$.

[12] The expression is $e^{-rt}\ \frac{m(t)[m(t)-b]}{2}\ N(t)^{(1+\eta)}$.

[13] In many cities' taxi market, customers must book in advance for a taxi, which exhibits no network effect.

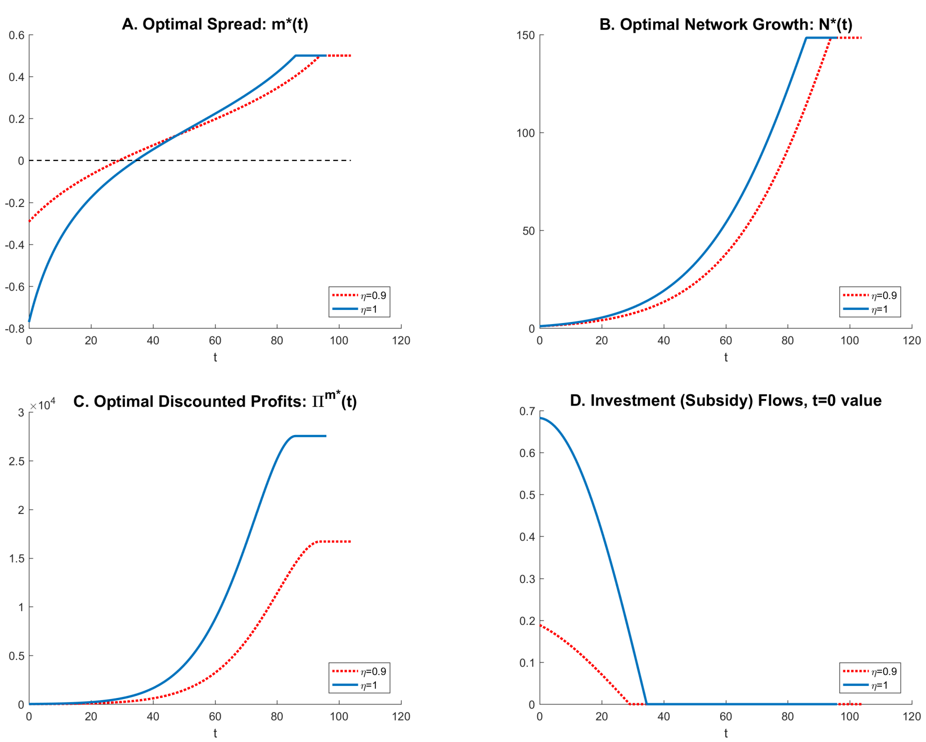


**Figure 2**

Parameters: $b = 1$, $r = 0.1$, $\lambda = 0.12$, $\bar{N} = e^5$, $\eta = 0.9$ or $1$.

The network effects are indispensable for a platform enterprise to exist and make profits. If on the contrary, the platform contains no network effects, i.e. $\eta = 0$, then the platform is exactly the same as the traditional market and it can never charge a commission. Once the platform charges a positive spread, the no-exit condition $[b - m(t)]^2 N(t)^\eta \geqslant b^2$ immediately breaks and all agents exit. Intuitively, when the platform brings no additional surplus to agents compared with the traditional market, there is no room for it to charge a commission. Thus, *ex ante* the entrepreneur has no incentive to launch the platform if there are no network effects in this market. If the network effects are weak, the commission that can be charged by the platform is low and it leads to low profitability of the platform enterprise. With some fixed launch costs or flow maintenance costs, the platform enterprise cannot survive either.

## B. Comparison with the Traditional Production Firms

Perhaps surprisingly, the optimal behavior of a platform enterprise exhibits similarities to a traditional production firm. They both invest first and profit later. The networks of the platform are analogous to the capital assets of a production firm. The platform first invests in the networks and then extract profits from the networks. Therefore, the platform generates negative cash flows early on and positive cash flows later, which has a similar pattern to a traditional production firm.

Some characters of the platform enterprise's investment this paper would like to emphasize and which may be different from a traditional production firm are as follows. 1) The optimal investment-stage and production-stage of the platform enterprise are clearly separated. The entrepreneur running the platform enterprise has the choice to switch between investment (subsidy) and production (commission) as frequently as he wishes, but he will never do so because it's suboptimal. 2) The investment for a platform enterprise takes place dynamically and gradually and a perfect timing is extremely important. Once the platform cannot follow the optimal timing, it will suffer from slower growth and a lower valuation. 3) There is no depreciation in the network, unlike most capital assets in production firms. Once the network is built, it lasts forever unless the agents exit.

If the entrepreneur does not have adequate internal funds to make investment, he has to raise external funds. Section 3 discusses the entrepreneur's financing issues in detail.

# 3 Financing Under Limited Enforcement

Section 2 builds up a dynamic model for the platform business. The optimal investment and pricing strategies of the entrepreneur are studied, assuming he is financially unconstrained. What if the entrepreneur needs to raise external financing and there exist financial frictions and financing costs, as are common to start-up firms? In this section, I analyze the financing and investment strategies of the entrepreneur when there exist financial frictions and financing costs. Moreover, to analyze the influences of financial market conditions, I relax the assumption of a fully-competitive capital market. Financiers may require positive profits for their investment. I examine how the required profits affect the incentives of the entrepreneur and thus his financing level, staging, and timing choices.

## 3.1 The Model

The type of financial friction I consider in this paper is limited enforcement, which constrains the entrepreneur's ability to make credible promises.[14] I assume the enforcement of contracts is limited as follows: the entrepreneur can abscond with funds in hand, instead of investing the funds in the networks. The entrepreneur has the incentive to abscond with funds once he has more funds in hand than what he expects to get from successfully managing the platform enterprise. The profits generated by the platform are assumed to be verifiable and paid out as dividends straight away. Thus, the entrepreneur can potentially embezzle the funds but not the operational cash inflows.

Specifically, the setting of the model is as follows: The platform enterprise is launched by an entrepreneur, who has skills but no money. There is a fixed cost $F_0$ to launch the platform enterprise, and any other costs for operating the platform are set to zero. The fixed cost has already been covered through the "angel investment" (or initial rounds of financing), and the entrepreneur is obliged to pay back $F$ (time-zero value) to the angel investors. To invest in the networks, the entrepreneur has to raise additional financing. Assume that there exists a fixed financing cost $C$ (time-zero value) for each round of financing. This fixed financing cost is a deadweight loss for raising a new round of capital, which can be understood as fees paid to the third parties for project valuation, endorsement, etc. So, each round the entrepreneur raises $(I_j + C)$ from financier $j$, where $I_j$ (time-zero value) is invested into the networks. Financier $j$ requires $W_j$ (time-zero value) as investment profits. That is, he requires a time-zero value $(I_j + C + W_j)$ back for this investment, where $W_j \geqslant 0$.[15]

Denote the total number of financing rounds by $n$. Let $W = \sum_{j=1}^{n} W_j$, representing the aggregate time-zero value of profits required by all financiers, and $I = \sum_{j=1}^{n} I_j$, representing the aggregate time-zero value of funds invested in the networks. Besides, the model allows the entrepreneur to choose the timing of each new round of financing. Therefore, he is able to allocate the funds optimally through time into the networks. That is to say, only the aggregate fund level $I$, but not any individual level $I_j$, affects the optimal investment strategy and the valuation of the platform.

