Optimal Disclosure of Value Distribution Information in All-pay Auctions

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Abstract

In this paper, we follow a Bayesian Persuasion approach to study the auction organizer's optimal disclosure of information about players' value distribution in a two-player all-pay auction setting. Players' private values (either high v_h or low v_l) are independently and identically distributed. There are two possible value distributions (i.e., two possible states), and none of the players knows the actual distribution. Before the auction starts, the organizer pre-commits to a public signal to reveal information about the prevailing value distribution. We find that there exists a cutoff for value ratio $v = v_h/v_l$, above which a monotone equilibrium arises under any prior belief about the state. In this circumstance, no disclosure is optimal. When value ratio v is below the cutoff, there exist exactly two threshold beliefs about the state that separate prior beliefs generating monotone and non-monotone equilibria. A prior belief would lead to a non-monotone equilibrium if and only if it lies in between. If the original prior μ_0 leads to a monotone equilibrium, then still no disclosure is optimal; otherwise, a partial disclosure, which generates a posterior distribution over the two threshold beliefs, is optimal.

Keywords: All-pay auction; Information disclosure; Public signal.

JEL classification: D44, D74, D82

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1 Introduction

All-pay auction, as a form of auction which requires all its bidding participants to pay their bid amounts regardless of whether they win or not, has been a useful tool to model competitions in many real-life environments. R&D races, sport competitions, lobbying, and election campaigns are just a few examples. As is well documented in the literature, auction participants may have private information, like abilities, values, and costs, and they need to form a belief about opponents' private information before making a move. The auction organizer can influence participants' beliefs through provision of information and induce them to behave in his favored way. For example, in the bidding for a procurement contract, the organizer can decide whether to reveal the list of competing suppliers. Such revelation of information about competitors on the list may enable a supplier to update his belief about competitors' provision costs and profitabilities, and lead to more aggressive bidding behaviors.

In this paper, we investigate the optimal design of information disclosure in an all-pay auction setting with two ex ante symmetric players competing for a single indivisible object. Players' private values, either v_h or v_l, are affiliated with a common unknown state of the world, which determines the distribution of values. There are two possible states, G and B, and the prior μ_0 is commonly known. The organizer pre-commits to a public signal about the state before the auction starts.

The issue of information disclosure in contests has been studied extensively in literature, for example, Fu *et al.* (2014), Denter *et al.*(2012), and Lu *et al.* (2018). However, to the best of our knowledge, most of the existing studies in this literature mainly focus on the comparison between no disclosure and full disclosure. From the contest organizer's perspective, the no-or-full disclosure comparison seems to be too restrictive. He also has the option to partially disclose information. Pioneered by Kamenica and Gentzkow (2011), the Bayesian persuasion approach serves as a useful tool to model partial disclosure in several recent studies (e.g. Zhang and Zhou (2016) and Chen (2019)). In this approach, a disclosure signal can be viewed as a conditional distribution, and the organizer's problem of choosing the optimal disclosure rule can be transformed as a problem of choosing the optimal distribution of posterior beliefs. In the paper, we also explore the optimal disclosure from the perspective of Bayesian persuasion. However, unlike those recent studies (Zhang and Zhou (2016) and Chen (2019)) which directly disclose information about players' values, this paper focus on the signals which are indirect in the sense that players update beliefs about opponents' values with the revealed information on state. Although players' private values are independently drawn from the same distribution, the uncertainty about value distribution creates affiliation between the values, i.e., they co-move positively. This indirect signal would enable the organizer to manipulate the value affiliation perceived by the players, thus affecting their bidding behaviors.

In the paper, we identified a monotonicity condition for the equilibrium of the all-pay auction to be monotone. This condition can be equivalently transformed to comparing the value ratio $v = v_h/v_l$ and belief ratio $p_s(v_l|v_l)/p_s(v_l|v_h)$, in which $p_s(v_l/v_h)$ is the high value player's belief of a low value opponent when the belief about state is μ_s. If the value ratio is larger than the belief ratio, then the monotonicity condition is satisfied, and a unique monotone symmetric equilibrium exists. Otherwise, a non-monotone symmetric equilibrium exists. Our equilibrium characterization shows that there is a cutoff v_0 for value ratio v, above which a monotone equilibrium always exists for any belief, and below which a non-monotone equilibrium may arise for some belief. Specifically, if $v < v_0$, there are exactly two beliefs, μ_1^v and μ_2^v, for a belief in between which a non-monotone equilibrium exists. That is, μ_1^v and μ_2^v are the beliefs that separate beliefs generating monotone and non-monotone equilibria. The cutoff v_0, as the maximum value of $p_s(v_l|v_l)/p_s(v_l|v_h)$, is solely determined by the two possible value distributions. It decreases as difference in distributions gets smaller and reaches one in the limit where the two distributions are exactly the same. In that case, we have an all-pay auction with unknown common value. The decreasing property of v_0 in the difference of distributions suggests that, the monotonicity condition is more likely to be satisfied when the two distributions get similar, and thus it's less likely to have a non-monotone equilibrium. Intuitively, a high affiliation level implies a tougher competition for high value players in the auction. With a greater affiliation (i.e., higher belief ratio), a high value player bids more aggressively, which leads to a lower rent for the player. Thus, a change from low value to high value, on the one hand, has a negative affiliation effect on the player's payoff, and on the other hand, has a positive value effect on the payoff. When $v \geq v_0$, the value effect is always greater than the affiliation effect, in which case the high value player has a non-negative payoff in the auction with any belief and always bid higher than a low value player. Thus, a monotone equilibrium arises. When $v < v_0$, for some beliefs, the affiliation effect is larger than the value effect, and the high value player has to mix his bid on an interval which encompasses the low value's bidding interval to ensure a non-negative payoff. Thus, a non-monotone equilibrium follows.

Our main result about information disclosure is that there exists a cutoff lower than v_0, $v_{00} = p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$, above which no disclosure is optimal and below which some partial disclosure is optimal [1]. To be more specific, if $v \geq v_{00}$, i.e., the value effect is greater than the affiliation effect with prior μ_0, the original auction game (with prior μ_0) has a unique monotone equilibrium, and no disclosure is optimal for the organizer. If $v < v_{00}$, i.e., the value effect is smaller than the affiliation effect with prior μ_0, the original game has a unique non-monotone equilibrium, and a partial disclosure which induces a posterior distribution over μ_1^v and μ_2^v is optimal for the organizer. The intuition is as follows. The identical and independent draw of players' private values and the uncertainty about value distribution result in affiliation of players' values. Since more information means less uncertainty about the distribution, information disclosure would lead to a lower value affiliation. Thus, when the organizer discloses information about the state (or the prevailing value distribution), what he actually does is manipulating the affiliation. When $v \geq v_{00}$, i.e., the initial affiliation effect is relatively lower than the value effect, the original auction game has a monotone equilibrium, in which the high value player always outbids the low value player. Thus, there is no efficiency loss in equilibrium. Since the affiliation level is already sufficiently low, any further revelation of information to decrease the affiliation wouldn't be of benefit to the organizer. Therefore, no disclosure is optimal in this circumstance. However, when $v < v_{00}$, that is, the initial affiliation effect is relatively larger than the value effect, the original auction has a non-monotone equilibrium in which the high value player's bidding interval overlaps with that of the low value player. In the equilibrium, a low value player might win against a high value player, which would result in efficiency loss [2]. Although the organizer can fully extract players' rent in a non-monotone equilibrium, the efficiency loss still negatively affects his revenue. By disclosing information to reduce affiliation to a proper level, the organizer can gain trade efficiency while still fully extracting players' surplus. Thus, in that case the organizer benefits from information disclosure. In optimal, the organizer would choose a disclosure policy which induces a distribution of posterior beliefs over μ_1^v and μ_2^v, i.e., the beliefs that separate beliefs generating monotone and non-monotone equilibria. Specifically,

[1] The result can be extended to a setting with more than two states.

[2] See Milgrom and Weber (1982), Krishna and Morgan (1997), and Chi, Murto, and Välimäki(2019).

at either μ_1^v or μ_2^v, there is no efficiency loss and players' rents remain zero in equilibrium. Any further reduction in affiliation would lead to a positive rent payment to a high value player, but brings no further gain in efficiency.