[14] Some previous papers consider limited enforcement as a type of financial frictions are Chien and Lustig (2009), Rampini and Viswanathan (2010).

[15] For simplicity of the problem, all the values in this section are denoted using the time-zero value.

Let $\Pi_0(I)$ denote the optimized time-zero value of all discounted profits as a function of $I$. $\Pi_0(I)$ can be regarded as the solution to a lump-sum constrained optimization problem when the lump-sum investment level is $I$. Each $\Pi_0(I)$ corresponds to a unique pricing strategy and network growth trajectory, as will be demonstrated in Proposition 2 below. Intuitively, $\Pi_0(I)$ increases in $I$ when $I \leqslant I^*$, where $I^*$ is the optimal time-zero value of investment derived in Section 2,[16] and $\Pi_0(I^*) = \Pi^{m^*}(0)$.

In this model, the aggregate investment level $I$ is endogenously chosen by the entrepreneur. To be exact, the amount of funds raised each round $I_j$, and the number of rounds $n$, are all endogenously determined by the entrepreneur's profit-maximization objective, subject to the incentive compatibility constraint for him not to embezzle funds and abscond, as well as the individual rationality constraint for him to launch the platform enterprise and raise external financing to invest in the networks.

The problem of the entrepreneur is summarized as follows:

$$
\begin{aligned}
& \max_{n, I_1, I_2, \ldots, I_n} \quad \Pi_0(I) - F - nC - W \\
s.t. \quad & \Pi_0(I) - F - nC - W \geqslant I_1 && (IC_1) \\
& \ldots \\
& \Pi_0(I) - F - nC - W \geqslant I_n && (IC_n) \\
& \Pi_0(I) - F - nC - W \geqslant \max\{\Pi_0(0) - F,\ 0\} && (IR) \\
& \text{where} \quad I = \sum_{j=1}^{n} I_j,\ W = \sum_{j=1}^{n} W_j,
\end{aligned}
\tag{P2}
$$

and $\Pi_0(I)$ is the solution to the following problem:

$$
\begin{aligned}
& \max_{m(t)} \int_0^\infty e^{-rt} \, \frac{m(t)[b - m(t)]}{2} \, N(t)^{(1+\eta)} \, dt \\
s.t. \quad & \dot{N}(t) = \begin{cases} \dfrac{\lambda}{2b}[b - m(t)]N(t), & if \ N(t) < \bar{N}, \\ 0, & if \ N(t) = \bar{N}, \end{cases} \\
\& \quad & [b - m(t)]^2 N(t)^\eta \geqslant b^2, \ \forall t, \\
\& \quad & \int_0^t e^{-rx} \, \frac{m(x)[m(x) - b]}{2} \, N(x)^{(1+\eta)} \, dx \leqslant I, \ \forall t.
\end{aligned}
$$

[16] $I^* = \int_0^{\tau^*} e^{-rt} \frac{m^*(t)[m^*(t)-b]}{2} N^*(t)^{(1+\eta)} dt$, where $\tau^*$ is the time point when the platform charges zero spread, or $m^*(\tau^*) = 0$. The entrepreneur will rationally raise $I \leqslant I^*$.

The entrepreneur maximizes the time-zero value of his expected profits from optimally and successfully managing the platform enterprise, subject to the incentive compatibility (IC) constraints and the individual rationality (IR) constraint. The incentive compatibility constraints say that the funds the entrepreneur receives each round should be no more than what he expects to gain from managing the platform enterprise. Otherwise, he would abscond with funds. The individual rationality constraint says that if the entrepreneur expects to get too little from raising funds and optimally investing in the networks, he would instead choose not to make the investment or not to launch the platform enterprise, *ex ante*.

## 3.2 Model Solutions and Analysis

To solve Problem (P2), we need to solve for $\Pi_0(I)$ first. As is defined, $\Pi_0(I)$ is the optimized time-zero value of all discounted future profits of the platform, with the constraint that the investment level is $I$. We can solve for the constraint-optimal pricing strategy $m(t)$ and the network growth function $N(t)$ to get $\Pi_0(I)$. A detailed derivation of $\Pi_0(I)$ is given in Proof of Proposition 2 in Appendix.

**Lemma 3** *The constraint-optimal pricing strategy $m(t)$ is non-decreasing.*

Lemma 3 shows that the platform's subsidy stages and commission stages are clearly divided. It is inefficient to charge a commission for a period and use the profits as internal funds to make subsidies. Therefore, the platform will never swing between subsidies and commission. The proof of Lemma 3 may be found in Appendix. Equipped with Lemma 3, we can solve for $m(t)$, $N(t)$, and $\Pi_0(I)$. The results are provided in Proposition 2.

**Proposition 2** *Let $a > 1$ and $\bar{N} \geqslant 4^{\frac{1}{\eta}}$. Define $\tau^*$ as the time point when the unconstrained optimal spread is zero, or $m^*(\tau^*) = 0$; define $I^*$ as the optimal aggregate subsidy amount, or $I^* = \int_0^{\tau^*} e^{-rt}\frac{m^*(t)[m^*(t)-b]}{2}N^*(t)^{(1+\eta)}dt$. When the available aggregate subsidy amount $I \leqslant I^*$, the constraint-optimal policy function $m(t)$, network* *growth $N(t)$, and the time-zero value of discounted future profits $\Pi_0(I)$, are as follows:*

$$m(t) = \begin{cases} \frac{b}{a}\sqrt{a-1}\tan(\frac{r}{2}\sqrt{a-1}\cdot t + c_3) + \frac{b}{a}(a-1) & , \ t \leqslant \underline{\tau} \\ 0 & , \ \underline{\tau} < t \leqslant \tilde{\tau} \\ m^*(t - \tilde{\tau} + \tau^*) & , \ t > \tilde{\tau} \end{cases}$$

$$N(t) = \begin{cases} e^{\frac{r}{1+\eta}t + c_4} \cdot [\cos(\frac{r}{2}\sqrt{a-1}\cdot t + c_3)]^{\frac{2}{1+\eta}} & , \ t \leqslant \underline{\tau} \\ e^{\frac{\lambda}{2}(t-\underline{\tau})} \cdot N(\underline{\tau}) & , \ \underline{\tau} < t \leqslant \tilde{\tau} \\ N^*(t - \tilde{\tau} + \tau^*) & , \ t > \tilde{\tau} \end{cases}$$

$$\Pi_0(I) = e^{-r\tilde{\tau}} \cdot \Pi^{m^*}(\tau^*) - I,$$

*where $c_3$, $c_4$, $\underline{\tau}$, $\tilde{\tau}$ are determined by terminal conditions:*
$N(0) = 1$, $N(\tilde{\tau}) = N^*(\tau^*)$, $m(\underline{\tau}) = 0$ *and* $\int_0^{\underline{\tau}} e^{-rt}\frac{m(t)[m(t)-b]}{2}N(t)^{(1+\eta)}dt = I$.

**Remark 2.1** *When the aggregate amount of available funds is less than the optimal amount, it is constraint-optimal for the platform to make highly aggressive subsidies early on and use up the available funds, and then wait with zero spread until it's optimal to start to charge a commission.*

The results highlight the importance of building up the networks at early stages. Timely and sufficient investment in the networks is crucial for the success and high valuation of a platform enterprise. If the entrepreneur misses the best timing to make investment, the platform enterprise will suffer from a lower valuation or even failure to survive. A prominent example is the failure of SideCar, as mentioned in the introduction of this paper. Therefore, it is rational for the entrepreneur running the platform to use up available funds to make heavy subsidies early on.

Figure 3 plots an example of constrained-optimal pricing strategy, network growth pattern, the value of discounted profits, and investment flows, compared with the unconstrained benchmarks the same as in Figure 2.

Having got $\Pi_0(I)$, I now set out to characterize the solutions for the entrepreneur's problem (P2). First, the entrepreneur's optimal financing patterns are summarized in Proposition 3.

**Proposition 3** *Each round of financing has the same time-zero value. Thus, the face value (time-t value) of the funds raised each round keeps increasing. The frequency of new rounds keeps decreasing in most of the economically meaningful cases, where a sufficient condition is $a > \frac{5}{4}$, and $\bar{N} < \bar{N}^*$ such that $m(0) > -b$.*

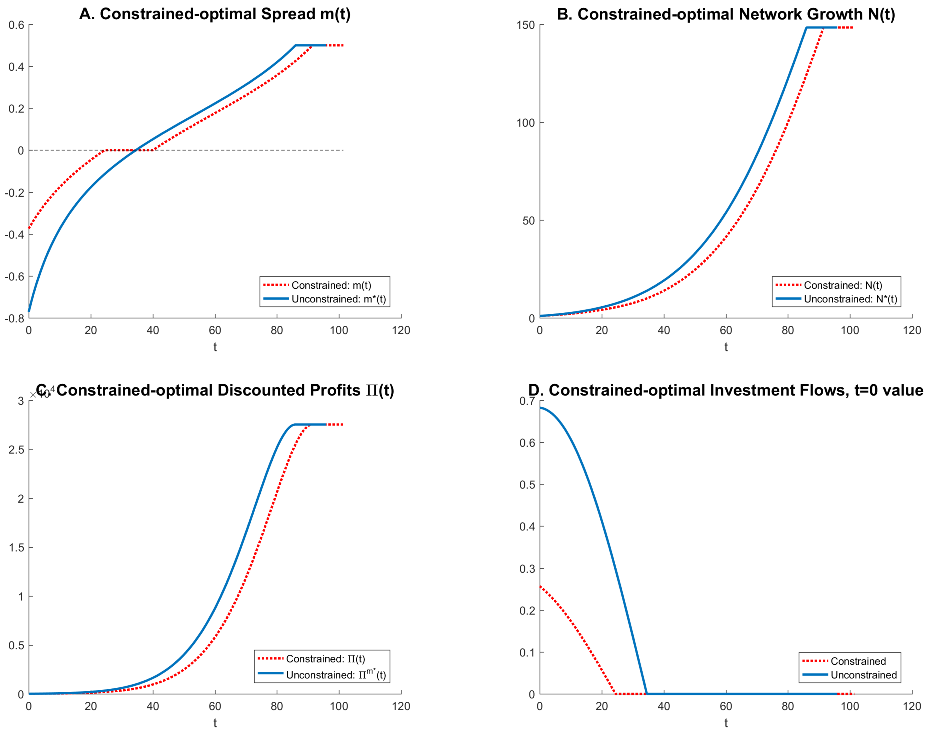


**Figure 3**
Parameters: $b = 1$, $r = 0.1$, $\lambda = 0.12$, $\bar{N} = e^5$, $\eta = 1$.
The constrained case has $I = 3.5$ and $\Pi_0(I) = 14.16$.

In the entrepreneur's problem stated above, it is obvious that setting $I_1 = I_2 = ... = I_n = \frac{I}{n}$ weakly dominates all other strategies, because it is the strategy where the incentive compatibility constraint is easiest to be satisfied. That is, the face value (time-t value) of each round of financing is increasing. Besides, with the decreasing flows of investment, as shown in Figure 2(D) and Figure 3(D), the time intervals between two rounds of financing are inclined to increase. A through proof of Proposition 3 is in Appendix.

How do the financing and investment decisions vary under different scenarios? In this paper, I analyze the effects of profitability and capital market conditions.

### A. Profitability and Rounds of Financing

**Proposition 4** *Ceteris Paribus, a platform enterprise with higher profitability* $(\frac{\Pi_0(I)-F}{I})$ *raises fewer rounds of financing.*

This proposition seems counter-intuitive, but it is actually a general result when the major financial friction is limited enforcement. All things equal, the more profitable the enterprise, the more profits left to the entrepreneur. Hence, he can credibly raise more funds within each round, which leads to fewer rounds of financing.

Proof of Proposition 4 is simple and direct. Using results in Proposition 3, we can rewrite the incentive compatibility constraints as: $\frac{\Pi_0(I)-F}{I} \geqslant n\frac{C}{I} + \frac{1}{n} + \frac{W}{I}$. Higher profitability makes the (IC) constraints easier to be satisfied, and thus leads to a smaller $n$ required, so as to maximize the entrepreneur's profits.

### B. Influences of Capital Market Conditions

In practice, the capital market is usually not fully competitive, and financiers may require positive profits to their investments (i.e. $W > 0$). If the capital market is abundant with money, financiers tend to require lower profits. On the other hand, if a lot of good projects are waiting to be financed in the capital market, the opportunity costs are higher and the required profits by financiers increase. The level of required profits by the financiers affects the entrepreneur's financing and investment decisions.

**Proposition 5** *Ceteris Paribus, the financing and investment levels ( $I$) weakly decrease with the required profits by the financiers ( $W$), while the number of financing rounds ( $n$) weakly increases with the required profits by the financiers ( $W$) as long as the deadweight-loss financing cost ( $C$) is not too high, where a sufficient condition is* $\frac{C}{I/n} < \frac{1}{n}$.

The underlying logic of Proposition 5 is similar to that of Proposition 4. The more profits required by the financiers, the fewer profits left to the entrepreneur, and thus the less funding the entrepreneur can credibly raise within one round. This results in a trade-off between raising lower aggregate level of funds to make less investment, and raising more rounds of financing to make adequate investment, depending on the cost of financing and relative profitability of the investment. Or mathematically, because $I$ is a continuous choice variable while $n$ is a discrete variable, the entrepreneur will optimally choose to decrease $I$ and increase $n$ by turns when $W$ continues increasing. When $W$ becomes excessively large, the participation constraint will bind and the entrepreneur will choose not to finance. Proof of Proposition 5 may be found in Appendix. Below is an example demonstrating the relationship between the required profits of financiers and the entrepreneur's financing decisions.