Our paper is closely related to the literature on equilibrium characterization in all-pay auctions. Hillman and Riley (1989) first characterizes the unique mixed strategy of two-player all-pay auction with complete information. Baye, Kovenock, and de Vries (1996) extends the set of equilibria to include asymmetric ones for the class of all-pay auction games with complete information. Amann and Leininger (1996) shows the existence and uniqueness of equilibrium for asymmetric two-player all-pay auction with incomplete information. Siegel (2014) studies the monotone equilibria in a two-bidder all-pay auction with multiple types. Liu and Chen (2016) consider a two-player all-pay auction model with correlated private value. They characterize both monotonic and non-monotonic symmetric Bayesian Nash equilibrium. However, they failed to prove the uniqueness of the non-monotone equilibrium mathematically. The most recent development on the characterization of equilibrium in all-pay auction is Chi, Murto, and Välimäki (2019). They construct a monotonicity condition, characterize both monotone and non-monotone equilibria, and establish the uniqueness of equilibrium in an all-pay auction with multiple bidders and affiliated signals.

Another strand of closely related literature is on information disclosure in contests. Zhou and Zhang (2016) examines the type-dependent probabilistic disclosure policies (i.e., Bayesian persuasion approach) in a two-player simultaneous contest. Chen, Kuang, and Zheng (2017a) investigate the type-dependent probabilistic disclosure policies in a two-player sequential contest. However, in these papers, one player's value is commonly observed while the other's is private information. This one-sided asymmetric information feature differs from that of our paper in which both players' values are private information. For the all-pay auction scenario, Lu, Ma, and Wang (2018) compares four different type-dependent non-probabilistic disclosure policies in a two-player all-pay auction. Kuang, Zhao, and Zheng (2019) completely characterizes the optimal type-dependent probabilistic disclosure policy in a two-player all-pay auction with correlated values. Chen (2019) consider the disclosure policy in a two-player all-pay auction with independent private values. He characterizes a public disclosure policy which is better than no disclosure but failed to identify the optimal one. Our paper differs from all of these papers in that the disclosure policy is indirect and state-dependent, rather than type dependent. This indirect disclosure captures the organizer's tradeoff between efficiency and players' rent when manipulating players' perceived value affiliation.

The rest of this paper is organized as follows. In Section 2 we set up the model. In Section 3 we analyze the equilibrium in posterior all-pay auction game. In Section 4, we solve the organizer's optimal design of information disclosure. We conclude in Section 5. The Appendix collects the technical proofs.

2 Model

Consider the following all-pay auction with incomplete information. There are two risk-neutral players competing for an object by submitting their bids simultaneously. The winning probability of player $i \in \{1,2\}$ under bid portfolio (x_1, x_2) is given by

$$p_i(x_1,x_2) = \begin{cases} 1 & \text{if } x_i > x_{-1}; \\ 0 & \text{if } x_i < x_{-1}. \end{cases}$$

If there is a tie, i.e., $x_1 = x_2$, the object is randomly allocated between the two players.

Prior to bidding, each player $i \in \{1,2\}$ privately learns his value of the object $v_i \in \{v_h, v_l\}$. The two possible values are ordered as $v_h > v_l$ to capture the idea that a player with v_h has a higher value for the object than a player with v_l. The two players' values are affiliated with a common unknown state of world, $\omega \in \Omega = \{G, B\}$. Let $\mu_0 \in \Delta(\Omega)$ denote the common prior over states. Specifically, the two players' values are independently drawn from the same binary distribution conditioned on the prevailing state ω, and we represent the conditional distributions with vector (α, β), where $\alpha = p(v = v_h|G)$ and $\beta = p(v = v_l|B)$. We assume that $\alpha \geq 1 - \beta$ with the idea that G is a good state under which it's more likely for a player to draw high value v_h compared to B (bad state).

The auction organizer pre-commits to a signal before the auction starts with the intention to maximize the expected total bids collected from both players. A signal consists of a finite realization space $\mathcal{S}$ and a family of distributions $\{\pi(\cdot|\omega)\}_{\omega\in\Omega}$. That is, conditioning on the prevailing state ω, the signal is realized according to distribution $\pi(\cdot|\omega) \in \Delta(\mathcal{S})$. If for any realization s and s', $\pi(s|\omega) = \pi(s'|\omega)$, then the signal is *uninformative* since the updated belief about state under any realization stays the same as the prior. If there exist two realizations s and s' such that $\pi(s|G) = 1$ and $\pi(s'|B) = 1$, then the signal is *fully informative* as the state can be directly inferred from the realized signal. Note that while the signal is conditional on the common state, eventually the players have to update beliefs about their opponents' private values. Thus, when a specific signal $s \in \mathcal{S}$ is realized, a player first updates his belief about state and then forms a belief about his opponent's value with the posterior belief about state using Bayes' rule. Denote the posterior belief about state as $\mu_s \in \Delta(\Omega)$. Due to the binary structure of value distribution, we write μ_s instead of $\mu_s(G)$ for notation simplicity in some scenarios.

The timing of the game is as follows:

1. The auction organizer pre-commits to a signal π.
2. Nature moves and the state of world is determined, say ω.
3. A signal realization $s \in \mathcal{S}$ is generated according to $\pi(\cdot|\omega)$, and players' private values are generated according to $p(\cdot|\omega) \in \Delta(\Omega)$.
4. The signal realization s is publicly observed, and each player privately learns his own value for the object. With observed s and privately learned value, a player forms a posterior belief μ_s, which leads to a new belief about his opponent's value.
5. The auction takes place, and the players place their bids simultaneously.

The game in stage 5 is an all-pay auction with affiliated private values, and the affiliation between players' values is determined by the posterior belief from signal realization in stage 3. The results in Chi, Murto, and Välimäki (2019) suggest that the level of affiliation in players' value in an all-pay auction may affect the structure of equilibrium, thus affecting the expected total bids. With the aim to maximize expected total bids, the organizer needs to choose his pre-committed signal optimally in stage 1. In the following, we first examine the posterior all-pay auction game in stage 5, and then proceed to solve the organizer's optimal design of signal in stage 1.

3 The Posterior All-pay Auction Game

In the posterior all-pay auction game, a player $i \in \{1, 2\}$ privately learns his value v_i and has a belief about his opponent's value, which is formed from posterior μ_s as following:

$$p_s(v|v_i) = \frac{\sum_{\omega \in \Omega} p(v|\omega) p(v_i|\omega) \mu_s(\omega)}{\sum_{\omega \in \Omega} p(v_i|\omega) \mu_s(\omega)}, \ \forall v \in \{v_l, v_h\}. \tag{3.1}$$

Since players' private values are independently drawn from the same distribution determined by the prevailing state, their values are affiliated, i.e., they co-move positively. This means that a player with high value (low value) is more likely to expect his opponent with high value (low value) than a player with low value (high value).

Claim 1. *Players' private values are affiliated. Specifically, given belief μ_s,*

$$p_s(v_i|v_i) \geq p_s(v_i|v_j). \tag{3.2}$$

Proof. See Appendix. □

Due to identical and independent drawn of players' private values, and different value distributions under different states, we find that players' values are affiliated. Since the values are binarily distributed, the inequality about players' beliefs in Claim 1 follows.

To characterize the equilibrium in the posterior all-pay auction game, we define following monotonicity condition similar to that in Siegel (2014).

Condition M: For $i \in \{1, 2\}$, $v_i p_s(v|v_i)$ increases in v_i for every $v \in \{v_h, v_l\}$.

That is, for each player i, the product of his value and his belief about opponent's value increases in his own value. Then we have $v_h p_s(v|v_h) \geq v_l p_s(v|v_l)$ for $\forall v \in \{v_h, v_l\}$. Because of the affiliation shown in Claim 1, the inequality holds automatically when $v = v_h$. But for $v = v_l$, the inequality holds only when the difference between v_h and v_l is large or the affiliation between low values is small (i.e., $p_s(v_l|v_h)$ is very close to $p_s(v_l|v_l)$). Let $v = v_h/v_l$. We define real-valued function $\phi(\mu_s) = vp_s(v_l|v_h) - p_s(v_l|v_l)$. The Condition M is equivalent to requiring $\phi(\mu_s) \geq 0$, which implies the increment in $v_i p_s(v_l|v_i)$ in response to player i's value change from v_l to v_h is positive.

Since the actual bid depends on a player's private value and belief about his opponent's value, we represent a strategy of player $i \in \{1, 2\}$ by a pair of cumulative distribution functions $F_i^s = (F_i^s(\cdot|v_h), F_i^s(\cdot|v_l))$, where $F_i^s(x|v)$ is the probability that player i bids at most x when his value is v and his belief about opponent's value is formed with μ_s. As a bid more than the value definitely generates a negative payoff in an all-pay auction, a player would never make that bid. Thus, without loss of generality we restrict our attention to strategies with $\text{supp}[F_i^s(\cdot|v_i)] \in [0, v_i]$. We analyze Bayesian Nash Equilibria. That is, if strategy profile F^s is an equilibrium, for each player i with private value v_i, $x \in \text{supp}[F_i^s(\cdot|v_i)]$ implies $x \in \arg\max u^s(v_i)$. Throughout this work, we focus our attention on *symmetric* equilibria in which players with the same private value employ the same bidding strategy, i.e., $F_i^s = F^s = (F^s(\cdot|v_h), F^s(\cdot|v_l))$. A symmetric equilibrium is *monotone* if for any $x \in \text{supp}(F(v_h,s))$ and $y \in \text{supp}(F(v_l,s))$, we have $y \leq x$. Otherwise, it's non-monotone[3]. Liu and Chen(2016) and Chi, Murto, and Välimäki (2019) establish the following result.