**Example.** *The parameters for the platform are the same as in Figure 2 and 3:* $b = 1$, $r = 0.1$, $\lambda = 0.12$, $\eta = 1$, $\bar{N} = e^5$. $F = 10$, $C = 0.1$. *The relationship between the required profits of financiers and the entrepreneur's financing decisions are shown in the table:*

| **Required Profits:** $W$ | **Financing Rounds:** $n$ | **Aggregate Financing Level:** $I$ |
|---|---|---|
| $0 - 0.43$ | 3 rounds | $I = I^*$ |
| $0.43 - 1.48$ | 3 rounds | $I < I^*$, and decreases with $W$ |
| $1.48 - 1.57$ | 4 rounds | $I = I^*$ |
| $1.57 - 2.34$ | 4 rounds | $I < I^*$, and decreases with $W$ |
| $2.34 - 2.84$ | 5 rounds | $I < I^*$, and decreases with $W$ |
| $2.84 - 3.15$ | 6 rounds | $I < I^*$, and decreases with $W$ |
| $3.15 - 3.32$ | 7 rounds | $I < I^*$, and decreases with $W$ |
| $3.32 - 3.43$ | 8 rounds | $I < I^*$, and decreases with $W$ |
| $3.43 - 3.47$ | 9 rounds | $I < I^*$, and decreases with $W$ |
| $> 3.47$ | Not to finance | $I = 0$ |

**Table 1: The Example**

# 4 Conclusion

This paper develops a tractable micro-founded dynamic platform model. The defining property of a platform market is the existence of cross-group network effects. Networks are analogous to capital-assets of the platform enterprise. The platform enterprise invests in the networks first and generates income from the networks later on. With inadequate internal funds, the enterprise has to raise external capital. The paper solves for the optimal financing and investment strategies of the entrepreneur in a world with financial frictions and financing costs. Meanwhile, the paper depicts the platform market growth patterns.

The key findings of this paper are that: 1) in face of financial constraints, a monopolistic platform should make aggressive subsidies by using up available funds to boost network growth early on; 2) the optimal spread charged by the platform is non-decreasing over time, i.e. per-transaction subsidies decrease over time and are followed by increasing fees; 3) with stronger network effects, the platform has a propensity to make more subsidies early one and enjoys a higher valuation and faster network growth; 4) staging is a natural choice to mitigate financial frictions and *ceteris paribus*, the number of financing rounds decrease with profitability of the platform and increases with required profits by financiers; 5) the value of funds raised each round increases and the financing frequency decrease for a platform enterprise.

There are some other interesting questions this paper has not discussed, which might be directions for future research. First, what if there exists platform competition or potential entry of new platforms? How will financial frictions and costs interact with competition and entry threat? Whether it will be a winner-takes-all equilibrium and whether the deep pocket matters most are still not that clear. Second, what if there exists uncertainty in this market, say, uncertainty on the network growth path (the law of motion)? How will the uncertainty affect optimal financing and investment decisions? It may also be interesting to study other forms of incentive problems, such as asymmetric information between the platform and financiers. I leave studies on these questions to future research.

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# Appendix

## Proof of Lemma 2

Assume the no-exit constraint does not bind, then $m^*(t)$ and $N^*(t)$ are as given in Proposition 1. Therefore, I only need to prove $m^*(t)$ and $N^*(t)$ given in Proposition 1 satisfy $[b - m^*(t)]^2 N^*(t)^\eta \geqslant b^2$ when $m^*(0) < 0$ and $\bar{N} \geqslant 4^{\frac{1}{\eta}}$. Since $m^*(t)$ will never be greater than the monopolistic price $\frac{b}{2}$, so we always have $b - m^*(t) > 0$. Thus, the no-exit constraint is equivalent to $[b - m^*(t)]N^*(t)^{\frac{\eta}{2}} \geqslant b$. Below is the proof.

Define $\theta_t = \frac{r}{2}\sqrt{a-1} \cdot t + c_1, \in (-\frac{\pi}{2}, \frac{\pi}{2})$. $\theta_t$ is increasing in $t$. Then, $\log\{[b - m^*(t)]N^*(t)^{\frac{\eta}{2}}\} = \log(1 - \sqrt{a-1}\tan\theta_t) + \frac{\eta}{1+\eta}[\log(\cos\theta_t)] + constant$. Denote it by $f(\theta_t)$. We then have:

$f(\theta_0) = \log[b - m^*(0)] > \log(b)$, and $f(\theta_{T^*}) = \log(\frac{b}{2}\bar{N}^{\frac{\eta}{2}}) \geqslant \log(b)$.

$$\begin{aligned}
f'(\theta_t) &= \frac{-\sqrt{a-1}}{1 - \sqrt{a-1}\tan\theta_t}(1 + \tan^2\theta_t) + \frac{\eta}{1+\eta}[-\tan\theta_t + \frac{1}{\sqrt{a-1}}] \\
&= \underbrace{\frac{1}{\sqrt{a-1} - (a-1)\tan\theta_t}}_{>0} \underbrace{\left[-(a-1)(1 + \tan^2\theta_t) + \frac{\eta}{1+\eta}(1 - \sqrt{a-1}\tan\theta_t)^2\right]}_{g(\theta_t)}.
\end{aligned}$$

Since $b - m^*(\theta_t) > 0$, we have $\sqrt{a-1} - (a-1)\tan\theta_t > 0$.

Define $g(\theta_t) = -(a-1)(1 + \tan^2\theta_t) + \frac{\eta}{1+\eta}(1 - \sqrt{a-1}\tan\theta_t)^2$. We get:

$$g(\theta_t) = -\frac{1}{1+\eta}(\sqrt{a-1}\tan\theta_t + \eta)^2 + (1 + \eta - a).$$

**Case 1.** $a \geqslant 1 + \eta$**:**

then we have $g(\theta_t) < 0$, thus $f'(\theta_t) < 0$, $f(\theta_t)$ is monotone decreasing. Since $f(\theta_{T^*}) \geqslant \log(b)$, we get $f(\theta_t) \geqslant \log(b)$, $\forall \theta_t \in [\theta_0, \theta_{T^*}]$. Thus, $\forall t \in [0, T^*]$, the no-exit constraint is satisfied.

**Case 2.** $a < 1 + \eta$**:**

Since $m^*(0) < 0$, we get $\tan\theta_0 < -\sqrt{a-1}$. And from $m^*(T^*) = \frac{b}{2}$, we get $\theta_{T^*} = \frac{2-a}{2\sqrt{a-1}}$. Thus, $\exists\ \hat{t} \in (0, T^*)$ where $m^*(\hat{t}) = 0$, $\tan\theta_{\hat{t}} = -\sqrt{a-1}$.

For $t \in [0, \hat{t}\,]$, $m^*(t) \leqslant 0$, the no-exist constraint is satisfied. We now want to prove that the no-exist constraint is always satisfied when $t \in (\hat{t}, T^*]$, i.e., when $\tan\theta_t \in (-\sqrt{a-1}, \frac{2-a}{2\sqrt{a-1}}]$.

We take derivative of $g(\theta_t)$:

$$g'(\theta_t) = -\frac{2\sqrt{a-1}}{1+\eta}(\eta + \sqrt{a-1}\tan\theta_t).$$

$g'(\theta_t)$ decreases in $\tan\theta_t$. So $g'(\theta_t) < g'(\theta_{\hat{t}}) = -\frac{2\sqrt{a-1}}{1+\eta}(\eta + 1 - a) < 0$, since we are in Case 2. That is, $g(\theta_t)$ is monotone decreasing for $\theta_t \in (\theta_{\hat{t}}, \theta_{T^*}]$.

$g(\theta_{\hat{t}}) = -\frac{1}{1+\eta}(1+\eta-a)^2 + (1+\eta-a) = \frac{a}{1+\eta}(1+\eta-a) > 0.$

$g(\theta_{T^*}) = -\frac{1}{1+\eta}(1+\eta-\frac{a}{2})^2 + (1+\eta-a) = -\frac{a^2}{4(1+\eta)} < 0.$

So, $f(\theta_t)$ first increases and then decreases. That is, $f(\theta_t)$ is quasi-concave on $(\theta_{\hat{t}}, \theta_{T^*}]$. Since $f(\theta_{\hat{t}}) \geqslant \log(b)$, $f(\theta_{T^*}) \geqslant \log(b)$, $f(\theta_t) \geqslant \log(b)$ is always satisfied on $(\theta_{\hat{t}}, \theta_{T^*}]$.

To summarize Case 1 and Case 2, the no-exit constraint does not bind for on the optimal path if $m^*(0) < 0$ and $\bar{N} \geqslant 4^{\frac{1}{\eta}}$. ■

## Proof of Proposition 1

Let $T^*$ denote the time when the platform reaches its maximum size, i.e., $N(T^*) = \bar{N}$. When $t > T^*$, the network size of each side is constant $\bar{N}$. So the optimal pricing strategy is the monopoly pricing: $m^*(t) = \frac{b}{2}$. Thus, $\Pi^{m^*}(t) = \frac{b^2}{8r}\bar{N}^{(1+\eta)}$ , $\forall t \geqslant T^*$. When $t \leqslant T^*$, it's a optimal control problem with fixed end points. We can apply techniques in calculus of variations to solve it.

First, define $h(t) = \log N(t)$, so $\dot{h}(t) = \frac{\dot{N}(t)}{N(t)} = \frac{\lambda}{2b}[b - m(t)]$. We can rewrite the problem as:

$$\max_{h(t)} \int_0^{T^*} (\frac{b}{\lambda})^2\, e^{-rt+(1+\eta)h(t)}\, \dot{h}(t)[\lambda - 2\dot{h}(t)]\, dt$$
$$s.t. \quad h(0) = 0,\ h(T^*) = \log\bar{N},\ \Pi(T^*) = \frac{b^2}{8r}\,\bar{N}^{(1+\eta)}.$$

Apply the Euler-Lagrange Equation, $\frac{\partial g}{\partial h} = \frac{d}{dt}(\frac{\partial g}{\partial \dot{h}})$, where

$$g(\ h(t),\ \dot{h}(t),\ t) = (\frac{b}{\lambda})^2\, e^{-rt+(1+\eta)h(t)}\, \dot{h}(t)[\lambda - 2\dot{h}(t)],$$

and we get

$$(1+\eta)\dot{h}^2(t) - 2r\dot{h}(t) + 2\ddot{h}(t) + \frac{r\lambda}{2} = 0. \tag{A-1}$$

Since $\dot{h}(t) = \frac{\lambda}{2b}[b - m(t)]$, $\ddot{h}(t) = -\frac{\lambda}{2b}\dot{m}(t)$. Plug them into (A-1):

$$\dot{m}(t) = \frac{(1+\eta)\lambda}{4b}m^2(t) + [r - \frac{(1+\eta)\lambda}{2}]\, m(t) + \frac{(1+\eta)\lambda b}{4} - \frac{br}{2}, \tag{A-2}$$

$$\Rightarrow \qquad \frac{(1+\eta)\lambda}{4b}\, dt = \frac{dm(t)}{m^2(t) + 2b[\frac{2r}{(1+\eta)\lambda} - 1]\, m(t) + b^2 - \frac{2rb^2}{(1+\eta)\lambda}}. \tag{A-3}$$

The denominator on the right-hand side of (A-3) is a quadratic form, whose discriminant is $\Delta = 4b^2[\frac{2r}{(1+\eta)\lambda} - 1]^2 - 4[b^2 - \frac{2rb^2}{(1+\eta)\lambda}] = \frac{4b^2}{a^2}(1-a)$, where $a \equiv \frac{\lambda(1+\eta)}{2r}$. So we can divide the situation into three cases according to the range of $\Delta$.

**Case 1.** $a > 1$**, that is** $\Delta < 0$**.**

Rewrite (A-3) as

$$\frac{ar}{2b}\, dt = \frac{dm(t)}{[m(t) - \frac{b}{a}(a-1)]^2 + \frac{b^2}{a^2}(a-1)}\ ,$$

and integrate each side of the equation. [1] Then we get when $t < T^*$,

$$m^*(t) = \frac{b}{a}\sqrt{a-1}\tan(\frac{r}{2}\sqrt{a-1}\cdot t + c_1) + \frac{b}{a}(a-1), \tag{A-4}$$

where $c_1$ is the constant of integration to be determined by terminal conditions.

To find $h^*(t)$ when $t < T^*$:

$$\begin{aligned} h^*(t) &= \int_0^t \dot{h}^*(t)\, dt \\ &= \int_0^t \frac{\lambda}{2b}[b - m^*(t)]\, dt \\ &= \int_0^t \frac{\lambda}{2a}[1 - \sqrt{a-1}\tan(\frac{r}{2}\sqrt{a-1}\cdot t + c_1)]\, dt \\ &= \frac{\lambda}{2a}t - \frac{\lambda}{ar}\log[\cos(\frac{r}{2}\sqrt{a-1}\cdot t + c_1)] + c_2 \\ &= \frac{r}{1+\eta}t + \frac{2}{1+\eta}\log[\cos(\frac{r}{2}\sqrt{a-1}\cdot t + c_1)] + c_2 \end{aligned}$$

[1]Formula: $\int \frac{1}{x^2+v^2}dx = \frac{1}{v}\ \tan^{-1}\ (\frac{x}{v})$.

where $c_2$ is also a constant to be determined by terminal conditions. [2]

Thus when $t < T^*$,

$$N^*(t) = e^{\frac{r}{1+\eta}t+c_2} \cdot [\cos(\frac{r}{2}\sqrt{a-1} \cdot t + c_1)]^{\frac{2}{1+\eta}} \ . \tag{A-5}$$

A quick way to derive the value function is to use the HJB equation:

$$r\Pi^{m^*}(h(t)) = \max_{m(t)} \{\frac{m(t)[b-m(t)]}{2}e^{(1+\eta)h(t)} + \frac{d\Pi}{dh}\dot{h}(t)\} \ . \tag{A-6}$$

Take F.O.C. with respect to $m(t)$:

$$\frac{d\Pi}{dh}(t) = \frac{2b}{\lambda}[\frac{b}{2} - m^*(t)]\, e^{(1+\eta)h^*(t)} \ .$$