Proposition 1. *In the posterior all-pay auction game with distribution of value distribution μ_s, there exists a unique symmetric equilibrium. Specifically,*

1. *if $\phi(\mu_s(G)) \geq 0$, the equilibrium is monotone, and players' equilibrium strategies are*

$$F^{s,m}(x|v_l) = \frac{x}{v_l p_s(v_l|v_l)} \text{ on } [0, v_l p_s(v_l|v_l)],$$

$$F^{s,m}(x|v_h) = \frac{x - v_l p_s(v_l|v_l)}{v_h p_s(v_h|v_h)} \text{ on } [v_l p_s(v_l|v_l), v_l p_s(v_l|v_l) + v_h p_s(v_h|v_h)];$$

2. *if $\phi(\mu_s(G)) < 0$, the equilibrium is non-monotone, and players' equilibrium strategies are*

$$F^{s,nm}(x|v_l) = x \cdot \frac{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}{v_h v_l[p_s(v_h|v_h) - p_s(v_h|v_l)]} \text{ on } [0, \underline{x}(s)],$$

$$F^{s,nm}(x|v_h) = \begin{cases} x \cdot \frac{v_l p_s(v_l|v_l) - v_h p_s(v_l|v_h)}{v_h v_l[p_s(v_h|v_h) - p_s(v_h|v_l)]} & \text{on } [0, \underline{x}(s)] \\ \frac{x - v_h p_s(v_l|v_h)}{v_h p_s(v_h|v_h)} & \text{on } [\underline{x}(s), v_h], \end{cases}$$

where $\underline{x}(s) = \frac{v_h v_l[p_s(v_h|v_h) - p_s(v_h|v_l)]}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}$.

Proposition 1 tells us that it dependes on the sign of $\phi(\mu_s(G))$ whether the unique equilibrium in the posterior all-pay auction game is monotone or non-monotone. When $\phi(\mu_s(G)) \geq 0$, the equilibrium is monotone, in which the low value type uniformly randomizes its bid on a lower interval (i.e., $[0, v_l p_s(v_l|v_l)]$) while the high value type uniformly randomizes on an upper interval (i.e., $[v_l p_s(v_l|v_l), v_h p_s(v_h|v_h)]$). Specifically, the two intervals are connected at $v_l p_s(v_l|v_l)$. The equilibrium is separating since players' value types can be inferred from almost any pair of bids. Notice that the highest possible bid the low value type $v_l p_s(v_l|v_l)$ is smaller than his value v_l. This is because that in a montone equilibrium , a low value type who makes his highest bid, wins only when his opponent is also low value

[3] It's impossible for the low value playe to win against the high value player with probability one in equilibrium. We would never have $y \geq x$ when $x \in \text{supp}(F(v_h,s))$ and $y \in \text{supp}(F(v_l,s))$. Thus, the equilibrium can't be monotone in an inversed pattern.

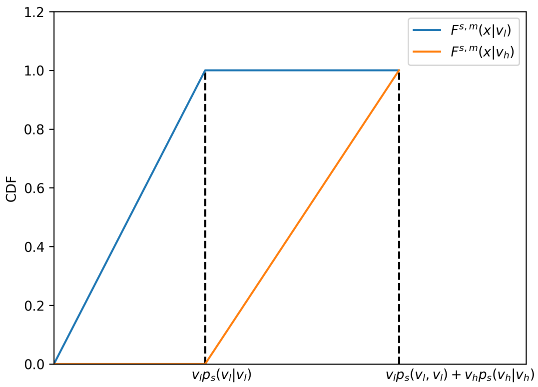

Figure 3.1: monotone equilibrium when $\phi(\mu_s(G)) \geq 0$

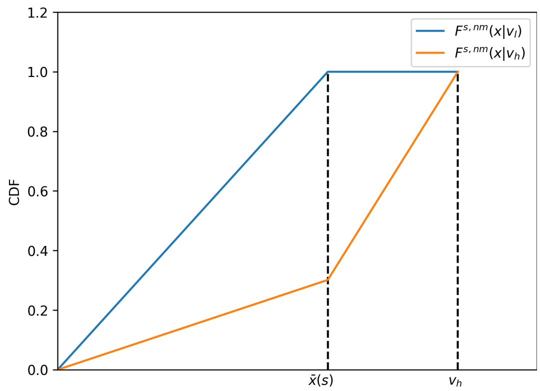

Figure 3.2: non-monotone equilibrium when $\phi(\mu_s(G)) < 0$

type (i.e., probability $p_s(v_l|v_l)$). Because of zero expected payoff, his expected gain from the highest bid, $v_l p_s(v_l v_l)$, must equal the bid itself. The highest bid for a high value type player is small than v_h, as he makes a positive payoff but his gain is exactly v_h.

When $\phi(\mu_s(G)) < 0$, i.e., the monotonicity condition is violated, the unique equilibrium is non-monotone. The low value type still makes zero payoff by uniformly randomizing on an interval starting from 0. The high value type's strategy now takes the form of piecewise-uniform randomization on interval $[0, v_h]$, which implies that he makes zero payoff as well. The highest bid for a high value type in the non-monotone equilibrium is unambiguously larger than that in a monotone equilibrium, which implies a tougher competition for the object when $\phi(\mu_s(G)) < 0$. We learn that $\phi(\mu_s(G)) < 0$ holds when the two value types are close or the affiliation is large. Either scenario would lead to a tougher competition, which consists with the intuition from the characterized equilibrium in Proposition 1.

Corollary 1. *In the posterior all-pay auction game with μ_s,*

1. *if $\phi(\mu_s(G)) \geq 0$, the expected total bids in equilibrium is*

$$R^m(\mu_s) = v_l p_s(v_l|v_l) + \Big(v_h p_s(v_h|v_h) + v_l p_s(v_l|v_l)\Big) \sum_{\omega \in \{G,B\}} \mu_s(\omega) p(v_h|\omega).$$

   The low value type makes zero payoff. The high value type's expected payoff is $v_l \phi(\mu_s(G)) = v_h p_s(v_l|v_h) - v_l p_s(v_l|v_l)$.

2. *if $\phi(\mu_s(G)) < 0$, the expected total bids in equilibrium is*

$$R^{nm}(\mu_s) = \underline{x}(s) + \frac{v_h(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)} \cdot \sum_{\omega \in \{G,B\}} \mu_s(\omega) p(v_h|\omega).$$

Both value types make zero payoff.

Proof. See Appendix. □

Corollary 1 summarizes the expected total bids and players' payoffs in the posterior all-pay auction no matter the equilibrium is monotone ($\phi(\mu_s) \geq 0$)or non-monotone ($\phi(\mu_s) < 0$). In a monotone equilibrium, the high value type's payoff is $v_l\phi(\mu_s)$, which is positive only when $\phi(\mu_s) > 0$. To see the intuition behind the high value type's positive payoff, we need to inspect the effect a value change from v_l to v_h. First of all, this value change is good news for a player since his gain from winning the object will be larger. But because of affiliation, this change also implies that his chance of facing a high value opponent higher, which implies a tougher competition. In the case where $\phi(\mu_s) \geq 0$, $v = v_h/v_l$ is larger in comparison to $p_s(v_l|v_l)/p_s(v_l|v_h)$. That is, the value effect of a change from v_l to v_h is larger than the affiliation effect of that change. Therefore, the high value type enjoys a positive payoff in a monotone equilibrium.

When the $\phi(\mu_s) < 0$, the high value type no longer has a positive payoff, which implies that the monotone equilibrium no longer holds. When $\phi(\mu_s) < 0$, the equilibrium is non-monotone, and both types have zero payoff since the value effect of a type change in this case is smaller than the affiliation effect of the change. The high value type at most is as well off as the low value type, which implies zero payoff for high value type. At first glance, it seems that the organizer can collect more revenue in a non-monotone equilibrium compared to a monotone equilibrium since players' rents are squeezed to zero. However, there exists efficiency loss in a non-monotone equilibrium as the low value may obtain the object when faced with a high value type. This efficiency loss in a non-monotone equilibrium restricts the organizer's ability of extracting surplus. Therefore, the effect of non-monotonicity of equilibrium on the organizer's expected total revenue is ambiguous.