Plug it into (A-6), and we get the value function:

$$\Pi^{m^*}(t) = \frac{1}{2r}N^*(t)^{(1+\eta)}[b - m^*(t)]^2 \ . \tag{A-7}$$

Since the value function is continuous at $T^*$, and $\Pi^{m^*}(T^*) = \frac{b^2}{8r}\bar{N}^{(1+\eta)}$, by applying (A-7), we get $m^*(T^*) = \frac{b}{2}$. Now we have three end points, $h(0) = 0$, $h(T^*) = \log \bar{N}$ and $m^*(T^*) = \frac{b}{2}$ to determine $c_1$, $c_2$ and $T^*$:

$$\begin{cases} \frac{2}{1+\eta}\log[\cos(c_1) + 1] + c_2 = 0 \\ \frac{r}{1+\eta}T^* + \frac{2}{1+\eta}\log[\cos(\frac{r}{2}\sqrt{a-1} \cdot T^* + c_1)] + c_2 = \log \bar{N} \\ \frac{1}{a}\sqrt{a-1}\tan(\frac{r}{2}\sqrt{a-1} \cdot T^* + c_1) + \frac{1}{a}(a-1) = \frac{1}{2} \end{cases} \tag{A-8}$$

The existence is demonstrated in Figure 2.

[2] Through out this paper, I define the domain of the trigonometric functions to be $(-\frac{\pi}{2}, \frac{\pi}{2})$, without loss of generality. Thus, $\cos(\frac{r}{2}\sqrt{a-1} \cdot t + c_1)$ is always positive.

**Case 2.** $a = 1$**, that is** $\Delta = 0$**.**

Rewrite (A-3) as

$$\frac{ar}{2b}\,dt = \frac{dm(t)}{m^2(t)} = -\,dm^{-1}(t)\ ,$$

$$\Rightarrow \qquad m(t) = -\frac{1}{\frac{ar}{2b}t - c_3}\ .$$

The constant $c_3$ must be positive, so that we can get a positive $m(t)$ when $t > 0$, as required by the problem. This leads to $m(0) = \frac{1}{c_3} > 0$. Since $\dot{m}(t) = \frac{ar}{2b}m^2(t) \geqslant 0$, we always have $m(t) > 0$. That is there is no investment in network in this case, or the platform always charges a positive commission on both sides.

**Case 3.** $0 < a < 1$**, that is** $\Delta > 0$**.**

Rewrite (A-3) as

$$\frac{ar}{2b}dt = \frac{dm(t)}{[m(t) - \alpha][m(t) - \beta]}\ ,$$

where

$$\alpha + \beta = \frac{2b}{a}(a-1) < 0, \quad \alpha\beta = \frac{b^2}{a}(a-1) < 0\ .$$

Let $\beta > 0$ and $\alpha < 0$, then:

$$\beta - \frac{b}{2} = \frac{b}{a}(\frac{a}{2} - 1\sqrt{1-a}) < 0, \qquad \forall\ 0 < a < 1.$$

Since $\dot{m}(t) = \frac{ar}{2b}[m(t) - \alpha][m(t) - \beta]$, we must have $m(0) > \beta > 0$. Otherwise, $m(t)$ can never grow to $\frac{b}{2}$ and will never satisfy the terminal conditions. To conclude, there is no investment in network in this case, or the platform always charges a positive commission on both sides. ■

## Proof of Lemma 3

When $a \leqslant 1$, $m^*(0) > 0$ and there is no subsidy stage. Thus the constraint never binds and the constraint-optimal pricing strategy is still $m^*(t)$, which is increasing.

When $a > 1$, let's prove by contradiction.

Suppose there is a segment of $m(t)$ that is decreasing. $m_{t_0-\Delta t}$ and $m_{t_0}$ are two adjacent points along that segment, $m_{t_0-\Delta t} > m_{t_0}$. Denote the corresponding network sizes by $N_{t_0-\Delta t}$ and $N_{t_0}$, and normalize $N_{t_0-\Delta t}$ to be 1. So, $N_{t_0} = N_{t_0-\Delta t} \cdot [1 + \frac{\lambda}{2b}(b - m_{t_0-\Delta t})\Delta t] = 1 + \frac{\lambda}{2b}(b - m_{t_0-\Delta t})\Delta t$, and the network size after these two periods is: $N_{t_0+\Delta t} = [1 + \frac{\lambda}{2b}(b - m_{t_0-\Delta t})\Delta t][1 + \frac{\lambda}{2b}(b - m_{t_0})\Delta t]$. Note that in this proof, we use discrete-time approximations. When $\Delta t \to 0$, they converge to the continuous-time results.

Define $\hat{m} = \frac{m_{t_0-\Delta t}+m_{t_0}}{2}$. Thus, $m_{t_0-\Delta t} > \hat{m} > m_{t_0}$. We would like to prove that, whenever there is a decreasing pricing strategy $(m_{t_0-\Delta t}, m_{t_0})$, we can find a non-decreasing strategy $(\hat{m}, \hat{m})$ that weakly dominates it. That is, we need to prove that, 1) $(\hat{m}, \hat{m})$ is within the entrepreneur's action space, 2) it leads to weakly faster network growth, and 3) it leads to weakly higher profits.

If $\hat{m} \geqslant 0$, it's obvious that the lump-sum subsidy level constraint won't bind after choosing the strategy $(\hat{m}, \hat{m})$. If $\hat{m} < 0$, since the strategy $(\hat{m}, \hat{m})$ generates weakly higher profits (i.e. make weakly lower subsidies) than the original strategy $(m_{t_0-\Delta t}, m_{t_0})$, as will be proven later, the lump-sum subsidy level constraint won't bind after these two periods, since the original strategy satisfy the constraint. And by backward induction, the constraint will also not bind in the intermediate period. In this way, we have proven that $(\hat{m}, \hat{m})$ is indeed within the entrepreneur's action space.

As has been shown, with strategy $(m_{t_0-\Delta t}, m_{t_0})$, the network size after these two periods is $N_{t_0+\Delta t} = [1 + \frac{\lambda}{2b}(b - m_{t_0-\Delta t})\Delta t][1 + \frac{\lambda}{2b}(b - m_{t_0})\Delta t]$. If we instead choose $(\hat{m}, \hat{m})$, the network size after two periods becomes $\hat{N}_{t_0+\Delta t} = [1 + \frac{\lambda}{2b}(b - \hat{m})\Delta t]^2$. $\hat{N}_{t_0+\Delta t} - N_{t_0+\Delta t} = \frac{\lambda^2 \Delta t^2}{4b^2}(\hat{m}^2 - m_{t_0-\Delta t} m_{t_0}) = \frac{\lambda^2 \Delta t^2}{16b^2}(m_{t_0-\Delta t} - m_{t_0})^2 > 0$. Thus, $(\hat{m}, \hat{m})$ indeed leads to faster network growth.