4 Information Disclosure

Now we move to information disclosure problem for the auction organizer. In Stage 5, the organizer's problem is to maximize the ex ante expected total revenue from the all-pay auction by designing the signal about state (or value distribution) optimally. Given signal π, for each signal realization s, a posterior about state is generated. Thus, a signal π actually can be viewed as a distribution τ of posteriors. And the probability of posterior μ_s in τ equals the probability of signal realization s in π. Following the conventional approach introduced by Kamenica and Gentzkow (2011), we transform the organizer's signal choice problem into the problem of choosing distribution of posteriors. Given disclosure policy π, let τ be the distribution of posteriors induced by π. The organizer's problem is transformed as

$$\max_{\tau} \quad \sum_{\mu_s} \tau(\mu_s) R(\mu_s)$$
$$\text{s.t.} \quad \sum_{\mu_s} \tau(\mu_s)\mu_s(\omega) = \mu_0(\omega).$$

$R(\mu_s)$ is the organizer's expected revenue in the posterior game induced by μ. The constraint is required by Bayes' plausible condition.

Given a signal realization s, a posterior about state (i.e., value distribution) μ_s is generated, and the risk-neutral auction organizer collects revenue $R(\mu_s)$ in the posterior all-pay auction game. By Corollary 1, we have either $R(\mu_s) = R^m(\mu_s)$ or $R(\mu_s) = R^{nm}(\mu_s)$, depending on whether $\phi(\mu_s(G)) \geq 0$ or not. Before proceeding to investigate the organizer's optimal information disclosure issue, we first explore the expression of $R(\mu_s)$ by examining the property of $\phi(\mu_s(G))$.

Lemma 1. *Define* $v_0 = 1 + \frac{(\sqrt{\alpha}-\sqrt{1-\beta})^2}{(1-\alpha)\beta}$*. Given posterior* μ*,*

1. *if* $v \geq v_0$*,* $\phi(\mu_s(G)) \geq 0$ *for* $\forall \mu_s(G) \in [0,1]$*, that is, for an all-pay auction with any* μ_s*, the equilibrium is always monotone.*

2. *if* $v < v_0$*, there exists an interval* $(\mu_1^v(G), \mu_2^v(G)) \subset [0,1]$ *such that* $\phi(\mu_s(G)) < 0$ *for* $\forall \mu_s(G) \in (\mu_1^v(G), \mu_2^v(G))$*. That is, for an all-pay auction with* $\mu_s(G) \in (\mu_1^v(G), \mu_2^v(G))$*, the equilibrium must be non-monotone; Otherwise, it is monotone.*

Lemma 1 tells us that when the two value types are sufficiently different such that $v \geq v_0$, we have $\phi(\mu_s) = \frac{v_h}{v_l} \cdot p_s(v_l|v_h) - p_s(v_l|v_l) \geq 0$ holds for all $\mu_s(G)$. By mathematical computation, we obtain that the affiliation level $p_s(v_l|v_l)/p_s(v_l|v_h)$ is single-peaked in $\mu_s(G)$ on interval $[0,1]$ and capped at v_0. Thus, when $v \geq v_0$, we always have $\phi(\mu_s) \geq 0$, that is, the monotonicity condition is always satisfied. In any posterior all-pay auction game induced by such a μ_s, the equilibrium is monotone, and the organizer extracts surplus $R(\mu_s) = R^m(\mu_s)$. However, when $v < v_0$, i.e., the two possible values are sufficiently close, since $p_s(v_l|v_l)/p_s(v_l|v_h)$ is single-peaked with peak at v_0, $\phi(\mu_s)$ changes its sign twice as

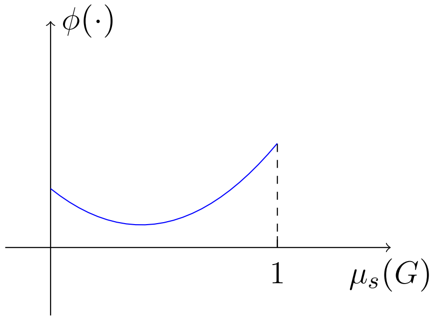

Figure 4.1: $v \geq v_0$

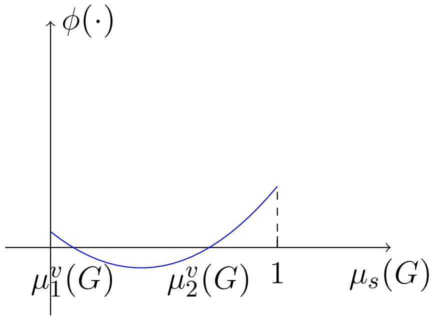

Figure 4.2: $v < v_0$

$\mu_s(G)$ moves along $[0, 1]$. For a posterior game induced by μ_s with $\phi(\mu_s) < 0$, we have $R(\mu_s) = R^{nm}(\mu_s)$ in equilibrium.

4.1 Sufficiently different types: $v \geq v_0$

Consider the scenario in which the two possible types, v_h and v_l, are sufficiently different, i.e., $v \geq v_0$. By Lemma 1, the equilibrium in a posterior game induced by any $\mu_s \in \Delta(\Omega)$ is monotone, which implies that $R(\mu_s) = R^m(\mu_s)$ for any μ_s. Thus, the organizer's problem can be formulated as

$$\begin{aligned} \max_{\tau} \quad & \hat{R}(\tau) = E_\tau R^m(\mu_s) \\ \text{s.t.} \quad & \sum_{\mu} \tau(\mu_s)\mu_s(\omega) = \mu_0(\omega), \forall \omega. \end{aligned} \tag{4.1}$$

Kamenica and Gentzkow (2011) establish the result that the maximum of $E_\tau R^m(\mu_s)$ is exactly the value of the concave closure of $R^m(\cdot)$ at prior μ_0. Therefore, we need to construct the concave closure of $R^m(\cdot)$. In this paper, there are only two states, G and B, it's without loss of generality to denote μ_s with $\mu_s(G)$. Thus, we are actually constructing the concave closure of $R^m(\mu_s(G))$ for $\mu_s(G) \in [0, 1]$.

Lemma 2. *$R^m(\mu_s(G))$ is concave in $\mu_s(G)$.*

Proof. See Appendix. □

Lemma 2 tells us that $R^m(\mu_s(G))$ is concave in $\mu_s(G)$. Thus, its concave closure is exactly itself. By the result established in Kamenica and Gentzkow (2011), the maximum of the organizer's expected total revenue $E_\tau R^m(\mu_s)$ equals $R^m(\mu_0)$, which can be achieved by distribution of posteriors τ^* with $\tau^*(\mu_0) = 1$. Obviously, τ^* is induced by an uninformative signal.

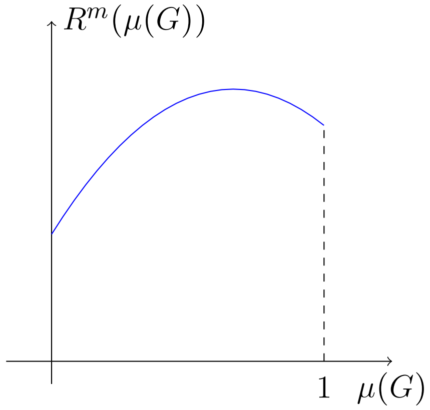

Figure 4.3: Expected revenue in posterior game: $v \geq v_0$

Proposition 2. *If the two value types are sufficiently different, i.e., $v \geq v_0$, the optimal signal is uninformative.*

When $v \geq v_0$, for whatever belief about state, the unique equilibrium in the all-pay auction is monotone. The high value player always bids more than a low value player, and the allocation of the object is efficient. By disclosing information, the trading efficiency is unaffected, but the affiliation of players' values is changed. Although the organizer could benefit from higher affiliation, the indeterministic nature of signal realization fails to guarantee a higher affiliation. Thus, an informative signal isn't necessarily good for the organizer.

4.2 Sufficiently close types: $v \leq v_0$

In the scenario when the two possible types are sufficiently different, i.e., $v < v_0$, there exists an open interval inside $[0, 1]$ on which $\phi(\mu_s(G)) < 0$. In a posterior game induced by such a μ_s, the unique equilibrium is non-monotone. For μ_s with $\mu_s(G)$ outside that interval, we have $\phi(\mu_s(G)) \geq 0$, and the unique equilibrium in the induced posterior game is monotone. Therefore, the organizer's expected revenue from a posterior game induced by μ_s in this scenario is

$$R(\mu_s(G)) = \begin{cases} R^{nm}(\mu_s(G)) & \text{if } \phi(\mu_s(G)) < 0; \\ R^m(\mu_s(G)) & \text{if } \phi(\mu_(G)) \geq 0. \end{cases}$$

Still, following the well-established result in Kamenica and Gentzkow (2011), we need to construct the concave closure of revenue function $R(\mu_s(G))$ on its whole range, i.e., $[0, 1]$. To facilitate the construction of the concave closure, we first examine the continuity of $R(\mu_s(G))$, the result of which is shown in the following lemma.

Lemma 3. *For the μ_s such that $\phi(\mu_s(G)) = 0$, $R^{nm}(\mu_s) = R^m(\mu_s)$.*

Proof. See Appendix. □

From Lemma 3, we learn that $R(\mu_s(G))$ is continuous in $\mu_s(G)$. That is, at the switching point of monotone and non-monotone equilibrium, players' equilibrium strategies generate the same level of expected revenue for the organizer.

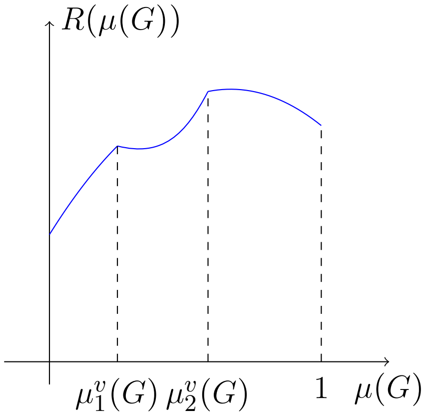

Figure 4.4: Expected revenue in posterior game: $v < v_0$

Now we have the continuity of revenue expression $R(\mu_s(G))$ and the concavity of expression $R^m(\mu_s(G))$ on $[0, 1]$ from Lemma 2, to construct the concave closure, we need to examine how $R^{nm}(\mu_s(G))$ changes on its domain,i.e., the interval on which $\phi(\mu_s(G)) < 0$. However, due to the complexity of revenue expression $R^{nm}(\mu_s(G))$, it's too complicated to identify the shape of $R^{nm}(\mu_s(G))$ with rigorous math on that range. But we know the values $R^{nm}(\mu_s(G))$ takes at two endpoints of its domain, i.e., μ_1 and μ_2, as a result of Lemma 3. Figure 4.4 depicts a shape that $R^{nm}(\mu_s)$ could possibly take.

Lemma 4. *For any μ_s such that $\phi(\mu_s) \leq 0$,*

$$R^{nm}(\mu_s(G)) \leq v_h + (v_h - v_l) \cdot \left[(\beta^2 - (1-\alpha)^2)\mu_s(G) - \beta^2\right].$$

The equality holds if and only if $\phi(\mu_s(G)) = 0$

Proof. See Appendix. □

If there is a linear function which dominates R^{nm} on the range and equals R^{nm} at the two endpoints, then it must be the concave closure of R^{nm}, since linear function is smallest concave function between two points. In Lemma 4, we find a linear function of $\mu_s(G)$, which dominants $R^{nm}(\mu_s(G))$ when $\phi(\mu_s) \le 0$ and equals $R^{nm}(\mu_s(G))$ when $\phi(\mu_s) = 0$. By definition, the linear function is the concave closure of R^{nm} on the range.

Since revenue expression $R^m(\mu_s(G))$ is concave in $\mu_s(G)$ on the whole range, i.e., $[0,1]$, it is also concave on the range where $\phi(\mu_s) \le 0$. By definition of concave closure, we have $R^m(\mu_s(G)) \ge v_h p_s(v_h) + v_l p_s(v_l) + (v_h - v_l) p_s(v_l, v_h)$. Then the concave closure of $R(\mu_s)$ can be easily identified.

Lemma 5. *Define* $\tilde{R} : [0,1] \to [0,+\infty)$ *as:*

$$\tilde{R}(\mu_s(G)) = \begin{cases} v_h p_s(v_h) + v_l p_s(v_l) + (v_h - v_l) p_s(v_l, v_h) & \text{if } \phi(\mu_s) < 0; \\ R^m(\mu_s(G)) & \text{if } \phi(\mu_s) \ge 0. \end{cases}$$

$\tilde{R}$ is the concave closure of R.

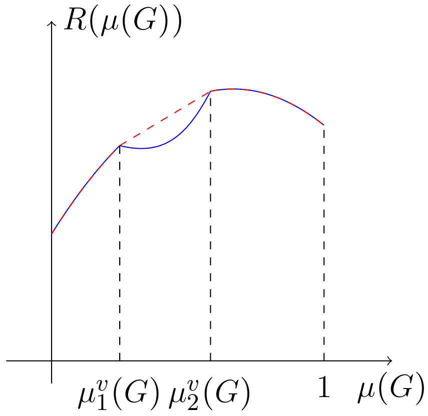

Figure 4.5: Concave closure $\tilde{R}$: $v < v_0$

In Lemma 5, we construct the concave closure of $R(\mu_s)$ for any μ_s. By the result in Lemma 3 and Lemma 4, the concave closure $\tilde{R}$ is continuous as is depicted in Figure 4.5. It coincides with R when $\phi(\mu_s) \ge 0$ and is above R when $\phi(\mu_s) < 0$. By the well-established result in Kamenica and Gentzkow (2011), the optimal signal always exists and the maximum expected total revenue that can be achieved equals $\tilde{R}(\mu_0(G))$, i.e., the value of concave closure at the prior belief μ_0. Specifically, if $\phi(\mu_0) \ge 0$, the maximum expected total revenue is exactly $R(\mu_0)$ since $\tilde{R}(\mu_0) = R(\mu_0)$, that is, the organizer cannot benefit from providing information. If $\phi(\mu_0) < 0$, the maximum achievable expected total revenue $\tilde{R}(\mu_0)$ is greater than $R(\mu_0)$, which implies that the organizer can benefit from disclosing information.

Because of the binary structure of state distribution, the concave closure of expected total revenue has a graphical representation. The distribution of posteriors induced by the optimal signal can be identified directly on the graph.

Proposition 3. *When the two types are relatively close, i.e., $v < v_0$,*

1. *if $\phi(\mu_0) \geq 0$, that is, no disclosure induces a monotone equilibrium, the organizer's optimal signal is uninformative, i.e., no disclosure.*

2. *if $\phi(\mu_0) < 0$, that is, no disclosure induces a non-monotone equilibrium, the organizer's optimal signal generates μ_1^v and μ_2^v.*

Proposition 3 tells us that whether to disclose information about state depends on monotonicity of equilibrium in the original all-pay auction game. If the original game generates a monotone equilibrium, the organizer can't do better by disclosing information. But if a non-monotone equilibrium arises in the original game, it's optimal for the organizer to disclose some information such that a monotone equilibrium arises in the auction game.

Notice that a non-monotone equilibrium arises when players' values are highly affiliated, and competition for the object is so fierce that both players' rents are fully dissipated. While in a monotone equilibrium, a high value player still receives some rent. In this sense, it seems that a non-monotone equilibrium is better than a monotone equilibrium for the organizer, which is against the optimal signal identified in Proposition **??**. However, this is not the case. In a non-monotone equilibrium, a low value type could win the object against a high value type as the supports of their bidding strategies overlap with each other. It leads to an efficiency loss, which is bad news for the organizer. In a monotone equilibrium, such efficiency loss doesn't exist. Therefore, the organizer would be better off in a monotone equilibrium if his gain from higher efficiency dominates his loss from higher rents to players. At the threshold beliefs (i.e., the beliefs at which the equilibrium turns to be non-monotone), the auction game has a monotone equilibrium, in which all types receive zero rent and the allocation is efficient. Thus, in this case where $\phi(\mu_0) < 0$, the optimal signal generates two posteriors, μ_1 and μ_2, and $\phi(\mu_1^v) = \phi(\mu_2^v) = 0$, i.e., the two beliefs that separate belief generating monotone and non-monotone equilibria.

When $\phi(\mu_0(G)) < 0$, we have $v < p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$. Summarizing the results in Proposition 2 and Proposition 3, we have following corollary:

Corollary 2. *When $v \geq p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$, no disclosure is optimal; when $v < p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$, the partial disclosure which generates a posterior distribution μ_1^v and μ_2^v is optimal.*

5 Conclusion

In this paper, we investigate the organizer's optimal public signal about value distribution in a two-player all-pay auction model with affiliated values. We restrict our analysis to two possible value distributions. We find that when the two private values are sufficiently close, i.e., $v < p_{\mu_0}(v_l|v_l)/p_{\mu_0}(v_l|v_h)$, it's optimal signal involves partial disclosure. Otherwise, no disclosure is optimal for the organizer.

The analysis in this paper can be extended to study the private signal about value distribution in a similar auction setting. The difficulty in such a study lies in the characterization of equilibrium in a posterior game. With private persuasion, the two players may observe different signal realization, which in effect doubles the players' types in the posterior all-pay auction game. Also, in this private signal setting, the organizer's signal is formulated differently (e.g. Arieli and Babichenko(2019)). We plan to explore the optimal private signal in our future work and examine whether the organizer can benefit from providing private signals.