The profits generated by the original strategy $(m_{t_0-\Delta t}, m_{t_0})$ during these two periods are $\Pi = \frac{1}{2} m_{t_0-\Delta t}(b - m_{t_0-\Delta t}) + \frac{1}{2} m_{t_0}(b - m_{t_0}) e^{-r\Delta t} N_{t_0}^{1+\eta}$, while the profits generated by the strategy $(\hat{m}, \hat{m})$ are $\hat{\Pi} = \frac{1}{2}\hat{m}(b - \hat{m})(1 + e^{-r\Delta t}\hat{N}_{t_0}^{1+\eta})$. We would like to prove that $\Pi - \hat{\Pi} < 0$.

For notational simplicity, denote $m_{t_0-\Delta t}$ and $m_{t_0}$ by $m_1$ and $m_2$, respectively. So, $-b \leqslant m_2 < \hat{m} < m_1 \leqslant b/2$, and $\hat{m} = (m_1 + m_2)/2$.

$$\begin{aligned}
& \Pi - \hat{\Pi} < 0 \\
\Leftrightarrow \quad & m_1(b-m_1)e^{r\Delta t} + m_2(b-m_2)[1 + \frac{\lambda}{2b}(b-m_1)\Delta t]^{1+\eta} \\
& - \hat{m}(b-\hat{m})e^{r\Delta t} - \hat{m}(b-\hat{m})[1 + \frac{\lambda}{2b}(b-\hat{m})\Delta t]^{1+\eta} < 0 \ , \\
\Leftrightarrow \quad & m_1(b-m_1)(1+r\Delta t) + m_2(b-m_2)[1 + \frac{\lambda(1+\eta)}{2b}(b-m_1)\Delta t] \\
& - \hat{m}(b-\hat{m})(1+r\Delta t) - \hat{m}(b-\hat{m})[1 + \frac{\lambda(1+\eta)}{2b}(b-\hat{m})\Delta t] < 0 \ ,
\end{aligned}$$

where we apply $\lim_{\Delta t \to 0} e^{r\Delta t} = 1 + r\Delta t, \ \lim_{x \to 0}(1+x)^{1+\eta} = 1 + (1+\eta)x,$

$$\begin{aligned}
\Leftrightarrow \quad & (2\hat{m}^2 - m_1^2 - m_2^2) + \\
& \Delta t \left\{ r[m_1(b-m_1) - \hat{m}(b-\hat{m})] + \frac{\lambda(1+\eta)}{2b}[m_2(b-m_2)(b-m_1) - \hat{m}(b-\hat{m})^2] \right\} < 0 \ , \\
\Leftrightarrow \quad & -\frac{(m_1-m_2)^2}{2} + \\
& \Delta t \, \frac{\lambda(1+\eta)}{2b}[bm_1(b-m_1) - b\hat{m}(b-\hat{m}) + m_2(b-m_2)(b-m_1) - \hat{m}(b-\hat{m})^2] < 0 \ ,
\end{aligned}$$

since $[m_1(b-m_1) - \hat{m}(b-\hat{m})] > 0$ and $a = \frac{\lambda(1+\eta)}{2r} > 1$.

With some algebra, we finally get that the above inequality is equivalent to:

$$-\frac{(m_1-m_2)^2}{2} - \Delta t \, \frac{\lambda(1+\eta)}{2b}(m_1 - \hat{m})[(b+m_2)(m_1-\hat{m}) + \hat{m}^2] < 0 \ ,$$

which is true. So we have proven that $\Pi - \hat{\Pi} < 0$.

Combining 1), 2), 3), we have proven Lemma 3. ■

## Proof of Proposition 2

The terminal conditions $N(T) = \bar{N}$ and $m(T) = \frac{b}{2}$ are the same as in the non-constrained problem, but with $T > T^*$. According to Lemma 3, $m(t)$ is non-decreasing. So, there exists $\tilde{\tau}$ such that $m(t) \leqslant 0$ when $t \leqslant \tilde{\tau}$ and $m(t) > 0$ when $t > \tilde{\tau}$.

According to the Principle of Optimality, if the terminal values are determined and there are no additional constraints, we can find the same optimal path by backward induction. That is to say, for $t \in [\tilde{\tau}, T]$, the optimal path will be exactly the same as in the non-constrained problem when $t \in [\tau^*, T^*]$, where $\tau^*$ satisfies $m^*(\tau^*) = 0$. Thus, we must have $m(\tilde{\tau}) = 0$, $N(\tilde{\tau}) = N^*(\tau^*)$, $\Pi(\tilde{\tau}) = \Pi^{m^*}(\tau^*)$.

Then, $\Pi_0(I) = e^{-r\tilde{\tau}}\Pi(\tilde{\tau}) - I = e^{-r\tilde{\tau}}\Pi^{m^*}(\tau^*) - I$.

The problem turns into finding minimum $\tilde{\tau}$ given aggregate subsidy level $I$. And its dual problem is to find minimum $I$ given $\tilde{\tau}$. Let's solve the dual problem:

$$\min_{m(t)} \int_0^{\tilde{\tau}} e^{-rt} \frac{m(t)[m(t) - b]}{2} N(t)^{1+\eta} dt$$

$$s.t. \qquad m(t) \leqslant 0, \ \forall t,$$

$$\dot{N}(t) = \frac{\lambda}{2b}[b - m(t)]N(t),$$

$$N(0) = 1, \ N(\tilde{\tau}) = N^*(\tau^*).$$

Again, define $h(t) = \log N(t)$, so $\dot{h}(t) = \frac{\dot{N}(t)}{N(t)} = \frac{\lambda}{2b}[b - m(t)]$. This problem is similar to the non-constrained problem, but with additional constraints, $m(t) \leqslant 0$, and different terminal conditions.

Because $m(t)$ is weakly increasing, let's denote the time $m(t)$ first touches zero by $\underline{\tau}$, $\underline{\tau} \leqslant \tilde{\tau}$. So, $m(t) < 0$ when $t \in [0, \underline{\tau})$ and $m(t) = 0$ when $t \in [\underline{\tau}, \tilde{\tau}]$. Thus, when $t \in [0, \underline{\tau})$, the additional constraint does not bind, and the solution to this problem follows the same Euler-Lagrange equation as in the non-constrained problem, only with different terminal conditions; when $t \in [\underline{\tau}, \tilde{\tau}]$, $m(t) = 0$, $\dot{h}(t) = \frac{\lambda}{2}$.