Appendix A

Proof of Claim 1: Since players' values are independently and identically drawn from the same distribution determined by state ω and $p_s(v_i|\omega) \neq p_s(v_i|\omega')$ for $\omega \neq \omega'$, we have

$$\begin{aligned} p_s(v_i, v_j) =& \sum_{\omega} p(v_i|\omega)p(v_j|\omega)\mu_s(\omega) \\ <& \sum_{\omega} p(v_i|\omega)\Big(\sum_{\omega} p(v_j|\omega)\mu_s(\omega)\Big)\mu_s(\omega) \\ =& p_s(v_i)p_s(v_j), \end{aligned}$$

for any v_i and v_j. Because there are only two possible values, for any $v_i \neq v_j$, the inequality above implies that

$$\frac{p_s(v_i, v_i) + p_s(v_i, v_j)}{p_s(v_i, v_j)} \geq \frac{p_s(v_i) + p_s(v_j)}{p_s(v_j)}.$$

Subtract 1 on both sides, and rearrange it, we obtain

$$p_s(v_i|v_i) = \frac{p_s(v_i, v_i)}{p_s(v_i)} \geq \frac{p_s(v_i, v_j)}{p_s(v_j)} = p_s(v_i|v_j).$$

Thus, Claim 1 follows.

Proof of Corollary 1:

1) When $\phi(\mu_s) \geq 0$, the equilibrium strategies for both value types are identified in Proposition **??**. It's observed that the two value types employ a uniform bidding strategies on two connected intervals. Thus, it's easy to obtain that the expected bids from the two types are

$$E^{s,m}(x|v_l) = \frac{1}{2} v_l p_s(v_l|v_l);$$

$$E^{s,m}(x|v_h) = \frac{1}{2} v_h p_s(v_h|v_h) + v_l p_s(v_l|v_l).$$

The expected total bids is

$$\begin{aligned} R^m(\mu_s) =& 2 \sum_{\omega \in \{G,B\}} \mu_s(\omega) \underbrace{\Big[p(v_h|\omega)E^{s,m}(x|v_h, s) + p(v_l|\omega)E^{s,m}(x|v_l, s)\Big]}_{\text{revenue when state is } \omega \text{ and posterior is } \mu_s} \\ =& \sum_{\omega \in \{G,B\}} \mu_s(\omega)\Big[p(v_h|\omega)(v_h p_s(v_h|v_h) + v_l p_s(v_l|v_l)) + v_l p_s(v_l|v_l)\Big] \\ =& v_l p_s(v_l|v_l) + \Big(v_h p_s(v_h|v_h) + v_l p_s(v_l|v_l)\Big) \sum_{\omega \in \{G,B\}} \mu_s(\omega)p(v_h|\omega). \end{aligned}$$

2) When $\phi(\mu_s) < 0$, in equilibrium the low value type's strategy is a uniform distribution on $[0, \underline{x}(s)]$, and the high type's strategy is a piecewise-uniform randomization on $[0, v_h]$. For a low value type player, it's very direct and easy to obtain his expected bid, which is

$$E^{s,nm}(x|v_l) = \frac{1}{2}\underline{x}(s),$$

For a high value type player, its piecewise-uniform randomization strategy is identified in Proposition **??**. Let

$$\bar{x}(s) = \frac{v_h v_l [p_s(v_h|v_h) - p_s(v_h|v_l)]}{v_l p_s(v_l|v_l) - v_h p_s(v_l|v_h)}.$$

The expected bid of a high value type player is

$$\begin{aligned} E^{s,nm}(x|v_h) &= \int_0^{\underline{x}(s)} x d\frac{x}{\bar{x}(s)} + \int_{\underline{x}(s)}^{v_h} x d\frac{x - v_h p_s(v_l|v_h)}{v_h p_s(v_h|v_h)} \\ &= \frac{1}{2\bar{x}(s)} \cdot \underline{x}^2(s) + \frac{1}{v_h p_s(v_h|v_h)} \cdot \frac{1}{2}(v_h^2 - \underline{x}^2(s)) \\ &= \frac{1}{2} \cdot \frac{1}{\bar{x}(s)} \cdot \underline{x}^2(s) + \frac{1}{v_h p_s(v_h|v_h)} \cdot \frac{1}{2}(v_h - \underline{x}(s))(v_h + \underline{x}(s)). \end{aligned}$$

Substitute the expression of $\underline{x}(s)$ into the the expression above, we have

$$E^{s,nm}(x|v_h) = \frac{1}{2\bar{x}(s)} \cdot \underline{x}^2(s) + \frac{1}{2} \cdot \frac{(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}(v_h + \underline{x}(s)).$$

On the other hand, by the definition of $\underline{x}(s)$ and $\bar{x}(s)$, we have

$$\underline{x}(s)/\bar{x}(s) = \frac{v_l p_s(v_l|v_l) - v_h p_s(v_l|v_h)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}.$$

Substitute it into the expression of $E^{s,nm}(x|v_h)$ we obtain

$$\begin{aligned} 2E^{s,nm}(x|v_h) &= \frac{v_l p_s(v_l|v_l) - v_h p_s(v_l|v_h)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)} \cdot \underline{x}(s) + \frac{(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}(v_h + \underline{x}(s)) \\ &= \underline{x}(s) + \frac{v_h(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}, \end{aligned}$$

which implies that

$$E^{s,nm}(x|(v_h,)) = \frac{1}{2}\underline{x}(s) + \frac{1}{2}\frac{v_h(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}.$$

Thus, the expected total bids is

$$\begin{aligned} R^{s,nm}(\mu_s) &= 2 \sum_{\omega \in \{G,B\}} \mu_s(\omega) \underbrace{\left[p(v_h|\omega)E^{s,nm}(x|v_h, s) + p(v_l|\omega)E^{s,nm}(x|v_l, s)\right]}_{\text{revenue when state is } \omega \text{ and posterior is } \mu_s} \\ &= \underline{x}(s) + \frac{v_h(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)} \cdot \sum_{\omega \in \{G,B\}} \mu_s(\omega) p(v_h|\omega). \end{aligned}$$

Proof of Lemma 1:

1. **Step 1**:

$$\phi(\mu_s) = v \cdot \underbrace{\frac{\alpha(1-\alpha)\mu_s(G) + \beta(1-\beta)(1-\mu_s(G))}{\alpha\mu_s(G) + (1-\beta)(1-\mu_s(G))}}_{p_s(v_l|v_h)} - \underbrace{\frac{(1-\alpha)^2\mu_s(G) + \beta^2(1-\mu_s(G))}{(1-\alpha)\mu_s(G) + \beta(1-\mu_s(G))}}_{p_s(v_l|v_l)}.$$

Rewrite function ϕ as

$$\phi(x) = (v-1)(1-\alpha) + (\alpha+\beta-1)(1-x)\left[\frac{v(1-\beta)}{\alpha x + (1-\beta)(1-x)} - \frac{\beta}{(1-\alpha)x + \beta(1-x)}\right]$$

The first order derivative is

$$\phi'(x) = (\alpha+\beta-1) \cdot \left\{\frac{(1-\alpha)\beta}{[(1-\alpha)x + \beta(1-x)]^2} - \frac{v\alpha(1-\beta)}{[\alpha x + (1-\beta)(1-x)]^2}\right\}.$$

Then $\phi'(x) \geq 0$ if and only if

$$\frac{\sqrt{(1-\alpha)\beta}}{(1-\alpha)x + \beta(1-x)} \geq \frac{\sqrt{v\alpha(1-\beta)}}{\alpha x + (1-\beta)(1-x)},$$

i.e.,

$$(\alpha+\beta-1)(\sqrt{(1-\alpha)\beta} + \sqrt{v}\sqrt{\alpha(1-\beta)})x \geq \sqrt{\beta(1-\beta)}(\sqrt{v}\sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}).$$

Since $\alpha+\beta \geq 1$, we always have $\sqrt{v}\sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)} \geq 0$. That is, the left-hand side of the inequality above is always positive. Thus, there exists x_0 such that $\phi'(x_0) = 0$. And it's also unqiue.

Step 2: $\phi'(x_0) = 0$ implies

$$\frac{\sqrt{(1-\alpha)\beta}}{(1-\alpha)x_0 + \beta(1-x_0)} = \frac{\sqrt{v\alpha(1-\beta)}}{\alpha x_0 + (1-\beta)(1-x_0)}. \tag{A.1}$$

$\phi(x_0) = 0$ implies

$$v \cdot \frac{\alpha(1-\alpha)x_0 + \beta(1-\beta)(1-x_0)}{\alpha x_0 + (1-\beta)(1-x_0)} = \frac{(1-\alpha)^2 x_0 + \beta^2(1-x_0)}{(1-\alpha)x_0 + \beta(1-x_0)}. \tag{A.2}$$

Combine equation (A.1) and (A.2), we have

$$\sqrt{v} \cdot \frac{\alpha(1-\alpha)x_0 + \beta(1-\beta)(1-x_0)}{(1-\alpha)^2 x_0 + \beta^2(1-x_0)} = \frac{\sqrt{\alpha(1-\beta)}}{\sqrt{(1-\alpha)\beta}}.$$

Rearrange it, we get

$$\begin{aligned}&\alpha^{1/2}(1-\alpha)^{3/2}\Big(\sqrt{v}\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\Big)x_0\\=&\beta^{3/2}(1-\beta)^{1/2}\Big(\sqrt{\alpha\beta}-\sqrt{v}\sqrt{(1-\alpha)(1-\beta)}\Big)(1-x_0).\end{aligned}\tag{A.3}$$

Rearrange equation (A.1), we have

$$\sqrt{\alpha(1-\alpha)}\Big(\sqrt{\alpha\beta}-\sqrt{v}\sqrt{(1-\alpha)(1-\beta)}\Big)x_0=\sqrt{\beta(1-\beta)}\Big(\sqrt{v}\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\Big)(1-x_0).\tag{A.4}$$

Combine equation (A.3) and (A.4), it follows that

$$\frac{(1-\alpha)\Big(\sqrt{v}\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\Big)}{\Big(\sqrt{\alpha\beta}-\sqrt{v}\sqrt{(1-\alpha)(1-\beta)}\Big)}=\frac{\beta\Big(\sqrt{\alpha\beta}-\sqrt{v}\sqrt{(1-\alpha)(1-\beta)}\Big)}{\Big(\sqrt{v}\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\Big)},$$

which is

$$\sqrt{v}\sqrt{(1-\alpha)\beta}=1-\sqrt{\alpha(1-\beta)}.$$

Thus we have

$$\begin{aligned}v_0=&\frac{1+\alpha(1-\beta)-2\sqrt{\alpha(1-\beta)}}{(1-\alpha)\beta}\\=&1+\frac{1-\beta+\alpha-2\sqrt{\alpha(1-\beta)}}{(1-\alpha)\beta}\geq 1.\end{aligned}$$

If $v\geq v_0$, then $\phi(x_0)\geq 0$. Since $\phi(x_0)$ is the minimum point of $\phi(x)$ on $[0,\infty)$, we have $\phi(\mu(G))\geq 0$ always holds.

2. If $v<v_0$, then the minimum point $\phi(x_0)<0$.

Claim 2. *If $v<v_0$, $\phi'(1)>0$ and $\phi(1)>0$.*

Proof. From the proof of Lemma 1,

$$\phi'(1)=(\alpha+\beta-1)\cdot(\frac{\beta}{1-\alpha}-\frac{v(1-\beta)}{\alpha}).$$

Thus, $\phi'(1)\geq 0$ if and only if $v\leq\frac{\alpha\beta}{(1-\alpha)(1-\beta)}=v_{00}$. Then we have

$$v_{00}-v_0=\frac{\alpha+\beta-1}{(1-\alpha)(1-\beta)}-\frac{(\sqrt{\alpha}-\sqrt{1-\beta})^2}{(1-\alpha)\beta}.$$

Rearrange it, we get

$$v_{00} - v_0 = \frac{\alpha+\beta-1}{(1-\alpha)(1-\beta)} - \frac{(\alpha+\beta-1)(\sqrt{\alpha}-\sqrt{1-\beta})}{(1-\alpha)\beta(\sqrt{\alpha}+\sqrt{1-\beta})}$$

$$v_{00} - v_0 = \frac{\alpha+\beta-1}{1-\alpha}\left\{\frac{1}{1-\beta} - \frac{\sqrt{\alpha}-\sqrt{1-\beta}}{\beta(\sqrt{\alpha}+\sqrt{1-\beta})}\right\}$$

Since

$$\begin{aligned}\beta(\sqrt{\alpha}+\sqrt{1-\beta}) - (1-\beta)(\sqrt{\alpha}-\sqrt{1-\beta}) &\geq 2\beta\sqrt{1-\beta} - (1-\beta)(\sqrt{\alpha}-\sqrt{1-\beta}) \\ &\geq \sqrt{1-\beta}\left\{\beta - \sqrt{1-\beta}\sqrt{\alpha}+1\right\} > 0,\end{aligned}$$

we have $v_{00} > v_0$. Thus, when $v < v_0 < v_{00}$, we have $\phi'(1) \geq 0$. On the other hand, $\phi(1) = (v-1)(1-\alpha) > 0$. □

By the proof of Claim 2 and the shape of ϕ, we learn that there exist x_1 and x_2 in $(0,1)$ such that $\phi(x_1) = 0$ and $\phi(x_2) = 0$. And for any $\mu_s(G) \in (x_1, x_2)$, $\phi(\mu_s) < 0$.

Proof of Lemma 2: Recall that

$$R^m(\mu_s) = (p_s(v_h|v_h)v_h + p_s(v_l|v_l)v_l)\Big[\mu_s(G)\alpha + (1-\mu_s(G))(1-\beta)\Big] + p_s(v_l|v_l)v_l.$$

Let $x = \mu_s(G)$, and

$$p_s(v_l|v_l) = \frac{(1-\alpha)^2 x + \beta^2(1-x)}{(1-\alpha)x + \beta(1-x)} = g(x);$$

$$p_s(v_h|v_h) = \frac{\alpha^2 x + (1-\beta)^2(1-x)}{\alpha x + (1-\beta)(1-x)} = h(x).$$

Then we have

$$\frac{R^m(x)}{v_l} = (h(x)\cdot v + g(x))\Big[x\alpha + (1-x)(1-\beta)\Big] + g(x).$$

The first order derivative is

$$\frac{dR^m(x)/v_l}{dx} = (h'(x)\cdot v + g'(x))\Big[x\alpha + (1-x)(1-\beta)\Big] + (h(x)\cdot v + g(x))(\alpha+\beta-1) + g'(x).$$

The second order derivative is

$$\frac{d^2R^m(x)/v_l}{dx^2} = (h''(x)\cdot v + g''(x))\Big[x\alpha + (1-x)(1-\beta)\Big] + 2(h'(x)\cdot v + g'(x))(\alpha+\beta-1) + g''(x).$$

On the other hand,

$$h'(x) = \frac{(\alpha+\beta-1)\alpha(1-\beta)}{[\alpha x + (1-\beta)(1-x)]^2},$$
$$h''(x) = \frac{(\alpha+\beta-1)^2\alpha(1-\beta)(-2)}{[\alpha x + (1-\beta)(1-x)]^3},$$

it follows that

$$h''(x)\Big[x\alpha + (1-x)(1-\beta)\Big] + 2h'(x)(\alpha+\beta-1) = 0.$$

Thus, we have

$$\begin{aligned}\frac{d^2R^m(x)/v_l}{dx^2} =& g''(x)\Big[x\alpha + (1-x)(1-\beta)\Big] + 2g'(x)(\alpha+\beta-1) + g''(x) \\ =& -g''(x)\Big[(1-\alpha)x + \beta(1-x)\Big] + 2g'(x)(\alpha+\beta-1) + 2g''(x).\end{aligned} \tag{A.5}$$

Since

$$g'(x) = \frac{-(\alpha+\beta-1)(1-\alpha)\beta}{[(1-\alpha)x + \beta(1-x)]^2},$$
$$g''(x) = \frac{-2(\alpha+\beta-1)^2(1-\alpha)\beta}{[(1-\alpha)x + \beta(1-x)]^3},$$

it follows that

$$-g''(x)[(1-\alpha)x + \beta(1-x)] + 2g'(x)(\alpha+\beta-1) = 0.$$

Then we have

$$\frac{d^2R^m(x)/v_l}{dx^2} = 2g''(x) \le 0.$$

Therefore, $R^m(\mu)$ is concave.

Proof of Lemma 3: When $\phi(\mu_s) = vp_s(v_l|v_h) - p_s(v_l|v_l) = 0$, we have $v_hp_s(v_l|v_h) = v_lp_s(v_l|v_l)$. At the belief μ_s,

$$\begin{aligned}\underline{x}_s =& v_l \cdot \frac{v_h[p_s(v_h|v_h) - p_s(v_h|v_l)]}{v_hp_s(v_h|v_h) - v_lp_s(v_h|v_l)} \\ =& v_l \cdot \Big[1 - \frac{(v_h - v_l)p_s(v_h|v_l)}{v_h - v_l - (v_hp_s(v_l|v_h) - v_lp_s(v_l|v_l))}\Big] \\ =& v_l(1 - p_s(v_h|v_l)) = v_lp_s(v_l|v_l).\end{aligned}$$

That is, at $\phi(\mu_s) = 0$, $F^{s,m}(\cdot|v_l)$ and $F^{s,nm}(\cdot|v_l)$ are the same.