Since we have already known the terminal conditions, the constraint-optimal path is then determined:

$$m(t) = \begin{cases} \frac{b}{a}\sqrt{a-1}\tan(\frac{r}{2}\sqrt{a-1}\cdot t + c_3) + \frac{b}{a}(a-1) & , \ t \leqslant \underline{\tau} \\ 0 & , \ \underline{\tau} < t \leqslant \tilde{\tau} \\ m^*(t - \tilde{\tau} + \tau^*) & , \ t > \tilde{\tau} \end{cases}$$

$$N(t) = \begin{cases} e^{\frac{r}{1+\eta}t + c_4} \cdot [\cos(\frac{r}{2}\sqrt{a-1}\cdot t + c_3)]^{\frac{2}{1+\eta}} & , \ t \leqslant \underline{\tau} \\ e^{\frac{\lambda}{2}(t-\underline{\tau})} \cdot N(\underline{\tau}) & , \ \underline{\tau} < t \leqslant \tilde{\tau} \\ N^*(t - \tilde{\tau} + \tau^*) & , \ t > \tilde{\tau} \end{cases}$$

$$\Pi_0(I) = e^{-r\tilde{\tau}} \cdot \Pi^{m^*}(\tau^*) - I,$$

where $c_3$, $c_4$, $\underline{\tau}$, $\tilde{\tau}$ are determined by terminal conditions:
$N(0) = 1$, $N(\tilde{\tau}) = N^*(\tau^*)$, $m(\underline{\tau}) = 0$ and $\int_0^{\underline{\tau}} e^{-rt}\frac{m(t)[m(t)-b]}{2}N(t)^{(1+\eta)}dt = I$. ■

## Proof of Proposition 3

The first part of Proposition 3 is already proven. Let's prove the second part.

The constrained-optimal time-t subsidy flows at time-zero value is:

$$\begin{aligned} & e^{-rt}\frac{m(t)[m(t)-b]}{2}N(t)^{(1+\eta)} \\ = & \frac{1}{2}e^{(1+\eta)c_4}(\frac{b}{a})^2[\sqrt{a-1}\tan\theta_t + a - 1][\sqrt{a-1}\tan\theta_t - 1]\cos^2\theta_t, \end{aligned}$$

where $\theta_t = \frac{r}{2}\sqrt{a-1}\cdot t + c_3$, $\theta_t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Let $f(\theta_t) = [\sqrt{a-1}\tan\theta_t + a - 1][\sqrt{a-1}\tan\theta_t - 1]\cos^2\theta_t$, then

$$\frac{df(\theta_t)}{d\theta_t} = 2(a-1)\sin(2\theta_t) + (a-2)\sqrt{a-1}\cos(2\theta_t).$$

Since we are considering the subsidy stage, $m(t) < 0$. Thus, $\tan(\theta_t) < -\sqrt{a-1}$. We can restrict the range of $\theta_t$ to be $(-\frac{\pi}{2}, 0)$, and then $\cos(2\theta_t) < 0$.

$$\frac{df(\theta_t)}{d\theta_t} < 0 \ \Leftrightarrow \ (a-2)\cot(2\theta_t) > -2\sqrt{a-1}. \tag{A-9}$$

**Case 1.** $a > 2$**:**

$$\frac{df(\theta_t)}{d\theta_t} < 0 \Leftrightarrow \cot(2\theta_t) > \frac{-2\sqrt{a-1}}{a-2}.$$

Since $\cot(2\theta_t) = \frac{1}{2}[\frac{1}{\tan(\theta_t)} - \tan(\theta_t)]$ is decreasing in $\tan(\theta_t)$, we get $\cot(2\theta_t) > \frac{1}{2}(\sqrt{a-1} - \frac{1}{\sqrt{a-1}}) > 0$, the above inequality indeed holds.

Thus, when $a > 2$, the constrained-optimal subsidy flows (time-zero value) is decreasing with $t$.

**Case 2.** $1 < a < 2$**:**

$$\frac{df(\theta_t)}{d\theta_t} < 0 \Leftrightarrow \cot(2\theta_t) < \frac{2\sqrt{a-1}}{2-a}.$$

If we consider the cases where $m(0) > -b$,[3] then we must have $\tan\theta_t > \frac{1-2a}{\sqrt{a-1}}$.

$$\cot(2\theta_t) = \frac{1}{2}[\frac{1}{\tan\theta_t} - \tan\theta_t] > \frac{1}{2}(\frac{2a-1}{\sqrt{a-1}} - \frac{\sqrt{a-1}}{2a-1}).$$

Then, a sufficient condition for Inequality (A-9) to hold is that:

$$\frac{1}{2}(\frac{2a-1}{\sqrt{a-1}} - \frac{\sqrt{a-1}}{2a-1}) < \frac{2\sqrt{a-1}}{2-a}. \tag{A-10}$$

Let $y = a - 1$, then $y \in (0, 1)$. After some algebra, (A-10) is equivalent to

$$(4y-1)(y+1)^2 > 0$$

Thus, the sufficient condition for (A-9) to hold is $1 < a < 2$ and $m(0) > -b$ in Case 2. When $\bar{N}$ is not too large, or $\exists\ \bar{N}^*$ such that when $\bar{N} < \bar{N}^*$, we have $m(0) > -b$.

When $a = 2$, Inequality (A-9) indeed holds.

To summarize, a sufficient condition for the constrained-optimal subsidy flows to be decreasing in $t$ is that: $a > \frac{5}{4}$, and $\bar{N} < \bar{N}^*$ such that $m(0) > -b$. ■

[3] In practice, platforms seldom give extremely large subsidies. For example, $p_D$ is always positive.

## Proof of Proposition 5

According to Proposition 3, we can rewrite the problem (P2) as:

$$
\begin{aligned}
&\max_{n,I}\ \Pi_0(I) - F - nC - W \\
s.t.\quad &\Pi_0(I) - F - nC - W \geqslant \frac{I}{n} && (IC) \\
&\Pi_0(I) - F - nC - W \geqslant \max\{\Pi_0(0) - F, 0\} && (IR)
\end{aligned}
$$

As we consider the case the entrepreneur participate in financing and subsidizing, let's omit the individual rationality constraint (IR) for a while. Then the Lagrangian function of this problem is:

$$\mathcal{L} = \Pi_0(I) - F - nC - W + \mu[\Pi_0(I) - F - nC - W - \frac{I}{n}].$$

Take F.O.C. with respect to $I$:

$$\Pi_0'(I) + \mu[\Pi_0'(I) - \frac{1}{n}] = 0$$

Since $\Pi_0(I)$ is weakly increasing in $I$, $\Pi_0'(I) \geqslant 0$. By definition, the Lagrangian multiplier $\mu \geqslant 0$. So we get $\Pi_0'(I) - \frac{1}{n} \leqslant 0$. That is, $\Pi_0(I) - \frac{I}{n}$ weakly decreases in $I$.

Let's rearrange the (IC) constraint as

$$\Pi_0(I) - \frac{I}{n} \geqslant F + nC + W.$$

Keep $F$ and $n$ constant and increases $W$. When the constraint does not bind, $W$ does not affect $I$. When the constraint start to bind, increasing $W$ must lead to an increase on the left hand side, and thus a decrease in $I$.

For the number of financing rounds $n$, *Ceteris Paribus*, increase in $W$ leads to weakly increase in $n$ as long as the financing cost $C$ is not to large. A sufficient condition is $\frac{C}{I/n} < \frac{1}{n}$, which can be got directly from $(\frac{I}{n} + nC)$ decreasing in $n$.

Because $I$ is a continuous choice variable while $n$ is a discrete one, the entrepreneur will decrease $I$ and increase $n$ by turns when $W$ is increasing, so as to maximize his expected profits. When $W$ is too large, the participation constraint will bind and he will choose not to finance and make no subsidies. ■