Now consider the strategies for a high value player in the two kinds of equilibrium. When $\phi(\mu_s) = 0$, we have

$$F^{s,nm}(x|v_h) = \begin{cases} 0 & \text{on } [0, v_lp_s(v_l|v_l)] \\ \frac{x - v_lp_s(v_l|v_l)}{v_hp_s(v_h|v_h)} & \text{on } [v_lp_s(v_l|v_l), v_h]. \end{cases}$$

Recall that

$$F^{s,m}(x|v_h) = \frac{x - v_l p_s(v_l|v_l)}{v_h p_s(v_h|v_h)} \text{ on } [v_l p_s(v_l|v_l), v_l p_s(v_l|v_l) + v_h p_s(v_h|v_h)].$$

The highest bid $v_l p_s(v_l|v_l) + v_h p_s(v_h|v_h) = v_h p_s(v_l|v_h) + v_h p_s(v_h|v_h) = v_h$ when $\phi(\mu_s) = 0$. That is, $F^{s,m}(\cdot|v_h)$ and $F^{s,nm}(\cdot|v_h)$ are the same at $\phi(\mu_s) = 0$.

Since the bidding strategies for the two value types are the same in the two equilibria at $\phi(\mu_s) = 0$, we must also have $R^m(\mu_s) = R^{nm}(\mu_s)$ at the belief μ_s.

Proof of Lemma 4: By Corollary 1, the organizer's expected revenue in a non-monotone equilibrium induced by μ_s is

$$\begin{aligned} R^{nm}(\mu_s) =& \underbrace{\frac{v_l v_h[p_s(v_h|v_h) - p_s(v_h|v_l)]}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)}}_{\underline{x}(s)} + \frac{v_h(v_h - v_l)p_s(v_h)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)} \\ =& v_l + \frac{(v_h - v_l)}{v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)} \cdot (v_h p_s(v_h) - v_l p_s(v_h|v_l)). \end{aligned}$$

Since

$$\begin{aligned} v_l \phi(\mu_s) =& v_h p_s(v_l|v_h) - v_l p_s(v_l|v_l) \\ =& v_h - v_l - (v_h p_s(v_h|v_h) - v_l p_s(v_h|v_l)), \end{aligned}$$

we have

$$\begin{aligned} R^{nm}(\mu_s) =& v_l + \frac{(v_h - v_l)}{v_h - v_l - v_l\phi(\mu_s)} \cdot (v_h p_s(v_h) - v_l p_s(v_h|v_l)) \\ =& v_l + (v_h - v_l)p_s(v_h) + \frac{(v_h - v_l)}{v_h - v_l - v_l\phi(\mu_s)} \cdot \Big\{(v_l + v_l\phi(\mu_s))p_s(v_h) - v_l p_s(v_h|v_l)\Big\}. \end{aligned}$$

Since $v_l + v_l\phi(x) = v_h p_s(v_l|v_h) + v_l p_s(v_h|v_l)$, we have

$$\begin{aligned} R^{nm}(\mu_s) =& v_l + (v_h - v_l)p_s(v_h) + \frac{(v_h - v_l)}{v_h - v_l - v_l\phi(\mu_s)} \cdot \Big\{\Big(v_h p_s(v_l|v_h) + v_l p_s(v_h|v_l)\Big)p_s(v_h) - v_l p_s(v_h|v_l)\Big\} \\ =& v_l + (v_h - v_l)p_s(v_h) + \frac{(v_h - v_l)}{v_h - v_l - v_l\phi(\mu_s)} \cdot \Big\{v_h p_s(v_l|v_h)p_s(v_h) - v_l p_s(v_h|v_l)p_s(v_l)\Big\}, \end{aligned}$$

which is

$$R^{nm}(\mu_s) = v_l + (v_h - v_l)p_s(v_h) + \frac{(v_h - v_l)}{v_h - v_l - v_l\phi(\mu_s)} \cdot (v_h - v_l)p_s(v_l, v_h).$$

Since $\phi(\mu_s) \leq 0$, it follows that

$$\begin{aligned} R^{nm}(\mu_s) \leq& v_l + (v_h - v_l)p_s(v_h) + \frac{(v_h - v_l)}{v_h - v_l} \cdot (v_h - v_l)p_s(v_l, v_h) \\ =& v_l + (v_h - v_l)p_s(v_h) + (v_h - v_l)p_s(v_l, v_h). \end{aligned}$$

The equality holds when $\phi(\mu_s) = 0$.

References

[1] Amann, E., & Leininger, W. (1996). Asymmetric all-pay auctions with incomplete information: the two-player case. *Games and Economic Behavior*, 14(1), 1-18.

[2] Barut, Y., & Kovenock, D. (1998). The symmetric multiple prize all-pay auction with complete information. *European Journal of Political Economy*, 14(4), 627-644.

[3] Baye, M. R., Kovenock, D., & De Vries, C. G. (1993). Rigging the lobbying process: an application of the all-pay auction. *American Economic Review*, 83(1), 289-294.

[4] Baye, M. R., Kovenock, D., & De Vries, C. G. (1996). The all-pay auction with complete information. *Economic Theory*, 8(2), 291-305.

[5] Chen, Z. C. (2019). Information Disclosure in Contests: Private versus Public Signals. *Available at SSRN* 3326462.

[6] Chen, Kuang, and Zheng (2017a).Bayesian Persuasion in All-pay Auction Contests. *working paper.*

[7] Chi, C. K., Murto, P., & Välimäki, J. (2019). All-pay auctions with affiliated binary signals. *Journal of Economic Theory*, 179, 99-130.

[8] Fu, Q., Jiao, Q., & Lu, J. (2011). On disclosure policy in contests with stochastic entry. *Public Choice*, 148(3-4), 419-434.

[9] Fu, Q., Jiao, Q., & Lu, J. (2014). Disclosure policy in a multi-prize all-pay auction with stochastic abilities. *Economics Letters*, 125(3), 376-380.

[10] Kuang, Zhao, and Zheng (2019). Information design in all-pay auction contests. *working paper.*

[11] Hillman, A. L., & Riley, J. G. (1989). Politically contestable rents and transfers. *Economics & Politics*, 1(1), 17-39.

[12] Liu, Z., & Chen, B. (2016). A symmetric two-player all-pay contest with correlated information. *Economics Letters*, 145, 6-10.

[13] Lu, J., Ma, H., & Wang, Z. (2018). Ranking Disclosure Policies In All-Pay Auctions. *Economic Inquiry*, 56(3), 1464-1485.

[14] Lu, J., & Parreiras, S. O. (2017). Monotone equilibrium of two-bidder all-pay auctions Redux. *Games and Economic Behavior*, 104, 78-91.

[15] Mathevet, L., Perego, J., & Taneva, I. (2017). On information design in games. Unpublished paper, Department of Economics, New York University.[1115].

[16] ilgrom, P. R., & Weber, R. J. (1982). A theory of auctions and competitive bidding. *Econometrica: Journal of the Econometric Society*, 1089-1122.

[17] Morath, F., & Münster, J. (2008). Private versus complete information in auctions. *Economics Letters*, 101(3), 214-216.

[18] Kamenica, E., & Gentzkow, M. (2011). Bayesian persuasion. *American Economic Review*, 101(6), 2590-2615.

[19] Kamenica, E. (2018). Bayesian persuasion and information design. *Annual Review of Economics*, 11.

[20] Konrad, K. A., & Morath, F. (2018). To deter or to moderate? Alliance formation in contests with incomplete information. *Economic Inquiry*, 56(3), 1447-1463.

[21] Kovenock, D., Morath, F., & Münster, J. (2015). Information sharing in contests. *Journal of Economics & Management Strategy*, 24(3), 570-596.

[22] Krishna, V., & Morgan, J. (1997). An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory*, 72(2), 343-362.

[23] Rentschler, L., & Turocy, T. L. (2016). Two-bidder all-pay auctions with interdependent valuations, including the highly competitive case. *Journal of Economic Theory*, 163, 435-466.

[24] Siegel, R. (2014). Asymmetric all-pay auctions with interdependent valuations. *Journal of Economic Theory*, 153, 684-702.

[25] Szech, N. (2011). Asymmetric all-pay auctions with two types. University of Bonn, *Discussion paper*, January.

[26] Zhang, J., & Zhou, J. (2016). Information disclosure in contests: A Bayesian persuasion approach. *The Economic Journal*, 126(597), 2197-2217.